Kinetic equations from Hamiltonian dynamics: Markovian limits

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Dynamical processes in macroscopic systems are often approximately described by kinetic and hydrodynamic equations. One of the central problems in nonequilibrium statistical mechanics is to understand the approximate validity of these equations starting from a microscopic model. We discuss a variety of classical as well as quantum-mechanical models for which kinetic equations can be derived rigorously. The probabilistic nature of the problem is emphasized: The approximation of the microscopic dynamics by either a kinetic or a hydrodynamic equation can be understood as the approximation of a non-Markovian stochastic process by a Markovian process.

CONTENTS

I. INTRODUCTION

A. Microscopic dynamics and kinetic equations

Dynamical processes in many-body systems are often approximately described by *kinetic equations*. Examples are the Boltzmann equation for a dilute gas, the Vlasov equation for a plasma, the Landau equation for a weakly interacting gas, the Euler equations and the Navier-Stokes equation for a compressible fluid, the Rayleigh-Boltzmann equation for neutron transport in solid materials, the Fokker-Planck equation for a Brownian particle, the diffusion equation, the Pauli master equation for atoms or molecules emitting and absorbing radiation, the laser equations, and the phonon Boltzmann equation for weakly interacting phonons. In many cases these kinetic equations serve as the basic equations for whole subfields of physics. Their common feature is that they describe the dynamics of single-particle distributions. The complicated interaction with other particles and/or the surroundings is approximately taken into account by collision terms, effective potentials, and the like. Of course, these approximations are valid only under certain physical conditions. But if these conditions are met, then the kinetic description is in excellent agreement with observed experimental facts.

Since we believe, not without reason, in the microscopic description of a many-body system, the use of kinetic equations poses the problem of how to understand their approximate validity, starting from microscopic dynamics. Ever since Boltzmann this has certainly been one of the central themes of nonequilibrium statistical mechanics. This review is intended to cover just one aspect of this problem, namely, those models for which the approximation by a kinetic equation can be rigorously controlled.

Let us outline the general program. One has to specify.

(i) The microscopic model

A model is regarded as *admissible*, if it satisfies the following: (a) The dynamics of the model are governed by Hamilton's equation of motion (by the Schrödinger

equation for a quantum-mechanical system). Therefore the model is specified by a Hamiltonian, supplemented, if necessary, by proper boundary conditions. (b) Initially (at $t=0$) a statistical state is given (i.e., either a probability measure on the phase space associated with the classical system or a statistical operator on the Hilbert space associated with the quantum-mechanical system). We emphasize that *all* statistical assumptions enter through the initial conditions.

On second thought, one would like to understand why certain statistical assumptions work so well, or, more ambitiously, to try to reduce or even to avoid statistical assumptions altogether. This is a rather difficult subject about which we have very little to say, although once in a while we will hope to provide some interesting piece of information. We propose here to take the point of view of statistical mechanics and to regard the justification of a statistical ensemble at the initial time as a separate problem.

(ii) The approximation

As already mentioned, kinetic equations are good approximations only under certain physical conditions. This is translated into the theoretical framework by performing a certain limit. For example, the Boltzmann equation is known to be valid for a dilute gas. Therefore, in order to derive the Boltzmann equation, one should let the density of the system go to zero, this being a condition on the initial state. However, the mean free path and the mean free time, which are the typical length and time scale of the system, will then tend to infinity. To obtain a well-defined limit, one therefore has also to adjust the length and time scale appropriately.

It will turn out that the approximations can always be chosen in such a way that the Hamiltonian and the initial state are scaled. So, if we denote the scaling parameter by ε, then for each ε one has a well defined microscopic dynamics, and—provided the scaling is appropriately chosen—in the limit as $\varepsilon \to 0$ one obtains a limiting dynamics governed by a kinetic equation. We emphasize that in many cases finding the proper approximation is already a nontrivial problem.

(iii) The proof of convergence

This is the more mathematical part of the program. Here one has to prove a theorem which assures, under certain conditions on the scaled Hamiltonian and the scaled initial states, the convergence to a limiting dynamics in a certain sense.

The kinetic equations mentioned in the beginning have a remarkable common feature: they are all first order in the time derivative, i.e., they have no memory terms, or the future state of the system is completely determined by its present state. This fact leads to a deeper *probabilistic* interpretation of the approximation leading to a kinetic equation.

Let us try to explain this point of view in the context of a well-known example. We consider the motion of a Brownian particle of mass M in a fluid. Let $q(t)$ be the position and $p(t)$ the momentum of the Brownian particle at time t. As usual, we assume that $q(0)=q$, that $p(0)=p$, and that initially the fluid is in thermal equilibrium conditioned on the Brownian particle being located at the position q. If the initial configuration of the fluid is specified, then the path $t \to (q(t), p(t))$ of the Brownian particle is determined. Since, however, a probability distribution over the initial configurations of the fluid is given, one has accordingly a probability distribution over all possible paths of the Brownian particle. Therefore, in a natural way, $(q(t), p(t))$ is regarded as a *stochastic process* starting at (q,p). The motion of the Brownian particle can be thought of as being governed by the stochastic differential equation

$$M\frac{d}{dt}q(t)=p(t), \quad \frac{d}{dt}p(t)=F(t), \tag{1.1}$$

with initial conditions $q(0)=q$, $p(0)=p$. $F(t)$ is a time-dependent random force whose statistical properties are *completely* determined by the initial fluid distribution, (q,p), and the dynamics of the system.

$(q(t), p(t))$ is a *non-Markovian* process due to recollisions with one and the same fluid particle, which give rise to a memory effect. The future of $q(t), p(t)$ depends on its whole past. Now, the idea is to consider physical conditions under which the memory effects become negligible. One such possible *Markovian approximation*, which goes back to Langevin, is to assume that $F(t)$ can be written

$$F(t)=-\gamma p(t)+\alpha w(t), \tag{1.2}$$

where $\gamma>0$ is a friction constant, $\alpha>0$, and $w(t)$ is Gaussian white noise. Under this assumption the process $(q(t), p(t))$ is Markovian, and the probability distribution $f(q,p,t)$ of $(q(t), p(t))$ is governed by the Fokker-Planck equation

$$\frac{\partial}{\partial t}f(q,p,t)=\left\{-\frac{1}{M}p\cdot\nabla_q+\gamma\nabla_p\cdot p+\frac{\alpha^2}{2}\Delta_p\right\}f(q,p,t). \tag{1.3}$$

A formal procedure (see Sec. II. B. 1) indicates that as $M\to\infty$, together with a proper rescaling of space, time, and momentum, the motion of the Brownian particle should indeed be governed by a Markov process defined by Eq. (1.3). The otherwise phenomenological parameters γ and α are then proportional to the time integral over the time-dependent force-force equilibrium correlation function of the fluid, i.e., to a microscopically defined quantity of the fluid. The proof that the properly scaled stochastic motion (1.1) converges to the stochastic motion defined through (1.2) has been given recently (see Sec. II.B.1).

The example just described turns out to be generic. All approximations leading to a kinetic description can be understood as approximating a non-Markovian stochastic process by a Markovian one. It seems therefore to be appropriate to call these approximations *Markovian limits*.

In recent years there has been considerable progress in proving the existence of Markovian limits. At this stage it seems therefore to be worthwhile to try to collect the results obtained in a more or less coherent framework, to see what has been achieved and where the open problems are. We will focus predominantly on classical continuous systems, since in this area our

understanding has advanced farthest, and since no coherent review seems to be available. On the other hand, such an undertaking would be rather incomplete were I not to mention, to some extent at least, classical lattice systems and quantum-mechanical systems. [Certain aspects of Markovian limits for quantum-mechanical systems have been reviewed by Gorini, Frigerio, Verri, Kossakowski, and Sudarshan (1978), by Davies (1976a, 1977b), and by Spohn and Lebowitz (1978). Martin (1979) discusses Markovian limits for three particular models.]

Already at this stage I want to emphasize the distinction between two classes of Markovian limits.

(a) There are Markovian approximations, where a *physical parameter* goes to *zero* (or to infinity). Typically, one has (i) a weak coupling limit (the coupling goes to zero), (ii) a low-density limit (the density goes to zero), (iii) a mean-field limit (the range of the potential goes infinity), and (iv) a Brownian motion limit (the mass of the Brownian particle goes to infinity).

Although there are still many relevant open, possibly rather difficult, problems left, it seems fair to say that one has reached a certain understanding of these limits.

(b) Physically, one studies a system at finite density and coupling strength, in which case the Markovian approximations just mentioned may become inappropriate. The fascinating fact is that there are other Markovian limits, although of a somewhat more subtle nature, where the physical parameters are kept *constant*. One summarizes these approximations as *hydrodynamic limits*.

For a fluid the point is to study only the slowly varying hydrodynamic fields, i.e., the density field, the velocity field, and the energy field, assuming that locally the velocity has approximately reached a Maxwellian distribution. The general belief is that on the time scale $\varepsilon^{-1}t$ and on the spatial scale $\varepsilon^{-1}q$ the Eulerian equations for a compressible fluid should be obtained, whereas on the time scale $\varepsilon^{-2}t$ and on the spatial scale $\varepsilon^{-1}q$ the Navier-Stokes equations should be obtained. More generally, hydrodynamic limits are understood as involving the system at *constant* physical parameters over long time and large spatial scales.

The hydrodynamic limit is poorly understood, the only progress in understanding it consisting of two absolutely remarkable results by Bunimovich and Sinai: For the two-dimensional Lorentz gas with a fixed periodic configuration of scatters (which has to satisfy a certain geometric property; see Sec. II.C.1) they show the validity of the analog of the Euler equations and of the diffusion approximation, which corresponds to the Navier-Stokes equations. A better understanding of the hydrodynamic limit seems to be the real future challenge.

Let me summarize somewhat the scope of this review. The first part discusses the Brownian particle in a fluid and in related models. These models have the advantage of conceptual and, under certain simplifying assumptions, also of physical, as well as mathematical, simplicity. Their study should always be considered from two points of view. On one hand, the Brownian particle in a fluid is of independent interest. This model illustrates many important concepts in nonequilibrium statistical mechanics, and its various Markovian limits are easily understood. In particular, we will discuss the hydrodynamic limit and the related problem of the existence of finite transport coefficients at some length. On the other hand, most ideas developed for the Brownian particle in a fluid immediately carry over to interacting particle systems, once the idea of a nonlinear Markovian process is accepted. Therefore the study of the Brownian particle in a fluid forms a natural gateway to the interacting particle systems in which one is ultimately interested. The second part discusses interacting particle systems, in particular the Landau equation, the Boltzmann equation, and the Vlasov equation. The classical continuous models seem to be rather standardized, and I believe that this list of Markovian limits is essentially complete.

The third part discusses classical lattice systems. Markovian approximations for harmonic lattices have been studied extensively from different points of view. I try to link their discussion to that about classical continuous systems. The weak coupling limit for anharmonic lattices is an interesting problem on which there exists a large physical literature. But it seems to be hard to extract even a consistent set of conjectures.

The final section discusses quantum-mechanical models. For quantum-mechanical models the whole idea of a Markovian approximation has to be thought over again. In particular, it is not so clear what the quantum analog of a stochastic process should be. However, one concept generalizes, namely, the concept of a classical Markovian semigroup. This leads to the theory of quantum-dynamical semigroups. Implicitly they have been used in physics for some time, but their mathematical properties have been studied only in recent years.

As a natural boundary condition for the selection of the material covered we consider the requirement that the model should be of Hamiltonian form and that the approximation by a kinetic equation should be controllable in a rigorous fashion. In this way we do not do justice to the large physical literature on the derivation of kinetic equations and their higher-order corrections. We also deliberately disregard the fact that the idea of a Markov approximation is also used when the underlying dynamics is already stochastic, in particular for stochastic differential equations. This field has been excellently reviewed by van Kampen (1976a), covering the more physical aspects, and by Papanicolaou (1977), covering the more mathematical aspects.

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B. Some classical continuous models

Here I want to introduce some classical models where the particles are allowed to move under their mutual interaction in some region $\Lambda \subset \mathbf{R}^3$ in space. (Classical lattice models and quantum-mechanical models will be introduced in Secs. IV and V.)

The model of ultimate interest is a system of N point particles interacting through a central pair potential V. The Hamiltonian of this system is given by

$$H = \sum_{j=1}^{N} \frac{1}{2m} p_j^2 + \sum_{i<j} V(q_i - q_j). \tag{1.4}$$

(q_i, p_i) stands for the position and the momentum of the ith particle with mass m. The equations of motion have to be supplemented with an appropriate condition at the boundary $\partial\Lambda$ of Λ.

There are other models which, for lack of a better name, we call system + reservoir models. One considers a test particle (= system) in a fluid (= reservoir). The Hamiltonian reads

$$H = \frac{1}{2M} p^2 + \sum_{j=1}^{N} V(q - q_j) + \sum_{j=1}^{N} \frac{1}{2m} p_j^2 + \sum_{i<j=1}^{N} U(q_i - q_j). \tag{1.5}$$

(q, p) stands for the position and the momentum of the test particle with mass M, while (q_i, p_i) denotes the position and the momentum of the ith fluid particle with mass m. V is the central interaction potential between the test particle and the fluid particles, and U is the central interaction potential amongst fluid particles. The system could be constructed in a somewhat fancier manner by allowing an external field, several system particles, etc. But a single free moving particle will suffice for our purposes.

We now distinguish three different cases. If $U = 0$, the reservoir is called *ideal*, and if $U \neq 0$, the reservoir is called *nonideal* (or interacting). In (1.5) one can formally take the limit of infinitely heavy fluid particles, $m \to \infty$. The fluid particles then simply stay wherever they were put initially. In this case one sets $U \equiv 0$. If $m < \infty$, then the system is called the *Rayleigh gas* [see Rayleigh (1891)], and if $m = \infty$, then the system is called the *Lorentz gas* [see Lorentz (1905)].

Usually, the reservoir is assumed to be in thermal equilibrium under the condition that the test particle be located at q. But other initial conditions will be considered as well.

Clearly, the test particle can be chosen identical to the fluid particles, i.e., with $M = m$ and $U = V$. Then one is back to the interacting particle system. Therefore, a full understanding of fluid + test particle models automatically includes an understanding of the fluid. On the other hand, (1.5) allows for simplifications such as $U = 0$ or $m = \infty$, which result in models much less complicated than interacting particle systems.

A little chart indicating the model, the Markovian limit, and the kinetic equation governing the limiting dynamics may be helpful:

Markovian limit	Lorentz gas Rayleigh gas	interacting particle system
weak coupling limit	linear Landau equation	Landau equation
low density limit	linear Boltzmann equation	Boltzmann equation
mean field limit	linear Vlasov equation effective Hamiltonian	Vlasov equation
Brownian motion limit	linear Fokker-Planck equation (Ornstein-Uhlenbeck process)	
hydrodynamic limit	analog of Euler equations	Euler equations of a compressible fluid
	diffusion equation	Navier-Stokes equations of a viscous and thermally conducting fluid

II. SYSTEM + RESERVOIR MODELS

A. The Lorentz gas

The Lorentz gas consists of a particle moving through infinitely heavy, randomly distributed scatterers. Let $x = (q, p) \in \mathbf{R}^3 \times \mathbf{R}^3$ denote the position and the momentum of the Lorentz particle. The mass of the Lorentz particle is set equal to one. We could restrict the motion of the Lorentz particle to some finite region, but it would be more convenient not to do so. Then $Q = (q_1, q_2, \dots)$ will denote a configuration of scatterers in $\mathbf{R}^3$, where q_j is the center of the jth scatterer. $(q_1, q_2, \dots)$ is either a finite or countable sequence in $\mathbf{R}^3$. In every bounded region there should be only a finite number of scatterers. So Q is assumed to be *locally finite*. Let $\mathcal{X}$ denote the space of all locally finite configurations. The interaction between the Lorentz particle and the scatterers is specified by a central potential V_ε of finite range. V_ε is assumed to be twice differentiable. ε is a scaling parameter which is introduced already here for convenience. Then the motion of the Lorentz particle is defined through the solution of Newton's equation of motion

$$\frac{d}{dt} q^\varepsilon(t, x, Q) = p^\varepsilon(t, x, Q),$$
$$\frac{d}{dt} p^\varepsilon(t, x, Q) = -\sum_j \nabla_q V_\varepsilon(q^\varepsilon(t, x, Q) - q_j), \tag{2.1}$$

$Q=(q_1, q_2, \dots)\in \mathcal{X}$, with initial conditions $q^\varepsilon(0,x,Q)=q$, $p^\varepsilon(0,x,Q)=p$. Since the scatterers are infinitely heavy, Q does not change in time. The sum in (2.1) makes sense, since V_ε is of finite range and since Q is locally finite. However, it may happen that the Lorentz particle reaches infinity in a finite time. For the distribution of scatterers to be considered below the set of such exceptional configurations is of measure zero. Although this is quite obvious, we emphasize that the joint system "Lorentz particle + scatterers" is of Hamiltonian form.

As initial distribution we choose $\delta_x\times\mu^\varepsilon\cdot\delta_x$ is the point measure concentrated at $x\in\mathbf{R}^6$. So the Lorentz particle starts at $x=(q,p)$. For simplicity μ^ε is chosen to be the ideal gas distribution with varying density, or equivalently to be the Poisson distribution. It is determined by the correlation functions

$$\rho_n^\varepsilon(q_1,\dots,q_n)=\prod_{j=1}^n \rho_1^\varepsilon(q_j). \tag{2.2}$$

$\rho_n^\varepsilon(q_1,\dots,q_n)$ is the expectation to find an n-tuple of scatterers at $q_1,\dots,q_n$.

In the physical literature it is customary to study the motion of the Lorentz particle through the *reduced* (or averaged) *dynamics*. The initial distribution of the Lorentz particle is assumed to be $f(x)\,dx$. Then for a fixed configuration Q the distribution of the Lorentz particle at time t is

$$f(x^\varepsilon(-t,x,Q))dx,\quad x^\varepsilon(t,x,Q)=(q^\varepsilon(t,x,Q),p^\varepsilon(t,x,Q))$$

and the averaged distribution of the Lorentz particle at time t is

$$f^\varepsilon(x,t)=\int \mu^\varepsilon(dQ) f(x^\varepsilon(-t,x,Q)). \tag{2.3}$$

The evolution $f(x)\to f^\varepsilon(x,t)\equiv(S_t^\varepsilon f)(x)$ defines the reduced dynamics. A few simple properties of the reduced dynamics follow immediately from its definition: $S_0^\varepsilon f=f$; S_t^ε is linear, i.e., $S_t^\varepsilon(f_1+f_2)=S_t^\varepsilon f_1+S_t^\varepsilon f_2$; S_t^ε preserves positivity, i.e., $f\geq 0$ implies $S_t^\varepsilon f\geq 0$; and S_t^ε preserves mass, i.e., $\int dx\,(S_t^\varepsilon f)(x)=\int dx f(x)$. In particular, if f is a probability density, then $S_t^\varepsilon f$ is a probability density at any time.

$f^\varepsilon(t)$ satisfies an integro-differential equation familiar from the theory of generalized master equations

$$\frac{\partial}{\partial t}f^\varepsilon(q,p,t)=-p\cdot\nabla_q f^\varepsilon(q,p,t)+\int_0^t ds\,K^\varepsilon(t-s)f^\varepsilon(q,p,s). \tag{2.4}$$

$K^\varepsilon(t-s)$ (which cannot computed explicitly) is some complicated operator acting on $f^\varepsilon(s)$. $(d/dt)f^\varepsilon(t)$ depends on the averaged distributions $f^\varepsilon(s)$ of the Lorentz particle for $0\leq s\leq t$. This constitutes a memory.

The observation basic to our whole enterprise is that the motion of the Lorentz particle can be scaled in such a way that the memory kernel $K^\varepsilon(t-s)$ almost reduces to a δ function in time. One may scale the strength of the interaction, the range of the potential, the density of scatterers, time, space, etc. For the Lorentz gas it will always be possible to find a scaling where only the potential and the density are scaled, as already indicated.

If the scaling is chosen appropriately for a Markovian approximation, then

$$\lim_{\varepsilon\to 0} S_t^\varepsilon f=S_t f\equiv f(t) \tag{2.5}$$

will exist and S_t will have the form $S_t=e^{Lt}$ with some linear L. The limiting $f(t)$ then satisfies the *kinetic equation*

$$\frac{d}{dt}f(t)=Lf(t), \tag{2.6}$$

with initial condition f. To rephrase: for $\varepsilon=0$, S_t satisfies the semigroup property $S_{t_1+t_2}=S_{t_1}S_{t_2}$, $t_1,t_2\geq 0$, whereas for $\varepsilon>0$ this was not the case because of the memory term. (2.5) constitutes an enormous simplification. In many cases L can be computed explicitly. Then solving the kinetic equation (2.6) one obtains the time evolution of the average distribution of the Lorentz particle in the limit—a rather hopeless task for finite ε.

$\int_A dx f^\varepsilon(x,t)$ is the probability of finding the Lorentz particle in the set $A\subset\mathbf{R}^6$ at time t, given that it started at time zero with distribution $f(x)\,dx$. One may be interested in a more complicated piece of information, such as the probability of finding the Lorentz particle at time t_1 in $A_1,\dots,$ at time t_n in A_n, given that it started at x initially. For this purpose it is then natural to regard the motion of the Lorentz particle as a *stochastic process*. For a fixed configuration Q the position and momentum of the Lorentz particle are determined through (2.1). But since the distribution of scatterers is random, the position and the momentum of the Lorentz particle at time t are also random. We denote these random variables by $X^\varepsilon(t)$,

$$X^\varepsilon(t):\ Q\to q^\varepsilon(t,x,Q),p^\varepsilon(t,x,Q). \tag{2.7}$$

By definition, $X^\varepsilon(0)=x$, i.e., $X^\varepsilon(t)$ starts at $x\in\mathbf{R}^6$. Let us introduce the path space Ω consisting of all functions $t\to(q(t),p(t))$. A point $\omega\in\Omega$ is a possible history (a possible path) of the Lorentz particle. Since to each configuration Q of scatterers there is a unique path $t\to x^\varepsilon(t,x,Q)$, the Poisson distribution μ^ε induces a probability measure P_x^ε on Ω. P_x^ε simply weighs the probability a certain set of paths is realized by the mechanical motion. The subscript x indicates that with probability one the path starts at x. $X^\varepsilon(t)$ may now be thought of as being defined on Ω, and the above-mentioned probability is

$$P_x^\varepsilon(X^\varepsilon(t_1)\in A_1,\dots,X^\varepsilon(t_n)\in A_n). \tag{2.8}$$

The fact that $X^\varepsilon(t)$ is *non-Markovian* can be seen directly without referring to (2.4). Clearly, the probability of finding the Lorentz particle in a region Δ at time $t+s$, given its precise path in the interval $[0,s]$, is different, in general, from the probability of finding the Lorentz particle in the region Δ at time $t+s$, given only its position and momentum at time s, since in the first case only those configurations of scatterers are taken into account which do not destroy the specified path of the Lorentz particle.

The idea of a Markovian approximation may now be understood on a deeper level than before: The non-

Markovian process $X^\varepsilon(t)$ is scaled in such a way that $X^\varepsilon(t)$ converges to a Markov process $X(t)$ as $\varepsilon \to 0$. Technically, the measure P_x^ε converges weakly on Ω to a measure P_x which determines a Markov process.

Let us suppose that the limiting Markov process is homogeneous in time and that it is given through the transition probability $p_t(x'|x)\,dx'$. [$\int_A dx' p_t(x'|x)$ is the probability of finding the particle in A at time t given its having started at x initially.] From the reduced dynamics,

$$f(x,t) = (S_t f)(x) = \int dx'\, p_t(x|x') f(x') .$$

So $p_t(x'|x)$ is the kernel of S_t. Then, up to some technical points, the convergence to a Markov process is defined by

$$\lim_{\varepsilon\to 0} P_x^\varepsilon(X^\varepsilon(t_1)\in A_1, \ldots, X^\varepsilon(t_n)\in A_n)$$
$$= \int_{A_1} dx_n \cdots \int_{A_1} dx_1 p_{t_n - t_{n-1}}(x_n|x_{n-1}) \cdots p_{t_1}(x_1|x)$$
$$\equiv P_x(X(t_1)\in A_1, \ldots, X(t_n)\in A_n) \tag{2.9}$$

for all measurable sets $A_1, \ldots, A_n$, and all $0 < t_1 < \cdots < t_n$ (weak convergence of all finite-dimensional distributions).

Thus, the kinetic equation should be understood as the forward equation of a Markov process. In a Markovian limit one then wants to approximate the stochastic dynamics of the Lorentz particle by the stochastic dynamics as governed through the kinetic equation.

An analogy from equilibirum statistical mechanics might be helpful at this point: Physically, the first and second equilibrium correlation functions already contain all the information one is interested in for practical purposes. However, from a theoretical point of view it has proved to be extremely useful to think in terms of probability distributions on the space of all possible configurations, i.e., in terms of Gibbs measures. In the same vein, for practical purposes the reduced dynamics suffices in most cases. Nevertheless, the full physical information is given by a probability distribution on the space of all possible histories of the Lorentz particle. So, under scaling, convergence of the process is the natural notion.

Before turning to specific examples of Markovian limits and their limiting Markov process, let me mention the neat survey article about the Lorentz gas by Hauge (1974), emphasizing the low density corrections, and the detailed numerical studies of the Lorentz gas by Bruin (1972), by Lewis and Tjon (1978), and by Alder and Alley (1978).

1. The weak coupling limit

The idea of the weak coupling limit is that, by some kind of central limit effect, very many, but weak collisions should lead to a diffusion type evolution. Therefore, the strength of the potential is scaled as

$$V_\varepsilon(q) = \varepsilon^{1/2} V(q) . \tag{2.10}$$

The momentum of the Lorentz particle is deflected on the order $\varepsilon^{1/2}$ in a collision. To have ε^{-1} collisions per unit time interval, the time t is scaled as

$$t_\varepsilon = \varepsilon^{-1} t . \tag{2.11}$$

In a t-time interval the Lorentz particle travels over a large distance. Not to lose sight of the particle, space also has to be scaled according to

$$q_\varepsilon = \varepsilon^{-1} q . \tag{2.12}$$

In particular, the initial position is scaled. The scaling (2.12) is such that the free motion $t_\varepsilon \to q_\varepsilon + p t_\varepsilon$ remains invariant. Equations (2.10) through (2.12) define the weak coupling limit for a constant density of scatterers, Instead of rescaling time and space, the density ρ_ε of scatterers may be scaled. This leads to the *equivalent* scaling: t, q unscaled

$$V_\varepsilon(q) = \varepsilon^{1/2} V(q/\varepsilon) , \tag{2.13}$$

$$\rho_\varepsilon = \varepsilon^{-3} \rho . \tag{2.14}$$

($\rho_\varepsilon = \varepsilon^{-d}\rho$ in d dimension.) Physically more intuitive seems to be a scaling where t, q, p is kept fixed and the potential and the density of scatterers are rescaled. But this is largely a matter of taste and of the physical application in mind.

Let $X^\varepsilon(t)$ be the process of the motion of the Lorentz particle under the scaling (2.13) and (2.14). Since a finite fraction, $(\rho \times \text{range } V)^3$, of the volume remains filled with scatterers, the Lorentz particle typically travels freely a time span ε and then interacts with a scatterer for another time span ε, in which its momentum is deflected on the order $\varepsilon^{1/2}$, etc. The force acting on the Lorentz particle in a collision is $\sim \varepsilon^{-1/2}$. Since the average force is zero, fluctuations in the force just produce an effect of the order unity. As $\varepsilon \to 0$ the scatterings should become independent. Since in a collision $|p|$ is conserved, and since the Poisson distribution does not single out any particular direction, one therefore expects that $p^\varepsilon(t)$ converges to $p(t)$, where $p(t)$ diffuses on the sphere with radius $|p|$ with a diffusion constant $D(|p|)$. By the central limit theorem $D(|p|)$ should be proportional to the average of the square of momentum change in one collision, which for small ε is given by

$$\langle (\Delta p)^2 \rangle \cong \varepsilon p \int_0^\infty dt \int dq\, F(q - pt) \cdot F(q) , \tag{2.15}$$

where $F = -\nabla_q V$ is the force. Therefore,

$$D(|p|) = \rho\alpha/|p| \tag{2.16}$$

with $\alpha = \frac{1}{2}\pi \int dk\, |k|\, |\hat{V}(k)|^2$, where $\hat{V}$ is the Fourier transform of V. Since the free motion is unchanged under scaling,

$$q^\varepsilon(t) = q + \int_0^t ds\, p^\varepsilon(s) , \tag{2.17}$$

and one expects $q^\varepsilon(t)$ to converge to $q(t) = q + \int_0^t ds\, p(s)$. (I learned this argument from S. Goldstein.)

A more formal second-order perturbation expansion leads to the same result. Under the scaling (2.13) and (2.14) let S_t^ε be the reduced dynamics. Formally,

$$\lim_{\varepsilon\to 0} (S_t^\varepsilon f)(x) = (S_t f)(x) \equiv f(x,t) , \tag{2.18}$$

where $f(x,t)$ satisfies the *linear Landau equation*

$$\frac{\partial}{\partial t}\rho(q,p,t)=\left\{-p\cdot\nabla_q+\rho\sum_{i,j=1}^{3}\frac{\partial}{\partial p_i}D_{ij}(p)\frac{\partial}{\partial p_j}\right\}f(q,p,t). \tag{2.19}$$

The three Cartesian coordinates of p are p_1, p_2, and p_3, and the diffusion matrix is given by

$$D_{ij}(p)=\alpha(1/2|p|)(\delta_{ij}-p_ip_j/|p|^2). \tag{2.20}$$

In spherical coordinates the diffusion term is

$$(\rho\alpha/|p|)\Delta_{|p|}f(q,|p|,\theta,\phi), \tag{2.21}$$

where $\Delta_{|p|}$ is the Laplace-Beltrami operator on the sphere with radius $|p|$, which is just the result we claimed before.

Let us now assume that the Lorentz particle is confined to a region $\Lambda\subset\mathbf{R}^3$, that the particle is specularly reflected upon reaching the boundary $\partial\Lambda$ of Λ, and that the density of the Poisson distribution of scatterers is not necessarily constant, i.e., that ρ is a function of q. Then the scalings (2.13) and (2.14) are still meaningful. Since the collision term of the linear Landau equation is local, in the weak coupling limit one obtains (2.19) with ρ replaced by $\rho(q)$ and with the boundary condition for $-p\cdot\nabla_q$ corresponding to specular reflection at $\partial\Lambda$. If the scaling (2.10) to (2.12) is adopted, then Λ and ρ also have to be scaled.

An interesting phenomenon occurs in the case the distribution of scatterers is not Poisson: We adopt the scalings (2.10) to (2.12) and assume that the scatterer distribution is given by the equilibrium state of a gas with a bounded, rapidly decreasing, central pair potential U at inverse temperature β and at fixed density ρ. β is chosen to be so small that the Mayer expansion converges, i.e., the system of scatterers will be deep inside the gaseous region. Let ρ_2 be the second correlation function and g be the pair correlation function of the equilibrium distribution of scatterers, $\rho_2(q_1,q_2)=\rho^2(1+g(q_1-q_2))$. Since on the q_ε-scale the Lorentz particle travels very far, it samples the correlations between scatterers. On the other hand, since g decays exponentially, the independence of scatterings in well separated regions is not destroyed. Therefore in this case only the diffusion constant changes through replacing α by

$$\alpha'=\frac{\pi}{2}\int dk(1+\rho\hat{g}(k))|k||\hat{V}(k)|^2. \tag{2.22}$$

2. Convergence of the weak coupling limit

Recently, Kesten and Papanicolaou (1979) proved the convergence of the weak coupling limit. They studied the more general problem of the motion of a mechanical particle in a time-independent stochastic force field which may be velocity dependent. (This general case is of interest for the motion of charged particles in inhomogeneous electric and magnetic fields.) Specializing their result to the particular case described in the previous section, one obtains:

Theorem 2.1: Let V be of finite range and three times continously differentiable with bounded derivatives. Let the distribution μ^ε of scatterers be a Poisson distribution with constant density $\rho_\varepsilon=\varepsilon^{-3}\rho$. Let $X^\varepsilon(t)$ be the stochastic motion of the Lorentz particle starting at x with the potential scaled as in (2.12). Then

$$\lim_{\varepsilon\to 0}X^\varepsilon(t)=X(t), \tag{2.23}$$

where $X(t)$ is the diffusion process corresponding to (2.19) starting at x. The convergence is in the sense of weak convergence of the corresponding path measures on $C((0,\infty),\mathbf{R}^3\times\mathbf{R}^3)$. For the validity of Theorem 2.1 it is important that the dimension $d\geq 3$. In one dimension $X^\varepsilon(t)$ converges trivially to free motion. In two dimensions the problem of convergence is open. The difficulty results from the recurrence of the limiting diffusion process.

The proof of Theorem 2.1 is technically much more involved than those of the low-density and mean field limits. An outline of the proof for a related problem can be found in Kesten and Papanicolaou (1978).

3. The low-density (Grad) limit

We consider a situation with a low density of scatterers. So the constant density ρ_ε of scatterers is scaled as

$$\rho_\varepsilon=\varepsilon\rho. \tag{2.24}$$

Then typical spatial and time variations of the distribution of the Lorentz particle are on the order of a mean free path and a mean free time, i.e., on the order $1/\rho_\varepsilon=1/\varepsilon\rho$. To obtain a nontrivial limit, therefore, space and time have to be scaled as

$$q_\varepsilon=\varepsilon^{-1}q,\quad t_\varepsilon=\varepsilon^{-1}t. \tag{2.25}$$

Again there is an *equivalent* scaling

$$\begin{aligned}&t,q\ \text{(unscaled)}\\ &V_\varepsilon(q)=V(q/\varepsilon),\\ &\rho_\varepsilon=\varepsilon^{-2}\rho.\end{aligned} \tag{2.26}$$

($\rho_\varepsilon=\varepsilon^{-d+1}\rho$ in d dimensions.) Note that the normalized differential cross section is invariant under the scaling (2.26). The volume occupied by scatterers tends to zero as $\varepsilon\to 0$.

Let $X^\varepsilon(t)$ be the motion of the Lorentz particle scaled according to (2.26). For $\varepsilon\to 0$ the probability of colliding with the same scatterer again goes to zero. The scatterings become independent. Therefore one expects $p^\varepsilon(t)$ to converge to $p(t)$, where $p(t)$ is the following jump process (a proof can be found in the next section). $p(t)=p$ for an exponentially distributed time with parameter $\pi\rho|p|$. Then the Lorentz particle collides with a scatterer, which causes the momentum to jump instantaneously from p to p_1 with probability $\sigma(p_1|p)dp_1$, the normalized differential cross section of the potential V. By conservation of energy, $\sigma(p_1|p)dp_1$ contains the δ function $\delta(|p|-|p_1|)$. Then $p(t)=p_1$ for an exponentially distributed time with parameter $\pi\rho|p_1|$, etc. By (2.17) one expects that $q^\varepsilon(t)$ converges to $q(t)=q+\int_0^t ds\,p(s)$. In terms of the reduced dynamics,

$$\lim_{\varepsilon\to 0}(S_t^\varepsilon f)(x)=(S_tf)(x)\equiv f(x,t), \tag{2.27}$$

where $f(x,t)$ satisfies the *linear Boltzmann equation*

$$\frac{\partial}{\partial t} f(q,p,t) = -p \cdot \nabla_q f(q,p,t) + \pi \rho |p| \Big(\int dp' \, \sigma(p'|p) f(q,p',t) - f(q,p,t) \Big) . \tag{2.28}$$

(2.28) is also known as the Rayleigh-Boltzmann equation, the Lorentz-Boltzmann equation, or the transport equation.

For actual calculations, the difficulty of computing the differential cross section remains. A particular simple case is the hard-core potential. In that case $\sigma(p'|p)dp'$ is the normalized uniform distribution on the sphere with radius $|p|$.

If we choose a nonideal scatterer distribution, as discussed already for the weak coupling limit, then, because the density goes to zero, the dependence on the pair potential is lost in the limit. Of course, by a different scaling of the equilibrium distribution of scatterers, one may, by force, so to speak, retain long-range correlations. In this case the stochastic motion of the Lorentz particle still converges as $\varepsilon \to 0$, but the Markov property is destroyed by fluctuations (see Sec. II.A.4).

Starting from the linear Boltzmann equation, we may take the weak coupling limit again. The weakening of the potential proportional to $\varepsilon^{1/2}$ is then compensated by decreasing the mean free path as ε, i.e., by increasing the collision rate as ε^{-1}. This can be seen by taking the weak coupling limit in two steps by scaling

$$V_{\varepsilon,\varepsilon'}(q) = \varepsilon^{1/2} V(q/\varepsilon') , \quad \rho_{\varepsilon,\varepsilon'} = (\varepsilon')^{-2} \varepsilon^{-1} \rho . \tag{2.29}$$

Then $\varepsilon' \to 0$ leads to the linear Boltzmann equation, and $\varepsilon \to 0$ subsequently leads to the linear Landau equation with the dependence (2.22) on the pair correlation function lost.

4. Convergence of the low-density limit

The proof of the convergence of the low-density limit will become particularly transparent in the case of a hard-core potential. Therefore, for this section only, we assume that the scatterers are hard spheres of radius ε from which the Lorentz particle is scattered by specular reflection.

A configuration of scatterers is assumed to be locally finite, i.e., in any bounded region there will be only a finite number of scatterers. Let us denote by $\aleph$ the space of all locally finite configurations. Then the distribution of scatterers is given by a probability measure μ^ε on $\aleph$. We assume that μ^ε will be determined by its correlation functions $\{\rho_n^\varepsilon | n = 0, 1, \ldots\}$. $\rho_n^\varepsilon(q_1, \ldots, q_n)$ is the probability density of finding an n-tuple of scatterers at $q_1, \ldots, q_n$. See Ruelle (1969) and Gallavotti, Lanford, and Lebowitz (1970) for details.

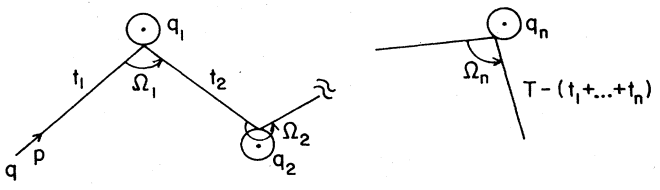


FIG. 1. Path of the Lorentz particle.

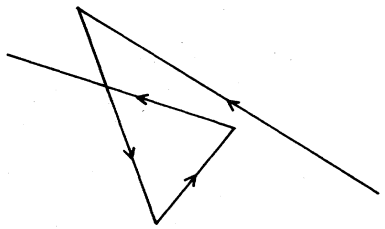


FIG. 2. Path in Γ_3^0 (mechanically possible path for some choice of ε).

We fix a finite, but arbitrary, time $T > 0$. (The extension to $T = \infty$ is a well-known measure theoretical procedure which we want to avoid here.) Then a typical path of the Lorentz particle has the form illustrated by Fig. 1. t_1 is the time of travel from q to the first collision point $q + pt_1$, $\Omega_1 \in S^2$ is the solid angle by which the particle is scattered in the first collision, t_2 is the time of travel from the first collision $q + pt_1$ to the second collision point $q + pt_1 + |p|\Omega_1 t_2$, etc. The path space has a particularly simple structure. We denote it by $\Gamma = \cup_{n \geq 0} \Gamma_n$, where $\Gamma_n \subset \mathbf{R}^{3n}$ is the set of all paths with exactly n collision points

$$\Gamma_n = \{(t_1, \Omega_1, \ldots, t_n \Omega_n) \in \mathbf{R}^{3n} | \, \Omega_j \in S^2, 0 \leq t_1 + \cdots + t_n \leq T\} .$$

The probability measure μ^ε induces the path measure P_x^ε on Γ. Both Γ and P_x^ε depend on T. Not to overload the notation we suppress this dependence.

Theorem 2.2: Let μ^ε be a probability measure on $\aleph$ determined by its correlation functions $\{\rho_n^\varepsilon | n \geq 0\}$ which satisfy

(C0) $\rho_n^\varepsilon(q_1, \ldots, q_n) = 0$ for $|q - q_j| \leq \varepsilon$, $j = 1, \ldots, n$.

(C1) There exist constants M and z such that for all $\varepsilon > 0$,

$$\varepsilon^{2n} \rho_n^\varepsilon(q_1, \ldots, q_n) \leq M z^n .$$

(C2) There exist continuous functions r_n on $\mathbf{R}^{3n}$ such that

$$\lim_{\varepsilon \to 0} \varepsilon^{2n} \rho_n^\varepsilon(q_1, \ldots, q_n) = r_n(q_1, \ldots, q_n) ,$$

uniformly on compact sets of $\{(q_1, \ldots, q_n) \in \mathbf{R}^n | q \neq q_j, q_i \neq q_j, i \neq j, i, j = 1, \ldots, n\}$. Then P_x^ε converges weakly to a probability measure P_x on Γ.

Remark: For the single time distribution this result

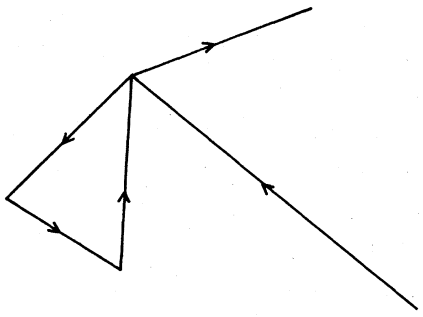


FIG. 3. Path in $\Gamma_4 \backslash \Gamma_4^0$ (mechanically impossible path for any choice of ε).

was proved by Gallavotti (1969, 1972). Using the method of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy, Spohn (1978) proved the result for a general class of potentials. Van Beijeren (unpublished) obtained by the same method the convergence of the single time distribution. The present proof was suggested to the author by O. E. Lanford. In spirit the proof is close to Gallavotti's ideas.

Remark: (C0) guarantees that the initial position q of the Lorentz particle is not overlapped by a scatterer and that, therefore, the dynamics become well defined with probability one. (C1) and (C2) replace (2.2) with scaling (2.26) for scatterer distributions which are not necessarily Poisson. For the uniform convergence we excluded sets on which it could fail—e.g., if the distribution μ^ε is such that the scatterers may not overlap, then uniform convergence can hold only on compact sets of $\{q_i \neq q_j, i \neq j\}$.

Proof: The key to the proof is to realize that there are hypersurfaces of "bad" points in Γ which should be avoided. Away from these bad points the convergence is controllable.

Let $\Gamma_n^0 = \{(t_1\Omega_1, \ldots, t_n\Omega_n) \in \Gamma_n | t_1 > 0, \ldots, t_n > 0$, each one of the n collision points having exactly one point in common with the path corresponding to $t_1\Omega_1, \ldots, t_n\Omega_n\}$, and let $\Gamma^0 = \cup_{n\geq 0}\Gamma_n^0$. Since (see Figs. 1, 2, and 3) $\Gamma_n \backslash \Gamma_n^0$ is of lower dimension than Γ_n, $\Gamma \backslash \Gamma_n^0$ has Lebesgue measure zero. We want to show that for each compact $A \subset \Gamma_n^0$,

$$\lim_{\varepsilon\to 0} P_x^\varepsilon(A) = P_x(A). \tag{2.30}$$

Since A is compact, we may as well choose $A \subset \Gamma_n^0$.

Let $\Lambda \subset \mathbf{R}^3$ be a bounded region containing q such that in time T the Lorentz particle cannot leave Λ. Then the probability of finding exactly n scatterers at $q_1, \ldots, q_n$ in Λ is given by the absolutely continuous probability measure

$$f_{n,\Lambda}^\varepsilon(q_1, \ldots, q_n)\frac{1}{n!}dq_1\cdots dq_n$$

$$= \sum_{m=0}^\infty \frac{(-1)^m}{m!}\int_{\Lambda^m} dq_1'\cdots dq_m'\,\rho_{n+m}^\varepsilon(q_1, \ldots, q_n, q_1', \ldots, q_m')$$

$$\times\frac{1}{n!}dq_1\cdots dq_n, \quad n = 0, 1, \ldots. \tag{2.31}$$

Γ_n^0 was constructed in such a way that for compact $A \subset \Gamma_n^0$, ε can be chosen so small that each of the n collisions is necessarily effected by a different scatterer. Note that the closer one comes to $\partial(\Gamma_n^0)$, the smaller ε has to be. Then, for ε so small that there are no recollisions for paths in A, $P_x^\varepsilon(A)$ is the probability of having exactly n scatterers in Λ such that a path in A is produced, plus the probability of having exactly $n+1$ scatterers in Λ such that n of them are located as to produce a path in A and such that the remaining scatterer has to lie outside a tube of radius ε around the path, plus $\cdots$. If one sums up all these contributions and takes the combinatorial factors into account, then one arrives at

$$P_x^\varepsilon(A) = \int_{\tilde{A}} dq_1\cdots dq_n \sum_{m=0}^\infty \frac{1}{m!}\int_{(\Lambda\backslash\Lambda_\varepsilon)^m} dq_1'\cdots dq_m'$$
$$\times f_{n+m,\Lambda}^\varepsilon(q_1, \ldots, q_n, q_1', \ldots, q_m')$$
$$= \int_{\tilde{A}} dq_1\cdots dq_n \sum_{m=0}^\infty \frac{(-1)^m}{m!}\int_{(\Lambda_\varepsilon)^m} dq_1'\cdots dq_m'$$
$$\times\rho_{n+m}^\varepsilon(q_1, \ldots, q_n, q_1', \ldots, q_m'). \tag{2.32}$$

$\tilde{A}$ is the set of all $q_1, \ldots, q_m$ such that a path in A is produced. Λ_ε is the tube of radius ε around the path corresponding to $(q_1, \ldots, q_n) \in \tilde{A}$. The equality uses a well-known identity between probability densities and correlation functions.

We consider now the position $q_1, \ldots, q_n$ of the scatterers as functions of $t_1\Omega_1, \ldots, t_n\Omega_n$ (see Fig. 1). The volume element transforms as $dq_1, \ldots, dq_n = \prod_{j=1}^n \{\pi|p|\varepsilon^2\, dt_j\, d\Omega_j\}$ because of isotropic scattering in three dimensions. (For smooth potentials, the differential cross section would enter at this stage.) Therefore,

$$P_x^\varepsilon(A) = \int_A \prod_{j=1}^n \{\pi|p|dt_j\, d\Omega_j\}\sum_{m=0}^\infty \frac{(-1)^m}{m!}\varepsilon^{-2m}$$
$$\times\int_{(\Lambda_\varepsilon)^m} dq_1'\cdots dq_m'\varepsilon^{2(n+m)}\rho_{n+m}^\varepsilon(q_1, \ldots, q_n, q_1', \ldots, q_m'). \tag{2.33}$$

By (C1) the sum (2.33) is bounded by

$$|A|\sum_{m=0}^\infty \frac{1}{m!}(\pi\varepsilon^2|p|T)^m\varepsilon^{-2m}Mz^{(n+m)} = |A|z^nMe^{\pi|p|Tz}. \tag{2.34}$$

Therefore, by Lebesgue's dominated convergence and by (C2),

$$\lim_{\varepsilon\to 0} P_x^\varepsilon(A) = \int_A \prod_{j=1}^n \{\pi|p|dt_j\, d\Omega_j\}$$
$$\times\sum_{m=0}^\infty \frac{(-1)^m}{m!}\pi^m\int_0^T ds_1\cdots\int_0^T ds_m r_{n+m}$$
$$\times(q+pt_1, \ldots, q+pt_1+\cdots+|p|\Omega_n$$
$$\times(T-t_1-\cdots-t_n), q(s_1), \ldots, q(s_m)) \equiv P_x(A). \tag{2.35}$$

$t \to q(t)$ is the path corresponding to $t_1\Omega_1, \ldots, t_n\Omega_n$.

Weak convergence follows now from (2.30) together with $P_x(\Gamma\backslash\Gamma^0) = 0$■.

There is a moral to the proof. If the scatterers are such that there is a finite probability for exact back scattering, then one still has convergence of $P_x^\varepsilon \to P_x$ on Γ^0 as $\varepsilon \to 0$, but $P_x(\Gamma^0) < 1$. A part of the path measure P_x corresponding to the linear Boltzmann equation lives on $\Gamma\backslash\Gamma^0$ and cannot be reached in the low-density limit. For a smooth potential the probability for back scattering is zero. But if the scatterers are small crosses all orientated at 45° with respect to P, then the phenomenon just described occurs.

Proposition 2.3. Let μ^ε satisfy the conditions of Theorem 2.2 and let $n(\Lambda)$ be the number of scatterers

in the bounded region Λ.

(i) If

$$\lim_{\varepsilon\to 0}\varepsilon^2 n(\Lambda)=\int_\Lambda dq_1\, r(q_1) \tag{2.36}$$

in probability, then

$$r_n(q_1,\ldots,q_n)=\prod_{j=1}^n r_1(q_j)\,. \tag{2.37}$$

(ii) If $r_2(q_1,q_2)=r_1(q_1)r_1(q_2)$, then $\varepsilon^2 n(\Lambda)$ converges in probability to (2.38), and therefore $r_n(q_1,\ldots,q_n)=\prod_{j=1}^n r_1(q_j)$.

Remark: The condition in (i) means that the rescaled number of particles has no fluctuations as $\varepsilon\to 0$.

Proof: By (C1) and (C2),

$$\lim_{\varepsilon\to 0}\mu^\varepsilon(\varepsilon^2 n(\Lambda_1)\cdots\varepsilon^2 n(\Lambda_n))=\int_{\Lambda_1\times\cdots\times\Lambda_n} dq_1\cdots dq_n\, r_n \times(q_1,\ldots,q_n)\,; \tag{2.38}$$

and by (C1), (C2), and the condition in (i),

$$\lim_{\varepsilon\to 0}\mu^\varepsilon(\varepsilon^2 n(\Lambda_1)\cdots\varepsilon^2 n(\Lambda_n))=\prod_{j=1}^n\left(\int_{\Lambda_j} dq_j\, r(q_j)\right). \tag{2.39}$$

This implies the factorization.

If $r_2(q_1,q_2)=r_1(q_1)r_1(q_2)$, then

$$\lim_{\varepsilon\to 0}\mu^\varepsilon(\varepsilon^4 n(\Lambda)^2)=\int_{\Lambda\times\Lambda} dq_1\, dq_2\, r_2(q_1,q_2) =\left(\int_\Lambda dq_1\, r(q_1)\right)^2, \tag{2.40}$$

which together with (2.38) for $n=1$ implies that the variance of $\varepsilon^2 n(\Lambda)-\mu^\varepsilon(\varepsilon^2 n(\Lambda))$ vanishes as $\varepsilon\to 0$ ■

Corollary 2.4. The limit process corresponding to the path measure P_x^ε on Γ is Markov for all T if and only if

$$\lim_{\varepsilon\to 0}\varepsilon^2 n(\Lambda)=\int_\Lambda dq_1\, r_1(q_1) \tag{2.41}$$

in probability. In this case, the forward equation of the process is the linear Boltzmann equation

$$\frac{\partial}{\partial t}f(q,p,t)=-p\cdot\nabla_q f(q,p,t) +r_1(q)\pi|p|\left(\int d\Omega' f(q,|p|\Omega',t)-f(q,p,t)\right). \tag{2.42}$$

Proof: Only for the factorization (2.39),

$$P_x(A)=\int_A\prod_{j=1}^n\{\pi|p|dt_j d\Omega_j\} \times r(q+pt_1)\cdots r(q+pt_1+\cdots+|p|\Omega_n \times(T-t_1-\cdots-t_n))\exp\left(-\pi|p|\int_0^T ds\, r(q(s))\right). \tag{2.43}$$

Equation (2.43) is known as Poisson formula. If one computes $E_x(f(x(t)))=f(x,t)$ using (2.43), then one recovers the usual time-dependent perturbation series for the equation adjoint to (2.42), where $\{p\cdot\nabla_q - r_1(q)\pi|p|\}f(q,p)$ is regarded as the unperturbed part ■

For a twice continuously differentiable, central scattering potential of finite range, a result analogous to Corollary 2.4 can be proved, provided that the correlation functions ρ_n^ε are, in addition, continuous. Boundary conditions may be added, as already discussed, for the weak coupling limit. The results are valid for any dimension $d\geq 2$.

However, for potentials of infinite range, even fastly decaying, the method developed so far does not seem to apply (see also Sec. II.A.5).

5. The mean-field limit

The spirit of mean-field approximation is to take into account external influence (e.g., scattering) by an effective mean force.

In order not to obtain a trivial answer we now choose a spatially varying density $\rho_1^\varepsilon(q)$ of the Poisson distribution. One assumes a weak potential by scaling

$$V_\varepsilon(q)=\varepsilon V(q)\,. \tag{2.44}$$

(Note that in contrast to the weak coupling limit the range of the potential is *not* scaled.) The density of scatterers is increased only very slowly as

$$\rho_1^\varepsilon(q)=\varepsilon^{-1}r(q)\,. \tag{2.45}$$

Since the range of the potential is not scaled, the scatterers will overlap and conservation of p is no longer expected. (2.44) and (2.45) define the mean-field limit.

One can also use the equivalent scaling

$$\rho_1^\varepsilon(q)=r(\varepsilon^{1/3}q)\,,\quad V_\varepsilon(q)=\varepsilon V(\varepsilon^{1/3}q)\,,\quad t_\varepsilon=\varepsilon^{-1/3}t,\quad q_\varepsilon=\varepsilon^{-1/3}q\,. \tag{2.46}$$

($\frac{1}{3}$ is replaced by $1/d$ in d dimensions.) Just considering the potential scaling, one sees that the mean-field limit corresponds to the motion of the Lorentz particle through scatterers with a weak, long-range potential. Note that for the Coulomb potential $V(q)=|q|^{-1}V_\varepsilon(q)=\varepsilon^{2/3}|q|^{-1}$. In this case the scaling corresponds to a small charge.

Let $X^\varepsilon(t)$ be the scaled process. It is easily verified that the force on the Lorentz particle converges in probability to $-\int dq'\, r(q')\nabla_q V(q-q')$. Therefore, one expects $X^\varepsilon(t)$ to converge in probability to $(q(t),p(t))$, where $q(t)$ and $p(t)$ will be the solutions of

$$\frac{d}{dt}q(t)=p(t)\,,\quad \frac{d}{dt}p(t)=-\nabla_q\int dq'\, r(q')V(q(t)-q')\,, \tag{2.47}$$

with initial conditions $q(0)=q$, $p(0)=p$ (this proof is worked out in the following section). Thus, in contrast to the case in the weak coupling and low-density limits, in the mean-field limit the motion of the Lorentz particle is deterministic and governed by the *effective* Hamiltonian

$$H_{\text{eff}}(q,p)=\tfrac{1}{2}p^2+\int dq'\, r(q')V(q-q')\,. \tag{2.48}$$

The mean-field limit is of some interest, since it

provides a tool for dealing with the long-range part of the interaction. The potential V is split as V_1+V_2, where V_1 is of finite range and V_2 of infinite range. Then, for the low-density limit, the potential is scaled as

$$V_{1,\varepsilon}(q)=V_1(q/\varepsilon),\quad V_{2,\varepsilon}(q)=\varepsilon^2 V_2(q)\,, \tag{2.49}$$

and the density as $\rho_1^\varepsilon(q)=\varepsilon^{-2}r(q)$. In the limit $\varepsilon\to 0$, V_1 leads to the collision term of the linear Boltzmann equation, whereas V_2 gives an effective force according to (2.47).

6. Convergence of the mean-field limit

The proof simplifies considerably in the case that the total number of scatterers is finite with probability one. μ^ε is then a probability measure on $U_{n\geq 0}\mathbf{R}^{3n}$.

Theorem 2.5: Let $n(\Lambda)$ be the number of scatterers in the set $\Lambda\subset\mathbf{R}^3$ and assume that

(C1) there exist constants M and z such that for all $\varepsilon>0$,

$$\mu^\varepsilon(\{[\varepsilon n(\mathbf{R}^3)]^m\})\leq Mz^m\,;$$

(C2) there exists a bounded measure r on $\mathbf{R}^3$ such that

$$\lim_{\varepsilon\to 0}\mu^\varepsilon(\{\varepsilon n(\Lambda)=r(\Lambda)\}),$$

$$\lim_{\varepsilon\to 0}\mu^\varepsilon(\{\varepsilon n(\Lambda)\varepsilon n(\Lambda')\})=r(\Lambda)r(\Lambda')$$

for all r-continuous sets $\Lambda,\Lambda'\subset\mathbf{R}^3$. Let $X^\varepsilon(t)$ be the stochastic motion of the Lorentz particle starting at x with the central scatterer potential $V_\varepsilon(q)=\varepsilon V(q)$, where $V\in C^2$ with bounded derivatives. Then $X^\varepsilon(t)$ converges in probability to $X(t)=(q(t),p(t))$. $q(t),p(t)$ is the solution of

$$\frac{d}{dt}q(t)=p(t)\,,\quad \frac{d}{dt}p(t)=\bar F(q(t))\,, \tag{2.50}$$

with initial conditions $x=(q,p)$. $\bar F$ is the average force $\bar F(q)=\int r(dq')F(q-q')$, with $F(q)=-\nabla_q V(q)$.

Remark: The proof follows Neunzert (1975) and Braun and Hepp (1977), who studied the derivation of the Vlasov equation (see Sec. III.D).

Proof: Let us denote $F(q,Q)=\sum_j F(q-q_j)$ for $Q=(q_1,q_2,\ldots)$. Then for $t\geq 0$,

$$\begin{aligned}|q^\varepsilon(t,Q)-q(t)|+|p^\varepsilon(t,Q)-p(t)|&\\ \leq\int_0^t ds\,|p^\varepsilon(s,Q)-p(s)|&\\ +\int_0^t ds\,\varepsilon|F(q^\varepsilon(s,Q),Q)-F(q(s),Q)|&\\ +\int_0^t ds\,|\varepsilon F(q(s),Q)-\bar F(q(s))|\,,&\end{aligned} \tag{2.51}$$

where we suppressed the dependence on the initial x. Since F is globally Lipshitz continuous, then

$$|F(q,Q)-F(q',Q)|\leq cn(\mathbf{R}^3)(Q)|q-q'|\,. \tag{2.52}$$

Inserting (2.52) in (2.51) and iterating, one obtains

$$\begin{aligned}|q^\varepsilon(t,Q)-q(t)|+|p^\varepsilon(t,Q)-p(t)|&\\ \leq\int_0^t ds\,|\varepsilon F(q(s),Q)-\bar F(q(s))|\,\exp[(cn(\mathbf{R}^3)(Q)+1)t]\,.&\end{aligned} \tag{2.53}$$

By Schwarz's inequality, we get

$$\begin{aligned}\mu^\varepsilon(|q^\varepsilon(t,\cdot)-q(t)|+|p^\varepsilon(t,\cdot)-p(t)|)&\\ \leq\int_0^t ds(\mu^\varepsilon((\varepsilon F(q(s),\cdot)-\bar F(q(s)))^2))^{1/2}&\\ \times(\mu^\varepsilon(e^{(cn(\mathbf{R}^3)+1)2t}))^{1/2}\,.&\end{aligned} \tag{2.54}$$

By assumption (C1) the second factor of (2.54) is bounded. Let $\rho_1^\varepsilon(dq_1)$ and $\rho_2^\varepsilon(dq_1,dq_2)$ be the first and second correlation measure of μ^ε. Assumption (C2) implies the weak convergence $\lim_{\varepsilon\to 0}\varepsilon\rho_1^\varepsilon(dq_1)=r(dq_1)$, $\lim_{\varepsilon\to 0}\varepsilon^2\rho_2^\varepsilon(dq_1,dq_2)=r(dq_1)r(dq_2)$. Therefore,

$$\begin{aligned}g^\varepsilon(q)&\equiv\mu^\varepsilon((\varepsilon F(q,\cdot)-\bar F(q))^2)\\ &=\varepsilon^2\int\rho_2^\varepsilon(dq_1,dq_2)F(q-q_1)F(q-q_2)\\ &+\varepsilon^2\int\rho_1^\varepsilon(dq_1)(F(q-q_1))^2\\ &-2\varepsilon\int\rho_1^\varepsilon(dq_1)F(q-q_1)\int r(dq_1)F(q-q_1)\\ &+\left(\int r(dq_1)F(q-q_1)\right)^2\end{aligned} \tag{2.55}$$

converges to zero as $\varepsilon\to 0$.

Since $g^\varepsilon(q)$ is bounded, this implies that the first factor of (2.54) tends to zero as $\varepsilon\to 0$. Therefore (2.54) tends to zero, proving the L^1-convergence of $X^\varepsilon(t)-X(t)$ ■

B. The Rayleigh gas

The Rayleigh gas consists of a test particle moving through a fluid. In contradistinction to what happens with the Lorentz gas, the scatterers now move and suffer recoil upon collision with the Rayleigh particle. The formal Hamiltonian is given by

$$H=\frac{1}{2M}p^2+\sum_j V(q-q_j)+\sum_j\frac{1}{2m}p_j^2+\sum_{i<j}U(q_i-q_j)\,. \tag{2.56}$$

q,p denotes the position and the momentum, and M is the mass of the Rayleigh particle. $x_j=(q_j,p_j)$ stands for the position and the momentum of the jth fluid particle. m is the mass of a fluid particle.

Initially, the Rayleigh particle is at q with momentum p, and the fluid is in some state. As for the Lorentz gas, the initial state of the fluid will be scaled appropriately. However, the choice of the sequence of fluid states is severely restricted by the fact that in order to obtain a Markovian behavior for the particle, the properly scaled number of particles in any bounded region should not fluctuate in the limit. Therefore, if $\rho_n^\varepsilon(t)$ denotes the nth correlation function of the fluid at time t, by Proposition 2.3 the sequence of fluid states has to satisfy

$$\lim_{\varepsilon\to 0}\varepsilon^n\rho_n^\varepsilon(x_1,\ldots,x_n,t)=\prod_{j=1}^n r(x_j,t)\,,\quad\text{a.e.}\,. \tag{2.57}$$

To discuss the scalings more precisely we will have to distinguish two cases.

(a) $U \equiv 0$, *ideal fluid.* In this case r will evolve freely, i.e., $r(x_j,t) = r(q_j - p_j t, p_j)$ with $r(x_j) = r(x_j, 0)$. All we said about the Lorentz gas basically carries over to the Rayleigh gas. (Of course, the proofs will have to be reconsidered.) The scaling of the potential V and of the density of the fluid is the same as for the Lorentz gas. The limiting processes will differ somewhat from the ones obtained for the Lorentz gas: First, the recoil of the fluid particle has to be taken into account. In particular, the conservation of energy is lost. Secondly, the density of the fluid may be time dependent, which will give rise to a nonhomogeneous Markov process (the transition probability depends not only on the time difference).

(b) $U \neq 0$, *nonideal fluid.* Now one has the freedom to scale U also. The condition (2.57) seems to leave open only the following two cases: (i) U is scaled away fast. Then one is back to (a). (ii) U is scaled according to the weak coupling, the low-density, or the mean-field limit. Then, as will be discussed at length in Sec. III, $r(x,t)$ evolves according to the Landau equation, the Boltzmann equation, or the Vlasov equation. The scaling of V may differ from the scaling of U. Depending on whether one scales V according to the weak coupling, the low-density, or the mean-field limit, one obtains either a diffusion process, or a jump process, or a deterministic process in the limit. As for (a), since the fluid density may depend on time, the limiting Markov process will be nonhomogeneous. The case of a nonideal fluid is, of course, more difficult than the case of an ideal fluid, since one must first control the corresponding limit of the interacting fluid.

Another possibility for the initial state of the fluid consists of requiring that the fluid be in *thermal equilibrium* at inverse temperature β and density ρ conditioned on the Rayleigh particle being located at q. Again, the density and the pair potential U have to be scaled properly. Since the equilibrium state is time invariant, the limiting Markov process will be homogeneous.

It should come as no surprise that, since another parameter, namely, the mass M of the Rayleigh particle, enters the Hamiltonian, there is yet another Markovian limit, associated with $M \to \infty$. This limit is called Brownian motion limit and will be discussed in the following section.

1. The Brownian motion limit

Brownian motion is presumably the oldest and most studied stochastic model in nonequilibrium statistical mechanics. [Nelson (1967) presents the subject beautifully.] This kind of phenomenon is observed for a heavy particle suspended in a fluid. To model this situation we assume that the Brownian particle is initially at q with momentum p and that the fluid is in thermal equilibrium at inverse temperature β and density ρ, conditioned on the Brownian particle being located at q. The mass M of the Rayleigh particle is increased as

$$M_\varepsilon = \varepsilon^{-2} M . \tag{2.58}$$

To keep the kinetic energy of the Rayleigh particle finite,

$$p_\varepsilon = \varepsilon^{-1} p . \tag{2.59}$$

Equivalently, the velocity scales as $v_\varepsilon = \varepsilon v$, i.e., the Rayleigh particle becomes very slow. Under (2.58) and (2.59) the velocity of the Rayleigh particle changes on the order ε in a collision. The number of collisions per unit time should therefore be on the order ε^{-2} which is achieved by scaling time as

$$t_\varepsilon = \varepsilon^{-2} t . \tag{2.60}$$

As for the other limits, the free motion should be left invariant under the scaling, which implies that space has to be scaled as

$$q_\varepsilon = \varepsilon^{-1} q . \tag{2.61}$$

(2.58) to (2.61) define the Brownian motion limit.

As before, there is an equivalent scaling which scales properties of the fluid rather than of the Rayleigh particle: Mass, momentum, space, and time of the Rayleigh particle remain unscaled. (2.61) is translated to scaling the potential as

$$V_\varepsilon(q) = V(q/\varepsilon) \tag{2.62}$$

and the density as

$$\rho_\varepsilon = \varepsilon^{-3} \rho . \tag{2.63}$$

To have a momentum transfer of the order ε in a collision, the mass of a fluid particle becomes small

$$m_\varepsilon = \varepsilon^2 m \tag{2.64}$$

and the kinetic energy of a fluid particle remains constant

$$p_{j,\varepsilon} = \varepsilon p_j . \tag{2.65}$$

Equivalently, the velocity scales $v_{j,\varepsilon} = \varepsilon^{-1} v_j$, i.e., the fluid particles become very fast.

Let $X^\varepsilon(t)$ be the process of the motion of the Rayleigh particle scaled according to (2.62) to (2.65). Then in the limit $\varepsilon \to 0$, the fluid particles are so fast that recollisions with the same fluid particle become relatively unlikely. The force on the Rayleigh particle consists of a systematic part, which slows down the Rayleigh particle proportional to its momentum, and a fluctuating part, which is almost like white noise. Therefore, one expects $X^\varepsilon(t)$ to converge to $X(t)$, where $X(t)$ is the Ornstein-Uhlenbeck process. This process corresponds to the *Fokker-Planck equation*

$$\frac{\partial}{\partial t} f(q,p,t) = \left\{ -\frac{1}{M} p \cdot \nabla_q + \frac{D\beta}{M} \nabla_p \cdot p + D\Delta_p \right\} f(q,p,t) \tag{2.66}$$

for the probability density $f(q,p,t)$ of the Rayleigh particle.

Formal derivations by Lebowitz and Rubin (1963) and by Lebowitz and Resibois (1965), using second-order perturbation theory, show that the diffusion constant D (in velocity space) is given by the time integral over the force autocorrelation function

$$D = \frac{1}{3} \int_0^\infty dt \langle (\Sigma F) \cdot (\Sigma F)(t) \rangle_0 . \tag{2.67}$$

Here $F=-\nabla_q V$ is the force and $\Sigma F=\sum_j F(q_j)$. (t) indicates the dynamics of the fluid given formally through the Hamiltonian

$$H=\frac{1}{2m}\sum_j p_j^2+\sum_{i<j}U(q_i-q_j)+\sum_j V(q_j). \tag{2.68}$$

This is the dynamics of the fluid in the presence of the Brownian particle kept fixed at the origin. $\langle\ \rangle_0$ denotes the equilibrium average with (2.68) as Hamiltonian in the thermodynamic limit at inverse temperature β and density ρ. Needless to say, even the existence of D is not known except in the case of an ideal fluid, $U=0$. In one dimension, Holley (1969) has shown the convergence of $X^\varepsilon(t)$ to the Ornstein-Uhlenbeck process (see Sec. II.B.4). For higher dimensions the convergence was proved recently by D. Dürr, S. Goldstein, and J. L. Lebowitz (1980) in the case of an ideal fluid with a general momentum distribution.

As for the weak coupling limit, the Brownian motion limit may be taken in two steps. One scales as

$$\begin{aligned} V_{\varepsilon,\varepsilon'}(q)&=V(q/\varepsilon'),\\ \rho_{\varepsilon,\varepsilon'}&=(\varepsilon\varepsilon'^2)^{-1}\rho,\\ p_{\text{fluid},\varepsilon}&=\varepsilon p_{\text{fluid}}\ . \end{aligned} \tag{2.69}$$

Then $\varepsilon'\to 0$ leads to the linear Boltzmann equation, and, subsequently, $\varepsilon\to 0$ leads to the Fokker-Planck equation (4.11). The second limit has been proved by Il'in and Khas'minskii (1964). In this order of limits, the dependence on U is lost.

2. Convergence of the low-density limit of a tagged particle in an ideal fluid

To prove the convergence of the low-density limit for the Rayleigh gas we use the same technique as for the Lorentz gas. We choose the simplest possible case: The Rayleigh particle interacts with the fluid particles via a hard-core potential, and the fluid is noninteracting and initially Poisson distributed with constant density. The remarks on continuous potentials, boundary conditions, Markov property, fluctuations, etc., made for the Lorentz gas also apply to the Rayleigh gas.

Let $\mathcal{X}$ be the space of configurations $(q_1,p_1,q_2,p_2,\dots)$ in $\mathbf{R}^6$ locally finite in the positional coordinates $(q_1,q_2,\dots)$. If the Rayleigh particle is initially located at q, then the initial state μ^ε of the fluid is assumed to be spatially Poisson with density

$$\varepsilon^{-2}\rho\chi_q^\varepsilon(q_1),$$

where χ_q^ε is the indicator function of the set $\{q_1\in\mathbf{R}^3\,\|\,q_1-q|\geq\varepsilon\}$ and to have independent momenta with distribution $h(p_1)dp_1$, which satisfies

$$\int dp_1 h(p_1)(1+|p_1|^3)<\infty. \tag{2.70}$$

The Rayleigh particle is a hard sphere of mass M and diameter ε, and the fluid particles are hard spheres of mass one and diameter ε. The fluid particles simply pass through each other, whereas the Rayleigh particle collides elastically with the fluid particles. From the results of Alexander (1975), it follows that the Rayleigh particle suffers only a finite number of collisions in a finite time with probability one. Simultaneous collisions with two or more fluid particles are of zero probability. Therefore the motion of the Rayleigh particle is well defined. Let us denote by $X^\varepsilon(t)$ the corresponding process with the Rayleigh particle starting at $x=(q,p)$.

Let us fix some finite $T>0$. Because of the hard collisions the paths have a simple structure. They are piecewise of the form $s\to q+(1/M)ps$ with a finite number of "pieces." Let t_1 be the time of flight from q to the first collision point, y_1 the momentum after the first collision, t_2 the time of flight from the first collision point to the second collision point, etc. For notational simplicity let $p=y_0$ and $T-t_1-\cdots-t_n=t_{n+1}$. Then the path space of the process $X^\varepsilon(t)$ is $\Gamma=\cup_{n\geq 0}\Gamma_n$, $(y_1,t_1,\dots,y_n,t_n)\in\Gamma_n\subset\mathbf{R}^{4n}$, with the restriction that $0\leq t_1+\cdots+t_n\leq T$. μ^ε induces the probability measure P_x^ε on Γ. We will show that P_x^ε converges weakly to P_x on Γ and that P_x is the measure of a Markovian random jump process.

The first thing to show is that the limit process is well defined.

Lemma 2.6: Let

$$\lambda(p)=\pi\rho\int dy\,h(y)\left|\frac{1}{M}p-y\right| \tag{2.71}$$

and

$$K(p'|p)=\lambda(p)^{-1}\left(\frac{M+1}{2M}\right)^2\rho \times\int dy\,h(y)\delta\left((p'-p)\cdot(p'-y)+\frac{M-1}{M}(p^2-p'^2)\right), \tag{2.72}$$

with h satisfying (2.70). Let $p(t)$ be the jump process constructed from $\lambda(p)$ as inverse waiting time and $dp'K(p'|p)$ as jump probability: $p(t)=p$ for $0\leq t<t_1$, where t_1 is exponentially distributed with parameter $\lambda(p)$. Independently of t_1, $p(t)$ jumps from p to dp_1, with probability $dp_1K(p_1|p)$. $p(t)=p_1$ for $t_1\leq t<t_1+t_2$, etc. [see Breiman (1968) for the notion of jump processes]. Then $p(t)$ is well defined in the sense that it has a finite number of jumps in any finite time interval.

Proof: Let p_n be the Markov chain with transition probability $dp'K(p'|p)$ and P_p be the corresponding path measure with $p_0=p$. According to Breiman (1968), Proposition 15.43, $p(t)$ is well defined if

$$\sum_{n=0}^{\infty}\frac{1}{\lambda(p_n)}=\infty,\quad \text{a.s.}. \tag{2.73}$$

Since $\lambda(p)\leq a+b|p|$, (2.73) is implied by

$$\limsup_n\frac{1}{n}|p_n|\leq 1,\quad \text{a.s.}. \tag{2.74}$$

Let $A_n=\{|p_n|\geq n\}$. Then, by the Lemma of Borel-Cantelli, $\sum_{n=1}^\infty P_p(A_n)<\infty$ implies (2.74). Now

$$P_p(A_n)\leq(1/n^2)E_p(|p_n|^2) \tag{2.75}$$

by Chebyshev's inequality. Therefore, (2.73) holds, provided that $E_p(|p_n|^2)\leq c$ independent of n.

The uniform bound on $E_p(|p_n|^2)$ results from conservation of energy. One computes that

$$E_p(p_1^2)=\frac{M^2+1}{(M+1)^2}p^2+\lambda(p)^{-1}\pi\rho\int dy\,h(y)\left|\frac{1}{M}p-y\right|\left(\frac{2M^2}{(M+1)^2}y^2+\frac{M(M-1)}{(M+1)^2}2p\cdot y\right). \tag{2.76}$$

By (4.15), $\lambda(p)^{-1}\int dy\,h(y)|(1/M)p-y\|\,y|^{\alpha}$, $\alpha=1,2$, is uniformly bounded in p. Therefore,

$$E_p(p_1^2)\leq\frac{M^2+1}{(M+1)^2}p^2+c_1+2\delta|p|\frac{1}{\delta}c_2 \leq\left(\frac{M^2+1}{(M+1)^2}+\delta^2\right)p^2+c(\delta). \tag{2.77}$$

Iterating (2.77) yields

$$E_p(p_n^2)\leq p^2\left(\frac{M^2+1}{(M+1)^2}+\delta^2\right)^n+c(\delta)\sum_{m=0}^{n-1}\left(\frac{M^2+1}{(M+1)^2}+\delta^2\right)^m. \tag{2.78}$$

Since $M^2+1\leq(M+1)^2$, δ can be chosen so small that $(M^2+1)/(M+1)^2+\delta^2<1$. Therefore, $E_p(|p_n|^2)$ is uniformly bounded in n for all $p\in\mathbb{R}^3$■

Theorem 2.7: Let P_x^{ε} be the path measure corresponding to the motion of the Rayleigh particle as introduced above. Then

$$\lim_{\varepsilon\to 0}P_x^{\varepsilon}=P_x \tag{2.79}$$

weakly on Γ. The limiting probability measure P_x is the path measure of the jump process $p(t)$ starting at p up to time T constructed in Lemma 2.6.

Proof: Let $A\subset\Gamma_n$ be compact. Let us split

$$P_x^{\varepsilon}(A)=\tilde{P}_x^{\varepsilon}(A)+\tilde{\tilde{P}}_x^{\varepsilon}(A),$$

where $\tilde{P}_x^{\varepsilon}(A)$ is the probability of paths in A such that each one of the n collisions is produced by a different fluid particle. We will show that $\lim_{\varepsilon\to 0}\tilde{P}_x^{\varepsilon}(A)=P_x(A)$. By the normalization of P_x, then $\lim_{\varepsilon\to 0}\tilde{\tilde{P}}_x^{\varepsilon}(A)=0$. For $(y_1,t_1,\ldots,y_n,t_n)=\gamma\in A$, let $\Lambda_{\varepsilon}(\gamma,p')$ be the tube of radius ε around the path defined by the collision points $q,q+[(1/M)y_0-p']t_1,\ldots,q+[(1/M)y_0-p']t_1+\cdots[(1/M)y_n-p']t_{n+1}$. A fluid particle with momentum p' will not collide with the Rayleigh particle moving along γ, if and only if at time $t=0$ it is outside $\Lambda_{\varepsilon}(\gamma,p')$. Let us consider γ as a function of the initial positions and momenta of the fluid particles. Then by the same argument as for the Lorentz gas

$$\tilde{P}_x^{\varepsilon}(A)=\int_{\{\gamma(q_1,p_1,\ldots,q_n,p_n)\in A\}}\prod_{i=1}^{n}\{dq_i\,dp_i\,\varepsilon^{-2}\rho\chi_q^{\varepsilon}(q_i)h(p_i)\}\times\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\prod_{i=1}^{m}\left(\int_{\mathbb{R}^3}dp_i'\int_{\Lambda_{\varepsilon}(\gamma,p_i')}dq_i'\,\varepsilon^{-2}\rho\times\chi_q^{\varepsilon}(q_i')h(p_i')\right). \tag{2.80}$$

(q_i,p_i) is the initial position and momentum of the fluid particle which gives rise to the ith collision.

In a collision, energy and momentum are conserved, i.e.,

$$(1/M)y_{i-1}^2+p_i^2=(1/M)y_i^2+\bar{p}_i^2,\qquad y_{i-1}+p_i=y_i+\bar{p}_i, \tag{2.81}$$

where $\bar{p}_i$ is the outgoing momentum of the fluid particle in the ith collision. Eliminating $\bar{p}_i$ yields

$$(y_i-y_{i-1})\cdot(y_i-p_i)+[(M-1)/2M](y_{i-1}^2-y_i^2)=0. \tag{2.82}$$

Therefore, to have a collision with given y_{i-1} and y_i, p_i has to lie in the plane defined by (2.82). This plane is denoted by $E(y_{i-1},y_i)$, the Cartesian coordinates in $E(y_{i-1},y_i)$ by e_i, and the Lebesgue measure on $E(y_{i-1},y_i)$ by de_i. Once $p_i\in E(y_{i-1},y_i)$ is fixed, q_i is determined by the time of flight $t_1+\cdots+t_i$. There is a one-to-one correspondence between $(q_1,p_1,\ldots,q_n,p_n)$ and $(y_1,t_1,e_1,\ldots,y_n,t_n,e_n)$.

In (2.80) the $(q_1,p_1,\ldots,q_n,p_n)$ coordinates are now transformed to the $(y_1,t_1,e_1,\ldots,y_n,t_n,e_n)$ coordinates. Elastic collision is defined by requiring that, in addition to (2.81), the momentum transfer be parallel to the vector from the center of the Rayleigh to the center of the fluid particle. Using this obtains for the volume element

$$\prod_{i=1}^{n}\{dq_i\,dp_i\}=\prod_{i=1}^{n}\left[dt_i\,dy_i\,de_i\,\varepsilon^2\left(\frac{M+1}{2M}\right)^2\frac{1}{|y_i-y_{i-1}|}\right]. \tag{2.83}$$

Inserting (2.83) in (2.80) yields

$$\tilde{P}_x^{\varepsilon}(A)=\int_A\int_{\Delta_n^{\varepsilon}}\prod_{i=1}^{n}\left\{dt_i\,dy_i\,de_i\left(\frac{M+1}{2M}\right)^2\times\frac{1}{|y_i-y_{i-1}|}\rho\chi_q^{\varepsilon}(q_i)h(p_i)\right\}\times\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\prod_{i=1}^{m}\left(\int_{\mathbb{R}^3}dp_i'\int_{\Lambda_{\varepsilon}(\gamma,p_i')}dq_i'\,\varepsilon^{-2}\rho\times\chi_q^{\varepsilon}(q_i')h(p_i')\right). \tag{2.84}$$

Here $(q_1,p_1,\ldots,q_n,p_n)$ is considered a function of $(y_1,t_1,e_1,\ldots,y_n,t_n,e_n)$. $\Delta_n^{\varepsilon}\subset X_{i=1}^{n}I(y_{i-1},y_i)$ is the set of all incoming momenta $(p_1,\ldots,p_n)$ of fluid particles such that there are no recollisions.

The second integration is estimated by

$$\pi\rho\int_{\mathbb{R}^3}dp'\,h(p')\sum_{i=0}^{n}\left|\frac{1}{M}y-p'\right|t_{i+1} \leq\pi\rho\int_{\mathbb{R}^3}dp'\,h(p')\left(|p'|\,t+\sum_{i=0}^{n}\frac{1}{M}|y_i|\,t_{i+1}\right)\leq c \tag{2.85}$$

on A, since A is compact and by using (2.70). Therefore, the sum is bounded by e^c. The first integration is then bounded by

$$\int_A\int_{X_{i=1}^{n}I(y_{i-1},y_i)}\prod_{i=1}^{n}\left[dt_i\,dx_i\,de_i\left(\frac{M+1}{2M}\right)^2\times\frac{1}{|y_i-y_{i-1}|}\rho h(p_i)\right]e^c. \tag{2.86}$$

Using (2.70) and (2.72) and since A is compact, the integral (2.86) is bounded. Clearly, $\Delta_n^{\varepsilon}\to X_{i=1}^{n}I(y_{i-1},y_i)$ up to a set of $\Pi_{i=1}^{n}de_i$ measure zero and, $\chi_q^{\varepsilon}(q')\to 1$ for $q'\neq q$ as $\varepsilon\to 0$. Furthermore,

$$\lim_{\varepsilon\to 0}\int_{\Lambda_{\varepsilon}(\gamma,p')}dq'\,\varepsilon^{-2}\chi_q^{\varepsilon}(q')=\pi\sum_{i=0}^{n}\left|\frac{1}{M}y_i-p'\right|t_{i+1}. \tag{2.87}$$

Therefore, by Lebesgue's dominated convergence,

$$\lim_{\varepsilon\to 0}\tilde{P}_x^\varepsilon(A)=\int_A\int_{\chi_{i=1}^n I(y_{i-1},y_i)}\times\prod_{i=1}^n\left[dt_i\,dy_i\,de_i\left(\frac{M+1}{2M}\right)^2\frac{1}{|y_i-y_{i-1}|}\rho h(p_i)\right]\times\exp\left[-\pi\rho\int dp'h(p')\left(\sum_{i=0}^n\left|\frac{1}{M}y_i-p'\right|t_{i+1}\right)\right],\tag{2.88}$$

which, after some manipulations, turns out to be equal to the path measure P_x ■

One may also consider a situation where the initial distribution of the fluid is not stationary. If at $t=0$ the correlation functions of the fluid are given by

$$\prod_{j=1}^n\{\varepsilon^{-2}\chi_\Lambda^\varepsilon(q_j)r(q_j,p_j)\},$$

with $r(q,p)$ continuous and

$$\int dp\sup_q\{r(q,p)(1+|p|^3)\}<\infty,\tag{2.89}$$

then the stochastic motion $X^\varepsilon(t)$ of the Rayleigh particle converges to $X(t)$. $X(t)$ is a Markov process which is governed by the forward equation

$$\frac{\partial}{\partial t}f(q,p,t)=-p\cdot\nabla_q f(q,p,t)+\int dp'\int dp_1\,r(q-p_1t,p_1)\left(\frac{M+1}{2M}\right)^2\times\delta\left((p-p')\cdot(p-p_1)+\frac{M-1}{M}(p'^2-p^2)\right)f(q,p',t)-\int dp_1\,r(q-p_1t,p_1)\pi\left|\frac{1}{M}p-p_1\right|f(q,p,t).\tag{2.90}$$

$X(t)$ is homogeneous in time if and only if $r(q,p)=h(p)$.

3. Convergence of the low-density limit for a tagged sphere in a hard-sphere fluid

We consider a hard sphere of mass M and diameter ε immersed in a fluid of hard spheres of mass one and diameter ε. The whole system is restricted to the bounded region Λ with smooth boundary $\partial\Lambda$. Particles collide elastically and are specularly reflected upon hitting the boundary. (Other boundary conditions can be handled easily). If one tries to prove the convergence of the low-density limit for this system following the lines of the last section, one realizes that, since now the history of a fluid particle before and after a collision with the tagged particle is so much more complicated, there is no simple way of writing the analog of (2.80). [In fact, it would be of great interest to obtain such an analogue for a hard-sphere fluid, which is a problem connected with extending Wild's sum (Wild, 1951) to the spatially inhomogeneous case. This might possibly shed some light on how to extend Lanford's analysis of the Boltzmann equation to longer times.] Therefore we will use here differential equation techniques as developed by Lanford for the derivation of the Boltzmann equation (see Sec. III.C). The nice point about the present proof is that this is one case where Lanford's theorem can be used iteratedly to show convergence for all times.

It is convenient to use the grand canonical prescription for the fluid. Initially the fluid is in thermal equilibrium at inverse temperature β and fugacity z_ε conditioned on the tagged particle being located at q. Therefore, the equilibrium measure has the densities

$$\left\{Z(q)^{-1}(z_\varepsilon)^n\chi_\Lambda^\varepsilon(q,q_1,\ldots,q_n)\prod_{i=1}^n h_\beta(p_i)\,|\,n\geq 0\right\},\tag{2.91}$$

where $Z(q)$ is the normalization constant. h_β is the normalized Maxwellian at inverse temperature β and $\chi_\Lambda^\varepsilon(q,q_1,\ldots,q_n)=0$, whenever either two arguments are closer than ε or one of the arguments is closer than $\frac{1}{2}\varepsilon$ to $\partial\Lambda$ or outside of Λ, and $\chi_\Lambda^\varepsilon(q,q_1,\ldots,q_n)=1$ otherwise. The fugacity is increased as $z_\varepsilon=\varepsilon^{-2}\rho$. (Since as $\varepsilon\to 0$ the fluid becomes ideal, although at an infinite particle density, this is basically the same as increasing the density as $\varepsilon^{-2}\rho$). Let $X^\varepsilon(t)$ be the stochastic process of the motion of the Rayleigh particle starting at $x=(q,p)$. By the results of Alexander (1975), $X^\varepsilon(t)$ is well defined. $p^\varepsilon(t)$ is a jump process with a finite number of jumps in any finite time interval, and $q^\varepsilon(t)=q+\int_0^t ds\,(1/M)p^\varepsilon(s)$.

Theorem 2.8: Let $X^\varepsilon(t)$ be the process corresponding to the motion of the tagged particle starting at $x=(q,p)$ with the hard-sphere fluid in conditioned equilibrium according to (2.91). Then

$$\lim_{\varepsilon\to 0}X^\varepsilon(t)=X(t)\equiv(q(t),p(t)),\tag{2.92}$$

in the sense of the convergence of all finite dimensional distributions

$$\lim_{\varepsilon\to 0}E_x^\varepsilon(f_1(X^\varepsilon(t_1))\cdots f_n(X^\varepsilon(t_n)))=E_x(f_1(X(t_1))\cdots f_n(X(t_n)))\tag{2.93}$$

for all n, $0\leq t_1<\cdots<t_n$, and bounded continuous functions $f_1,\ldots,f_n$ of compact support. (With respect to x the convergence is uniform on compact sets in $\Lambda\times\mathbb{R}^3$.) $p(t)$ is the Markov jump process of Lemma 2.6 with $h=h_\beta$ and specular reflection at $\partial\Lambda$ added, and $q(t)=q+\int_0^t ds\,(1/M)p(s)$.

Proof: (The proof uses the method of time-dependent correlation functions. This technique will be explained in Sec. III.C.1 in the context of the Boltzmann equation. The reader is kindly advised first to consult this section.) We consider the joint correlation functions of the tagged particle and the fluid, $(\rho_0(x),\rho_1(x,x_1),\ldots)$, $x=(q,p)$, $x_j=(q,p)$. $(x,x_1,\ldots,x_n)$ is the expectation to find the tagged particle at x and an n-tuple of fluid particles at $x_1,\ldots,x_n$. The evolution of the correlation functions is denoted by $\{(V_t^\varepsilon\rho)_n|n\geq 0\}$, where ρ is considered the vector $(\rho_0,\rho_1,\ldots)$.

One obtains

$$E_x^\varepsilon(f(X^\varepsilon(t)))=(\rho_{\mathrm{eq},0}^\varepsilon(x))^{-1}(V_{-t}^\varepsilon(f\rho_{\mathrm{eq}}^\varepsilon))_0(x).\tag{2.94}$$

$\rho_{\mathrm{eq}}^\varepsilon=(\rho_{\mathrm{eq},0}^\varepsilon,\rho_{\mathrm{eq},1}^\varepsilon,\ldots)$ are the unconditioned equilibrium correlation functions of the fluid +tagged particle system at inverse temperature β and fugacity z_ε. $f\rho_{\mathrm{eq}}^\varepsilon$ is shorthand for $(f(x)\rho_{\mathrm{eq},0}^\varepsilon(x),f(x)\rho_{\mathrm{eq},1}^\varepsilon(x,x_1),\ldots)$.

The idea is to study the convergence of $V_{-t}^\varepsilon(f\rho_{\mathrm{eq}}^\varepsilon)$ using Theorem 3.1. One has to check the two conditions

(C1) and (C2) [Eqs. (3.52) and (3.53)]. Now, using the invariance of the equilibrium measure, we obtain

$$|(V_t^\varepsilon (f\rho_{\rm eq}^\varepsilon))_n(x,x_1,\ldots,x_n)| \leq (\sup_x |f(x)|)h_\beta(p)\prod_{j=1}^n \{h_\beta(p_j)\}\bar{\rho}_{\rm eq}^\varepsilon(q,q_1,\ldots,q_n)\,. \tag{2.95}$$

Here $\bar{\rho}_{{\rm eq},n}^\varepsilon$ are the spatial parts of the unconditioned equilibrium correlation functions of the fluid + tagged particle at fugacity z_ε, for which it is known (Ruelle, 1969) that

$$\bar{\rho}_{{\rm eq},n}^\varepsilon(q,q_1,\ldots,q_n) \leq (z_\varepsilon)^n\,. \tag{2.96}$$

Therefore, $V_t^\varepsilon(f\rho_{\rm eq}^\varepsilon)$ satisfies the bound (C1) for *all* times. Furthermore,

$$\lim_{\varepsilon\to 0}\varepsilon^{2n}(f\rho_{{\rm eq},n}^\varepsilon)(x,x_1,\ldots,x_n) = f(x)h_\beta(x)\prod_{j=1}^n\{\rho h_\beta(x_j)\}\,, \tag{2.97}$$

uniformly on compact sets of $\Gamma_{n+1}(0)$. [The definition of $\Gamma_n(s)$ in Sec. III.C uses n particles of unit mass. Here we use the obvious alteration, where the first particle has mass M.]

Let S_t be the backward semigroup of the limiting Markov process $q(t)$, $p(t)$, and $\bar{S}_t$ be the forward semigroup of the Markov process $p(t)$, $q(t) = q - \int_0^t ds\,(1/M)p(s)$. For continuous functions of compact support they satisfy the detailed balance property

$$(S_t f)h_\beta = \bar{S}_t(fh_\beta)\,. \tag{2.98}$$

By Theorem 3.1 and Property 2,

$$\lim_{\varepsilon\to 0}\varepsilon^{2n}(V_{-t}^\varepsilon(f\rho_{\rm eq}^\varepsilon))(x,x_1,\ldots,x_n) = (\bar{S}_t(fh_\beta))(x)\prod_{j=1}^n\{\rho h_\beta(x_j)\}\,, \tag{2.99}$$

uniformly on compact sets of $\Gamma_{n+1}(-t)$ for $0\leq t\leq t_0(\rho,\beta)$. Since (C1) is valid for all times, Theorem 3.1 can be applied again with $V_{-t_0}^\varepsilon(f\rho_{\rm eq}^\varepsilon)$ as initial condition, etc. Therefore, (2.99) is valid for all times. In particular, using (2.94) and (2.98),

$$\lim_{\varepsilon\to 0}E_x^\varepsilon(f(X^\varepsilon(t))) = (h_\beta(x))^{-1}(\bar{S}_t(fh_\beta))(x) = (S_t f)(x) = E_x(f(X(t)))\,, \tag{2.100}$$

uniformly on compact sets of $\Lambda\times\mathbb{R}^3$ for all $t\geq 0$.

We turn to the second finite distribution. One obtains

$$E_x^\varepsilon(f_1(X^\varepsilon(t_1))f_2(X^\varepsilon(t_1+t_2))) = (\rho_{{\rm eq},0}^\varepsilon(x))^{-1}(V_{-t_1}^\varepsilon(f_1(V_{-t_2}^\varepsilon(f_2\rho_{\rm eq}^\varepsilon))))_0(x)\,, \tag{2.101}$$

$t_1, t_2 > 0$. For fixed t_2, we regard $f_1(V_{-t_2}(f_2\rho_{\rm eq}^\varepsilon))$ as initial condition. As before, $V_{-t_1}^\varepsilon(f_1(V_{-t_2}^\varepsilon(f_2\rho_{\rm eq}^\varepsilon)))$ satisfies the bound (C1) for all $t_1\geq 0$. By (2.99) $(f_1(V_{-t_2}^\varepsilon(f_2\rho_{\rm eq}^\varepsilon)))_n$ satisfies (C2) with $\Gamma_{n+1}(-t_2)$. Therefore, by Theorem 3.1 and Property 2,

$$\lim_{\varepsilon\to 0}\varepsilon^{2n}(V_{-t_1}^\varepsilon(f_1(V_{-t_2}^\varepsilon(f_2\rho_{\rm eq}^\varepsilon))))_n(x,x_1,\ldots,x_n) = (\bar{S}_{t_1}(f_1(\bar{S}_{t_2}(f_2h_\beta))))(x)\prod_{j=1}^n\{\rho h_\beta(x_j)\}\,, \tag{2.102}$$

uniformly on compact sets of $\Gamma_{n+1}(-t_2-t_1)$ for all $t_1\geq 0$, where we have already used the iteration argument. Using (2.101) and (2.98)

$$\lim_{\varepsilon\to 0}E_x^\varepsilon(f_1(X^\varepsilon(t_1))f_2(X^\varepsilon(t_1+t_2))) = (S_{t_1}(f_1(S_{t_2}f_2)))(x) = E_x(f_1(X(t_1))f_2(X(t_1+t_2)))\,, \tag{2.103}$$

uniformly on compact sets of $\Lambda\times\mathbb{R}^3$.

Convergence of the higher finite-dimensional distributions is proved in the same way ■

4. One-dimensional hard rod systems

The Rayleigh gas in one dimension has been studied in some detail. One considers an infinite system of hard rods of mass m on the line and inserts at the point q a test hard rod of mass M. All rods have zero length. The system evolves by elastic collisions. Initially the fluid is Poisson distributed with density ρ, and the fluid particles have independent momenta $\pm\alpha$ with probability $\frac{1}{2}$. Holley (1969) proves under these assumptions that the motion of the Rayleigh particle converges to the Ornstein-Uhlenbeck process in the Brownian motion limit. The extension of this result to two dimensions by Hennion (1973) does not start from the mechanical model.

In the case $M = m$, so all rods have the same mass, a number of other results have been obtained. Assuming that the Rayleigh particle starts at the origin, one considers only the spatial part $q(t)$ of its stochastic motion, which is scaled as

$$q^\varepsilon = \varepsilon q(\varepsilon^{-2}t)\,. \tag{2.104}$$

Equation (2.104) is the hydrodynamic limit to be discussed in the next section. If the fluid is initially Poisson distributed with density ρ and has independent momenta identically distributed as $h(dp)$ with $\int h(dp)|p| = \langle|p|\rangle < \infty$, $\int h(dp)p = 0$, then Spitzer (1969) proves that $q^\varepsilon(t)$ converges to the Wiener process with diffusion constant $D = \langle|p|\rangle^{1/2}\rho^{-1/2}$ as $\varepsilon\to 0$. In particular, in the limit the spatial part $f(q,t)$ of the probability distribution of the Rayleigh particle is governed by the diffusion equation

$$\frac{\partial}{\partial t}f(q,t) = \tfrac{1}{2}D\Delta f(q,t)\,. \tag{2.105}$$

An interesting, apparently open, problem is the extension of this result to the case $M\neq m$. If the distance between fluid particles is still independent but no longer exponential, then $q^\varepsilon(t)$ still converges to a Gaussian process but with a covariance differing from the one of the Wiener process (Major and Szàsz, 1979). Szàsz (1979) also studies the joint motion of two Rayleigh particles inserted at different points in the fluid. Scaling as in (2.104) he again obtains a Gaussian process with possibly nontrivial dependence between the two particles, depending on how their initial distance is scaled.

There are some other results on the infinite equal mass hard rod system on the line which, although not directly connected to Markovian limits, I want to mention briefly. The hard rod system has very good ergodic properties: As before, the stationary mea-

sure is spatially Poisson with constant density and has independent momenta identically distributed as $h(dp)$. If all rods have the same length, Sinai (1972) and Aizenman, Goldstein, and Lebowitz (1974) have shown this system to be a K system; and if, in addition, zero momentum is excluded, then Aizenman, Goldstein, and Lebowitz (1974) have shown this system to be a Bernoulli flow. A system with a mixture of hard rods of different lengths is a K system (Aizenman, 1975). The self- and total-equilibrium time correlation functions have been computed by Jespen (1965), Lebowitz and Percus (1967), and Lebowitz, Percus, and Sykes (1968), for hard rods of equal length and by Aizenman, Lebowitz, and Marro (1978) for a mixture of hard rods of different lengths.

C. The hydrodynamic limit

For the limits discussed so far, a Markovian approximation was obtained by letting a certain physical parameter (i.e., the interaction strength, the fluid density, or the inverse mass) go to zero. Physically one is interested in analyzing properties of the motion of the tagged particle beyond these limiting situations at *constant* interaction strength, fluid density, and mass. Amazingly enough, in this situation there is also a Markovian approximation, although of a somewhat more subtle nature than the ones before: For long times one expects the *spatial* density of the tagged particle to be governed by the diffusion equation. The hydrodynamic limit is just the scaling appropriate to destile this long-time behavior of the tagged particle.

Let us again use the simplicity of the Lorentz gas. The intuitive idea behind the hydrodynamic limit is that after a few mean free times the Lorentz gas is already very close to *local* equilibrium, which means that at each point in space the direction of the momentum is almost uniformly distributed, and that subsequently the spatial density of this local equilibrium state evolves slowly according to the diffusion equation. For the Lorentz gas the diffusion equation is all there is to hydrodynamics, since the only conserved quantity is the particle number (different energies do not couple).

To motivate the scaling appropriate for the hydrodynamic limit and to provide some intuition we start with a very simple and well-known stochastic model: A particle travels on the real line. Its position is denoted by $q(t)$ and its momentum by $p(t)$, $|p(t)|=1$. The mass of the particle equals one. At every integer point of the lattice there is a scatterer. In between scatterers the particle moves freely. Whenever the particle reaches a scatterer, it is transmitted with probability $1-w$ and reflected with probability w independently of its past history. If w is close to zero, the momentum of the particle is only rarely reversed and it will travel straight for long time periods, whereas for w close to one the particle stays almost at its original position with continuous momentum reversals.

One can define something like a weak coupling limit for this stochastic model. The probability of reflection is scaled as

$$w_\varepsilon = \lambda\varepsilon\,, \tag{2.106}$$

and time and space as

$$t_\varepsilon = \varepsilon^{-1}t\,,\quad q_\varepsilon = \varepsilon^{-1}q\,. \tag{2.107}$$

Equivalently, $w_\varepsilon = \varepsilon\lambda$ and the scatterers are placed at $\varepsilon\mathbf{Z} = \{\ldots, -\varepsilon, 0, \varepsilon, \ldots\}$. If $q^\varepsilon(t)$, $p^\varepsilon(t)$ denotes the motion scaled according to (5.1) and (5.2), then it is easily verified that $p^\varepsilon(t)$ converges to $\bar{p}(t)$, where $\bar{p}(t)$ is the jump process on $\{-1, 1\}$ with waiting time λ^{-1} and

$$\bar{q}(t) = q + \int_0^t ds\,\bar{p}(s)\,. \tag{2.108}$$

A probability distribution evolves according to

$$\frac{\partial}{\partial t} f(q,p,t) = -p\circ\nabla_q f(q,p,t) + \lambda\{f(q,-p,t) - f(q,p,t)\}\,. \tag{2.109}$$

The idea of the hydrodynamic limit is to study the long-time behavior of the spatial part of the motion for *fixed* w. Since diffusion is expected, the motion is scaled in such a way that the asymptotic mean-square displacement

$$\lim_{t\to\infty}\frac{1}{t}\sum_{p=\pm1}\tfrac{1}{2}E^\varepsilon_{q,p}((q^\varepsilon(t)-q)^2) = D(w) \tag{2.110}$$

is kept constant. The long-time behavior of the motion, so to speak, is preserved under scaling (2.110). Since by (2.110) the particle travels in a time span t roughly a distance $t^{1/2}$, for long times, the appropriate scaling is

$$q^\varepsilon(t) = q + \varepsilon\int_0^{\varepsilon^{-2}t} ds\,p(s)\,, \tag{2.111}$$

or, equivalently,

$$q^\varepsilon(t) - q = \varepsilon(q(\varepsilon^{-2}t) - q)\,. \tag{2.112}$$

Then $q^\varepsilon(t)$ jumps on the lattice $\varepsilon\mathbf{Z}$ (the stepsize equals ε) ε^{-2} times per unit time interval. As is well known from the symmetric random walk, such a situation should lead to diffusion. Indeed, one can prove—e.g., with the methods described by Papanicolaou (1975)—that independently of the initial momentum distribution $q^\varepsilon(t)$ converges to $q^0(t)$ as $\varepsilon\to 0$, where $q^0(t)$ is Brownian motion starting at q with diffusion constant $D(w) = (1-w)/w$. In particular, the distribution $f(q,t)$ of $q(t)$ is governed by the diffusion equation

$$\frac{\partial}{\partial t} f(q,t) = \tfrac{1}{2}D(w)\Delta f(q,t)\,. \tag{2.113}$$

Also for the $\bar{q}(t), \bar{p}(t)$ process the hydrodynamic limit can be studied. Then (2.108) is scaled according to (2.111) as

$$\bar{q}^\varepsilon(t) = q + \varepsilon\int_0^{\varepsilon^{-2}t} ds\,\bar{p}(s)\,. \tag{2.114}$$

Then, in the limit $\varepsilon\to 0$, $\bar{q}^\varepsilon$ converges to $\bar{q}^0$, where $\bar{q}^0$ is Brownian motion starting at q with diffusion constant $D = \lambda^{-1}$. Thus, the long-time behavior of the weak coupling approximation is the same as the long-time behavior for finite w, although with the *wrong* diffusion constant. Clearly,

$$\lim_{\varepsilon\to 0}\varepsilon D(w_\varepsilon) = \lambda^{-1} = D\,. \tag{2.115}$$

The weak coupling approximation, therefore, yields the lowest nonvanishing order in w of the diffusion constant $D(w)$. This is a general fact to be discussed in detail in the next section.

It is clear now how to proceed for the Lorentz gas. For simplicity we choose a hard-core potential. The Lorentz particle starts at q with a certain momentum distribution. The scatterer distribution is Poisson with density ρ conditioned not to overlap q. Since $|p(t)|$ is conserved, it is convenient to fix $|p(t)|=1=|p|$. The other energies follow then by scaling of t. The process $p(t)$ is a (non-Markovian) jump process on the unit sphere, which is stationary for the uniform distribution on the unit sphere.

Because of the mechanical motion

$$q(t)=q+\int_0^t ds\, p(s), \tag{2.116}$$

which as in the foregoing example is scaled as

$$q^\varepsilon(t)=q+\varepsilon\int_0^{\varepsilon^{-2}t} ds\, p(s). \tag{2.117}$$

Equation (2.117) defines the hydrodynamic limit. The hope is that, independently of the initial distribution, $q^\varepsilon(t)$ will converge to $q^0(t)$ as $\varepsilon\to 0$, where $q^0(t)$ is Brownian motion in three dimensions with diffusion constant D.

The initial velocity distribution should approach its stationary distribution as $t\to\infty$. Therefore the diffusion constant D is defined by the asymptotic mean square displacement starting at q with a uniform velocity distribution

$$\lim_{t\to\infty}\frac{1}{t}\int dp\,\frac{1}{4\pi}\,\delta(|p|-1)E_{q,p}((q(t)-q)^2)=dD, \tag{2.118}$$

where d is the dimension of the space (usually $d=3$). Using the stationary nature of $p(t)$, one obtains

$$D=\frac{2}{d}\int_0^\infty dt\int dp\,\frac{1}{4\pi}\,\delta(|p|-1)E_{q,p}(p(t)\cdot p), \tag{2.119}$$

which is the well-known Einstein formula (Einstein, 1905). Equations (2.119) and (2.67) are examples of the Green-Kubo formula which relate the transport coefficient to the time integral over the corresponding current correlation function.

To prove the hydrodynamic limit a necessary prerequisite is to show that the diffusion constant D is finite, which by (2.119) means to show the integrable decay of the velocity autocorrelation function. This is an open problem. Series expansions by van Leeuwen and Wejland (1967), and Wejland and van Leeuwen (1968), mode-mode coupling arguments by Ernst and Wejland (1971), and computer experiments by Bruin (1972), by Lewis and Tjon (1978), and by Alder and Alley (1978) indicate a long-time decay of the velocity autocorrelation function as $t^{-(\alpha/2+1)}$ in d dimensions.

For a real fluid the analog of the diffusion equation is the hydrodynamic equations of the Navier-Stokes form. As is well known from the Chapman-Enskog expansion, they already constitute the second-order approximation in an expansion in terms of the mean free path. The first-order approximation is the Euler equations of a compressible fluid, which just reflect the conservation laws. For the Lorentz gas the Euler equations are trivial, namely,

$$\frac{\partial}{\partial t}f(\boldsymbol{q},t)=0. \tag{2.120}$$

The corresponding scaling is

$$q^\varepsilon(t)=q+\varepsilon\int_0^{\varepsilon^{-1}t} ds\, p(s), \tag{2.121}$$

or, equivalently,

$$q_\varepsilon=\varepsilon^{-1}q,\quad t_\varepsilon=\varepsilon^{-1}t. \tag{2.122}$$

On this scale diffusion is suppressed. In fact, Nishida (1979) shows that if the five hydrodynamic fields of a Boltzmann fluid (the one-particle distribution evolves according to the Boltzmann equation) are scaled as in (2.122), then they converge to the solutions of the Euler equations as $\varepsilon\to 0$.

We remark that for a higher density of scatterers trapping may occur which will invalidate the diffusion approximation altogether. Alder and Alley (1978) studied this region in their computer experiments. For the related wind tree models there are some analytical results by Gates (1972a, b) and Aarnes (1974). Lieb and Hauge (1974) use a Peierls type of argument to show trapping for the Lorentz gas with overlapping scatterers at high enough densities.

1. The diffusion approximation for the Lorentz gas with a periodic configuration of scatterers

Bunimovich and Sinai (1979,1980) prove under certain, purely geometrical, assumptions the validity of the diffusion approximation (2.111). I want to explain here the results without attempting to enter the complexity of the proofs, see Sinai (1980) for a review.

One considers a certain configuration of hard disks on a square with periodic boundary conditions, i.e. on a two dimensional torus. (The results are, in fact, valid for arbitrary convex scatterers). This configuration has to satisfy:

(i) The disks do not touch or overlap.

(ii) Starting from any point of the square outside the disks in an arbitrary direction, there is a uniform upper bound on the time until the first collision (as before, speed one of the Lorentz particle is assumed).

One now repeats this elementary square periodically over the whole plane and thereby obtains the Lorentz gas with a *periodic, fixed* configuration of scatterers. The simplest configuration satisfying (i) and (ii) we could imagine is the following: The centers of the disks with radius $R, \sqrt{\frac{3}{4}}\,a<2R<a$, are located at the points of a triangular lattice with nearest neighbor distance a. [If $2R\leq\sqrt{\frac{3}{4}}\,a$, (ii) is violated, and if $2R\geq a$, (i) is violated.]

A stationary probability measure for the mechanical motion on the torus with scatterers is the uniform spatial distribution outside scatterers and the uniform velocity distribution. We denote this probability measure by $dqdp$.

Sinai (1970) [see also the lectures by Gallavotti (1975)] has shown that this system has very good ergodic properties. The ergodicity of the system alone implies the validity of the "Euler equation"

$$q^{\varepsilon}(t) = q + \varepsilon \int_0^{\varepsilon^{-1}t} ds\, p(s) \to q \tag{2.123}$$

as $\varepsilon \to 0$ for $dq\,dp$—almost all initial conditions (q,p).

If the particle starts initially with the distribution $g(q,p)\,dqdp$ with differentiable density $g(q,p)$ in the elementary square at the origin, then Bunimovich and Sinai (1979) show that

$$q^{\varepsilon}(t) = \varepsilon \int_0^{\varepsilon^{-2}t} ds\, p(s) \to b(t) \tag{2.124}$$

as $\varepsilon \to 0$, where $b(t)$ is the two dimensional Wiener process with diffusion matrix D. In particular, this result implies the existence of a finite diffusion constant in the sense of (2.118) and the validity of the diffusion equation in the following sense: If $f(q,t)$ denotes the spatial distribution of the Lorentz particle at time t, given that it had the initial distribution $g(q,p)\,dqdp$ in the elementary square at the origin, then

$$f^{\varepsilon}(q,t) \equiv \varepsilon^{-2} f(\varepsilon^{-1}q, \varepsilon^{-2}t) \to (2\pi \det Dt)^{-1} \exp\left(-\frac{1}{2}\sum_{i,j=1}^{2} q_i D_{ij}^{-1} q_j\right) \tag{2.125}$$

weakly as $\varepsilon \to 0$. (q_1, q_2) are the cartesian coordinates of q.

As an aside, I might note that according to Bloch's theorem a quantum-mechanical particle travels *freely* in such a periodic potential.

D. The physical meaning of the weak coupling and low-density limits

The hydrodynamic limit can be studied also for the diffusion process corresponding to the linear Landau equation (2.19) and the continuous random walk process corresponding to the linear Boltzmann equation (2.28). When these processes are scaled according to (2.117), their spatial part $q^{\varepsilon}(t)$ converges to Brownian motion in three dimensions with diffusion constants D_w and D_l, respectively (Nelson, 1967; Papanicolaou, 1975). To a large extent the success of the linear Landau and the linear Boltzmann equations presumably comes from the fact that these equations describe qualitatively correctly the long-time behavior of the Lorentz gas, although with the wrong diffusion constant.

The computation of the diffusion constants D_w and D_l is straightforward, since p happens to be an eigenvector of the collision operator. For the linear Landau equation one obtains

$$D_w = (1/2\pi)|p|^3/p\alpha\,, \tag{2.126}$$

with α given either by (2.16) or by (2.22) in the case of a nonideal scatterer distribution. $|p|^3$ results from averaging over the uniform momentum distribution on the sphere of radius $|p|$. For the linear Boltzmann equation, when conservation of energy and the fact that V is central are used, the differential cross section can be rewritten as

$$dp'\,\sigma(p'|p) = (1/4\pi|p|^2)\delta(|p| - |p'|)\sigma_{|p|}(\phi' \cdot \phi)\,, \tag{2.127}$$

with $\phi' = p'/|p'|$, $\phi = p/|p|$. Let

$$\alpha(|p|) = 1 - \frac{1}{2}\int_{-1}^{1} du\, \sigma_{|p|}(u)u > 0\,, \tag{2.128}$$

with normalization $\int_{-1}^{1} du\, \sigma_{|p|}(u) = 2$. [Note that for purely forward scattering $\alpha(|p|) = 0$, since $\sigma_{|p|}(u)$ is then concentrated at $u = 1$, and that for purely backward scattering $\alpha(|p|) = 2$, since $\sigma_{|p|}(u)$ is then concentrated at $u = -1$.] Then the diffusion constant D_l for the linear Boltzmann equation is given by

$$D_l = (1/2\pi^2\rho)|p|/\alpha(|p|)\,. \tag{2.129}$$

Let us consider the Lorentz gas with a smooth scatterer potential V. If the scatterer distribution is Poisson with density $\rho(q') = \rho e^{-\beta V(q'-q)}$, then $p(t)$ starting at q with Maxwellian distribution $h_\beta(p)$ is stationary. Therefore, in a natural way, the diffusion constant is considered as a function of V, ρ, and β,

$$D(V,\rho,\beta) = \frac{2}{d}\int_0^{\infty} dt\, \langle p(t)\cdot p(0)\rangle_\beta\,, \tag{2.130}$$

where $\langle\ \rangle_\beta$ indicates the average $\int dp\, h_\beta(p) E_{q,p}(\cdot)$. If hydrodynamics is valid, then

$$\langle (q(t) - q)^2\rangle_\beta \sim 3tD(V,\rho,\beta) \tag{2.131}$$

for large t.

Now scaling (2.131) according to the weak coupling limit

$$\langle (q^{\varepsilon}(t) - q)^2\rangle_\beta \sim 3t\varepsilon D(\varepsilon^{1/2}V,\rho,\beta)\,. \tag{2.132}$$

As $\varepsilon \to 0$, the left-hand side converges to $\langle (q^0(t) - q)^2\rangle_\beta$, where $q^0(t)$ is the spatial part of the diffusion process corresponding to the linear Landau equation, for which it is known that $\langle (q^0(t) - q)^2\rangle_\beta \sim 3tD_w(\rho,\beta)$ for large t. Therefore one expects that

$$\lim_{\varepsilon\to 0} \varepsilon D(\varepsilon^{1/2}V,\rho,\beta) = D_w(\rho,\beta) \equiv \int dp\, h_\beta(p)\frac{|p|^3}{2\pi\alpha\rho}\,. \tag{2.133}$$

Thus the weak coupling limit describes the behavior of the Lorentz particle at zero coupling, however extrapolated from the region of small coupling in a nontrivial (i.e., including the prefactor ε) way. Similarly, one concludes that the linear Landau equation gives the leading ε dependence of other physical quantities for small coupling strength $\varepsilon^{1/2}$.

The physical meaning of the low-density limit is analyzed in the same way. Scaling (2.131) according to (2.25), we obtain

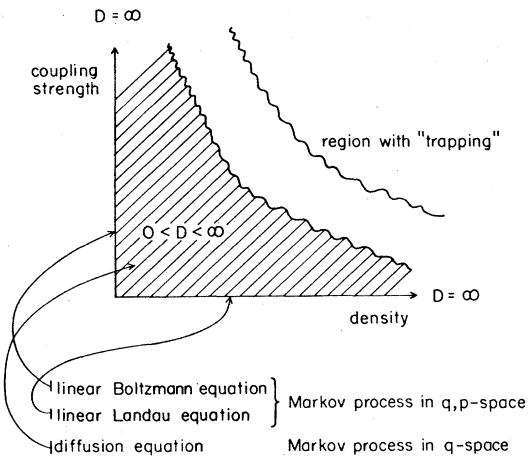
$$\langle (q^{\varepsilon}(t) - q)^2\rangle_\beta \sim t\varepsilon D(V,\varepsilon\rho,\beta)\,. \tag{2.134}$$

By the same argument as before, one expects that

$$\lim_{\varepsilon\to 0} \varepsilon D(V,\varepsilon\rho,\beta) = D_l(V,\beta) \equiv \int dp\, h_\beta(p)\frac{|p|}{2\pi^2\rho\alpha(|p|)}\,. \tag{2.135}$$

Thus the linear Boltzmann equation describes the motion of the Lorentz particle at zero-density nontrivially extrapolated from the low-density region.

We summarize the situation by means of the following diagram:



D denotes the diffusion constant

We should add a word of warning: For some models the diagram may be completely degenerate. For example, in the case of the two-dimensional wind-tree model with overlap of scatterers allowed, van Beijeren and Hauge (1972) argue that, because of the formation of paths which almost retrace themselves, the mean-square displacement grows more slowly than t. Their result is confirmed by a computer study of Wood and Lado (1971). This means that for this model D always equals zero. Despite this fact, the linear Boltzmann equation can be derived in the low-density limit and will predict a finite diffusion constant.

E. The problem of the existence of transport coefficients

The proof of the existence of finite transport coefficients is one of the outstanding problems in non-equilibrium statistical mechanics. For ideal systems, i.e., ideal fluids, and ideal crystals, the transport coefficients, such as thermal conductivity and diffusivity, are infinite (Rieder, Lebowitz, and Lieb, 1967; Casher and Lebowitz, 1971). The one-dimensional Rayleigh gas has a finite diffusion constant (see Sec. II.B.4). An impurity in a one-dimensional harmonic chain with nearest-neighbor coupling has a finite diffusion constant (see Sec. IV.A). For a one-dimensional harmonic chain of length L with random masses, the thermal conductivity grows as $\sqrt{L}$ [Papanicolaou (1976); Verheggen (1979); see also O'Conner and Lebowitz (1974) and O'Connor (1975)]. A real breakthrough was the proof of Bunimovich and Sinai of the existence of a finite diffusion constant for a two-dimensional Lorentz gas with a periodic configuration of scatterers (see Sec. II.C.1).

As pointed out in the previous section the existence of a finite diffusion constant for the Lorentz gas is equivalent to an integrable decay of the velocity autocorrelation function. Another way to attack the problem is to use a *kinetic* definition of the diffusion constant (Lebowitz and Spohn, 1978). The idea is to choose a set-up as in an actual diffusion experiment. We consider a slab of height L in the q_3 direction and infinitely extended otherwise. The slab is filled with scatterers which are distributed according to a Poisson distribution with density ρ. Outside the slab there are no scatterers. At the bottom wall there is a constant incident flux of Lorentz particles with isotropic momentum distribution $(1/4\pi|p|^2)\delta(|p|-|p'|)dp'$. The Lorentz particles diffuse through the slab. Some of them will leave the slab at the top wall, some of them at the bottom wall. [In a more mechanical picture, one could imagine the half-space $q_3 \leq 0$ filled with an ideal gas of density $\bar{\rho}$ and with an isotropic momentum distribution $(1/4\pi|p|^2)\delta(|p|-|p'|)dp'$. For a fixed configuration of scatterers the system then would evolve according to Hamiltonian dynamics.] In the steady state, which is reached in the limit as $t \to \infty$, there will be a steady flux $j(L)$ of particles from bottom to top wall depending on the height L of the slab. By Fick's law

$$j(L) = (1/L)D \tag{2.136}$$

for L sufficiently large compared to the mean free path. Experimentally, (2.136) defines the diffusion constant.

In the case of hard-sphere scatterers there is an intuitive expression for the steady state. Let $p/|p| = \phi$ and let $d\phi$ be the normalized surface measure of the sphere S^2 in three dimensions. Let $P^{-}_{\text{abs}}(q, \phi; L)$ be the probability for the Lorentz particle to reach (in other words, to be *absorbed* at) the bottom wall, given that it started at q in direction ϕ. If initially q was overlapped by a scatterer, the probability of the Lorentz particle's being absorbed is zero. Then the steady state $f(q, \phi)\,dq d\phi$ (density $\frac{1}{2}$ of the incident flux) is given by

$$f(q, \phi)\,dq\,d\phi = P^{-}_{\text{abs}}(q, -\phi; L)\,dq\,d\phi\,. \tag{2.137}$$

By translation invariance of the Poisson distribution, $f(q, \phi)$ is independent of q_1 and q_2, $q = (q_1 q_2, q_3)$. (A similar expression holds for a smooth scatterer potential. Again the basic quantity which enters is the probability of the Lorentz particle to be absorbed at the bottom wall, given that it started at q with momentum p.)

Therefore, the steady-state current $j(L, R, \rho)$, depending on the radius R of the hard sphere scatterers and on their density ρ, at the point q through a cross-sectional area parallel to the 1-2 plane is given by

$$j(L, R, \rho) = |p| \int_{S^2} d\phi \cos\phi_1 P^{-}_{\text{abs}}(q, -\phi; L)\,, \tag{2.138}$$

where ϕ_1 is the component of ϕ in the q_3 direction. By conservation of mass and by symmetry of the Poisson distribution, the steady-state current is independent of q. Therefore, setting $q = (q_1, q_2, L)$ in (2.138), the steady-state current is just proportional to the probability of getting all the way through the slab—as expected. The diffusion constant $D(R, \rho)$ is then *kinetically* defined by

$$D(R, \rho) = \lim_{L \to \infty} L j(L, R, \rho)\,. \tag{2.139}$$

Clearly, the problem is to show that the limit (2.139) exists and that it is different from zero and infinite. Basically, one has to get a handle on the absorption probabilities for a non-Markovian process. This is a poorly understood subject, and it should be no surprise to learn that the existence of the limit (2.139)

is an open problem. Physically, one would like to argue in the following way: Divide the slab into N horizontal layers of constant thickness L/N. Roughly, if the Lorentz particle is in the layer i, after a typical mean time interval, it will have an equal probability of being in either the layer $i+1$ or the layer $i-1$. Because of long-range dynamical correlations this will not be strictly true. There will always be some memory left. In any event, roughly, the Lorentz particle performs a symmetric random walk for which it is well known that the probability of a particle's being absorbed at $i=1$ before reaching $i=N$, given that it started at i, is $(N-i-1)/N$, which gives the desired $1/L$ dependence.

For fixed height L the diffusion current $j(L,R,\rho)$ close to the Boltzmann-Grad limit was investigated by Lebowitz and Spohn (1978). This is a nontrivial problem, since the time of absorption is unbounded, whereas convergence of the process always involves a largest, although arbitrary, time. One finds the expected result, namely, that under the scaling (2.26),

$$\lim_{\varepsilon\to 0} j(L,\varepsilon R,\varepsilon^{-2}\rho)=j(L,R) \tag{2.140}$$

exists and that $j(L,R)$ is the diffusion current computed with the same boundary conditions as before, but with the dynamics of the Lorentz particle inside the slab governed by the linear Boltzmann equation. Furthermore, Aizenman and Spohn (1979) prove that

$$j(L,R)=\frac{|p|}{2\pi^2\rho R^2}\frac{1}{L}\left[1+O\left(\frac{1}{\rho L}\right)\right], \tag{2.141}$$

i.e., for the linear Boltzmann equation the limit (2.139) exists and the kinetically defined diffusion constant coincides with the one defined by the Green-Kubo formula. For a fixed interval $[L_1,L_2]$, $\rho^{-1}\ll L_1$, one can choose ε small enough (and $\rho_\varepsilon=\varepsilon^{-2}\rho$) such that for $L_1<L<L_2$, Fick's law of diffusion is satisfied to within an error ε. Experimentally, so to speak, the result is fine. However, the interchange of the limits $L\to\infty$ and $\varepsilon\to 0$ has not been proved. In principle, there could be a nonuniform dependence on L.

Using the kinetic definition of the diffusion constant, it can also be argued that physically the Boltzmann-Grad limit indeed corresponds to a low-density situation. If Fick's law of diffusion is valid, then according to (2.139),

$$j(L,R,\rho)\sim D(R,\rho)(1/L), \tag{2.142}$$

which under the scaling (2.25) becomes

$$j((1/\varepsilon)L,R,\varepsilon\rho)\sim D(R,\varepsilon\rho)\varepsilon/L. \tag{2.143}$$

By (2.140), where an equivalent scaling is used,

$$\lim_{\varepsilon\to 0} j\left(\frac{1}{\varepsilon}L,R,\varepsilon\rho\right)=j(L,R)=\frac{D_l}{R^2}\frac{1}{L}\left[1+O\left(\frac{1}{\rho L}\right)\right]. \tag{2.144}$$

Therefore, one expects that

$$\lim_{\varepsilon\to 0}\varepsilon D(R,\varepsilon\rho)=\frac{1}{R^2}D_l, \tag{2.145}$$

which is the result obtained already in (2.135).

The kinetic definition of the diffusion constant immediately poses the problem of whether it yields a result identical to the one obtained from the Green-Kubo formula. We have not been able to settle this question by means of simple formal manipulations. In our opinion, the equivalence of the two definitions is just another facet of the hydrodynamic limit. If $q^\varepsilon(t)$ in (2.117) converges in a sufficiently strong sense to Brownian motion, then one would also expect that the absorption probabilities of $q^\varepsilon(t)$ are close to those of Brownian motion which are known to behave as $1/L$.

F. Fluctuations

So far we have studied average (with respect to the distribution of scatterers) properties of the Lorentz particle. However, many results should in fact be true for a typical configuration of scatterers. This also corresponds more closely to the physical situation, where one thinks of one typical configuration of scatterers being realized, say, by the impurities of a crystal. The averaging over scatterer configurations is then just a device to obtain typical properties of the Lorentz gas. An example is given by the mean field limit: If we choose a sequence $(q_1,\ldots,q_N)$ of scatterer positions such that

$$\frac{1}{N}\sum_{j=1}^{N} f(q_j)\to\int f(q)r(dq)$$

as $N\to\infty$ for all continuous functions f, then in the limit $\varepsilon\to 0, N=\varepsilon^{-1}$, the motion of the Lorentz particle is governed by the linear Vlasov equation (2.47) (cf. Theorem 2.5). Here we want to discuss two other examples.

The first one can be extracted out of Braun and Hepp (1977). In Sec. II.A.6 it was shown that in the mean-field limit

$$q^\varepsilon(t)\to q(t),\quad p^\varepsilon(t)\to p(t) \tag{2.146}$$

in probability as $\varepsilon\to 0$, where $q(t),p(t)$ are the solutions of (2.47) with initial conditions q,p. Let us consider the fluctuations of position and momentum around their deterministic path $q(t),p(t)$, i.e., the "fluctuation observables"

$$\begin{aligned}\xi^\varepsilon(t)&=\varepsilon^{-1/2}(q^\varepsilon(t)-q(t)),\\ \eta^\varepsilon(t)&=\varepsilon^{-1/2}(p^\varepsilon(t)-p(t)).\end{aligned} \tag{2.147}$$

Let

$$\xi^\varepsilon(\chi_\Lambda)=\varepsilon^{1/2}[n(\Lambda)-\mu^\varepsilon(n(\Lambda))] \tag{2.148}$$

be the fluctuations in the number of scatterers in Λ. In addition to the assumptions of Theorem 2.5 we assume that $\xi^\varepsilon(\chi_\Lambda)$ converge jointly to a Gaussian process with mean zero and covariance kernel $r_f(dq,dq')$. In particular, for the covariance

$$\lim_{\varepsilon\to 0}\mu^\varepsilon(\xi^\varepsilon(\chi_\Lambda)\xi^\varepsilon(\chi_{\Lambda'}))=\int_{\Lambda\times\Lambda'} r_f(dq,dq'). \tag{2.149}$$

Furthermore, the scatterer potential V should be four times differentiable with uniformly continuous derivatives. Then in the mean-field limit

$$\xi^\varepsilon(t),\eta^\varepsilon(t)\to\xi(t),\eta(t) \tag{2.150}$$

as a process. $\xi(t),\eta(t)$ is a Gaussian stochastic process defined as the solution of the stochastic differential

equation

$$\frac{d}{dt}\xi(t)=\eta(t),$$
$$\frac{d}{dt}\eta(t)=A(q(t))\xi(t)+w(t), \tag{2.151}$$

with initial conditions $\xi(0)=0$, $\eta(0)=0$. $A(q)$ is the matrix with matrix elements

$$A_{ij}(q)=\frac{\partial}{\partial q_i}\bar{F}_j(q)=-\int r(dq')\left(\frac{\partial^2}{\partial q_i \partial q_j}V\right)(q-q'), \tag{2.152}$$

$q=(q_1,q_2,q_3)$. $w(t)$ is a Gaussian stochastic process with mean zero and covariance

$$\langle w_i(t)w_j(s)\rangle=\int r_f(dq',dq'')F_i(q(t)-q')\times F_j(q(s)-q''), \tag{2.153}$$

$i,j=1,2,3$. For small ε, we may set heuristically

$$q^\varepsilon(t)=q(t)+\varepsilon^{1/2}\xi(t),\quad p^\varepsilon(t)=p(t)+\varepsilon^{1/2}\eta(t). \tag{2.154}$$

The fluctuations around the deterministic path are Gaussian with a tendency to spread in the course of time.

Our second example refers to the Lorentz gas in the low-density limit. We consider hard-sphere scatterers with Poisson distribution conditioned on the Lorentz particle being located at q. The initial distribution of the Lorentz particle is $g(x)\,dx=\delta_q h(p)dp$ with the momentum distribution being absolutely continuous with respect to Lebesgue. For a bounded and continuous observable f, the mean

$$G^\varepsilon: Q\to\int dx f(x)g(x^\varepsilon(-t,x,Q)) \tag{2.155}$$

is considered as a random variable on the space of configurations. Then the distribution of G^ε tends to a δ function in the limit $\varepsilon\to 0$.

Proposition 2.9: Under the assumptions of Theorem 2.2 and Corollary 2.4,

$$\lim_{\varepsilon\to 0}G^\varepsilon=\int dx f(x)g(x,t) \tag{2.156}$$

in probability. $g(x,t)$ is the solution of the linear Boltzmann equation (2.28) with initial conditions g.

Proof: We have to show that

$$\lim_{\varepsilon\to 0}\int dp\,dp'\,h(p)h(p')\mu^\varepsilon(dQ)\times f(x^\varepsilon(t,q,p,Q))f(x^\varepsilon(t,q,p',Q))=\left[\int dx f(x)g(x,t)\right]^2. \tag{2.157}$$

Equation (2.157) is regarded as coming from a Lorentz gas with two particles both starting at q but with different momenta p and p'. In principle, one should now repeat the proof of Theorem 2.2. If $p\neq p'$, then one considers paths for which each collision comes from a separate scatterer. Since the fluctuation in the number of scatterers vanishes, the motion of the two particles becomes independent and this contribution converges to $E_{q,p}(f(X(t)))E_{q,p'}(f(X(t)))$ as $\varepsilon\to 0$. [$X(t)$ is the stochastic process corresponding to the linear Boltzmann equation.] By normalization the contribution of the remaining paths has to go to zero. Since the set $\{p=p'\}$ has $dp\,dp'$—measure zero, the assertion follows by dominated convergence■

Close to the limit, up to an exceptional set of small measure, all configurations yield practically the same result as the linear Boltzmann equation. Note that the motion of the Lorentz particle is deterministic for a fixed configuration of scatterers, whereas the motion is stochastic for the linear Boltzmann equation. However, with regard to expectation values over some absolutely continuous initial momentum distribution, there is practically no difference. A specific example is the diffusion current discussed in the previous section [for which the same fluctuation result is proved in Lebowitz and Spohn (1978)]. For small densities, the diffusion current $j(L,\varepsilon R,\varepsilon^{-2}\rho,Q)$ for a fixed configuration Q is practically the same for most configurations. The current fluctuates only very little around the diffusion current $j(L,R)$ computed from the linear Boltzmann equation.

III. INTERACTING PARTICLE SYSTEMS

A. Nonlinear Markov processes

An interacting particle system is specified by its Hamiltonian

$$H=\sum_{j=1}^n \tfrac{1}{2}p_j^2+\sum_{i<j}V_\varepsilon(q_i-q_j). \tag{3.1}$$

$(q_j,p_j)=x_j$ stands for the position and the momentum of the jth particle. The mass of a particle is set equal to one. $V_\varepsilon\neq 0$ is a central, twice continuously differentiable potential of finite range. Since, as for the Lorentz gas, the potential will be scaled later on, have already introduced here the scaling parameter ε. The system is assumed to be enclosed in the bounded region Λ with smooth boundary $\partial\Lambda$. (Under some circumstances it may be desirable to admit an unbounded region Λ. The infinite volume limit would have to be discussed separately.) It turns out to be convenient not necessarily to fix the number of particles. Then the classical phase space is $\Gamma=\cup_{n\geq 0}(\Lambda\times\mathbf{R}^3)^n$. A state of the system is described by a probability measure on Γ. The initial state of the system is denoted by μ^ε, where the scaling parameter ε is introduced here already for convenience. The solutions of Hamilton's equation of motion with Hamiltonian (3.1) and specular reflection at $\partial\Lambda$ define the evolution of the jth particle

$$t\to x_j^\varepsilon(t,x)=(q_j^\varepsilon(t,x),p_j^\varepsilon(t,x)), \tag{3.2}$$

depending on the initial condition $x=(x_1,\ldots,x_n)\in\Gamma$ and on ε through V_ε.

On a macroscopic level, physically one is interested in the average number of particles in an arbitrary cell $\Delta\subset\Lambda\times\mathbf{R}^3$ at time t. In other words, the object of interest is the time-dependent *one-particle correlation function* $\rho_1^\varepsilon(x_1,t)$, defined by

$$\int_\Delta dx_1\rho_1^\varepsilon(x_1,t)=\int_\Gamma \mu^\varepsilon(dx)n(\Delta,t)(x) \tag{3.3}$$

for all Borel sets $\Delta \subset \Lambda \times \mathbf{R}^3$, where $n(\Delta, t)$ denotes the number of particles in Δ at time t. [In general, $\rho_1^\varepsilon(x_1, t)dx_1$ is a measure on $\Lambda \times \mathbf{R}^3$.] Of course, for various quantities the two-particle correlation function will be important. Also, instead of studying the average number of particles in Δ at time t one may ask for the actual number of particles in Δ at time t. All these refinements will have to be discussed later on.

Let us adopt the point of view of *reduced dynamics*. For a given initial one-particle correlation function $\rho_1^\varepsilon(x_1)$ we simply assume that the system is completely uncorrelated, i.e., that the correlation functions of the initial measure are

$$\rho_n^\varepsilon(x_1, \ldots, x_n) = \prod_{j=1}^{n} \rho_1^\varepsilon(x_j) . \tag{3.4}$$

At a later time, of course, correlations will build up. But the evolution

$$S_t^\varepsilon : \rho_1^\varepsilon(x_1) \to \rho_1^\varepsilon(x_1, t) \tag{3.5}$$

is well defined for any initial ρ_1^ε. S_t^ε preserves positivity and normalization. However, in contrast to the Lorentz gas, S_t^ε is *nonlinear*.

With the same reasoning as for the Lorentz gas we adopt the following scalings:

(i) Weak coupling limit,

$$V_\varepsilon(q) = \varepsilon^{1/2} V(q/\varepsilon) , \qquad \rho_1^\varepsilon(x_1) = \varepsilon^{-3} r(x_1) ; \tag{3.6}$$

(ii) Low-density limit,

$$V_\varepsilon(q) = V(q/\varepsilon) , \qquad \rho_1^\varepsilon(x_1) = \varepsilon^{-2} r(x_1) ; \tag{3.7}$$

(iii) Mean-field limit,

$$V_\varepsilon(q) = \varepsilon V(q) , \qquad \rho_1^\varepsilon(x_1) = \varepsilon^{-1} r(x_1) . \tag{3.8}$$

Therefore, also at time t, $\rho_1^\varepsilon(x_1, t) \sim \varepsilon^{-\nu} (\nu = 1, 2, 3)$. One then expects that

$$\lim_{\varepsilon \to 0} \varepsilon^\nu \rho_1^\varepsilon(x_1, t) = r(x_1, t) \equiv (S_t r)(x_1) \tag{3.9}$$

exists and that the change of the limiting scaled one-particle correlation function $r(x_1, t)$ should depend only on the present one-particle correlation function,

$$\frac{\partial}{\partial t} r(x_1, t) = (Lr)(x_1, t) . \tag{3.10}$$

L is a nonlinear operator and S_t should be a semigroup of nonlinear transformations. On general grounds, L has to be quadratic and has other properties which to elucidate is one of the objectives of this section.

For system + reservoir models it turned out to be extremely useful to study the probability distribution of all possible histories of the tagged particle. Here we try to adopt a similar point of view for an interacting particle system. (A different approach will be discussed in Sec. III.E.) Physically, we will study the dynamics of the system of particles through the collection of all its *time-dependent self-correlation functions*.

Let Ω be the space of all possible paths $t \to (q(t), p(t)) \in \Lambda \times \mathbf{R}^3$. The motion of the jth particle $t \to x_j^\varepsilon(t, x)$ defines a path in Ω. For any given initial condition $x = (x_1, \ldots, x_n) \in \Gamma$, one has then n distinct paths in Ω, one for each particle. We define a measure $P^{(x)}$ on Ω by giving weight one to each one of the n paths. The process corresponding to the probability measure $(1/n)P^{(x)}$ is denoted by $X^{(x)}(t)$. Clearly, $X^{(x)}(t)$ is deterministic. For every initial $x \in \Gamma$ its time evolution can be reconstructed from $X^{(x)}(t)$ immediately. The analog of $X^{(x)}(t)$ for the Lorentz gas is the motion of the Lorentz particle for a fixed configuration of scatterers. $X^{(x)}(t)$ is, so to speak, one possible history of the interacting particle system.

We define a probability measure P^ε on Ω by averaging $P^{(x)}$ over all initial configurations

$$P^\varepsilon = \left(\int dx_1 \rho_1^\varepsilon(x_1) \right)^{-1} \int_\Gamma \mu^\varepsilon(dx) P^{(x)} . \tag{3.11}$$

Since $\int \mu^\varepsilon(dx) P^{(x)}(\Omega) = \int dx_1 \rho_1^\varepsilon(x_1)$ by definition, P^ε is normalized to one. The process corresponding to the probability measure P^ε is denoted by $X^\varepsilon(t)$. (This is the analog of averaging over all configurations of scatterers in the case of the Lorentz gas.)

The significance of the process $X^\varepsilon(t)$ is the following: The single time distribution

$$P^\varepsilon(X^\varepsilon(t) \in A) = \left(\int dx_1 \rho_1^\varepsilon(x_1) \right)^{-1} \int_A dx_1 \rho_1^\varepsilon(x_1, t) \tag{3.12}$$

is the (normalized) one-particle correlation function, the quantity of central interest. [The analogy to the Lorentz gas should be clear. There $P_x^\varepsilon(X^\varepsilon(t) \in A)$ also defined the reduced dynamics.] The multitime distributions

$$P^\varepsilon(X^\varepsilon(t_1) \in A_1, \ldots, X^\varepsilon(t_n) \in A_n) \tag{3.13}$$

give the probability of finding the *same* particle at time t_1 in $A_1, \ldots,$ at time t_n in A_n. This is just the definition of the time-dependent self-correlation functions.

If $X^\varepsilon(t)$ is conditioned to start at $y \in \Lambda \times \mathbf{R}^3$, then

$$P^\varepsilon(\cdot \mid X^\varepsilon(0) = y) \tag{3.14}$$

describes the motion of a test particle (which has the same physical properties as all the other particles) immersed in the fluid, where the test particle starts at y and the fluid has the distribution μ^ε conditioned on one of the fluid particles being located at y. For example, if n is fixed and $\mu^\varepsilon = f(x_1, \ldots, x_n)dx_1 \ldots dx_n$, then an arbitrary fluid particle is picked as a test particle and initially fixed at y and the remaining $(n-1)$ fluid particles have the initial distribution

$$f(y, x_1, \ldots, x_{n-1})dx_1 \ldots dx_{n-1} \times \left(\int dx_1 \ldots dx_{n-1} f(y, \ldots, x_{n-1}) \right)^{-1} .$$

The motion of a test particle in a fluid was studied all the way along in Sec. II. Here, the fluid is interacting and initially not in thermal equilibrium. But the reasoning from Sec. II should carry over. One expects that the scaled process $X^\varepsilon(t)$ with measure $P^\varepsilon(\cdot \mid X^\varepsilon(0) = y)$ will converge to a Markov process $X(t)$

as $\varepsilon \to 0$. However, the limiting Markov process will be nonhomogeneous in time, since the fluid evolves by itself. In Sec. II the forward equation of the limiting Markov process depended linearly on the scaled one-particle correlation function of the scatterers. Assuming the same behavior here, the limiting Markov process $X(t)$ should be governed by the forward equation

$$\frac{\partial}{\partial t} f(y,t) = L_{r(t)} f(y,t) . \tag{3.15}$$

$L_{r(t)}$ is a linear operator acting on f depending itself linearly on the scaled limiting one-particle correlation function $r(t)$ of the fluid.

How does $r(t)$ itself develop in time? This is easily answered by noting the identity

$$P^\varepsilon(X^\varepsilon(t+\tau)\in A) = \int dy\, P^\varepsilon(X^\varepsilon(t+\tau)\in A | X^\varepsilon(t)=y) \times P^\varepsilon(X^\varepsilon(t)=y) .$$

Inserting (3.12), we obtain

$$\varepsilon^\nu \int_A dx_1 \rho_1^\varepsilon(x_1, t+\tau) = \varepsilon^\nu \int dy\, P^\varepsilon(X^\varepsilon(t+\tau)\in A | X^\varepsilon(t)=y)\rho_1^\varepsilon(y,t) . \tag{3.16}$$

By (3.9), $\varepsilon^\nu \rho_1^\varepsilon(x_1, t+\tau)$ converges to $r(x_1, t+\tau)$, whereas the right-hand side of (3.16) converges to $\int_A dx_1 f(x_1, t+\tau)$, where $f(t+\tau)$ is the solution of (3.15) with initial conditions $r(t)$ at time t. Taking the derivative at $\tau = 0$, one obtains the evolution equation

$$\frac{d}{dt} r(t) = L_{r(t)} r(t) \equiv L r(t) . \tag{3.17}$$

If the test particle has the same initial distribution as all the other fluid particles, then (3.15) yields the nonlinear evolution equation for the one-particle correlation function.

The structure just uncovered was studied on an abstract level first by McKean (1966, 1967) and called by him a *nonlinear Markov process*. This is a process in which the transition mechanism depends on the present state of the system itself. Since the understanding of this structure is of some importance, let us depart for a while to the abstract setting.

To avoid measure theoretical complications, let us choose a process $X(t)$ with finite state space S. The path space Ω then will consist of all right continuous, piecewise constant functions from $\mathbf{R}_+$ to S with a finite number of jumps in any finite time interval. A *Markov* process P on Ω is defined by

$$P(X(t+\tau)=x | X(s), s\le t) = P(X(t+\tau)=x | X(t)) \tag{3.18}$$

for all $x\in S$ and any $t, \tau \ge 0$. The Markov processes usually encountered are really a family of Markov processes $\{P_x | x\in S\}$ having the following structure. To each $x\in S$ there exists a Markov process P_x with

$$P_x(X(0)=y) = \delta_{xy} \tag{3.19}$$

(the process starts at x) and

$$P_x(X(t)=z | X(s)=y) = P_x(X(t+\tau)=z | X(s+\tau)=y) \tag{3.20}$$

for all $x, y, z \in S$, $0\le s\le t$; $\tau \ge 0$ (homogeneity in time). The transition probabilities $p_t(y|x)$ are defined by

$$p_t(y|x) = P_x(X(t)=y) \tag{3.21}$$

and satisfy, by Eq. (3.20), the semigroup property

$$p_{t_1+t_2}(z|x) = \sum_{y\in S} p_{t_2}(z|y) p_{t_1}(y|x) . \tag{3.22}$$

The (forward) Markov semigroup S_t is defined by

$$(S_t f)(y) = \sum_{x\in S} p_t(y|x) f(x) \tag{3.23}$$

on $\mathfrak{L}^1(S)$ and can be written, by (3.22), as $S_t f = e^{Lt}$.

As is well known, L has the structure

$$(Lf)(x) = \sum_{y\in S} (W(x|y)f(y) - W(y|x)f(x)), \tag{3.24}$$

where $W(y|x)\ge 0$ is the transition probability per unit time from x to y. Conversely, if L is of the form (3.24), then $(e^{Lt}f)(y) = \sum_x p_t(y|x)f(x)$ defines transition probabilities from which the measure P_x is build up by the Chapman-Kolmogorov equations as

$$P_x(X(t_1)=x_1, \ldots, X(t_n)=x_n) = P_{t_n - t_{n-1}}(x_n|x_{n-1}) \cdots p_{t_1}(x_1|x) . \tag{3.25}$$

A *nonlinear Markov process* on Ω is again a family of Markov processes $\{P_f\}$, but is now indexed by probability distributions f on S [i.e., $f(x)\ge 0, \sum_{x\in S} f(x) = 1$]. P_f has the properties

$$P_f(X(0)=x) = f(x) \tag{3.26}$$

(the initial distribution is f) and

$$P_f(X(t)=y | X(s)=x) = P_{S_s f}(X(t-s)=y | X(0)=x) \tag{3.27}$$

(homogeneity in time) for all $x, y\in S$ and $0\le s\le t$, where

$$(S_t f)(x) = P_f(X(t)=x) . \tag{3.28}$$

Let me explain the meaning of (3.27). The distribution of the system (≡ fluid) evolves as $f \to S_t f$. If the initial distribution of the system is f, then the probability of reaching y at time t, given x at time s (for a test particle to reach y at time t, given it started at x at time s), equals the probability of reaching y at time $t-s$, given x at time 0, *provided* the initial distribution of the system is corrected to $S_s f$. $S_t f$ satisfies the semigroup property

$$\begin{aligned}(S_{t_1}S_{t_2}f)(x) &= P_{S_{t_2}f}(X(t_1)=x) \\ &= \sum_{y\in S} P_{S_{t_2}f}(X(t_1)=x | X(0)=y)(S_{t_2}f)(y) \\ &= \sum_{y\in S} P_f(X(t_1+t_2)=x | X(t_2)=y) P_f(X(t_2)=y) \\ &= P_f(X(t_1+t_2)=x) = (S_{t_1+t_2}f)(x) ,\end{aligned} \tag{3.29}$$

where the Markov property of P_f was used in the last step. However, S_t is in general nonlinear. Formally, one can still write

$$\frac{d}{dt} S_t f = L S_t f , \tag{3.30}$$

where L is a nonlinear operator on $\mathfrak{L}^1(S)$. However, from (3.30) alone, one cannot recover the nonlinear

Markov process. To a nonlinear forward equation one can associate many nonlinear Markov processes. What is needed is the forward equations for each P_f. Clearly, if in addition

$$P_f = \sum_{x \in S} P_x f(x) \tag{3.31}$$

is assumed, then one is back to the familiar (linear) case described before.

The simplest example of a nonlinear Markov process is provided by the Carleman equation (Carleman, 1957). In that case, $S = (-1, 1)$, and the forward equation for P_f is

$$\frac{\partial}{\partial t} g(x,t) = (f(x,t) + f(x,t))(g(-x,t) - g(x,t)), \tag{3.32}$$

where $f(x,t)$ is the solution of

$$\frac{\partial}{\partial t} f(x,t) = (f(x,t) + f(-x,t))(f(-x,t) - f(x,t)), \tag{3.33}$$

with initial condition $f(x)$. The formal analogy with the (spatially homogeneous) Boltzmann equation is of course no coincidence, since (3.33) was meant as a simple model for understanding the structure of the latter. [The illuminating article by McKean (1975) is strongly recommended. There the main point is a study of hydrodynamics for the Carleman equation. This is the nonlinear analog of the problem discussed in Sec. II.C.]

Let me summarize where we now stand: We consider an interacting particle system characterized by its interaction potential V_ε and with initial state μ^ε corresponding to the correlation functions $\{\rho_n^\varepsilon | n \geq 0\}$. The potential and the initial states are scaled as (3.6), (3.7), or (3.8). Then to each $r \geq 0$ the process $X_{(r)}^\varepsilon(t)$ with path measure $P_{(r)}^\varepsilon$ is constructed as above. One expects that weakly on Ω,

$$\lim_{\varepsilon \to 0} P_{(r)}^\varepsilon = P_r , \tag{3.34}$$

and that the collection $\{P_r | r \geq 0\}$ will form a nonlinear Markov process. In other words, the processes $X_{(r)}^\varepsilon(t)$ should converge to a nonlinear Markov process as $\varepsilon \to 0$.

B. The Landau equation

Landau (1936) studied a weakly coupled gas in order to understand the evolution of a plasma and arrived at an equation for the one-particle density. An excellent source on the Landau equation is the books by Balescu (1963, 1975).

From our experience with the Lorentz gas and from what I will say about the Boltzmann equation in the following section, the way to proceed is rather clear. One chooses a sequence of initial states μ^ε such that their density increases as ε^{-3} (ε^{-d} in d dimensions) and such that they are essentially uncorrelated, i.e., such that the nth correlation function of μ^ε satisfies

$$\lim_{\varepsilon \to 0} \varepsilon^{2n} \rho_n^\varepsilon(x_1, \ldots, x_n) = \prod_{j=1}^n r(x_j) . \tag{3.35}$$

μ^ε evolves according to the dynamics given through the Hamiltonian (3.1) with potential $\varepsilon^{1/2} V(q/\varepsilon)$. Then the correlation functions $\{\rho_n^\varepsilon(t) | n \geq 0\}$ of $\mu^\varepsilon(t)$, the time evolved measure at time t, should satisfy

$$\lim_{\varepsilon \to 0} \varepsilon^{3n} \rho_n^\varepsilon(x_1, \ldots, x_n, t) = \prod_{j=1}^n r(x_j, t) . \tag{3.36}$$

$r(x,t)$ is the solution of the Landau equation

$$\begin{aligned} \frac{\partial}{\partial t} r(q,p,t) &= -p \cdot \nabla_q r(q,p,t) \\ &+ \int d\bar{p}\, r(q,\bar{p},t) \Big(\sum_{j=1}^3 \frac{\partial}{\partial \bar{p}_j} a_j(p-\bar{p}) \Big) r(q,p,t) \\ &+ \int d\bar{p} r(q,\bar{p},t) \\ &\times \Big(\sum_{i,j=1}^3 \frac{\partial}{\partial p_i} D_{ij}(p-\bar{p}) \frac{\partial}{\partial p_j} \Big) r(q,p,t) \end{aligned} \tag{3.37}$$

with initial datum $r(q,p)$, where D_{ij} is the diffusion matrix (2.20) and a_j is the drift term,

$$a_j(p) = -\alpha p_j / |p|^3 . \tag{3.38}$$

The Landau equation has properties which formally are similar to those of the Boltzmann equation; in particular, the usual H theorem is valid and the Maxwellians are the only stationary solution of the collision term.

The evolution of a test particle in the fluid is governed, in the weak coupling limit, by the linear Landau equation

$$\begin{aligned} \frac{\partial}{\partial t} f(q,p,t) = \Big\{ &-p \cdot \nabla_q + \int d\bar{p}\, r(q,\bar{p},t) \Big(\sum_{j=1}^3 \frac{\partial}{\partial \bar{p}_j} a(p-\bar{p}) \Big) \\ &+ \int d\bar{p}\, r(q,\bar{p},t) \\ &\times \Big(\sum_{i,j=1}^3 \frac{\partial}{\partial p_i} D_{ij}(p-\bar{p}) \frac{\partial}{\partial p_j} \Big) \Big\} f(q,p,t) . \end{aligned} \tag{3.39}$$

(3.39) together with (3.37) has the structure of a nonlinear Markov process.

The Landau equation seems to be rather neglected in the mathematical physics literature; in comparison to the Boltzmann equation, at least, it is rather surprising how little is known rigorously about it. To my knowledge, one of the few results is the proof of the existence and uniqueness of the solution of the spatially homogeneous Landau equation in a finite time interval by Arsen'ev and Peskov (1978).

C. The Boltzmann equation

In 1872 Boltzmann introduced an equation, later to bear his name, which describes the time evolution of the one-particle density of a dilute gas. The Boltzmann equation is still the cornerstone of the kinetic theory of gases. From the beginning the problem of the relation of the Boltzmann equation to the underlying dynamics was pressing. [The Ehrenfest article (Ehrenfest and Ehrenfest, 1911) gives a vivid account on the conceptual difficulties raised by the irreversible character of the Boltzmann equation.] On a formal level, Grad's careful analysis (Grad, 1958) seems to be the most satisfactory answer. He also introduced the limit in which the Boltzmann equation becomes exact. The

close relationship between the full BBGKY hierarchy for hard spheres and the Boltzmann equation was pointed out by Cercignani (1972). Lanford, 1975, gave a proof that, for short times, the solutions of the BBGKY hierarchy converge to the solutions of the Boltzmann hierarchy. This is one of the gems in the field. We will give here only a short summary emphasizing certain aspects. The reader is urged to consult the articles by Lanford (1975, 1976a) and the thesis of King (1975).

1. Convergence of the solution of the BBGKY hierarchy to the solution of the Boltzmann hierarchy

We consider a system of hard spheres of diameter ε and unit mass inside a bounded region Λ with smooth boundary $\partial\Lambda$. [The restriction of Λ being bounded will be lifted below. The results referred to later on extend to positive potentials scaled as $V_\varepsilon(q)=V(q/\varepsilon)$ with certain regularity properties (King, 1975).] The number of particles is not necessarily fixed. So the classical phase space is $\Gamma=\cup_{n\geq 0}(\Lambda\times\mathbf{R}^3)^n$. Not all points of the phase space can be realized because of the hard-core exclusion.

The spheres (particles) are elastically reflected amongst themselves and at the boundary $\partial\Lambda$. For pair collisions and collisions with the wall the dynamics are thereby well defined. For grazing collisions and triple and higher collisions the dynamics simply remain undefined. Alexander (1975) has shown that the (complicated) set of initial phase points which lead to such higher collisions at any later or previous time is of Lebesgue measure zero. Therefore, if the initial distribution of hard spheres is absolutely continuous with respect to Lebesgue, such exceptional sets are of probability zero and can be discarded.

Let the initial state μ^ε of the system be specified by the absolutely continuous probabilities of finding exactly n particles at $dx_1\cdots dx_n$, $\{f_n(x_1,\ldots,x_n)\times(1/n!)dx_1\cdots dx_n|n\geq 0\}$. The scaling parameter ε has already been introduced for convenience. Then the correlation functions $\{\rho_n^\varepsilon|n\geq 0\}$ corresponding to this state are defined by

$$\rho_n^\varepsilon(x_1,\ldots,x_n)=\sum_{m=0}^{\infty}\frac{1}{m!}\int_{(\Lambda\times\mathbf{R}^3)^m}dy_1\cdots dy_m f_{n+m}^\varepsilon\times(x_1,\ldots,x_n,y_1,\ldots,y_m)\,. \tag{3.40}$$

The time evolution of a state of the hard-sphere system is studied by means of the time evolution of the corresponding correlation functions. A straightforward computation which is, however, nontrivial to justify rigorously (Cercignani, 1972; Lanford, 1978) leads to the following evolution equation:

$$\frac{\partial}{\partial t}\rho_n^\varepsilon(x_1,\ldots,x_n,t)=H_n^\varepsilon\rho_n^\varepsilon(x_1,\ldots,x_n,t)+\varepsilon^2\left\{\sum_{j=1}^{n}\int_{\mathbf{R}^3}dp_{n+1}\int_{S^2}d\hat\omega\,\hat\omega\cdot(p_{n+1}-p_n)\times\rho_{n+1}^\varepsilon(x_1,\ldots,x_n,q_j+\varepsilon\hat\omega,p_{n+1},t)\right\}. \tag{3.41}$$

Here $\hat\omega$ is a unit vector in $\mathbf{R}^3$ and $d\hat\omega$ the surface measure of the unit sphere S^2 in three dimensions. H_n^ε describes the evolution of n hard spheres of diameter ε inside Λ. (11.2) is the *BBGKY hierarchy* for hard spheres. The solutions of the BBGKY hierarchy are denoted by

$$\rho_n^\varepsilon(x_1,\ldots,x_n,t)=(V_t^\varepsilon\rho^\varepsilon)_n(x_1,\ldots,x_n) \tag{3.42}$$

for the initial vector of correlation functions $\rho^\varepsilon=(\rho_0^\varepsilon,\rho_1^\varepsilon,\ldots)$.

At this stage we can formally lift the restriction that Λ has to be a bounded region. So Λ may be, for example, a slab or the whole three-dimensional space. It is also clear that specular reflection at $\partial\Lambda$ is only one choice out of many possible boundary conditions; we could consider, for example, a stochastic boundary condition at $\partial\Lambda$ corresponding to a wall with a certain temperature. All these boundary conditions would be included in the definition of H_n^ε.

We want to study the low-density (Grad) limit of the solutions of the BBGKY hierarchy. The hard-core diameter was scaled already as ε. As for the Lorentz gas, the density of particles should then increase as ε^{-2}. (On this scale the mean free path $\sim\varepsilon^2\varepsilon^{-2}$ remains constant, whereas the volume occupied by spheres $\sim\varepsilon^3\varepsilon^{-2}$ tends to zero as $\varepsilon\to 0$.) Therefore for each hard-sphere diameter ε one chooses an initial state with correlation functions ρ_n^ε such that $\rho_n^\varepsilon\sim\varepsilon^{-2n}$. With this in mind, we define the *rescaled correlation functions*

$$r_n^\varepsilon(x_1,\ldots,x_n)=\varepsilon^{2n}\rho_n^\varepsilon(x_1,\ldots,x_n)\,. \tag{3.43}$$

Then (3.41) reads

$$\frac{d}{dt}r_n^\varepsilon(t)=H_n^\varepsilon r_n^\varepsilon(t)+C_{n,n+1}^\varepsilon r_{n+1}^\varepsilon(t)\,, \tag{3.44}$$

where the collision term in curly brackets is abbreviated as $C_{n,n+1}^\varepsilon$. Regarding the sequence $\{r_n^\varepsilon|n\geq 0\}$ as vector r^ε one can write (3.44) compactly as

$$\frac{d}{dt}r^\varepsilon(t)=H^\varepsilon r^\varepsilon(t)+C^\varepsilon r^\varepsilon(t)\,, \tag{3.45}$$

where H^ε is a diagonal matrix with entries H_n^ε and C^ε is a matrix with entries $C_{n,n+1}^\varepsilon$ and zero otherwise.

Let us consider H^ε as the unperturbed part of the operator $H^\varepsilon+C^\varepsilon$ and C^ε as the perturbation. The time-dependent (Dyson) perturbation series for the solution of (3.45) then reads

$$r^\varepsilon(t)=\sum_{m=0}^{\infty}\int_{0\leq t_1\leq\cdots\leq t_m\leq t}dt_m\cdots dt_1 S^\varepsilon(t-t_m)\times C^\varepsilon\cdots C^\varepsilon S^\varepsilon(t_1)r^\varepsilon\,, \tag{3.46}$$

where r^ε stands for $r^\varepsilon(0)$, and where $(S^\varepsilon(t)r^\varepsilon)_n=(e^{H^\varepsilon t}r^\varepsilon)_n=e^{H_n^\varepsilon t}r_n^\varepsilon$ gives the evolution of n hard spheres of diameter ε inside Λ, always including the specular reflection at $\partial\Lambda$. Solutions of the BBGKY hierarchy are always understood in the sense of (3.46). Of course, one has to say in what sense (3.46) converges.

To see what the formal limit $\varepsilon\to 0$ of (3.46) should be, let us consider the typical term

$$S^{\varepsilon}(s)C^{\varepsilon}_{n,n+1}S^{\varepsilon}(t)r^{\varepsilon}_{n+1}, \tag{3.47}$$

with $t, s > 0$. Expression (3.47) describes the evolution of n hard spheres for a time s, adjoining the $(n+1)$st sphere with momentum p_{n+1} and position such as to touch the jth sphere, the evolution of $n+1$ spheres for a time t, and finally the sum over all $j = 1, \ldots, n$, momenta p_{n+1} and possible touching points $q_j + \varepsilon\hat{\omega}$. Besides the possible collision at time s at the point, where the jth and $(n+1)$st sphere are joined, other collisions may occur. These are, however, of zero probability in the limit $\varepsilon \to 0$ and (3.47) converges to

$$S(s)C_{n,n+1}S(t)r_{n+1}, \tag{3.48}$$

where $S(t)$ is the free motion including specular reflection at the boundary and

$$\begin{aligned}(C_{n,n+1}r_{n+1})(x_1, \ldots, x_n) &= \sum_{j=1}^{n} \int_{\hat{\omega}\cdot(p_j - p_{n+1}) \geq 0} dp_{n+1} d\hat{\omega}\, \hat{\omega}\cdot(p_j - p_{n+1}) \\ &\times \{r_{n+1}(x_1, \ldots, q_j, p'_j, \ldots, q_j, p'_{n+1}) \\ &- r_{n+1}(x_1, \ldots, q_j, p_j, \ldots, q_j, p_{n+1})\}. \end{aligned} \tag{3.49}$$

p'_j, p'_{n+1} are outgoing momenta to the incoming momenta p_j, p_{n+1} and momentum transfer in direction $\hat{\omega}$. Therefore, formally, as $\varepsilon \to 0$ (3.46) goes over to

$$\begin{aligned} r(t) = \sum_{m=0}^{\infty} \int_{0 \leq t_1 \leq \cdots \leq t_m \leq t} dt_m \cdots dt_1\, S(t - t_m) \\ \times C \cdots C S(t_1) r. \end{aligned} \tag{3.50}$$

Differentiating (3.50) with respect to t one obtains the *Boltzmann hierarchy* for hard spheres

$$\begin{aligned}\frac{\partial}{\partial t} r_n(x_1, \ldots, x_n, t) &= -\sum_{j=1}^{n} p_j \cdot \nabla_{q_j} r_n(x_1, \ldots, x_n, t) \\ &+ (C_{n,n+1} r_{n+1})(x_1, \ldots, x_n, t). \end{aligned} \tag{3.51}$$

$-p_j \cdot \nabla_{q_j}$ includes the specular reflection at $\partial\Lambda$.

For $t \leq 0$ the same reasoning leads again to (3.51), but with the sign of the collision term reversed.

To prove (that $r^{\varepsilon}(t)$ defined by (3.46) indeed converges to $r(t)$ defined by (3.50) as $\varepsilon \to 0$ we need two conditions.

First, the initial correlation functions r^{ε} have to be bounded uniformly in ε. This guarantees the uniform convergence of the perturbation series (3.46) for some interval $|t| \leq t_0$. [Heuristically, the finite radius of convergence comes from the following fact: For $n = 1$, in (3.46) the time integration yields $t^m(1/m!)$, whereas the m collision operators yield roughly $m!$. For a better result, cancellations have to be taken into account.] If h_{β} denotes the normalized Maxwellian at inverse temperature β, then a suitable choice for this bound is as follows:

(C1) There exist a pair (z, β) such that

$$\begin{aligned}\varepsilon^{2n} |\rho^{\varepsilon}_n(x_1, \ldots, x_n)| &= |r^{\varepsilon}_n(x_1, \ldots, x_n)| \\ &\leq M \sum_{j=1}^{n} \{z h_{\beta}(x_j)\} \end{aligned} \tag{3.52}$$

for all $\varepsilon < \varepsilon_0$ with a positive constant M independent of ε.

Second, r^{ε}_n has to converge to r_n in such a way that the series (3.46) converges term by term to the series (3.50). For the initial phase point $x^{(n)} = (x_1, \ldots, x_n) \in (\Lambda \times \mathbf{R}^3)^n$ let $q_j(t, x^{(n)}), j = 1, \ldots, n$, be the position of the jth point particle at time t under the free motion. Then

$$\begin{aligned}\Gamma_n(t) = \{x^{(n)} = (x_1, \ldots, x_n) \in (\Lambda \times \mathbf{R}^3)^n | q_i(s, x^{(n)}) \\ \neq q_j(s, x^{(n)})\end{aligned}$$

for $i \neq j = 1, \ldots, n$ and $-t \leq s \leq 0$, if $t \geq 0$, $0 \leq s \leq -t$, if $t \leq 0\}$.

In words, $\Gamma_n(t)$ is the restriction of the n-particle phase space to the set of all phase points that under free backward streaming over a time t, if t is positive, (or free forward streaming over a time $|t|$, if t is negative) do not lead to a collision between any pair of particles, regarded as point particles. By this restriction only a set of Lebesgue measure zero is excluded from $(\Lambda \times \mathbf{R}^3)^n$.

Note that (i) $\Gamma_n(t)$ depends only on the free motion; (ii) $\Gamma_n(t) \supset \Gamma_n(t')$ for $t' = \alpha t$, $\alpha \geq 1$; (iii) $\Gamma_n(t) \neq \Gamma_n(-t)$; and (iv) that $x^{(n)} \in \Gamma_n(t)$ is equivalent to $\bar{x}^{(n)} \in \Gamma_n(-t)$, where $x^{(n)}$ is the phase point obtained from $x^{(n)}$ under the reversal $p_j \to -p_j$. In particular, $\Gamma_n(t)$ is not invariant under reversal of velocities.

The suitable choice of convergence is then as follows:

(C2) There exists a continuous function r_n on $(\Lambda \times \mathbf{R}^3)^n$ such that

$$\lim_{\varepsilon \to 0} \varepsilon^{2n} \rho^{\varepsilon}_n = \lim_{\varepsilon \to 0} r^{\varepsilon}_n = r_n, \tag{3.53}$$

uniformly on all compact sets of $\Gamma_n(s)$ for some $s \geq 0$.

Theorem 3.1 (Lanford): Let $\{\rho^{\varepsilon}_n | n \geq 0\}$ be a sequence of initial correlation functions (not necessarily coming from a positive measure) of a fluid of hard spheres of diameter ε inside a region Λ and let $\{\rho^{\varepsilon}_n | n \geq 0\}$ satisfy (C1) and (C2). Let $\rho^{\varepsilon}_n(t)$ be the solution of the BBGKY hierarchy with initial conditions ρ^{ε}_n and let $r_n(t)$ be the solution of the Boltzmann hierarchy with initial conditions r_n.

Then there exists a $t_0(z, \beta) > 0$ such that for $0 \leq t \leq t_0(z, \beta)$ the series (3.46) and (3.50) converge pointwise and such that $\rho^{\varepsilon}_n(t)$ satisfies a bound of the form (C1) with $z' > z$, $\beta' < \beta$. Furthermore,

$$\lim_{\varepsilon \to 0} \varepsilon^{2n} \rho_n(t) = \lim_{\varepsilon \to 0} r^{\varepsilon}_n(t) = r_n(t), \tag{3.54}$$

uniformly on compact sets of $\Gamma_n(s + t)$.

For $-t_0(z, \beta) \leq t \leq 0$, (3.54) holds, provided that $s \leq 0$ and that in the Boltzmann hierarchy the collision term $C_{n,n+1}$ is replaced by $-C_{n,n+1}$.

$t_0(z, \beta)$ may be chosen as $\frac{1}{5}\sqrt{3}/\pi z\sqrt{\beta}$. The second factor has physically the meaning of the mean free time of a Boltzmann gas in equilibrium at inverse temperature β and density z.

Lanford (1976a) shows how to construct initial states satisfying (C1) and (C2). One chooses a continuous one-particle density r bounded by the Maxwellian and a finite partition Δ^{ε}_i of $\Lambda \times \mathbf{R}^3$, which becomes finer and finer as $\varepsilon \to 0$. Then in each cell, Δ^{ε}_i particles are uniformly distributed such that their hard-core exclusion

is respected and such that their number is $\varepsilon^{-2}\int_{\Delta_i^\varepsilon} dx_1 \times r(x_1)$, up to a relative error of the order ε. This defines an initial state with correlation functions ρ_n^ε. Under certain regularity assumptions on the sequence of partitions, (C1) and (C2) with $s=0$ are fulfilled.

The reader may wonder how Lanford's theorem escapes the conflict between the reversible character of the BBGKY hierarchy and the irreversible character of the Boltzmann hierarchy. The point is that if for the sequence of states $V_t^\varepsilon \rho^\varepsilon$, $0 < t \leq \frac{1}{2} t_0(z,\beta)$, we reverse all momenta at time t and thereby form a new sequence of initial states $\bar{\rho}^\varepsilon$, then $\bar{\rho}^\varepsilon$ does not satisfy the condition (C2) of Lanford's theorem and therefore the theorem is not (and, moreover, should not be) applicable to this sequence of initial states.

We describe now three interesting properties of the Boltzmann hierarchy. The first one is the well-known "propagation of chaos."

Property 1: If the initial conditions of the Boltzmann hierarchy factorize

$$r_n(x_1,\ldots,x_n) = \sum_{j=1}^n r(x_j)\,, \tag{3.55}$$

then the solutions with this inital condition stay factorized

$$r_n(x_1,\ldots,x_n,t) = \sum_{j=1}^n r(x_j,t)\,. \tag{3.56}$$

$r(x,t)$ is the solution of the *Boltzmann equation*

$$\begin{aligned}\frac{\partial}{\partial t} r(q,p,t) &= -p\cdot\nabla_q r(q,p,t) \\ &+ \int_{\hat{\omega}\cdot(p-p_1)\geq 0} dp_1\, d\hat{\omega}\, \hat{\omega}\cdot(p-p_1) \\ &\times\{r(q,p',t) r(q,p_1',t) \\ &- r(q,p,t) r(q,p_1,t)\}\,,\end{aligned} \tag{3.57}$$

with initial conditions $r(q,p)$.

Property 1 implies the validity of the Boltzmann equation in the following sense. Assume that the initial correlation functions $\{\rho_n^\varepsilon | n \geq 0\}$ satisfy the bound (C1) and that

$$\lim_{\varepsilon\to 0} \varepsilon^{2n} \rho_n^\varepsilon(x_1,\ldots,x_n) = \prod_{j=1}^n r(x_j)\,, \tag{3.58}$$

uniformly on compact sets of $\{(x_1,\ldots,x_n)\in(\Lambda\times\mathbf{R}^3)^n | q_i\neq q_j,\ i\neq j=1,\ldots,n\}$ with some continuous r. Then, for $0\leq t\leq t_0(z,\beta)$,

$$\lim_{\varepsilon\to 0} \varepsilon^2 \rho_1^\varepsilon(x_1,t) = r(x_1,t)\,, \tag{3.59}$$

uniformly on compact sets of $\Lambda\times\mathbf{R}^3$, where $r(x_1,t)$ is the solution of the Boltzmann equation (3.57) with initial condition $r(x_1)$.

The second property comes from considering one of the fluid particles as a test particle. This property was used already in the proof of Theorem 2.8.

Property 2: If the initial conditions of the Boltzmann hierarchy are of the form

$$r_n(x_1,\ldots,x_n) = f(x_1)\prod_{j=2}^n r(x_j)\,, \tag{3.60}$$

then its solutions are

$$r_n(x_1,\ldots,x_n,t) = f(x_1,t)\prod_{j=2}^n r(x_j,t)\,. \tag{3.61}$$

$r(x,t)$ is the solution of the Boltzmann equation (3.57) with initial condition $r(x)$, and $f(x,t)$ is the solution of the *linear Boltzmann equation*

$$\begin{aligned}\frac{\partial}{\partial t} f(q,p,t) &= -p\cdot\nabla_q f(q,p,t) \\ &+ \int_{\hat{\omega}\cdot(p-p_1)\geq 0} dp_1\, d\hat{\omega}\, \hat{\omega}\cdot(p-p_1) \\ &\times\{r(q,p_1',t) f(q,p',t) \\ &- r(q,p_1,t) f(q,p,t)\}\,,\end{aligned} \tag{3.62}$$

with initial condition $f(q,p)$.

The third property will be important for discussing fluctuations.

Property 3: If the initial conditions of the Boltzmann hierarchy are of the form

$$r_n(x_1,\ldots,x_n) = \Big(\sum_{j=1}^n f(x_j)\Big)\prod_{j=1}^n \{z h_\beta(x_j)\}\,, \tag{3.63}$$

then its solutions are

$$r_n(x_1,\ldots,x_n,t) = \Big(\sum_{j=1}^n f(x_j,t)\Big)\prod_{j=1}^n \{z h_\beta(x_j)\}\,. \tag{3.64}$$

$f(x,t)$ is the solution of the *linearized Boltzmann equation*

$$\begin{aligned}\frac{\partial}{\partial t} f(q,p,t) &= -p\cdot\nabla_q f(q,p,t) \\ &+ z\int_{\hat{\omega}\cdot(p-p_1)\geq 0} dp_1\, d\hat{\omega}\, \hat{\omega}\cdot(p-p_1) h_\beta(p_1)\,, \\ &\{f(q,p_1',t) + f(q,p',t) - f(q,p_1,t) - f(q,p,t)\} \\ &\equiv (Lf)(q,p,t)\,,\end{aligned} \tag{3.65}$$

with initial condition $f(q,p)$.

Property 3 is proved by inserting the *Ansatz* (3.64) in the Boltzmann hierarchy and by then using repeatedly the fact that the collision operator acting on the Maxwellian vanishes. Note that (3.65) corresponds to linearizing the Boltzmann equation around the stationary Maxwellian as $h_\beta(1+f)$.

It should be understood that Properties 1–3 are subject to the restrictions of Theorem 3.1; in particular, the initial conditions have to satisfy the bound (C1) and the results are valid only up to $t_0(z,\beta)$.

2. Convergence to a nonlinear Markov process

With the support of Theorem 3.1 it is only a small step to prove that in the low-density limit the process $P^{(\mu\varepsilon)}$ constructed in Sec. III.A converges to a nonlinear Markov process as $\varepsilon\to 0$—within the limitations imposed by Theorem 3.1, of course.

Let Ω be the path space already introduced for the Rayleigh gas: $\Omega = \cup_{n\geq 0}\Omega_n$, where $(y_1,t_1,\ldots,y_n,t_n)\in\mathbf{R}^{4n}$ is in Ω_n, if $0\leq t_1+\cdots+t_n\leq T=t_0(z,\beta)$. On Ω, a probability measure is constructed in the following way: Let μ^ε be the initial distribution of the hard spheres with diameter ε and let $\{\rho_n^\varepsilon | n\geq 0\}$ be the corresponding correlation functions. For each initial configuration $x = (x_1,\ldots,x_n)\in\Gamma_n$ the motion of the jth fluid particle t

$\to (q_j(t,x), p_j(t,x))$, $0 \le t \le T$, defines in the obvious way a path in Ω. This path is given weight one. Since x is a configuration of n fluid particles, a measure $P^{(x)}$ on Ω is thereby defined with total mass n. Then

$$P^\varepsilon = \left(\int dx_1 \rho_1^\varepsilon(x_1) \right)^{-1} \int_\Gamma \mu^\varepsilon(dx) P^{(x)} . \tag{3.66}$$

We denote a process corresponding to P^ε by $X^\varepsilon(t) = (q^\varepsilon(t), p^\varepsilon(t))$. $p^\varepsilon(t)$ is a jump process on $\mathbf{R}^3$ with a finite number of jumps in any finite time interval. Because of the mechanical motion,

$$q^\varepsilon(t) = q(0) + \int_0^t ds p^\varepsilon(s) . \tag{3.67}$$

Theorem 3.2: Let μ^ε be a sequence of initial states with correlation functions $\{\rho_n^\varepsilon | n \ge 0\}$ satisfying condition (C1) and

$$\lim_{\varepsilon \to 0} \varepsilon^{2n} \rho_n^\varepsilon(x_1, \ldots, x_n) = \prod_{j=1}^n r(x_j) \tag{3.68}$$

uniformly on compact sets of $\Gamma_n(0)$ with some continuous r, $r(x_1) \le z h_\beta(x_1)$.

Then

$$X^\varepsilon(t) \to X(t) , \tag{3.69}$$

as $\varepsilon \to 0$ in the sense that all finite-dimensional distributions of $X^\varepsilon(t)$ converge weakly to the finite-dimensional distributions of $X(t)$. $X(t)$ is a Markov process.

Let P_r be the path measure corresponding to the limiting Markov process $X(t)$. Then $\{P_r | r \le z h_\beta\}$ form a nonlinear Markov process in the sense that

$$P_r(X(t) \in A | X(s) = x) = P_{r(s)}(X(t-s) \in A | X(0) = x) , \tag{3.70}$$

$0 \le s \le t \le T$. $r(s)$ is the solution of the Boltzmann equation (3.57) with initial condition r.

Proof: Let $V_t^\varepsilon \rho$ be the solution of the BBGKY hierarchy (3.41) in the sense of the perturbation series (3.46) with initial conditions $\rho = (\rho_1, \rho_2, \ldots)$. Let $V_t r$ be the solution of the Boltzmann hierarchy (3.51) with initial conditions r. If f is a bounded and continuous function on $\Lambda \times \mathbf{R}^3$, then $f\rho$ is shorthand for $(f(x_1)\rho_1(x_1), f(x_1) \times \rho_2(x_1, x_2), \ldots)$.

We have to show the convergence of

$$E^\varepsilon(f_1(X_1^\varepsilon(t_1)) \cdots f_n(X_n^\varepsilon(t_n))) \tag{3.71}$$

for continuous $f_1, \ldots, f_n$ of compact support, $0 \le t_1 \le \cdots \le t_n \le T$. From Sec. III. A, we have

$$E^\varepsilon(f(X^\varepsilon(t))) = \left(\varepsilon^2 \int dx_1 \rho_1^\varepsilon(x_1) \right)^{-1} \times \varepsilon^2 \int dx_1 f(x_1) (V_t^\varepsilon \rho^\varepsilon)_1(x_1) . \tag{3.72}$$

Therefore, by Theorem 3.1,

$$\lim_{\varepsilon \to 0} E^\varepsilon(f(X^\varepsilon(t))) = \left(\int dx_1 r(x_1) \right)^{-1} \times \int dx_1 f(x_1) (V_t r)_1(x_1) . \tag{3.73}$$

A straightforward calculation shows that

$$E^\varepsilon(f_1(X^\varepsilon(t_1)) f_2(X^\varepsilon(t_2))) = \left(\varepsilon^2 \int dx_1 \rho_1^\varepsilon(x_1) \right)^{-1} \times \int dx_1 f_2(x_1) \varepsilon^2 (V_{t_2 - t_1}^\varepsilon (f_1 (V_{t_1}^\varepsilon \rho^\varepsilon)))_1 (x_1) , \tag{3.74}$$

$0 \le t_1 \le t_2 \le t_0(z, \beta)$. By Theorem 3.1, $\varepsilon^{2n}(V_{t_1}^\varepsilon \rho^\varepsilon)_n$ converges to $(V_{t_1} r)_n$, uniformly on compact sets of $\Gamma_n(t_1)$. The same is true for $\varepsilon^{2n} f(V_{t_1}^\varepsilon \rho^\varepsilon)_n$. Since $t_1 \le t_0(z, \beta)$, there exists a pair $z'\beta'$ such that $f(V_{t_1}^\varepsilon \rho^\varepsilon)$ satisfies (C1) with z', β'. Therefore, $f(V_{t_1}^\varepsilon \rho^\varepsilon)$ satisfies the conditions of Theorem 3.1. Since $t_2 \le t_0(z, \beta)$, one has $t_2 - t_1 \le t_0(z', \beta')$. From Theorem 3.1 one concludes, then, that

$$\lim_{\varepsilon \to 0} E^\varepsilon(f_1(X^\varepsilon(t_1)) f_2(X^\varepsilon(t_2))) = \left(\int dx_1 r(x_1) \right)^{-1} \int dx_1 f_2(x_1) (V_{t_2 - t_1}(f_1(V_{t_1} r)))_1 (x_1) . \tag{3.75}$$

Convergence of the higher-order correlation functions is proved in the same way.

We denote by $S_{t,s}$ the propagator corresponding to the solution of the linear Boltzmann equation (3.62), where $r(x,t)$ is the prescribed solution of the Boltzmann equation with initial condition $r(x)$. Then, using Properties 1 and 2 and the fact that the initial distribution of $X(t)$ is $r(x_1) / \int dx_1 r(x_1)$, one obtains

$$\text{Eq. (3.73)} = \left(\int dx_1 r(x_1) \right)^{-1} \int dx_1 f(x_1) (S_{t,0} r)(x_1) \tag{3.76}$$

and

$$\text{Eq. (3.75)} = \left(\int dx_1 r(x_1) \right)^{-1} \times \int dx_1 f_2(x_1) (S_{t_2, t_1}(f_1(S_{t_1, 0} r)))(x_1) , \tag{3.77}$$

and the analogous expression for the higher correlation functions. This proves that $X^\varepsilon(t)$ converges to $X(t)$, where $X(t)$ is the Markov process with (3.62) as a forward equation and with initial distribution $r(x_1) / \int dx_1 \times r(x_1)$.

Equation (3.70) follows from the form of the linear Boltzmann equation ∎

D. The Vlasov equation

The Vlasov equation (Vlasov, 1938) describes the motion of an interacting particle system in the mean field limit, thereby taking into account the influence of weak long-range forces. Often a collision term of Landau type is added. The Vlasov equation is of central importance in plasma physics and in stellar dynamics.

The convergence of the mean field limit was shown by Neunzert (1975 and 1978) and independently by Braun and Hepp (1977). Their results are essentially complete and point for point match the heuristic discussion of Sec. III. A; in particular, the convergence to a nonlinear process can be proved for this limit. One simplifying feature comes from the fact that the limiting nonlinear Markov process is deterministic.

The correct limit can be easily guessed from either Sec. II. A.5 or from the BBGKY hierarchy. We consider

a (finite) particle system in $\mathbf{R}^3$ interacting via the twice continuously differentiable pair potential V_ε. V_ε is scaled as in Sec. II. A.5 as $V_\varepsilon = \varepsilon V$. $F = -\nabla V$ is the force. Then the *BBGKY hierarchy* for the time-dependent correlation functions reads

$$\frac{\partial}{\partial t}\rho_n^\varepsilon(x_1,\ldots,x_n,t) = \left\{-\sum_{j=1}^n p_j\cdot\nabla_{q_j} - \varepsilon\sum_{z<j}^n F(q_z-q_j)\cdot\nabla_{p_j}\right\} \times\rho_n^\varepsilon(x_1,\ldots,x_n,t) - \varepsilon\sum_{j=1}^n\int dx_{n+1}F(q_j-q_{n+1})\cdot\nabla_{p_j}\rho_{n+1}^\varepsilon(x_1,\ldots,x_{n+1},t). \tag{3.78}$$

In the mean-field limit the force between two individual particles goes to zero, whereas the force on a particular particle exerted collectively by all others stays finite. As in Sec. II A.5, this is achieved by increasing the density as ε^{-1}. Therefore

$$\lim_{\varepsilon\to 0}\varepsilon^n\rho_n^\varepsilon = r_n. \tag{3.79}$$

Then, as $\varepsilon\to 0$, (3.78) formally converges to the *Vlasov hierarchy*

$$\frac{\partial}{\partial t}r_n(x_1,\ldots,x_n,t) = \left\{-\sum_{j=1}^n p_j\cdot\nabla_{q_j}\right\}r_n(x_1,\ldots,x_n,t) - \sum_{j=1}^n\int dx_{n+1}F(q_j-q_{n+1})\cdot\nabla_{p_j}r(x_1,\ldots,x_{n+1},t). \tag{3.80}$$

If initially the factorization

$$r_n(x_1,\ldots,x_n) = \prod_{j=1}^n r(x_j)$$

is assumed, then the scaled one-particle correlation function of the fluid evolves according to the Vlasov equation

$$\frac{\partial}{\partial t}r(q,p,t) = -p\cdot\nabla_q r(q,p,t) - \int dq_1\,dp_1 r(q_1,p_1,t)F(q-q_1)\cdot\nabla_p \times r(q,p,t), \tag{3.81}$$

with initial condition $r(q,p)$. The existence and uniqueness of solutions of the Vlasov equation have been established by Neunzert (1975) and, independently, Dobrushin (1979). A test particle in the fluid evolves according to the effective Hamiltonian

$$H_{\text{eff}}(q,p,t) = \tfrac{1}{2}p^2 + \int dq_1\,dp_1 r(q_1,p_1,t)V(q-q_1)$$

as

$$\frac{d}{dt}q(t) = p(t),$$

$$\frac{d}{dt}p(t) = \int dq_1\,dp_1 r(q_1,p,t)F(q(t)-q_1). \tag{3.82}$$

Equation (3.82), together with (3.81), defines a (deterministic) nonlinear Markov process.

One can prove the convergence of the solution of the BBGKY hierarchy to the solutions of the Vlasov hierarchy in the mean field limit for arbitrary times and for a large class of potentials (Braun and Hepp, 1977). A particularly nice result is the following, since it does not use any statistical assumptions at the initial time.

Theorem 3.3 (Neunzert, 1975; Braun and Hepp, 1977): Let V be a twice differentiable potential with bounded derivatives. Given $N(\varepsilon)$ particles interacting via the potential εV, let

$$\mu_t^\varepsilon(dx) = N(\varepsilon)^{-1}\sum_{j=1}^{N(\varepsilon)}\delta(x-x_j(t)). \tag{3.83}$$

$\delta(x-x_0)$ is the point measure concentrated at $x_0\in\mathbf{R}^3\times\mathbf{R}^3$. $x_j(t)$ is the position and the momentum of the jth fluid particle at time t. [$\mu_t^\varepsilon(dx)$ is the normalized sum of $N(\varepsilon)$ delta functions concentrated at the locations of the fluid particles at time t in the one-particle phase space.] If $N(\varepsilon)\varepsilon = 1$ and if

$$\lim_{\varepsilon\to 0}\mu_0^\varepsilon(dx) = r_0(dx) \tag{3.84}$$

weakly on $\mathbf{R}^6$, then

$$\lim_{\varepsilon\to 0}\mu_t^\varepsilon(dx) = r_t(dx) \tag{3.85}$$

weakly on $\mathbf{R}^6$ and $r_t(dx)$ is the weak solution of the Vlasov equation with initial conditions $r_0(dx)$.

If $r_0(dx)$ has a differentiable density, $r_0(dx) = r_0(x)dx$, then so has $r_t(dx) = r(x,t)dx$ and $r(x,t)$ is the classical solution of the Vlasov equation with initial conditions $r_0(x)$.

Mürman (1978) considers a degenerate type of mean field limit. The central finite range potential V is scaled as $V_\varepsilon(q) = V(q/\varepsilon)$, and the number of particles increases as ε^{-3} (in three dimensions). In the limit the BBGKY hierarchy formally converges to the Vlasov hierarchy with a δ potential. $\{F(q_j-q_{n+1})$ in (3.80) is replaced by $[\int dq\,V(q)]\delta(q_j-q_{n+1})\nabla_{q_{n+1}}.\}$ Under certain assumptions Mürman shows the existence of a subsequence of the solutions of the BBGKY hierarchy which converges weakly in suitable spaces to a solution of the degenerate Vlasov hierarchy.

E. Fluctuations

So far we have studied the dynamics of the fluid by choosing an arbitrary fluid particle as a test particle and by then considering its stochastic motion through the changing fluid as in the case of the Lorentz gas. If the test particle has the same initial distribution as all the other fluid particles, then this procedure yields the dynamics of the one-particle correlation function. Physically, the fluid is studied through its *time-dependent self-correlation functions*.

There is a complementary point of view, which is advocated by Hepp and Lieb (1973c) in a somewhat different context (see Sec. V. E) and by Lanford (1975), and which I want to develop now. Here the dynamics of the fluid are studied via its *time-dependent total correlation functions*. [A definition of these correlation functions can be found, among other places, in Appendix A of Lebowitz and Percus, (1967)].

Let us consider an interacting particle system in the bounded region Λ. We scale the potential and the initial state according to the weak coupling limit, the low-density limit, or the mean-field limit. Let $\Delta \subset \Lambda \times \mathbf{R}^3$ be a bounded region. Then the *average* number of particles in Δ at time t is $\int_\Delta dx_1 \rho_1^\varepsilon(x_1, t)$. If the molecular chaos assumption (the assumption that the limiting scaled correlation functions factorize) is satisfied, then, under some further conditions,

$$\lim_{\varepsilon \to 0} \varepsilon^\nu \int_\Delta dx_1 \rho_1^\varepsilon(x_1, t) = \int_\Delta dx_1 \, r(x_1, t) , \tag{3.86}$$

where $r(x_1, t)$ is the solution of the Landau equation, the Boltzmann equation, or the Vlasov equation. So the properly scaled average number of particles in Δ at time t can be computed from the solution of the kinetic equation. (The way the scaling was introduced in Secs. III. B, III. C, and III. D, $\nu = 3$ corresponds to the weak coupling limit, $\nu = 2$ corresponds to the low-density limit, and $\nu = 1$ corresponds to the mean field limit.)

The *actual* number of particles in Δ at time t will, of course, differ for different initial configurations of particles. Since a probability distribution of the initial configurations is given, the number of particles in Δ at time t is a random variable on Γ. To discuss its properties we have to introduce some notation. For a (measurable) one-particle function f: $\Lambda \times \mathbf{R}^3 \to \mathbf{R}$, let $X^\varepsilon(f)$ be the following sum function on Γ:

$$X^\varepsilon(f)(x_1, \ldots, x_n) = \sum_{j=1}^n f(x_j) , \tag{3.87}$$

and let $X^\varepsilon(f, t)$ be the function $X^\varepsilon(f)$ evolved under the dynamics with pair potential V_ε. $X^\varepsilon(f, t)$ is a random variable on $(\Gamma, \mu^\varepsilon)$, where μ^ε is the initial measure. $X^\varepsilon(f, t)$ depends on ε through the dynamics and through μ^ε. If f is chosen to be the indicator function χ_Δ of the set Δ [$\chi_\Delta(x_1) = 1$ if $x_1 \in \Delta$, $\chi_\Delta(x_1) = 0$ otherwise], then $X^\varepsilon(\chi_\Delta, t)$ is the number of particles in Δ at time t.

Let us compute the variance of $\varepsilon^\nu X^\varepsilon(f, t)$ for a continuous f of compact support,

$$\begin{aligned}
&\mu^\varepsilon(|\varepsilon^\nu X^\varepsilon(f, t) - \mu^\varepsilon(\varepsilon^\nu X^\varepsilon(f, t))|^2) \\
&= \varepsilon^{2\nu} \Big\{ \int dx_1 \, dx_2 \rho_2^\varepsilon(x_1, x_2, t) f(x_1) f(x_2) \\
&\quad + \int dx_1 \rho_1^\varepsilon(x_1, t) |f(x_1)|^2 - \Big(\int dx_1 \rho_1^\varepsilon(x_1, t) f(x_1) \Big)^2 \Big\} .
\end{aligned} \tag{3.88}$$

Since for the chosen scaling $\varepsilon^{2\nu} \rho_2^\varepsilon(x_1, x_2, t) \to r(x_1, t) \times r(x_2, t)$ as $\varepsilon \to 0$, the variance of $\varepsilon^\nu X^\varepsilon(f, t)$ converges to zero as $\varepsilon \to 0$. Thus, the distribution of $\varepsilon^\nu X^\varepsilon(f, t)$ becomes sharply peaked around $\int dx_1 \, r(x_1, t) f(x_1)$ for small ε. This result offers the following interpretation of a Markovian limit for an interacting system: For small ε and for a typical initial particle configuration $(x_1, \ldots, x_n)$ (typical with respect to the initial measure μ^ε),

$$X^\varepsilon(f, t)(x_1, \ldots, x_n) \cong \varepsilon^{-\nu} \int dx_1 \, r(x_1, t) f(x_1) . \tag{3.89}$$

So the kinetic equation yields information about typical initial configurations rather than about average quantities. In passing, let me note that (3.88) offers a neat solution to an old and puzzling problem: Inasmuch as the kinetic equation becomes valid, the actual and the average number of particles in the region Δ at time t coincide.

On the other hand, assume that for an initial measure μ^ε with correlation functions satisfying (C1) and (C2), the variance of $\varepsilon^\nu X^\varepsilon(f)$ tends to zero as $\varepsilon \to 0$,

$$\lim_{\varepsilon \to 0} \mu^\varepsilon(|\varepsilon^\nu X^\varepsilon(f) - \mu^\varepsilon(\varepsilon^\nu X^\varepsilon(f))|^2) = 0 \tag{3.90}$$

for all continuous f of compact support. Then, by Proposition 2.3, necessarily

$$r_n(x_1, \ldots, x_n) = \prod_{j=1}^n r_1(x_j) . \tag{3.91}$$

Thus no fluctuations in the scaled number of particles in arbitrary regions are equivalent to the assumption of molecular chaos. The theorems of Lanford and of Braun and Hepp imply then that if initially the scaled number of particles does not fluctuate, it will also not fluctuate at a later time.

To study the fluctuations in more detail one has to magnify them tremendously: One subtracts out the average value and studies the fluctuations only on a scale proportional to $\varepsilon^{-\nu/2}$. Therefore one defines the fluctuation field

$$\xi^\varepsilon(f, t) = \varepsilon^{\nu/2} [X^\varepsilon(f, t) - \mu^\varepsilon(X^\varepsilon(f, t))] . \tag{3.92}$$

Conjecture 3.4: Under further assumptions on the initial state μ^ε, on the scaled potential V_ε, and on f,

$$\xi^\varepsilon(f, t) \to \xi(f, t) \tag{3.93}$$

as $\varepsilon \to 0$, where $\xi(f, t)$ is a Gaussian random field. [This means that the joint distribution of $\xi^\varepsilon(f_1, t_1), \ldots, \times \xi^\varepsilon(f_n, t_n)$ converges weakly to a Gaussian distribution.]

Therefore one expects that

$$\varepsilon^\nu X^\varepsilon(f, t) \cong \int dx_1 \, r(x_1, t) f(x_1) + \varepsilon^{\nu/2} \xi(f, t) \tag{3.94}$$

for small ε. The scaled number of particles in a region Δ evolves to first order deterministically according to the kinetic equation, and its fluctuations around the deterministic path are Gaussian. This picture is quite familiar from other areas of statistical physics; see, for example, the survey article by van Kampen (1976b).

It is believed that the Gaussian random field $\xi(f, t)$ has the following structure. If one writes formally $\xi(f, t) = \int dq \, dp \, f(q, p) \xi(q, p, t)$, then $\xi(q, p, t)$ should satisfy the linear, time-dependent stochastic differential equation

$$\frac{\partial}{\partial t} \xi(q, p, t) = (L_{r(t)} \xi)(q, p, t) + F(q, p, t) . \tag{3.95}$$

$L_{r(t)}$ is the time-dependent linear operator obtained by linearizing the kinetic equation at $r(t)$, and $F(q, p, t)$ is a Gaussian white noise fluctuating force with mean zero.

There are two instances where the formal picture can be justified rigorously.

Theorem 3.5: (Braun and Hepp 1977). Let $V_\varepsilon = \varepsilon V$, where V is four times differentiable with uniformly con-

tinuous derivatives, and let the sequence of initial states μ^ε be such that its correlation functions $\varepsilon^n \rho_n^\varepsilon$ converge weakly to the product $\prod_{j=1}^n r$ and such that

$$\varepsilon^{1/2}\{X^\varepsilon(f) - \mu^\varepsilon(X^\varepsilon(f))\} \equiv \xi^\varepsilon(f) \tag{3.96}$$

converges to a Gaussian random field $\xi(f)$ as $\varepsilon \to 0$ for all three times differentiable f with uniformly continuous derivatives. [The joint distribution of $\xi^\varepsilon(f_1), \ldots, \xi^\varepsilon(f_n)$ converges weakly to a Gaussian.] Then

$$\xi^\varepsilon(f,t) \to \xi(f,t) \tag{3.97}$$

as $\varepsilon \to 0$ in the sense of weak convergence of all finite dimensional distributions. $\xi(f,t)$ is a Gaussian random field determined by the stochastic differential equation (3.95) with initial conditions $\xi(f)$. The fluctuating force is identically zero, and $L_{r(t)}$ is the linearized Vlasov operator

$$\begin{aligned}(L_{r(t)}\xi)(q,p) = &-p\cdot\nabla_q \xi(q,p) - \int dq'\,dp'\, r(q',p',t)\\ &\times F(q-q')\cdot\nabla_p \xi(q,p)\\ &- \int dq'\,dp'\, \xi(q',p') F(q-q')\cdot\nabla_p r(q,p,t),\end{aligned} \tag{3.98}$$

where $r(t)$ is the solution of the Vlasov equation (3.81) with initial condition r.

For the low-density limit, one has only a very restricted result (van Beijeren, Lanford, Lebowitz, and Spohn, 1979). One considers a system of hard spheres of diameter ε in thermal equilibrium. μ^ε is then the grand canonical equilibrium distribution at inverse temperature β and fugacity $z_\varepsilon = \varepsilon^{-2}z$. (This corresponds to increasing the density as ε^{-2}.) The distribution of $\varepsilon^2 X^\varepsilon(f,t)$ concentrates at $\int dq\,dp\, z h_\beta(p) f(q,p)$. The system approaches the state of an ideal gas in equilibrium as $\varepsilon \to 0$. As regards the fluctuations, at least the covariance of the fluctuation field can be shown to converge as $\varepsilon \to 0$.

Theorem 3.6: Let $f, g \in L^2(\Lambda \times \mathbf{R}^3, z h_\beta(p)dq\,dp) = \mathcal{H}$ with scalar product $\langle\ |\ \rangle_{\beta,z}$, and let $0 \le t - s \le t_0(ez,\beta)$. Then, we have

$$\lim_{\varepsilon\to 0} \langle \xi^\varepsilon(f,s)\xi^\varepsilon(g,t)\rangle_{\beta,z_\varepsilon} = \langle g \,|\, e^{L(t-s)} f\rangle_{\beta,z}, \tag{3.99}$$

where L is the linearized Boltzmann operator (3.65). [$\{e^{Lt}\,|\,t \ge 0\}$ is a contraction semigroup on $\mathcal{H}$ (Klaus, 1975).]

If one assumes the limiting fluctuation field to exist and to be Gaussian, then its covariance (3.99) implies that it is of the form (3.95). Since $r(t) = z h_\beta$, $L_{r(t)}$ is the linearized Boltzmann operator. The fluctuating force $F(q,p,t)$ has the covariance

$$\begin{aligned}&\int dp\,d\bar{p}\, f(p) g(\bar{p}) \langle F(q,p,t) F(\bar{q},\bar{p},t)\rangle\\ &= \tfrac{1}{2}\delta(t-\bar{t})\delta(q-\bar{q})\\ &\times \int dp\,dp_1\,d\hat{\omega}\,\hat{\omega}\cdot(p-p_1) z h_\beta(p_1) z h_\beta(p) \hat{\omega}\cdot(p-p_1) \ge 0\\ &\times (f(p_1') + f(p') - f(p_1) - f(p))\\ &\times (g(p_1') + g(p') - g(p_1) - g(p)).\end{aligned} \tag{3.100}$$

One notices that the covariance has the form $-\langle f\,|\,(L + L^*)g\rangle_{\beta,z}$ as derived from the fluctuation-dissipation theorem.

Outline of the proof: One notes that

$$\langle \xi^\varepsilon(f,s)\xi^\varepsilon(g,t)\rangle_{\beta,z_\varepsilon} = \varepsilon^2 \int dx_1\, \rho_1^\varepsilon(x_1,t) g(x_1), \tag{3.101}$$

where $\rho_1^\varepsilon(x_1,t) = (V_t^\varepsilon \rho^\varepsilon)_1(x_1)$, with

$$\begin{aligned}\rho_n^\varepsilon(x_1,\ldots,x_n) = &\left(\sum_{j=1}^n f(x_j)\right)\rho_{\mathrm{eq},n}^\varepsilon(x_1,\ldots,x_n)\\ &+ \int dx_{n+1} f(x_{n+1})(\rho_{\mathrm{eq},n}^\varepsilon(x_1,\ldots,x_{n+1})\\ &- \rho_{\mathrm{eq},n}^\varepsilon(x_1,\ldots,x_n)\rho_{\mathrm{eq},1}^\varepsilon(x_{n+1})),\end{aligned} \tag{3.102}$$

and $\rho_{\mathrm{eq}}^\varepsilon$ the equilibrium correlation functions. An equilibrium estimate shows that the second term vanishes in the low-density limit. To the first term one applies Lanford's theorem using Property 3∎

IV. LATTICE MODELS

A. The harmonic lattice

A general crystal lattice is specified by a basic underlying lattice and by a unit cell (which may contain more than one particle). For simplicity, and since we are not going to do real lattice dynamics, anyhow, let us choose the simple cubic lattice $\mathbf{Z}^d$ and one particle per unit cell (Bravais lattice). Then $q_j \in \mathbf{R}^\nu$ denotes the displacement of the jth particle from its equilibrium position $j \in \mathbf{Z}^d$ and $p_j \in \mathbf{R}^\nu$ its conjugate momentum. For physical systems, usually $d = \nu = 3$. In the harmonic approximation the equations of motion read

$$\begin{aligned}\frac{d}{dt} q_i(t) &= \frac{1}{m_i} p_i(t),\\ \frac{d}{dt} p_i(t) &= -\sum_{j\in\mathbf{Z}^d} V_{ij} q_j(t).\end{aligned} \tag{4.1}$$

m_i is the mass of the jth particle, and $\frac{1}{2}\sum_{i,j} V_{ij} q_i q_j$ is the potential energy which is assumed to be positive with $V_{ij} = V_{ji}$. (V_{ij} is a $\nu \times \nu$ matrix.) For the study of Markovian limits Poincaré recurrences have to be avoided, which is achieved by considering an infinite harmonic lattice. We describe briefly how the dynamics of the infinite harmonic lattice are constructed.

Let M be the matrix with matrix elements $m_j \delta_{ij}$, and let V be the matrix with matrix elements V_{ij}. Then the equations of motion read schematically

$$\frac{d}{dt}\begin{pmatrix} p(t)\\ q(t)\end{pmatrix} = \begin{pmatrix} 0 & -V\\ M^{-1} & 0\end{pmatrix}\begin{pmatrix} p(t)\\ q(t)\end{pmatrix}, \tag{4.2}$$

where $q(t)$ stands for the vector $(q_j(t)\,|\,j \in \mathbf{Z}^d)$ and $p(t)$ for the vector $(p_j(t)\,|\,j \in Z^d)$. Equation (4.2) should be solved for a class of initial data which is large enough to support interesting states, but which is small enough to guarantee uniqueness of solutions. A convenient growth condition is to require that q_j, p_j are polynomially bounded. Then the classical *phase space* of the crystal is $s'(\Gamma)$, the space of polynomially bounded sequence over $\Gamma = \mathbf{Z}^d \cup \cdots \cup \mathbf{Z}^d$ (2ν times). $s'(\Gamma)$ is the dual of $s(\Gamma)$, the space of sequences decreasing faster

than any polynomial. To construct the dynamics on $s'(\Gamma)$ the idea is to solve the equations dual to (4.2):

$$\frac{d}{dt}\begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix}=\begin{bmatrix}0 & M^{-1}\\ -V & 0\end{bmatrix}\begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix} \tag{4.3}$$

is $s(\Gamma)$. Sufficient conditions for the solutions of (4.3) to stay in $s(\Gamma)$ are, e.g., (i) M is bounded above and below and (ii) $|V_{ij}|\leq be^{-c|i-j|}$ with suitable constant $b, c>0$. Then, if T_t^* denotes the solutions of (4.3), their dual T_t are solutions of (4.2) and therefore define the *time evolution* (phase flow) of the harmonic crystal on $s'(\Gamma)$.

The thermal *equilibrium state* of the crystal at inverse temperature β is described by the *Gaussian* measure on $s'(\Gamma)$ with mean zero and convariance

$$\beta^{-1}\begin{bmatrix}M & 0\\ 0 & V^{-1}\end{bmatrix}. \tag{4.4}$$

For this measure to be well defined, (4.4) has to be a continuous, strictly positive, bilinear form on $s(\Gamma)$. (4.4) is obtained by taking the infinite volume limit through a sequence of finite volume canonical equilibrium states with tied down boundary conditions. (If $\Lambda\subset\mathbf{Z}^d$ is a bounded region, then having tied down boundary conditions will mean $q_j=0$ for $j\in\mathbf{Z}^d\setminus\Lambda$.) For other boundary conditions other equilibrium states may be reached. These are still Gaussian but not mean zero. Additionally, there are many other Gaussian measures invariant under the dynamics T_t (Spohn and Lebowitz, 1977).

The approach described here is found in the article by Lanford and Lebowitz (1975) and also in the thesis by van Hemmen (1976).

Customarily, one particle, say the one indexed by zero, is regarded as a test particle. For the study of heat flow and other problems, it is convenient to consider several test particles. So we choose a bounded simply connected region $\Lambda\subset Z^d$ and consider the particles in Λ as the system (of test particles) and the particles in $\mathbf{Z}^d\setminus\Lambda$ as the infinite reservoir. Initially, the positions and the momenta of the particles in Λ are fixed at $x=(q_i,p_i\,|\,i\in\Lambda)$ and the reservoir is in thermal equilibrium conditioned on the system particles being at x. Then, as for the Lorentz gas, the motion of the system is described by the stochastic process $X^\varepsilon(t)$. ε is a scaling parameter coming from scaling the interaction, the mass, etc. Since the dynamics are linear, and since the equilibrium measure is Gaussian, $X^\varepsilon(t)$ is a *Gaussian stochastic process*; but *non-Markovian*, in general. In the spirit of our enterprise, the process $X^\varepsilon(t)$ is scaled in such a way that

$$X^\varepsilon(t)\to X(t) \tag{4.5}$$

as $\varepsilon\to 0$, where $X(t)$ is Markovian. Since $X(t)$ is Gaussian and Markovian, its generator has a linear drift term and a constant diffusion term. The task of proving (4.5) is greatly simplified by the fact that for a sequence of Gaussian processes it is sufficient to show the convergence of the mean and the covariance,

$$\begin{aligned}&\lim_{\varepsilon\to 0}E_x(X^\varepsilon(t))=E_x(X(t)),\\ &\lim_{\varepsilon\to 0}E_x(X^\varepsilon(t)X^\varepsilon(s))=E_x(X(t)X(s))\end{aligned} \tag{4.6}$$

for all t, s (Nelson, 1967).

Two cases have been studied in some detail.

(i) The single impurity

In this case, $\nu=1, d$ arbitrarily, and V_{ij} is a nearest-neighbor interaction potential with coupling constant κ, i.e., $V_{ii}=2\kappa$ and $V_{ij}=-\kappa$, if i and j are nearest neighbors, and $V_{ij}=0$ otherwise. The mass of all the reservoir particles is equal to m, and the mass of the impurity at the origin is M. The reservoir is in thermal equilibrium at inverse temperature β conditioned on the impurity being located at q_0. In a certain sense this model is exactly soluble. Therefore, the effect of a finite reservoir, the duration of Poincaré recurrences, memory effects, approach to equilibrium, etc., can be studied quite explicitly (Hemmer, 1959; Rubin, 1960, 1961; Mazur and Montroll, 1960; Cukier and Mazur, 1971; Morita and Mori, 1976). Certainly, this model helped to understand some of the features of nonequilibrium statistical mechanics.

In one dimension the scaling (2.58) to (2.61) for the impurity leads to the Ornstein-Uhlenbeck process determined through the stochastic differential equation

$$\begin{aligned}&\frac{d}{dt}q_0(t)=\frac{1}{M}p_0(t),\\ &\frac{d}{dt}p_0(t)=-\frac{2\sqrt{\kappa m}}{M}p_0(t)+F(t),\end{aligned} \tag{4.7}$$

with initial conditions q_0, p_0 (Rubin, 1961). $F(t)$ is Gaussian white noise with mean zero and covariance $\langle F(t)F(t')\rangle=4\sqrt{\kappa m}\,\beta^{-1}\delta(t-t')$. For fixed mass $M\geq m$, scaling the positional part of the process according to (2.117) leads to the Wiener process with diffusion constant $2D=(2\beta\sqrt{m\kappa})^{-1}$ independent of M (Morita and Mori, 1976). Ford, Kac, and Mazur (1965) scale to a long-range interaction while keeping the mass of the impurity fixed. This scaling also leads to the Ornstein-Uhlenbeck process.

(ii) The singular and weak coupling limits

These limits are motivated by quantum-mechanical models. They can be handled by the methods of Davies (1974a, 1974b, 1976b). Some examples have been worked out by Spohn and Lebowitz (1977).

The interaction matrix V is split into

$$V=V^S+V^{SR}+V^R \tag{4.8}$$

by combining all couplings between system and reservoir particles in V^{SR} and by combining all couplings between system particles themselves and external potentials on system particles in V^S. The interaction between system and reservoir is assumed to be weak, $V_\varepsilon^{SR}=\varepsilon V^{SR}$. To obtain a nontrivial effect for the system, time has to be rescaled as

$$t_\varepsilon=\varepsilon^{-2}t. \tag{4.9}$$

Since the coupling to the reservoir is weak, the system particles will make far excursions and will oscillate

rapidly relative to each other on the t-time scaled. There are *two* different ways to compensate for this.

One possibility is to increase the system masses and to decrease the interaction between system particles. If M^S denotes the (diagonal) mass matrix of system particles, then

$$M_\varepsilon^S = \varepsilon^{-2} M^S \tag{4.10}$$

and

$$V_\varepsilon = \varepsilon^2 V^S + \varepsilon V^{SR} + V^R . \tag{4.11}$$

Equations (4.9) to (4.11) define the *singular coupling limit*. The origin of the name can be seen by going to the t-time scale. Then

$$\begin{aligned} V_\varepsilon &= V^S + \varepsilon^{-1} V^{SR} + \varepsilon^{-2} V^R , \\ M_\varepsilon &= M^S + \varepsilon^2 M^R . \end{aligned} \tag{4.12}$$

The motion of the reservoir and the couplings between the reservoir and the system are scaled in such a way as to produce in time δ-correlated Gaussian forces (= white noise) acting on the system.

Computing the expectations (4.6), one observes that one has to prove the existence of

$$\lim_{\varepsilon \to 0} P T_t^\varepsilon P . \tag{4.13}$$

T_t^ε is the time evolution defined by (4.2), with V_ε and M_ε as in (4.12), and P projects onto the system. In words, one has to study, for small ε, the dynamics of the system particles with V_ε and M_ε as in (4.12) for the particular initial condition, where all reservoir particles are at rest at their equilibrium position. Under certain conditions on V and M, (4.13) can be proved using the methods of Davies (1976b). In the limit the effect of the reservoir is *local* in the following sense: Only to the dynamics of those system particles which are directly coupled to the reservoir are a friction term and a stochastic force as in (4.7) added. The friction coefficients and the diffusion matrix depend on the details of the reservoir and the interaction.

The other way to compensate for the fast motion of the system on the t-time scale is simply to subtract out the uncoupled motion of the system. (This method is frequently used for quantum-mechanical systems.) The dissipation due to the coupling to the reservoir is studied relative to the uncoupled motion. We keep the masses fixed and scale the interaction as

$$V_\varepsilon = V^S + \varepsilon V^{SR} + V^R . \tag{4.14}$$

We denote the stochastic motion of the system with this scaling as $X^\varepsilon(t)$ and the deterministic dynamics of the uncoupled system by T_t^0. (T_t^0 are the solutions of (4.2) with M^S and V^S inserted.) Then one wants to study

$$\lim_{\varepsilon \to 0} T^0_{-\varepsilon^{-2}t} X^\varepsilon(\varepsilon^{-2} t) . \tag{4.15}$$

The rescaling of time as $t_\varepsilon = \varepsilon^{-2} t$ is introduced since the dissipation is expected to be of the order ε^2. Equation (4.15) is a Gaussian process, since it is a linear combination of Gaussian processes.

Computing the expectations (4.6) one observes that essentially one has to prove the existence of

$$\lim_{\varepsilon \to 0} T^0_{-\varepsilon^{-2}t} P T^\varepsilon_{\varepsilon^{-2}t} P . \tag{4.16}$$

Here, T_t^ε is the dynamics of the harmonic crystal with V_ε as in (4.14). Again the Davies methods can be used to prove (4.16) under certain conditions on V and M [see Spohn and Lebowitz (1977)]. In the limit, the effect of the reservoir is *nonlocal*: Every system particle diffuses in position and momentum space. Physically the origin of nonlocality can be understood in the following way: An excitation of the system travels back and forth inside the system many ($\sim \varepsilon^{-2}$) times until it dissipates into the reservoir. In the meantime the excitation has completely forgotten from where it originated.

B. The anharmonic lattice

I include this section only to remind the reader of yet another open problem.

If one goes beyond the harmonic approximation for a crystal, then anharmonic terms are included in the interaction. A typical example is the crystal with weak anharmonic nearest-neighbor interaction defined by the Hamiltonian function

$$\sum_j \frac{1}{2m} p_j^2 + \sum_{\langle i,j \rangle} \{ (q_i - q_j)^2 + \varepsilon (q_i - q_j)^4 \} , \tag{4.17}$$

where $\langle i, j \rangle$ denotes a pair of nearest neighbors. For infinite anharmonic crystals with strong restoring forces the existence of the time evolution for a large class of initial data has been proved by Lanford, Lebowitz, and Lieb (1977). Their results have been extended, as to include (4.17), in addition, by Marchioro, Pellegrinotti, Pulvirenti, and Suhov (1979) and by Marchioro, Pellegrinotti, and Pulvirenti (1979). One would like to study the evolution of initially "uncorrelated" states for weak anharmonicities. Our previous study of interacting particle systems suggests the following analogy to the propagation of molecular chaos: If the initial state of the crystal is a Gaussian probability measure, then, in the limit of small coupling, at a later time it will still be a Gaussian measure with mean and covariance determined by a nonlinear equation.

In the treatment in the physical literature of problems of this kind (Peierls, 1955; Prigogine, 1963), the resulting nonlinear equation has been called a phonon-Boltzmann equation. This stems from the following picture: The harmonic crystal is thought of as a gas of noninteracting normal modes (≡ phonons). The anharmonicities then introduce some interaction between phonons. For small ε the interaction is weak and one expects some kind of Boltzmann (or rather Landau) equation to be valid.

At present, even to extract a consistent set of conjectures from the literature seems to be hard.

V. QUANTUM-MECHANICAL MODELS

A. Some general remarks

It is clear how to write the quantum-mechanical version of the models discussed so far. Physical questions, too, such as determining mean square displacement of a test particle, are easily formulated. However, one should not draw the analogies too closely. There are many quantum-mechanical models without a classical counterpart. A good example is provided by

the quantum theory of radiation [see, e.g., Louisell (1973)], which in a very practical sense is the main field of application of the weak coupling limit. There an atom is coupled to the electromagnetic field (a coupling which in comparison to the other physical parameters is indeed weak, namely, of the order of the fine-structure constant), and one is interested in the transition rates between different bound states. Additionally, the classical model may behave qualitatively differently from its quantum-mechanical counterpart. For example, for the Lorentz gas with a periodic configuration of scatterers the mean-square displacement of the classical Lorentz particle is proportional to t, whereas the mean-square displacement of the quantum-mechanical Lorentz particle is proportional to t^2 (as for a free particle) because of coherence effects.

The broad classification of models is the same as in the classical case. One has interacting particle systems. For these only mean-field-type limits can be handled. This leads then to reversible, but nonlinear, evolution equations (Sec. V.E). In particular, the problem of the quantum-mechanical (nonlinear) Boltzmann equation is open, despite extensive investigations on the formal level (Kadanoff and Baym, 1962; Fuijita, 1966).

The models primarily considered are of the system + reservoir type. If $\mathcal{H}_S$ denotes the Hilbert space associated with the system, and $\mathcal{H}_R$ the Hilbert space associated with the reservoir, then the formal Hamiltonian of such a model is

$$H = H_S \otimes 1 + H_{SR} + 1 \otimes H_R \tag{5.1}$$

on $\mathcal{H}_S \otimes \mathcal{H}_R$, where H_{SR} is some interaction term. An important distinction with no classical counterpart is between models where the system Hamiltonian H_S has a discrete spectrum and those where H_S has a continuous spectrum. In physical terms it is the distinction between a spatially confined particle (≡ bound states) and a spatially unconfined particle (≡ states in the continuum, extended states), i.e., where the particle can move in all of $\mathbf{R}^3$. The Markovian limit for systems with an H_S with a discrete spectrum, is, in essence, understood, whereas the case of an H_S with a continuous spectrum still presents certain difficulties.

For classical models we argued that the dynamics of the test particle (≡ system) is in a natural way regarded as a stochastic process. A Markovian limit is then a certain scaling for which the process becomes Markovian. Could the same point of view be adopted for a quantum-mechanical system?

Let us first consider the single-time correlation functions, or, equivalently, the *reduced dynamics*. We denote by $T(\mathcal{H}_S)$ the set of all trace class operators on $\mathcal{H}_S$. Then a state of the system is described by a density matrix $\rho \in T(\mathcal{H}_S)$ with $\rho \geq 0$ and $\mathrm{tr}_S \rho = 1$. Suppose that at $t=0$ the state of the joint system is $\rho \otimes \rho_R$, where ρ is an arbitrary state of the system and ρ_R a fixed state of the reservoir, e.g., ρ_R is the thermal equilibrium state at a certain temperature. If tr_R denotes the partial trace of the reservoir, then the reduced dynamics T_t^ε corresponding to the scaled Hamiltonian H^ε is defined by

$$T_t^\varepsilon \rho = \mathrm{tr}_R (e^{-iH^\varepsilon t} \rho \otimes \rho_R e^{iH^\varepsilon t}) \tag{5.2}$$

for all $\rho \in T(\mathcal{H}_S)$. By definition, T_t^ε is linear, $T_t^\varepsilon \rho \geq 0$, if $\rho \geq 0$, and $\mathrm{tr}_S (T_t^\varepsilon \rho) = \mathrm{tr}_S \rho$. In particular, a state is mapped onto a state.

If the scaling is appropriate for a Markovian limit, then one expects, as in the classical case, that

$$\lim_{\varepsilon \to 0} T_t^\varepsilon = T_t , \tag{5.3}$$

and that T_t satisfies the semigroup property

$$T_{t_1 + t_2} = T_{t_1} T_{t_2} , \tag{5.4}$$

$t_1, t_2 \geq 0$. $\{T_t \mid t \geq 0\}$ is the quantum-mechanical analog of a Markov semigroup. The structure of T_t has been investigated only in recent years. We review some of the results in the following section.

The higher-order correlation functions are organized in a particular way. One imagines that at times $0 < t_1 < t_2 \cdots < t_n$ measurements are performed on the system. As a generalization of the von Neumann projection postulates, one assumes that the effect of a measurement is given by

$$\rho \to A \rho A^* , \tag{5.5}$$

where A is a bounded operator on $\mathcal{H}_S$. (More generally, the effect of a measurement is given by a completely positive map $\rho \to \sum_j A_j \rho A_j^*$.) Then the nth time correlation function is constructed by evolving the state $\rho \otimes \rho_R$ for a time span t_1, performing a measurement at time t_1, evolving the new state for a time span $t_2 - t_1$, etc., i.e.,

$$\begin{aligned} \mathrm{tr}[&(A_n \otimes 1) U^\varepsilon (t_n - t_{n-1}) \cdots U^\varepsilon (t_2 - t_1)(A_1 \otimes 1) \\ &\times U^\varepsilon (t_1)(\rho \otimes \rho_R) \\ &\times U^\varepsilon (t_1)^* (A_1 \otimes 1)^* \\ &\times U^\varepsilon (t_2 - t_1)^* \cdots U^\varepsilon (t_n - t_{n-1})^* (A_n \otimes 1)^*] , \end{aligned} \tag{5.6}$$

where $U^\varepsilon(t) = e^{-iH^\varepsilon t}$ is the dynamics corresponding to the scaled Hamiltonian H^ε. By the polarization identity

$$\begin{aligned} B \rho A^* = \tfrac{1}{4} \{ &(A+B)\rho(A^*+B^*) - (A-B)\rho(A^*-B^*) \\ &- i(A+iB)\rho(A^*-iB^*) + i(A-iB)\rho(A^*+iB^*) \} , \end{aligned} \tag{5.7}$$

an arbitrary time-ordered correlation function is obtained in this fashion.

If the scaling is appropriate for a Markovian limit, then one expects, again as in the classical case, that

$$\begin{aligned} \lim_{\varepsilon \to 0} \mathrm{tr} [&(A_n \otimes 1) U^\varepsilon (t_n - t_{n-1}) \cdots U^\varepsilon (t_2 - t_1) \\ &\times (A_1 \otimes 1) U^\varepsilon (t_1)(\rho \otimes \rho_R) U^\varepsilon (t_1)^* (A_1 \otimes 1)^* U^\varepsilon (t_2 - t_1)^* \cdots \\ &\times U^\varepsilon (t_n - t_{n-1})^* (A_n \otimes 1)^*] \\ &= \mathrm{tr}_S [A_n (T_{t_n - t_{n-1}} \cdots T_{t_2 - t_1} (A_1 (T_{t_1} \rho) A_1^*) \cdots) A_n^*] , \end{aligned} \tag{5.8}$$

where T_t is the semigroup of the limiting reduced dynamics.

The structure (5.8) is generally assumed in the physical literature [Haake (1973), for example]. Lindblad (1979a, and 1979b, cf. also 1979c) uses (5.8) as the definition of a quantum stochastic process. Equation [(5.8) is the quantum version of the Chapman-Kolmogorov equations.] There are also somewhat different pro-

posals by Accardi (1975 and 1976), extended by Frigerio (1980) and by Lewis (1980), and by Davies (1976a).

B. Quantum-dynamical semigroups

Quantum-dynamical semigroups are the generalization of Markov semigroups to noncommutative algebras. Large parts of the theory have been developed on the level of von Neumann and C^*-algebras. Since we want to apply the theory to quantum systems with a finite number of degrees of freedom, we restrict ourselves to the case of all bounded operators on a separable Hilbert space.

There is associated a separable Hilbert space $\mathfrak{K}$ to the quantum system consideration. Let $B(\mathfrak{K})$ be the algebra of all bounded operators on $\mathfrak{K}$, and let $T(\mathfrak{K})$ be the set of all trace class operators on $\mathfrak{K}$. $T(\mathfrak{K})$ is a Banach space under the trace norm

$$\|\rho\|_1 = \mathrm{tr}|\rho| . \tag{5.9}$$

States of the system are described by density matrices $\rho \in T(\mathfrak{K})$ satisfying $\rho \geq 0$ and $\mathrm{tr}\rho = 1$.

We imagine the system somehow coupled to its surroundings. Then the Hamiltonian dynamics $e^{-itH}\rho e^{itH}$ appropriate for an isolated system are to be replaced by a more general class of linear dynamical maps T_t: $T(\mathfrak{K}) \to T(\mathfrak{K})$, $t \geq 0$. If the total probability is conserved, then the minimal requirements on T_t are (i′) that T_t be positive, i.e., $\rho \geq 0$ implies $T_t\rho \geq 0$, and (ii) that T_t preserve the trace, i.e., $\mathrm{tr}[T_t\rho] = \mathrm{tr}\rho$. (i′) and (ii) just ensure that a state is mapped onto a state. The Markovian assumption[1] is formalized by the semigroup property (iii):

$$T_{t_1+t_2}\rho = T_{t_1}T_{t_2}\rho , \tag{5.10}$$

$t_1, t_2 \geq 0$. This allows us to write $T_t = e^{Lt}$, where L is a linear operator on $T(\mathfrak{K})$, provided there is some continuity in t. Typically one requires strong continuity (iv):

$$\lim_{t\to 0} \|T_t\rho - \rho\|_1 = 0 . \tag{5.11}$$

We translate (i) to (iv) to the Heisenberg picture. One defines the dual time evolution T_t^* by

$$\mathrm{tr}[A(T_t\rho)] = \mathrm{tr}[(T_t^*A)\rho] \tag{5.12}$$

for all $\rho \in T(\mathfrak{K}), A \in B(\mathfrak{K})$. Then (i) to (iii) translate to

(i′) T_t^* is positive,

(ii) $T_t^*1 = 1$,

(iii) $T^*_{t_1+t_2} = T_{t_1}^*T_{t_2}^*$.

The appropriate continuity to require is

(iv) $\lim_{t\to 0} \mathrm{tr}[\rho(T_t^*A - A)] = 0$

for all $\rho \in T(\mathfrak{K}), A \in B(\mathfrak{K})$ (ultraweak continuity). Furthermore, one would like to have (as the notation suggests) T_t^* the dual of a map on $T(\mathfrak{K})$. This is guaranteed by

(v) T_t^* is normal.

This means that if $\mathrm{tr}(\rho A_n) \to \mathrm{tr}(\rho A)$, then also $\mathrm{tr}(\rho T_t^*A_n) \to \mathrm{tr}(\rho T_t^*A)$ (ultraweak continuity of T_t^*).

The example of the previous section suggests a stronger kind of positivity, called complete positivity. The importance of this property has been realized only in recent years (Kraus, 1971; Gorini, Kossakowski, and Sudarshan, 1976; Lindblad, 1976a). Complete positivity makes the mathematical theory really go. Physically it is *not* an additional assumption—at least, if one believes that the system, together with its surroundings, is described by Hamiltonian dynamics. Imagine that there is another system associated with the Hilbert space $\mathfrak{K}_n$ of finite dimension, n, which is regarded as dynamically uncoupled to the system of interest. Then the dynamics of the joint system are described by $T_t^* \otimes 1_{(n)}$ on $B(\mathfrak{K}) \otimes B(\mathfrak{K}_n)$. One would also expect $T_t^* \otimes 1_{(n)}$ to be positive. However, in general, this is not the case. One then calls the map T_t^* *n-positive* if $T_t^* \otimes 1_{(n)}$ is positive, and calls T_t^* *completely positive*, if T_t^* is n-positive for all n. Since one can always imagine the system of interest to be coupled to an n-level system in such a trivial way, it is natural to strengthen (i′) to (i):

(i) T_t^* is completely positive.

For $\dim\mathfrak{K} = 2$, linear maps on $T(\mathfrak{K})$ are 4×4 matrices. In this case, Gorini, Kossakowski, and Sudarshan (1976) parametrized the positive and completely positive semigroups explicitly and found that complete positivity is a considerable restriction.

In passing we note that (i) up to (v) imply that T_t and T_t^* are contractions, i.e.,

$$\|T_t\rho\|_1 \leq \|\rho\|_1 , \quad \|T_t^*A\| \leq \|A\|$$

(i) up to (v) define a *(quantum) dynamical semigroup*.

Two problems have been studied in detail. (a) We know that $T_t = e^{Lt}$. What is the form of L? Can we see directly from the infinitesimal generator L whether or not it generates a dynamical semigroup? (b) Can one say something about ergodic properties of T_t, in particular, about

$$\lim_{t\to\infty} T_t\rho$$

from the knowledge of the generator L?

The first problem was answered by Lindblad (1976a) for dynamical semigroups with a bounded generator (which is equivalent to the norm continuity of the semigroup) and independently by Gorini, Kossakowski, and Sudarshan (1976) for a finite dimensional Hilbert space [cf. also the simplified proof by Paravicini and Zecca (1977)].

Theorem 5.1: $T_t = e^{Lt}$ is a quantum-dynamical semigroup with bounded generator L if and only if

$$L\rho = -i[H,\rho] + \sum_{j\in I} [V_j\rho, V_j^*] + [V_j, \rho V_j^*] , \tag{5.13}$$

where $H = H^*$, $V_j \in B(\mathfrak{K})$, $j \in I$. The decomposition into the two parts in (5.13) is not unique. There are other

[1] For a classical Markov process the single-time distribution evolves by a semigroup. The converse is not true, however. There are examples of non-Markovian processes with the single-time distribution evolving by a semigroup. This situation may occur also for a quantum stochastic process (Lindblad, 1979c).

canonical forms of the generator which are sometimes useful. L^* can always be written

$$L^*(A) = KA + AK^* + \Phi(A) \tag{5.14}$$

with bounded $K \in B(\mathcal{H})$ and bounded completely positive map Φ: $B(\mathcal{H}) \to B(\mathcal{H})$ such that $K + K^* + \Phi(1) = 0$. Another form is the one given by Gorini *et al.* (1976), which is valid for a finite (N-) dimensional Hilbert space. Choose a basis $\{(1/\sqrt{N})\mathbf{1}, F_i \mid i = 1, 2, \ldots, N^2 - 1\}$ in $T(\mathcal{H})$ such that $\mathrm{tr}F_i = 0$, $\mathrm{tr}F_i^* F_j = \delta_{ij}$. Then

$$L\rho = -i[H, \rho] + \frac{1}{2} \sum_{i,j=1}^{N^2-1} c_{ij} \{[F_i, \rho F_j^*\} + [F_i \rho, F_j^*]\}, \tag{5.15}$$

where $H = H^*$ and c_{ij} is a (complex) positive matrix. For a given L, H is uniquely determined by $\mathrm{tr}H = 0$ and the c_{ij}'s are uniquely determined by the choice of the F_i's.

For unbounded generators, L is of the standard form (5.14) on some appropriate domain in all known cases. Davies (1977c) shows that semigroups with a certain extension property are all of the standard form. It should be remarked that, as is well known from the commutative case, for unbounded L, in general, one can construct many dynamical semigroups with L as generator, depending on the choice of the boundary conditions, i.e., on the choice of the domain.

The ergodic properties of dynamical semigroups are interesting and nontrivial extensions of the classical ideas on time-continuous Markov chains. Let us denote by $\{\cdot\}'$ the commutant of the set $\{\cdot\}$ in $B(\mathcal{H})$, e.g., $\{A\}'$ is the set of all operators commuting with A. Then Frigerio (1978) proves the following result, [cf. also Davies (1970), Evans (1977), Frigerio (1977), Watanabe (1978), and Frigerio and Spohn (1978)].

Theorem 5.2: Let $T_t = e^{Lt}$ be a dynamical semigroup with bounded generator L of the form (5.13). If there exists an invariant state $\rho_0, T_t\rho_0 = \rho_0$, with $\overline{\mathrm{Ran}\rho_0} = \mathcal{H}$ (i.e., no zero eigenvalues), then the following two properties are equivalent: (i) ρ_0 is the unique stationary state and (ii) $M(T) = \{H, V_j, V_j^*, j \in I\}' = \{\mathbf{C}1\}$.

If the Hilbert space $\mathcal{H}$ is finite dimensional, then there always exists at least one stationary state. However, it might have zero eigenvalues and it is not known whether Theorem 3.8 applies in that case. For an infinite dimensional Hilbert space it may happen that all initial states move out to "infinity" and that no stationary state exists at all.

Approach to equilibrium is not guaranteed by the existence of a unique stationary state, since purely oscillating solutions may occur. Using in a stronger way the effect of the dissipative term, Frigerio (1978) proves the following result, [cf. also, Davies (1970), Spohn (1976), Spohn (1977a)].

Theorem 5.3: Let $T_t = e^{Lt}$ be a dynamical semigroup with bounded generator of the form (5.13). If T_t has at least one stationary state ρ_0 and if the linear span $\mathrm{lin}\{V_i, i \in I\}$ is a self-adjoint set with $\{V_i, i \in I\}' = \{\mathbf{C}1\}$, then $\overline{\mathrm{Ran}\rho_0} = \mathcal{H}$ and

$$\lim_{t \to \infty} \mathrm{tr}(AT_t\rho) = \mathrm{tr}(A\rho_0) \tag{5.16}$$

for all $A \in B(\mathcal{H})$ and states $\rho \in T(\mathcal{H})$.

If there are several stationary states, their structures can still be analyzed. If $M(T)$ of Theorem 5.2 is not trivial, then there exists a family $\{P_n\}$ of pairwise orthogonal projections with $\sum_n P_n = 1$ such that $M(T) \cap M(T)' = \{P_n\}''$. Two extreme cases together with all their linear combinations may occur. If $\{V_i \mid i \in I\}' = M(T) = \{P_n\}'$, then for all $A \in B(\mathcal{H})$ and all states $\rho \in T(\mathcal{H})$,

$$\lim_{t \to \infty} \mathrm{tr}(AT_t\rho) = \sum_n \mathrm{tr}(\rho P_n A P_n). \tag{5.17}$$

In particular, all states of the form $P_n \rho P_n = \rho$ are stationary. If $\{V_i \mid i \in I\}' = M(T) = \{P_n\}''$, then for all $A \in B(\mathcal{H})$ and states $\rho \in T(\mathcal{H})$,

$$\lim_{t \to \infty} \mathrm{tr}(AT_t\rho) = \sum_n \mathrm{tr}(P_n \rho P_n)\, \mathrm{tr}(\rho_0 P_n A P_n) \times [\mathrm{tr}(P_n \rho_0 P_n)]^{-1}. \tag{5.18}$$

In that case, $P_n \rho_0 P_n / \mathrm{tr}(P_n \rho_0 P_n)$ are stationary states. Note that the asymptotic effect of T_t is a reduction of the wave packet corresponding to the measurement of an observable $B = \sum_n b_n P_n$.

We end this section with some, in a certain sense, exactly soluble dynamical semigroups. Other examples of semigroups will naturally arise in the following section.

For self-adjoint H,

$$\frac{d}{dt}\rho(t) = -\tfrac{1}{2}[H, [H, \rho(t)]] \tag{5.19}$$

has the solution

$$T_t = (2\pi t)^{-1/2} \int_{-\infty}^{\infty} ds\, e^{-s^2/2t} e^{-iHs} \rho e^{iHs}, \tag{5.20}$$

sometimes called Gaussian semigroups. An example of a similar type is

$$\frac{d}{dt}\rho(t) = U^* \rho(t) U - \rho(t) \tag{5.21}$$

with unitary U. Its solution is

$$T_t\rho = \sum_{n=0}^{\infty} \frac{t^n}{n!} e^{-t} U^{*n} \rho U^n. \tag{5.22}$$

Kossakowski (1972) studied in detail dynamical semigroups of this structure.

Another class of "exactly soluble" semigroups has received considerable attention recently (Davies, 1972; Lindblad, 1976b; Evans and Lewis, 1977; Davies, 1977a; Damoen, Vanheuverzwijn, and Verbeure, 1977, 1979; Vanheuverzwijn, 1978; Evans, 1978; Frigerio, Gorini, and Pulé, 1979), and these are now known as quasifree dynamical semigroups. Recall that if for a quantum particle in one dimension the Hamiltonian is quadratic in $x, id/dx$, then the explicit solution is easily found (e.g., free particle, harmonic oscillator, etc.). Interestingly enough, if one adds a dissipative term of the form (5.13) which is quadratic in $x, id/dx$ (the V_i's are linear combinations of x and id/dx), then one can still write down the solution in closed form (Lindblad, 1976b). More generally, if for a Bose or Fermi field the Hamiltonian is quadratic in the creation and annihilation operators, the solution is obtained in terms of unitary transformation on the one-particle space of the

system. If a dissipative term is added which is quadratic in the fields as in (5.13). (the V_i's are linear combinations of the creation and annihilation operators), then the solution is obtained in terms of a bounded semigroup on the one-particle space of the system together with a numerical factor to guarantee positivity. Specific examples of quasifree dynamical semigroups, such as the Bloch equation for spin relaxation and the harmonic oscillator with emission and absorption of one energy quantum only [cf. Haake, 1973], have been used in physical applications for some time.

C. System coupled to a thermal reservoir

1. The weak coupling limit

First I will describe the standard model and then will comment on possible extensions later.

The dynamics of the uncoupled system, associated with the separable Hilbert space $\mathcal{H}$, is given by a self-adjoint Hamiltonian H on $\mathcal{H}$.

The reservoir is assumed to be an ideal Fermi gas in a stationary quasifree state, typically in thermal equilibrium. The construction is well known [see, e.g., Davies, (1976a)]. Let $\mathfrak{A}$ be the CAR algebra over the complex reservoir one-particle Hilbert space $\mathcal{H}_R$. This is the C^* algebra generated by the bounded operators $a(f)$ which are conjugate linearly with respect to $f\in\mathcal{H}_R$ and which satisfy the anticommutation relations

$$\begin{aligned} &a(f)a^*(g)+a^*(g)a(f)=\langle g|f\rangle 1\,,\\ &a(f)a(g)+a(g)a(f)=0 \end{aligned} \tag{5.23}$$

for all $f,g\in\mathcal{H}_R$. A quasifree evolution on $\mathfrak{A}$ is defined by

$$\alpha_t a(f)=a(e^{-iht}f)\,, \tag{5.24}$$

where h is the one-particle Hamiltonian of the reservoir. The formal Hamiltonian of the reservoir is then

$$H_R=\sum_{i,j}\langle f_i|hf_j\rangle a^*(f_i)a(f_j)\,, \tag{5.25}$$

with f_j an orthonormal basis in $\mathcal{H}_R$. Typically, $\mathcal{H}_R=L^2(\mathbb{R}^3,dk)$ and $(e^{-iht}f)(k)=e^{-ik^2t}f(k)$. States on $\mathfrak{A}$ for which all truncated correlation functions of order larger than two vanish are called quasifree. They are defined by

$$\omega_R(a^*(f_m)\cdots a^*(f_1)a(g_1)\cdots a(g_n))=\delta_{nm}\det\{\langle Rf_i|g_j\rangle\}\,, \tag{5.26}$$

where $0\leq R\leq 1$ is the defining operator. The equilibrium state at inverse temperature β and chemical potential μ corresponding to the one-particle Hamiltonian h is given by

$$R=(e^{\beta(h-\mu)}+1)^{-1}\,. \tag{5.27}$$

The dynamics of the uncoupled reservoir are assumed to be the quasifree evolution α_t, and the initial state ω_R of the reservoir is assumed to be stationary and quasifree. This implies that the defining operator R has to satisfy

$$e^{-iht}Re^{iht}=R\,, \tag{5.28}$$

which, in particular, is true for the thermal equilibrium state (5.27). The system-reservoir interaction is of the form

$$Q\otimes\Phi\,, \tag{5.29}$$

where $\Phi=a^*(f)+a(f)$ for some $f\in\mathcal{H}_R$ and $Q=Q^*\in B(\mathcal{H})$. Then the average force on the system vanishes

$$\omega_R(\Phi)=0\,. \tag{5.30}$$

Let us assume that the coupling between the system and reservoir is weak, i.e., $\varepsilon Q\otimes\Phi$. Then, as pointed out already for the harmonic lattice, there are two types of weak coupling limits. The first one treats the motion of the system on the order one and the dissipation on the order ε^2, whereas the second one scales in such a way that the uncoupled motion of the system and dissipation are of the same order of magnitude. The limits correspond to the scaling

$$H^\varepsilon=H\otimes 1+\varepsilon Q\otimes\Phi+1\otimes H_R \tag{5.31}$$

and

$$H^\varepsilon=\varepsilon^2H\otimes 1+\varepsilon Q\otimes\Phi+1\otimes H_R\,. \tag{5.32}$$

Let

$$h(t)=\omega_R(\Phi\alpha_t\Phi)\,, \tag{5.33}$$

which is a time-dependent equilibrium correlation function of the reservoir. The crucial condition for the existence of the weak coupling limit (in either form) is

$$\int_0^\infty dt\,|h(t)|\,(1+t)^\delta<\infty \tag{5.34}$$

for some $\delta>0$. For the usual free dynamics in three dimensions the typical fall-off of an equilibrium correlation function is $t^{-3/2}$, so (5.34) is satisfied. Equation (5.34) shows that in order for the weak coupling limit to exist, the reservoir has to be infinitely extended. (This is a necessary, but certainly not a sufficient condition.) For a spatially confined system (5.34) cannot be satisfied.

Let us first consider the scaling (5.31). If initially system and reservoir are uncorrelated, then the reduced dynamics T_t^ε is defined by

$$\mathrm{tr}(AT_t^\varepsilon\rho)=(\rho\otimes\omega_R)(\alpha_t^\varepsilon A\otimes 1) \tag{5.35}$$

for all $A\in B(\mathcal{H})$, $\rho\in T(\mathcal{H})$, where α_t^ε is the dynamics of the coupled system corresponding to the Hamiltonian (5.31).

Since the dissipation is of order ε^2, the evolution has to be studied on a time scale ε^{-2}.

Theorem 5.4 (Davies, 1976b): Under the assumption (5.34),

$$\lim_{\varepsilon\to 0}\sup_{0\leq\varepsilon^2t\leq\tau}\|T_t^\varepsilon\rho-e^{(-i[H,\cdot]+\varepsilon^2K)t}\rho\|_1=0 \tag{5.36}$$

for all $\rho\in T(\mathcal{H})$, where

$$K\rho=\int_0^\infty dt\{h(t)^*[Q(-t),\rho Q]+h(t)[Q\rho,Q(-t)]\}\,, \tag{5.37}$$

with $Q(t)=e^{itH}\rho e^{-ith}$ and $e^{-i[H,\cdot]t}\rho=e^{-iHt}\rho e^{iHt}$.

In parentheses, it should be remarked that the usual second-order perturbation theory applied to the differential form of the equations of motion leads to

$$K_1\rho=\int_0^\infty dt\{h(t)^*[Q,e^{itH}\rho Qe^{-itH}]+h(t)[e^{itH}Q\rho e^{-itH},Q]\} \tag{5.38}$$

[cf. Haake (1973), Agarwal (1973)]. In general, the semigroup $\exp\{(-i[H,\cdot\,]+\varepsilon^2K_1)t\}$ neither preserves positivity nor satisfies an approximation as (5.36) (Dümcke and Spohn, 1979).

From simple examples it can be seen that the semigroup $\exp\{(-i[H,\cdot\,]+\varepsilon^2K)t\}$ does not preserve positivity, in general, and therefore is not a candidate for a bona fide limiting dynamics. A way out is to study the dissipation relative to the uncoupled motion of the system. Let $T_t^0\rho=e^{-it[H,\cdot\,]}\rho$. Then one has to investigate the limit of $T^0_{-\varepsilon^{-2}t}T^{\varepsilon}_{\varepsilon^{-2}t}$ as $\varepsilon\to 0$. From (5.36),

$$T^0_{-\varepsilon^{-2}t}e^{(-i\varepsilon^{-2}[H,\cdot\,]+K)t}\rho=\rho+\int_0^t ds\, T^0_{-\varepsilon^{-2}s}KT^0{}_{\varepsilon^{-2}s}T^0_{-\varepsilon^{-2}s}\times e^{(-i\varepsilon^{-2}[H,\circ\,]+K)s}\rho\,. \tag{5.39}$$

Therefore one expects that

$$\lim_{\varepsilon\to 0}T^0_{-\varepsilon^{-2}t}e^{(-i\varepsilon^{-2}[H,\cdot\,]+K)t}\rho=e^{K^{\natural}t}\rho\,, \tag{5.40}$$

with

$$K^{\natural}\rho=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt\,T^0_{-t}KT^0_t\rho\,. \tag{5.41}$$

Indeed, if the spectrum of H is discrete, then Theorem 5.4 holds with K replaced by $K^{\natural}$, equivalently, for $t\geq 0$,

$$\lim_{\varepsilon\to 0}T^0_{-\varepsilon^{-2}t}T^{\varepsilon}_{\varepsilon^{-2}t}\rho=e^{K^{\natural}t}\rho \tag{5.42}$$

in trace norm.

In general, the limit (5.41) exists only if the spectrum of H is discrete. So, a Markovian limit, where free motion and dissipation are on different scales, is a somewhat delicate object, to say the least, if the spectrum of H is continuous [cf. Davies (1978) for an illuminating discussion].

From (5.42) it is clear that $e^{K^{\natural}t}$ is completely positive and that $\{e^{K^{\natural}t}\,|\,t\geq 0\}$ is a dynamical semigroup. It is not hard to compute $K^{\natural}$ explicitly. Let $H=\sum_n\varepsilon_nP_n$ be the spectral decomposition of H, let

$$Q(\omega)=\sum_{\varepsilon_m-\varepsilon_n=\omega}P_nQP_m \tag{5.43}$$

and

$$\int_0^\infty dt\,e^{i\omega t}h(t)=\tfrac{1}{2}\hat{h}(\omega)+is(\omega)\,, \tag{5.44}$$

with real $\hat{h}(\omega)$, $s(\omega)$. Then

$$K^{\natural}\rho=\sum_{\omega}\{-is(\omega)[Q^*(\omega)Q(\omega),\rho]+\hat{h}(\omega)([Q(\omega)\rho,Q(\omega)^*]+[Q(\omega),\rho Q(\omega)^*])\}\,. \tag{5.45}$$

The ω sum runs over all energy differences. Since $\hat{h}(\omega)\geq 0$ as the Fourier transform of a two-point function, (5.45) is seen to be of the general form (5.15).

The time evolution generated by $K^{\natural}$ is well understood. Let us assume that the reservoir is in thermal equilibrium at inverse temperature β, that $\hat{h}(\omega)>0$, and that $e^{-\beta H}$ is of trace class. If the system Hamiltonian H and the coupling operator Q are sufficiently incompatible, i.e., if $\{H,Q\}'=\{\mathbf{C}1\}$, then as $t\to\infty$, every initial state approaches the canonical equilibrium state $e^{-\beta H}/\mathrm{tr}e^{-\beta H}$ (Spohn, 1977a). In the energy representation off-diagonal elements of a statistical operator decay exponentially with oscillations superimposed. If the spectrum H is nondegenerate, then the statistical operators diagonal in the energy representation form a set of states invariant under $e^{K^{\natural}t}$. On these states $e^{K^{\natural}t}$ induces a classical Markov jump process which is governed by the Pauli master equation

$$\frac{d}{dt}p_n(t)=\sum_m\{W_{mn}p_m(t)-W_{nm}p_n(t)\}\,, \tag{5.46}$$

where $W_{nm}=\mathrm{tr}(P_mQP_nQ)\hat{h}(\varepsilon_m-\varepsilon_n)$, i.e., the transition probabilities per unit time are given by the golden rule. (5.46) implies that the diagonal elements of a statistical operator reach their equilibrium values exponentially fast.

There are a number of straightforward extensions. The interaction term may be a finite sum $\sum_i Q_i\otimes\Phi_i$ and Φ could be a product of field operators. The reservoir could be chosen as a quasifree Bose system. The system may be coupled to several reservoirs at different temperatures simultaneously (Spohn and Lebowitz, 1978). The generator is then a sum of generators, one for each reservoir. A much less simple task is the case of a reservoir which is not quasifree. A particular model has been treated by Davies and Eckmann (1975). The weak coupling limit can be extended to a time-dependent system Hamiltonian (Davies and Spohn, 1978).

Let us return to the scaling (5.32), where the free motion of the system and the dissipation are of the same order of magnitude, namely, ε^2. One should immediately go over to the rescaled time by $t_\varepsilon=\varepsilon^{-2}t$. Let T_t^{ε} be the reduced dynamics defined as in (5.35) but with H scaled as

$$H^{\varepsilon}=H\otimes 1+\varepsilon^{-1}Q\otimes\Phi+\varepsilon^{-2}1\otimes H_R\,. \tag{5.47}$$

Theorem 5.5 (Palmer, 1976b): If H is bounded, and if $h(t)$ satisfies (5.34), then

$$\lim_{\varepsilon\to 0}T_t^{\varepsilon}\rho=e^{Lt}\rho\,, \tag{5.48}$$

the limit being understood in trace norm, where

$$L=-i[H,\rho]+is[Q^2,\rho]+h[Q,[Q,\rho]]\,, \tag{5.49}$$

with $\int_0^\infty dt\,h(t)=h+is$, h,s real.

Note that the spectrum of H may be continuous. The dissipative part of L is independent of H. This is analogous to what we found for the Lorentz and Rayleigh gas. The friction acts *locally*, in the sense that the dissipative part acts only on states in support of Q. This should be contrasted with $K^{\natural}$, which does not act locally, since it is intertwined with H in a complicated way.

The higher-order correlation functions follow the expected pattern (R. Dümcke, oral communication). For the scaling (5.47), (5.8) is valid under the same assumptions as Theorem 5.5. For the scaling (5.31) one has to remember that the Markov approximation is taken relatively to the uncoupled motion of the system. So the measurement $\rho\to A\rho A^*$ at time t should be replaced by $\rho\to A(-t)\rho A(-t)^*$, where $A(t)=e^{iHt}Ae^{-iHt}$ is A evolved under the uncoupled system dynamics. This then leads to

$$\lim_{\varepsilon\to 0}\mathrm{tr}[(A_n(-\varepsilon^{-2}t_n)\otimes 1)U^\varepsilon(\varepsilon^{-2}(t_n-t_{n-1}))\cdots U^\varepsilon(\varepsilon^{-2}(t_2-t_1))(A_1(-\varepsilon^{-2}t_1)\otimes 1)U^\varepsilon(\varepsilon^{-2}t_1)(\rho\otimes\omega_R)U^\varepsilon(\varepsilon^{-2}t_1)^*$$
$$\times(A_1(-\varepsilon^{-2}t_1)\otimes 1)^*U^\varepsilon(-\varepsilon^{-2}(t_2-t_1))^*\cdots U^\varepsilon(\varepsilon^{-2}(t_n-t_{n-1}))^*(A_n|-\varepsilon^{-2}t_n)\otimes 1)^*]$$
$$=\mathrm{tr}_S[A_n(e^{K^\dagger(t_n-t_{n-1})}\cdots(e^{K^\dagger(t_2-t_1)}(A_1(e^{K^\dagger t_1}\rho)A_1^*))\cdots)A_n^*], \quad (5.50)$$

provided that the spectrum of H is discrete and that (5.34) holds.

2. The singular coupling limit

The singular coupling limit was used by Hepp and Lieb (1973c), in their work on the laser, and studied in detail by Gorini and Kossakowski (1976) and by Frigerio and Gorini (1976).

The system + reservoir Hamiltonian is assumed to be of the form

$$H^\varepsilon = H\otimes 1 + Q\otimes\Phi(f^\varepsilon) + 1\otimes H_R\,, \quad (5.51)$$

where the reservoir is a quasifree Fermi (or Bose) field, and where $\Phi(f^\varepsilon)=a^*(f^\varepsilon)+a(f^\varepsilon)$, as before. With an appropriate scaling of f^ε and, possibly, of the reservoir state ω_R^ε, one can let the two-point function

$$\omega_R^\varepsilon(\Phi(f^\varepsilon)\alpha_t\Phi(f^\varepsilon)) \quad (5.52)$$

converge to a δ-function at $t=0$ as $\varepsilon\to 0$. In this *singular coupling limit* one expects the motion of the system to be Markovian.

Under the scaling (5.47), the reservoir two-point function

$$\varepsilon^{-2}\omega_R(\Phi\alpha_{\varepsilon^{-2}t}\Phi) \quad (5.53)$$

tends to a δ function at $t=0$. Therefore, for this kind of weak coupling limit precisely the phenomenon expected for the singular coupling limit happens. Indeed, Palmer (1976b) has shown that scaling H^ε as in (5.51) is equivalent to scaling the Hamiltonian in the form (5.47). Therefore in the limit as $\varepsilon\to 0$ one obtains the evolution (5.48).

3. N-level system in a low-density gas

This problem has been studied by Palmer (1976a) in his thesis. The analysis is not quite complete, and properties of the limiting Markovian dynamics remain to be investigated.

One considers an N-level system immersed in an infinite ideal Fermi gas. Collisions between the system and the gas particles are assumed to be elastic

$$H = H_S\otimes 1 + Q\otimes\{a^*(f_1)a(f_{-1})+a^*(f_{-1})a(f_1)\}+1\otimes H_R\,. \quad (5.54)$$

$f_1, f_{-1}\in L^2(\mathbf{R}^3,dx)$ is the one-particle space of the Fermi gas. Initially the reservoir is in thermal equilibrium at fugacity $z_\varepsilon=\varepsilon^2$ and inverse temperature β. Note that the interaction conserves the reservoir particle number.

The idea is to use the representation space of the thermal equilibrium state, i.e., to work in the Hilbert space provided by the GNS construction. Let $\mathfrak{F}_1,\mathfrak{F}_2$ be two copies of the fermion Fock space over $L^2(\mathbf{R}^3,dx)$ with annihilation operators a_1, a_2 and vacuum vectors Ω_1, Ω_2. Let N_i be the number operator and $(-1)^{N_i}=\theta_i$, $i=1,2$. Then it is known (Araki and Wyss, 1964; Balslev, Manuceau, and Verbeure, 1968; Balslev and Verbeure, 1968) that the representation space is $\mathfrak{F}_1\otimes\mathfrak{F}_2$. The equilibrium state is represented by $\Omega_1\otimes\Omega_2$ and the annihilation operator by

$$a(f)=a_1(T_1f)\otimes 1+\theta_1\otimes a_2^*(T_2f)\,, \quad (5.55)$$

with some bounded linear operators T_1, T_2 depending on z_ε and β. On $\mathfrak{F}_1\otimes\mathfrak{F}_2$, H is represented by H^ε which depends on ε through T_1 and T_2. By expanding in ε, one obtains

$$H^\varepsilon = H_S\otimes 1+Q\otimes(A+\varepsilon B+\varepsilon^2C)+1\otimes H_R+O(\varepsilon^3)\,, \quad (5.56)$$

where A, B, C, and H_R are bounded operators acting on $\mathfrak{F}_1\otimes\mathfrak{F}_2$. The initial state of the reservoir is $\Omega_1\otimes\Omega_2$. Dropping the term of order ε^3 one is back to a weak coupling type of problem. Palmer shows that the reduced dynamics of the system with H^ε as in (5.56) [with the term $O(\varepsilon^3)$ dropped] has a limit as $\varepsilon\to 0$ on the time scale $\varepsilon^{-2}t$.

4. The polaron

The polaron (Fröhlich, 1954; Feynman, 1955; Feynman and Hibbs, 1965) is an electron moving in an ideal solid, which is represented as a Bose field (with one component for simplicity) over the lattice $\mathbf{Z}^d$. The formal Hamiltonian of the polaron reads

$$H^\varepsilon = -\frac{1}{2m}\Delta+\varepsilon^2E\cdot x+\varepsilon\sum_{n\in\mathbf{Z}^d}V(x-n)(a^*(n)+a(n))$$
$$+\omega\sum_{n\in\mathbf{Z}^d}a^*(n)a(n)\,. \quad (5.57)$$

The Hilbert space of the electron is $L^2(\mathbf{R}^d,dx)$. Δ stands for the Laplacian. m is the mass of the electron. E is a constant external electric field. $V(x)$ is a smooth, rapidly decreasing interaction potential. $a^*(n), a(n)$ stands for the creation and annihilation of the nth oscillator. Note that we simply have a lattice of independent oscillators (which might pass as a model for the optical mode). So far, the analysis has not been extended to a harmonic solid with couplings.

Let T_t^ε be the reduced dynamics (in the Heisenberg) picture defined by

$$T_t^\varepsilon A=\omega_\beta(e^{itH^\varepsilon}(A\otimes 1)e^{-itH^\varepsilon}) \quad (5.58)$$

for any bounded operator A on $L^2(\mathbf{R}^d,dx)$ with ω_β the thermal equilibrium state of the ideal solid at inverse temperature β. (T_t^ε may be defined by first restricting the interaction to a finite sum over the bounded region $\Lambda\subset\mathbf{Z}^d$ and then taking the infinite volume limit $\Lambda\to\mathbf{Z}^d$.)

We want to study T_t^ε for small coupling $\varepsilon\to 0$. Note that this problem is quite different from the ones we had before. The electron is allowed to move in the whole d-dimensional space (the spectrum of its Hamiltonian is continuous) and interacts with the reservoir

at infinitely many different points. The results mentioned before cannot be applied to the polaron.

Theorem 5.6 (Spohn, 1977b): Let $C_0(\mathbf{R}^d)$ be the Banach space of bounded continuous functions on $\mathbf{R}^d$ vanishing at infinity equipped with the sup-norm. On $C_0(\mathbf{R}^d)$ let e^{At}, $t \geq 0$, be the semigroup generated by

$$(Af)(p) = E\frac{\partial}{\partial p} f(p) + \int K(dp'|p)\{f(p') - f(p)\}, \qquad (5.59)$$

where the kernel $K(dp'|p)$ is defined by

$$K(dp'|p) = |\hat{V}(p-p')|^2 \frac{1}{2\omega}\left[(1-e^{-\beta})^{-1}\delta\left(\frac{1}{2m}p^2 - \frac{1}{2m}p'^2 + \omega\right)\right.$$
$$\left. + (e^{\beta}-1)^{-1}\delta\left(\frac{1}{2m}p^2 - \frac{1}{2m}p'^2 - \omega\right)\right]. \qquad (5.60)$$

$\hat{V}$ is the Fourier transform of V. e^{At} is a contraction and preserves positivity.

Let p stand for the momentum operator. For $f \in C_0(\mathbf{R}^d)$, let $f(p), (e^{At}f)(p)$ be regarded as bounded operators on $L^2(\mathbf{R}^d, dx)$. If $d \geq 3$, then there exists a $t_0 > 0$ such that for all $0 \leq t \leq t_0$:

$$\lim_{\varepsilon \to 0} T^{\varepsilon}_{\varepsilon^{-2}t} f(p) = (e^{At}f)(p) \qquad (5.61)$$

weakly as operators on $L^2(\mathbf{R}^d, dx)$.

Theorem 5.6 proves only the convergence of the momentum distribution. It is an open problem how to scale and how to prove convergence of the spatial distribution (or of the whole statistical operator).

The polaron illustrates a typical quantum effect. Classically, one obtains a diffusion equation in the weak coupling limit (cf. Secs. II.A.1 and II.A.2). Quantum mechanically, however, in the limit, the motion is described by a jump process: The electron interacts in a unit time interval weakly with many oscillators. Mostly nothing happens. But once in a while a *finite* quantum of energy is transmitted causing a finite change in the momentum. [The two terms in (5.60) correspond to the emission and the absorption of one energy quantum.]

I should comment somewhat on the technique of proving Theorem 5.6. For the weak and singular coupling limits described before, one always uses the sufficiently fast decay of the reservoir correlation functions in order to control the limit. For the polaron the estimates rely on the spreading of the freely moving wave packet, which is on the order $t^{-d/2}$. The restriction $d \geq 3$ comes then from the fact that quantities as $\langle \varphi | e^{i\Delta t}\psi \rangle$ should be integrable as a function of time.

Instead of (5.57) one can study the motion of an electron through random impurities with

$$H^{\varepsilon} = -\frac{1}{2m}\Delta + \varepsilon^2 E\cdot x + \varepsilon \sum_{n \in \mathbf{Z}^d} v(n) V(x-n), \qquad (5.62)$$

where $v(n)$ is a Gaussian random field with mean zero and rapidly decreasing covariance $\langle v(n)v(m)\rangle = g(n-m)$. For this model Martin and Emch (1975) investigated $e^{-iH^{\varepsilon}t}\psi$ in the weak coupling limit. Their ideas were used by Spohn (1977b) to extend the results to the form of Theorem 5.6.

D. Effective Hamiltonians

The mean-field limit (cf. Sec. II.A.5) is an example of the stochastic motion's becoming deterministic in the Markovian limit. Here we take up this theme again, which is of independent interest in quantum mechanics, since it allows one to obtain the static (potential) interaction between quantum mechanical particles from a field theoretical model (Davies, 1979a).

One considers two nonrelativistic particles with Hamiltonian

$$H_1 + H_2 = -(1/2m_1)\Delta_1 - (1/2m_2)\Delta_2 \qquad (5.63)$$

on $L^2(\mathbf{R}^3, dx_1) \otimes L^2(\mathbf{R}^3, dx_2)$. They interact through a free boson field with mass $m \geq 0$. The field Hamiltonian F acts on the one-particle space as

$$(F\psi)(k) = (m^2 + k^2)^{1/2}\psi(k). \qquad (5.64)$$

The particles and the field are coupled by

$$A = \sum_{j=1}^{2} \int_{\mathbf{R}^3} dx\, V_j(x_j - x) \otimes \{a^*(f_x) + a(f_x)\}. \qquad (5.65)$$

Here $a^*(f)$, $a(f)$ are the boson field operators and $f_x(k) = f(k)e^{-ikx}$ is a suitable chosen test function shifted by x.

One assumes that the coupling to the field is weak and that the particles are very heavy. So the properly scaled total Hamiltonian reads

$$H^{\varepsilon} = \varepsilon^{-2}(\varepsilon^2 H_1 + \varepsilon^2 H_2 + \varepsilon A + F), \qquad (5.66)$$

which is just of the form encountered before. In this static limit the interaction of the two particles is taken into account by an effective potential V. If W denotes the two-point function of the field in the vacuum state Ω,

$$W(x-y) = \langle \Omega | \{a^*(fy) + a(fy)\} F^{-1} \{a^*(f_x) + a(f_x)\}\Omega \rangle, \qquad (5.67)$$

then, up to a constant, the effective potential is given by

$$V(x_1 - x_2) = -\frac{1}{4\pi^3}\int_{\mathbf{R}^6} dx\, dy\, \hat{V}_1(x_1 - x) W(x-y) V_2(x_2 - y). \qquad (5.68)$$

So, in the limit, one expects the dynamics of the particle to be governed by

$$\bar{H} = -(1/2m_1)\Delta_1 - (1/2m_2)\Delta_2 + V(x_1 - x_2). \qquad (5.69)$$

Indeed, Davies (1979a) proves that for all

$$\psi \in L^2(\mathbf{R}^3, dx_1) \otimes L^2(\mathbf{R}^3, dx_2),$$
$$\lim_{\varepsilon \to 0} e^{-itH^{\varepsilon}}\psi \otimes \Omega = (e^{-it\bar{H}}\psi) \otimes \Omega. \qquad (5.70)$$

E. The mean-field limit for interacting quantum systems

The standard model for a mean-field limit is a lattice spin system, where each pair of spins interacts with the same strength independently of their relative location. The strength of the interaction decreases proportionately to the size of the system. This system can also be

thought of as a usual interacting particle system disregarding statistics, where the strength of the interaction decreases in inverse proportion to the number of particles. We will show that, if the initial state is a product state, then in the limit it will remain a product state. The single-site dynamics are given by a nonlinear equation of Hartree type, i.e., by an effective Hamiltonian which itself depends linearly on the state of the single site at time t.

On the other hand, as with the physical analogue of the classical system discussed in Sec. III.D, one may consider either bosons or fermions interacting by a pair potential. If by appropriate scaling, the mutual interaction between any pair of particles can be neglected, but the "collective" interaction is retained, then an initially quasifree state will stay quasifree. The one-particle density (two-point function) is governed by the Hartree-Fock equation including the exchange term. The convergence of the mean-field limit with statistics included is an open problem.

Let us first consider the mean-field limit for the lattice model. Since there is no geometry, the sites are labeled $1,\ldots,N$. At each site sits a quantum-mechanical system associated with the Hilbert space $\mathfrak{K}$. The Hilbert space at site j is denoted by $\mathfrak{K}_j=\mathfrak{K}$. As usual, the quantum system at site j is called the spin at site j, although $\mathfrak{K}$ need not to be finitely dimensional. (We will give an example of more complicated single-site structure later on.) Let A be a bounded self-adjoint operator on $\mathfrak{K}$ and V a bounded self-adjoint operator on $\mathfrak{K}\otimes\mathfrak{K}$. Let $A_j=A$ be considered an operator acting on $\mathfrak{K}_j$ and $V_{ij}=V$ an operator acting on $\mathfrak{K}_i\otimes\mathfrak{K}_j$, $i\neq j$. Then for a system with N lattice sites the Hamiltonian is given by

$$H_N=\sum_{j=1}^{N}A_j+\frac{1}{N}\sum_{i\neq j=1}V_{ij}. \tag{5.71}$$

Each pair of spin interacts via V. In order to have $H_N\approx N$, the strength of the interaction is scaled as $1/N$.

Let $\mathrm{tr}_{[n,N]}$ denote the partial trace over the Hilbert spaces labeled by $n,n+1,\ldots,N$, and tr_n the partial trace over the nth Hilbert space.

Theorem 5.7: If the initial state of the system is a product state

$$\rho_N=\rho\otimes\cdots\otimes\rho\quad(N\text{ times}), \tag{5.72}$$

with $\rho\in T(\mathfrak{K})$, then

$$\lim_{N\to\infty}\mathrm{tr}_{[n+1,N]}(e^{-iH_Nt}\rho_N e^{iH_Nt})=\rho(t)\otimes\cdots\otimes\rho(t), \tag{5.73}$$

in trace norm on $T(\otimes_{j=1}^{n}\mathfrak{K}_j)$ for all n. $\rho(t)\in T(\mathfrak{K})$ is the solution of the Hartree equation

$$i\frac{d}{dt}\rho(t)=[A,\rho(t)]+\mathrm{tr}_2\{[V_{12}+V_{21},\rho(t)\otimes\rho(t)]\}, \tag{5.74}$$

with initial condition ρ.

Proof: Let $\rho^{(N)}\in T(\otimes_{j=1}^{N}\mathfrak{K}_j)$ be invariant under permutations of the labeling and let

$$\rho_n^{(N)}=\mathrm{tr}_{[n+1,N]}\rho^{(N)}. \tag{5.75}$$

Then $e^{-iH_Nt}\rho^{(N)}e^{iH_Nt}$ is also invariant under permutations of the labeling and $\rho_n^{(N)}(t)=\mathrm{tr}_{[n+1,N]}(e^{-iH_Nt}\rho^{(N)}e^{iH_Nt})$ satisfies the differential equation

$$i\frac{d}{dt}\rho_n^{(N)}(t)=\left(\sum_{j=1}^{n}A_j+\frac{1}{N}\sum_{i\neq j=1}^{n}V_{ij},\rho_n^{(N)}(t)\right)+\left(\frac{N-n}{N}\right)\sum_{j=1}^{n}\mathrm{tr}_{n+1}\{[V_{j\,n+1}+V_{n+1\,j},\rho_{n+1}^{(N)}(t)]\}, \tag{5.76}$$

which is written more compactly as

$$\frac{d}{dt}\rho_n^{(N)}(t)=-i[H_n^{(N)}(t),\rho_n^{(N)}(t)]+\left(\frac{N-n}{N}\right)C_{n,n+1}\rho_{n+1}^{(N)}(t), \tag{5.77}$$

defining the n-particle Hamiltonian $H_n^{(N)}$ and the collision operator $C_{n,n+1}$.

Let $S_n^{(N)}(t)\rho_n=e^{-iH_n^{(N)}t}\rho_n e^{iH_n^{(N)}t}$. Then the formal time-dependent perturbation series reads

$$\rho_n^{(N)}(t)=\sum_{m=0}^{N-n}\int_{0\le t_m\le\cdots\le t_1\le t}dt_m\cdots dt_1 S_n^{(N)}(t-t_1)\times\frac{N-n}{N}C_{n_1n+1}\cdots\frac{N-n-m+1}{N}C_{n+m-1,n+m}\times S_{n+m}^{(N)}(t_m)\rho_{n+m}^{(N)}. \tag{5.78}$$

Let $\|\cdot\|_n$ denote the trace norm on $T(\otimes_{j=1}^{n}\mathfrak{K}_j)$. Since $S_n^{(N)}(t)$ preserves the $\|\cdot\|_n$ norm, the mth term of the series (5.78) is bounded by

$$(t^m/m!)n(n+1)\cdots(n+m-1)(4\|V\|)^m\|\rho_{n+m}^{(N)}\|_{n+m}. \tag{5.79}$$

If one assumes

(C1) $\|\rho_n^{(N)}\|_n\le a^n$,

then the series (5.78) converges in trace norm for $|t|\le t_0$ with $t_0<4\|V\|a$.

Let $S_n(t)\rho_n=e^{-iH_nt}\rho_n e^{iH_nt}$, with $H_n=\sum_{j=1}^{n}A_j$. Clearly, $S_n^{(N)}(t)\rho_n$ converges to $S_n(t)\rho_n$ in trace norm as $N\to\infty$. If one assumes

(C2) $\lim_{N\to\infty}\|\rho_n^{(N)}-\rho_n\|_n=0$,

then $\rho_n^{(N)}(t)$ converges as $N\to\infty$ in trace norm to

$$\rho_n(t)=\sum_{m=0}^{\infty}\int_{0\le t_m\le\cdots\le t_1\le t}dt_1\cdots dt_m S_n(t-t_1)\times C_{n,n+1}\cdots C_{n+m-1,n+m}S_{n+m}(t_m)\rho_{n+m} \tag{5.80}$$

for $|t|\le t_0$.

Let $\rho^{(N)}$ be a statistical operator. Then $\|\rho_n^{(N)}(t)\|_n=\|\rho_n^{(N)}\|_n$ by preservation of positivity and trace. Therefore, if for the initial state the bound (C1) is satisfied, it remains valid for all times, and the argument just given can be iterated to prove convergence of $\rho_n^{(N)}(t)$ as $N\to\infty$ for all times.

One checks that for the particular initial state ρ_N the conditions (C1) and (C2) are satisfied with $\rho_n=\rho\otimes\cdots\otimes\rho$. Differentiating (5.80) with respect to t, one obtains the limiting hierarchy equations

$$i\frac{d}{dt}\rho_n(t)=\left(\sum_{j=1}^{n}A_j,\rho_n(t)\right)+\sum_{j=1}^{n}\mathrm{tr}_{n+1}\{[V_{j\,n+1}+V_{n+1\,j},\rho_{n+1}(t)]\}. \tag{5.81}$$

For these the product property is preserved in time■

An example of Theorem 5.7 is the strong coupling BCS model. In the quasispin formulation the Hamiltonian reads

$$H_N = \varepsilon \sum_{j=1}^{N} \sigma_j^3 - \frac{\mu}{N} \sum_{i \neq j=1}^{N} \sigma_i^+ \sigma_j^- . \tag{5.82}$$

σ^3, σ^+, σ^- are the Pauli spin matrices on $\mathbf{C}^2$ satisfying $\{\sigma^+, \sigma^-\} = 1$, $\{\sigma^+, \sigma^+\} = 0 = \{\sigma^-, \sigma^-\}$, $\sigma^3 = [\sigma^+, \sigma^-]$. Therefore we have the identification

$$\mathcal{H} = \mathbf{C}^2, \quad A = \varepsilon\sigma^3, \quad V = \tfrac{1}{2}(\sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+) .$$

The Hartree equation then becomes

$$i \frac{d}{dt} \rho(t) = \varepsilon [\sigma^3, \rho(t)] + [\mathrm{tr}\{\sigma^+ \rho(t)\}\sigma^- + \mathrm{tr}\{\sigma^- \rho(t)\}\sigma^+, \rho(t)] . \tag{5.83}$$

From a somewhat different point of view this model has been studied by Thirring and Wehrl (1967), Thirring (1968), and van Hemmen (1978). Buffet and Martin (1978) studied the BCS model, where, in addition, each spin is weakly coupled to its own thermal reservoir. This leads to an interesting dynamical behavior in the limit as $N \to \infty$.

Another extensively studied example is the Dicke-Haken-Lax laser model. Strictly speaking, this model is not covered by Theorem 5.7, since A and V are unbounded. Without reservoirs, in the rotating wave approximation, the Hamiltonian reads

$$H_N = \nu \sum_{j=1}^{N} a_j^* a_j + \varepsilon \sum_{j=1}^{N} \sigma_j^3 + \frac{\lambda}{N} \sum_{i \neq j=1}^{N} (a_i^* \sigma_j^- + a_i \sigma_j^+) . \tag{5.84}$$

a^*, a are the creation and annihilation operator of a harmonic oscillator acting on $\mathcal{H}_{\mathrm{osc}}$ (one boson). They satisfy $[a, a^*] = 1$. Therefore we have the following identification:

$$\mathcal{H} = \mathcal{H}_{\mathrm{osc}} \otimes \mathbf{C}^2, \quad A = \nu a^* a + \varepsilon\sigma^3, \quad V = \lambda(a^*\sigma^- + a\sigma^+) .$$

The Hartree equation then becomes

$$\frac{d}{dt} \rho(t) = -i[\nu a^* a + \varepsilon\sigma^3, \rho(t)] - i\lambda[\mathrm{tr}\{a^* \rho(t)\}\sigma^- + \mathrm{tr}\{a\rho(t)\}\sigma^+, \rho(t)] - i\lambda[\mathrm{tr}\{\sigma^- \rho(t)\}a^* + \mathrm{tr}\{\sigma^+ \rho(t)\}a, \rho(t)] . \tag{5.85}$$

To take losses of the radiation field and the pumping of the two-level atoms into account, one couples each photon mode and each spin to its own reservoir. We perform now *first* the limit $N \to \infty$. Then the single-site dynamics are governed by the Hartree equation (5.85) with the term $-i[H_{SR} + 1 \otimes H_R, \rho(t)]$ added due to the coupling to the reservoir (see Sec. V.C). Then the singular coupling limit is taken as described in Secs. V.C.1 and V.C.2. The nonlinearity of the uncoupled system motion does not enter the dissipative part. In the limit the following terms have to be added to the Hartree equation (5.85):

$$\kappa([a, \rho(t)a^*] + [a\rho(t), a^*]) + \gamma(\tfrac{1}{2} + \eta)([\sigma^+, \rho(t)\sigma^-] + [\sigma^+\rho(t), \sigma^-]) + \gamma(\tfrac{1}{2} - \eta)([\sigma^-, \rho(t)\sigma^+] + [\sigma^-\rho(t), \sigma^+]) . \tag{5.86}$$

The first term drives the photon mode to its ground state with rate κ^{-1}. The two other terms pump the spin to

$$\rho_{st} = (\tfrac{1}{2} + \eta)\sigma^+\sigma^- + (\tfrac{1}{2} - \eta)\sigma^-\sigma^+ .$$

Averaging (5.85) and (5.86) over a, a^*, σ^-, σ^+, σ^3 (and possibly also a^*a), one obtains the laser equations as a closed set of first-order differential equations. It should be noted that the solutions of these equations have a very complicated structure. For a suitable choice of parameters the laser equations are, after some rescaling and subtracting out of phase factors, equivalent to the equations of the Lorenz system (Haken, 1975). [Lorenz (1963) approximated the hydrodynamic equations for convection in a fluid layer which is heated from below and cooled from above. He obtained a set of three first-order (in time) ordinary differential equations with a quadratic nonlinearity. Although rather innocent looking, these equations display many nontrivial and hardly understood mathematical phenomena. Recently, the Lorenz system has been studied extensively, [cf. Lanford (1976b) for a survey.]

A third example is an interacting particle system with Hamiltonian

$$H_N = -\sum_{j=1}^{N} \frac{1}{2m} \Delta_j + \frac{1}{2} \sum_{i \neq j=1} V(x_i - x_j) , \tag{5.87}$$

acting on $L^2(\Lambda^N)$ for some region $\Lambda \subset \mathbf{R}^d$. For the Laplacian, suitable boundary conditions at $\partial\Lambda$ are assumed. If V is bounded, then the proof of Theorem 5.7 can be modified to apply to the Hamiltonian in (5.87). An initial state of the form (5.72) means that statistics are disregarded. In the limit as $N \to \infty$ the evolution of the reduced density matrix is given by the usual *Hartree equation*

$$i \frac{d}{dt} \rho(t) = \left[-\frac{1}{2m} \Delta + V_{\rho(t)}, \rho(t) \right] . \tag{5.88}$$

The effective potential V_ρ is

$$V_\rho(x) = \int dy\, V(x - y)\rho(y, y) , \tag{5.89}$$

where $\rho(y, y')$ is the integral kernel corresponding to ρ.

The dynamics of various mean-field models were studied in great detail by Hepp and Lieb (1973c, 1973d, 1975). Their point of view has been described already in Sec. III.E in the classical context. Let B be a bounded operator on $\mathcal{H}$, and let $B_j = B$ be considered as an operator acting on $\mathcal{H}_j$. Then Hepp and Lieb consider the intensive observables

$$\frac{1}{N} X^{(N)}(B) = \frac{1}{N} \sum_{j=1}^{N} B_j \tag{5.90}$$

on $\otimes_{j=1}^{N} \mathcal{H}_j$ and their time evolution

$$\frac{1}{N} X^{(N)}(B, t) = e^{iH_N t} \frac{1}{N} X^{(N)}(B) e^{-iH_N t} . \tag{5.91}$$

Under the same assumptions as in Theorem 5.7,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{tr}_{[1,N]}\{\rho_N X^{(N)}(B,t)\} = \operatorname{tr}_1\{\rho(t)B\}, \tag{5.92}$$

where $\rho(t)$ is the solution of the Hartree equation (5.74). The variance of $(1/N)X^{(N)}(B,t)$ vanishes as $N\to\infty$,

$$\lim_{N\to\infty} \operatorname{tr}_{[1,N]}\left\{\rho_N \left| \frac{1}{N} X^{(N)}(B,t) - \operatorname{tr}_{[1,N]}\left\{\rho_N \frac{1}{N} X^{(N)}(B,t)\right\}\right|^2\right\} = 0. \tag{5.93}$$

So, in the limit intensive observables do not fluctuate. Additionally, intensive observables commute in the limit as $N\to\infty$. In this sense the intensive observables become classical and evolve deterministically as $N\to\infty$.

Since intensive observables do not fluctuate in the limit, one is naturally led to consider the fluctuation observables

$$\xi^{(N)}(B,t) = (1/\sqrt{N})(X^{(N)}(B,t) - \operatorname{tr}_{[1,N]}\{\rho_N X^{(N)}(B,t)\}). \tag{5.94}$$

Formally, the commutator $[\xi^{(N)}(B_1,t_1), \xi^{(N)}(B_2,t_2)]$ is of order one. The fluctuations should therefore remember the underlying quantum-mechanical structure. Indeed, Hepp and Lieb (1973c) show that under the same assumptions as in Theorem 5.7,

$$\lim_{N\to\infty} \xi^{(N)}(B,t) = \xi(B,t), \tag{5.95}$$

where $\xi(B,t)$ is a quasifree Bose field. So quantum-mechanically a quasifree Bose field substitutes for the classical Gaussian random field. More precisely, $\xi(B)$ is a Bose field over the one-particle Hilbert space obtained by the closure of $B(\mathfrak{K})$, the set of bounded operators on $\mathfrak{K}$, equipped with the scalar product

$$\langle A|B\rangle = \operatorname{tr}\{\rho A^* B\} - \operatorname{tr}\{\rho A^*\}\operatorname{tr}\{\rho B\}.$$

$\xi(B)$ is in the Fock vacuum. The time evolution of the field is quasifree and determined through the linearized Hartree equation

$$\xi(B,t) = \xi(B(t)), \tag{5.96}$$

where $B(t)$ is the solution of

$$\frac{d}{dt} B(t) = i[A + \operatorname{tr}_2\{\rho(t)(V_{12}+V_{21})\}, B(t)] + i\operatorname{tr}_2\{(V_{12}+V_{21})[B(t),\rho(t)]\}, \tag{5.97}$$

with initial condition B. Here, $\rho(t)$ is the solution of the Hartree equation (5.74) with initial condition ρ. The convergence (5.95) is then understood as the convergence of all moments

$$\lim_{N\to\infty} \operatorname{tr}_{[1,N]}\{\rho_N \xi^{(N)}(B_1,t_1)\cdots\xi^{(N)}(B_N,t_N)\} = \langle \xi(B_1,t_1)\cdots\xi(B_N,t_N)\rangle \tag{5.98}$$

where $\langle\ \rangle$ denotes the vacuum expectation.

The fluctuation theory exactly follows the classical pattern. If the mean-field model is coupled to reservoirs, as in the case of the laser, the time evolution of the Bose field is more complicated and is determined, in addition to the linearized Hartree equation, by the quantum analog of fluctuating forces.

I would note that the equilibrium properties of mean field models have been studied by Hepp and Lieb (1973a, 1973b), de Vries and Vertogen (1974), Fannes, Sisson, Verbeure, and Wolfe (1976), and Fannes, Spohn, and Verbeure (1979) and that they have been exhaustively reviewed by van Hemmen (1978).

If we want to include statistics in the third example with Hamiltonian (5.87), then Theorem 5.7 is not applicable. A formal argument of how to proceed is as follows. Assuming either Bose or Fermi statistics, the Hamiltonian in (5.87) is written in second-quantized form as

$$H = \int dx\, dy\, h(x,y) a^*(x) a(y) + \int dx\, dy\, V(x-y) a^*(x) a^*(y) a(y) a(x), \tag{5.99}$$

with $h(x,y) = h(y,x)$, acting on the Fock space over $L^2(\Lambda)$. For a state ω the correlation functions ω_n are defined by

$$\omega_n(x_1,\ldots,x_n;\ y_1,\ldots,y_n) = \omega(a^*(x_n)\ldots a^*(x_1)a(y_1)\ldots a(y_n)) \tag{5.100}$$

Their time evolution is governed by the quantum-mechanical BBGKY hierarchy

$$\begin{aligned} i\frac{\partial}{\partial t}\omega_n(x_1,\ldots,x_n;y_1,\ldots,y_n;t) &= \sum_{j=1}^{N}\left\{\int dx\, h(x,x_j)\omega_n(x_1,\ldots,x_{j-1},x,\ldots,x_n;y_1,\ldots,y_n;t) - \int dy\, h(y_j,y)\omega_n(x_1,\ldots,x_n;y_1,\ldots,y_{j-1},y,\ldots,y_n;t)\right\} \\ &+ \sum_{i<j=1}^{n}(V(x_i-x_j)-V(y_i-y_j))\omega_n(x_1,\ldots,x_n;y_1,\ldots,y_n;t) \\ &+ \sum_{j=1}^{n} 2\int dx\,(V(x_j-x)-V(y_j-x))\omega_{n+1}(x_1,\ldots,x_n,x;y_1,\ldots,y_n,x;t). \end{aligned} \tag{5.101}$$

We assume that initially the system is in a quasifree state with defining operator R corresponding to the integral kernel $R(x,y)$ [cf. 5.26]. Then

$$\omega_n(x_1,\ldots,x_n,y_1,\ldots,y_n;0) = \operatorname{per}_{\det}\{R(x_i,y_j)\}. \tag{5.102}$$

For bosons ω_n is the permanent of the $n\times n$ matrix

$\{R(x_i, y_j)\}$ and for fermions ω_n is the determinant of the same matrix.

In a mean-field limit the force between any pair of particles should tend to zero, whereas the total force on one particular particle should stay finite. For bosons this is achieved by scaling

$$V_\varepsilon = \varepsilon V, \tag{5.103}$$

$$R_\varepsilon = \varepsilon^{-1} R, \tag{5.104}$$

just as in the classical case. For fermions, (5.104) does not make sense because of the Pauli exclusion principle. The proper mean-field scaling is open in that case.

In the mean-field limit the BBGKY hierarchy formally goes over to

$$i\frac{\partial}{\partial t}\omega_n(x_1,\ldots,x_n;y_1,\ldots,y_n,t) = \sum_{j=1}^{n}\left\{\int dx\, h(x,x_j)\omega_n(x_1,\ldots,x_{j-1},x,\ldots,x_n;y_1,\ldots,y_n;t) - \int dy h(y_j,y)\omega_n(x_1,\ldots,x_n;y_1,\ldots,y_{j-1},y,\ldots y_n;t)\right\}$$
$$+\sum_{j=1}^{n} 2\int dx\{V|x_j-x) - V(y_j-y)\}\omega_{n+1}(x_1,\ldots,x_n,x;y_1,\ldots,y_n,x;t). \tag{5.105}$$

The *Ansatz*

$$\omega_n(x_1,\ldots,x_n;y_1,\ldots,y_n;t) = \det\nolimits^{\text{per}} R(x_i,y_j,t) \tag{5.106}$$

satisfies (5.105). Therefore a state which is initially quasifree stays quasifree. The evolution of the defining operator $R(t)$ with integral kernel $R(x,y,t)$ is governed by the *Hartree-Fock equation*

$$i\frac{d}{dt}R(t) = [h + V_{R(t)}, R(t)]. \tag{5.107}$$

Equation (5.107) should be considered a nonlinear evolution equation on $T(L^2(\Lambda))$. h is the operator on $L^2(\Lambda)$ with integral kernel $h(x,y)$, and the effective potential V_R is the operator with integral kernel

$$\delta(x-y)\int dx' 2V(x'-x)R(x',x') \pm V(x-y)R(x,y). \tag{5.108}$$

The plus sign stands for bosons and the minus sign for fermions. The second term in (5.108) is known as exchange term. The existence and uniqueness of solutions of the Hartree-Fock equation have been studied by Bove, da Prato, and Fano (1974, 1976) by Davies (1979b), and by Ginibre and Velo (1979).

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