Critical phenomena at low temperature*

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This paper is a subjective review of experimental progress in the field of continuous phase transitions over the last decade or so, and is based primarily upon the author's own work and experiences in the field.

CONTENTS

I. INTRODUCTION

I was first introduced to low-temperature physics nearly two decades ago as a beginning graduate student in the chemistry department of the University of California at Berkeley. During the intervening years I have witnessed a number of major developments in this field, as a result of the work of extremely outstanding experimentalists and theorists. I am indeed deeply honored that my own contributions have been deemed sufficiently important for me to be chosen as the recipient of one of the two 1978 Fritz London Memorial Awards.

My thesis project at Berkeley consisted of a thermodynamic study of the rotational states of o–H_2 in the solid state under pressure up to several Kbars (Ahlers and Orttung, 1964). This problem was suggested by K. S. Pitzer, and completed under the direction of Norman Phillips, who had a number of other students doing high-precision low-temperature calorimetry. In addition to Phillips' work, there was a long-standing tradition of high-precision thermodynamic measurements in the G. F. Giauque Low Temperature Laboratory at Berkeley. The daily contact with Phillips' and Giauque's students instilled in me a firm conviction that, given the right circumstances, qualitatively new physics can emerge from highly quantitative measurements pertinent to problems which, at a less quantitative level, had been examined and perhaps considered "solved" at a much earlier time. This appreciation for quantitative measurements has largely stayed with me throughout my scientific career, and to a great extent, I believe, is responsible for whatever contributions I may have made to low-temperature physics.

Having completed my Ph.D. I left Berkeley in 1963 and joined the staff of Bell Telephone Laboratories in Murray Hill, New Jersey. At first I worked on the equation of state of solid ^4He, an interest which had evolved during my student years. But the stimulating atmosphere at Bell Laboratories soon channeled my activities in new directions. I consider myself very fortunate that, in large part because of interactions with my colleagues, I started experiments in the field of continuous phase transitions, an area of statistical mechanics which was to witness remarkable advances in the years to come. Similarly, in the early 1970's it was to a large extent the influence of my co-workers, particularly of Paul Fleury, that initiated my interest in convection and turbulence in fluid flow, a topic in nonequilibrium statistical mechanics that has seen a great revival in the physics community in recent years and that remains a fascinating, although largely unsolved, problem to this day.

Although the award citation included my work in both critical phenomena and fluid turbulence, I shall summarize in the remainder of this lecture only that part of my activities which falls into the former category. The reasons for this are twofold. First, although there is a certain correspondence between phase transitions and hydrodynamic instabilities, any connection with the evolution of weak turbulence is not at all clear. Second, the phase transition problem has reached a certain level of maturity which warrants a review from a somewhat broader perspective. Investigations of chaotic fluid flow, on the other hand, are still largely in their infancy and, although recent experiments have provided us with a large number of pieces, the puzzle has not nearly been assembled.

In no sense will this account be an exhaustive review of experimental investigations of critical phenomena. It will primarily summarize my own contributions, and the very extensive work of others will be referred to only when it has a direct bearing upon my activities. It will also be somewhat superficial in that only the major results will be described;[1] but I will attempt to present

*Based on the author's Tenth Fritz London Memorial Award Lecture given August 23, 1978 at the 15th International Conference on Low Temperature Physics in Grenoble, France.

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[1]An in-depth review of static critical phenomena near the superfluid transition in ^4He is given by Ahlers (1978). A more extensive review of properties near the superfluid transition in ^4He and ^3He–^4He mixtures is given by Ahlers (1976). A summary of results for the evolution of turbulence in convecting ^4He I is given by Ahlers and Behringer (1978).

these results in the perspective of the time at which they were obtained, and to relate them to concurrent theoretical developments.

II. CRITICAL PHENOMENA

A. Background

The study of critical phenomena is now over 100 years old, and usually is thought to have started with the discovery of critical opalescence near the liquid–gas critical point of carbon dioxide by Andrews (1869). This experimental work was followed almost immediately (Van der Waals, 1873) by the first of the several mean field theories which were to evolve over the next 50 years or so. Most of these theories were designed to describe the critical behavior of a particular type of physical system, such as the liquid-gas critical point, the Curie point of a ferromagnet, or the order-disorder transition in alloys. A crucial common feature of all of them, however, was that the underlying assumptions resulted in a thermodynamic potential of the system which was an analytic function of an order parameter η and of the temperature T. This analyticity yielded identical results for the *essential* feature of the critical behavior in vastly different physical systems because these features were a consequence of the existence of a series expansion for a free energy in T and η. The most general, and perhaps simplest and most elegant, formulation of a theory of continuous phase transitions based on an analytic thermodynamic potential is due to Landau (1937a, b, c, d, 1965). It is not the purpose of this account to elaborate upon theoretical developments; but a brief discussion of the Landau theory in its simplest form will acquaint the reader with the concepts of critical exponents and universality, and will tend to put into perspective some later developments in the field. A more detailed discussion of the Landau theory is given by Stanley (1971).

The Landau theory assumes that the Helmholtz potential $A(T, \eta)$ can be expanded about $T = T_c$ (T_c is the critical temperature) and $\eta = 0$ in the Taylor series

$$A(T, \eta) = a_0(T) + a_2(T)\eta^2 + a_4(T)\eta^4 + \cdots . \tag{1}$$

In Eq. (1), odd powers of η are omitted because a change in the sign of the order parameter does not affect the free energy of the system. The functions $a_i(T)$ are analytic in T, and can be expanded as

$$a_i(T) = a_{i0} + a_{i1}(T - T_c) + \cdots . \tag{2}$$

Only the coefficients of the lowest-order nonvanishing terms need be retained. For a_0, we thus have $a_0 = a_{00}$ because $A(T, \eta)$ is finite and nonzero at T_c, where η vanishes. We obtain $a_2(T)$ by considering the inverse isothermal susceptibility

$$\chi^{-1}(T, \eta) = (\partial^2 A/\partial\eta^2)_T = 2a_2 + 12a_4\eta^2 . \tag{3}$$

Since χ diverges at T_c where $\eta = 0$, we must have $a_{20} = 0$ and thus to leading order $a_2 = a_{21}(T - T_c)$. Similarly, to leading order, one can show that $a_4 = a_{40}$. Now, Eq. (1) becomes

$$A(T, \eta) = a_{00} + a_{21}(T - T_c)\eta^2 + a_{40}\eta^4 + \cdots . \tag{4}$$

The equation of state is

$$H = (\partial A/\partial\eta)_T = 2a_{21}(T - T_c)\eta + 4a_{40}\eta^3 + \cdots . \tag{5}$$

In the absence of an external field, $H = 0$ and for $T < T_c$ we have

$$\eta = (a_{21}/2a_{40})^{1/2}(T_c - T)^{1/2} + \cdots \tag{6}$$

for the order parameter as a function of the temperature along the coexistence curve [for $T > T_c$, the stable solution of Eq. (5) with $H = 0$ is $\eta = 0$]. Equation (6) to leading order is usually written as

$$\eta = \eta_0 |t|^{\beta} , \tag{7}$$

where

$$t \equiv (T - T_c)/T_c .$$

The parameter β is one of the critical exponents which have played such an important role throughout the recent history of the field. We see that the Landau theory predicts $\beta = 1/2$. Similarly, writing the susceptibility as

$$\chi = \chi_0 |t|^{-\gamma} , \tag{8}$$

we have $\gamma = 2\beta = 1$ from Eqs. (3) and (7). These simple examples illustrate two general results of the Landau theory which have been of considerable conceptual importance in later years although they are not applicable in detail to most real physical systems. The first is the prediction that, regardless of the details of a particular system, there is only one set of values for the critical exponents, and thus only a single class of critical behavior. This universality of the Landau theory is admittedly somewhat trivial because there is only a *single* universality class; but later, when it became apparent that the theory was not applicable in detail, the concept of universality became entirely nontrivial and provided an important unifying scheme for the classification of phase transitions. Much of the experimental work to be discussed below has been devoted to its study. The second result of the Landau theory which I wish to mention is the prediction that the critical exponents are given by ratios of small integers. The notion of rational critical exponents seems to have survived for a long time after it was generally recognized that the exponent *values* given by the theory do not apply to real systems. Thus even in recent years β has been quoted to have such values as $\frac{1}{3}$, $\frac{3}{8}$, or $\frac{5}{16}$.

Rather quantitative experimental measurements on liquid–gas critical points existed already near the turn of the century, and they provided substantial evidence for the existence of departures from the Van der Waals equation of state.[2] However, the conflict between theory and experiment was to some extent ignored for many years to come. Nonetheless, the compilation of quantitative experimental data in the 1940s by Guggenheim (1944, 1967) was in part responsible for a revival of interest in the field and for the beginning of what may now be considered the modern era of the study of criti-

[2]A recent review of the early history of the field, with an emphasis on the interactions between experiment and theory, is given by Levelt Sengers (1974).

cal phenomena. Guggenheim's examination of data for the shape of the coexistence curve near liquid–gas critical points made it widely known that β had an experimental value rather closer to $\frac{1}{3}$ than to the classical result $\frac{1}{2}$. The other major simultaneous development in the field was of course the exact solution of the two-dimensional Ising model by Onsager (1944) which had yielded a logarithmically divergent specific heat,[3] in disagreement with the Landau theory.

By the early 1960s, when I first became aware of the field, the renewed interest in continuous phase transitions had resulted in considerable experimental and theoretical activity.[4] This effort was stimulated by a realization that critical phenomena involved very general, and still in some sense universal, properties in spite of, and perhaps associated with, the breakdown of mean field theory. The theoretical work initially led to the development of a new phenomenological theory, often known as scaling.[5] We shall not discuss the scaling theory in any detail here. The interested reader will find an excellent account of it in the book by Stanley (1971). In essence, the Landau assumption of analyticity of a thermodynamic potential is replaced in this theory by the more general postulate that the free energy is a homogeneous function of its arguments. This homogeneity assumption is less restrictive than the analyticity postulate; and consequently the predictions of the new theory are less specific. The values of the exponents are no longer given; but the theory predicts relationships, known as scaling laws, between two or more exponents describing the singularities of different properties near the same critical point. The theory also predicts the general form of the equation of state. Another major parallel theoretical effort concentrated on obtaining numerical solutions by series expansion technique for specific model systems such as the three-dimensional Ising and Heisenberg models [see Fisher (1967) for a review].

Experimental efforts during this period were directed towards more and more refined measurements of critical point parameters such as exponents, as well as towards measurements of the equation of state over a broader range of temperatures and fields. This was also the time when the newly developed techniques of laser light scattering and neutron scattering were first being applied to the study of both spatial and temporal fluctuations near critical points. These experiments resulted in important new information (see, e.g., Heller, 1967); but I shall not discuss them further here because they made relatively little contact with my own work.

By the middle 1960s, it became evident in part from the experiments, but especially from the model calculations, that widely differing physical systems often but not always shared the same critical behavior. The conviction evolved that there exists a small number of universality classes to which critical points can be assigned; but a systematic method of classification was still lacking. The experimental results for the exponents were used to test the predicted scaling laws. This was an important undertaking because the scaling theory was a phenomenological theory, and therefore it was natural to ask whether its predictions were exact or whether they were merely a good *approximation* to the behavior of real systems.

Some of the early experiments were misleading with regard to both the scaling predictions and the question of universality in spite of their high level of sophistication. For instance, liquid–gas critical points almost always yielded values near 0.355 for β (Levelt Sengers and Sengers, 1975; Sengers and Levelt Sengers, 1978), whereas numerical calculations for the three-dimensional Ising model yielded results rather closer to 0.32 (see, for instance, Fisher, 1967). We now believe that these two systems belong to the same universality class, and that β for liquid–gas critical points is appreciably lower than the early experiments had indicated (Hocken and Moldover, 1976; Balzarini and Ohrn, 1972). In retrospect, the reason for this problem is clear. The universality of critical behavior manifests itself only when T_c is approached very closely. For finite values of the reduced temperature t there are contributions to the thermodynamic functions which are not included in pure power laws like Eqs. (7) and (8). These contributions may be singular and may remain large even for very small t, although they vanish for $t=0$. A fit of the data at nonzero t to Eq. (7), for instance, will in that case result in systematic errors for β. We shall return to the experimental study of these higher-order singular terms later on when we consider measurements of the superfluid density near T_λ in ^{4}He.

The above example illustrates the need for making measurements extremely close to the critical temperature. For most systems, this becomes very difficult not only because of limitations imposed by measurement techniques, but perhaps primarily because the samples must be sufficiently homogeneous to avoid "rounding" of the phase transition on the scale of the temperature resolution of the measurements. The problem of obtaining sufficiently good samples has indeed been a major one since the revival of experimental interest in the field because the requirements are so severe.

Sample inhomogeneities have been responsible also for apparent conflicts between experimental results for exponents on the one hand and scaling predictions on the other. This problem is well illustrated by specific heat measurements for magnetic systems. We expect a singularity of type $C_p \sim |t|^{-\alpha}$ as $T \to T_c$ from above, and of type $C_p \sim |t|^{-\alpha'}$ as $T \to T_c$ from below T_c.[6] Scaling predicts that $\alpha = \alpha'$, whereas many early experiments yielded $\alpha > \alpha'$. Often, the reason was an incorrect choice of a representative T_c, provoked by "rounding"

[3]The exponent describing the specific heat singularity is given the symbol α. Since $\lim_{\alpha \to 0} [\alpha^{-1}(t^{-\alpha} - 1)] = -\ln(t)$, a logarithmic specific heat corresponds to $\alpha = 0$.

[4]For reviews of the field up the middle 1960s, see Heller (1967); Fisher (1967); Kadanoff *et al.* (1967).

[5]Various formulations of the scaling theory are given by Essam and Fisher (1963), Widom (1965), Domb and Hunter (1965), Kadanoff (1965), Patashinskii and Pokrovskii (1966), and Griffiths (1967).

[6]We follow the usual convention of identifying exponents pertinent to properties below T_c by a prime.

of the transition and a consequent shift of the maximum in C_p to temperatures below the T_c for the pure system. In addition, in some cases possible asymmetry about T_c in the size of higher-order singular contributions to C_p may also have contributed to the problem.

I have tried to outline so far some of the objectives of experimental work on continuous phase transitions, and some of the problems faced by the experimentalists, in the middle 1960s. Two major objectives seem to have been to test the scaling theory, and to provide more and more detailed information which could be used in the development of a theory for the assignment of critical points to universality classes. Two major problems were the need for extremely homogeneous materials, and the influence of higher-order singular terms on the values of parameters derived from data. The latter was actually not widely appreciated at that time. Although theorists knew that in principle very complicated confluent singularities can exist (Fisher, 1967), a certain optimistic belief prevailed that nature in some sense would be simple and that an analysis of data in terms of pure power laws like Eqs. (7) and (8) would yield good results for the parameters which describe the asymptotic behavior of the measured variable. The problems associated with sample inhomogeneities were much more generally appreciated, but in most cases there were no easy solutions.

B. Sample inhomogeneities and the superfluid transition

Even the best solid materials have phase transitions which are obviously rounded over a range of about 10^{-4} in the reduced temperature. Liquid–gas critical points are appreciably affected by the gravitational field over a similar temperature range (see, for instance, Barmatz *et al.*, 1975; Sengers and Levelt Sengers, 1978; Moldover *et al.*, 1979). Departures from the behavior of the uniform system on the scale of highly quantitative experiments extend to even larger values of $|t|$. It became apparent in the early 1960s that the superfluid transition in ^4He provides the near-ideal system that is needed to test some of the predictions of the theory in the greatest possible detail. For that system there are no strains because the sample is a liquid, and the only significant impurity is ^3He, which occurs naturally only to the extent of 1 part in 10^7, and which at that level has a negligible effect. The major source of an inhomogeneity is the gravitational field. Its effect is much smaller, however, than it is near-liquid–gas critical points. For liquid–gas critical points, the density difference between the two phases is the order parameter, and therefore the pressure (or chemical potential) is the field which is the thermodynamic conjugate to the order parameter. This implies that the compressibility is the strongly divergent susceptibility given in Eq. (8). Thus the compressibility becomes extremely large and density gradients in the fluid under the influence of gravity are appreciable and vary with the height. For the superfluid transition, the compressibility has only a weak singularity (of type $t^{-\alpha}$) and in practice remains nearly constant even when the transition is approached very closely. Therefore the gravitational field induces a small, nearly-constant density gradient in samples of nonzero height. This results in a height-dependent transition temperature, but the effect is numerically almost negligible. Thus, at saturated vapor pressure, T_λ is expected to be shifted by only 1.3 μK for a 1 cm change in fluid height. Working with a sample which is 1 mm tall, one would have a two-phase system over a range of only 6×10^{-8} in $|t|$. Outside this range, it is not difficult to use the known pressure dependence of T_λ to correct data for the slight departures from the properties of a uniform system. The material is in fact so nearly uniform that to this day the temperature resolution and stability which would be necessary to fully exploit its potential have not yet been developed.

Although liquid helium has unparalleled advantages for experimental investigations, its potentials in some sense are quite limited. First, it represents only one type of critical behavior, and a study of the difference between distinct classes of critical points has to rely in addition on investigations using less favorable materials. Further, only a small number of relevant response functions are accessible to experiment. This is because the order parameter of the system is a property associated with the ground-state wave function which is not readily accessible to experiment. The field conjugate to this order parameter also cannot be varied in the laboratory. Nonetheless, we shall see that important, highly quantitative experimental information about continuous phase transitions has been derived from work on this system.

C. Heat capacity near the superfluid transition

In 1965, when I first became interested in doing experiments near T_λ, we already had the very beautiful specific heat measurements of Buckingham, Fairbank, and Kellers[7] which had been part of the reason for presenting the 6th Fritz London Award to William Fairbank (coincidentally, this work was presented for the first time in 1958 at the 5th International Conference on Low Temperature Physics, which was the conference at which the *first* London Award was given to N. Kurti). They yielded the famous "logarithmic" singularity in the specific heat at constant pressure C_p which can be expressed by

$$C_p^+ = A\log t + B \tag{9a}$$

for $T>T_\lambda$ and by

$$C_p^- = A'\log(-t) + B' \tag{9b}$$

for $T<T_\lambda$. Here $t=T/T_\lambda-1$. The experiment indicated that to a good approximation $A=A'$. This equality between the amplitudes for a logarithmic C_p is one of the consequences of the scaling theory; but of course the experiment had preceded the theory by several years. The experimental result was very similar to the exact theoretical prediction for the two-dimensional Ising model (although the Ising model has a completely symmetric specific heat with $B=B'$). A few years after the λ-point measurement, the divergence of the specific heat at constant volume C_V at liquid–gas critical points

[7]This work was reported in a number of places, including Fairbank *et al.* (1958), Kellers (1960), Buckingham and Fairbank (1961), Fairbank (1963), and Fairbank and Kellers (1966).

was discovered (Bagatskii, Voronel', and Gusak, 1962), and those data were interpreted initially in terms of a logarithmic singularity. The feeling evolved that a logarithmic specific heat was perhaps a common feature of many different types of critical points,[8] although alternate interpretations of the liquid-gas specific heat were considered already in the middle 1960s (Fisher, 1964a, b). Since the measurements by Fairbank and co-workers, some progress had been made in the techniques of high-resolution thermometry and thermal isolation; and in view of the new scaling predictions and the expectations of universal behavior near critical points it was very exciting to attempt a new, perhaps even more precise set of measurements. There was one problem with ^{4}He, however. As mentioned earlier, the transition temperature is pressure dependent and thus the gravitational field induces an inhomogeneity in samples of finite height. Estimates based on the phase diagram indicated that this effect would be small and that corrections could easily be applied. On the earth's surface and at vapor pressure, one would expect isothermal ^{4}He II and ^{4}He I to coexist over a narrow temperature interval, with an interface between the two phases which moves vertically by 0.79 cm per 1 μK temperature change. But in 1966, a theoretical paper appeared which claimed that this classical situation would not prevail, and that the two phases could not coexist even in the presence of gravity because a proximity effect would prevent it. This prediction turned out to be based on an incorrect application of the Landau theory; but it provoked me to carry out a preliminary experiment on a very tall sample of ^{4}He. This experiment (Ahlers, 1968a) showed clearly the coexistence of the two phases, and demonstrated an interface movement of 0.81 ± 0.03 cm/μK, in excellent agreement with the phase diagram. Of course, one does expect the existence of a proximity effect in this system; but a correct application of the Landau theory indicates that this results in a "thickness" of about 10^{-2} cm for the interface (Hohenberg, 1968); and to this day experimental skills and ingenuities have not been sufficient to detect this thickness.

During this work on the gravity effect I benefited greatly from the theoretical guidance which was so generously given by my colleague Pierre Hohenberg. I consider myself very fortunate indeed that Pierre has been closely associated with my work ever since; and he has never tired of explaining to me those consequences of theoretical developments which were pertinent to my experiments.

It was time now to seriously attempt the specific heat measurements; but while the apparatus for this was being designed, Ferrell and co-workers (1967, 1968) and Halperin and Hohenberg (1967, 1969) developed a scaling theory for transport properties which predicted, among other things, that the thermal conductivity λ of ^{4}He I should diverge upon approaching T_λ from above according to the power law $\lambda \sim t^{-\nu/2}$, where ν is the exponent of the correlation length ξ for spatial fluctuations in the order parameter.[9] Qualitative indicators of a divergence of λ already were given by the measurements of Kerrisk and Keller (1967, 1968); but quantitative results were lacking. A thermal conductivity cell was therefore incorporated in the specific heat apparatus. I will return to the results for λ later on; but the thermal conductivity measurements also made it possible to determine T_λ with the very high precision of 10^{-7} K; and this was important for the interpretation of the C_p measurements because it reduced the number of parameters to be determined from these data by one.

Some of the results (Ahlers, 1969, 1971) for the heat capacity at saturated vapor pressure C_s, which is virtually equal to C_p, are shown as solid circles in Fig. 1. The measurements of Fairbank *et al.* are shown as open circles. The agreement is obviously excellent. Near T_λ, where errors due to temperature resolution dominate, the new data are more precise by only perhaps a factor of two or three; but for $|t| > 5 \times 10^{-5}$ the improvement in precision is about an order of magnitude. To our great surprise, these more precise data, when fitted to a logarithmic singularity, no longer permitted $A = A'$ as predicted by theory. The only way to reconcile these results with the predictions was to abandon the logarithmic functional form, and to fit the data to the more general power law

$$C_p^+ = (A/\alpha)t^{-\alpha} + B \tag{10a}$$

for $t > 0$, and to

$$C_p^- = (A'/\alpha')|t|^{-\alpha'} + B' \tag{10b}$$

for $t < 0$. The logarithimic singularity then corresponds to the specific case $\alpha = \alpha' = 0$; but equal amplitude ratios are predicted only for vanishing exponents. When the data were fitted to power laws, they yielded very slightly negative values of α and α', and $A/A' = 1.1$. The small but negative exponents imply that C_p becomes very large but remains finite at T_λ. This was a result which seemed difficult to accept on intuitive grounds; but intuition failed us there and we shall see below that we now have convincing experimental and theoretical reasons to believe in a negative α. The very small value of $\alpha = \alpha'$ also seemed rather disturbing because of the strong prejudices for the idea that critical exponents should be expressible as ratios of small integers. This prejudice to a large extent had its roots in the Landau theory which I discussed above; but it was further supported by the feeling that it would be very difficult to construct a theory which would yield irrational exponents. A well known theorist said to me at the time: "How can you expect me to make a theory which predicts one-fiftieth, or even one one-hundredth?" Well, as we shall see shortly, there is such a theory now and the notion of ratios of small integers for critical exponents finally seems to have disappeared.

[8] This is well illustrated by the contents of the 1965 conference on critical phenomena held in Washington, D. C. (Green and Sengers, 1966).

[9] No confusion should arise from the use of the symbol λ both for the thermal conductivity, and as a subscript of T to denote the superfluid transition temperature T_λ.

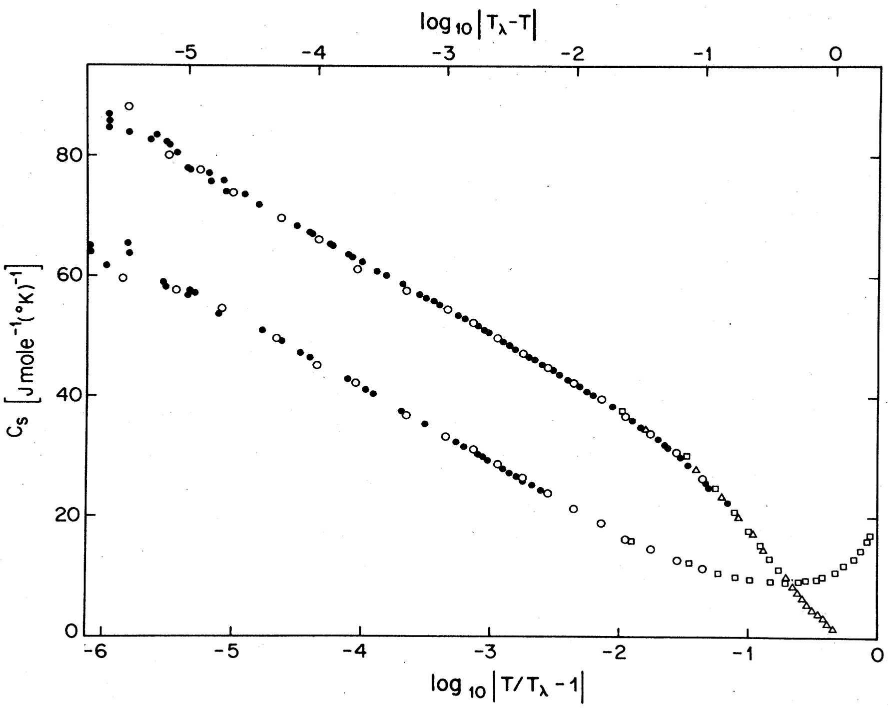

FIG. 1. Heat capacity at saturated vapor pressure of liquid ^{4}He near T_λ as a function of $\log_{10}|T/T_\lambda - 1|$. The upper set of data is for $T < T_\lambda$, and the lower set is for $T > T_\lambda$. Open circles—Buckingham and Fairbank (1961). Solid circles—Ahlers (1971). After Ahlers (1971).

D. The superfluid transition and universality

Already around 1970, the conviction that critical behavior involved rather general phenomena had culminated in a *concrete* formulation of a hypothesis of universality.[10] According to the new postulate, all phase transition problems can be divided into a small number of different classes, with each class determined by the dimensionality of the system and the symmetries of the ordered state. Within a class, exponents like α or α' and certain amplitude ratios like A/A' should be constant and thus independent of the details of the interactions of the particular system. Simple examples of members of the same universality class are all Ising models, regardless of the spin or underlying lattice type. Here the type of lattice for instance is an irrelevant detail of the interactions. A Heisenberg model, however, would belong to a different class because the symmetry of the ordered state is different. For the Heisenberg system the order parameter (magnetization for instance) is a vector, where as for the Ising system it is a scalar. Physically, the irrelevance of the details of the interactions of a particular system arises from the fact that critical phenomena are the result of fluctuation in the order parameter over a spatial range which becomes larger and larger as the critical point is approached. The associated averaging over domains which are large compared to the scale of the detailed interactions renders the details unimportant; but it remains important whether the fluctuating quantity is, for instance, a scalar or a vector. For liquid helium, universality meant that A/A' and α should be independent of pressure and of the ^{3}He concentration because these parameters do not change the symmetry of the ordered state. Again, the superfluid transition provided a flexible testing ground of this theoretical idea. There was a problem, however, for in ^{4}He α and A/A' pertain to C_p; and under pressure it is difficult to measure C_p because the size of the container or the amount of sample would have to be varied as the temperature is changed. Instead, the heat capacity at constant volume C_v was measured, and C_p was obtained from C_v, by making the appropriate corrections (Ahlers, 1973). This worked well at small P, where $C_p - C_v$ was small, and for $P < 15$ bars yielded universal parameters. But at larger pressure the experiment seemed to yield pressure-dependent values of A/A'. The reasons for this are still not entirely clear, but it is likely that this result was in error and attributable to an underestimate of the uncertainties in the rather large corrections which were necessary to convert C_v to C_p. The question of universality along the entire λ line in ^{4}He therefore was not settled by the C_v measurements, but rather considerably later by precise determinations of the thermal expansion coefficient β_p (Mueller *et al.*, 1975, 1976). Since β_p is an asymptotically linear function of C_p near T_λ, β_p has the same exponents and amplitude ratios as C_p. A direct measurement of β_p is easier than one of C_p. This work

[10]Early statements of the hypothesis of universality may be found in Fisher (1966), Watson, (1969a, b), and Jasnow and Wortis (1968). More recent references are Kadanoff (1971), Griffiths (1970), Betts *et al*. (1971), and Stauffer *et al*. (1972).

was started in 1973, when I was fortunate enough to be able to spend a year at the Kernforschungsanlage in Jülich, West Germany, working in collaboration with Frank Pobell and Karl Mueller. By this time, we already knew from measurements of the superfluid density to be described in Sec. F below that an analysis of the data in terms of simple power laws like Eq. (10) did not always give those values of critical point parameters like α and A/A' which pertain in the limit as $|t|$ vanishes. Thus the data were fitted to the function

$$\beta_p = (A/\alpha)|t|^{-\alpha}[1 + D|t|^x] + B \tag{11}$$

for $t > 0$, and to the same function with primed parameters for $t < 0$. Here $x > 0$, and thus the term $D|t|^x$ vanishes at T_λ. For the measurements at elevated pressure, the temperature variable t must of course be defined in terms of $T_\lambda(P)$. The values obtained for $\alpha = \alpha'$, A/A', and D/D' are shown in Fig. 2. The error bars are standard errors (68% confidence limits). Since most of them overlap the mean values which are indicated by the dashed lines, universality is strongly supported by these measurements along the λ line in ^{4}He. The best universal values of the parameters are

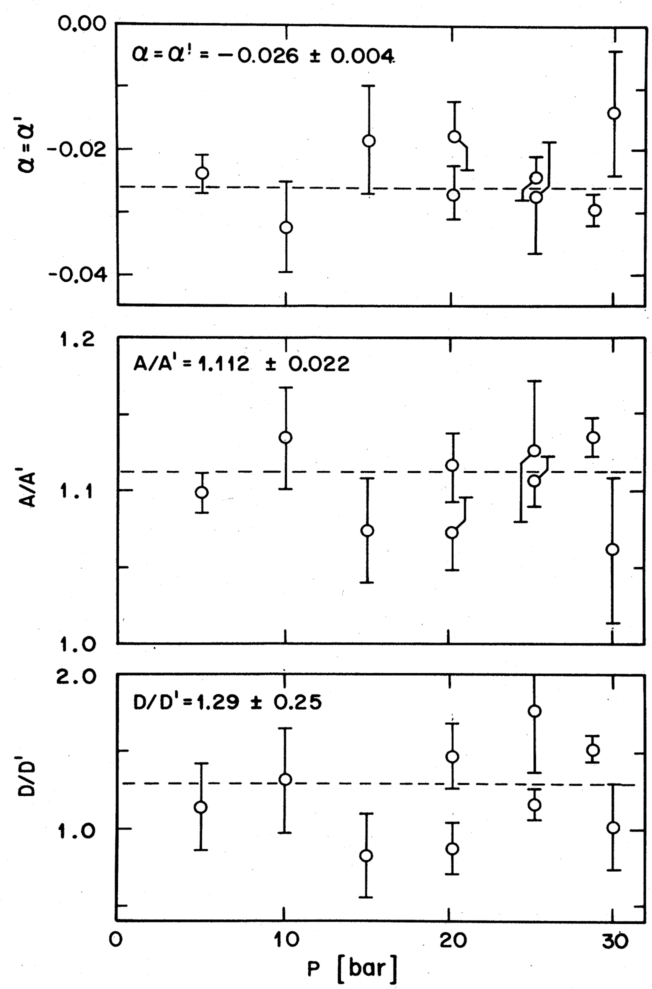

FIG. 2. The exponent $\alpha = \alpha'$, the ratio of the leading amplitudes A/A', and the ratio of the amplitudes of the confluent singularity, for the thermal expansion coefficient β_p near T_λ as a function of pressure. After Mueller *et al*. (1976).

$$\alpha = \alpha' = -0.026 \pm 0.004\,, \tag{12a}$$

$$A/A' = 1.112 \pm 0.022\,, \tag{12b}$$

$$D/D' = 1.29 \pm 0.25\,. \tag{12c}$$

Since α is negative, these results confirm the earlier conclusion that C_p remains finite at T_λ. But the measurements extrapolate to a very large, only mildly pressure-dependent value near 30 k_B for $C_p(T_\lambda)$.

E. Effective exponents

Equation (11) above, which was used to analyze the thermal expansion data, includes the confluent singular term $D|t|^x$ in addition to the leading power-law singularity (see also Sec. F below). One can hardly overstate the difficulties which are involved in extracting meaningful parameters from a fit of experimental data to such a complicated function. This problem of data analysis is reviewed in some detail elsewhere (Ahlers, 1978). Only the most precise data extending to the smallest possible values of $|t|$ are at all suitable for the purpose. In practice, this means that measurements near most other critical points besides the superfluid transition will yield errors for the parameters of Eq. (11) which are so large that different universality classes could not be distinguished. This explains the popularity in the past of fits to the much simpler pure power laws. The results can be extremely misleading, however. For instance, a pure power law fit of β_p will result in values of α which vary from about −0.01 at vapor pressure to about 0.07 at 30 bars (Ahlers, 1978), even when only data with $|t| \lesssim 3 \times 10^{-3}$ are used.[11] This pressure dependence is of course not a variation in the exponent which describes the asymptotically dominant singularity, but rather a variation in an effective exponent $\tilde{\alpha}$ which has no particular significance except that it reflects the strength of the confluent singular terms. These effective exponents have been discussed recently by Aharony and Ahlers (1980) and by Chang and Houghton (1980) (see also Ahlers, 1978). It is interesting to note that many early experiments on liquid–gas critical points have yielded "universal" values for $\tilde{\beta}$ and $\tilde{\gamma}$ when the data were fitted to pure power laws (Levelt Sengers and Sengers, 1975). The values of $\tilde{\beta}$ and $\tilde{\gamma}$ were not those, however, which are now believed to be appropriate for the leading singularity (Hocken and Moldover, 1976). The apparently "universal" nature of $\tilde{\beta}$ and $\tilde{\gamma}$ is thus an indication that many liquid–gas critical points obey a law of corresponding states in the sense that they all have similar values for the equivalent of the amplitude D in Eq. (11). This law of corresponding states is of course not the same as universality, because within the universality postulate individual amplitudes like D and A are permitted to depend upon the details of the interactions for the particular system. The dependence of $\tilde{\alpha}$ upon the pressure along the superfluid transition line indicates that this transition does not obey a law of corresponding states and that D is a function of P. However, there is no evidence against *universality* because α and A/A' are pressure inde-

[11]For many other systems with phase transitions, measurements which are not affected by sample imperfections are obtainable only for $|t| > 3 \times 10^{-3}$.

pendent. For small P, D is nearly equal to 0 and therefore $\tilde{\alpha}$ is nearly equal to α. I do not know why D should vanish near $P=0$; but that it does is a lucky accident because much of the early work on liquid helium was done at vapor pressure. Analyses in terms of pure power laws gave nearly the correct results for the leading singularities. It is also interesting to note that D is nearly equal to zero for the spin-1/2 Ising model, but greater than zero for other spins (Saul *et al.*, 1975). So far, theory can tell us very little about the size of nonuniversal parameters like D.

While the work on the thermodynamics of the superfluid transition which I have described was slowly progressing, a major new theoretical development occurred in the field when K. G. Wilson (1971) applied renormalization group methods to the problem of continuous phase transitions.[12] The renormalization group theory (RGT) of critical phenomena is now believed by many to be exact. It provides a derivation of the results of scaling and universality from a more microscopic starting point. It also supplies the methods for deciding which symmetry properties of the ordered state are relevant for the determination of the universality class, and it makes it possible at least in principle to calculate the values of exponents and amplitude ratios. In a sense, the stakes became considerably higher for the experimentalist after the advent of the RGT, and the claim that the existence and nature of *singularities* in systems with *regular* microscopic interactions had been explained in a *fundamental* way deserved the most careful experimental scrutiny. These theoretical developments certainly had a highly stimulating effect upon our investigations of the λ point.

F. Superfluid density and confluent singularities

So far, I have discussed the experiments pertinent to the specific heat exponent α and amplitude ratios A/A' and D/D'. The other parameter which is accessible to experiment and relevant to critical phenomena is the superfluid density ρ_s. The exponent ζ of ρ_s is related to α through a scaling law (Josephson, 1966). An accurate determination of ζ in addition to α therefore would provide the opportunity to test a scaling prediction at an unprecedented level of accuracy. In addition, the ratio T/ρ_s is proportional to the correlation length ξ for spatial fluctuations in the order parameter, with the proportionality constant given by theory. Measurements of ρ_s therefore yield both the amplitudes ξ_0 and the exponent ν' of ξ. As we shall see below, a knowledge of ξ_0 will enable us to test additional predictions of universality.

Historically, the ρ_s measurements actually preceded the determination of α from the thermal expansion measurements. I would therefore like to go back a few years, to the beginning of 1970, when Dennis Greywall joined the staff of Bell Laboratories. Dennis and I collaborated for two years on the measurement of second-sound velocities in ^{4}He and ^{3}He-^{4}He mixtures. For pure ^{4}He, the second-sound velocity u_2 is related to the superfluid density ρ_s by two-fluid hydrodynamics, and is given by[13]

$$u_2^2 = S^2 T \rho_s / \rho_n C_p \,. \tag{13}$$

Here S is the entropy, and $\rho_n = \rho - \rho_s$ is the normal-fluid density. We used the measured u_2 and Eq. (13) to determine the superfluid fraction ρ_s/ρ (Greywall and Ahlers, 1972, 1973). From previous measurements at vapor pressure by Tyson and Douglass (1966), and by Clow and Reppy (1966), we already knew that to a good approximation near T_λ

$$\rho_s/\rho = k\,|t|^{\zeta} \tag{14}$$

with ζ close to $\frac{2}{3}$. In order to have a sensitive graphical representation of the data, we therefore plotted the nearly constant $(\rho_s/\rho)t^{-2/3}$ vs t on logarithmic scales. This is shown in Fig. 3. The results surprised us for two reasons. First, we had expected from an examination of the earlier measurements at several pressures but not very near T_λ by Romer and Duffy (1969) that to a good approximation ρ_s/ρ would obey a law of corresponding states in the sense that the data at all pressures as a function of t would fall on a single curve. The applicability of a law of corresponding states to ρ_s/ρ has been shown with greater precision than that of the Romer and Duffy results more recently by Maynard *et al.* (1976), but again only for temperatures not very close to T_λ. We see from Fig. 3 that this law does not hold in the immediate vicinity of the transition. There is of course no known fundamental reason why it should, and its applicability at lower temperatures, although remarkable, may well be only approximate.[14] The second source of surprise was the considerable curvature exhibited by the data in Fig. 3 for the higher pressures. If it were possible to represent ρ_s/ρ by a power law like Eq. (14), the measurements should fall on a straight line, with a slope equal to $\zeta - \frac{2}{3}$. The

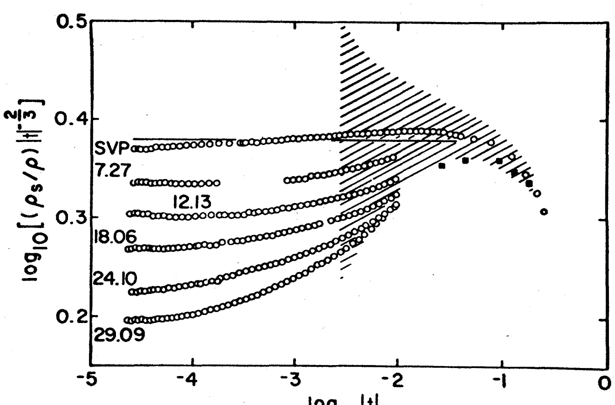

FIG. 3. High-resolution plot of the superfluid fraction ρ_s/ρ, as a function of $t = T/T_\lambda - 1$ on logarithmic scales. The data for ρ_s/ρ were multiplied by $|t|^{-2/3}$ in order to remove most of their temperature dependence. The numbers near the data indicate the pressure in bar. After Greywall and Ahlers (1972).

[12]For reviews of the renormalization group theory, see, for instance, K. G. Wilson and J. Kogut (1974) and M. E. Fisher (1974).

[13]See, for instance, I. M. Khalatnikov (1965).

[14]I already discussed departures from a corresponding states law earlier in this paper in connection with the effective specific heat exponent $\tilde{\alpha}$; but those results were not yet known to us when we made the ρ_s measurements.

curvature at the higher pressures therefore indicates that a pure power law is not a good representation of ρ_s/ρ, even for values of $|t|$ as small as, say, 10^{-4}. Thus it became necessary to interpret the results in terms of the more complicated function

$$\rho_s/\rho = k|t|^{\zeta}[1+g(t)], \tag{15a}$$

where $g(t)$ is singular and vanishes at T_λ. When is was assumed that $g(t)$ has the form

$$g(t) = d|t|^x \tag{15b}$$

those data yielded

$$x = 0.5 \pm 0.1. \tag{16}$$

These measurements therefore clearly indicate that there are correction terms to the leading power law singularly, and that these terms have a singular temperature dependence. They provided part of the motivation for the use of confluent singularities for the thermal expansion analysis [Eq. (11)] which I discussed above. Since there is no reason to assume that the superfluid transition is a special case, the ρ_s results demonstrate that confluent singular terms in general have to be assumed to contribute to properties near critical points. The presence of such terms makes the comparison of measurements with theoretical predictions extremely difficult because of the additional parameters which they introduce into the analysis.

Prior to our measurements of ρ_s, strong indications for the existence of confluent singularities had already been obtained from high-temperature series expansions for the Ising model.[15] Simultaneously with our ρ_s measurements, the existence of confluent singularities near phase transitions, as well as some of their properties, were predicted from the RGT (Wegner, 1972), and this additional theoretical information has somewhat reduced the problem involved in the interpretation of experiments.

G. Exponent values and scaling

The amplitude d of the confluent singular contribution to ρ_s is smallest at vapor pressure. This is clear from Fig. 3 where the data labeled SVP nearly fall on a straight line. For that reason, those data permit the most accurate determination of the *leading* exponent ζ. When fitted to Eq. (15), they give

$$\zeta = 0.6749 \pm 0.0007. \tag{17}$$

The results at higher pressure are consistent with this value and therefore with universality; but they do not give ζ as accurately. The information about x [Eq. (16)] of course comes primarily from the high-pressure data where the term dt^x contributes appreciably to ρ_s. The result Eq. (17), together with the result Eq. (12a) for α, can be used to test the scaling law (Josephson, 1966)

$$\nu' = (2-\alpha)/3 \tag{18}$$

because $\zeta = \nu'$. For the rhs of Eq. (18), we get 0.6753 $\pm$ 0.0013, consistent with the experimental ζ within the very small errors. This comparison of exponents for different quantities via scaling is the only one I know of with values determined from fitting to functions which include confluent singular terms; and it therefore provides important new quantitative support for the theory. It is especially interesting that the particular scaling law Eq. (18) is confirmed so well by the data because the high temperature series expansion results for the Ising model have long been regarded as inconsistent with it (see, for instance, Camp *et al.*, 1976). The Ising exponents are, however, still under active investigation.

[15]M. Wortis, private communication, and Saul *et al.* (1975).

In addition to providing a derivation of scaling and universality, the RGT in principle can also be used to calculate the values of exponents and of the universal amplitude combinations. These calculations usually have to be carried out by approximate mathematical techniques, however, and thus are subject to errors which are difficult to estimate. Only the exponents have so far been calculated with relatively high accuracy[16] (LeGuillou and Zinn-Justin, 1977), and their values are compared with the experimental results in Table I. Although the differences are small compared to the uncertainties of most experimentally determined exponents for systems other than helium, the theoretical and experimental values in Table I do differ by considerably more than their estimated errors. It is somewhat questionable at this time to what extent the discrepancy may be due to underestimates of the uncertainties. However, it seems unlikely that the *experimental* errors were appreciably underestimated because the values of α and ζ, although they come from two completely independent experiments, satisfy the scaling law Eq. (18) well within the quoted errors. It seems that the possibility of a fundamental problem with the applicability of the theory in its present form and on a highly quantitative level to real physical systems cannot be discarded altogether. Additional evidence for similar problems comes from the high-temperature series expansions for the Ising model (see, for instance, Camp *et al.*, 1976), which yield, for example, a value of 1.250 for γ, to be compared with the RGT value of 1.241 ± 0.002. In this case, however, I already mentioned that the size of the errors for the series results are still the subject of a current debate.

TABLE I. Experimental and theoretical results for the measurable exponents and amplitude combinations near T_λ in ^4He.

Parameter	Experiment	RGT
$\alpha = \alpha'$	-0.026 ± 0.004	-0.007 ± 0.006 [a]
ζ	0.6749 ± 0.0007	0.669 ± 0.002 [a]
x	0.5 ± 0.1	0.522 ± 0.017 [a]
A/A'	1.112 ± 0.022	
D/D'	1.29 ± 0.25	
$u\alpha$ [b]	0.527 ± 0.016	

[a] LeGuillou and Zinn-Justin (1977).
[b] At saturated vapor pressure.

[16]See Bervillier and Godrèche (preprint), however, for the calculation of an amplitude combination which happens not to be accessible to measurement in liquid helium.

In addition, the series exponents do not satisfy the scaling law Eq. (18) and the comparison of the exponent values with the RGT is clouded by that problem.

Table I also contains a summary of the experimental results for those universal amplitude combinations which are accessible to measurement in liquid helium.

H. Superfluid density in ^{3}He–^{4}He mixtures

In collaboration with Greywall, the second sound measurements were extended also to ^{3}He–^{4}He mixtures (Ahlers and Greywall, 1972, 1974; Ahlers, 1976). In the region near the tricritical point, the results along the coexistence curve were consistent with $\rho_s \sim (T - T_t)^{1.0}$, where T_t is the tricritical temperature. Along the λ line, we found that the amplitude $k(X)$ vanishes upon approaching X_t (X is the ^{3}He concentration), with an exponent of about 0.34. Both of these results follow from RGT calculations (Riedel and Wegner, 1972).

I. Two-scale-factor universality

We have seen that the scaling law Eq. (18) is satisfied by the experimental values of α and ζ at a highly quantitative level, and that the lack of pressure dependences of A/A', α, and D/D' are consistent with universality. The minor differences between the theoretical and experimental values of α and ζ require additional investigation, and cannot be regarded as concrete evidence for a breakdown of the theory at this time. Unfortunately, quantitative agreement does not extend to all pertinent parameters which are measureable near T_λ. An additional quantity which is predicted to be universal is the free energy, divided by the temperature, for a volume of fluid which is equal to the cube of the correlation length (Stauffer *et al.*, 1972; Hohenberg *et al.*, 1976). This parameter is given by a combination of the specific heat amplitude A' and the superfluid density amplitude k, and can be written as

$$\alpha u = \left(\frac{m_4}{\hbar}\right)^6 k_B^2 \frac{A' T^3}{V\rho^3 k^3}. \tag{19}$$

The experimental values of αu are shown in Fig. 4. They increase by a factor of 1.5 as the pressure changes from 0 to 30 bars. Whereas a variation of perhaps 15% could be attributed to experimental errors, it is difficult to see how the experiments could be off by 50% at high P. However, in view of the other very excellent agreements between the RGT predictions and experiment, it seems prudent to reserve judgement on the significance of this problem until additional, more quantitative measurements of k and A' have been made.

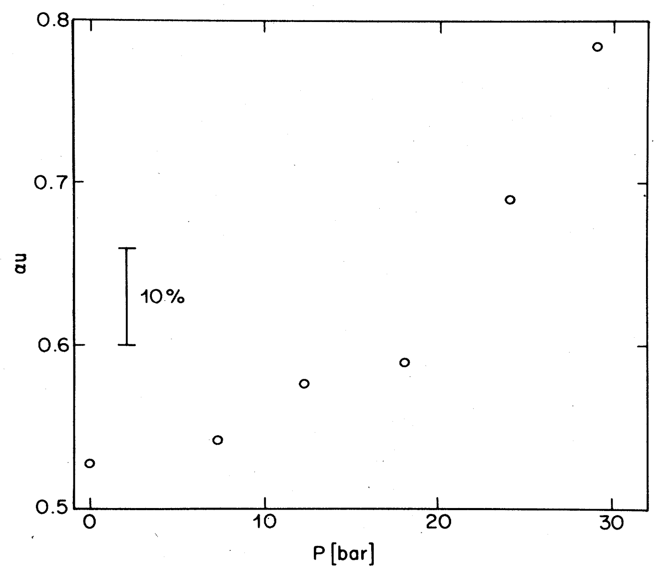

FIG. 4. Experimental values of the parameter αu, defined by Eq. (19), as a function of pressure. From theory, αu is expected to be universal and thus independent of pressure.

J. Other universality classes of the n-vector model

The work which I have described so far has yielded a considerable amount of quantitative information about the behavior of thermo-hydrodynamic parameters near T_λ in ^{4}He. But the superfluid transition represents only one particular universality class. Specifically, the order parameter in this case is isotropic and has two degrees of freedom (a magnitude and a phase). Thus it is representative of the case $n = 2$ of the n-vector model. In order to learn about the differences in the critical point parameters of different universality classes, it is necessary to make measurements on systems other than superfluid helium in spite of the experimental disadvantages. Therefore, in 1971 and in collaboration with Avinoam Kornblit, we started a program of specific heat measurements near magnetic phase transitions. The first material chosen for these investigations was $RbMnF_3$, of which very high quality crystals were available, and which had long been regarded as the best representative of isotropic Heisenberg systems. A Heisenberg magnet without Ising or cubic anisotropy corresponds to the case $n = 3$ of the n-vector model. We found that appreciable rounding of the specific heat due to inhomogeneities in the sample was confined to $|t| < 10^{-4}$. Although this is extremely good for solid materials, it of course cannot rival the sharpness of the superfluid transition which is limited only by gravitational effects in the region $|t| < 10^{-7}$. Nonetheless, we were able to determine A/A' and α with reasonable accuracy, and found (Kornblit *et al.*, 1973; Kornblit and Ahlers, 1973)

$$\alpha = \alpha' = -0.14 \pm 0.02 \tag{20a}$$

and

$$A/A' = 1.4 \pm 0.1. \tag{20b}$$

These values are clearly quite different from those for ^{4}He, as was expected.

The third representative of the isotropic n-vector model which is accessible to laboratory experiments is the case $n = 1$. Examples are Ising magnets or liquid–gas critical points. These systems have been widely studied by others, and have yielded the values[17] $\alpha = 0.10$ and $A/A' = 0.54$. Thus we see a monotonic trend in α and A/A' with spin dimensionality. This is shown explicitly in Fig. 5, where the experimental values of $\alpha = \alpha'$ and A'/A are shown as a function of n^{-1}. The

[17] For a summary, see, for instance, Barmatz *et al.* (1975).

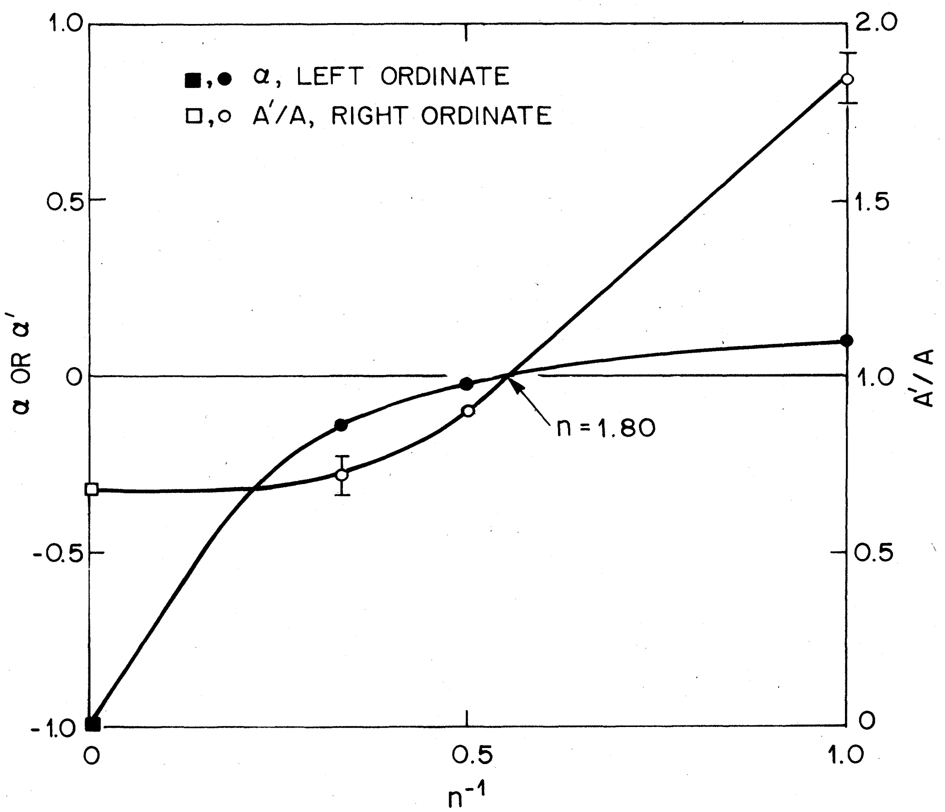

FIG. 5. Experimental (circles) and exact theoretical (squares) results for α (solid symbols, left ordinate) and A'/A (open symbols, right ordinate) as a function of the inverse spin dimensionality n^{-1}. The smooth lines through the data cross $\alpha = 0$ and $A'/A = 1$ when $n = 1.80$.

squares at $n^{-1} = 0$ represent exact theoretical results for the case $n = 0$ (Stanley, 1971; Abe and Hikami, 1975). We see that the four known values of A/A' and α fall on smooth, monotonic curves. These curves cross $\alpha = 0$ and $A/A' = 1$ for $n = 1.80$; and thus only for this physically meaningless case would we obtain the logarithmic singularity with equal amplitudes above and below the transition which was originally envisioned for the superfluid transition in ^{4}He. From the RGT we would expect a smooth variation of α and A/A' with n because the theory gives the exponents and amplitude ratios as continuous functions of n and the dimensionality d without any known singularities near $d = 3$. The numerical values given by the theory (LeGuillou and Zinn-Justin, 1977) for α at $n = 1$, 2, and 3 coincide with the experimental points in Fig. 5 on the scale of that figure.

K. Dipolar Ising systems

Our investigation of magnetic transitions was extended to a number of other materials, primarily to study the effect of dipolar interactions upon the critical behavior (Kornblit and Ahlers, 1975; Ahlers and Kornblit, 1975; Kornblit *et al.*, 1978). This work was undertaken because it had been predicted from the RGT that the universality classes of dipolar systems would be different from those of the corresponding isotropic short-range-force n-vector model, mainly because of the anisotropy of the dipolar interactions. Particularly noteworthy are the experimental results, obtained in collaboration with A. Kornblit and H. J. Guggenheim (Ahlers *et al.*, 1975) for the dipolar Ising ferromagnet $LiTbF_4$. It was first called to my attention through a seminar given at Bell Laboratories by J. Als-Nielsen that the dipolar Ising system is particularly interesting from the viewpoint of the RGT. The theory predicts that for every universality class there is a marginal dimensionality d^* which separates Landau-like behavior for large d from "scaling" behavior for small d. For $d = d^*$, the critical behavior is given correctly to leading order in the temperature and the field by the Landau theory; but the RGT predicts that there will be corrections to the leading singularity which are proportional to fractional powers of the logarithm of the reduced temperture t or the field. For the isotropic n-vector model, $d^* = 4$, and therefore these logarithmic terms are not accessible to experiment; but in the case $n = 1$ (Ising) the anisotropy of the dipolar forces reduces d^* by one. Thus, in that case, $d^* = 3$ and the logarithmic terms are accessible in the real physical world. It is remarkable that these theoretical predictions largely had been made already in 1969 by Larken and Khemelnitskii (1969), prior to Wilson's (1971) work on the RGT. These logarithms for $d = d^*$ are now recognized to be one of the central results of the RGT; but even for $d^* = 3$ they are very difficult to measure because the very weak temperature dependence of the fractional power of a logarithm is usually nearly negligible compared to that of a strong leading singularity. The best chance of observing them quantitatively existed in specific heat measurements because the Landau specific heat has only a discontinuity instead of a stronger leading singularity at T_c. Our results for $LiTbF_4$ could be fitted by expressions which to leading order in the logarithms are given by

$$C_p^+ = A \ln^z(a/t) \tag{21a}$$

and

$$C_p^- = A' \ln^z |a/t| + B' . \tag{21b}$$

The data yielded

$$A/A' = 0.24 \pm 0.01 \tag{22a}$$

and

$$z = 0.34 \pm 0.03 . \tag{22b}$$

The theoretical predictions for the dipolar Ising system are 1/4 and 1/3 respectively, in remarkable agreement with the experiment. The agreement is particularly significant because in this case the theoretical values are exact and not dependent upon approximate methods of calculation.

L. Dynamics of the superfluid transition

As a final topic, I would now like to turn briefly from static properties to the dynamics of the superfluid transition. As mentioned above, measurements of the thermal conductivity λ of ^{4}He I near T_λ were started in 1967 when the dynamic scaling theory had predicted a divergence in λ with a critical exponent equal to $\nu/2$. Since ν is expected to be equal to the exponent ζ for ρ_s, we have $\nu/2 = 0.337$. The theory also predicted certain correction terms which are dependent upon the specific heat, however, and these terms introduced some complications into the interpretation of the data. Nonetheless, when these terms were included in the analysis, the measurements yielded a leading exponent equal to 0.334 ± 0.005 (Ahlers, 1968b), in remarkable agreement with the prediction. This agreement was illusory, however. More precise measurements, and especially experiments at higher pressures carried out in 1971 (Ahlers, 1971), revealed that the thermal conductivity cannot be represented within experimental error by a power law with the specific heat corrections given by dynamic scaling. Systematic deviations of the data from the theoretical expression indicated that additional singularities like $g(t)$ in Eq. (15a) had to be included to fit the results. This is illustrated in Fig. 6 by the upper set of data. The dashed lines were obtained by fitting the results for $t \leq 10^{-5}$ to the dynamic scaling prediction, with the exponent treated as an adjustable parameter. For $10^{-5} \leq t \leq 10^{-3}$, the measured values are higher than the dashed lines, and the excess $\Delta\lambda$ is shown as the lower set of points in the figure. It is apparent that $\Delta\lambda$ is singular since it has an exponent less than unity. This evidence of confluent singular terms somewhat preceded their discovery in the superfluid density; but at the time we did not realize that their occurrence was a general phenomenon and instead thought that they were a peculiarity of transport properties. When a single additive term at^z, $z > 0$ was included in the analysis, the most precise data yielded a leading exponent equal to about 0.40 and significantly higher than the value of $\nu/2$. The early measurements at vapor pressure (Ahlers, 1968b) were not precise enough to warrant a detailed analysis in

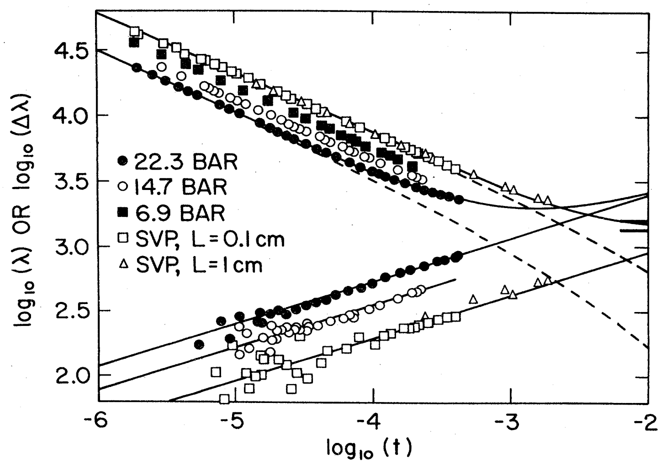

FIG. 6. Thermal conductivity λ (upper set of data), and $\Delta\lambda$ (lower set of data), in erg sec^{-1}cm^{-1}K^{-1}, as a function of t on logarithmic scales. The dashed lines are an extrapolation of a fit for $t \leq 10^{-5}$ to the dynamic scaling prediction. The difference between this extrapolation and the data is $\Delta\lambda$.

terms of this very complicated function with many parameters; but within their scatter they were consistent with the later measurements (Ahlers, 1971).

The conflict between the experimental value of the leading exponent of λ and the theoretical prediction has not yet been fully resolved; but recent applications of renormalization group methods to the dynamics of the superfluid transition have yielded much additional information (Hohenberg and Halperin, 1977). From these calculations, it follows that there are not one, but in fact four separate origins of confluent singular contributions to λ, each generating terms with their own exponents. At least one of these exponents is predicted to be quite small, and therefore the corresponding term may remain sizable even for very small t. In addition, when the theory predicts terms of order, say, t^x, it also would yield terms of order t^{nx} where n is a positive integer. If x is small (say of order 0.1), clearly one cannot neglect terms with $n > 1$ as is usually done. An analysis of the data in terms of combinations of power laws may therefore be meaningless in such a complicated case, and it may be necessary to go back one step in the theory and compare the data directly to numerical integrations of the recursions relations (the usual power laws are the result of linearizations and expansions of these relations). Attempts to do this are at the present under way in collaboration with Hohenberg and Kornblit. We do not yet know the result of such an analysis; but the predictions for the dynamics are so complicated that a definitive test of the theory at a highly quantitative level may never be possible. It is somewhat reassuring, however, that *both* theory and experiment indicate a complicated behavior. It is of course unfortunate that these complications exist particularly for the system most suitable for high-precision experimental work.

Thermal conductivity measurements were also extended to ^{3}He–^{4}He mixtures near T_λ (Ahlers, 1970). In this case, there are two modes, and dynamic scaling (Halperin and Hohenberg, 1969) predicted that the diffusivity of at least one of them should diverge as T_λ was approached from higher temperatures. The experimental results, some of which are shown in Fig. 7, revealed that the thermal conductivity remained finite at T_λ, although it was singular. This behavior has now been explained on the basis of the RGT (Siggia, 1977). Since λ, and thus the thermal diffusivity D_T, remained finite, it followed from the scaling theory that the mass diffusivity D had to diverge. Frank Pobell and I were able to demonstrate the existence of this divergence during my stay in Jülich (Ahlers and Pobell, 1974). We found an exponent for D which, although not very accurate, was consistent with the predicted $\nu/2$. An interesting problem in this area that remains to be studied experimentally is the manner in which the finite thermal conductivity of the mixtures evolves into a divergent one as the concentration vanishes (Siggia, 1977).

Another transport property which can be measured near T_λ is the damping constant D_2 of second sound. Dynamic scaling predicted that $D_2 \sim |t|^{-\nu/2}$ as T_λ is approached from below. Over a decade ago, when dynamic scaling was first proposed, J. A. Tyson (1968) reported experimental values for D_2 which were generally consistent with the predicted divergence. Recently, however, RGT calculations (see, for instance, Hohenberg and Halperin, 1977) have yielded not only the exponent, but also a universal ratio of amplitudes which made it possible to estimate the amplitude D_0 in the equation

$$D_2 = D_0 t^{-\nu/2}.$$

The theoretical estimate of D_0 was considerably smaller than the early experimental value. This disagreement between theory and experiment was rather serious because it involved not the kind of subtle quantitative features which I discussed for the thermal conductivity, but rather consisted of a gross difference in the predicted and measured value of D_0. This problem was eliminated very recently (Ahlers, 1979) by new measurements of D_2 which were smaller than the early ones by a factor of about five and in good agreement with the predictions. Even these new data are not very precise, however, and we are not yet in a position to ask the kind of detailed questions involving confluent singularities that I discussed for the thermal conductivity.

III. SUMMARY AND FUTURE DIRECTIONS

Looking back over the last decade or two, one cannot help but notice how turbulent a path was followed by the experimental study of continuous phase transitions. There have been a number of occasions when early conclusions derived from the experiments were misguided. I have mentioned a few examples (universality of liquid-gas critical points with $\beta = 0.355$; a universal logarithmic specific heat, a pressure dependent specific heat amplitude ratio A/A' along the λ-line, etc.). In no sense, however, do these occasional erroneous interpretations cast a shadow upon the quality of the experimental work. I think it is fair to say that experimental studies of critical phenomena generally have been highly quantitative, very innovative, and extremely careful throughout and even before the modern era of the field. Rather, the blind alleys into which experimentalists on occasion were led illustrate the extreme difficulties which were involved in making real, substantial progress. These difficulties are in a sense attributable to the nature of the theoretical approach that was dictated by the problem. The theory predicts the asymptotic behavior of the system, i.e., the behavior in the limit as the temperature approaches T_c and as the relevant field vanishes. By their very nature, experiments can only be done at nonzero values of $T - T_c$, and the condition of zero field can be rigorously satisfied only when special symmetry properties of the system under in-

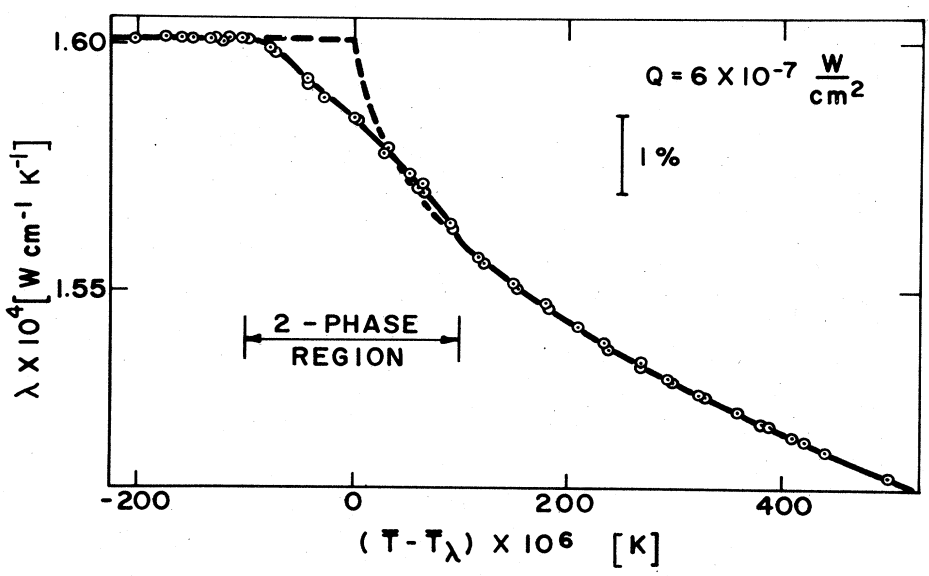

FIG. 7. The thermal conductivity λ of a ^3He-^4He mixture with a molar ^3He concentration $X = 0.15$ near the superfluid transition. The indicated two-phase region exists because the thermal gradient used in the measurement induces a concentration gradient and thus a transition temperature gradient.

vestigation require it. More specific manifestations of these problems are the influence of higher-order, or confluent, singularities, and the effect of sample inhomogeneities, which I have discussed in some detail.

In spite of the great difficulties which are involved in the interpretation of experiments, progress over the last decade or so really has been quite substantial. Let me list some of it. We know from experiment that confluent singularities exist and can be important, and to some extent we have learned how to include them in the data analysis. It is well established that the specific heat of three-dimensional systems is not logarithmic, and we have overcome the preconceived notion that critical exponents are rational. We have learned to appreciate the significance of universal amplitude ratios in addition to the exponents, and have at least a few highly accurate values of exponents and amplitude ratios. These results give us a very detailed test of at least one of the scaling laws [Eq. (18)]. The difference in the critical point parameters for different universality classes is in several cases well established and, for instance, the trend of α and A/A' with spin dimensionality of the isotropic n-vector model is known from experiment (see Fig. 5). There is good quantitative experimental evidence for the fractional powers of logarithms which occur in systems of marginal dimensionality. Although the dynamics of critical points has not been investigated as extensively as the statics, it is clear that certain transport coefficients diverge in a manner which is at least semiquantitatively in agreement with theoretical estimates. There have been, of course, many other advances in the field which I have not discussed because I was not deeply involved in them. Examples are very beautiful studies of multicritical phenomena, highly illuminating experiments on two-dimensional systems, particularly on helium films, and measurements on critical phenomena in random systems.

There remains the question of where we will go from here. The field has become very large and diversified, and no doubt there will be progress in many directions. There will be investigations of more and more exotic and complicated types of critical phenomena for which theoretical predictions will evolve. Much of that work will be extremely valuable, and it will tend to be exciting because many of the results will be qualitatively new. But I believe that it is equally as important to continue refining the kind of highly quantitative experiments on simple systems that I have described here. In practice, this can perhaps be done only for a very few systems, including the superfluid transition in liquid ^{4}He. For the superfluid transition, we must, of course, reinvestigate experimentally the problem of two-scale-factor universality. It should be possible to reduce the experimental uncertainty in the free energy per correlation volume by an order of magnitude without the development of extensive new experimental methods. If the pressure dependence of the parameter αu shown in Fig. 4 persists thereafter, then we do have a serious problem which must be addressed by the theorists. If that problem somehow gets resolved, then we are in a position with this particular system to utilize two more orders of magnitude in temperature resolution before serious problems with sample inhomogeneities due to the earth's gravitational field are encountered. There is good reason to believe that the technology of nK thermometry and temperature stability can be developed within a reasonable time span. Eventually, it should therefore be possible to test the scaling prediction Eq. (18) to say, ± 1 in the *fourth* digit behind the decimal point rather than in the *third*. At the same time one should be able to obtain α and ν with an uncertainty of perhaps $\pm 1 \times 10^{-4}$. This would be most helpful in clarifying the somewhat marginal disagreement with the theoretical exponent values which is illustrated by the numbers in Table I. I believe that this kind of highly quantitative work is very important. Either it will strengthen our belief that the RGT is an exact theory of phase transitions in real, physical systems, or it will demonstrate that the theory provides only a very good approximation to critical phenomena in the real world.[18]

[18] I ignore here the fact that true singularities exist only in infinite systems. The finiteness of real samples need not come into play even when the resolution in t is of order 10^{-9}.

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