Symmetry defects and broken symmetry. Configurations Hidden symmetry*

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This paper is an introduction to the study of spontaneous symmetry breaking and topological classification of defects. The latter topic has aits foundation in the former one; both subjects requires some mathematics not familiar to many physicists: group action and homotopy theory. These mathematics are introduced from examples and their main results, to be used for physics, are explained. It is hoped that this paper will enable the non specialist to read the physics literature which has very recently appeared on topological classification of symmetry defects. Some appendices (A,C,D,F,G) gives more mathematical details; the other appendices (B,E,H) explain some applications to different fields of physics, while mesomorphic phases are treated in the text (He^3 superfluid phases have been treated by Mermin, 1979).

CONTENTS

Within the last three years a new topic seems to have arisen in condensed matter physics: application of homotopy theory to the study of defects in ordered media.[1] Of course physicists studying crystal dislocations by the Burgers circuit or the Volterra construction were using homotopy just as Molière's Monsieur Jourdain was "speaking in prose" (Kléman, 1977). But the explicit use of powerful topological methods has brought a new understanding to the subject and produced some theoretical predictions.

Homotopy theory has also been used explicitly by physicists in other fields: for the "kinks" of Finkelstein (Finkelstein, 1966; Finkelstein and Rubinstein, 1968), for the classification of t'Hooft–Polyakov monopoles (Tyupkin *et al.*, 1975; Monastyrskii and Perelomov, 1975), and for instantons (Beliavin *et al.*, 1975). A common mathematical technique can be applied successfully to these different domains of physics because they all share in the occurrence of broken symmetry. (This is at least my personal opinion.)

Of course one could write a book with the same title as this paper. Although the paper gives, I hope, a fairly complete list of references for this new topic born three years ago, its purpose is not to give a review of the existing literature, but rather to enable (and encourage) the reader to go to the original papers. For this I try to present here the simple, but fundamental concepts underlying the application of homotopy theory to symmetry defects, broken symmetry, and hidden symmetry. I also list the results obtained; it must be clear to the reader that the subject is not closed; it is just opening, and many questions will appear in this paper. What is sure is that the topological approach will become the classical introduction to the study of defects in ordered media. Will it be fruitful? I would guess so; I hope to help the reader to form his own opinion.

I. BROKEN SYMMETRY

A. Historical introduction

In most civilizations, the earliest pottery decoration is generally in geometric patterns; since prehistoric times men have been very sensitive to symmetry con-

*A complementary review of homotopy theory was published by N. D. Mermin in Rev. Mod. Phys. 51, 591.

[1]A brief bibliography tracing the rise of this topic would include the following references: Rogula, 1976; Toulouse and Kléman, 1976; Toulouse, 1976; Volovik and Mineev, 1976b, Kléman, Michel and Toulouse, 1977; Kléman, 1977; Toulouse, 1977; Poenaru and Toulouse, 1977; Kléman and Michel, 1977; Kukula, 1977; Michel, 1977; Volovik and Mineev, 1977a, 1977b; Cross and Brinkman, 1977; Garel, 1978; Kléman and Michel, 1978; Mermin, 1978; Mermin *et al.*, 1978; Bouligand *et al.*, 1978; Poenaru and Toulouse, 1979; Volovik and mineev, 1979.

cepts. The 17 crystallographic groups in two dimensions appeared in decorations (e.g., in the Granada Alhambra) long before their list was established scientifically. Also, from high school the majority of us were taught how to exploit the symmetry of a problem: in general it greatly simplifies the search for solutions. What a shock then to discover that the equilibrium state of the ozone molecule O_3, made of three identical oxygen atoms, is not an equilateral triangle but only an isosceles with an angle of 63° 20′ (±30′) and two of 58° 20′ (±30′).

Of course symmetries play a great role in science, and it is fundamental to discover and to use the symmetry of the laws of nature (Galilean invariance, Poincaré invariance, gauge invariance, etc.). However, the symmetry of physical states might be much smaller: For instance, the interactions between atoms or ions are Euclidean invariant (i.e., invariant by translations and rotations), but crystals do exist and their groups of invariance (the 230 classes of crystallographic groups) contain only discrete translations and rotations. Indeed symmetrical problems might have solutions with different and lesser symmetries. During its evolution (e.g., through phase transitions) a physical system may pass from a symmetric to a less symmetric state: symmetry is spontaneously broken.

Some examples of broken symmetry are fairly obvious. Consider a straight homogeneous iron bar whose constant section is a circle. Compress it by two forces, equal but opposite, directed along the bar and applied at its extremities. The symmetry group of the problem is $D_{\infty h}$; it is generated by the group of rotations around the axis of the bar $[=C_\infty = SO(2)]$ and the reflection h through the plane[2] perpendicular to the bar at its middle, or reflections v through a plane containing the axis of the bar. (Note that the group contains all reflections v and that the product of a reflection v and the reflection h is the rotation of π around the intersection line of the two planes.) When the compression becomes large enough, the bar will no longer stay straight but will flex, and its equilibrium shape is then a sine curve. This solution is also contained in the partial differential equation governing the problem of elasticity, as demonstrated by Euler in the eighteenth century.

As far as I know, the next historical example was found by Jacobi in 1834 (Jacobi, 1834). Consider a rotating mass of an incompressible fluid in gravitating self-interaction. Newton had already proved that the equilibrium figure is an oblate ellipsoid axially symmetric through the rotation axis; he applied it to the Earth and got a good approximation of its shape. The symmetry group of its vertical angular momentum is $C_{\infty h}$. However, when one puts the problem in equations (see Appendix B) one sees that the shape of the surface depends only on the square of the angular momentum, so the symmetry group of this problem is also $D_{\infty h}$. When the angular momentum J is large enough, another type of solution appears and is more stable: the equilibrium shape is a three-unequal-axis ellipsoid.[3] With increasing J other types of solutions with convection currents can appear (see Appendix A for their symmetry); such solutions were first studied by Riemann and Dedekind and they compete with the hydrostatic equilibrium in the rotating frame. Poincaré (1885) found a new broken symmetry at higher J: one half of the long equatorial axis becomes longer than the other half (there is no center of symmetry left). [Later E. Cartan (1922) proved the stability of this solution.] Poincaré had even found an enumerable infinity of successive symmetry breakings. Recently it was shown (Constantinescu *et al.*, 1979) that the Jacobi solution is only the first of another infinite family of symmetry breakings, from $D_{\infty h}$ into D_{nh} (one for each n).

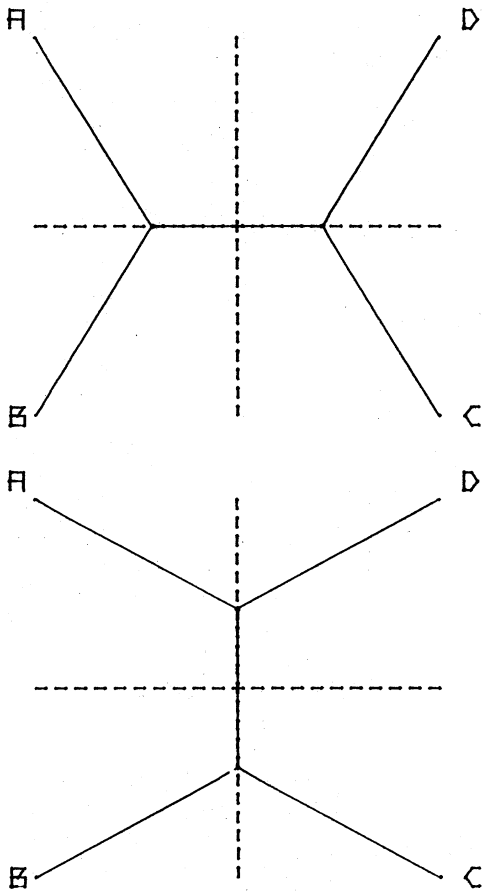

FIG. 1. Among all possible graphs joining the four vertices A, B, C, D of a square, (a) and (b) are the shortest ones. The symmetry group of the square is C_{4v}. The graphs (a) and (b) have less symmetry: their little group is C_{2v}. They form an orbit of C_{4v}; i.e., they are transformed into each other by any operation of C_{4v} which does not belong to C_{2v} (for instance the rotation by $\pi/2$ around the center of the square $ABCD$).

B. A nutshell example of symmetry breaking

The above two examples conclude my historical introduction. They and a third easy problem that I suggest

[2]The traditional notation for these groups assumes the bar to be vertical and this plane, therefore, to be horizontal. See Appendix A for the complete description of closed subgroups of $O(3)$.

[3]Its symmetry group is D_{2h}: all these groups are defined in Appendix A.

to the reader are sufficient to illustrate the simple but fundamental notions of group theory we shall need to use throughout this article.

Problem: Four towns are situated at the vertices of a square with μ miles per side. What is the shortest road system which could join these four towns?

I leave to the reader to prove that a shortest solution is given by Fig. 1(a). It represents $(1+\sqrt{3})\mu = 2.732\mu$ miles of road (instead of $2\sqrt{2}\,\mu = 2.828\,\mu$ for the two diagonals of the square). Of course another solution (Fig. 1(b)) is obtained from Fig. 1(a) by a rotation of $\pi/2$. We also remark that the problem has the symmetry of the square: C_{4v} (C_4 generated by the rotation of $2\pi/4$ and 4 symmetry axes) while the solutions have only the symmetry C_{2v} (rotations of a multiple of $2\pi/2 = \pi$ and 2 orthogonal symmetry axes). It is by using the rotation of $\pi/2$ (or any operation of C_{4v} which is not in C_{2v}) that we pass from one to the other equivalent solution.

We see that if a problem has a symmetry group G, the set of all its solutions is *invariant* under G, but a given solution s might be invariant only under a subgroup G_s of G; G_s is called the isotropy group or the little group of s. Then by the action of the group G on s one can build a family of solutions: it is called the orbit of s and denoted by $G(s)$. By construction this set is invariant under G. We remark that if G is a finite group and $|G|$ and $|G_s|$ are, respectively, the number of elements of G and of G_s, the number of elements of $G(s)$ is $|G|/|G_s|$; in our third example of the road system, $|C_{4v}| = 8$, $|C_{2v}| = 4$; so we have two solutions. When G and G_s are topological groups we have an example of a principal fiber bundle; as we shall explain later G is the bundle, G_s the fiber, and $G(s)$ the base. When G is a Lie group and G_s a closed subgroup, the dimensions of the manifolds $G, G_s, G(s)$ satisfy Eq. (3′) below. I leave the reader to verify the nature of broken symmetry in our examples.

Example 1 (the iron bar): $G_s \sim C_{2v}$ generated by the reflection h and the rotation of π around the horizontal line in the plane which is perpendicular to the bar. $G(s)$ can be represented by a horizontal circle, centered on the midpoint of the bar; the topology of such a circle is often denoted by S_1, the one-dimensional sphere.[4]

Example 2 (Jacobi's rotating ellipsoid): G_s is D_{2h} (defined in Appendix A), and $G(s)$ can be represented by a horizontal circle positioned at the extremes of the long axis of the equatorial ellipse of the Jacobi ellipsoid with the two points on a same diameter identified. This is P_1, the one-dimensional projective space. (The definition of P_n from S_n is similar.)

Often the set $\mathcal{S}$ of solutions is infinite; the solutions depend on parameters (e. g., J^2 in the third example) so $\mathcal{S}$ has generally a topology; it can even be a smooth manifold or a vector space. The fundamental concepts describing such group actions appear through all physics. We review them here.[5]

C. The basic concepts of group action

Given mathematical objects with the same mathematical structures (e.g., a set, a topological space, a smooth manifold, a group, a vector space, etc.) a map f from one object M to another M' which preserves the structure is denoted by $M \xrightarrow{f} M'$ and is simply called nowadays a *morphism* (before, there were as many names as structures: map, continuous map, smooth map, homomorphism, linear map). When the morphism is one-to-one onto (one says "bijective"), it is inversible and it becomes an isomorphism. When $M' = M$ the set of isomorphisms form a group $\mathrm{Aut}M$, the group of automorphisms on M. An *action* of G on M is defined by a homomorphism $G \xrightarrow{f} \mathrm{Aut}M$. For example, when M is the n-dimensional real vector space $\mathcal{E}_n$, $\mathrm{Aut}\mathcal{E}_n = GL(n,R)$ the n-dimensional general linear group on R; it contains all inversible $n \times n$ matrices; an action of G on $\mathcal{E}_n$ is a linear representation of G. The first question to settle is, when are two G actions equivalent? Given two actions $G \xrightarrow{f} \mathrm{Aut}M$ and $G \xrightarrow{f'} \mathrm{Aut}M'$ on two mathematical objects of the same nature, one says that a morphism $M \xrightarrow{\phi} M'$ is equivariant if for all $g \in G$ (denoted by $\forall g \in G$)

$$\phi \circ f(g) = f'(g) \circ \phi \,. \tag{1}$$

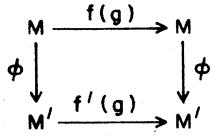

Diagram 1

Another way to express (1) is to say that the diagram 1 of morphisms is commutative. The two G actions f and f' are *equivalent* if and only if there exists an equivariant isomorphism between them. (The reader can check that for linear representations this is the usual definition of equivalence.)

A warning: on the same set one may consider several structures. For instance G acts naturally on itself by $G \xrightarrow{\Phi} \mathrm{Aut}G$ where $\Phi(g)x = gxg^{-1}$; $\Phi(g)$ is an inner automorphism; $\mathrm{Im}\Phi$ (the image of the homomorphism Φ) is the group of inner automorphisms, and $\mathrm{Ker}\Phi$ (the kernel of Φ is the set of elements of G which yield the trivial automorphism) is $\mathcal{C}(G)$, the center of G. The orbit $G(x)$, i.e., the set $\{gxg^{-1}, g \in G\}$ is called the conjugation class of x, and the little group G_x is called (for this action) the centralizer $\mathcal{C}_G(x)$. But one can just consider the set $\tilde{G}$ of elements of G, forgetting the group law ($\mathrm{Aut}\tilde{G}$ is just the group of permutations of the elements of G); then an interesting action of G on $\tilde{G}$ is that by left translations $G \xrightarrow{\tau} \mathrm{Aut}\tilde{G}$, i.e., $\tau(g)x = gx$. Then G is a unique orbit for this action (one says that the action is *transitive*) and, for any x, the little group G_x is equal to $\{1\}$, the trivial subgroup (one says the

[4] A warning to physicists: many physicists refer to the surface in the n-dimensional Cartesian space $\sum_{i=1}^{n} x_i^2 = 1$ as an n-dimensional sphere; here we follow the mathematician's convention of calling this $(n-1)$-dimensional surface the $(n-1)$-dimensional sphere S_{n-1}.

[5] Most of these concepts are taught in many junior high schools in the U. S. Those who know them can skip to the next subsection. Older physicists should learn them for speaking with their children and teaching their students!

action is *free*).

Given G actions on $M, M', M'', \ldots$, for any object one can build from some M's, there is a "natural" G action on this object. Here we can only give some references telling the reader what must be "natural." (Bourbaki, 1958; Cartan and Eilenberg, 1956; MacLane, 1975). Let us treat two simple examples. If G acts on the set $\tilde{G}$ it acts on all subsets of $\tilde{G}$: The set of subsets of $\tilde{G}$ is denoted generally by $\Phi(\tilde{G})$. In the first G action we defined (that by inner automorphisms), the elements of the orbit of a subset H are called the conjugate subsets: gHg^{-1}, $(g \in G)$, and the little group of H is called the normalizer of H and denoted by $N_G(H)$. When H is a subgroup of G and $N_G(H) = G$, H is an invariant subgroup of G. $N_G(H)$ is therefore the largest subgroup of G which has H as invariant subgroup. In the second G action, that by left translations when H is a subgroup, the elements gH of the orbits of H are called the left cosets of H; and the little group G_H is H itself. In that case we shall denote the orbit of H by $[G:H]$. When H is an invariant subgroup of G, there is a natural group law on $[G:H]$; this group is usually denoted by G/H and is called the quotient group of G by H.

Consider now an arbitrary G action on M and the orbit $G(m)$ of m, an arbitrary element of M. Little groups of elements of an orbit are conjugated: indeed, if we simply denote by $g \cdot m$ the transform of m by g, we have

$$G_{g \cdot m} = g G_m g^{-1}. \tag{2}$$

Is it possible that different points of an orbit have the same little group? The answer is $G_{g \cdot m} = G_m$ iff (if and only if) $g \in N_G(G_m)$ (the normalizer of G_m in G). As an exercise the reader should apply this result to our three examples of symmetry breaking.

(1) $N_{D_{\infty h}}(C_{2v}) = D_{2h}$, so a flexed bar and one rotated by π around the vertical axis have the same symmetry group.

(2) D_{2h} is its own normalizer in $D_{\infty h}$; all other ellipsoids have different symmetry groups, all conjugates of D_{2h}.

(3) C_{2v} is an invariant subgroup of C_{4v}: the two solutions have the same symmetry group.

It is easy now to have a complete classification of G orbits because the orbits of the type $G(H)$ in the action by left translation of G on the set of left cosets of a subgroup are prototypes of each equivalent class of orbit. Indeed let $G(m)$ be an arbitrary orbit of G in an arbitrary action; we define the equivariant isomorphism ϕ (between sets) with the orbit $G(G_m)$ by $\phi(g \cdot m) = gG_m$ and it becomes easy to prove the fundamental theorem.

Theorem 1: The types of G orbits are in bijective correspondence with the conjugation classes of subgroups of G.

So we shall simply denote a type of orbit by $[G:H]$; remember that it is equivalent to $[G:H']$ iff $H' = xHx^{-1}$, i.e., H' and H are conjugate.

There is a partial ordering, by inclusion, among the subgroups of a group G (G is the largest, $\{1\}$ the smallest), hence also a partial ordering of the conjugation classes of subgroups of a compact (or finite) group. This yields, in the inverse order, a partial ordering of orbit types: indeed, as we saw for finite groups

$$\text{number of points of orbit } [G:H] \equiv |G(H)| = |G|/|H|; \tag{3}$$

if G is a Lie group:

$$\dim[G:H] = \dim G - \dim H. \tag{3'}$$

All these concepts are illustrated by the simple example of Fig. 2 and Diagram 2.

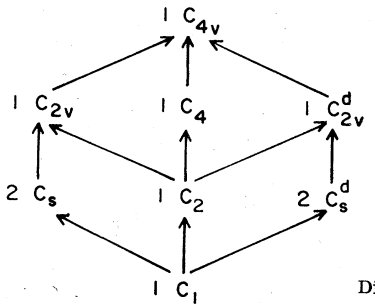

Diagram 2

It is time to give one last definition concerning group action; one calls the union of all orbits of the same type in one action "the stratum," i.e., m and m' of M are on the same stratum iff G_m and $G_{m'}$ are conjugate. The first thing to do in studying a group action is to decompose it into strata and orbits. For instance the action of the full Lorentz group $[O(3,1)]$ on p space, the four-dimensional real vector space of energy momenta, has four strata:

(1) The timelike vectors $p^2 > 0$ have orbits of the type $[O(3,1):O(3)]$ (two-sheet hyperboloids).

(2) The spacelike vectors $p < 0$ have orbits of the type $[O(3,1):O(2,1)]$ (one-sheet hyperboloids).

(3) The lightlike vectors $p^2 = 0$, $p \neq 0$ have one orbit only, the surface of the light cone minus its vertex, of type $[O(3,1):E(2)]$ where $E(2)$ is the Euclidean group in two dimensions (up to an isomorphism).

(4) The null vector $p = 0$, invariant under the whole Lorentz group, has one orbit of one point.

In Appendix C we apply these fundamental concepts to the study of some properties of compact Lie group actions. In Appendix D we give other examples of decomposition of linear group actions into orbits and strata. We also introduce the "natural" equivariant algebras which arise from these actions. With the background of definitions and fundamental concepts given here, and always assumed in the literature on group actions, the reader can understand the wording of the many theorems on group actions he can find in the mathematics texts (Bredon, 1972; Palais, 1960; Montgommery, 1951; Mostow, 1957; Michel, 1971) or physical papers (e.g., Michel, 1970, 1972; Michel and Radicati, 1970, 1971a, 1917b).

To return to symmetry breaking: let us consider a G-invariant problem and $\mathcal{S}$ its set of solutions (e.g., the equilibrium states of a system). When one varies one parameter of the problem (e.g., temperature T or pressure p, or as in Sec. I.B, angular momentum square J^2) one follows a trajectory in $\mathcal{S}$; when this trajectory

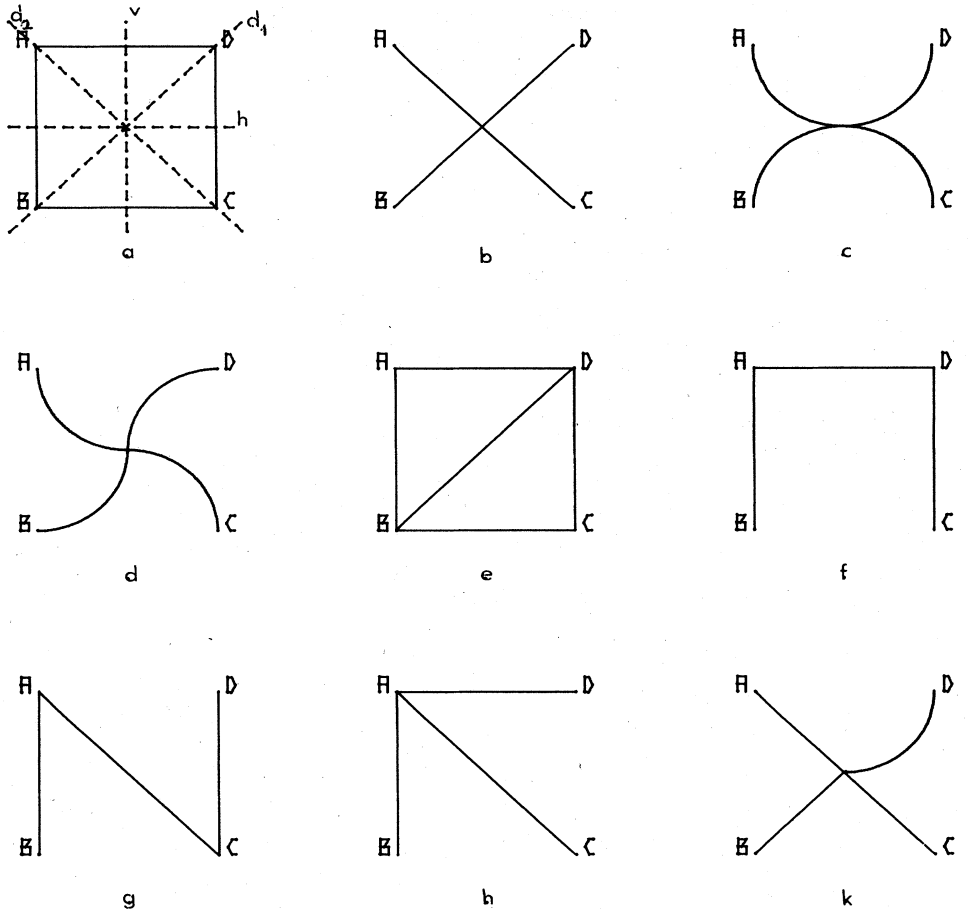

FIG. 2. The symmetry group C_{4v} of the square (a) contains the reflections s_h, s_v, s_1, s_2 through the four symmetry axes h, v, d_1, d_2 and the four rotations I, r, r^2, $r^3=r^{-1}$ around the center of 0 of the square and by an angle 0, $\pi/2$, π, $3\pi/2$, respectively. The five conjugation classes of elements are: I, r^2, r and r^3, s_h and s_v, and s_1 and s_2. The group C_{4v} has three subgroups of four elements: $C_{2v}(I, r^2, s_h, s_v)$, $C_4(I, r, r^2, r^3)$, and $C_{2v}^d(I, r^2, s_1, s_2)$. These subgroups are their own conjugate; i.e., they are invariant subgroups. That is also true of $C_2(I, r^2)$, center of C_{4v}. The four other subgroups with two elements—$C_s^h(I, s_h)$, $C_s^v(I, s_v)$, $C_s^1(I, s_1)$, $C_s^2(I, s_2)$—are conjugated by pairs; we denote by C_s the conjugate class of C_s^h and C_s^v and by C_s^d that of C_s^1 and C_s^2. The group C_{4v} is the normalizer of its invariant subgroups. The subgroups C_s^h and C_s^v have a common normalizer C_{2v} while C_{2v}^d is the normalizer of both C_s^1 and C_s^2. The partially ordered set of conjugation classes of subgroups of C_{4v} is shown in Diagram 2; on the left of each class symbol is given the number of subgroups it contains (when this number is 1, we use the same symbol for the subgroup and for its conjugation class). To each element of this set corresponds a type of symmetry for the graphs joining the four vertices of a square; each one of these eight symmetry types is illustrated by one graph of the figure according to the table:

graph	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(k)
symmetry class	C_{4v}	C_{2v}	C_4	C_{2v}^d	C_s	C_2	C_s^d	C_1
number of elements in the orbit	1	2	2	2	4	4	4	8
little group	C_{4v}	C_{2v}	C_4	C_{2v}^d	C_s^v	C_2	C_s^2	C_1
other different little group of the orbit					C_s^h			C_s^1

passes from one stratum to the other there is a symmetry change. Most often, neighboring strata can be relatively ordered, that is, the symmetry changes one way from a subgroup H to a group G in accordance with the Curie principle (P. Curie, 1894), "the symmetry of causes must be found in the symmetry of their effects"; but going the other way along the trajectory, the symmetry is "spontaneously" broken from G to H.

It is easy to show (e.g., Palais, 1960; Bredon, 1972; Michel, 1970) that for the continuous actions of topological groups, the strata with the larger isotropy groups (i.e., smaller orbits) appearing in these actions, are closed. So generally by going to a limit (e.g., $T\to 0$ or $T\to\infty$) for a parameter the trajectory inside a stratum may go to its boundary, i.e., a closed stratum with higher symmetry. Indeed symmetries are stable (this helps to make science possible) and symmetry breaking is the exception. However we misled the reader

when we spoke of a trajectory; more correctly, when we vary the external parameters as a continuous function of one variable λ, then for critical values of λ, we observe that a *bifurcation* can appear to an orbit of solutions with lower symmetry. This is the case in the two first broken symmetry examples we presented; for instance in the second, when J^2 increases from zero, one has the trajectory of axially symmetric solutions; on this trajectory, at the critical J^2 value, start also the orbits of triaxial ellipsoids, so the set of solutions is like an umbrella.

When symmetry breaking appears at a bifurcation, it is irrelevant to know which direction the system will take; in practice the direction taken is due to inhomogeneities, impurities, fluctuations.[6] What is important is to know that there exist different strata of solutions and the interesting question is which orbit type they contain or, what is equivalent, *into which subgroups H can the symmetry G be broken*?

This is one of the questions the Landau theory of second-order phase transitions (Landau, 1938; Landau and Lifshitz, 1958) tried to solve. (Although this phenomenological theory does not give the right critical exponents, it does give good predictions on the nature of symmetry breaking.)

D. Variational problems and symmetry breaking

The Landau theory is an example of the following type of problems: their solutions are given by the extrema of a G-invariant function or a functional, depending also on external parameters. In the Landau theory, one looks for extrema of the free-energy thermodynamic potential F, functional of the crystal density $\rho(x)$, dependant on temperature T and pressure p and invariant under G_0 the crystallographic group of the higher symmetry phase. Assume that at T_0, p_0 the absolute minimum of F is at $\rho_0(x)$. What can happen when T decreases from T_0? The different minima $F(\rho_i(x))$ vary with T, so we denote them by $F(\rho_i(x); T))$, and one of the other minima may become equal to that at ρ_0, i.e., $F(\rho_0(x); T_0) = F(\rho_1(x); T_1)$ at a temperature T_1 and becomes the new absolute minimum when $T < T_1$. This situation describes a first-order phase transition [or, as some say now, it is an example of Thom's catastrophe (Thom, 1974; Poston and Stewart, 1978). *A priori* there are no relations between the symmetry group G_0 and G_1 of the respective minima $F(\rho_0, T_0)$, $F(\rho_1, T_1)$ and no prediction can be made on the symmetry change. (Note, however, that the Morse theory does impose some relations on the extrema of most functions on a compact manifold) (e.g., Michel and Mozrzymas, 1977).

Another situation may also appear: the minimum $F(\rho_0(x); T))$, as a function of T, may remain the lowest minimum but its symmetry may change at a critical temperature T_1. Of the two groups G_0 and G_1, one must be a subgroup of the other, and the Landau theory gives a restrictive list of possible subgroups of symmetry breaking.

Consider the simplest model of this type of problem. F depends only on one real variable X and one parameter T:

$$F = \tfrac{1}{4}X^4 + \tfrac{1}{2}\alpha(T - T_0)X^2, \quad \alpha > 0. \tag{4}$$

$F(X) = F(-X)$, i.e., F is invariant under the two-element group Z_2 which changes X into $-X$. So one expects an extremum at $X = 0$ (one stratum with one orbit around one point; all other points $X \neq 0$ form the other stratum, with two-point orbits). For $T \geq T_0$ this extremum $F(0) = 0$ is a minimum, but it is a maximum for $T < T_0$; and for each such value of T there are two minima (not invariant under Z_2 but forming an orbit of Z_2)

$$X = \overline{\alpha(T_0 - T)}, \quad X = \overline{-\alpha(T_0 - T)}. \tag{5}$$

[This simple model provides a qualitative explanation for symmetry breaking in many phenomena, e.g., spontaneous magnetization where X is the magnetic susceptibility, but then the critical exponent β in $(T_0 - T)^\beta$ is $\frac{1}{3}$ instead of $\frac{1}{2}$.] See Fig. 3.

The fact that $X = 0$ is an extremum is easy to generalize: R is replaced by an arbitrary manifold M, and Z_2 by an arbitrary compact Lie group G. Then one has the following theorem (Michel, 1971).

Theorem 2: All G-invariant real valued functions on M have in common orbits of extrema: these critical orbits are completely characterized by the property of being *isolated in their strata*, i.e., in their neighborhood there are no orbits of the same type. (Hints for the proof of this theorem are given in Appendix C.)

Of course each function may have other orbits of extrema.[7] However, in nearly all the physics papers I have read, presenting a variational model of symmetry breaking, the solution was given by such a critical orbit; the solution thus obtained was not a success of the model (to vary any other function would have given the same extremum), but a verification of this useful theorem.

It is worth noting that most of the directions of symmetry breaking in the inner symmetry space of the fundamental interactions of physics are on critical orbits. (Michel 1970, 1972; Michel and Radicati, 1970, 1971a, 1971b). This is presented in Appendix E.

E. Bifurcation theory and symmetry breaking

The simple example of a function F to be varied, given by Eq. (4), is also a simple example of bifurcation of solutions (see Fig. 3). When one goes along the flat bottom $F = 0$ of the valley formed by F for $T > T_0$ and decreasing T value, one meets at $T = T_0$ a bifurcation

[6] It may also happen that Nature does not choose; this occurs when the orbit $[G:H]$ is finite. For instance, in a second-order phase transition from a monocrystal with G symmetry, into a less symmetrical crystal $(H < G)$, the resulting crystal may have "macles," i.e., subdomains which in general pave more or less regularly the whole crystal. Each subdomain corresponds to a point of the orbit, i.e., to a piece of crystal with symmetry H, and the different points of the orbit (each one representing different "positions of this crystal) are equally represented. Examples are known with 24 point orbits; they are frequent in natural rocks in the case of two point orbits (twinning), e.g., Dauphineite in quartz.

[7] If a G-invariant function has an extremum at x, it has the same extremum at all points of $G(x)$, the G orbit of x.

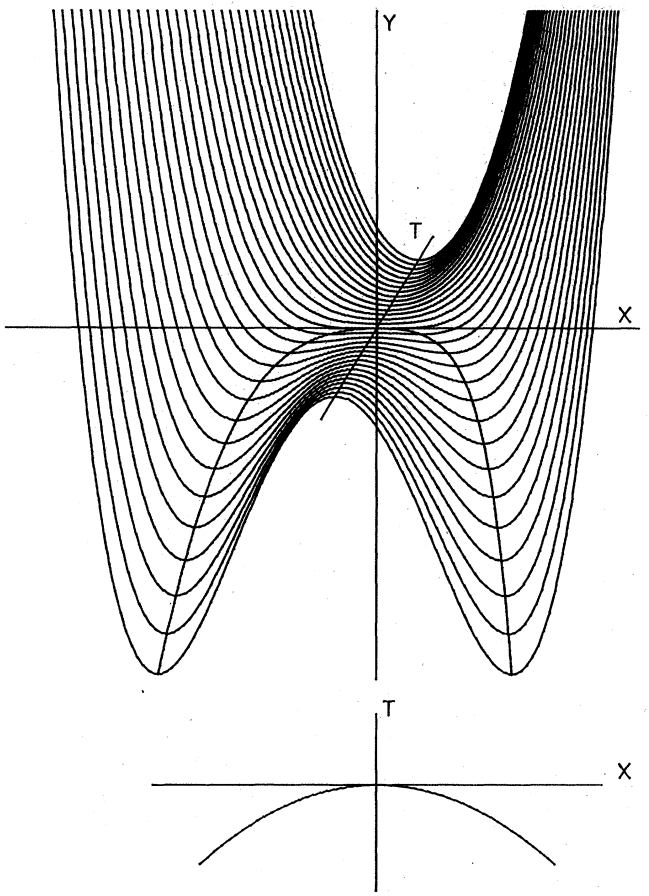

FIG. 3. This figure represents the simplest model of spontaneous symmetry breaking in phase transition: F is a thermodynamic potential (e.g., Gibbs free energy) depending on the temperature T and on another external parameter X (e.g., magnetization) but is independent from the sign of X. Following the minimum of the function $F(X, T)$ for a fixed value of T, when the parameter T decreases from a value larger than T_0, one meets at $T = T_0$ a bifurcation between two valleys. Actually, the function $F(X, T)$ plotted in (a) is that of Eq. (4). Its minima in the plane X, T are plotted in (b).

into two valleys[8] sloping down at the same rate [substituting in Eq. (4) the value of x from Eq. (5)],

$$F = -\tfrac{1}{4}\, 2^2 (T_0 - T)^2 \text{ for } T < T_0 . \tag{6}$$

Bifurcations can be studied in a larger setting than variational problems. Indeed bifurcation theory is part of the framework of integral-differential equations; even more generally one considers a nonlinear differentiable map S between Banach spaces B_1, B_2 (which might be identical) depending on a point λ of a parameter space, and the solutions of the problem must satisfy $S(u, \lambda) = 0$. Let us assume that u_0 is a solution at $\lambda = \lambda_0$ and compute the Frechet derivative

$$F_{u_0}(v, \lambda) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [S(u_0 + \epsilon v, \lambda) - S(u_0, \lambda)] .$$

For each v, λ pair, we have a morphism $B_1 \to B_2$. If $F_{u_0}(0, \lambda t)$ is not inversible there is a bifurcation at the solution u_0, λ_0 and in the "good" cases the subspace $\mathrm{Ker} F_{u_0}(0, \lambda_0)$ is finite dimensional and tangent to the manifold of solutions in B_1 in the neighborhood of u_0. To my astonishment it seems that only recently a systematic study of equivariant bifurcation theory has been made (Prodi and Ambrosetti, 1973; Burger, 1963; Sattinger, 1977, 1978a, 1978b). The symmetry group G acts linearly on B_1 and B_2 and trivially on the λ space, and S is an equivariant map. Then G acts on $\mathrm{Ker} F_{u_0}(0, \lambda_0)$ by a linear representation which is irreducible if one excepts the case of accidental degeneracy (for a similar, detailed study of example 2, the rotating ellipsoid, see Constantinesu *et al.*, 1979). Hence, *a priori*, the broken symmetry subgroups will be little groups of "irreps" (irreducible linear representations of G). Assuming good hypotheses, e.g., that F_{u_0} is analytic in λ, Sattinger (1977) found that the breaking directions in $\mathrm{Ker} F_{u_0}(0, \lambda_0)$ are idempotents of the G-equivariant algebras I had the occasion to consider earlier with Radicati (Michel 1970; Michel and Radicati, 1970, 1971a, 1971b), showing that the breaking directions of the internal symmetry of hadronic physics are idempotents (or nilpotents) of such algebras (see Appendix E). The Gell-Mann (1962) and Biedenharn (1963) d algebras on the adjoint representation of SU(n) are an example of such algebras, as explained in Appendix D.

Our three examples of symmetry breaking, as well as those of the Landau theory, can all be considered as applications of equivariant bifurcation theory. In all cases the symmetry is broken into an isotropy group of an irrep. I do not know of any published list of the isotropy subgroups of the irreps of the rotation group SO(3) and the three-dimensional group O(3). Such a list is so useful that I give it in Appendix A.

What are the other mathematical schemes for symmetry breaking?

F. Other symmetry breaking mechanisms. The decomposition into pure states

Two other mathematical processes for obtaining broken symmetry in physics are (1) going to the thermodynamic limit, i.e., replacing a system that has a very large but finite number of constituents by an infinite system, and (2) renormalization in field theory.

One can cite historical papers for obtaining broken symmetry by either process (Peierls, 1936, Lee, 1969; Lee and Gervais, 1969) and a review paper for each process (Frohlich, 1978; Symanzik, 1970). These two mechanisms have not yet been related to the study of symmetry defects. Surely this will happen. I shall skip them here mainly on the grounds of present incompetence.

The processes of symmetry breaking that we review last apply to classical and quantum statistical dynamics, quantum mechanics, field theory, and more generally all domains of physics in which the results of observations can be described by expectation values of the operators

[8] It is true that the delta of a river displays many bifurcations, but valley bifurcations are rather rare on Earth. The best known example is probably that of Orinoco in Venezuala (Amazonas): one arm of the bifurcation flows into the Casiquiare of the Amazon basin.

of an algebra $\mathfrak{a}$, the algebra of observables. This algebra is Abelian for classical statistical mechanics, non-Abelian in the other cases.

Consider *quantum mechanics*: the pure states are represented by normed ($\langle x|x\rangle = 1$) vectors $|x\rangle$, of a Hilbert space $\mathfrak{H}$; Hermitian operators on $\mathfrak{H}$ form the algebra of observables; $\langle x|A|x\rangle = \mathrm{tr}A|x\rangle\langle x|$ is the expectation value of the observable A for the state $|x\rangle$. We recall that when A is a positive operator, by definition its expectation values for all $|x\rangle$ are positive. Most often, one has only partial information on the state of the system; in that case one must use a density operator ρ to describe it; ρ is a (discrete or continuous) linear combination with positive coefficients of rank one projectors such as $|x\rangle\langle x|$. It is only when we have a complete knowledge of the state of the system that $\rho = |x\rangle\langle x| = \rho^2$; then the system can be represented by the normed vector $|x\rangle$ defined up to a phase. In this latter case the state is said to be a pure state; otherwise it is a mixture.

Let us consider the example of spin polarization. As long as all information we have and all observations we make on the ground state of an atomic nucleus of spin $j \neq 0$ are rotationally invariant, this state is described by a density operator ρ_0 invariant by rotation. Since the total Hilbert space of state vectors for this nucleus can be written as the linear product of Hilbert spaces for each type of coordinate, let us concentrate on the $2j+1$ dimensional space $\mathfrak{H}_j$ corresponding to spin coordinates; on this space $\rho_0 = [1/(2j+1)]I$; indeed the rotation group [or more exactly its covering SU(2) when j is a half-integer] acts on ρ_0 by $\rho_0 \to D_j(r)\rho D_j(r)^{-1}$, where $r \to D_j(r)$ is the $2j+1$ dimensional irrep of SU(2). This action can be extended to the whole special unitary group SU($2j+1$) by $\rho \to U\rho U^{-1}$. The pure states $\rho = \rho^2$ form one orbit [SU($2j+1$) : $U(2j)$] homeomorphic to $P(2j, C)$, the complex projective space in $2j$ dimensions, so its real dimension is $4j$. The set of all density operators is the polarization domain $\mathfrak{D}_j$ (Minnaert, 1971; Doncel, Michel, and Minnaert, 1972; Doncel *et al.*, 1973). This set is the complex hull of the orbit of pure states in the real vector space ϵ_j [of dimension $(2j+1)^2$] of Hermitian operators acting on $\mathfrak{H}_j$. The pure states density operators are the extremal points[9] of the convex polarization domain $\mathfrak{D}_j$.

We now transpose these well known remarks on quantum mechanics to the general situation of an algebra $\mathfrak{a}$ of observables (see above, with quantum and classical statistical mechanics or field theory as examples). Technically $\mathfrak{a}$ is chosen as a C^* algebra with a unit I; we do not need to go into too much detail here: $\mathfrak{a}$ is an associative algebra built on a complex vector space with a norm such that $\|AB\| = \|A\|\|B\|$, an anti-involution $(AB)^* = B^*A^*$ satisfying $\|A^*A\| = \|A\|^2$. (C is a one-dimensional C^* algebra with $\|z\| = |z|$ and $z^* = \bar{z}$, the complex conjugate of z.)

The positive observables are of the form AA^* (or A^*A). A state is a normed 1 positive linear form ϕ on $\mathfrak{a}$, i.e., $\phi(\lambda A + \mu B) = \lambda\phi(A) + \mu\phi(B)$, $\phi(AA^*) > 0$, $\phi(I) = 1$ (in the particular case of quantum mechanics, $\phi(A) = \mathrm{Tr}\rho A$, where ρ is the density operator representing the state ϕ). The set $\mathfrak{D}$ of such ϕ is convex (i.e., if $\phi_i \in \mathfrak{D}$, $\alpha_i \geq 0$, $\sum\alpha_i = 1$, then $\sum\alpha_i\phi_i \in \mathfrak{D}$) and its extremal points are the pure states.[10]

From this general scheme there is a way to reconstruct a Hilbert space by the well known GNS (Gelfand-Neimark-Segal) construction sketched in Appendix F. One generally starts from a state $\Phi \in \mathfrak{D}$ which will play the role of vacuum, or lowest-energy state $|\Omega\rangle$ in $\mathfrak{H}_\Phi$, and one also obtains a linear representation on $\mathfrak{H}_\Phi$, $A \to \pi_\Phi(A)$ of the algebra $\mathfrak{a}$ of observables. If Φ is a pure state, the representation π_Φ is irreducible. Let $G \xrightarrow{f} \mathrm{Aut}\,\mathfrak{a}$ be the action of the symmetry group G on the algebra of observables. Such an action can be transposed to the set of states $\mathfrak{D}$. A pure state ψ may not be invariant by G. Then one can consider its orbit $G(\psi)$. If there exists a G-invariant measure[11] $d\mu(g)$ on this orbit and if the total measure of the orbit is finite then, normalizing $\int_{G(\psi)} d\mu(g) = 1$, by averaging over the orbit one obtains a G-invariant state

$$\psi_0 = \int_{G(\psi)} \psi\, d\mu(g) \tag{7}$$

which is of course a mixture.

The GNS construction starting from the G-invariant mixture state ψ_0 yields a reducible representation $A \to \pi_{\psi_0}(A)$ of $\mathfrak{a}$ on the Hilbert space $\mathfrak{H}_{\psi_0}$ with fundamental state $|\Omega_0\rangle$, while the same construction for any state $g\cdot\psi$ of the orbit $G(\psi)$ would have yielded an irrep $\pi_{g\psi}$ of $\mathfrak{a}$ on $\mathfrak{H}_{g\cdot\psi}$. As in Eq. (7) we have direct integral decomposition:

$$\mathfrak{H}_{\psi_0} = \int_{G(\psi)}^{\oplus} \mathfrak{H}_{g\cdot\psi}\, d\mu(g)\,;$$

for every (8)

$$A \in \mathfrak{a},\ \pi_{\psi_0}(A) = \int_{G(\psi)}^{\oplus} \pi_{g\cdot\psi}(A)\, d\mu(g)\,.$$

One has the choice of working with the reducible repre-

[9] Pure polarization states are "different" if they cannto be transformed into each other by the orthogonal group $O(3)$ when the particle is at rest. However, the decomposition of the manifold $P(2j, C)$ of pure polarization states into $O(3)$ orbits does not seem intuitive to most physicists for spin $j > 1/2$. Consider the simplest case $j = 1$, with $\mathfrak{H}_j$ three dimensional; let ξ_i $i = 1, 2, 3$, the coordinates of $|x\rangle$ in an orthonormal basis $\sum\bar{\xi}_i\xi_i = \langle x|x\rangle = 1$. The pure states depend on four real parameters [indeed real dim $P(2, C) = 4$]. So there is an infinity of $O(3)$ orbits, which can be labeled by the $O(3)$ invariant $|\sum_i \xi_i^2| = \mu$. There are three strata, the open dense stratum (see Appendices C and D) $0 < \mu < 1$ with an infinity of orbits, and two strata of one orbit each, the orbit $\mu = 1$ of longitudinally polarized states and the orbit $\mu = 0$ of circularly polarized states. The statement is often found in good books on quantum mechanics that you can transform spin-1 pure polarization states into each other by rotation and/or space reflections. Not only is this statement wrong, but it is only for the special cases $\mu = 0$, 1 that one can find a (real) triedra such that the polarization state vector is an eigenstate vector of J_3 [the eigenvalue is $\pm(1-\mu)$].

[10] They may be only primary states instead of pure states; we do not explore these technicalities here.

[11] This is always the case when G is discrete or compact or Abelian or a semidirect product of them, such as the Euclidian group $E(3) = T_3 \square O(3)$. It might not be true for some orbits when G is a noncompact Lie group such as the Lorentz or Poincaré group.

sentation and the Hilbert space $\mathcal{H}_{\psi_0}$ with a degenerate vacuum (i.e., a full G orbit of the fundamental state) or of choosing one of the irreps, for example, that on the Hilbert space $\mathcal{H}_{\psi}$, on which the G symmetry is explicitly broken, since the automorphisms $f(g)$ of $\mathfrak{A}$, when $g \notin G_{\psi}$, cannot be implemented as operators on $\mathcal{H}_{\psi}$ (they transform $\mathcal{H}_{\psi}$ into $\mathcal{H}_{g.\psi}$). As an example, take an infinite system of atoms whose fundamental equilibrium state (at given T, p) is a crystal: then any Euclidean-invariant equilibrium state will be a mixture. However, one generally prefers to fix the position of the crystal. Its fundamental state is then a pure phase, not Euclidean invariant, since its symmetry group is crystallographic, i.e., a discrete subgroup H of $E(3)$ such that the orbit $[E(3):H]$ is compact.

Conversely, given a G action on the C^* algebra of observables OL and a G-invariant state ψ_0 which is a mixture, how does ψ_0 decompose into pure states? With a minimum of physical assumptions[12] in this general mathematical frame, one can prove (Kastler and Robinson, 1966; Robinson and Ruelle, 1967; Haag *et al.*, 1969; Ruelle, 1970; Kastler *et al.*, 1972; Emch, 1972) that any G-invariant equilibrium state ψ_0 can be decomposed into pure states by an integral on $\mathfrak{D}$, the set of states, with a G-invariant finite measure $d\mu$ (normalized to 1). The group G acts on the support of the measure $d\mu$ either *transitively* (i.e., the support is a G orbit) or by an *ergodic* action (the support can be decomposed into orbits, all of zero measure, except one whose measure is one). We have described the transitive case. This is the only possible case when G is compact; then every closed subgroup H of G can appear in symmetry breaking. There are selection rules on H when G is a noncompact Lie group, as the example of $E(3)$ treated in the next section will show. Moreover there is the second possibility, that of "ergodic states." No general classification of such states exists as yet, although more and more are actually being found and studied.

G. The mesomorphic phases of matter

The symmetry breaking scheme explained in the preceding section has been applied to $E(3)$, the three-dimensional Euclidean group in Kastler *et al.* (1972); it classified the possible little groups of transitive states. All orbits corresponding to smooth actions of $E(3)$ (i.e., those with closed isotropy groups) carry an invariant measure, which is finite if (and only if) the orbit $[E(3):H]$ is compact. These equilibrium states of matter are often called mesomorphic phases; in the approximation of infinite extension their symmetry group is a subgroup H of $E(3)$, defined up to a conjugation and such that $[E(3):H]$ is compact. When H is discrete, the phase is a crystal. For the purpose of classifying crystal symmetries, crystallographers were not interested in the characteristic lengths of the crystal, but they wanted to distinguish eventually between left-handed and right-handed crystals. So they classed the H's up to a conjugation in the connected affine group $\mathrm{Aff}_0(3)$.[13] This yielded 230 crystal symmetries (they are not all represented in nature). By the same classification there are an infinity of other H subgroups. They can be put in families according to the topology of their largest connected subgroup H_0 and their intersection $H \cap T = T_H$ with the translation subgroup of $E(3)$.

TABLE I. The broad classes of the symmetry groups H of mesomorphic states of matter: $[E(3):H]$ compact, H_0 = largest connected subgroup of H, $T_H = H \cap T$. The family label of Kléman and Michel (1978) to which we refer for more details.[a]

Family	T_H	H_0	
I_a	R^3	$R^3 \times U(1)$	Ordinary nematics
I_b	R^3	R^3	Exceptional nematics
II_a	$R^2 \times Z$	R^3	Cholesterics (chiral)
II_b	$R^2 \times Z$	$R^2 \times U(1)$	Smectics A
$II_{c,d}$	$R^2 \times Z$	R^2	Smectics C (can be chiral: II.d)
V	R^2	R^2	Chiral smectics C (chiral)
III	$R \times Z^2$	R	Rod lattices (e.g., lyotropics)
IV	Z^3	$\{1\}$	Crystals

[a]An independent classification of the symmetry groups of mesomorphic phases has been given by Goshen *et al.* (1975): it also gives an earlier reference to Hermann.

Each broad class had already a generic name that we give in Table I.

Very few symmetry classes are represented in nature [although families $I_b, II_{c,d}$, V each have an infinity of subgroups H nonconjugated in $\mathrm{Aff}_0(3)$]. The polar molecules which have no symmetry planes can form the chiral families; conversely those families of phases are never observed for chemicals made from nonpolar molecules. Let us give for the nonspecialist a brief description of the mesomorphic families.

In *nematics*, the molecules are aspherical; their positions are distributed at random as in a liquid, but they are aligned. In *ordinary nematics* H is the semidirect product $R^3_{\square} D_{\infty h}$. That is, the orientation of the molecules causes them to yield only axially symmetric quadrupole (second-rank tensor) effects (e.g., light refraction index, magnetic susceptibility, etc.), even when the molecules have no axial symmetry. Probably, near the solidification temperature, very aspherical molecules may rotate less easily, and one expects "*exceptional nematics*" (e.g., birefringent quadrupoles with three unequal axes). An exceptional nematic with symmetry $R^3_{\square} D_{3h}$ may have been observed (Billard *et al.*, 1978).

Cholesterics are constructed of polar molecules; their symmetry group H contains all the translations in a plane and, with a perpendicular axis, a continuous helicoidal group. They appear frequently in biological tissues. In *smectics* the molecules are distributed in parallel monomolecular or bimolecular layers, and they are aligned either perpendicularly (*smectics A*) or obliquely (*smectics C*) to the layers. In *chiral smectics C* inside each layer the polar molecules are oriented with a constant oblique angle, but the azimuth of this orientation turns by a constant angle θ from one layer

[12]The main physical assumption made is that of asymptotic Abelianness (Kastler and Robinson, 1966) for the space directions, i.e., two observables with compact support commute when one of the supports is translated to infinity by a space translation.

[13]The affine group is the inhomogeneous general linear group (including the translations). We denote it by Aff(3) and its connected component by $\mathrm{Aff}_0(3)$.

to the next; the division in subfamilies II_d or V depends upon whether θ/π is rational or not. Family III is illustrated by the hexagonal rod lattice of lyotropics (Luzzati and Tardieu, 1974; Chandrasekhar *et al.*, 1977) and also by the lattice of vortex lines in a type-II superconductor (Essman and Traüble, 1967).

The "ergodic" $E(3)$ broken symmetry states have not as yet been classified. The classification of the helimagnetic crystals or of the modulated crystals is understood (Kastler *et al.*, 1972; Janner, 1977; Janner, 1979). However, many incommensurate phases in crystals are presently found (Aalst *et al.*, 1976; Tizumi *et al.*, 1977; Pouget *et al.*, 1979) and they will surely require a deeper study of the general problem of symmetry. There are also quite a number of smectic phases, labeled $B, E, H, \ldots$, which are locally like crystals, but in which the order correlation disappears rapidly along the direction orthogonal to the layers. How shall we describe their order? Are they ergodic states? Many questions are unanswered in this active field of research.

H. Concluding remarks to Part I

Euclidean symmetry is only a part of the geometrical symmetry in physics: the culminating role of geometry is found in general relativity. Moreover, there are nongeometrical invariances, related to internal, dynamical symmetries: that is the case of gauge invariance in supraconductivity, suprafluidity, and electrodynamics, in the unified theory of weak and electromagnetic interactions. We all hope that these well understood cases of symmetry breaking will be helpful for understanding the approximate symmetries of hadrons (isotopic spin, SU(3) flavors, charm, and new flavors suggested by the upsilon) broken by the weak interactions, the electromagnetic interaction, and the quark masses along well defined patterns.

II. TOPOLOGICAL CLASSIFICATION OF SYMMETRY DEFECTS AND CONFIGURATIONS

A. Symmetry defects and configurations

By the end of Sec. I we reached an understanding of the "transitive" equilibrium states; these ideas, when applied to the breaking of Euclidean symmetry, yielded a classification of known mesomorphic states of matter, with room for new phases with different symmetries in the already known families of Table I. To reach this classification we had to make an idealization that the ordered phase was extending indefinitely in space in order to be globally invariant under an allowed subgroup H of $E(3)$. That idealization is not bad if the sample to be studied is large enough so that its symmetry can be recognized. Actually, in nature, samples are not only limited in size, but also far from "perfect," i.e., they have a lot of defects whose very existence makes them a topic of physics. So we are led, after the study of broken symmetries, to study imperfect symmetries.

Consider an ordered medium, with symmetry group H, broken from G, occupying a spatial domain V. If in every subdomain $\Omega' \subset V$, the state of the phase $S_{\Omega'}$ can be obtained from the state S_Ω in a given subdomain $\Omega \subset V$ by an operation of the symmetry group H (H is defined by idealizing the subdomains $\Omega, \Omega', \ldots$ as infinite[14]), this sample of a phase of matter occupying V is said to be in a perfect state.

If this is not the case, there is a domain Ω' whose local state $S_{\Omega'}$ cannot be obtained from the local state S_Ω by a transformation of H. However, since it is the same phase, we assume[15] that there is an element $g \in G$ (the unbroken symmetry group) which transforms the local state S_Ω into the local state $S_{\Omega'}$. Note that all elements of the coset gH will transform it too. In short, given the state S_Ω, with broken symmetry group $H < G$, the local states $S_{\Omega'}$ at domains small enough can be labeled by the left cosets of gH, i.e., as explained in Sec. I.C, by the points of the orbit $[G:H]$. For example, in the case of a mesomorphic phase, $G = E(3)$, the different cosets gH will rotate and/or translate the local state S_Ω, and the set of *positions* of the state S_Ω is the orbit $[G:H]$; another state $S_{\Omega'}$ must be one of these positions. Hence to describe the imperfect state of the ordered medium we define a function ϕ from the occupied domain V to the orbit of local states (i.e., positions for a mesomorphic phase) $[G:H]$.[16] The *perfect* state is described by a *constant* function (i.e., a point of $[G:H]$). If by continuous deformations[17] the function can be made constant, we say that the medium is in a nearly perfect state. More likely, in practice, the function ϕ cannot be defined everywhere on V. Those subsets of V on which ϕ cannot be defined are called *defects*. Still there might be continuous deformations of the function ϕ, which extend it over a defect; we then say that the defect is *topologically unstable*. As we shall see, it might even happen that ϕ can be made continuous everywhere (i.e., no defects in V), but still cannot be deformed continuously into a constant; we then say that the global state of the medium in V is in a topologically stable *configuration*.[18]

There is a full-fledged mathematical theory which tells when a continuous function defined on a subdomain W of V can be continued everywhere on V. Of course this will also depend on the space $[G:H]$ in which the function takes its value. We need this theory. Two basic books on it are Hilton (1961) and Steenrod (1957).

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[14] This would not be necessary if pseudogroups were being used, but I decided to avoid their use in this paper.

[15] This is the natural assumption when G is a local gauge group. When G is the Euclidian group, the local deformations of mesomorphic phases may not be restricted geometrically to translations and rotations. I do not know yet how to include linear, even diffeomorphic geometrical deformations in the abstract scheme presented here. See, however Sec. (5a) of Kléman, Michel, and Toulouse (1977).

[16] As we explained in Ftn. 6 such composite ordered media occur naturally in crystals when the orbit $[G:H]$ is finite.

[17] We shall soon give a precise definition of this concept.

[18] In the literature "configurations" are also called "kinks" and "textures." Since these words have been used for 20 years with a precise but completely different meaning in the study of dislocations (the defects arising in crystals from lack of translational invariance), we avoid them here.

B. Why we need homotopy for studying topological stability of symmetry defects

We consider two topological spaces T, X and the continuous functions $f, g, \ldots$ from T to X. By definition, f can be continuously deformed into g (one can also say f is *homotopic* to g) if there exists a continuous function $F(t, \alpha)$ in the two variables $t \in T$, $0 \leq \alpha \leq 1$, valued in X such that $F(t, 0) = f(t)$, $F(t, 1) = g(t)$.[19] We shall simply note "f homotopic to g" by $f \sim g$. Of course $f \sim f$; if $f \sim g$, then the continuous function $F(t, \beta)$ with $\beta = 1 - \alpha$ shows that $g \sim f$. Moreover if $f \sim g$ and $g \sim h$ then $f \sim h$: (the reader should either verify this or read the proof in Hilton (1961) or Steenrod (1957).

Since the relation "to be homotopic" is reflexive, symmetric, and transitive, it is an equivalence relation, and the continuous functions from T to X fall into equivalence classes called "homotopy" classes. As an illustration, let us study a sufficient condition for the existence of a unique homotopy class. A topological space Y is contractible if the identity function $Y \xrightarrow{I} Y$ is homotopic to a constant function c_0 (i.e., every $y \in Y, c_0(y) = y_0$ a fixed point of Y); the continuous function $Y \times \Lambda \xrightarrow{F} Y$ deforms I continuously to c_0 [i.e., $F(y, 1) = y, F(y, 0) = y_0$] and shows that Y can be continuously shrunk to one of its points. This is the case of R^n, for instance. If Y is a contractible topological space, and X an arbitrary topological space, it is easy to prove that the continuous functions from X to Y form only one homotopy class; indeed given $X \xrightarrow{f} Y$, the function $X \times \Lambda \xrightarrow{G} Y$ defined by $G(x, \alpha) = F(f(x), \alpha)$ where $Y \times \Lambda \xrightarrow{F} Y$ has been defined above, satisfies $G(x, 1) = f(x), G(x, 0) = y_0$.

As an immediate application, if the orbit of local states $[G:H]$ is contractible, there are no topologically stable defects or configurations. It is a simple exercise to check that if $Z = Y \times C$ is the topological product of Y and the contractible space C, then the homotopy classes of continuous functions from X to Z and X to Y are in bijective (one-to-one onto) correspondence. If we denote by $M(X, Y)$ the set of continuous functions[20] from X to Y and by $\mathcal{H}M(X, Y)$ the set of homotopy classes, we then have

$$C_{\text{contractible}} \Rightarrow \mathcal{H}M(X, Y \times C) = \mathcal{H}M(X, Y) \tag{9}$$

(= between sets means that there is a well defined and "natural" bijective map between them).

Similarly if X is contractible the continuous function $X \xrightarrow{f} Y$ is equal to $f = f_0 I_x = f_0 F(x, 1)$ where the continuous function $X_x \xrightarrow{F} X$, as defined above, satisfies $F(x, 1) = x$, $F(x, 0) = x_0$, so f is homotopic to the constant function $f(x) = f(x_0)$.[21] This is not helpful for our study of defects, since defects make V noncontractible. As an exercise one should check that

$$C_{\text{contractible}} \Rightarrow \mathcal{H}M(X \times C, Y) = \mathcal{H}M(X, Y)\,. \tag{10}$$

This expression suggests that the defects we want to study can be shrunk geometrically to isolated points (dimensions $d = 0$), lines ($d = 1$), or walls ($d = 2$). A line, topologically equivalent to a segment, with both ends inside V, can be shrunk continuously to a point; thus topologically stable line defects must either be closed (topologically equivalent to a ring S) or have their two extremities on other line or wall defects or on the surface ∂V of V (for instance, the walls of the container of a liquid phase).[22]

Consider an isolated point defect; from our knowledge of the function ϕ around it, we must be able to decide if ϕ can be continued at the point. For instance, taking the point as the origin of coordinates assumes that the continuous function ϕ is known on the domains $D_{\epsilon_1 \epsilon_2}: 0 < \epsilon_1 \leq x_1^2 + x_2^2 + x_3^2 \leq \epsilon_2$, with ϵ_1 arbitrarily small. One feels intuitively that if ϕ is homotopic to a constant function, it can be extended continuously in the complex ball $x_1^2 + x_2^2 + x_3^2 \leq \epsilon_2$ since it is trivial to do it for a constant function, and conversely, if ϕ is not homotopic to a constant function, it is impossible to extend it continuously over the point defect. This intuition is right and can be made mathematically rigorous. We also remark that $D_{\epsilon_1 \epsilon_2}$ is a topological product $S_2 \times [\epsilon_1, \epsilon_2]$ and since the segment $[\epsilon_1, \epsilon_2]$ is contractible, according to Eq. (9) the homotopy class of ϕ is given by our knowledge of this function on a sphere S_2 enclosing the point defect.[23]

Similarly, topological stability of line defects will be studied by enclosing each such defect with a sphere S_1 (i.e., $x_1^2 + x_2^2 = 1$); the study of wall defects requires only two points (one on each side of the wall), i.e., "enclosing" the wall with the sphere S_0, i.e., $x_1^2 = 1$. More generally, *a d-dimensional defect in a ν-dimensional space is topologically stable if the $[G:H]$ valued function ϕ (which describes the imperfect state of the phase) restricted on a $S_{\nu-d-1}$ sphere enclosing the defect is not homotopic to a constant.*

Of course we are mainly interested in $\nu = 3$, but there is also a great interest in $\nu = 2$ phases (surfaces) and $\nu > 3$ might be interesting in some mathematical models of statistical mechanics; $\nu > 3$ is also used for studying the symmetry of uncommensurable crystals (Janner, 1977, 1979).

Finally, isolated d-dimensional topologically stable defects in a ν-dimensional ordered phase can be classified by the homotopy class of $\phi|_{S_n}$ in $\mathcal{H}M(S_n, [G:H])$ with

$$n = \nu - d - 1\,. \tag{11}$$

[19] These are equalities of continuous functions from T to X; one also says that F is defined on the topological space $T \times \Lambda$, as the topological product of T and Λ, the closed unit interval $[0,1]$; then for a fixed $\alpha = \alpha_0$, $F(t, \alpha_0) = F|_{T \times \{\alpha_0\}}$, the restriction of F to the subspace $T \times \{\alpha_0\}$ of $T \times \Lambda$.

[20] This set can be given a topology and thus made a topological space.

[21] As we shall see, this does not forbid the existence of configurations when V is contractible, but it shows that we have to state the concept precisely. See Sec. II D.

[22] These details of our assumption are admittedly arbitrary. Why not then remove the whole line on the surface and squeeze it to a point? These conventions seem the best fitted for our study when surface effects can be neglected: Then the lines extend to infinity. See Ftn. 39 for surface defects.

[23] It might be useful to specify that $\sum_{i=1}^{n} x_i^2 = 1$ is the equation of a sphere S_{n-1}, which is itself the boundary of the ball B_n defined by $\sum_{i=1}^{n} x_i^2 \leq 1$. Indeed, S_{n-1} is an $(n-1)$-dimensional manifold.

C. Some highlights of homotopy theory

There is no point in reproducing here homotopy theory which is explained in many books and many courses (Hilton, 1961; Steenrod, 1957; Gray, 1976). Some popularization of it has been made by physicists.[24] It is more important to explain here the main trends of the theory and the specific results we need to apply in this paper. However, a full understanding of mathematical results requires that we know their proof. All proofs missing here can be studied by reading at most 20 pages of two "old" mathematics books: *The Topology of Fiber Bundles* by Steenrod (1957) (and most physicists probably need an introduction to the basic concepts of fiber bundles), and the small book by Hilton (1961). Here I shall also indicate some broad features, that the reader will not find immediately or explicitly in books. Milnor's book (Milnor, 1965) proves some homotopy theorems and is strongly recommended for its pedagogy.

To classify symmetry defects by homotopy class, as we explained at the end of Sec. II.B, might seem to be amusing but not necessarily very fruitful for physics if it does not help us to make predictions about the physical properties of the defects. This is not the case, as we shall show. The reason is that each set $\mathcal{HM}(S_n, [G:H])$ of homotopy classes has a natural mathematical structure, that of a group with an action on it.[25]

When the order parameter space is a topological group[26] G whatever the definition space X is (see Sec. II.B), there is a natural group structure on the set $M(X, G)$ of all continuous functions from X to G. This group law on $M(X, G)$ is denoted by $*$ and defined by

$$(g*f)(x) = g(x)\cdot f(x)\,, \tag{12}$$

where $\cdot$ is the group law on G. The unit of this group is the constant valued function $f(x) = 1 \in G$. One has to prove that if $f' \sim f, g' \sim g$ then $g'*f' \sim g*f$ (easy to verify), in order that Eq. (12) also define a group law on $\mathcal{HM}(X, G)$. For every X, such a group is defined—for instance when $X = S_n$.

We are interested in the case in which $[G:H]$ is not a group, so we have to explain how the natural group structure is defined on $\mathcal{HM}(S_n, X)$ for any n and any topological space X. We have first to deal with two technical points: we have to choose provisionally a fixed point $x_0 \in X$, and it is technically advantageous to remark that an n-dimensional cube $\Gamma_n = \{t_i, 0 \leq t_i \leq 1, 1 \leq i \leq n\}$ with all the points on its boundary $\partial\Gamma_n$ (i.e., points with at least one $t_i = 0$ or 1) identified, is topologically "isomorphic" to a sphere.[27] Then the set

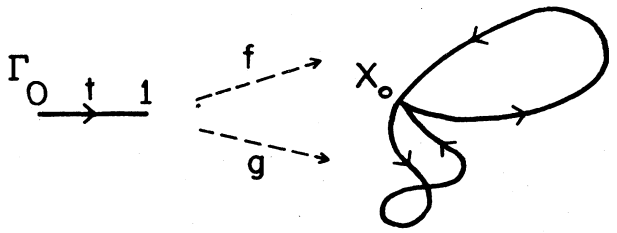

FIG. 4. Given the images $f(\Gamma)$ and $g(\Gamma)$ of the oriented segment $\Gamma[0, 1]$ by two continuous maps f and g which satisfy $f(0) = f(1) = x_0 = g(0) = g(1)$, one can consider their union as an image of Γ by a continuous map $g*f$ which does satisfy $g*f(0) = g*f(1) = x_0$ (the fact that there is a $\lambda \in \Gamma$ such that $g*f(\lambda) = x_0$ is irrelevant).

$M(\Gamma_n, \partial\Gamma_n; X, x_0)$ of continuous maps f from Γ_n to X with $f(\partial\Gamma_n) = x_0$ can be identified with $M(S_n, X; x_0)$ of maps f' from the sphere S_n to X such that x_0 belongs to the image of f.

Given two such functions f, g, one defines the "product"

$$(g*f)(t_1, t_2, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & 0 \leq t_1 \leq \tfrac{1}{2} \\ g(2t_1 - 1, t_2, \ldots, t_n) & \tfrac{1}{2} \leq t_1 \leq 1\,. \end{cases} \tag{13}$$

For $n = 1$ this means that, as a function of t_1, the closed path $g*f(t_1)$ is obtained by running successively along the two paths, first $f(t)$, then $g(t)$ with double speed (see Fig. 4).[28]

This $*$ product induces a group law on the set $\mathcal{HM}(S_n, X; x_0)$ of homotopy classes. Let us prove first that it is compatible with the homotopy equivalence relation, i.e., $f \sim f'$ and $g \sim g' \Rightarrow g*f \sim g'*f'$. We use the shorthand t for $t_1, t_2, \ldots, t_n$, and consider the continuous functions $F(t, \alpha), G(t, \alpha) \in M(S_n \times \Lambda, X; x_0)$; for each value of α, they are functions of t; in particular $F(t, 0) = f(t)$, $F(t, 1) = f'(t)$, $G(t, 0) = g(t)$, $G(t, 1) = g'(t)$. Defining for each value of α, $H(t, \alpha) = G(t, \alpha)*F(t, \alpha)$ one verifies that $H(t, \alpha) \in M(S_n \times \Lambda, X; x_0)$ and that $H(t, 0) = (g*f)(t)$, $H(t, 1) = (g'*f')(t)$. Although the $*$ product has no neutral element and is not associative on $M(S_n, X; x_0)$ it is not difficult to verify that the law induced on $\mathcal{HM}(S_n, X; x_0)$ has a neutral element, the homotopy class of constant functions; that it is associative and that the class of $f(1 - t_1, t_2, \ldots, t_n)$ is the inverse class of that of $f(t)$. (The complete proof is in Hilton, 1961, or Steenrod, 1957). This group formed with the homotopy classes of $M(S_n, X; x_0)$ is traditionally denoted by $\pi_n(X, x_0)$ and called the nth homotopy group of the space X ("pointed" by x_0).

The definition of $\pi_1(X, x_0)$ is known to many physicists. For example, if $X = R^2 - \{0\}$, i.e., a plane minus a point, the homotopy class will be labeled by the algebraic number of turns around this point, so its π_1 (one also says its fundamental group or its Poincaré group) is Z, the additive group of the integer. Since $R^2 - \{0\}$ is homeomorphic to the topological product $S_1 \times R$, one

[24] See in particular the pedagogical paper of Mermin in Rev. Mod. Phys. (1979) and also Thomas (1978); I myself provided a four-page summary in Michel (1977).

[25] And on the union, for all n, of these sets, there is a graded Lie algebra structure (Poenaru and Toulouse, 1979).

[26] This is the case in our study of symmetry defects when H is an invariant subgroup; then $[G:H]$ has a natural group structure, that of the quotient group G/H.

[27] Isomorphisms between topological spaces, i.e., bijective (one-to-one onto) continuous maps, whose inverses are also continuous, are called homeomorphisms.

[28] Since the variables $t_2 \cdots t_n$ are not directly involved in Eq. (13), everything which is proven for $n = 1$ can be immediately extended to any n, thanks to the technical use of Γ_n, $\partial\Gamma_n$ for S_n.

has $\pi_1(S_1, x_0) = Z$. (It is easy to check it directly.) Any circle S_1 on S_k, $k > 1$, can be shrunk to a point, and this is true also of $S_n, n < k$. It is not hard to prove generally that

$$\pi_n(S_k, x_0) = \begin{cases} 0 & n < k \quad (14) \\ Z & n = k \quad (14') \end{cases}$$

(cf. Hilton, 1961; Steenrod, 1957; Milnor, 1965; Gray, 1976; Thomas, 1978),[29] and the reader can also convince himself easily that any image of S_n on a circle S_1 can be shrunk to a point for $n > 1$

$$n > 1 \quad \pi_n(S_1) = 0 . \quad (15a)$$

The corresponding problem for $\pi_n(S_k), n > k$ is much more difficult. (It is not yet solved for all pairs n, k.)

A general result is that for any topological space X,

$$\text{for } n > 1, \quad \pi_n(X, x_0) \text{ is Abelian} . \quad (15b)$$

(Indeed a rotation of π in the plane t_1, t_2 induces a map $\Gamma_n \to \Gamma_n$ homotopic to the identity, but which transforms $g * f$ into $f * g$.) One should also verify that when X is a topological group G, then the group laws defined in the homotopy classes by Eqs. (12) or (13) are identical. Indeed with the convention that $f(t_1, t_2, \ldots, t_n) = 1 \in G$, when $t_1 \leq 0$ or $t_1 \geq 1$ is added to the convention $f(\partial \Gamma_n) = 1 \in G$,

$$H(t_1, t_2, \ldots, t_n; \alpha) = g((1+\alpha)t_1 - \alpha, t_2, \ldots, t_n) \cdot f((1+\alpha)t_1, t_2, \ldots, t_n) \quad (16a)$$

reduces to the product $g * f$ of Eq. (12) for $\alpha = 0$ and to that of Eq. (13) when $\alpha = 1$. Moreover, for topological groups:

$$\pi_1(G; 1) \text{ is Abelian} . \quad (16b)$$

Indeed $K(t, \alpha) = f(\alpha t_1) \cdot g(t_1) \cdot f(t_1) \cdot f^{-1}(\alpha t_1)$ for $\alpha = 0$ and $\alpha = 1$ is equal to $g * f$ and $f * g$, respectively.

We shall give explicit values of $\pi_n (n = 1, 2, 3)$ when G is a Lie group.

Finally we can also define $\pi_0(x)$ as the set of connected components of X. For a topological group G, the connected component of $1 \in G$ is an invariant subgroup G_0 of G; consequently π_0 is a group

$$\pi_0(G) = G/G_0 . \quad (17)$$

For our problem, we need to know the homotopy groups of the orbit $[G:H]$, and we are led to the question: can we determine them from our knowledge of the homotopy groups of G and of H? The answer is yes. This answer is based on fundamental aspects of modern mathematics which I shall merely popularize here.[30]

All mathematical objects with the same mathematical nature—(e.g., topological spaces, groups, pointed sets, i.e., sets with a specified element, etc.—) and all "morphisms" between these objects, i.e., all maps preserving the mathematical structure [respectively, continuous maps, group homomorphisms, map from X to Y with $f(x_0) = y_0$, etc.] form a category.

If $\mathfrak{F}$ and $\mathfrak{G}$ are two categories, a covariant functor F is a function $\mathfrak{F} \xrightarrow{F} \mathfrak{G}$ such that any map $X \xrightarrow{f} Y$ in $\mathfrak{F}$ is transformed into a map $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathfrak{G}$, this function F preserving the composition of maps, i.e., $F(g \circ f) = F(g) \circ F(f)$, and if I_x is the identity map on X, $F(I_x) = I_{F(X)}$. Hence more generally, F transforms isomorphisms into isomorphisms, and commutative diagrams of morphisms of $\mathfrak{F}$ into commutative diagrams of morphisms of $\mathfrak{G}$.[31] We verify that for a fixed n, the π_n are covariant functors defined on the category of pointed topological spaces and valued in the category of Abelian groups when $n > 1$, the category of groups for $n = 1$, and the category of pointed sets for $n = 0$.

Indeed, given two pointed topological spaces: $x_0 \in X$, $y_0 \in Y$, and a continuous map $X \xrightarrow{\phi} Y$ satisfying $\phi(x_0) = y_0$, one defines a map: $\tilde{\phi}_n : M(S_n, X; x_0) \to M(S_n, Y; x_0)$ by $\tilde{\phi}_n(f) = \phi \circ f$. It is compatible with the homotopy equivalence; then one has to verify that the corresponding map between $\pi_n(X, x_0)$ and $\pi_n(Y, y_0)$ is a group homomorphism, generally denoted by ϕ_n^*. Moreover if ϕ is an isomorphism $X \xrightarrow{\Phi} Y$, then ϕ_n^* is a group isomorphism.

In a category of "pointed" objects $\mathfrak{T}$ (in which each

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[29] Milnor (1965), *Topology from the differentiable viewpoint*, is not strictly a book on homotopy. However, it does prove Eq. (14′), whose complete proof is not given in Steenrod (1957). As we shall observe at the end of this paper, it is not obvious that the function ϕ should be required only to be continuous; physics also seems to require some differentiability. Milnor's very pedagogical, 64-page book studies *smooth* (i.e., differentiable) maps between manifolds, while homotopy theory usually uses continuous maps between topological spaces. But Milnor does explain that results for smooth maps can also be valid for continuous maps when the latter can be approximated (uniformly) by the former. It will be useful here to recall the definition of the Brouwer degree. Let f be a smooth map between two n-dimensional compact manifolds: $X \xrightarrow{f} Y$. If the Jacobian df/dx of f at x does not vanish, the corresponding value $y = f(x) \in Y$ is called a regular value of f; for such regular values y, the set $f^{-1}(y)$ [i.e., $\{x \in X, f(x) = y\}$] is finite; by convention when $f^{-1}(y)$ is empty, we also say that y is a regular value of f. Then the set of regular values of any smooth map $X \to Y$ is dense in Y. Assume that X and Y are orientable and orient them; then $\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(df/dx)$ is independent of the regular value y and it is the Brouwer degree of f. Hopf has proved that, if $Y = S_n$, two maps f_1, f_2: $X \to S_n$ are smoothly homotopic if and only if they have the same degree. Moreover every continuous map $X \to S_n$ can be uniformly approximated by a smooth map. So here and for most physical applications we can use differentiable maps. This, however, is to ignore a subtlety: in this approximation of differentiable maps it might be impossible to also require inversibility. Indeed Milnor proved in 1956 that on the topological space S_7 there are several nonequivalent differentiable manifold structures. This extends to most topological spheres of higher dimension.

[30] I checked and found that such popularization is made presently in the best high schools in the U. S. and Europe, and that the corresponding general ideas belong to undergraduate studies in many universities. I feel they should belong to the scientific culture of physicists in this last quarter of the twentieth century.

[31] Particular families of commutative diagrams of a category can be themselves the objects of a new category whose particular families of commutative diagrams can form the objects of a new category whose .... A contravariant functor reverses the arrows, i.e., if $X \xrightarrow{f} Y$, then $F(X) \xleftarrow{F(f)} F(Y)$. One can also define functors of several variables.

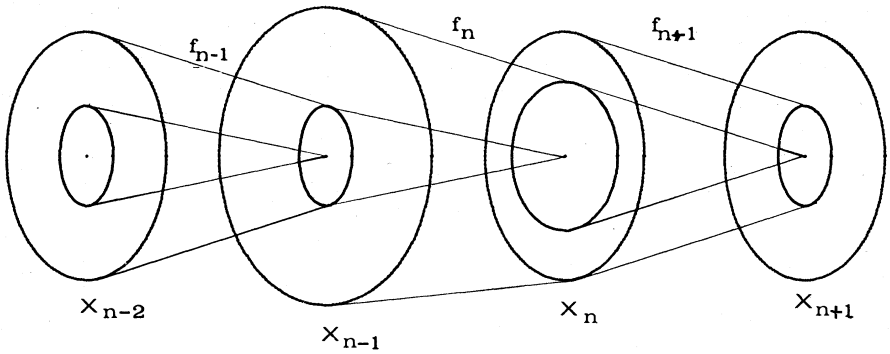


FIG. 5. This is a visualization of an exact sequence of a pointed set [Eqs. (18) and (18′)] or of any richer mathematical structure with a distinguished element: group, vector space ··· .

object has a distinguished point, either naturally—e.g., a group neutral element 1 in multiplicative notation, 0 in additive notation for Abelian groups—or by convention—$x_0 \in X$, base point of the topological space X), the kernel Ker$f$ of the morphism $f: X \to Y$ is the inverse image of $y_0: f^{-1}(y_0) \subset X$, i.e., the set $\{x \in X, f(x) = y_0\}$. The kernel is a substructure of X and so an object of $\mathcal{T}$. Given a sequence of morphisms in $\mathcal{T}$

$$\cdots X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \to \cdots \tag{18}$$

this sequence is exact if for all n,

$$\mathrm{Im} f_n = \mathrm{Ker} f_{n+1}. \tag{18'}$$

Figure 5 is an illustration of an exact sequence between four objects. When 1 is the one-point "pointed set," the exact sequences

$$1 \to A \xrightarrow{i} B, A \xrightarrow{p} B \to 1, 1 \to A \xrightarrow{f} B \to 1 \tag{19}$$

mean that the morphism i is injective (one to one), the morphism p is surjective (onto), and f is bijective (i.e., injective and surjective) so f is an isomorphism.

For example, when H is a topological subgroup of a topological group G,

$$1 \to H \to G \to [G:H] \to 1 \tag{20}$$

is an exact sequence of pointed topological spaces where 1 is a one-point space, the distinguished points of H and G are their units, and H is the distinguished point of the sets $[G:H]$ of left cosets of H. When $H \lhd G$ (H an invariant subgroup of G), then we have the exact sequence of groups:

$$1 \to H \xrightarrow{i} G \xrightarrow{p} G/H \to 1. \tag{21}$$

A diagrammatic definition of a topological product of pointed topological spaces is given in Diagram 3.

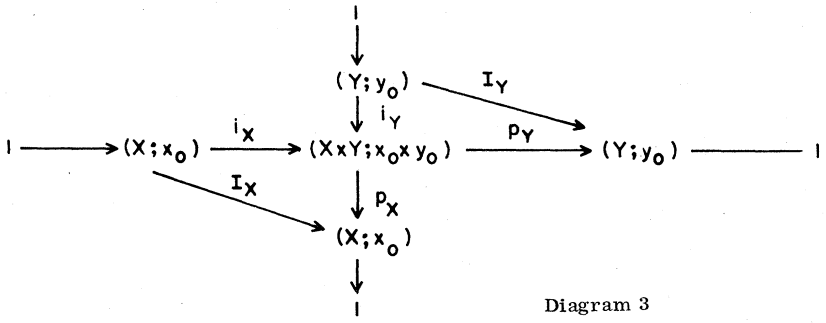


Diagram 3

Equation (20) can also be interpreted as a principal fiber bundle G with base $[G:H]$ and fiber H. Indeed a (topological) fiber bundle E of base B is a surjective continuous map $E \xrightarrow{p} B \to 1$ such that for every point $b \in B$, the inverse images $p^{-1}(b) \subset E$ are all homeomorphic to a topological space, the fiber F. If the homeomorphisms $p^{-1}(b) \to p^{-1}(b')$ are obtained by the action of a group G on E, one has a G fiber bundle. Then G acts on B, and p is G equivariant. Fiber bundles are mathematical objects more and more needed by physicists. A well known example of a nontrivial fiber bundle is a Möbius strip. (The basis is a circle S_1, the fiber a line segment.) There is no global continuous section, i.e., no continuous map $B \xrightarrow{s} E$ such that $p \circ s = I_B$, the identity on B. If such a section s exists, it must be injective. A topological product $X \times Y$ has two fiber bundle structures of base X and Y with, respectively, the injections i_X and i_Y of Diagram 3 as global continuous sections. In the case of groups [Eq. (21)] if there is a continuous section s which is a group homomorphism, then G is the semidirect product of H and G/H [we will denote it by $G = H_{\square}(G/H)$] and, as pointed topological space, $G = H \times (G/H)$. Moreover G is a direct product of the groups H and G/H when $s(G/H)$ is an invariant subgroup of G and is in the centralizer $C_G(H)$ of H in G, i.e., the subgroup of G whose elements commute with every element of H.

To return to the main features of homotopy theory,

the functor π_n transforms the exact sequence (20) into the sequence of group homomorphisms

$$1 \to \pi_n(H) \xrightarrow{i^*} \pi_n(G) \xrightarrow{p^*} \pi_n([G:H]) \to 1 \text{ (not exact)}. \quad (22)$$

The sequence (22) is not exact in general, but it is not difficult to show its exactness at $\pi_n(G)$, i.e., $\mathrm{Im}i^* = \mathrm{Ker}p^*$. To make it exact at $\pi_n(H)$ and $\pi_n([G:H])$ one proves (see Steenrod, 1957) that the sequence has to be changed into an exact sequence

$$\to \pi_{n+1}[G:H] \to \pi_n(H) \to \pi_n(G) \to \pi_n[G:H] \to \pi_{n-1}(H) \to . \quad (23)$$

This is true for all n. In summary, the homotopy groups of Eq. (20) satisfy the long exact sequence (we shall need it only for connected G's, so it stops at $\pi_0(G) = 1$):

$$\to \pi_3([G:H]) \to \pi_2(H) \to \pi_2(G) \to \pi_2([G:H]) \to \pi_1(H) \to \pi_1(G)$$
$$\to \pi_1([G:H]) \to \pi_0(H) \to 1 . \quad (24)$$

A fascinating result! The functor π_n transforms the commutative diagram of Diagram 3 into a similar diagram of group homomorphisms. Since $i_{A^*} \circ p_{A^*} = I_A$ ($A = X$ or Y), where the i_{A^*} are injective and the p_{A^*} are surjective, one can prove that a sequence of aligned arrows is exact at $(X \times Y; x_0 \times y_0)$. This implies that

$$\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y) . \quad (25)$$

(We recall that $\times$ between topological spaces means the topological product, while $\times$ between groups means the direct product of groups.)

We have dropped the base point in the notation. Can we really drop it? What happens if we change the base point of X from x_0 to x_1? Any closed loop f (i.e., an image of a map from S_1) containing x_0 can be transformed into a closed loop containing x_1 by choosing a path $\overline{x_0x_1}$ and counting it both ways (see Fig. 6). So to each homotopy class $\mathcal{H}M(S_1, X; x_0)$ corresponds a class $\mathcal{H}M(S_1, X; x_1)$ by $f \to \overline{x_1x_0} \cup f \cup \overline{x_0x_1}$. However, this correspondence may depend on the chosen path. Indeed, if we choose another path $\overline{\overline{x_0x_1}}$, the new element of $\pi_1(X, x_1)$ corresponding to f, i.e., $\overline{\overline{x_1x_0}} \cup f \cup \overline{\overline{x_0x_1}}$, is a conjugate of the class of $\overline{x_1x_0} \cup f \cup \overline{x_0x_1}$ by the element $\overline{x_1x_0}\ \overline{\overline{x_0x_1}}$ of $\pi_1(X_1, x_1)$. More generally, to the image f of the sphere S_n we can add a path $\overline{x_0x_1}$. The result is again an image of S_n, containing x_1; the choice of two paths $\overline{x_0x_1}$ and $\overline{\overline{x_0x_1'}}$ will define an action of $\pi_1(X, x_0)$ on $\pi_n(X, x_1)$. [For this, one has to show that this action respects the group structure of $\pi_n(X, x_1)$.] We then conclude that, if we forget the base point, homotopy groups are defined up to an isomorphism, and the choice of different base points corresponds to actions $\pi_1(X) \to \mathrm{Aut}\pi_n(X)$. (When $n = 1$ the action is by inner automorphism.) Hence, without base points, the homotopy classes are only defined up to an orbit of π_1 on π_n (conjugation classes when $n = 1$). However when G is a topological group, $\pi_1(G)$ is Abelian, so the action of π_1 on itself is trivial. This is also the case for all actions $\pi_1(G) \to \mathrm{Aut}\pi_n(G)$ when G is a Lie group.[32] In applications, when H is not an invariant subgroup, $[G:H]$ cannot be given a group structure; an isolated defect can only be labeled by an orbit of π_1 on π_n. But with several defects, more detailed information can be expected, as we shall see.

In Appendix G we apply Eq. (24) to the computation of all homotopy groups that we need.

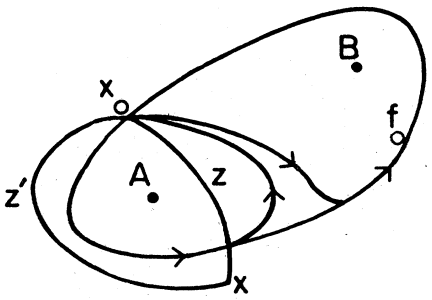


FIG. 6. The space X is the plane minus the two points A and B. Given any base point, say, x_0 let a (respectively, b) be the elements of π_1 corresponding to a loop around A (respectively, B) with positive orientation. An arbitrary closed continuous curve is described by an element of π_1: $\ldots\ b^{n_2}a^{m_2}b^{n_1}a^{m_1}$ ($m_1,\ n_i \in Z$) which specifies that this curve has turned $|m_1|$ times around A (the sign of m_1 corresponds to the common orientation of these turns), then n_1 times around B, then m_2 times around A $\ldots$. One says that π_1 is isomorphic ($\sim$) to $F(2)$—the "free" group with two generators. Since this is true for any base point, $\pi_1(X, x_0) \sim \pi_1(X, x_1) \sim F(2)$. A path x_1x_0, say, $\overrightarrow{x, zx_0}$, between x_0 and x_1 allows us to transform a curve f_0 with base point x_0 into a curve with base point x_1:

$$\Phi(f_0) = \overrightarrow{x_1zx_0} \cdot f \cdot \overrightarrow{x_0zx_1},$$

and a curve with base point x_1 into a curve with base point x_0

$$\psi(f_1) = \overrightarrow{x_0zx_1} \cdot f_1 \cdot \overrightarrow{x_1zx_0} .$$

These correspondences define the two isomorphisms

$$\pi_1(X, x_0) \xrightarrow{\Phi} \pi_1(X, x_1)$$

and

$$\psi = \Phi^{-1}.$$

Another path $x_1z'x_0$ defines another isomorphism Φ'. The correspondence $\Phi(f_0) \to \Phi' f_0)$, which can be written explicitly

$$\Phi'(f_0) = \overrightarrow{x_1z'x_0} \cdot f_0 \cdot \overrightarrow{x_0z'x_1}$$
$$= \overrightarrow{x_1z'x_0zx_1} \cdot \overrightarrow{x_1zx_0} \cdot f_0 \cdot \overrightarrow{x_0zx_1} \cdot \overrightarrow{x_1zx_0z'x_1}$$
$$= (\overrightarrow{x_1zx_0z'x_1})^{-1} \cdot \Phi(f_0) \cdot (\overrightarrow{x_1z'x_0zx_1}),$$

defines an inner conjugation in $\pi_1(X, x_1)$ by the element defined by $x_1zx_0z'x_1$, i.e., the inner automorphism $\Phi' \circ \Phi$ of $\pi_1(X x_1)$.

In this figure f_0 is homotopic to a counterclockwise turn around A followed by a counterclockwise turn around B (see the dotted lines), so it corresponds to $ba \in \pi_1(X_1x_0)$. Similarly $\overrightarrow{x_1zx_0} \cdot f_0 \cdot \overrightarrow{x_0zx_1} \to ba$ and $\overrightarrow{x_1z'x_0} \cdot f_0 \cdot \overrightarrow{x_0z'x_1} \to ab$. Since $\overrightarrow{x_1zx_0z'x_1} \to a$, one has indeed $ab = a(ba)a^{-1}$.

D. The previsions and successes of the classification of defects by Homotopy Theory

To my knowledge, the first paper written explicitly on this topic was that of Rogula (1976). Its title was: "Large deformations of crystals, Homotopy and Defects." This paper was unnoticed by the next publications on the subject (Toulouse and Kléman, 1976; Toulouse, 1976, 1977; Kléman, 1977; Kléman *et al.*, 1977; Volovik and Mineev, 1976; Poenaru and Toulouse, 1977; Kléman and Michel, 1977; Kukula, 1977). However, I do not believe that this paper takes the right

[32] See Steenrod (1957). There is, however, a nontrivial action of $\pi_0(G)$ on $\pi_n(G)$; but we will not have to use it.

approach to the subject. (The π_1 it defines and does not compute is huge; it has $GL(3, Z)_{\square}Z^3$ as an homomorphic image and is independent of the crystallographic group.) Toulouse and Kléman (1976) introduced a general scheme for all ordered phases and made a prediction (not yet verified!) that the vortex lines of the superfluid phase A of ^{3}He will annihilate by pairs, just as the line defects in nematics do, and not as the vortex lines of ^{4}He superfluids do, since they carry an integer n which is a topologically conserved quantum number. They classified defects by the homotopy groups of the manifold of "internal states" which contains the set of Landau parameters [as they are, for instance, defined in Landau and Lifschitz statistical mechanics (Landau and Lifschitz, 1958)]. In fact the manifold of internal states to which they referred is the orbit $[G:H]$ of "positions" (when G is the Euclidean group) or of "internal states" (when G is more general, e.g., when G contains some gauge transformation). Of course physicists studying deformations and defects of ordered phases, for instance with the use of the Burgers circuit or the Volterra construction, were using homotopy just as "Monsieur Jourdain was speaking in prose," as is well explained in Kléman (1977); but the explicit use of homotopy for topological classification of defects resulted in interesting and new applications. Some were immediately made by Kléman, Michel, Poénaru, and Toulouse (Toulouse, 1976, 1977; Kléman *et al.*, 1977; Poenaru and Toulouse, 1977, 1979; Kléman and Michel, 1977; Michel, 1977) in France[33] and by Volovik and Mineev (1976, 1977a, 1977b) in the USSR.[34,35] In the last 12 months the list of publications on this subject has more than doubled; it includes Kléman's book (1978) and several papers by Mermin (1978, 1979; Mermin, Mineev, and Volovik, 1978). Here I give only a summary of published applications of the homotopy theory to the classification of topologically stable defects. This will be a guide to the original literature that the reader, by now, should be able to enjoy reading. Up until now the two main fields of application have been superfluid helium and mesomorphic phases.

The gauge group $U(1)$ is completely broken in local states of the superfluid ^{4}He phase, so the orbit of internal states is $[U(1):\{1\}] = S_1$ (just a phase!). Equations (14), (15) [i.e., $\pi_n(S_1) = \{1\}$ except $\pi_1(S_1) = Z$], and (11) show that the only topologically stable defects are lines characterized by one integer. Both groups, Toulouse–Kléman (1976) and Volovik–Mineev (1977a), used this simple and known example to illustrate their general scheme before applying it (Toulouse and Kléman, 1976; Volovik and Mineev, 1976b, 1977a) to the superfluid ^{3}He-A phase for which they both predicted that the only stable defects are line defects annihilating in pairs: indeed, for this phase, the orbit of internal states is $[\mathrm{SO}(3):\{1\}] = P(R, 3)$, so $\pi_0 = \{1\}$, $\pi_1 = Z_2$, $\pi_2 = \{1\}$. This prediction is different from those made before [see references in Volovik and Mineev (1976b), including one of their own publications]; it has not yet been checked experimentally. Moreover Volovik and Mineev (1972, 1976b) also studied the defects of the ^{3}He-B superfluid. The orbit of internal states is $[U(1) \times SO(3):\{1\}] = S_1 \times P(R, 3)$ so $\pi_0 = \{1\}$, $\pi_1 = Z \times Z_2$, $\pi_2 = \{1\}$, i.e., we have only line defects, which are grouped in two types. These authors pointed out that, if the sample size of the superfluid ^{3}He is small enough (10^3 to 10^2 the coherence length, i.e., 10^{-3} cm), so that the spin-orbit interaction can be neglected, more topologically stable defects can appear, including point defects (see Table II for a summary of all results).

For mesomorphic phases, the case of ordinary nematics is the simplest to study (Toulouse and Kléman, 1976; Kléman, Michel, and Toulouse, 1977; Volovik and Mineev, 1976b; 1977a): the orbit of internal states is $[E(3):R^3 \times D_{\infty h}] = [O(3):D_{\infty h}] = [\mathrm{SO}(3):D_\infty] = P(R, 2)$ so $\pi_0 = \{1\}$, $\pi_1 = Z_2$, $\pi_2 = Z$ (see Appendix G). That is, there are line defects which annihilate by pairs ($\pi_1 = Z_2$). They are easy (Fig. 7) to see in a microscope; indeed, the etymology of nematics is from the greek $\nu\tilde{\eta}\mu\alpha$ for "thread." The point defects are also easy to see; what is their topological character? As we explained in the previous section, when the base point is forgotten, the homotopy classes of continuous maps of S_n are not classified by the elements of π_n but only by the orbits of the action of π_1 on π_n. Here, when the orbit space is $P(2, R)$, $\pi_1(P(2, R)) = Z_2 = \{1, \alpha\}$ acts on $\pi_2 = Z$ by $\alpha \cdot n = -n$ (since $\mathrm{Aut}Z = Z_2$, this is the only nontrivial action). Physically, the base point is irrelevant, and an isolated point defect in an ordinary nematic can only be given the absolute value of an integer, as a topological quantum number, e.g., $|n| = 1$ for the most usual case. However, when two point defects A and B are near each other, their relative sign can, in principle, be observed: for instance, if $|n_A| = |n_B| = 1$, and if they coalesce, will they annihilate ($n_A + n_B = 0$) or form a new point defect with $|n| = |n_A + n_B| = 2$? Moreover, as explained by Volovik and Mineev (1977a), if a line defect is passed between two point defects, their relative sign is changed [see the end of Sec. II.D and Fig. 3 of Volovik and Mineev (1977a)].

More generally, for all mesomorphic phases, there exist topologically stable point defects if, and only if, the global symmetry group H of the phase contains all rotations around an axis [i.e., H has a subgroup $\mathrm{SO}(2) = U(1)$]. This is easily proven in Appendix G; more precisely, in that case $\pi_2([E(3):H]) = Z$. According to Table I this occurs only for ordinary nematics and for smectics A. For this latter phase, π_1 is the semidirect product $Z_{\square}Z_2$ (see Kléman and Michel, 1977, or Appendix G); all its Z_2 subgroups also act nontrivially on $\pi_2 = Z$. So we can repeat for the topologically stable point defects of smectics A what has been said about the

[33] My first work on the subject was done, and lectures on it presented, at the University of Montreal-CRM summer school, June 1976, and at the University of Michigan, Ann Arbor. The work of Kléman, Michel, and Toulouse (1977) was a preprint of this latter institution (October 1976), submitted to *Physical Review Letters* and refused.

[34] Although their first letter (Volovik and Mineev, 1976b) was submitted six months after that of Toulouse and Kléman, they did this work independently.

[35] I heard from Professor P. W. Anderson that a Princeton student, J. Kukula, gave in his senior thesis (1977) the topological classification of defects in cholesterics; this work was not published, since the beautiful preprint by Volovik and Mineev had appeared in the meantime.

TABLE II. Topological classification of defects in superfluids and mesomorphic phase. Groups $\pi_n([G:H]), n=0,1,2,3$. [a]

Phase	G	H	π_0	π_1	π_2	π_3
^{4}He	U(1)	1	1	Z	1	1
^{3}HeA	SO(3)×SO(3)×U(1)	SO(3)×U(1)	1 [b]	Z_2	1	Z
^{3}HeA (small size)	SO(3)×SO(3)×U(1)	SO(2)×U(1)×Z_2	1	Z_4	Z	$Z\times Z$
^{3}HeB	U(1)×SO(3)	1×SO(2)	1	Z	Z	Z
^{3}HeB (small size)	U(1)×SO(3)	1	1	$Z\times Z_2$	1	Z
	Mesomorphic			Superfluids		
Nematics ordinary	E(3)	$R^3_{\square} D_{\infty h}$	1	Z_2	Z	Z
Nematics exceptional	E(3)	$R^3_{\square} F$	1	$\bar{F}_0$	1	Z
Cholesterics (chiral)	$E_0(3)$	$R^2_{\square}$ ($R_{\text{hel}}{}_{\square} D_2$)	1	$\bar{D}_2=Q$	1	Z
Smectics A	$E(3)$	$(R^2\times Z)_{\square} D_{\infty h}$	1	$Z_{\square} Z_2$	Z	Z
Smectics C	$E(3)$	$(R^2\times Z)_{\square} C_{2h}$	1	$Z_{\square} Z_4$	1	Z
Smectics C (chiral)	$E_0(3)$	$(R^2\times Z)_{\square} C_2$	1	$Z_{\square} Z_4$	1	Z
Rod lattices usual	$E(3)$	$(R\times Z^2)_{\square} D_{6h}$	1	$Z^2_{\square} \bar{D}_6$	1	Z
Rod lattices exceptional	$E(3)$	$\text{Ext}(R\times Z^2, F)=H$	1	$\text{Ext}(Z^2, \bar{F}_0)$	1	Z
Crystals	$E(3)$	$\text{Ext}(Z^3, F)=H$	$\begin{cases} Z_2 \text{ if } F=F_0 \\ 1 \text{ if } F>F_0 \end{cases}$	$\text{Ext}(Z^3, \bar{F}_0)$	1	Z

[a] G is the broken symmetry group of physics, H the subgroup of (preserved) symmetry of the phase, $\times$ is direct, $\square$ semidirect, group product. $E(3)=R^3_{\square} O(3)$ is the Euclidean group in 3 dimensions, $E_0(3)=R^3_{\square} SO(3)$ its connected component subgroup. F is a finite subgroup of $O(3)$; $F_0=F\cap SO(3)$. For the list and specific notation (e.g., D_2, D_6, C_{2h}, C_2) see Appendix A (where $D_{\infty h}$ is also defined). Let θ be the covering homomorphism $SU(2)\xrightarrow{\theta} SO(3)\to 1$, then $\bar{F}_0=\theta^{-1}(F_0)$ (e.g., $\bar{C}_2=Z_4, \bar{D}_2=Q$, the quaternionic group generated by $i\tau_k$ where τ_k are the Pauli matrices). $\text{Ext}(A,K)$ means a group M which has the Abelian group A as invariant subgroup and $K=M/A$. All M are defined when the action $K\to \text{Aut}A$ is given; here $K=F$ or $\bar{F}_0$ the action is imposed by $M<E(3)$ or $<\overline{E_0(3)}=R^3_{\square} SU(2)$. Note that $SO(2)\sim U(1)$. Topologically stable (t.s.) wall defects exist only in some crystals; t.s. line defects and t.s. print defects are classified, respectively, by the homotopy groups π_1 and π_2 of the orbit $[G:H]$, while its π_3 classifies the configurations.

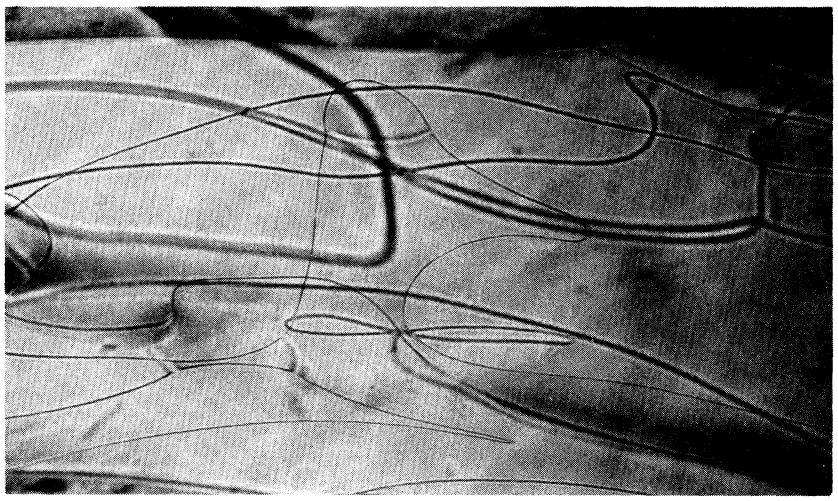
[b] In Volovik and Mineev (1977a), slightly after their earlier work (1976), these authors added the prediction of surface defects, represented by $\pi_0(SO(3)\times Z_2)=Z_2$.

point defects of ordinary nematics, with the following qualifications: the elements of π_1 can be written (n,ε) with $\varepsilon^2=1$; they obey the group law $(n,\varepsilon)(n',\varepsilon')=(n+\varepsilon\cdot n', \varepsilon\varepsilon')$; outside the identity $(0,1)$ there is an infinity of conjugation classes with two elements $(n,1)$ and $(-n,1)$, and two conjugation classes with an infinity of elements; we denote these classes (even, -1) and (odd, -1) and label them "e" and "o". Therefore the topological types of isolated line defects for smectics A can be denoted by $|n|$, e, o. The relative sign of two points defects changes when ones passes an e or an o line defect between them.

While for superfluids and ordinary nematics $\pi_1([G:H])$ is Abelian, for all other mesomorphic phases (except for crystals with the lowest symmetry among the 230 symmetry classes and for the few hypothetical lowest symmetry classes of exceptional nematics) the first homotopy group is non-Abelian. As we already explained, isolated line defects are then classified by the conjugation classes of π_1. When two line defects A and B are observed, there are more observables than their conjugation classes $\hat{a}$ and $\hat{b}$: indeed, when a is chosen arbitrarily in the conjugation class $\hat{a}$, the representative b of B in the conjugation class $\hat{b}$ is fixed up to a conjugation by $\alpha\in\mathfrak{C}(a)$, the centralizer of a in π_1; so ab and $a\alpha b\alpha^{-1}$ are conjugates and they are also conjugate to ba (take $\alpha=a^{-1}$): all these conjugate products define the conjugation class of the line defect obtained when A and B coalesce. Moreover, if, for any $\alpha\in\mathfrak{C}(a)$, a and $\alpha b\alpha^{-1}$ commute, i.e., $a\alpha b\alpha^{-1}a^{-1}\alpha b^{-1}\alpha^{-1}=1$, then $ab=ba$ and this commutation (or noncommutation) property is independent of the choice of representatives when the classes of a, b, and ab are known. Poenaru and Toulouse (1977) have established the physical interpretation of this mathematical property: the commutativity or the noncommutativity corresponds to the possibility or the impossibility for the two line defects' to cross each other.[36]

In Table II, for all families of mesomorphic phases we characterize the global symmetry group H and give the corresponding homotopy groups $\pi_n([E(3):H])$ for $n=0,1,2,3$. We have still to clarify the physical meaning of $\pi_0([G:H])$ and of $\pi_3([G:H])$. As we have seen, π_0 will classify topologically stable surface defects. If the broken symmetry group G is connected, the orbit $[G:H]$ of positions or of local states is also connected since it is the image of G by a continuous map; then

[36] These authors have also generalized this mathematical property to p- and q-dimensional defects in a $(p+q+1)$-dimensional space by considering the graded Lie algebra constructed by Whitehead on the set of all homotopy groups of $[G:H]$.



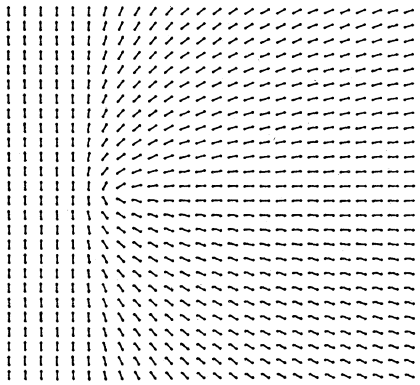


FIGURE 7B

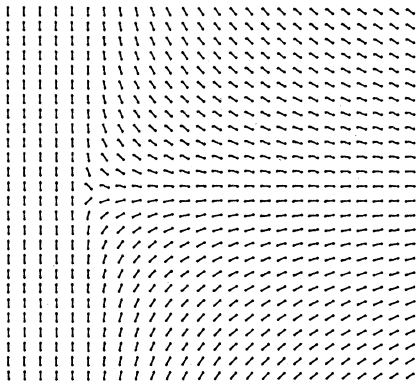


FIGURE 7C

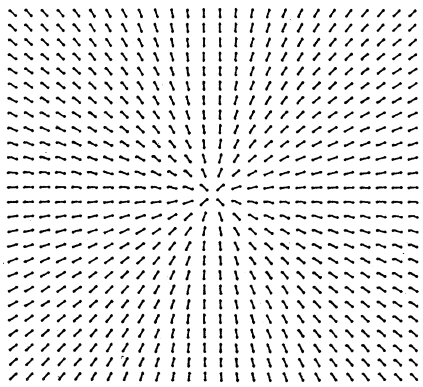


FIGURE 7D

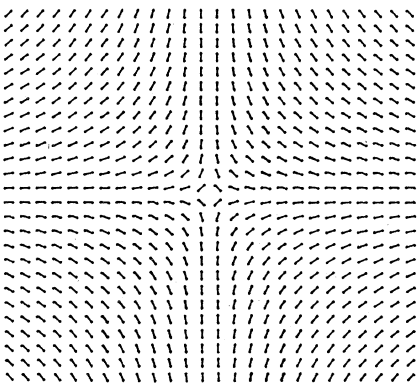


FIGURE 7E

FIG. 7. Line defects in nematics. This photograph has been made by M. Kléman. Here we see clearly two kinds of line defects: the thin lines, which represent topologically stable defects (index $\pm 1/2$), and the thick lines with bright sides, which represent topologically unstable defects with a probable index ± 1. We have defined in Ftn. 26 the Brouwer degree of a map $S_n \to S_n$: it is very intuitive in the simplest case $S_1 \xrightarrow{h} S_1$, since degree (h) is the algebraic number of turns on the circle S_1 of the image $h(\varphi)$ when the azimuth φ makes one positive turn (from 0 to 2π). Similarly one defines the index of the zero $v(x_0)$ of a continuous vector field on a two-dimensional surface: in any local coordinate system in the neighborhood of x_0, it is the degree of the correspondence h: azimuth of x (near x_0) $\to$ azimuth of $v(x)$; these two azimuths are defined modulo 2π. In the case of a field of directors $n(x)$, corresponding to the local orientation (in a plane) of a nematic, the azimuth of $n(x)$ is defined modulo π and the custom in physics is to define the index of a zero $n(x_0)$ multiples of 1/2. Examples of index 1/2, $-1/2$, 1, and -1 are given in Fig. 7(b), 7(c), 7(d), and 7(e), respectively. These figures represent the field of orientation of a nematic near a line defect and in a plane perpendicular to it. The index is conserved when the perpendicular plane is moved along the line defect. It is a good exercise to verify that all half-integer indices correspond to topologically stable line defects while the integer indices describe topologically unstable line defects.

$\pi_0([G:H])=0$: there are no wall defects. This is the case for superfluid states and more generally for all liquid or liquid crystal phases; as a general rule singularity surfaces in liquid do not seem to be stable, as is predicted by the homotopy classification for mesomorphic phases made with chiral molecules. In these cases the isotropic liquid phase has a rotating power for optical polarization, so its symmetry group cannot contain reflections: it is $E_0(3)$, the connected Euclidian group, generated by rotations and translations. These optically active chemical compounds often have chiral mesomorphic phases (chiral smectic C or cholesterics), whose rotating power can be larger by several orders of magnitude than that of the isotropic phases. Homotopic classification would predict wall defects if nonchiral molecules [$G=E(3)$, e.g., racemics] could form chiral phases [i.e., with a global symmetry group $H<E_0(3)$]. No such phases have yet been observed: they seem to be fundamentally unstable. The situation is very different for crystals. Crystal atoms or ions have no chirality. Among the 230 crystallographic classes in three dimensions, 65 correspond to crystallographic groups $H<E_0(3)$ [among these 65 there are 11 pairs of isomorphic groups, conjugate in Aff(3) but not in $\mathrm{Aff}_0(3)$]. Crystals belonging to these 65 crystallographic classes do have topologically stable wall defects, called "twin boundaries": indeed they annihilate by pairs as required by $\pi_0([E(3):H])=Z_2$.

We define at the end of Sec. II. A "topologically stable configurations." The continuous function $\phi: V \to [G:H]$, describing the local states of the medium in the volume V, is defined everywhere on V but it is not homotopic to a constant. This can happen when the domain V occupied by the phase is not contractible—for instance, if V is the volume between two coaxial vertical cylinders, bounded by two horizontal planes. Consider this volume filled with a nematic whose molecular long axis (the director) is horizontal and everywhere orthogonal to the horizontal radial direction: hence on the boundary ∂V this axis is tangential; this is actually the case in many physical situations where the nematic molecules prefer to be parallel to the wall of the container. We have described a configuration; it is topologically stable and has no defects. If V is contractible, we have proved, in the same fashion as Eq. (29), that every continuous function $V \xrightarrow{\phi} X$, valued in an arbitrary topological space X, is homotopic to a constant. This would exclude the existence of topologically stable configurations if we were not *imposing some boundary conditions* (making V equivalent to a noncontractible space) which are naturally suggested by physics. Here we define and discuss what we shall call the "normal" configurations: those which may exist inside an ordered medium otherwise in a perfect state. Consider a (contractible) domain V occupied by an ordered phase in a perfect state everywhere outside a ball B_3 of radius a, so that the function $\phi: V \to [G:H]$ describing the local states is constant on $V-B$. Of course V can be extended (continuously) over B with the same constant value. Given a function ϕ, constant on the boundary $\partial B_3 = S_2$ of the ball B_3, are there any ways to extend it continuously inside B_3 which are not homotopic to the constant? Since the ball B_3, with all points of its surface identified, is homeomorphic to the sphere S_3, the homotopy classes of the continuous extensions of ϕ are given by $\pi_3([G:H])$. For mesomorphic phases all the $\pi_3([G:H])$ are isomorphic to Z (see Table II).

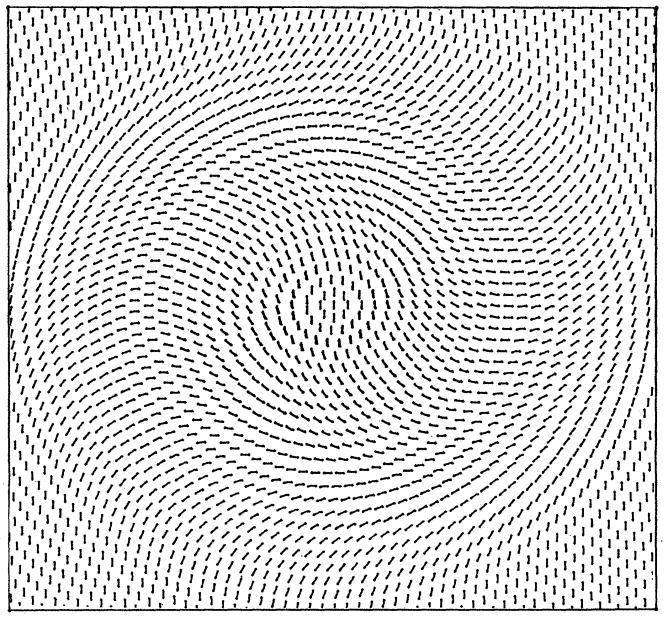


FIG. 8. Example of a configuration in a two-dimensional "nematic" vertically oriented outside the configuration. With the origin of coordinates at the center of the figure, the orientation at the point $\mathbf{r}$ makes an angle $f(r)$ with the vertical, where $f(0)=0$ and $f(r)=\pi$ for $r \geq a$. This configuration is not topologically stable; it can be continuously transformed back into the perfect "nematic" vertically oriented. Note indeed that $[G:H]=S_1$ and $\pi_2(S_1)=1$.

Here is a classical construction of a configuration belonging to the homotopy class $n \in Z$. Take as the origin of coordinates the center of B_3, the ball of radius a. Choose a continuous function $f(t)$ such that $f(0)=0$ and $f(t)=1$ for $t \geq 1$; such a function can even be infinitely differentiable, e.g.,

$$f(t)=\begin{cases} 1-e^{-t^2/(1-t^2)} & 0 \leq t \leq 1 \\ 1 & 1 \leq t \end{cases}. \tag{26}$$

Then, the position of the local state at the point $\mathbf{r}$ is obtained from that of the local state everywhere constant in $V-B_3$ by a rotation of axis $\mathbf{r}$ and angle $2\pi n f(r/a)$.[37]

As Table II shows, the values of π_3 homotopy groups predict no such configurations in superfluid ^{4}He but do predict them for the two superfluid phases of ^{3}He. These configurations may have been observed, but not the line defects, yet.

[37] We have just defined smooth maps $S_3 \xrightarrow{g_n} P(R,3) \sim \mathrm{SO}(3)$. The reader can easily verify, from the definition we gave of the Brouwer degree in Ftn. 26, that deg $(g_n)=2n$. Since S_3 is the universal, double covering of $P(R,3)$, the homotopy class is n. We could have defined $g_n = \theta \circ h_n$: $S_3 \xrightarrow{h_n} \mathrm{SU}(2) \xrightarrow{\theta} \mathrm{SO}(3)$; in Appendix G we define θ and we give an explicit identification of S_3 (given by B_3 with all boundary points identified) with the elements of SU(2). With this identification, h_1 is the identity map and $\deg(h_n)=n$.

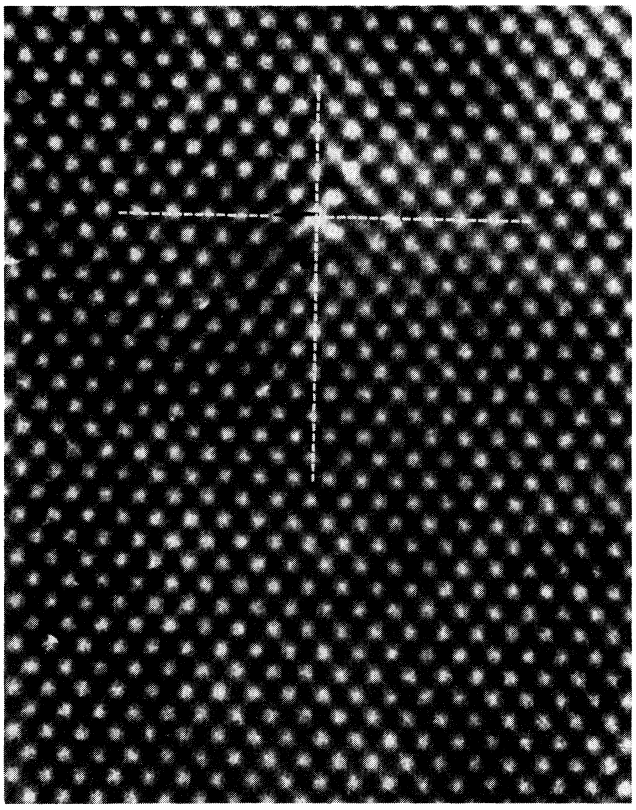


FIG. 9. Dislocation in a germanium crystal. This electron microscope photograph has been obtained by A. Bourret (CENG, Grenoble). The plane of the picture is perpendicular to the 110 axis; the Burgers vector of the dislocation is $1\bar{1}0$.

E. Limitations and extensions of the topological stability

It is outside the scope of this paper even to summarize the abundant literature on defects and configurations of the superfluid phases of ^{3}He before and after the explicit introduction of homotopy.[38] But it can be said that homotopy has become essential in the study of this subject. Not only has it corrected past errors, but it does help, by the tool of the long exact homotopy sequence, to relate defects inside the phase to defects on the surface (bojums).[39]

The study of defects in the mesomorphic phases is much older. The importance of crystal dislocations (Fig. 9) has made of them a whole chapter in solid state physics, and, as we mentioned in the introduction, implicit use of homotopy methods has been made to study them (Volterra construction, Burgers circuit). One could say that, for a given crystal symmetry, a clear-cut classification of the topologically stable defects is provided by the homotopy groups of $[E(3):H]$ where H is the crystallography group. But this is only of academic value: the energetic considerations are much more important for the existence and stability of these defects than their topological classification. The situation is quite different for liquid crystals; to deform them requires very little energy. Their defects show up immediately under a microscope; they are easy to create, to move, to coalesce. Historically, it is from their study that G. Friedel (1922) deduced the symmetry of some liquid crystal phases (e.g., from the most easily seen defects of smectics, the homofocal conics, he deduced their layered structure). Nematics defects were easily classified. However, the classification of defects in cholesterics was not achieved until 1970 by J. Friedel and Kléman (1970). It is now clearly understood that the explicit homotopy method ($\pi_1 = Q$) will surely be the one to use from now on for pedagogical purposes. The experimental study of defects and configurations in cholesterics and smectics led to a rich variety of phenomena whose classification was more akin to a zoological approach (see, for instance, the series of six beautiful papers by Bouligand (1972a, 1972b, 1973a, 1973b, 1974a, 1974b). The scale of the observation is relevant for the topological classification of defects: e.g., a cholesteric seen in a scale small compared to the helicoïdal period (but large compare to the molecular size) is just a smoothly deformed nematic. This scale effect has also been taken in account in Table II for superfluid ^{3}He. The homotopy approach has shed new light on the subjects of cholesterics and line defects in nonchiral smectics C, as shown by the studies of Bouligand (1978) and Bouligand and Kléman (1979), respectively. However the situation is not so clear for smectics A; the point defects do not seem to have been observed. New mesomorphic phases, with symmetries not yet observed, will surely be discovered, and the homotopy classification of their defects will *a priori* be useful.

The topological classification of defects and configurations of ordered phases is less than 4 years old. It has still to be applied to many other ordered media in which topological stability is relevant and not completely eclipsed by energetic considerations. This is a very broad domain of applications: superconducting phases, magnetic phases, two-dimensional phases—e.g., crystallike, superfluid phases, magnetic bubbles and lines, even fingerprints.... There are two other obvious extensions of the domain of applications: First are ordered phases in the presence of external fields which produce dynamical effects. Such external fields generally decrease the symmetry of the phase, inducing a new classification of the topological stability of defects and configurations. Second, we must study the transitions between two ordered phases with global

[38] Representative articles from before the introduction of homotopy are Ambegaokar *et al.*, 1974, 1975; Mermin, 1976; Chechetkin, 1976; Volovik and Mineev, 1976a; Maki and Kumar, 1977. Articles written since the introduction of homotopy include Toulouse and Kléman, 1976; Volovik and Mineev, 1976b, 1977a, 1977b, 1979; Kukula, 1977; Cross and Brinkman, 1977; Anderson and Toulouse, 1977; Shankar, 1977; Mermin, 1977a, 1977b; Bailin and Love, 1978, 1979; Garel, 1978; Stein *et al.*, 1978; Mermin *et al.*, 1978; Volovik, 1978. This list is by no means intended as complete. See the excellent review by Mermin (1979).

[39] Indeed one has also to study the defects on the surface, both from the energetic and from the topological point of view. For the latter, one can use relative homotopy: instead of the base point x_0 one considers a subspace A of X (e.g., the boundary ∂X); the elements of $\pi_n(X;A)$ are the homotopy classes of functions $\Gamma_n \xrightarrow{f} X$ such that $f(\partial\Gamma_n) \subset A$. One proves (e.g., Hilton, 1961) for the homotopies $\pi(A)$, $\pi(X)$, $\pi(X;A)$ a long exact sequence similar to Eq. (24).

symmetry groups H_1 and H_2; then, as in Kléman and Michel (1977), we must ask the question: what happens to the defects of phase 1 after the transition to phase 2? This is especially relevant when the transition is second order with spontaneous symmetry breaking from H_1 to H_2. Relative homotopy and the long exact sequence are the natural tools for answering this and similar questions (e.g., defects in a phase with core made from another phase).

Of course, like any other method, the homotopy approach must have limitations. There are several of these; to know them will help us to evaluate better what can be expected from homotopy and how the method may be used to greatest advantage.

(1) Homotopy is only a crude topological classification. When a function ϕ: $\Omega \to [G:H]$ is homotopic to a constant, and, when Ω is not simply connected (and this is always the case when the volume V itself is not simply connected), there might be other topological obstructions to extend the function ϕ from Ω to V. In that case there are other topologically stable defects given, not by homotopy, but by classes of cohomology of Ω with coefficients in $\pi_1([G:H])$. When this homotopy group is not Abelian the general mathematical theory is difficult.

(2) In the usual case homotopy is applied to arbitrary continuous deformations. But some deformations of our ϕ function might require too much energy: for instance, the layers in many smectics are mono- or bimolecular, so their mutual distance cannot be increased beyond some elastic limits. If very strong continuous deformations are imposed upon the medium they may produce, temporarily, an increasing set of new defects; note that, if ϕ is still continuous on S_1 or S_2 considered in V, this does not change its homotopy class and there occur no contradictions to the proposed scheme for the topological classification of defects. However, one may wonder if there is not a better scheme based on a more refined homotopy theory, using smooth deformations and preserving some differential structure or some integrability conditions. This is the case for the homotopy of foliations,[40] a field in which much progress has been made recently; but no complete classification has yet been achieved for foliations of codimension one, and particularly for measured foliations which seem required by the smectics, except for the dimension 2 (Poenaru, 1978) (this result is essentially due to Thurston). See also a relevant paper of Thom (1978). Let $\mathbf{n}(x)$ be a unit normal vector at x to the smectic layer. The integrability condition for the existence of layers is $\mathbf{n} \cdot \mathrm{rot}\mathbf{n} = 0$ [see, e.g., Kléman (1978), Chap. 5]. To have parallel layers (measured foliation) $\mathrm{rot}\mathbf{n} = 0$ is required. It is a theorem of Reeb (1952)[41] on foliation that no nontrivial smectic configuration can satisfy either integrability condition. Similarly one can establish the impossibility of smooth, topologically nontrivial configurations in crystals.

(3) Obviously, topological methods should not be based only upon our knowledge of the symmetry groups G and H, but also upon the dynamical equations. An example of this combined approach is the work of Finkelstein and Weil (1978) on plasmas and magnetic fields, which has produced a suggestion to explain the spiraling of galaxies by the configurations (they say kinks) of the magnetic field. For Hamiltonian dynamics, as well as for arbitrary differential equations, topology offers a rich variety of differential applications [for an interesting elementary book on this subject, read Arnold (1974)]. However, the situation is quite different for systems of partial differential equations (Thom, 1978). Meanwhile, there has been intense activity in non-Abelian gauge field theories and other domains of nonlinear physics, mainly in an effort to find soliton-type solutions (they are topological configurations) and solutions with given types of singularities (Dirac monopoles, merons). Most of these problems require topological considerations, mainly homotopy, and physicists are becoming increasingly eager to use these mathematical tools.

(4) Finally, topological stability is limited by quantum effects (tunneling). In ordered media local states have fluctuations which increase when one approaches T_c, the critical temperature for the transition to the more symmetric (i.e., the isotropic) phase, and topologically stable defects have a much larger probability of disappearing (Kléman *et al.*, 1977), for example, in the annealing process. Instantons are topological configurations in space-time, which correspond to quantum effects. Also there are intimate relations between topological quantum numbers, quantization of Dirac monopoles and vortex lines in superfluids and superconductors.

F. Hidden symmetry

Defects in solids are technologically very important. They are naturally found in all materials and they have a great influence on most of the physical properties of the crystals. Symmetry defects and configuations appear also in many natural materials and biological tissues, as well as in purely synthetic phases of matter; they form a fascinating subject of physics which will greatly expand. Homotopy is only one, and will continue to be only one, of the theoretical tools used by physicists for the study of this subject; but it comprises such fundamental concepts that it is absolutely indispensable.

In conclusion I should like to emphasize one of the more important implications of the simple relation between the type of symmetry breaking, that is from G to H, and the topological classification of defects and configurations by the homotopy groups $\pi_n([G:H])$, $0 \leq n \leq 3$; its converse (from H to G) can also be used efficiently. This has already been done historically; more and more it will be from topological classification of defects and configurations that one will obtain the symmetry H of new mesomorphic phases, with the hope that many new subgroups $H < E(3)$ will appear in the different broad classes of Table I.

This is even more important when the invariance group G is not known, as is the case for the internal symmetry of the fundamental interactions. Spectacular progress has already been made since 1961: with con-

[40] For reviews on this subject see Reeb (1952), Haefliger (1962).
[41] This was pointed out to me by V. Poenaru.

tributions mainly by Goldstone, Higgs, Glashow, Salam, Ward, Weinberg, and t'Hooft, the local gauge invariance $U(1)$ of quantum electrodynamics has been extended to $U(2)$. This unification for leptons of the electromagnetic and weak interaction has to be used identically for the charged leptons ε, μ, τ, except for one parameter, their masses. This $U(2)$ group is broken into $U(1)$.

Just as van der Waals forces between molecules are secondary effects of the Coulomb interaction between nuclei and electrons, the present strong interactions between baryons and mesons are probably secondary effects of the fundamental interaction between the quarks and antiquarks which constitute them. This interaction has an exact symmetry, the SU(3) local gauge group; it is mediated by an octet of massless colored gluons. Presently, we know only that the number f of colored triplets of quarks and of antiquarks is at least five. Quarks have different masses; and the flavors of f quarks are quantum numbers of an unknown broken group: the unified weak and electromagnetic interaction breaks this group in a very specific manner (e.g., Cabibbo angle) not completely known. Most particle physicists think that the exact invariance group $H = \mathrm{U}(3)$ is the preserved subgroup of the most fundamental invariance group, G, which acts on the degrees of freedom outside space-time. Any choice of candidate for G predicts the existence of possible singularities and configurations (among them t'Hooft-Polyakov monopoles and instantons).

Why not also include gravitation, the only other known interaction (and the first observed by man!)? It may be that we shall then have to replace the Lie algebra of the invariance group by a graded Lie algebra generating a "supersymmetry." All this is based on the hypothesis that the physical laws of our universe come from a broken symmetry solution of a much more symmetrical physics that we are trying to discover.

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APPENDIX A: CLOSED SUBGROUPS OF SO(3) AND O(3); LITTLE GROUPS OF THEIR IRREDUCIBLE REPRESENTATIONS

In this appendix we use the concepts of orbits and little groups (or isotropy groups) which are defined in Sec. I.C of this paper.

We first establish the list, up to a conjugation, of the finite subgroups of SO(3), the three-dimensional rotation group. Let G be one of them, and nontrivial: its order, i.e., its number of elements, is $|G| > 1$. A nontrivial rotation acts on the unit sphere, leaving on it two fixed points that we call the poles of the rotation axis.

Let $\{p\}$ be the set of poles of G; the group G acts itself on $\{p\}$ dividing it into N orbits that we label with the index α. A little group G_{p_α} can only contain rotations around the pole axis: the rotations by angles $2\pi k/\nu_\alpha$, $0 \leq k < \nu_\alpha$ form the cyclic group $C_{\nu_\alpha} \sim G_{p_\alpha}$. Since all little groups of an orbit are conjugated, the

$$n_\alpha = |G|/\nu_\alpha \tag{A1}$$

poles of the orbit α have same multiplicity $\nu_\alpha - 1 > 0$. By summation over the orbits one obtains

$$\sum_{\alpha=1}^{N} n_\alpha(\nu_\alpha - 1) = 2(|G| - 1). \tag{A2}$$

These two equations can be summarized as

$$\nu_\alpha \geq 2, N - \sum_{\alpha=1}^{N} \frac{1}{\nu_\alpha} = 2 - \frac{2}{|G|}. \tag{A3}$$

Since $|G|$ and ν_α are integers larger than one and N is a positive integer, the only possible values of N are 2 or 3 and the only possible solutions for ν_α, n_α are

$\nu_\alpha > 1$	n	n	2	2	n	2	3	3	2	3	4	2	3	5
n_α	1	1	n	n	2	6	4	4	12	8	6	30	20	12
$\|G\|$	n		$2n$			12			24			60		
G	C_n		D_n			T			O			Y		
$N_{SO(3)}G$	D_∞		D_{2n}			O			O			Y		

We have already said that C_n is the cyclic group generated by the rotation of angle $2\pi/n$; if one adds a rotation by π around an orthogonal axis one generates D_n. The groups T, O, Y are, respectively, the symmetry groups [$<$SO(3)] of a tetrahedron, a cube or an octahedron, a dodecahedron or an icosahedron; the first n_α is the number of edges, the other two n_α are the number of vertices and faces.

Finally, up to a conjugation, there are two infinite closed subgroups of SO(3): the group of all rotations around an axis, C_∞ and, when one adds the rotations by π around the orthogonal axes, D_∞. From the complete list we obtained, we remark that two isomorphic closed subgroups of SO(3) are conjugated.

We define in Sec. I.C the normalizer $N_G(H)$ of a subgroup H of G; since we will need them, we also gave the list of the normalizers of the closed subgroups of SO(3).

We proceed now to the study of the closed subgroups of the three-dimensional orthogonal group $O(3)$, which is generated by SO(3) and the inversion through the origin: $-I$. That operation commutes with all rotations of SO(3), so $O(3)$ is the direct product:

$$O(3) = \mathrm{SO}(3) \times Z_2(I, -I). \tag{A4}$$

For each closed subgroup H of SO(3), we can form the direct product:

$$G = H \times Z_2(I, -I). \tag{A5}$$

Their traditional notation (due to Schönflies) is

H		C_n	D_n	T	O	Y	C_∞	D_∞
G	n even	C_{nh}	D_{nh}	T_h	O_h	Y_h	$C_{\infty h}$	$D_{\infty h}$
	n odd	S_{2n}	D_{nd}					

When $G<O(3)$ does not contain $-I$, if we multiply by $-I$ the elements of G which are not in $H=G\cap SO(3)$, we obtain a group of SO(3) of same order $|G|$. Conversely, given a closed subgroup of SO(3) which has a subgroup H of index two (i.e., $|H'|=2|H|$, when H' is finite), if we multiply by $-I$ the elements of H' which are not in H, we obtain the other closed subgroups of $O(3)$ not in SO(3):

$$\begin{array}{lccccc} H' & C_{2n} & D_n & D_{2n} & 0 & D_\infty \\ H & C_n & C_n & D_n & T & C_\infty \\ G\begin{cases} n \text{ even} \\ n \text{ odd} \end{cases} & \begin{matrix} S_{2n} \\ C_{nh} \end{matrix} & \begin{matrix} C_{nv} \\ C_{nv} \end{matrix} & \begin{matrix} D_{nd} \\ D_{nh} \end{matrix} & T_d & C_{\infty v}. \end{array}$$

The n-fold rotation axis is thought to be vertical; the index h indicates a reflection through the horizontal plane; the groups C_{nv}, D_{nh}, D_{nd} have n reflections through vertical planes containing the n-fold rotation axis; these vertical planes contain the horizontal axes of rotations by π in D_{nh} and they are bisectors of them in D_{nd}. The group S_{2n} is cyclic; it is generated by the product of $-I$ and of a rotation by an angle π/n. The group T_d is the complete symmetry group of a tetrahedron.

One can show (e.g., Mostow, 1957) that every closed subgroup H of a compact (or finite) group G appears as a little group of a finite-dimensional orthogonal representation. However, only a partial list of these subgroups (defined up to a conjugation) may appear as little groups of the orthogonal irreducible[42] (on the real) representations of G. This partial list gives a first family of selection rules for the possible spontaneous symmetry breakings occuring in Landau theory (see Sec. I.D) or bifurcation theory (Sec. I.E). As an example, consider the group D_∞. Its irreducible representations are all real; they are either two-dimensional and labeled $[m]$ where m is a positive integer:

$$[m]:\ r_\omega \to \begin{vmatrix} \cos m\omega & -\sin m\omega \\ \sin m\omega & \cos m\omega \end{vmatrix},\quad \rho_\omega \to \begin{vmatrix} \cos m\omega & \sin m\omega \\ \sin m\omega & -\cos m\omega \end{vmatrix} \tag{A6}$$

or one-dimensional and labeled [+] or [−]

$$[+]:\ r_\omega \to 1, \rho_\omega \to 1;\quad [-]:\ r_\omega \to 1, \rho_\omega \to -1. \tag{A6'}$$

We have denoted by r_ω the rotation by ω around the vertical axis and by ρ_ω the rotation by π around the horizontal axis of azimuth $\omega/2$. The kernel of the representation $[m]$ is C_m; given a point $x\neq$ of azimuth $\omega/2$ in the representation space, it is invariant under ρ_ω. Hence the C_m's never appear as little groups of irreducible representations of D_∞; only the subgroups D_m, C_∞, D_∞ do appear.

The irreducible representations of SO(3) are all real; up to an equivalence there is only one such representation for each odd dimension $2l+1$, where l is the angular momentum. It seems rather extraordinary that the list of their little groups is not available in the physics literature, but the only attempt I know to produce it is rather recent (Ovrut, 1978); it is obtained by a complicated method and the only explicit results are given for $0\leq l\leq 4$; there seems to be an error for $l=3$. Another paper, Mickelson and Niederle (1970) deals with the complex representations of SU(2) ($2l$ integer ≥ 0); they are complex representations of SO(3) when l is integer. However, as is not the case for finite groups, the complex version of a real representation of a compact group contains in general more little groups than that real representation (for $l=1$ compare the result of this appendix to that of Sec. I.F, where we deal with the polarization states of a spin 1 particle). Projective representations are useful in quantum mechanics, and Bacry (1974) gave a very elegant and powerful method to obtain the list of little groups of the projective space of the irreducible representations of SU(2) (there is an error for the group T which appears for all integers $l\geq 4$ and for $l=2$).[43]

Before studying SO(3) we make some general remarks about little groups of real representations of compact Lie groups. Any such representation $g\to\Delta_l(g)$ yields, by restriction, a real representation of any subgroup $H<G$; one usually calls the degree of subduction, $c_l(H)$, the multiplicity of the trivial representation of H in $\Delta_l|_H$; if H is finite:

$$c_l(H)=\frac{1}{H}\sum_{g\in H}\chi_l(g),\ \text{with } \chi_l(g)=\mathrm{tr}\Delta_l(g). \tag{A7}$$

In all cases, $c_l(H)$ is invariant by conjugation in G. Of course H cannot be a little group when $c_l(H)=0$. We denote by $\mathcal{E}^H$ the set of vectors of the representation space $\mathcal{E}$, invariant under H; they form a vector subspace of dimension $c_l(H)$. Obviously:

Lemma 1. In a real representation l of the compact group G, if there is a subgroup H', strictly larger than H: $H'>H$, such that $c_l(H')=c_l(H)$, then H is not a little group.

Indeed every vector invariant under H is also invariant under the larger group H'. The converse of this lemma is true only for finite groups:

Lemma 2. In a representation l of a finite group, if for all subgroups $H'>H$, $c_l(H')<c_l(H)$, then H is a little group.

This is also true for some subgroups H of a compact group G:

Lemma 2'. In a compact Lie group G, if the set of subgroups larger than a given subgroup H is finite or enumerable, H is a little group of any representation of G such that $H'>H$ implies $c(H')<c(H)$.

Indeed, the space $\mathcal{E}^{H'}$ of vectors of $\mathcal{E}$ invariant by H' is a strict vector subspace of $\mathcal{E}^H$. Since the $\mathcal{E}^{H'}$ are at most enumerable, they cannot fill $\mathcal{E}^H$; so there are

[42] Every real linear representation of a compact group G is equivalent to an orthogonal representation. If such a representation is irreducible on the real, it might be reducible on the complex into two complex representations. Conversely, given an irreducible complex representation of G nonequivalent to a real representation, by taking its direct sum with its complex conjugate one obtains a representation equivalent to an orthogonal representation irreducible on the real.

[43] It is easy to restrict Bacry's results to real projective irreducible representations: one requires that the $2l$-point constellation has the origin as symmetry center; since changes of sign are allowed, the list obtained is not that of little groups.

vectors of $\mathcal{E}^H$ which are not invariant by any subgroup $H' > H$.

This lemma applies to all closed subgroups of SO(3) but the C_n. For the latter, the representations of D_∞ already gave us a counterexample: for the irreducible representation $[m]$, we have

$$c_m(C_m) = 2,\ c_m(D_m) = 1;\ n > m,\ c_m(C_n) = c_m(D_n) = 0\,. \tag{A8}$$

Lemma 2′ applies to D_m: it is a little group. It does not apply to C_m since the (whole conjugate class of) subgroups D_{km} contain C_m as subgroup; and we have seen that C_m is not a little group.

The character $\chi_l(g)$ of the irreducible representation l of SO(3) is $1 + 2\sum_{k=1}^{l} \cos k\omega$ where ω is the angle of the rotation g. We use (A7) to compute the subduction degrees of the finite subgroups

$$c_l(C_n) = 1 + 2E(l/n),\ c_l(D_n) = \tfrac{1}{2}[1 + (-1)^l] + E(l/n)\,, \tag{A9}$$

where

$$E(x) = (\text{largest integer} \leq x) \tag{A9′}$$

$$c_l(T) = b(l) + 2E\left(\frac{l}{3}\right),\quad c_l(O) = b(l) + E\left(\frac{l}{3}\right) + E\left(\frac{l}{4}\right),$$

$$c_l(Y) = b(l) + E\left(\frac{l}{3}\right) + E\left(\frac{l}{5}\right). \tag{A9″}$$

with

$$b(l) = [3 + (-1)^l - 2l]/4\,. \tag{A9‴}$$

It is easy to find (and probably well known to the reader) that

$$c_l(C_\infty) = 1,\quad c_l(D_\infty) = [1 + (-1)^l]/2\,. \tag{A10}$$

SO(3) is the little group of the 0 vector in all its representations and it is the unique little group of its trivial representation $l = 0$. We now assume $l > 0$ and apply lemma 2′; we obtain, from the values of the subduction degrees and the fact that $D_{kn} > D_n$, the results of Table AI, except for the C_n. For those groups we just look for the possibility to find a vector which has one of them as little group. When $l = 1$ and odd, take vectors x and y with little groups C_∞ and D_n, $1 < n \leq l$, respectively; then $x+y$ has C_n for its little group. When $l > 2$ and even, choose x and y such that their little groups are, respectively, D_n and D_{2n}, $2 < 2n \leq l$, with the same "vertical" rotation axis and their azimuths are irrational relative to one another; then $x+y$ has C_n as little group. Finally, when $l \geq 3$, we can choose three vectors x, y, z in the subspaces of the representation [1], [2], [3], of a subgroup D_∞ such that $x+y+z$ has the trivial little group C_1.

TABLE AI. Little groups of irreducible representations of $SO(3)$.

SO(3)	$l \geq 0$ (all l)
D_∞	even $l > 0$
C_∞	odd l
Y	$l = 6, 10, 12$ and $l > 14$ except $17, 19, 23, 29$
O	$l \geq 4$ except $5, 7, 11$
T	$l \geq 3$ except $4, 5, 8$
D_n	$l \geq n$
C_n	odd $l \geq n > 1$, even $l \geq 2n > 2$.

The irreducible representations of $O(3)$ are labeled 1+ or 1− depending whether the symmetry through the origin is represented by $+1$ or -1. In the former case, the little groups are obtained from those of Table AI, by taking their direct product with $Z_2(I, -I)$. We leave to the reader the study of the representations 1−.

In Table AII, we give explicitly the list of nonconjugate little groups for the irreducible representations of SO(3) with $0 \leq l \leq 6$.

APPENDIX B: ELLIPSOIDAL FIGURES OF EQUILIBRIUM OF A ROTATING FLUID AND BIFURCATIONS FROM IT

We consider the simple case: an isolated mass m of an incompressible fluid of density ρ. We look for equilibrium states whose shape is an ellipsoid with semi-axis lengths a_1, a_2, a_3 so $m = 4\pi a_1 a_2 a_3 \rho/3$, and which are in static equilibrium in the rotating frame. One proves that the rotation axis is one of the symmetry axes of the ellipsoid: we label it "3" and denote by Ω_0 the angular velocity. The equilibrium surface is an equipotential: potential energy + kinetic energy = constant

$$G\left(A^{(0)} - \sum_{i=1}^{3} A_i^{(0)} x_i^2\right) + \tfrac{1}{2}\Omega_0^2 (x_1^2 + x_2^2) = C\,, \tag{B1}$$

where G is the gravitational constant and

$$A_{ij\ldots}^{(k)} = a_1 a_2 a_3 \int_0^\infty \frac{u^k\, du}{\sqrt{D}\,(u + a_i^2)(u + a_j^2)\ldots} \tag{B2}$$

with $D = (u + a_1^2)(u + a_2^2)(u + a_3^2)$. If the volume occupied by the fluid is $S(x) \geq 0$, the angular momentum J is

$$J = \Omega_0 \rho \int_{S \geq 0} (x_1^2 + x_2^2)\, d^3x\,. \tag{B3}$$

Equation (B1) can be identified with the equation

$$S(x) = \sum_{i=1}^{3} \frac{x_i^2}{a_i^2} - 1 = 0 \tag{B4}$$

TABLE AII. Little groups of the irreducible representations of SO(3) with $0 \leq l \leq 6$.

l	SO(3)	D_∞	C_∞	Y	O	T	D_6	D_5	D_4	D_3	D_2	C_5	C_4	C_3	C_2	C_1
0	x															
1	x		x													
2	x	x									x					
3	x		x			x				x	x			x	x	x
4	x	x			x				x	x	x				x	x
5	x		x					x	x	x	x	x	x	x	x	x
6	x	x		x	x	x	x	x	x	x	x			x	x	x

whose semiaxis lengths a_1, a_2, a_3 satisfy:

$$(\Omega^2 - 2A_1^{(0)})a_1^2 = (\Omega^2 - 2A_2^{(0)})a_2^2 = -2A_3^{(0)}a_3^2 , \tag{B5}$$

where Ω is the dimensionless $\Omega_0/\pi G\rho$. The first equality can also be written:

$$(a_1^2 - a_2^2)(\Omega^2 - 2A_{12}^{(1)}) = 0 . \tag{B6}$$

It is verified by all axially symmetric ellipsoids ($a_1 = a_2$). Those which moreover satisfy (B5) are called the MacLaurin ellipsoids. However, Eq. (B6) has non-axially symmmetric solutions when $\Omega^2 = 2A_{12}^{(1)}$. They form a one-parameter family of three unequal axis ellipsoids, the Jacobi ellipsoids. The bifurcation point of the Jacobi sequence from the MacLaurin sequence is defined by

$$a_1 = a_2 \quad and \quad \Omega^2 = 2A_{12}^{(1)} . \tag{B7}$$

Bertin and Radicati (1976) showed that this bifurcation with symmetry breaking could also be well described by the Landau theory of second-order phase transitions (Landau, 1938). Poincaré (1885) found a bifurcation from the Jacobi sequence with breaking of the symmetry through the center; he also found an infinity of bifurcations, each one from the sequence defined by the previous bifurcation. Constantinescu, Michel, and Radicati (1979), with the use of group theoretical methods, showed how to obtain symmetry breaking bifurcations and found an infinity of them from the MacLaurin sequence; they occur when the following equation is satisfied:

$$a_3^2 A_3^{(0)} = a_1^{2m} A_{111\ldots11}^{(0)} \quad (m \text{ indices } 1) . \tag{B8}$$

At each bifurcation a new sequence starts with a spontaneous symmetry breaking from $D_{\infty h}$ to D_{mh}.

It can also be predicted that some symmetry may never be broken: a theorem of Liechenstein (1933) states that the symmetry through the plane orthogonal to the rotation axis (the h of $D_{\infty h}$) cannot spontaneously disappear.

Many books have been written on the equilibrium states of a rotating fluid. Appell (1932) and Chandrasekhar (1969) are almost complementary. In his lectures, Bardeen (1972) has extended this problem to general relativity.

APPENDIX C: INVARIANT VECTOR FIELDS IN THE SMOOTH ACTION OF COMPACT LIE GROUPS ON MANIFOLDS

We give some properties of the smooth action (i.e., infinitely differentiable) of a compact Lie group G (a finite group is a particular case of it) on a finite-dimensional real smooth manifold M (of course these assumptions are not all necessary for many of the results presented here). In that case, the orbits are closed submanifolds of M. A linear action is an interesting example of this situation: M is a real vector space $\mathcal{E}$ carrying a linear representation of G. This case is rather general: indeed, Mostow (1957) has shown that when there is a finite number of strata in the action of G on M, there is an injective equivariant map $M \xrightarrow{i} \mathcal{E}$, where $\mathcal{E}$ is a real vector space carrying an orthogonal representation of G (Mostow proved also that if M is compact, the number of strata is finite).

Taking the average by G of any Riemann metric on M endows this manifold with a G-invariant metric (one also says that G acts isometrically on M); in the particular case where $M = \mathcal{E}$, any linear representation of G can be made orthogonal. Then one proves that, given any orbit $G(m)$, there exists a G-invariant neighborhood $V = G \cdot V \supset G(m)$, such that for any point $x \in V$, there is a unique nearest point of the orbit; we denote it by $r(x)$. Since the correspondence $x \to r(x)$ is obtained from an invariant metric, the map r is equivariant

$$\forall g \in G, \quad g \cdot r(x) = r(g \cdot x) . \tag{C1}$$

Furthermore, if $g \in G_x$; then $g \cdot r(x) = r(x)$, i.e.,

$$\forall x \in V, \quad G_x \leqslant G_{r(x)} . \tag{C2}$$

This means that in a neighborhood V of the orbit $G(m)$, all little groups are, up to a conjugation, equal to or smaller than those of the orbit. By definition of the stratum, the equality occurs only for the points of the stratum $S(m)$. Let $\mathcal{S}$ be the set of strata, i.e., the set of conjugation classes of little groups appearing in the action $G \to \mathrm{Aut}M$. We have seen that this set is partially ordered; if H is a minimal little group, and $G_m = H$, then $V \subset S(m)$, and one proves that $S(m)$ is open. When G is finite, it is easy to prove (see, e.g., Michel and Mozrzymas, 1977) that $S(m)$ is open and dense and $G_m = \mathrm{Ker} f$. When G is compact it can still be proven that $S(m)$ is open dense, so there is a unique minimal little group, up to a conjugation, in $\mathcal{S}$ [keep in mind that in general it is not an invariant subgroup of G, so it varies with m in the stratum $S(m)$]. Finally, one defines locally the slice $N(m) = r^{-1}(m) \subset V$, i.e., the set of points $x \in V$, such that $r(x) = m$.

The isometric action of G on M defines at each point m an orthogonal representation of the little group G_m on $T_m(M)$, the tangent plane to M at m. This representation is reducible since, among the subspaces of $T_m(M)$, it leaves invariant $T_m(G(m))$, $T_m(S(m))$, $T_m(N(m))$, the tangent planes to the orbit, the stratum, and the slice at m. Note that $T_m(N(m))$ is the subspace orthogonal to $T_m(G(m))$; we write it as a direct sum of orthogonal subspace

$$T_m(M) = T_m(G(m)) \overset{\perp}{\oplus} T_m(N(m)) . \tag{C3}$$

With the definitions

$$F(m) = T_m(S(m)) \cap T_m(N(m)), \quad K(m) = T_m(N(m)) \cap T_m \times (S(m))^{\perp} \tag{C4}$$

we have the following direct sum of orthogonal subspaces:

$$T_m(M) = T_m(G(m)) \overset{\perp}{\oplus} F(m) \overset{\perp}{\oplus} K(m) , \tag{C5}$$

$$T_m(S(m)) = T_m(G(m)) \overset{\perp}{\oplus} F(m) . \tag{C5'}$$

We can now make precise the nature of the linear representation of G_m on the tangent plane $T_m(M)$. From (C2) we see that this representation is trivial on $F(m)$: all vectors of $F(m)$ are invariant by G_m, whereas $K(m)$ has no invariant vectors. The representation of G_m on $T_m \times (G(m))$, the tangent plane to the orbit, depends only on G and G_m: for example, when G is a semisimple Lie group, this representation is obtained from the adjoint

representation of G (this is the natural representation of G on the vector space of its Lie algebra $\mathcal{G}$) by the action of G_m on the space $\mathcal{G}_m^{\perp}$ ($\perp$ defined by the Cartan Killing metric on $\mathcal{G}$), where $\mathcal{G}_m$ is the Lie algebra of G_m; for instance, if G_m contains a Cartan subgroup H of G (see Appendix D) there is no G_m invariant vector $\neq 0$ in $T_m(G(m))$. Of course, for the generic stratum, $K(m)=0$. If $F(m)=0$, the orbit $G(m)$ is isolated in its stratum, that is, there is a neighborhood V of the orbit $G(m)$ which contains no other orbit of the same type.

A smooth G-invariant vector field on M is a smooth function from M to $T(M)$, the tangent bundle of M; it defines a correspondence $m \to v(m) \in T_m(M)$ such that

$$\forall g \in G_m, \quad g \cdot v(m) = v(m) . \tag{C6}$$

So $v(m)$ is in $T_m(S(m))$: every G-invariant vector field is tangent at each point $m \in M$ to the stratum $S(m)$.

If the vector field is the gradient field of a G-invariant function, it is moreover orthogonal to the orbit (since the function is constant on each orbit) so it is in $F(m)$. Hence when $F(m)=0$, i.e., on orbits isolated in their stratum, the gradient of every G-invariant function vanishes. The converse is also true (Michel, 1971a). This leads to Theorem 2 of the paper.

For finite groups $T_m(G(m))=0$ this theorem applies to any G-invariant vector field; this is also true when m is a fixed point ($G_m=G$) of the compact Lie group G. We do not make here a complete list of the cases when every G-invariant vector field has to vanish at m; we have seen that this is the case when G_m contains a Cartan subgroup; this is also the case if the Euler characteristic of the orbit $G(m)$ does not vanish (e.g., in the case of even-dimensional spheres).

We already noticed that the strata corresponding to maximal little groups are closed. If M is compact, these strata are also compact. When such a stratum contains only a finite number of orbits, these orbits are critical (see Theorem 2). If not, the restriction of any invariant function f to such a stratum must have at least one orbit of maxima and one of minima; since the gradient of f is tangent to the stratum, these two orbits are orbits of extrema for the whole function f. Hence the following theorem.

Theorem C. In the smooth action of the compact Lie group G on the compact manifold M every G-invariant real smooth function on M has at least an extremum on every stratum corresponding to maximal elements in the set of conjugation classes of little groups which appear in the action.

APPENDIX D: INVARIANT POLYNOMIALS AND EQUIVARIANT ALGEBRAS ON AN ORTHOGONAL REPRESENTATION OF A GROUP

Given an action of G on M we can consider its set of orbits: it is usually called the orbit space and denoted by M/G. When G is compact and M is a manifold, the G-invariant Riemann metric on M induces a natural metric on M/G. By the canonical map $M \to M/G$, which assigns each orbit in M to its representation point in M/G, the image of the generic stratum is open dense. In the action of the rotation group SO(3) on the five-dimensional phase space of the three particles produced in a decay at rest, the orbit space is the (two-dimensional) Dalitz plot. There are two strata, generic and exceptional. The generic stratum corresponds to the interior of the Dalitz plot; the three momenta of the particles form a triangle, and the little group is trivial. The boundary of the Dalitz plot is the image of the exceptional stratum; the three momenta are collinear, and the little group is SO(2).

We consider now, as a very interesting example, the action of a simple compact Lie group G on its Lie algebra $\mathcal{G}$: this is the adjoint linear representation of G. The Cartan Killing bilinear form defines its invariant orthogonal product. The simplest case is the rotation group SO(3) acting on the three-dimensional space $\mathcal{E}$ (the Lie algebra law is the vector product that we denote $x \wedge y$). Every direction of $\mathcal{E}$ can be transformed into any other direction, i.e., the unit sphere is an orbit of G [the little group is SO(2)]. This is an exceptional situation: in the general case, there are exceptional strata. The generic, open dense stratum has for little group a Cartan subgroup H: it is a maximal Abelian subgroup of G, so it is isomorphic to $U(1)^r$ [the direct product of r groups $U(1)$], and r is called the rank of the group G. The slice (see Appendix C) $N(m)$ at a point m of the generic stratum can be extended globally; it is the normal subspace to the orbit and a Cartan subalgebra $\mathcal{H}$, as well as a Lie algebra of the Cartan subgroup $H=G_m$. (More generally, at every $m \in \mathcal{G}$, $x \in \mathcal{G}_m$, $x \wedge m = 0$).

Every smooth invariant function on $\mathcal{G}$ is a smooth function of the invariant polynomials. These polynomials form a ring (their sums and their products are invariant polynomials) generated by r polynomials θ_k, $1 \leq k \leq r$: one of them is the Cartan Killing scalar product, $\theta_1(x)=(x,x)$. More precisely, and this is not true in general for other linear representations of G, every invariant polynomial p is a polynomial in the θ_k's. On the generic stratum the θ_k's are algebraically independent and therefore their gradients $d\theta_k/dx$ are linearly independent. The semialgebraic equations of the exceptional strata are given by inequalities and/or equalities among polynomials in the θ_k's; they yield linear relations among the gradients $d\theta_k/dx$. On the one-dimensional strata, all gradients are collinear to x, and the intersection of these strata with the unit sphere are critical orbits: they correspond to maximal little groups.

There is some freedom in the choice of the basic polynomials θ_α, but their degree is well defined

$2,3,4,\ldots,r+1$ for $A_r = \mathrm{SU}(r+1)$,
$2,4,6,\ldots,2r$, $r \geq 2$, for $B_r = \mathrm{SO}(2r+1)$ and $C_r = Sp(2r)$,
$2,4,6,\ldots,2(r-1)$ and r, $r \geq 4$ for $D_r = \mathrm{SO}(2r)$,
$2,6$ for G_2, $2,6,8,12$ for F_4; $2,5,6,8,9,12$ for E_6,
$2,6,8,10,12,14,18$ for E_7; $2,8,12,14,18,20,24,30$ for E_8.

There is a unique (up to a factor) invariant of degree 2, namely (x,x); only the groups SU(n), $n \geq 3$ have a third degree invariant. The quotient group

$$W = N_G(H)/H , \tag{D1}$$

where $N_G(H)$ is the normalizer of a Cartan subgroup, is called the Weyl group of G. It is a finite group whose

order is the product of the degrees of the θ_k's. It acts on the Cartan subalgebra $\mathfrak{C}$, is generated by reflections through hyperplanes, and has exactly $s-r$ such reflections where s is the sum of the degrees of the θ_k's. The intersections of the orbits and strata of G acting on $\mathcal{G}$ with the Cartan subalgebra $\mathfrak{C}$ are exactly the orbits and the strata in the action of W on $\mathfrak{C}$, and the restriction of the θ_α's to $\mathfrak{C}$ are the generators of the W-invariant polynomials on $\mathfrak{C}$. (This situation cannot be generalized to many linear representations. For instance, if the dimension of the orbits of the generic stratum is smaller than or equal to half the dimension of the representation, the slice of a generic point, which always cuts locally each G orbit at one point, contains, when it is extended to the whole subspace normal to the orbit, a continuous part of at least one orbit.)

We consider now the case of SU(n). Its n^2-1 dimensional Lie algebra can be realized as the vector space of Hermitian n by n traceless matrices; the adjoint action of SU(n) is defined by

$$u\in \mathrm{SU}(n),\ uu^*=I,\ x\in \mathcal{S}U(n),\ x^*=x,\ \mathrm{tr}x=0\,, \tag{D2}$$

$$x\to uxu^{-1}\,. \tag{D2'}$$

The Lie algebra law is

$$x_\wedge y=-\tfrac{1}{2}i(xy-yx)\,. \tag{D3}$$

The n-1 basic invariant polynomials can be chosen as

$$\theta_k(x)=\frac{1}{k+1}\mathrm{tr}x^{k+1}\quad 1\leqslant k\leqslant n-1\,. \tag{D4}$$

A Cartan subalgebra, i.e., a maximal Abelian subalgebra, is, from (D3), a maximal set of commuting matrices, so it can be chosen to be diagonal and, because of the trace condition, it forms an n-1 dimensional vector space $\mathfrak{C}$. The Weyl group is $\mathfrak{S}_n$, the permutation group of n objects: it permutes the eigenvalues of the diagonal matrices. Hence the generic stratum is the set of matrices with n distinct eigenvalues; its little groups are isomorphic to $U(1)^{n-1}$. The different strata are defined by the multiplicities $k_1, k_2, \ldots, k_m, \sum_{i=1}^m k_i=n$, of the eigenvalues; their little groups are $S(U(k_1)\times U(k_2)\times\ldots\times U(k_m))$ whose elements are the direct sum of block diagonal k_i by k_i unitary matrices u_i satisfying $\prod_{i=1}^m \det u_i=1$. The one-dimensional stratum contains the Hermitian matrices with only two distinct eigenvalues α,β such that $k\alpha+(n-k)\beta=0$ and the corresponding little group is $S(U(k)\times U(n-k))$; note that $S(U(1)\times U(n-1))=U(n-1)$.

Let us go now to a general situation, where an orthogonal representation of a group G has a third degree invariant:

$$\theta(g\cdot x)=\theta(x)\ \text{and}\ \theta(\lambda x)=\lambda^3\theta(x)\,. \tag{D5}$$

It is well known how to make a bilinear completely symmetrical scalar product by polarization of a quadratic form:

$$(x,y)=(1/2!)[(x+y,x+y)-(x,x)-(y,y)]\,. \tag{D6}$$

Similarly, from the $\theta(x)$ of (D5) one obtains a completely symmetric trilinear form

$$\begin{aligned}\tilde{\theta}(x,y,z)=(1/3!)[&\theta(x+y+z)-\theta(x+y)-\theta(y+z)\\&-\theta(z+x)+\theta(x)+\theta(y)+\theta(z)]\end{aligned} \tag{D7}$$

which is G invariant

$$\tilde{\theta}(g\cdot x,g\cdot y,g\cdot z)=\tilde{\theta}(x,y,z)\,. \tag{D8}$$

In an orthogonal space, every linear form such as $z\to\tilde{\theta}(x,y,z)$ can be written as a scalar product

$$\tilde{\theta}(x,y,z)=(x_\vee y,z)\,. \tag{D9}$$

The map $(x,y)\to x_\vee y$ defines an algebra on the space $\mathcal{S}$ of the group representation since the most general definition of an algebra is by a homomorphism $\mathcal{S}\otimes\mathcal{S}\to\mathcal{S}$. Moreover, the G invariance ensures the G equivariance of this algebra: it has G as group of automorphisms.

Let us apply this general method to obtain equivariant symmetric algebras of SU(n), $n\geqslant 3$, the only simple Lie groups having a third degree invariant in their adjoint representation. It is $\theta=\mathrm{tr}x^3/3$. One can normalize the algebra to be

$$x_\vee y=\frac{\sqrt{n}}{2}(xy+yx)-\frac{I}{\sqrt{n}}\mathrm{tr}xy\,. \tag{D10}$$

This is the "d algebra" introduced by Gell-Mann (1962) and Biedenharn (1963). As we have seen, if x belongs to a one-dimensional stratum, i.e., $x/\sqrt{(x,x)}$ is on a critical orbit of the unit sphere, it has only two distinct eigenvalues; so it must satisfy a second-degree equation. Such an equation has to be SU(n) equivariant so its only possible form is

$$x_\vee x=\lambda x\,, \tag{D11}$$

i.e., x is an idempotent (nilpotent when $\lambda=0$) of the symmetric algebra $\vee$. As noticed by Michel and Radicati (1970, 1971a,b) these idempotents are favored as directions of symmetry breaking for the internal symmetry of hadronic physics (see Appendix E). Sattinger (1977) also found these idempotents as symmetry breaking directions in the theory of bifurcations. These $SU(n)$ equivariant symmetrical algebras have been studied by many physicists [see, e.g., Michel and Radicati (1973), who also give a list of previous references]. Let us give some results here. The roots of the SU(n) Lie algebra are the matrices r with the characteristic equation

$$r^{n-2}(r^2-1)=0\,. \tag{D12}$$

They are of unit length. For $n>2$ the unit vectors:

$$q=r_\vee r(n-2)^{-1/2} \tag{D13}$$

are the pseudoroots of SU(n). They play for the symmetric algebra $\vee$ a role similar to that of the roots for the Lie algebra $\wedge$. Moreover, they are idempotent of the symmetric algebra:

$$q_\vee q=\frac{n-4}{\sqrt{n-2}}q\,. \tag{D14}$$

Their little group is $S(U(n-2)\times U(2))$.

Of course, from the quartic invariant, one can form by polarization a completely symmetrical trilinear algebra $\mathcal{S}\otimes\mathcal{S}\otimes\mathcal{S}\to\mathcal{S}$ and so on... [see, e.g., Mott, 1975, Michel, O'Raifeartaigh, and Wali (1977) and also Appendix E].

APPENDIX E: DIRECTIONS FOR BREAKING OF THE INNER SYMMETRY OF THE FUNDAMENTAL INTERACTIONS

We recall first the definition of tensor operators in quantum mechanics. Let $g \to U(g) = U^{-1*}(g)$, the unitary representation of the invariance group G on the Hilbert space $\mathfrak{H}$ of state vectors. We denote by $\mathfrak{L}(\mathfrak{H})$ the vector space of linear operators in $\mathfrak{H}$. The group G acts linearly on $\mathfrak{L}(\mathfrak{H})$ by

$$g \cdot A = U(g) A U(g)^* . \tag{E1}$$

It transforms Hermitian operators into Hermitian operators. Given a linear representation $g \to \Delta(g)$ of G on $\mathcal{E}$, a tensor operator T of variance $\mathcal{E}$ is a G-equivariant linear map $\mathcal{E} \to \mathfrak{L}(\mathfrak{H})$, i.e., it satisfies the commutative diagram of linear maps:

$$\begin{array}{ccc} & E & \longrightarrow & L(H) \\ \forall g \in G & \Delta(g) \downarrow & & \downarrow U(g)(\cdot)U(g)^* \\ & E & \longrightarrow & L(H) \end{array}$$

Note that a "tensor operator" is not an operator on $\mathfrak{H}$, but $\mathrm{Im}\, T$, the image of T, is a set of operators which is called "irreducible tensorial set of operators" (e.g., the book of Fano and Racah) when the representation of G on $\mathcal{E}$ is irreducible. Given two operators T_1, T_2 of variance $\mathcal{E}$, $\mathcal{E}_2$, we form new tensor operators, defined by

$$\forall a_i \in \mathcal{E}_i,\ (T_1 \oplus T_2)(a_1 \oplus a_2) = T_1(a) + T_2(a),\ T_1 \oplus T_2 \text{ has variance } \mathcal{E}_1 \oplus \mathcal{E}_2 ;$$

$$\forall a_i \in \mathcal{E}_i,\ (T_1 \otimes T_2)(a_1 \otimes a_2) = T_1(a)\, T_2(a),\ T_1 \otimes T_2 \text{ has variance } \mathcal{E}_1 \otimes \mathcal{E}_2 .$$

We give now some applications to the physics of fundamental interactions. This subject is in full evolution. The following assumptions are the most commonly made: the elementary constituents of matter are quarks and leptons which have spin 1/2. There is an exact symmetry group

$$G_c = U(3) \tag{E2}$$

which corresponds to the conservation of color and electric charge. There is much confusion on this precise point in the literature; one generally sees that $G_c = \mathrm{SU}(3) \times U(1)$. Let us explain the difference between $U(n)$ and the direct product $\mathrm{SU}(n) \times U(1)$, repeating essentially the reference Michel (1962).

The direct product $G = G_1 \times G_2$ of the groups G_1, G_2 of elements $x_i, y_i, \ldots, i = 1, 2$, is the group whose elements are pairs (x_1, x_2) and the multiplication law is

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 y_1, x_2 y_2) . \tag{E3}$$

Conversely, if $G_1 \triangleleft G$, $G_2 \triangleleft G$ (reads: invariant subgroup $G \sim G_1 \times G_2$), if and only if $G_1 \cap G_2 = \{1\}$ and every element of G_1 commutes with every element of G_2. The group of unitary $n \times n$ matrices is divided by $U(n)$. There is a group homomorphism $U(n) \xrightarrow{\det} U(1) = \{e^{i\varphi}\}$ since the determinant of a product of matrices is the product of their determinant. By definition $\mathrm{SU}(n)$ = Ker"det"; it is the subgroup of $U(n)$ whose matrices have determinant 1. The center $\mathfrak{C}(U(n))$ is the set of matrices zI_n multiple of the unit $n \times n$ matrix I_n with $|z| = 1$ and, by definition of the center, every matrix of $\mathfrak{C}(U(n))$ commute with those by $\mathrm{SU}(n)$. However, $U(n)$ is not the direct product $U(1) \times \mathrm{SU}(n)$ because $\mathfrak{C}(U(n))$ and $\mathrm{SU}(n)$ have a nontrivial intersection:

$$\mathfrak{C}(\mathrm{SU}(n)) = Z_n = \{c_k I_n,\ c_k = e^{2\pi i k/n},\ 0 \le k < n\} . \tag{E4}$$

So $U(n)$ is isomorphic to the quotient group:

$$U(n) \sim \frac{U(1) \times \mathrm{SU}(n)}{Z_n} \tag{E5}$$

or we could divide all groups by Z_n, to obtain a direct product

$$\frac{U(n)}{Z_n} = \frac{\mathfrak{C}(U(n))}{Z_n} \times \frac{\mathrm{SU}(n)}{Z_n} . \tag{E6}$$

The representation of $\mathrm{SU}(n)$ can be labeled by "Young diagrams" with $n-1$ lines, i.e., by $n-1$ integers

$$p_1 \ge p_2 \ge p_3 \cdots \ge p_{n-1} \ge 0 . \tag{E7}$$

For instance, for the n-dimensional representation, $p_1 = 1$ and $p_i = 0$, $1 < i < n$, while for its complex conjugate $p_i = 1$. For the adjoint representation (see Appendix D) $p_1 = 2$, $p_i = 1$, $1 < i < n$, it is self-conjugated. More generally, the complex conjugate irrep (irreducible representation) of $p_1, \ldots, p_{n-1}$ is $p_i' = p_1 - p_{n+1-i}$; $1 \le i \le n-1$. The trivial representation corresponds to $p_i = 0$. By the Schur lemma, in an irrep of a group, the elements of its center are represented by a multiple of the unit matrices. For $\mathrm{SU}(n)$, $c_k I_n$ of (E4) is represented by

$$\Delta_{p_1 \ldots p_{n-1}}(c_k I_n) = e^{2\pi i k r/n} I, \quad r = \sum_{i=1}^{n-1} p_i . \tag{E8}$$

For instance, when r is a multiple of n, as is the case in the adjoint representation, the center $\mathfrak{C}(\mathrm{SU}(n))$ is represented trivially, so Δ is a representation of $\mathrm{SU}(n)/Z_n$, which is called the adjoint group. For $n = 2$ the adjoint group is $\mathrm{SO}(3) \sim \mathrm{SU}(2)/Z_2$. The irreps of the direct product are the tensor product of the irreps of the factors. For instance, the representations of $\mathrm{SU}(n) \times U(1)$ are labeled by the p_i's satisfying (E7) and the integer q which caracterizes the irrep $u \to e^{iq\varphi}$ of $U(1)$. Since $U(n)$ is an homomorphic image of $U(1) \times \mathrm{SU}(n)$ [Eq. (E5)] every irrep of $U(n)$ is an irrep of the direct product $U(1) \times \mathrm{SU}(n)$ but the inverse is not true. As Eqs. (E4) and (E8) show, the irreps $q, p_1 \ldots p_{n-1}$, of $U(1) \times \mathrm{SU}(n)$, which are irreps of $U(n)$, satisfy

$$q \equiv r \ \text{modulo}\ n . \tag{E9}$$

Note that if $u \to \Delta(u)$ is an irrep of $U(n)$, $(\det u)^k \Delta(u)$ is another irrep with $q' = q + kn$; this is compatible with (E9). Hence the fact that the fundamental colored states, i.e., the quarks, have an electric charge which is $\frac{2}{3}$ or $-\frac{1}{3}$ that of the fundamental electric charge carried by uncolored states [trivial representation of $\mathrm{SU}(3)$], is well described by the $U(3)$ symmetry group of chromodynamics and not at all by the group $U(1) \times \mathrm{SU}(3)$.

All other symmetries of the fundamental particles are broken, but the directions of breaking belong generally to exceptional strata of the group representation, and one can guess that this feature may also be true in the theory unifying all interactions. We do not know yet how

many quark flavors there are in Nature; five are already known: u, d, s(strangeness), c(charm), b(beauty or bottom). We will consider here the four best known: u, d, s, c. These four quark flavors form an orthonormal basis of the fundamental representation of dimension 4 of SU(4). We shall choose the basis c, u, d, s. In this basis, the electric charge operator is

$$q = \frac{1}{3}\begin{pmatrix} c & u & d & s \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{E10}$$

Since $\mathrm{tr} q = \frac{2}{3} = 0$, q is not in the Lie algebra of SU(4) but in that of $U(4)$.

The electromagnetic current of the quarks is the value of a tensor operator in this direction q of the adjoint representation of $U(4)$ while the values of the same tensor operator in the directions

$$c_\pm = c_1 \pm i c_2,$$

$$c_1 = \begin{pmatrix} 0 & 0 & -\sin\theta & \cos\theta \\ 0 & 0 & \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta & 0 & 0 \\ \cos\theta & \sin\theta & 0 & 0 \end{pmatrix}$$

$$c_2 = \begin{pmatrix} 0 & 0 & i\sin\theta & -i\cos\theta \\ 0 & 0 & -i\cos\theta & -i\sin\theta \\ -i\sin\theta & i\cos\theta & 0 & 0 \\ i\cos\theta & i\sin\theta & 0 & 0 \end{pmatrix} \tag{E11}$$

are the vector part of the charged weak current. The axial vector parts of these currents are the values of another tensor operator in these same directions $c_\pm$. The angle θ is (E11) in the Cabbibbo angle. There exist also particular hadronic currents in the direction

$$c_3 = c_1 \wedge c_2 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}. \tag{E12}$$

[The Lie algebra law $\wedge$ and the law of the symmetric algebra $\vee$ have been defined in Eqs. (D3) and (D10).] Note that there is no neutral current between the d and s quark. This fact is verified with a high accuracy in the K_2^0 decay and it was one of the bases of the hypothesis of the c quark and the first line and first column structure of (E11) (Glashow *et al.*). Then we verify the relations [see (D3)]

$$q \vee q = 0, \quad c_i \vee c_i = 0, \quad i = 1, 2, 3., \; c_1 \vee c_2 = 0. \tag{E13}$$

Hence

$$c_\varepsilon \vee c_{\varepsilon'} = 0, \; \varepsilon^2 = \varepsilon'^2 = 1. \tag{E14}$$

Every matrix with only two distinct eigenvalues satisfies a second degree equation. For a non-Hermitian raceless matrix x of the Lie algebra of $SU(n)$ with eigenvalues λ_1, λ_2 of multiplicity n_1, n_2 we have the relation (see Appendix D)

$$n_1 + n_2 = n, \; n_1\lambda_1 + n_2\lambda_2 = 0, \quad (x, x) = 1 \Leftrightarrow \frac{1}{2}(n_1\lambda_1^2 + n_2\lambda_2^2) = 1; \tag{E15a}$$

$$x \vee x = \eta x \quad \text{with } \eta = \sqrt{\frac{2}{n_1 n_2}}(n_2 - n_1). \tag{E15b}$$

The little group of x is the subgroup of matrices of $SU(n)$ commuting with x. It is

$$S(U(n_1) \times U(n_2)) = \{u_1 \in U(n_1), u_2 \in U(n_2), \det u_1 \det u_2 = 1\} \tag{E16}$$

[and not $\mathrm{SU}(n_1) \times S(U(n_2) \times U(1))$, as too often found in the literature]. Note that if $n_2 = 1$, $n_1 = n - 1$, $S(U(n-1) \times U(1)) = U(n-1)$. Such directions in the Lie algebra, i.e., in the adjoint representation of $SU(n)$, are those of the minima of the degree four, $SU(n)$ invariant polynomials built on this representation. Such polynomials are simply the generalization of the polynomial of Eq. (4): they have $x = 0$ and are positive when (x, x) is large enough. The restriction to degree four is necessary for the renormalizability of the theory. The matrix equation $dP/dx = 0$ giving the extrema is a third degree equation so, for the group irreps as the adjoint representation of $SU(n)$ or $SO(n)$ whose space is realized by matrices, these matrices have at most three distinct eigenvalues. For the $SU(n)$ adjoint representation, the x extrema an minima only for matrices with two distinct eigenvalues. These polynomials introduced by Landau (1938), for second-order phase transitions in crystals, are called, in particle physics, Higgs polynomials. For a study of their minima for a arbitrary representation of a compact Lie group G, see Michel (1979).

The Higgs polynomials on an m-dimensional representation is built with a scalar field with m internal degrees of freedom. So to break symmetry spontaneously one has to pay the price of introducing m Higgs bosons. If H is the little group of a minimum and the dimension of G and H are, respectively, g and h, the orbit $[G:H]$ has dimension g-h. In the broken symmetry scheme the mass matrix of the Higgs bosons is proportional to $\partial^2 P/\partial x_i \partial x_j$, the Hessian of the polynomial. This mass matrix has a kernel: the tangent plane to the orbit; the corresponding g-h massless particles are the Goldstone bosons.

Finally we have to deal with the case of local gauge theory. The compact invariance group G depends on space time $G(x)$. The gauge field $A^\mu(x)$ is valued in $\mathcal{G}$, the Lie algebra of G, so it has g internal components and forms a representation of the Lie algebra $\mathcal{G}$ at x.

The local gauge invariance of the theory requires one to replace derivative $\partial_\mu = \partial/\partial x_\mu$ by the covariant derivative D_μ defined by

$$D_\mu A_\nu(x) = \partial_\mu A_\nu + e[A_\mu(x), A_\nu(x)], \tag{E17}$$

where e is the coupling constant of the theory. Similar-

ly, for every field $\phi(n)$ which is transformed into $\Delta(g)\phi(x)$ by the representation of G, there is a corresponding representation Λ of the Lie algebra and the covariant derivative for ϕ is

$$D_\mu\phi(n) = \partial_\mu\phi(r) + e\Lambda(A_\mu(x))\phi\,. \tag{E18}$$

For a realistic model one should also introduce at least a matter of field (Dirac spinor) with a similar covariant derivative. The simplest Lagrangian with spontaneous symmetry breaking is built only with the vector gauge field A_μ and the Higgs fields ϕ; its density is:

$$-\mathcal{L} = \tfrac{1}{4}(F_{\mu\nu}, F^{\mu\nu}_{\mu}) + \tfrac{1}{2}(D_\mu\phi, D_\mu\phi) + P(\phi)\,, \tag{E19}$$

where

$$F_{\mu\nu} = \frac{1}{e}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu, A_\nu].$$

$(\ ,\ )$ is the G invariant scalar product on the Lie algebra and on the space of the representation of ϕ and $P(\phi)$ is the degree four Higgs polynomial. The only mass term is in $P(\phi)$, so we start with g gauge vector massless bosons and m scalar particles. Thanks to local gauge transformation $A_\mu \to A_\mu + \partial\phi/\partial\mu$ the g-h Goldstone scalar fields can be merged with the g-h gauge vector fields in the space orthogonal to the Lie subalgebra $\mathfrak{K}$ of H so the spontaneously broken symmetry theory has, finally, h massless gauge vector bosons generating the Lie algebra of $\mathfrak{K}$, the preserved symmetry group, g-h massive vector bosons, and $m-(g-h)$ massive Higgs scalars.

An example of such theory with $G_W = U(2)$, $g = 4$ is the unified theory of weak and electromagnetic interactions for the leptons. This model was elaborated successively by Glashow (1961), Salam and Ward (1964), Weinberg (1967) and its renormalizability was established by t'Hooft (1972). After the symmetry breaking $H = U(1)$, so $h = 1$. The corresponding massless vector boson is the photon. The $g-h=3$ massive vector bosons are the $W^+W^-Z^0$ mediating the parity-violating $V-A$ weak interaction. The Higgs bosons are in the spin $\frac{1}{2}$ irrep of $U(2)$; it has a unique stratum with little group $U(1)$, complex dimension 2 and therefore real dimension 4, so $m=4$ and $m-g+h=1$, which means that the theory requires only one massive Higgs boson, which is not yet identified in Nature. The matter fields are the right helicity electron which is a singlet of SU(2), and a doublet of SU(2) formed by the left helicity electrons and neutrinos, and the corresponding antiparticle fields (e_l^- and $e_r^+, \bar{\nu}_e$). Because $U(2)$ is not a simple group, there are two coupling constants f and g; they are generally defined through the observables $e = fg|(f^2+g^2)|^{-1/2}$, the electric charge and $\tan\theta_w = f/g$, where θ_w is the Glashow-Weinberg angle.

This model can be extended to the quarks u and d, and there exists two other copies of it, one for $\mu\nu_\mu cs$ and the other for $\tau\nu_\tau bt$, where τ and b are the more recently discovered leptons and quarks and t is an expected 6th quark flavor.

A candidate group for unification of chromodynamics and weak and electromagnetic interaction is $SU(5)$ whose first breaking, in the 24-dimensional adjoint representation, could be into $S(U(3)\times U(2))$, the two factors inside the bracket being the color symmetry group $G_c = U(3)$ and the group $G_w = U(2)$ we have just presented. The condition of determinant merges the two electromagnetic subgroups into a unique one. The problem of unification of all four known fundamental interactions (weak, electromagnetic, chromatic, and gravitational) is better and better understood. Its solution will surely involve non-Abelian gauge theories but it might be in a form very different from what is studied now.

APPENDIX F: THE GELFAND-NEIMARK-SEGAL CONSTRUCTION

The GNS (Gelfand-Neimark-Segal) construction is well known by some physicists and yet ignored by a majority. It deals with a basic problem of quantum physics: how to obtain a Hilbert-space $\mathfrak{K}$ and a representation by operators on $\mathfrak{K}$, of the algebra $\mathfrak{a}$ of observables, from a given state ρ. In the text we have defined the C^* algebra $\mathfrak{a}$. A state is a linear form $\mathfrak{a}\to C$, i.e.,

$$\forall A, B\in\mathfrak{a},\ \rho(\alpha A+\beta B) = \alpha\rho(A)+\beta\rho(B)\,, \tag{F1}$$

which is positive:

$$\forall A\in\mathfrak{a},\ \rho(A^*A)\geq 0\,. \tag{F2}$$

For convenience one also requires ρ to be normalized: $\rho(I)=1$. Note that the set $\mathfrak{e}$ of states is convex.

Then $\rho((A+\lambda B)^*(A+\lambda B))\geq 0$ yields the Schwartz inequality:

$$|\rho(A^*B)|^2 \leq \rho(A^*A)\rho(B^*B)\,. \tag{F3}$$

So $\rho(A^*B)$ would be a Hermitian scalar product on $\mathfrak{K}$ except that it may vanish for nonzero vectors. Let $\mathfrak{K}$ be the set of K such that $\rho(K^*K)=0$; then, from (F3)

$$\forall X\in\mathfrak{a},\ \rho(X^*K)=0,\ \text{e.g.,}\ \rho((AK)^*AK)=0\,. \tag{F4}$$

In short, $\mathfrak{a}\mathfrak{K}=\mathfrak{K}$: one says that $\mathfrak{K}$ is a left ideal of $\mathfrak{a}$. Moreover, for any

$$K_1, K_2\in\mathfrak{K}, \rho((A+K_1)^*(B+K_2)) = \rho(A^*B). \tag{F5}$$

So ρ induces a true Hermitian scalar product on the quotient vector space $\mathfrak{K}=\mathfrak{a}/\mathfrak{K}$ defined by the exact sequence of vector spaces

$$\{0\}\to\mathfrak{K}\to\mathfrak{a}\xrightarrow{\phi}\mathfrak{K}\to\{0\}. \tag{F6}$$

Denoting $\phi(A)$ shortly by ϕ_A, the Hermitian scalar product on $\mathfrak{K}$ is

$$\langle\phi_A, \phi_B\rangle = \rho(A^*B) \tag{F7}$$

and after topological completion, $\mathfrak{K}$ is a Hilbert space. For each $A\in\mathfrak{a}$ we can define a linear operator $\pi(A)$ on $\mathfrak{K}$:

$$\forall A, B\in\mathfrak{a},\ \pi(A)\phi_B = \phi_{AB}\,. \tag{F8}$$

One easily proves that

$$\pi(\alpha A+\beta B) = \alpha\pi(A)+\beta\pi(B),\ \ \pi(A)\pi(B)=\pi(AB),$$
$$\pi(A^*) = \pi(A)^*\,, \tag{F9}$$

so π is a * representation of the algebra $\mathfrak{a}$ on $\mathfrak{H}$. Since ρ is normed, ϕ_I is a unit vector and the expectation value of any $\pi(X)$ for ϕ_I is simply the expectation value of X for the state ρ:

$$\langle \phi_I, \pi(X)\phi_I \rangle = \rho(X) . \tag{F10}$$

Physically, one generally chooses for the state ρ the vacuum or the fundamental state of the system; when it is not an extremal state of $\mathfrak{C}$, the representation π is reducible. This generally occurs when there is spontaneous symmetry breaking and the vacuum is then degenerate.

APPENDIX G: COMPUTATION OF $\pi_n[G:H]$ for $n = 0,1,2,3$

As we explained, $\pi_0([G:H])$ is the number of connected components of this orbit. The image projected by a continuous map of a connected topological space is connected. So when G is connected $\pi_0([G:H]) = \{1\}$. More generally it is a pointed set, homomorphic image of the discrete group G/G_0, where G_0 is the connected component of G containing the identity. Each connected component of the orbit $[G:H]$ has the same homotopy. So to compute the other homotopy groups π_n, $n>0$, we need only to consider the connected orbit $[G_0:H']$ where

$$H' = G_0 \cap H . \tag{G1}$$

For most of our examples we could have tried to guess π_1 and π_2 by intuitive geometrical arguments. We feel that it is better for the reader to enable him to learn on these simple examples how to use the powerful tool provided by the long homotopy exact sequence [Eq.(24)]. It is also a strong motivation to study its proofs; this requires the study of very few pages of Hilton (1961) or Steenrod (1957). Before applying Eq. (24), we need to know the lowest homotopy groups of G. For superfluid and mesomorphic phases we need only the cases $G=O(3)$ and $G=E(3)$.

(i) $G=O(3)$. Then $G_0 = \mathrm{SO}(3)$. As every physicist knows, SO(3) is the homomorphic image of SU(2) by the mapping θ[where θ is given by Eq. (G3)]:

$$\{1\} \to Z_2 \to \mathrm{SU}(2) \xrightarrow{\theta} \mathrm{SO}(3) \to \{1\} \tag{G2}$$

with:

$$\theta[\exp(-i\tau\cdot \mathbf{n}\omega/2) = \text{rotation of axis n and angle } \omega. \tag{G3}$$

Since $\{\tau\cdot \mathbf{n}\omega\}$ is the set of all 2×2 traceless Hermitian matrices, SU(2) is the set of all 2×2 unitary matrices of determinant 1 when $0 \leq \omega \leq 2\pi$. Such a matrix $u \in \mathrm{SU}(2)$ can also be written

$$u = \begin{vmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{vmatrix} \text{ with } \det u = |z_1|^2 + |z_2|^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1 . \tag{G4}$$

This shows that the group manifold of SU(2) is the sphere S_3. This sphere can also be obtained by the following construction: every $u \in \mathrm{SU}(2)$ is represented in the ball B_3, by the point $\omega\vec{\mathbf{r}}$, and all points of the boundary ∂B_3 (i.e., $\omega = 2\pi$) are identified since they represent $u = -I$. Since SO(3) is the quotient group $\mathrm{SU}(2)/Z_2 = \mathrm{SO}(3)$, where $Z_2 = \{I, -I\}$ is the center $\mathfrak{C}$ ((SU(2)), its group manifold $[\mathrm{SU}(2):Z_2]$ is obtained by identifying in S_3 the opposite points u, $-u$, since such pairs of points have the same image under the mapping θ. This description shows that, by definition of the real projective space, the manifold $[\mathrm{SO}(3);\{1\}]$ of SO(3) is $P(3,R)$.

From the very definition of the homotopy groups, for a discrete set of points Δ, $n>0$, $\pi_n(\Delta) = \{1\}$. So if Δ is a discrete subgroup of a connected group G, the long exact sequence (24) when applied to

$$\{1\} \to \Delta \to G \to [G:\Delta] \to 1 \tag{G5}$$

breaks into the short exact sequences $n>1$, $\{1\} \to \pi_n(G) \to \pi_n[G:\Delta]) \to \{1\}$, i.e.,

$$n>1, \quad \pi_n(G) = \pi_n([G:\Delta]) \tag{G6}$$

and

$$\{1\} \to \pi_1(G) \to \pi_1([G:\Delta]) \to \Delta \to \{1\} \tag{G6'}$$

since $\pi_0(\Delta) = \Delta$ and $\pi_0(G) = 1$ (G is connected).

Since (G2) is an example of (G5) we have

$$n>1, \pi_n(\mathrm{SO}(3)) = \pi_n(\mathrm{SU}(2)) = \pi_n(S_3) . \tag{G7}$$

We have referred the reader to any basic book for the proof of Eq. (14), $n<k$, $\pi_n(S_k) = \{1\}$ and $\pi_n(S_n) = Z$. So we deduce

$$\pi_3(\mathrm{SO}(3)) = Z, \quad \pi_2(\mathrm{SO}(3)) = \{1\}, \quad \pi_1(\mathrm{SO}(3)) = Z_2; \tag{G8}$$

the last result is obtained from (G7') with $\pi_1(G) = \pi_1(S_3) = \{1\}$.

In Appendix A we gave the list of the closed subgroups H of $O(3)$. From the remark made at the beginning of this appendix, $n>0$, $\pi_n([O(3):H]) = \pi_n([\mathrm{SO}(3):H'])$ where $H' = H \cap \mathrm{SO}(3)$ is either a discrete group

$$\Delta \in \{C_n, n \geq 1; D_n, n \geq 2; T, O, Y\}$$

or a one-dimensional Lie group

$$C_\infty \sim \mathrm{SO}(2) \sim U(1), \quad D_\infty . \tag{G9}$$

Then, by application of (G6) again,

$$\pi_3([\mathrm{SO}(3)]:\Delta) = Z, \quad \pi_2([\mathrm{SO}(3)]:\Delta) = \{1\} . \tag{G10}$$

For the π_1, we note that $[\mathrm{SO}(3):\Delta] = [\mathrm{SU}(2):\bar{\Delta}]$, where

$$\bar{\Delta} = \theta^{-1}(\Delta) , \tag{G11}$$

the preimage of Δ under θ defined in (G2). The application of (G6') to SU(2) and $\bar{\Delta}$ yields

$$\pi_1([\mathrm{SO}(3)]:\Delta) = \bar{\Delta} . \tag{G12}$$

For instance, when Δ is the four-element group D_2 generated by the three rotations of π around the three orthogonal axes $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$, (G3) shows that $\bar{D}_2$ is generated by $\pm i\tau_1$, $\pm i\tau_2$, $\pm i\tau_3$ and its other elements are I, $-I$. This eight-element group is called the quaternionic group and is often denoted by Q.

Obviously, $[\mathrm{SO}(3):\mathrm{SO}(2)]$, the orbit of a vector for the rotation group, is S_2. Since $\mathrm{SO}(2) \sim S_1$ and $\pi_n(S_1) = \{1\}$, $n>1$ we deduce from the long exact sequence (24)

$$n>2, \pi_n(S_3)=\pi_n(S_2)=\pi_n([\mathrm{SO}(3):\mathrm{SO}(2)]) \quad \text{(G13)}$$

and verify, from $\pi_2(\mathrm{SO}(3))=\{1\}$, that $\pi_2(S_2)=\pi_1(S_1)$

$$\pi_2(S_2)=\pi_2[\mathrm{SO}(3):\mathrm{SO}(2)]=Z;$$

$$\pi_1([\mathrm{SO}(3):\mathrm{SO}(2)])=\pi_1(S_2)=\{1\}. \quad \text{(G13')}$$

In three-dimensional space, if we replace a vector by a "director," i.e., a nonoriented direction, by definition of the projective space, we obtain as orbit of SO(3)

$$P(2,R)=[\mathrm{SO}(3):D_\infty]\ . \quad \text{(G14)}$$

One can consider S_2 as a fiber bundle, whose base is $P(2,R)$ and whose fibers have two points. More generally, a fiber bundle $\overline{X}$, whose fiber is a discrete set of points Δ, is called a covering of its basis X. Moreover, if $\overline{X}$ is simply connected, i.e., $\pi_1(\overline{X})=\{1\}$, then for every continuous surjective map $Y\xrightarrow{\alpha}X$ one can find a surjective map $\overline{X}\xrightarrow{\overline{\alpha}}Y$ such that $\alpha\circ\overline{\alpha}=p$, the projection $\overline{X}\xrightarrow{p}X$ of the bundle; by an easy corollary, $\overline{X}$ is found to be unique (up to an isomorphism) so $\overline{X}$ is called the universal covering of X. The long exact sequence applies also for general fiber bundles [with some care for the π_0 part; see Steenrod (1957)]. We have for $n>1$, $\pi_n(\overline{X})=\pi_n(X)$ and $\pi_1(X)=\Delta$. In particular

$$[\mathrm{SO}(3):D_\infty]=P(2,R),\ \pi_3(P(2,R))=Z,\ \pi_2(P(2,R))=Z,$$
$$\pi_1(P(2,R))=Z_2\ . \quad \text{(G15)}$$

For superfluid helium phases, we need only the case where G is $U(1)$, SO(3) or a direct product of such factors; in the latter case we use Eq. (25), $\pi_n(X\times Y)=\pi_n(X)\times\pi_n(Y)$ in every term of the long exact sequence.

(ii) $G=E(3)$. This is the case we need for mesomorphic phases.

The connected Euclidian group $E_0(3)$ is the semidirect product

$$E_0(3)=R^3_{\square}\mathrm{SO}(3); \quad \text{(G16)}$$

so as a topological space, as we explained in Sec. IIC, it is the topological product $R^3\times\mathrm{SO}(3)$; since $\pi_n(R^3)=\{1\}$, (R^3 is contractible),

$$\pi_n(E_0(3))=\pi_n(\mathrm{SO}(3)). \quad \text{(G17)}$$

Similarly, the homotopy groups π_n ($n>0$) of the closed subgroups H of $E(3)$ are those of the subgroups of SO(3). The characterization of these subgroups H is partially given in Table I and more completely in Table II. We need again to consider only $H'=H\cap E_0(3)$. It is also convenient to consider the universal covering of $E_0(3)$:

$$\overline{E_0}(3)=R^3_{\square}\mathrm{SU}(2),\ \overline{E_0}(3)\xrightarrow{\hat{\theta}}E_0(3)\to 1 \quad \text{(G18)}$$

and its subgroup

$$\overline{H'}=\hat{\theta}^{-1}(H')\ . \quad \text{(G19)}$$

Let $\overline{H'_0}$ be the connected component of $\overline{H'}$ which contains $\{1\}$. Table I shows that except for ordinary nematics and smectics A, $\overline{H'_0}=R^k$ ($0\leq k\leq 3$), so

$$n>0,\ \pi_n(\overline{H'})=\{1\},\ \pi_0(\overline{H'})=\overline{H'}/\overline{H'_0}=H'/H'_0\ . \quad \text{(G20)}$$

Since $[E_0(3):H']=[\overline{E_0}(3):\overline{H'}]$, the long exact sequence, applied to

$$1\to\overline{H'}\to\overline{E_0}(3)\to[\overline{E_0}(3):\overline{H'}]\to 1, \quad \text{(G21)}$$

will yield the homotopy of the orbit of local states $[E_0(3):H']$. Indeed from (G16) and (G7) when

$$H'_0=R^k,\ n>1,\ \pi_n([\overline{E_0}(3):\overline{H'}])=\pi_n(\overline{E_0}(3))=\pi_n(S_3) \quad \text{(G22)}$$

and from $\pi_1(\overline{E_0}(3))=\pi_0(E_0(3))=0$

$$\pi_1([\overline{E_0}(3):\overline{H'}])=\pi_0(\overline{H'})\ , \quad \text{(G23')}$$

which explicitly reads (using (G17), (G8), (G20))

$$M=[E_0(3):H'],\ \pi_3(M)=Z,\ \pi_2(M)=\{1\},\ \pi_1(M)=H'/H'_0\ . \quad \text{(G23)}$$

We are left with the two cases (ordinary nematics and smectics A) in which H' contains a group $C_\infty\sim U(1)$ of rotations around an axis. Note that $\theta^{-1}(U(1))$ is again isomorphic to $U(1)$. In that case (G21) must be modified for π_1; indeed [see Eqs. (14), (15)]

$$\pi_1(\overline{H'})=Z,[=\pi_1(U(1))]. \quad \text{(G24)}$$

Since the long exact sequence applied to (G21) contains trivial $\{1\}$ for $\pi_n(H')$, $n>1$, $\pi_2(\overline{E_0}(3))$, $\pi_1(\overline{E_0}(3))$, we obtain directly for $M=[E_0(3):H']$, using Eqs. (G17) and (G8)

$$n>2,\ \pi_n(M)=\pi_n(S_3),\ \pi_2(M)=Z,\ \pi_1(M)=\overline{H'}/\overline{H'_0}\ . \quad \text{(G25)}$$

(iii) *G is a Lie group, H a closed subgroup.* This will cover most of the physical applications. At the beginning of this appendix we explained how to compute $\pi_0(G/H)$. For the other π's we need only to consider G *connected*. Assume that it is a real Lie group (a complex Lie group is a real Lie group with twice this dimension). Iwasawa (1949) proved that every element g of a semisimple real Lie group[44] G can be written as a product $g=kan$, $k\in K$, $a\in A$, $n\in N$ where K, A, N are three disjoint (except for 1 in common!) subgroups such that: K is a maximal compact subgroup (all such subgroups are conjugate), A is an R^k subgroup, and N is a nilpotent subgroup (all elements x of its Lie algebra satisfy $x^q=0$ for a given integer q). Hence, topologically,

$$G=K_G\times A_G\times N_G=K\times R^{l+m},\ \pi_n(G)=\pi_n(K)\ . \quad \text{(G26)}$$

Similarly for the connected H_0, $H_0=K_H\cdot A_H\cdot N_H$. One can choose K_G containing K_H. Then

$$n>0,\ \pi_n([G:H])=\pi_n([K_G:H']) \text{ with } H'=H\cap K_G. \quad \text{(G27)}$$

The structure of the compact (finite-dimensional) Lie groups is well known. It is the quotient of a direct product of simple, simply connected, Lie groups $K_1,\ldots,K_n$ and $U(1)$'s by a finite (discrete) group F in the center of this direct product

$$K_G=\Big(U(1)^k\times\prod_{i=1}^{r}K_i\Big)\Big/F\ . \quad \text{(G28)}$$

[44] For a general real Lie group the Iwasawa decomposition may not hold, but the topological decomposition of (G26) is still true.

Equation (G28) also says that the direct product is a covering of K_G. The universal covering of it is

$$R^k \times \prod_{i=1}^{r} K_i \xrightarrow{\theta} U(1)^k \times \prod_{i=1}^{r} K_i \to 1$$

so

$$n>1,\ \pi_n(K_G)=\prod_{i=1}^{r}\pi_n(K_i),\ \pi_1(K_G)=\theta^{-1}(F)\ . \tag{G29}$$

The known homotopy groups of semisimple connected Lie groups have been tabulated. For two review papers see Samuelson (1952) and Borel (1955). But we need here only the second and third homotopy group. Cartan (1936) has proved that the second homotopy group of all compact Lie groups is trivial (Hint: $\mathfrak{K}$ is the Lie algebra of K, and the exponential map $\mathfrak{K}\to K$ is surjective; so every $g\in K$ belongs to at least a one-parameter subgroup and the set R of regular elements, i.e., those g belonging to only one parameter subgroup, is contractible; one proves that the set of nonregular elements is of codimension 3.) So for all Lie groups

$$\pi_2(G)=0\ . \tag{G30}$$

Bott (1954) has proved that for any K_i, $\pi_3(K_i)=Z$, so

$$\pi_3(G)=\pi_3(K_G)=Z^r\ . \tag{G30'}$$

There are four infinite series of simple simply connected Lie groups and five exceptional ones (G_2, F_4, E_6, E_7, E_8). The four series of K_i are (where n is the rank of the group):

$n>0$, $A_n=\mathrm{SU}(n+1)$, $n>1$, $B_n=$ spinor $(2n+1)$:

$n>1$, $C_n=$ symplectic $(2n)$;

$D_n=$ spinor $(2n)$ (with the isomorphism $B_2=C_2$, $A_3=D_3$),

where spinor (m) is the double covering of the special orthogonal group $\mathrm{SO}(m)$.

It is easy to show that the maximal compact connected subgroup of $GL(n,c)$ or $SL(n,c)$ is $\mathrm{SU}(n)$ of Aff (n) or $E(n)$ or $GL(n,R)$ or $SL(n,R)$ is $\mathrm{SO}(n)$; so

$$\pi_3(GL(n,R))=Z,\ \pi_2(GL(n,R))=\{1\},\ \pi_1(GL(n,R))=Z_2. \tag{G31}$$

Stewart (1960) proved that the group $\mathrm{Diff}_0(R^n)$ of diffeomorphisms of R_n also has $\mathrm{SO}(n)$ as a maximal compact subgroup, so (G31) can even be extended to this group.

APPENDIX H: CONFIGURATIONS IN NON-ABELIAN GAUGE FIELDS (MONOPOLES, INSTANTONS)

t'Hooft (1974) and Polyakov (1974) have independently proved that there exist static solutions of the equations derived from the Lagrangian (E19), which are everywhere regular, with total finite energy and carrying a magnetic monopole. These solutions are solitons whose stability is due to a topological selection rule. This was explicitly proven by Tyupkin *et al.* (1975) and Monastirskii and Perelomov (1975).

At infinity in the space directions, the Higgs field value has to be on the orbit of the lowest minimum of the Higgs polynomial $V(\phi)$; this defines a continuous function from S_2, the sphere of directions in three-dimensional space, to the orbit $[G:H]$, where H is the subgroup of preserved symmetry. This function belongs to a homotopy class of $\pi_2([G:H])$ and one expects t'Hooft and Polyakov monopoles if this class is not the zero element of $\pi_2([G:H])$ (functions homotopic to a constant).

So from the knowledge of the broken gauge group G and its subgroup H of preserved symmetry, one can deduce the existence or nonexistence of t'Hooft-Polyakov monopoles. One could also give the order of magnitude of their mass (137 times that of the W-Z bosons) if they existed in the unified weak-electromagnetic theory). For instance, if H is the invariant subgroup of G, $[G:H]$ becomes a Lie group G/H, and as Cartan (1934) showed (see Appendix G) $\pi_2(G/H)=0$. So there are no such magnetic monopoles in the unifield theory of Glashow-Salam and Ward-Weinberg: indeed $G=U(2)$, $H=U(1)$, and $G/H=\mathrm{SO}(3)$.

In Appendix G one explains how to compute $[G:H]$ for arbitrary compact Lie groups. Let

$$\overline{G}_0 \xrightarrow{\theta} G_0 \to 1 \tag{H1}$$

be the universal covering of the connected subgroup. G_0 of G (G_0 is the group carried by the connected component of the unit element of G) and

$$\overline{H}=\theta^{-1}(H)\ . \tag{H2}$$

Then

$$\pi_2([G:H])=\pi_2([\overline{G}_0:\overline{H}]) \tag{H3}$$

and from the long exact homotopy sequence and $\pi_2(\overline{G}_0)=0=\pi_1(\overline{G}_0)$, we obtain

$$\pi_2([G:H])=\pi_1(\overline{H})\ . \tag{H4}$$

Since $U(3)$ is the exact symmetry gauge group of the fundamental interactions, in any grand unification of these interactions with a *semisimple* gauge group, $\overline{U(3)}$ will be isomorphic to $U(3)$ and then we do expect the existence of t'Hooft-Polyakov magnetic monopoles with integer magnetic charges.

We now explain the topological stability of instantons. To simplify we consider the case of pure gauge vector fields, with the Lagrangian $\frac{1}{4}(\mathbf{F}_{\mu\nu},\mathbf{F}^{\mu\nu})$ (see Appendix E for notations). For the vacuum, i.e., $F_{\mu\nu}(x)=0$, the gauge field $A_\mu(x)$ is of the form

$$A_\mu(x)=g^{-1}(x)\partial_\mu g(x)\ , \tag{H5}$$

where $g(x)$ is a continuous function defined on space-time and valued in the group. Assume that at times

$$t_i,\ i=1,2,\ F_{\mu\nu}(\mathbf{x},t_i)=0 \tag{H6}$$

and that at any time t

$$t_1\le t\le t_2\ . \tag{H7}$$

$F_{\mu\nu}(\mathbf{x},t)$ vanishes at infinity ($|\mathbf{x}|\to\infty$). These physical conditions require that $A_\mu(\mathbf{x},t)$ be of the form (H5) on a domain which is homeomorphic to a sphere S_3; then the function $g(x)$ belongs to a homotopy class of $\pi_3(G)$. As we tell in Appendix G, for any compact Lie groups

$$\pi_3(G)=Z^r\ , \tag{H8}$$

where r is the number of simple factors in G. If $g(x)$ belong to a nontrivial homotopy class, it cannot be con-

tinued everywhere inside the ball whose boundary is S_3, i.e., in space-time between the times t_1 and t_2; hence $A_\mu(x)$ cannot be everywhere of the form (H5), and somewhere or sometime t between t_1 and t_2 one must have $F_{\mu\nu}(x) \neq 0$. This is incompatible with energy conservation in classical physics; quantum-mechanically, transition between vacuum at t_1 and vacuum at t_2 is possible, but the nontriviality in $\pi_3(G)$ of the homotopy class of $g(x)$ corresponds to the presence of instantons.

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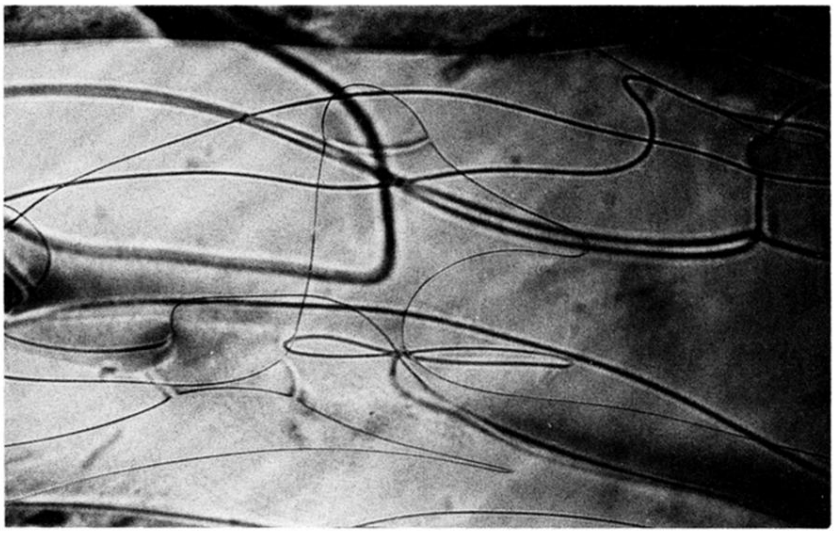

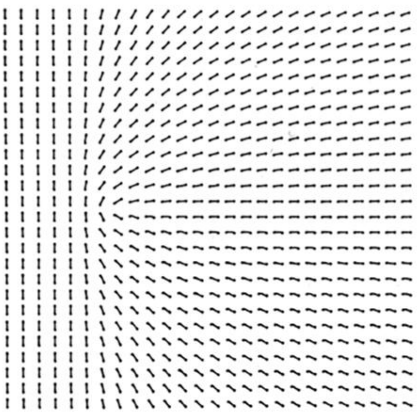

FIGURE 7B

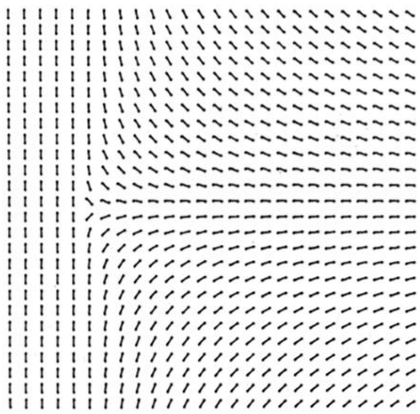

FIGURE 7C

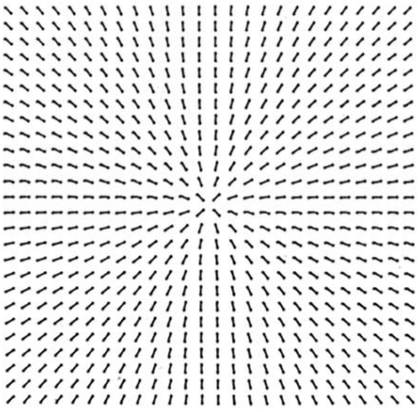

FIGURE 7D

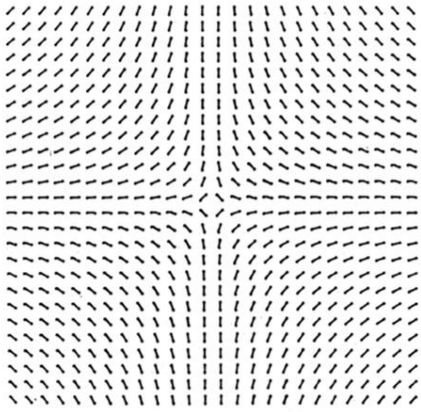

FIGURE 7E

FIG. 7. Line defects in nematics. This photograph has been made by M. Kléman. Here we see clearly two kinds of line defects: the thin lines, which represent topologically stable defects (index $\pm 1/2$), and the thick lines with bright sides, which represent topologically unstable defects with a probable index ± 1. We have defined in Ftn. 26 the Brouwer degree of a map $S_n \rightarrow S_n$: it is very intuitive in the simplest case $S_1 \xrightarrow{h} S_1$, since degree (h) is the algebraic number of turns on the circle S_1 of the image $h(\varphi)$ when the azimuth φ makes one positive turn (from 0 to 2π). Similarly one defines the index of the zero $v(x_0)$ of a continuous vector field on a two-dimensional surface: in any local coordinate system in the neighborhood of x_0, it is the degree of the correspondence h: azimuth of x (near x_0) $\rightarrow$ azimuth of $v(x)$; these two azimuths are defined modulo 2π. In the case of a field of directors $n(x)$, corresponding to the local orientation (in a plane) of a nematic, the azimuth of $n(x)$ is defined modulo π and the custom in physics is to define the index of a zero $n(x_0)$ multiples of 1/2. Examples of index 1/2, $-1/2$, 1, and -1 are given in Fig. 7(b), 7(c), 7(d), and 7(e), respectively. These figures represent the field of orientation of a nematic near a line defect and in a plane perpendicular to it. The index is conserved when the perpendicular plane is moved along the line defect. It is a good exercise to verify that all half-integer indices correspond to topologically stable line defects while the integer indices describe topologically unstable line defects.

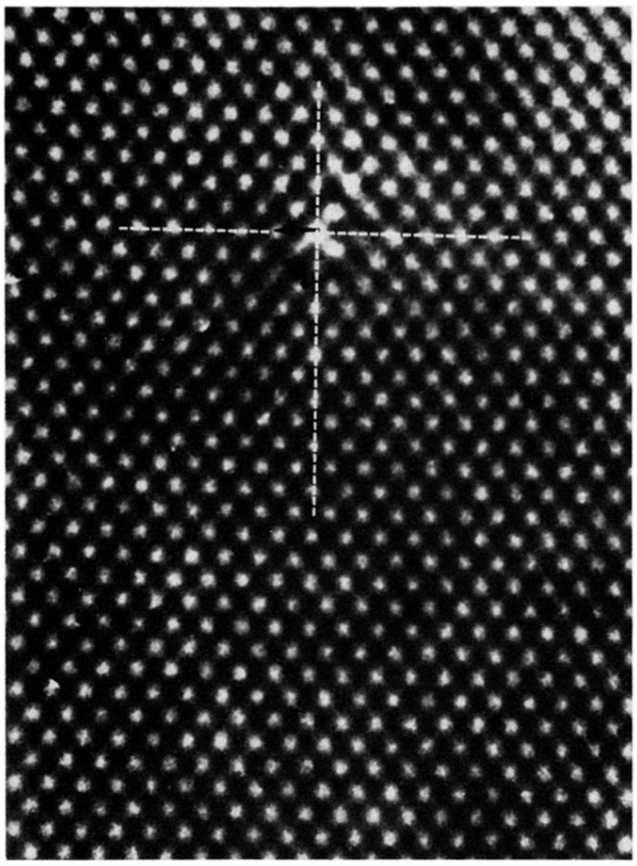

FIG. 9. Dislocation in a germanium crystal. This electron microscope photograph has been obtained by A. Bourret (CENG, Grenoble). The plane of the picture is perpendicular to the 110 axis; the Burgers vector of the dislocation is $1\bar{1}0$.