Asymptotic freedom in deep inelastic processes in the leading order and beyond*†

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The present status of quantum chromodynamics formalism for inclusive deep-inelastic scattering is reviewed. Leading-order and higher-order asymptotic freedom corrections are discussed in detail. Both the formal language of operator product expansion and renormalization group and the intuitive parton model picture are used. Systematic comparison of asymptotic freedom predictions with deep-inelastic data is presented. Extensions of asymptotic freedom ideas to other processes such as massive μ-pair production, semi-inclusive deep-inelastic scattering, e^+e^- annihilation, and photon–photon scattering are briefly discussed. The importance of higher-order corrections is emphasized.

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*Based on the lectures given by the author at the 6th International Workshop on Weak Interactions, Iowa, 1978, and the academic training lectures presented at Fermilab, February, 1979.

†Dedicated to my mother.

I. INTRODUCTION

A. Preliminary remarks

Quantum chromodynamics (QCD) is the most promising candidate for a theory of strong interactions. It has the property of asymptotic freedom which seems to be consistent with the deep-inelastic data, and it provides a possibility of confining quarks and gluons. The quark and gluon confinement in QCD has not yet been proven. On the other hand, the theoretical structure of asymptotic freedom in deep-inelastic scattering, in the leading order and in the next to the leading order in the effective strong interaction quark–gluon coupling constant, seems to be well understood by now. Also a great effort has been made in comparing asymptotic freedom predictions with the experimental data. We think it is an appropriate time to review the present situation.

The progress in understanding the structure of asymptotic freedom in deep-inelastic scattering proceeded in several steps during the last six years. Just after the discovery of asymptotic freedom (Gross and Wilczek, 1973a, b; Politzer, 1973),[1] all calculations relevant for the leading behavior of the moments of the deep-inelastic structure function were performed (Georgi and Politzer, 1974; Gross and Wilczek, 1974; Bailin, Love, and Nanopoulos, 1974). Three years later these results were put in a form useful for phenomenological applications (de Rujula, Georgi, and Politzer, 1974; Altarelli, Parisi, and Petronzio, 1976; Glück and Reya, 1977a, b; Buras, 1977; Buras and Gaemers, 1978; Hinchliffe and Llewellyn-Smith, 1977a; Altarelli and Parisi, 1977; Tung, 1975, 1978; Fox, 1977).[2] Until recently almost all asymptotic freedom phenomenology has been based on the leading-order formulas. During the last two years, the structure of the higher-order asymptotic freedom corrections to deep-inelastic scattering has been finally understood and completed (Zee, Wilczek, and Treiman, 1974; Caswell, 1974; Jones, 1974; Floratos, Ross, and Sachrajda, 1977, 1979; Bardeen, Buras, Duke, and Muta, 1978; Altarelli, Ellis, and Martinelli, 1978) and some phenomenological applications of these higher-order results have been made.

Parallel to the development in deep-inelastic scattering there has been a lot of progress in the extension of asymptotic freedom ideas to other than deep-inelastic processes and it is appropriate to present in this review some of the results of these studies.

B. Outline

The main purpose of this review is to present

(i) the leading order of asymptotic freedom and its phenomenological implications together with comparison with deep-inelastic data,

(ii) the structure of higher-order asymptotic freedom corrections and their effect on leading-order results.

We shall also briefly discuss

(iii) leading-order and higher-order asymptotic freedom corrections to other than deep-inelastic processes.

This review is organized in a rather unconventional way, which we shall try to justify below. Section II will be what one could call a handbook of parton model and asymptotic freedom formulas relevant for deep-inelastic scattering. We begin this section by recalling basic ideas behind the simple parton model with Bjorken scaling and we quote some of its well-known formulas which will be useful in the subsequent sections. We then present systematically all asymptotic freedom expressions (leading and next-to-the-leading order) necessary for the study of the scaling violations in deep-inelastic scattering. This section ends with a general structure of present day asymptotic freedom phenomenology in the form of a procedure. This will, it is hoped, enable anybody to make her (his) own QCD fit to deep-inelastic data. One might think that it is a bad idea to begin a review with a vast array of formulas.

[1]The fact that QCD is asymptotically free was first presented (but not published) by 't Hooft at the 1972 Marseille Conference on Yang–Mills Fields.

[2]See also Novikov *et al*. (1977).

In the standard reviews, one usually relegates them to an appendix or to the last section of the text. I think, however, that such an exposition of the formulas and of the general structure of asymptotic freedom at the beginning will give the reader a good feel for the whole subject and will enable her (him) to begin her (his) own research in this field without reading too much.

The derivations, discussions, explanations, and intuitive interpretations of the formulas of Sec. II are contained in the main part of the review, namely, in Secs. III to VIII. Section III deals with QCD as the field theory of colored quarks and gluons. The basic tools necessary to study QCD implications for deep-inelastic scattering are systematically presented here. After recalling the Feynman rules for QCD, we discuss briefly the concepts of regularization and renormalization. In particular we illustrate with examples dimensional regularization ('t Hooft and Veltman, 1972) and the minimal subtraction scheme ('t Hooft, 1973). Subsequently we discuss renormalization group equations in general. Next we present the operator product expansion and its relation to the moments of deep-inelastic structure functions. Finally we derive renormalization group equations for the Wilson coefficient functions and show with examples how to calculate anomalous dimensions. This section may be omitted by experts and pedestrian readers, without loss of continuity.

In Sec. IV we present the formal approach to deep-inelastic scattering based on the operator product expansion and renormalization group. We deal here explicitly with the mixing of gluon and singlet fermion operators. The main result of this section is an expression for the moments of an arbitrary structure function in terms of the Wilson coefficient functions with an explicit Q^2 dependence calculated in the *leading order*.

In Sec. V we turn to a more intuitive approach to asymptotic freedom which, on the one hand, is a simple extension of parton model ideas and, on the other hand, is equivalent to the formal approach developed in Sec. IV. The main results of this section are the equations for the Q^2 dependence of effective parton distributions. We discuss various properties of these equations and give their approximate analytic solutions.

In Sec. VI we list various implications of asymptotic freedom for deep-inelastic processes. Subsequently, we confront these predictions with the recent high-energy ep, μp, νN, and $\bar{\nu}N$ data.

In Sec. VII we discuss asymptotic freedom *beyond the leading order*. This section is rather formal. We discuss first the nonsinglet case because it is simpler than the singlet one. The renormalization dependence and independence of various quantities is dealt with in some detail. A discussion of the meaning of the parameter Λ, the sole scale parameter of the theory (except for masses), is also given. Corrections to various parton model sum rules and relations are presented. After a phenomenological application of nonsinglet formulas we turn in Sec. VIII to the singlet case which we present in detail. We discuss some phenomenological implications of the singlet formulae for deep-inelastic data. We also present parton model formulas for these higher-order corrections. We end Sec. VIII by discussing longitudinal structure functions.

In Sec. IX we discuss briefly the extension of asymptotic freedom ideas to other processes such as massive μ-pair production, semi-inclusive deep-inelastic scattering, e^+e^- annihilation, and photon–photon scattering.

Finally in Sec. X we make a few concluding remarks. The paper ends with two appendixes, where the basic formulas of the dimensional regularization and the relations between parton distributions and the matrix elements of local operators are given.

In the last six years there have been many very good reviews on asymptotic freedom (e.g., Politzer, 1974; Gross, 1976; Ellis, 1976; Gaillard, 1977; Altarelli, 1978a; Nachtmann, 1977; Llewellyn-Smith, 1978a; Ross, (1979).[3] The new topic discussed here, which has not been presented in the reviews above (except for some discussion in the review by Ross), are the higher-order corrections (Secs. VII, VIII). We have also attempted to present the whole material in a form easy for phenomenological applications. While completing this review we received a very nice review article by Peterman (1979) who also discusses, among other topics, higher-order asymptotic freedom corrections in some detail. Although unavoidably there is some overlap between Peterman's and this review, the structure and presentation of the reviews is quite different.

II. PARTON MODEL AND ASYMPTOTIC FREEDOM FORMULAS

A. Preliminaries

1. Deep-inelastic structure functions

It is well recognized by now that deep-inelastic processes, as depicted in Fig. 1, are excellent means to study the inner structure of hadrons. The basic quantities used to discuss these processes are the structure functions W_2, W_3, and W_L, which for spin-averaged processes[4] are defined by the following equation

$$\begin{aligned} W_{\mu\nu} &= \int d^4z e^{iqz} \langle p | [J_\mu(z), J_\nu(0)] | p \rangle_{\text{spin averaged}} \\ &= e_{\mu\nu}[\nu W_L(\nu, Q^2)/2x] + d_{\mu\nu}[\nu W_2(\nu, Q^2)/2x] \\ &\quad - i\varepsilon_{\mu\nu\alpha\beta}(p_\alpha q_\beta/\nu)\nu W_3(\nu, Q^2)\,, \end{aligned} \tag{2.1}$$

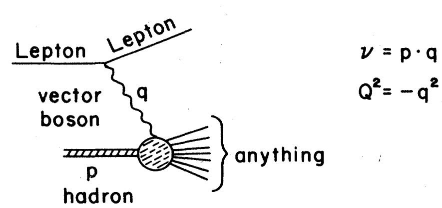


FIG. 1. Deep-inelastic lepton–hadron scattering.

[3]See also Zaharov (1976), Novikov *et al*. (1978), and Field (1979).

[4]In this review we restrict our discussions to spin-averaged processes. Asymptotic freedom effects in deep-inelastic scattering on polarized targets have been discussed by Ahmed and Ross (1975b, 1976), Altarelli and Parisi (1977), and Kodaira *et al*. (1979a, b). In particular Kodaira *et al*. calculate higher-order QCD effects. See also Gupta, Paranjape and Mani (1979) and Kodaira (1979).

where J_μ stands either for the electromagnetic current (ep, μp scattering) or a weak current ($\nu, \bar{\nu}$ scattering). For electromagnetic processes $W_3 = 0$. The tensors $e_{\mu\nu}$ and $d_{\mu\nu}$ are defined as follows:

$$e_{\mu\nu} = g_{\mu\nu} - (q_\mu q_\nu / q^2), \tag{2.2}$$

and

$$d_{\mu\nu} = -(p_\mu p_\nu / \nu^2) q^2 + (p_\mu q_\nu + p_\nu q_\mu)/\nu - g_{\mu\nu}. \tag{2.3}$$

Kinematical variables are defined in Fig. 1. For the purpose of subsequent sections, we prefer to deal with the longitudinal structure function νW_L rather than with W_1. νW_L and W_1 are related to each other as follows:

$$\nu W_L = \nu W_2 - 2xW_1. \tag{2.4}$$

The dependence of the structure functions on the variables ν and Q^2 is dictated by the underlying theory of strong interactions. The main object of this review is to study W_L, W_2, and W_3 in the framework of asymptotically free gauge theories (Politzer, 1973, 1974; Gross and Wilczek, 1973a, b).[5] First, however, let us recall how the structure functions in question behave in a simple parton model.

2. Bjorken scaling and its intuitive interpretation

As we indicated in Eq. (2.1), the structure functions depend generally on both ν and Q^2. However, if ν and Q^2 are sufficiently large so that all mass scales can be neglected, the dimensionless structure functions νW_2, νW_3, W_1, and νW_L will depend only on

$$x = (Q^2/2\nu), \tag{2.5}$$

i.e., we shall have Bjorken scaling (Bjorken, 1969)

$$\nu W^i_{2,3,L} \to F^i_{2,3,L}(x), \tag{2.6}$$

$$W^i_1 \to F^i_1(x). \tag{2.7}$$

Here i stands for a process considered $i = \nu N$, $\bar{\nu}N$, ep, μp, etc. The simple parton model was introduced by Feynman (1969) as an intuitive picture of Bjorken scaling wherein (i) target mass effects, (ii) quark mass effects, (iii) interactions between quarks (partons), (iv) $\langle p_\perp^2 \rangle$ of partons and other possible scales are *neglected*. This beautiful model is so well known to experimentalists and theorists that there is no need to describe it here in detail. A few comments and a collection of the most important parton model formulas are, however, necessary.

In the parton model one imagines that a photon, W boson, or Z^0 scatters off a free, pointlike constituent parton (q_i) as shown in Fig. 2(a). The corresponding virtual Compton amplitude is presented in Fig. 2(b). In this picture, x is the fraction of the proton momentum carried by the parton q_i. On a more quantitative level (Bjorken and Paschos, 1969, 1970; Kuti and Weisskopf, 1971; Feynman, 1972) one introduces parton distributions (quark, antiquark) $q_i(x)$ and $\bar{q}_i(x)$ which

[5]Early discussions of QCD prior to the discovery of asymptotic freedom can be found in particular in the papers by Nambu (1966), Fritzsch and Gell-Mann (1971), Bardeen, Fritzsch, and Gell-Mann (1972), Fritzsch and Gell-Mann (1972), Fritzsch, Gell-Mann, and Leutwyler (1973), and Weinberg (1973a, b).

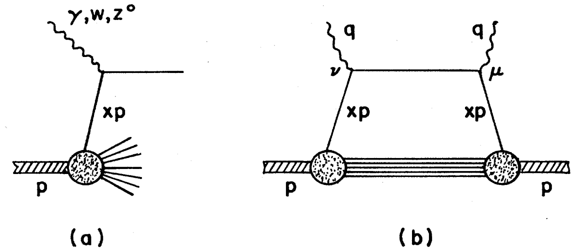


FIG. 2. Deep-inelastic scattering in the parton model. (a) vector boson–parton scattering, (b) corresponding virtual Compton amplitude. The indices μ and ν are the current indices as in Eq. (2.1).

measure the probability for finding a parton of type i in a proton with the momentum fraction x. Then, for instance,

$$F_2^{ep}(x) = \sum_i e_i^2 x [q_i(x) + \bar{q}_i(x)], \tag{2.8}$$

where e_i stands for the charge of the ith parton.

Similarly all deep-inelastic structure functions and various relevant cross sections can be expressed in terms of parton distributions weighted by the appropriate electromagnetic or "weak" charges. In the following we shall recall the rules for construction of these parton model formulas and subsequently list the most important expressions.

B. Basic formulas of the parton model

1. Parton distributions

In a four-quark model (u, d, s, c) (Glashow, Iliopoulos, and Maiani, 1970) we decompose the proton into valence part

$$V(x) = u_v(x) + d_v(x), \tag{2.9}$$

the noncharmed sea

$$S(x) \equiv u_s(x) + d_s(x) + \bar{u}(x) + \bar{d}(x) + s(x) + \bar{s}(x), \tag{2.10}$$

the charmed sea

$$C(x) \equiv c(x) + \bar{c}(x), \tag{2.11}$$

and we introduce a gluon distribution $G(x)$. The $u(x)$ and $d(x)$ distributions are then given as follows:

$$u(x) = u_v(x) + u_s(x), \tag{2.12}$$

and

$$d(x) = d_v(x) + d_s(x). \tag{2.13}$$

In what follows it will be convenient to denote generally any quark and antiquark distribution corresponding to the ith flavor, by $q_i(x)$ and $\bar{q}_i(x)$, respectively, and introduce the following combinations:

$$q(x) = \sum_i q_i(x), \tag{2.14}$$

$$\bar{q}(x) = \sum_i \bar{q}_i(x), \tag{2.15}$$

$$\Sigma(x) = q(x) + \bar{q}(x) = V(x) + S(x) + C(x), \tag{2.16}$$

$$\Delta_{ij}(x) = q_i(x) - q_j(x), \tag{2.17}$$

and

$$\overline{\Delta}_{ij}(x)=\overline{q}_i(x)-\overline{q}_j(x)\,. \tag{2.18}$$

Notice that

$$V(x)=q(x)-\overline{q}(x)\,. \tag{2.19}$$

The distributions $\Delta_{ij}(x)$, $\overline{\Delta}_{ij}(x)$, and $V(x)$ are nonsinglets under flavor symmetry SU(4), whereas $\Sigma(x)$ and $G(x)$ are singlets. The distinction between nonsinglet and singlet distributions will be very important when we come to discuss asymptotic freedom effects.

2. Electromagnetic structure functions

Taking the standard charge assignment $(u,d,s,c)=(\frac{2}{3},-\frac{1}{3},-\frac{1}{3},\frac{2}{3})$ into account one obtains

$$F_2^{ep}(x)=\tfrac{5}{18}x\,\Sigma(x)+\tfrac{1}{6}x\Delta^{ep}(x) \tag{2.20}$$

and

$$F_2^{eN}(x)=\tfrac{5}{18}x\,\Sigma(x)+\tfrac{1}{6}x\Delta^{eN}(x)\,, \tag{2.21}$$

where N denotes an isoscalar target and the nonsinglet distributions $\Delta^{ep}(x)$ and $\Delta^{eN}(x)$ are given as follows:

$$\Delta^{eN}(x)=[\overline{c}(x)-\overline{s}(x)]+[c(x)-s(x)]\,, \tag{2.22}$$

$$\Delta^{ep}(x)=\Delta^{eN}(x)+[u(x)-d(x)]+[\overline{u}(x)-\overline{d}(x)]\,. \tag{2.23}$$

3. ν and $\overline{\nu}$ cross sections

In order to write similar expressions for the $\nu,\overline{\nu}$ processes one needs a model for weak interactions. All the formulas below are for the Weinberg–Salam–GIM model (Weinberg, 1967; Salam, 1968; Glashow, Iliopoulos, and Maiani, 1970) in which the quarks are grouped in left-handed doublets and right-handed singlets

$$\begin{bmatrix} u \\ d_{\theta_C}\end{bmatrix}_L,\quad \begin{bmatrix} c \\ s_{\theta_C}\end{bmatrix}_L\quad u_R,d_R,c_R,s_R\,. \tag{2.24}$$

Here $d_{\theta_C}=d\cos\theta_C+s\sin\theta_C$ and $s_{\theta_C}=s\cos\theta_C-d\sin\theta_C$, with θ_C being the Cabibbo angle. Generalizations of the formulas below to more flavors of quarks are straightforward.

We quote first formulas for the differential cross sections $d\sigma/dxdy$ on isoscalar targets. Here $y=\nu/E$. In the parton model the cross section $d\sigma/dxdy$ is written as follows:

$$\frac{d\sigma}{dxdy}=\sum_i\left(\frac{d\sigma}{dxdy}\right)_i 2x[q_i(x)\text{ or }\overline{q}_i(x)]\,, \tag{2.25}$$

where $(d\sigma/dxdy)_i$ is the elementary cross section for scattering of $W^{\pm}$ or Z off a quark or antiquark.

If quarks are spin-$\frac{1}{2}$ particles as one usually assumes then in the Weinberg–Salam model the explicit formulas for the elementary cross sections are given (in units of G^2ME/π) as follows:

$$\left(\frac{d\sigma}{dxdy}\right)_i^{\mathrm{CC}}=\begin{cases}1 & \text{for } \nu q_i,\overline{\nu}\overline{q}_i\\ (1-y)^2 & \text{for } \nu\overline{q}_i,\overline{\nu}q_i\end{cases} \tag{2.26a}$$

$$\left(\frac{d\sigma}{dxdy}\right)_i^{\mathrm{NC}}=\begin{cases}\delta_1^2+\delta_3^2(1-y)^2 & \text{for } \nu q_i(\frac{2}{3}),\overline{\nu}\overline{q}_i(-\frac{2}{3})\\ \delta_1^2(1-y)^2+\delta_3^2 & \text{for } \overline{\nu}q_i(\frac{2}{3}),\nu\overline{q}_i(-\frac{2}{3})\\ \delta_2^2+\delta_4^2(1-y)^2 & \text{for } \nu q_i(-\frac{1}{3}),\overline{\nu}\overline{q}_i(\frac{1}{3})\\ \delta_2^2(1-y)^2+\delta_4^2 & \text{for } \overline{\nu}q_i(-\frac{1}{3}),\nu\overline{q}_i(\frac{1}{3})\end{cases}\,. \tag{2.26b}$$

CC and NC stand for the charged current and neutral current processes, respectively. The number in the parenthesis denotes the charge of the quark or antiquark. The "couplings" δ_i have the following dependence on the Weinberg angle θ_W:

$$\delta_1=\tfrac{1}{2}-\tfrac{2}{3}\sin^2\theta_W$$

$$\delta_2=-\tfrac{1}{2}+\tfrac{1}{3}\sin^2\theta_W$$

$$\delta_3=-\tfrac{2}{3}\sin^2\theta_W$$

$$\delta_4=\tfrac{1}{3}\sin^2\theta_W\,. \tag{2.27}$$

Using Eqs. (2.25)–(2.27) one can construct $d\sigma/dxdy$ for any process of interest. We quote the formulas for *isoscalar* targets. In order to simplify discussion, we neglect threshold effects due to charm production. We shall include these effects later. We then have for charged current processes

$$\left(\frac{d\sigma}{dxdy}\right)_{\mathrm{CC}}^{\nu}=(G^2ME_\nu/\pi)x\{[q(x)+s(x)-c(x)]+(1-y)^2[\overline{q}(x)+\overline{c}(x)-\overline{s}(x)]\} \tag{2.28}$$

and

$$\left(\frac{d\sigma}{dxdy}\right)_{\mathrm{CC}}^{\overline{\nu}}=(G^2ME_{\overline{\nu}}/\pi)x\{[\overline{q}(x)+\overline{s}(x)-\overline{c}(x)]+(1-y)^2[q(x)+c(x)-s(x)]\}\,. \tag{2.29}$$

For neutral current processes we obtain

$$\left(\frac{d\sigma}{dxdy}\right)_{\mathrm{NC}}^{\nu}=(G^2ME/\pi)x\{V(x)[\delta_1^2+\delta_2^2+(\delta_3^2+\delta_4^2)(1-y)^2]+[\overline{u}(x)+\overline{d}(x)][\delta_1^2+\delta_2^2+\delta_3^2+\delta_4^2][1+(1-y)^2]+[s(x)+\overline{s}(x)][\delta_2^2+\delta_4^2][1+(1-y^2]+[c(x)+\overline{c}(x)][\delta_1^2+\delta_3^2][1+(1-y)^2]\} \tag{2.30}$$

and

$$\left(\frac{d\sigma}{dxdy}\right)_{\mathrm{NC}}^{\overline{\nu}}=\left(\frac{d\sigma}{dxdy}\right)_{\mathrm{NC}}^{\nu}[(1-y)^2\leftrightarrow 1]\,. \tag{2.31}$$

Simple expressions can be obtained for the sums and the differences of the cross sections above. We have, for instance,

$$\left(\frac{d\sigma}{dxdy}\right)_{\mathrm{CC}}^{\nu+\overline{\nu}}=(G^2ME/\pi)x\{\Sigma(x)[1+(1-y)^2]+\Delta^{eN}(x)[-1+(1-y)^2]\} \tag{2.32}$$

and

$$\left(\frac{d\sigma}{dxd}\right)_{\rm CC}^{\nu-\bar\nu}=(G^2ME/\pi)xV(x)[1-(1-y)^2]\,, \tag{2.33}$$

where $\Delta^{eN}(x)$ is given by Eq. (2.22).

4. ν and $\bar\nu$ structure functions

The ν and $\bar\nu$ structure functions $F_2^{\nu,\bar\nu}(x)$, $F_3^{\nu,\bar\nu}(x)$, and $F_1^{\nu,\bar\nu}(x)$ are related to the cross sections $d\sigma/dxdy$ as follows:

$$\left(\frac{d\sigma}{dxdy}\right)^{\binom{\nu}{\bar\nu}}=(G^2ME/\pi)\{(1-y)F_2^{\nu,\bar\nu}(x) + xy^2F_1^{\nu,\bar\nu}(x)\pm(1-y/2)xyF_3^{\nu,\bar\nu}(x)\}. \tag{2.34}$$

Comparing (2.28)–(2.31) with (2.34) we obtain

$$2xF_1^{\nu,\bar\nu}(x)=F_2^{\nu,\bar\nu}(x)=x\Sigma(x) \tag{2.35}$$

and

$$xF_3^{\nu,\bar\nu}(x)=xV(x)\mp x\Delta^{eN}(x) \tag{2.36}$$

for *charged currents*, and

$$2xF_1^{\nu,\bar\nu}(x)=F_2^{\nu,\bar\nu}(x)=x\Sigma(x)[\delta_1^2+\delta_2^2+\delta_3^2+\delta_4^2] + x\Delta^{eN}(x)[\delta_1^2+\delta_3^2-\delta_2^2-\delta_4^2] \tag{2.37}$$

and

$$xF_3^{\nu,\bar\nu}(x)=xV(x)[\delta_1^2+\delta_2^2-\delta_3^2-\delta_4^2] \tag{2.38}$$

for *neutral currents*.

Notice that $F_3^{\nu,\bar\nu}(x)$ for both neutral current and charged current processes behaves as a nonsinglet whereas $F_2^{\nu,\bar\nu}$ for *charged* current processes behaves as a pure singlet. $F_2^{\nu,\bar\nu}$ for neutral current processes contains, similarly to electromagnetic structure functions [Eqs. (2.20) and (2.21)], both singlet and nonsinglet contributions.

5. Basic properties of the simple parton model

There are many consequences of parton model ideas which have been extensively discussed in the literature (e.g., Feynman, 1972; Llewellyn-Smith, 1972; Landshoff and Polkinghorne, 1972; Close, 1979). We mention only a few of them. First there is Bjorken scaling[6] in x for the structure functions and in x and y for the $d\sigma/dxdy$ cross sections. This means, for instance, that $\langle y\rangle$ and $\sigma_{\bar\nu}/\sigma_\nu$ are energy independent and the moments of the structure functions

$$\int_0^1 dx\, x^{n-2}F_i(x)\equiv M_i(n)\quad \begin{array}{l} n=2,3,\ldots\\ i=1,2,3,L\end{array} \tag{2.39}$$

are Q^2 independent.

Furthermore there exist certain sum rules and relations between structure functions which deserve attention. These are in particular:

(i) Adler sum rule (Adler, 1966)

$$\int_0^1\frac{dx}{x}[F_2^{\bar\nu p}-F_2^{\nu p}]=2\,, \tag{2.40}$$

(ii) Gross-Llewellyn-Smith sum rule (Gross and Llewellyn-Smith, 1969)

$$\int_0^1 dx[F_3^{\bar\nu p}+F_3^{\nu p}]=+6\,, \tag{2.41}$$

(iii) Bjorken sum rule (Bjorken, 1967)

$$\int_0^1 dx[F_1^{\bar\nu p}-F_1^{\nu p}]=1\,, \tag{2.42}$$

and

(iv) Callan–Gross relation (Callan and Gross, 1969)

$$F_2=2xF_1 \tag{2.43}$$

or consequently $F_L=0$.

We should also remark that the parton model has been extended to other than deep-inelastic processes. Famous examples are

$$\begin{aligned}&e^+e^-\to\pi+\text{anything}\\&e^+e^-\to\text{hadrons}\\&pp\to\mu^+\mu^-+\text{anything}\end{aligned} \tag{2.44}$$

and large $p_\perp$ processes.

The building blocks of all parton model formulas for processes listed under (2.44) are again quark distributions (and fragmentation functions) which also enter the deep-inelastic formulas. Consequently in the parton model there exist many relations between various different processes. This fact as well as the simplicity and intuitive picture behind the parton model already attracted many physicists ten years ago. In spite of the successes of the parton model in the past, this model now seems to be too simple to explain the data. In fact, although Bjorken scaling is well satisfied for $0.15\leq x\leq 0.25$ over the relevant (deep-inelastic) Q^2 range explored by present experiments ($2\leq Q^2\leq 100$ GeV^2), for $x<0.15$ and $x>0.25$ definite Q^2 dependence is seen in the data for ep and μp scattering. Similar scaling violations have been observed in high-energy $\nu,\bar\nu$ processes. In addition, the ratio $R=\sigma_L/\sigma_T$ as measured in ep scattering is definitely different from zero, contrary to Eq. (2.43). All these facts indicate that we have to go beyond the simple parton model if we want to understand the data.

6. Beyond the simple parton model

Even before the discovery of scaling violations in deep-inelastic scattering theorists found a beautiful interacting (gauge-) field theory—quantum chromodynamics, with its property of asymptotic freedom and calculable pattern of scaling violations. As we shall see in the course of this review, this theory has not only much better theoretical background than the simple parton model but also fits the existing data better. In addition in spite of a very heavy mathematical machinery the predictions of the theory in question have a very simple intuitive interpretation similar to the simple parton model but much richer.

It is perhaps useful to get a general overview and list how the simple parton model properties are modified in asymptotically free gauge theories (ASF).

First we write symbolically the ASF predictions for the moments of the structure functions as follows:

[6] We neglect mass effects for the moment.

$$\int_0^1 dx\, x^{n-2} F(x,Q^2) = A_n [\ln Q^2]^{-d_n}\left(1+\frac{f_n}{\ln Q^2}+\cdots\right), \quad (2.45)$$

where A_n, d_n, and f_n are numbers to be discussed in the subsequent sections. Then in the leading order [first term on the rhs (right-hand side) of Eq. (2.45)], all parton model formulas of this section remain unchanged except that now the parton distributions depend on both x and Q^2. In particular, all sum rules [e.g., (2.40)–(2.43)] are satisfied. The Q^2 dependence of parton distributions is calculable.

During the past year it became clear that parton model relations between various processes (deep-inelastic scattering, Drell–Yan process, etc.) also remain unchanged in the leading order.

On the other hand, if next-to-the-leading terms are taken into account [e.g., the second term in Eq. (2.45)], sum rules [e.g., (2.41)–(2.43)] are violated. One also expects, beyond the leading order, corrections to the parton model relations connecting various processes.

C. Basic formulas of asymptotic freedom

In this section we shall collect all asymptotic freedom formulas relevant for phenomenological study of deep-inelastic scattering. The derivations, discussions, and intuitive interpretations of the formulas below, can be found in Secs. III–VIII.

1. Leading order

In the leading order of asymptotic freedom all parton model formulas of Sec. II.B remain unchanged, except that now parton distributions depend on Q^2. In quantum chromodynamics the Q^2 dependence of parton distributions is given by certain equations, which we present now.

a. Effective coupling constant

Contrary to the simple parton model, which corresponds to a free field theory, QCD is an interacting field theory. The interactions between quarks and gluons can be described by the effective coupling constant $\bar{g}^2(Q^2)$, which satisfies the following equation:

$$\frac{d\bar{g}^2}{dt} = \bar{g}\beta(\bar{g}); \; \bar{g}(t=0)=g, \quad (2.46)$$

where

$$t = \ln\frac{Q^2}{\mu^2}, \quad (2.47)$$

and g is the renormalized coupling constant. Furthermore $\beta(g)$ is a renormalization group function and μ^2 is the subtraction scale at which the theory is renormalized. The presence of this scale is at the origin of scaling violations. The notion of $\beta(g)$ and μ^2 will be given in Sec. III. Here it suffices to say that $\beta(g)$ can be calculated in perturbation theory. We have

$$\beta(g) = -\beta_0\frac{g^3}{16\pi^2} - \beta_1\frac{g^5}{(16\pi^2)^2}+\cdots. \quad (2.48)$$

The parameters β_0 and β_1 have been calculated by Politzer (1973), Gross and Wilczek (1973a), Caswell (1974), and Jones (1974). In QCD [SU(3) gauge theory] they are given as follows:

$$\beta_0 = 11 - \tfrac{2}{3}f \quad (2.49a)$$

and

$$\beta_1 = 102 - (38/3)f. \quad (2.49b)$$

Here f is the number of flavors.

Keeping only the first term on the rhs of Eq. (2.48) and inserting $\beta(g)$ into Eqs. (2.46) one obtains the leading-order formula for $\bar{g}^2(Q^2)$:

$$\bar{g}^2(Q^2) = \frac{16\pi^2}{\beta_0\ln(Q^2/\Lambda^2)}. \quad (2.50)$$

The scale parameter Λ is related to μ and g as follows:

$$\Lambda^2 = \mu^2\exp[-(16\pi^2/\beta_0 g^2)]. \quad (2.51)$$

Λ is a free parameter which is to be found by comparing QCD predictions with experimental data. It follows from Eq. (2.50) that the effective coupling constant decreases with increasing Q^2 and vanishes for $Q^2=\infty$. This is what we mean by asymptotic freedom.

b. Intuitive approach

In the intuitive approach to asymptotic freedom (Kogut and Susskind, 1974)[7] to be discussed in detail in Sec. V one imagines that by increasing Q^2 of the photon or W boson or equivalently by probing the inner structure of the hadron at smaller distances one can resolve the quark into a quark and a gluon, the gluon into a quark–antiquark pair, and the gluon into two gluons. These three basic processes are shown in Fig. 3. It follows immediately from this picture that the parton distributions depend on Q^2. On a more quantitative level (Parisi, 1976; Altarelli and Parisi, 1977; Dokshitser, Dyakonov, and Troyan, 1978a) one introduces "splitting" functions $P_{ij}(z)$ which are the measure of the variation (with Q^2) of the probability of finding a parton i inside the parton j with the fraction of the parent momentum $z=x_i/x_j$. Then the equations which determine the Q^2 dependence of the parton distributions are given as follows:

$$\frac{d\Delta_{ij}(x,t)}{dt} = \frac{\bar{\alpha}(Q^2)}{2\pi}\int_x^1\frac{dy}{y}\Delta_{ij}(y,t)P_{qq}\left(\frac{x}{y}\right), \quad (2.52)$$

$$\frac{d\Sigma(x,t)}{dt} = \frac{\bar{\alpha}(Q^2)}{2\pi}\int_x^1\frac{dy}{y} \times\left[\Sigma(y,t)P_{qq}\left(\frac{x}{y}\right) + 2fG(y,t)P_{qG}\left(\frac{x}{y}\right)\right], \quad (2.53)$$

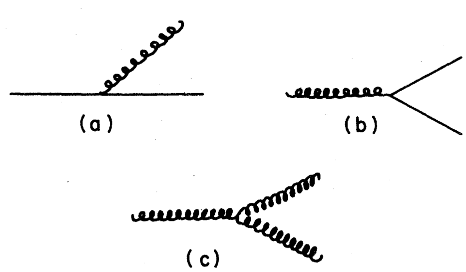


FIG. 3. Basic processes in the intuitive approach.

[7]This intuitive approach applies to all renormalizable field theories (Polyakov, 1971; Kogut and Susskind, 1974).

$$\frac{dG(x,t)}{dt}=\frac{\overline{\alpha}(Q^2)}{2\pi}\int_x^1\frac{dy}{y}\times\left[\Sigma(y,t)P_{Gq}\left(\frac{x}{y}\right)+G(y,t)P_{GG}\left(\frac{x}{y}\right)\right], \tag{2.54}$$

where Σ, Δ_{ij}, and t have been defined in Eqs. (2.16), (2.17), and (2.47), respectively. Furthermore

$$\overline{\alpha}(Q^2)\equiv\frac{\overline{g}^2(Q^2)}{4\pi} \tag{2.55}$$

and $\overline{\Delta}_{ij}(x,t)$ defined in Eq. (2.18) satisfies Eq. (2.52).

The functions $P_{ij}(z)$ are explicitly given in QCD (Altarelli and Parisi, 1977; Dokhshitser, Dyakonov, and Troyan, 1978a). They are

$$P_{qq}(z)=\frac{4}{3}\left[\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right], \tag{2.56}$$

$$P_{qG}(z)=\tfrac{1}{2}[z^2+(1-z)^2], \tag{2.57}$$

$$P_{Gq}(z)=\frac{4}{3}\left[\frac{1+(1-z)^2}{z}\right] \tag{2.58}$$

and

$$P_{GG}(z)=6\left[\frac{z}{(1-z)_+}+\frac{(1-z)}{z}+z(1-z)+\left(\frac{11}{12}-\frac{f}{18}\right)\delta(1-z)\right]. \tag{2.59}$$

The distribution $(1-z)_+^{-1}$ is defined by the following equation:

$$\int_0^1 dz\frac{f(z)}{(1-z)_+}\equiv\int_0^1 dz\frac{f(z)-f(1)}{(1-z)}, \tag{2.60}$$

where $f(z)$ is any function regular at the end points. For $z<1, (1-z)_+=1-z$. The properties of the splitting functions $P_{ij}(z)$ and the solutions of Eqs. (2.52)–(2.54) are discussed in Sec. V.

Finally we want to comment on how the integrodifferential equations above can be used in the phenomenological applications. One assumes or takes from the data the distributions $\Delta_{ij}(x,Q_0^2)$, $\Sigma(x,Q_0^2)$, and $G(x,Q_0^2)$ at a certain value of $Q^2=Q_0^2$. These distributions serve as the boundary conditions for Eqs. (2.52)–(2.54) which can be solved numerically. For practical purposes, before writing a computer program it is useful to get rid of terms $(1-z)_+^{-1}$ by employing the following formula:

$$\int_x^1 dz\, z\frac{H(x/z)}{(1-z)_+}=H(x)\ln(1-x)+\int_x^1\frac{dz}{(1-z)}[zH(x/z)-H(x)], \tag{2.61}$$

where $H(x)$ is any function regular at the end points.

c. *Formal approach*

In the formal approach to asymptotic freedom (Gross and Wilczek, 1974; Georgi and Politzer, 1974) to be discussed in detail in Sec. IV one uses the operator product expansion (Wilson, 1969) for the product of currents which enter Eq. (2.1). We write symbolically

$$J(z)J(0)=\sum_{i,n}\tilde{C}_n^i(z^2)z_{\mu_1}\cdots z_{\mu_n}O_i^{\mu_1\cdots\mu_n}, \tag{2.62}$$

where the sum runs over spin-n, twist-2 operators[8] such as the fermion nonsinglet operator $O_{\rm NS}^{\mu_1\cdots\mu_n}$ and the singlet fermion and gluon operators $O_\psi^{\mu_1\cdots\mu_n}$ and $O_G^{\mu_1\cdots\mu_n}$, respectively. Explicit expressions for these operators are given in Sec. III. $\tilde{C}_n^i(z^2)$ are the Wilson coefficient functions. We next define the reduced matrix elements, A_n^i, of the operators in question as follows:

$$\langle p|O_i^{\mu_1\cdots\mu_n}|p\rangle=A_n^i p_{\mu_1}\cdots p_{\mu_n}. \tag{2.63}$$

Then we can write (Christ, Hasslacher, and Mueller, 1972)

$$\int_0^1 dx\,x^{n-2}F_k(x,Q^2)=\sum_i A_n^i(\mu^2)C_{k,n}^i(Q^2/\mu^2,g^2), \tag{2.64}$$

where $F_k(x,Q^2)$ is an arbitrary deep-inelastic structure function ($k=1,2,3,L$) and $C_{k,n}^i(Q^2/\mu^2,g^2)$ are Fourier transforms of the coefficient functions in Eq. (2.62). Notice that in writing (2.64) we have been more explicit than in Eq. (2.62), indicating that the coefficient functions depend on the structure functions involved, and that the coefficient functions can be calculated in perturbation theory in g. We have also indicated that the reduced matrix elements A_n^i depend on μ^2.

As discussed in Sec. III, there is a set of nonsinglet operators corresponding to various λ^κ in Eq. (3.55). Since these operators neither mix under renormalization with each other nor with the singlet operators, the Q^2 dependence of their coefficient functions is in common. Therefore in this review any linear combination of nonsinglet operators will be denoted for simplicity by a common symbol $O_{\rm NS}^{\mu_1\cdots\mu_N}$ and the corresponding reduced matrix elements and coefficient functions by $A_n^{\rm NS}(\mu^2)$ and $C_{k,n}^{\rm NS}(Q^2/\mu^2,g^2)$, respectively. *It should, however, be kept in mind that $A_n^{\rm NS}(\mu^2)$ depend generally on the process and the structure function considered.* This dependence is discussed in Appendix B.

The Q^2 dependence of the Wilson coefficient functions is governed by certain equations called renormalization group equations which for the coefficient functions of nonsinglet operators take the following simple form:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-\gamma_{\rm NS}^n(g)\right]C_{k,n}^{\rm NS}\left(\frac{Q^2}{\mu^2},g^2\right)=0. \tag{2.65}$$

Here $\gamma_{\rm NS}^n(g)$ is the anomalous dimension of the spin-n nonsinglet operator and $\beta(g)$ has been defined in Eq. (2.46). The renormalization group equations are discussed in Secs. III and IV. Here it suffices to give the solution of Eq. (2.65) which is

$$C_{k,n}^{\rm NS}\left(\frac{Q^2}{\mu^2},g^2\right)=C_{k,n}^{\rm NS}(1,\overline{g}^2)\exp\left[-\int_{\overline{g}(\mu^2)}^{\overline{g}(Q^2)}dg'\frac{\gamma_{\rm NS}^n(g')}{\beta(g')}\right]. \tag{2.66}$$

The 1 on the rhs of Eq. (2.66) means simply $Q^2=\mu^2$.

$C_{k,n}^{\rm NS}(1,\overline{g}^2)$ and $\gamma_{\rm NS}^n(g)$ can be calculated in perturbation theory. Up to and including next-to-leading order corrections we have

[8]Twist = dimension − spin. Here we neglect contributions of higher-twist operators whose coefficient functions are suppressed relative to the twist-2 operators by powers of Q^2.

$$C^{\rm NS}_{k,n}(1,\bar g^2)=\begin{cases}\delta^k_{\rm NS}\left(1+\frac{\bar g^2}{16\pi^2}B^{\rm NS}_{k,n}\right) & k=1,2,3\\ \delta^L_{\rm NS}\left(0+\frac{\bar g^2}{16\pi^2}B^{\rm NS}_{L,n}\right) & k=L\end{cases} \tag{2.67}$$

and

$$\gamma^n_{\rm NS}(g)=\gamma^{(0),n}_{\rm NS}\frac{g^2}{16\pi^2}+\gamma^{(1),n}_{\rm NS}\frac{g^4}{(16\pi^2)^2}, \tag{2.68}$$

where $B^{\rm NS}_{k,n}$, $\gamma^{(0),n}_{\rm NS}$, and $\gamma^{(1),n}_{\rm NS}$ are known numbers to be specified below. $\delta^k_{\rm NS}$ are constants which depend on weak and electromagnetic charges. Specific examples of $\delta^k_{\rm NS}$ are given in Appendix B. Perturbative expansion for $\beta(g)$ is given in Eq. (2.48).

In the leading order one keeps only the first terms on the rhs of Eqs. (2.48), (2.67), and (2.68). Then using (2.64), (2.66), and (2.50) one obtains a general expression for the Q^2 evolution of the moments of any nonsinglet structure function[9]

$$\int_0^1 dx\,x^{n-2}F^{\rm NS}_k(x,Q^2)=\begin{cases}\delta^k_{\rm NS}A^{\rm NS}_n(Q_0^2)\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d^n_{\rm NS}} & k=1,2,3\\ 0 & k=L\end{cases}, \tag{2.69a}$$

where

$$d^n_{\rm NS}=\frac{\gamma^{(0),n}_{\rm NS}}{2\beta_0} \tag{2.70}$$

and we have put $\mu^2=Q_0^2$. The parameters $\gamma^{(0),n}_{\rm NS}$ have been calculated by Georgi and Politzer (1974) and Gross and Wilczek (1974) and are given as follows:

$$\gamma^{(0),n}_{\rm NS}=\frac{8}{3}\left[1-\frac{2}{n(n+1)}+4\sum_{j=2}^n\frac{1}{j}\right]. \tag{2.71}$$

Except for the value of Λ, the only unknown parameters in Eq. (2.69a) are the $A^{\rm NS}_n(Q_0^2)$. They can be determined from experiment by measuring $F^{\rm NS}_k(x,Q_0^2)$ at an arbitrary value of $Q^2=Q_0^2$.[10] Since the value of Q_0^2 in Eq. (2.69a) is arbitrary, as required by the renormalization group equations, it is often convenient to get rid of Q_0^2 by writing Eq. (2.69a) as follows:

$$\int_0^1 dx\,x^{n-2}F^{\rm NS}_k(x,Q^2)=\delta^k_{\rm NS}A^{\rm NS}_n[\ln(Q^2/\Lambda^2)]^{-d^n_{\rm NS}},\ k=1,2,3. \tag{2.69b}$$

Here $A^{\rm NS}_n$ are (independent of Q_0^2) constants which are related to $A^{\rm NS}_n(Q_0^2)$ by Eq. (4.19).

[9]For $k=1$ and 3 the power $n-2$ on the lhs of Eq. (2.69) should be replaced by $n-1$. Dependent on the structure function and process considered, Eq. (2.69a) and the following equations in this section apply either for even or odd values of n. The situation is summarized in Eq. (2.124) and explained in Sec. VII.E.3. In order to obtain predictions for all moments of n (odd and even) analytic continuation in n has to be made. This is trivial in the leading order but nontrivial in the next-to-leading order (see Footnote 33).

[10]The arbitrariness of Q_0^2 in Eq. (2.69a) is, however, restricted to sufficiently large values of Q_0^2 for which perturbative calculations can be trusted.

For singlet structure functions the situation is more complicated because, as discussed in Sec. III, the operators O^n_ψ and O^n_G mix under renormalization. The Q^2 dependence of the corresponding Wilson coefficient functions $C^\psi_{k,n}$ and $C^G_{k,n}$ is governed by two coupled renormalization group equations

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right]C^i_{k,n}\left(\frac{Q^2}{\mu^2},g^2\right)=\sum_j\gamma^n_{ji}(g^2)C^j_{k,n}\left(\frac{Q^2}{\mu^2},g^2\right)\quad i,j=\psi,G. \tag{2.72}$$

Here $\gamma^n_{ji}(g^2)$ are the elements of the anomalous dimension matrix. They have the following perturbative expansion:

$$\gamma^n_{ji}(g^2)=\gamma^{(0),n}_{ji}\frac{g^2}{16\pi^2}+\gamma^{(1),n}_{ji}\frac{g^4}{(16\pi^2)^2}+\cdots. \tag{2.73}$$

We shall discuss the solution of Eq. (2.72) in Sec. IV. Here it is sufficient to give the generalization of Eqs. (2.69a,b) to any singlet structure function $F^s_k(x,Q^2)$. In the *leading order* we have

$$\int_0^1 dx\,x^{n-2}F^s_k(x,Q^2)=\begin{cases}\delta^k_\psi A^+_n(Q_0^2)\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d^n_+}+\delta^k_\psi A^-_n(Q_0^2)\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d^n_-} & k=1,2\\ 0 & k=L\end{cases}, \tag{2.74a}$$

where δ^k_ψ are constants which depend on weak and electromagnetic charges. Specific examples of δ^k_ψ are given in Appendix B. δ^k_ψ enter the definitions of $C^\psi_{k,n}(1,\bar g^2)$ and of $C^G_{k,n}(1,\bar g^2)$, the coefficient functions of the operators O^n_ψ and O^n_G at $Q^2=\mu^2$, as follows:

$$C^\psi_{k,n}(1,\bar g^2)=\begin{cases}\delta^k_\psi[1+(\bar g^2/16\pi^2)B^\psi_{k,n}] & k=1,2\\ \delta^L_\psi[0+(\bar g^2/16\pi^2)B^\psi_{L,n}] & k=L\end{cases} \tag{2.75}$$

and

$$C^G_{k,n}(1,\bar g^2)=\begin{cases}\delta^k_\psi[0+(\bar g^2/16\pi^2)B^G_{k,n}] & k=1,2\\ \delta^L_\psi[0+(\bar g^2/16\pi^2)B^G_{L,n}] & k=L\end{cases}. \tag{2.76}$$

Here only the leading and the next-to-leading order terms have been shown. Furthermore $A^\pm_n(Q_0^2)$ are unknown constants which must be taken from experiment at one value of $Q^2=Q_0^2$. They are certain combinations [see Eqs. (5.29) and (5.30)] of the reduced matrix elements $A^\psi_n(Q_0^2)$ and $A^G_n(Q_0^2)$.

Finally the powers $d^n_\pm$ are given by

$$d^n_\pm=\lambda^n_\pm/2\beta_0, \tag{2.77}$$

where $\lambda^n_\pm$ are the eigenvalues of the one-loop anomalous dimension matrix:

$$\lambda^n_\pm=\frac{\gamma^{0,n}_{\psi\psi}+\gamma^{0,n}_{GG}\pm[(\gamma^{0,n}_{\psi\psi}-\gamma^{0,n}_{GG})^2+4\gamma^{0,n}_{\psi G}\gamma^{0,n}_{G\psi}]^{1/2}}{2}. \tag{2.78}$$

The leading-order formula (2.74a) is obtained by keeping only the first terms on the rhs of Eqs. (2.48), (2.73), (2.75), and (2.76). The parameters $\gamma^{0,n}_{ij}$ have been calculated by Georgi and Politzer (1974) and Gross and Wilczek (1974) and are given as follows:

$$\gamma_{\psi\psi}^{0,n}=\gamma_{\rm NS}^{0,n}=\frac{8}{3}\left[1-\frac{2}{n(n+1)}+4\sum_{j=2}^{n}\frac{1}{j}\right] \tag{2.79a}$$

$$\gamma_{\psi G}^{0,n}=-4f\frac{(n^2+n+2)}{n(n+1)(n+2)} \tag{2.79b}$$

$$\gamma_{G\psi}^{0,n}=-\frac{16}{3}\frac{(n^2+n+2)}{n(n^2-1)} \tag{2.79c}$$

$$\gamma_{GG}^{0,n}=6\left[\frac{1}{3}-\frac{4}{n(n-1)}-\frac{4}{(n+1)(n+2)}+4\sum_{j=2}^{n}\frac{1}{j}\right]+\frac{4}{3}f\,. \tag{2.79d}$$

It should be remarked that the nondiagonal elements of the anomalous dimension matrix, $\gamma_{\psi G}^{0,n}$ and $\gamma_{G\psi}^{0,n}$, depend on the normalization of quark and gluon operators, and only the product $\gamma_{\psi G}^{0,n}\gamma_{G\psi}^{0,n}$ is a physical quantity. In particular the nondiagonal elements in the papers by Georgi and Politzer (1974) and by Gross and Wilczek (1974) differ from each other but the product $\gamma_{\psi G}^{0,n}\gamma_{G\psi}^{0,n}$ is the same. Equations (2.79b, c) are from Gross and Wilczek.

In analogy with Eq. (2.69b) we can write Eq. (2.74a) as follows:

$$\int_0^1 dx\, x^{n-2}F_k^s(x,Q^2)=\delta_\psi^k A_n^+\left[\ln\frac{Q^2}{\Lambda^2}\right]^{-d_+^n}+\delta_\psi^k A_n^-\left[\ln\frac{Q^2}{\Lambda^2}\right]^{-d_-^n}, \tag{2.74b}$$

where $A_n^\pm$ are (independent of Q_0^2) constants which are related to $A_n^\pm(Q_0^2)$ by Eq. (4.43).

This completes the presentation of the formal approach in the leading order of asymptotic freedom. The main formulas are the Eqs. (2.69) and (2.74). They describe the Q^2 evolution of the nonsinglet and singlet structure functions in terms of three sets of unknown numbers $A_n^{\rm NS}(Q_0^2)$, $A_n^\mp(Q_0^2)$ (or $A_n^{\rm NS}, A_n^\mp$) and the scale parameter Λ. These unknown numbers and Λ are to be taken from the data.

d. Marriage of the intuitive and the formal approach

Let us denote the moments of the parton distributions $\Sigma(x,Q^2)$, $\Delta_{ij}(x,Q^2)$, and $G(x,Q^2)$ by

$$\langle\Delta_{ij}(Q^2)\rangle_n\equiv\int_0^1 dx\,x^{n-1}\Delta_{ij}(x,Q^2)\,, \tag{2.80}$$

$$\langle\Sigma(Q^2)\rangle_n\equiv\int_0^1 dx\,x^{n-1}\Sigma(x,Q^2)\,, \tag{2.81}$$

$$\langle G(Q^2)\rangle_n\equiv\int_0^1 dx\,x^{n-1}G(x,Q^2) \tag{2.82}$$

and introduce the variable

$$\bar{s}=\ln\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]. \tag{2.83}$$

It can be shown (see Sec. V for details) by using parton model formulas on the lhs (left-hand side) of Eqs. (2.69a) and (2.74a) that the formal Eqs. (2.69a) and (2.74a) are equivalent to the following equations for the Q^2 evolution of the moments of the parton distributions (Altarelli, Parisi, and Petronzio, 1976; Glück and Reya, 1977a,b; Buras, 1977; Hinchliffe and Llewellyn-Smith, 1977a; and Novikov *et al.*, 1977)

$$\langle\Delta_{ij}(Q^2)\rangle_n=\langle\Delta_{ij}(Q_0^2)\rangle_n\exp[-d_{\rm NS}^n\bar{s}]\,, \tag{2.84}$$

$$\langle\Sigma(Q^2)\rangle_n=\{(1-\alpha_n)\langle\Sigma(Q_0^2)\rangle_n-\tilde{\alpha}_n\langle G(Q_0^2)\rangle_n\}\exp[-d_+^n\bar{s}] + \{\alpha_n\langle\Sigma(Q_0^2)\rangle_n+\tilde{\alpha}_n\langle G(Q_0^2)\rangle_n\}\exp[-d_-^n\bar{s}]\,, \tag{2.85}$$

$$\langle G(Q^2)\rangle_n=\{\alpha_n\langle G(Q_0^2)\rangle_n-\varepsilon_n\langle\Sigma(Q_0^2)\rangle_n\}\exp[-d_+^n\bar{s}] + \{(1-\alpha_n)\langle G(Q_0^2)\rangle_n+\varepsilon_n\langle\Sigma(Q_0^2)\rangle_n\}\exp[-d_-^n\bar{s}]\,, \tag{2.86}$$

$$\alpha_n=\frac{\gamma_{\psi\psi}^{0,n}-\lambda_+^n}{\lambda_-^n-\lambda_+^n}\,,\quad \tilde{\alpha}_n=\frac{\gamma_{\psi G}^{0,n}}{(\lambda_-^n-\lambda_+^n)}\,,\quad \varepsilon_n=\frac{\gamma_{G\psi}^{0,n}}{\lambda_-^n-\lambda_+^n} \tag{2.87}$$

and $d_\pm^n$, $\lambda_\pm^n$, and $d_{\rm NS}^n$ are given by Eqs. (2.77), (2.78), and (2.70), respectively. The numerical values of $d_\pm^n$, $d_{\rm NS}^n$, α_n, $\tilde{\alpha}_n$, and ε_n can be found in Tables I and II. Equations (2.84)–(2.86) are very simple to use. Once the quark and gluon distributions are fixed at $Q^2=Q_0^2$ and $\langle\Sigma(Q_0^2)\rangle_n$, $\langle G(Q_0^2)\rangle_n$, and $\langle\Delta_{ij}(Q_0^2)\rangle_n$ are calculated according to (2.80)–(2.82), the rhs of Eqs. (2.84)–(2.86) are known for $Q^2\neq Q_0^2$ in terms of the single parameter Λ. This parameter can be found by fitting the lhs of the equations in question to the data (see, however, the discussion in Sec. VII). We shall demonstrate in Sec. V that Eqs. (2.84)–(2.86) are equivalent to the integrodifferential Eqs. (2.52)–(2.54).

This completes the presentation of the asymptotic freedom formulas in the leading order.

TABLE I. Numerical values of the parameters $d_{\rm NS}^n$, $d_\pm^n$, $\bar{R}_{2,n}^{\rm NS}$, $\bar{R}_{2,n}^{\pm}$, and $\bar{R}^{3,n}$ for $f=3$ and $f=4$. The table is from Bardeen and Buras (1979b).

f	n	$d_{\rm NS}^n$	d_-^n	d_+^n	$R_{2,n}^{\rm NS}$	$R_{2,n}^-$	$R_{2,n}^+$	$R_{3,n}$
3	2	0.395	0.000	0.617	1.951	−4.344	3.726	−0.271
	4	0.775	0.760	1.638	7.956	9.078	17.07	6.756
	6	1.000	0.996	2.203	13.19	12.81	30.43	12.36
	8	1.162	1.160	2.587	17.64	17.53	41.72	17.01
	10	1.289	1.287	2.882	21.50	21.44	51.41	20.99
4	2	0.427	0.000	0.747	2.098	−8.117	4.799	−0.124
	4	0.837	0.817	1.852	8.117	0.811	18.17	6.917
	6	1.080	1.074	2.460	13.34	12.99	31.63	12.52
	8	1.255	1.252	2.875	17.78	17.65	43.01	17.15
	10	1.392	1.390	3.192	21.63	21.57	52.78	21.12

TABLE II. Numerical values of the parameters which enter the formulas for the Q^2 evolution of parton distributions [Eqs. (2.85), (2.86), (2.137)–(2.144) for $f=4$ and $\overline{\mathrm{MS}}$ scheme].

n	2	4	6	8	10
α_n	0.429	0.980	0.996	0.998	0.999
$\tilde{\alpha}_n$	0.429	0.170	0.091	0.061	0.045
ε_n	0.571	0.113	0.048	0.029	0.020
Z_+^n	2.35	−2.14	−3.02	−3.42	−3.62
Z_-^n	0.00	1.95	2.17	2.27	2.35
Z_{NS}^n	1.65	2.05	2.16	2.25	2.33
K_{+-}^{ψ}	0.00	2.79	7.49	11.4	14.7
K_{-+}^{ψ}	5.90	6.87	0.221	0.070	0.034
K_{+-}^{G}	0.00	−0.056	−0.033	−0.020	−0.013
K_{-+}^{G}	−4.42	−343.0	−49.9	−39.9	−37.0

2. Higher-order corrections

In the literature most of the discussions of higher-order asymptotic freedom corrections have been done in the formal approach of Sec. II.C.1.c. We shall begin with this approach. Higher-order asymptotic freedom formulas, expressed in terms of parton distributions, will be given at the end of this section.

a. Effective coupling constant

The solution of Eq. (2.46) with $\beta(g)$ given by Eq. (2.48) can be expanded in inverse powers of $\ln(Q^2/\Lambda^2)$ with the result

$$\frac{\bar{g}^2(Q^2)}{16\pi^2}=\frac{1}{\beta_0\ln(Q^2/\Lambda^2)}-\frac{\beta_1}{\beta_0^3}\frac{\ln\ln(Q^2/\Lambda^2)}{\ln^2(Q^2/\Lambda^2)}+O\left(\frac{1}{\ln^3(Q^2/\Lambda^2)}\right). \tag{2.88}$$

Here and following (Buras, Floratos, Ross, and Sachrajda, 1977)[11] Λ has been chosen so that there are no further terms of order $1/(\ln^2 Q^2/\Lambda^2)$. A little algebra shows that μ^2, Λ^2, and g^2 are related to each other by

$$\Lambda^2=\mu^2\exp\left[-\frac{16\pi^2}{\beta_0 g^2}-\frac{\beta_1}{\beta_0^2}\ln(\beta_0 g^2)\right]. \tag{2.89}$$

Equations (2.88) and (2.89) are generalizations of Eqs. (2.50) and (2.51), respectively. In what follows we want to present the corresponding generalizations of Eqs. (2.69) and (2.74). The derivation of the formulas below is presented in great detail in Secs. VII and VIII.

b. Nonsinglet structure functions

For nonsinglet structure functions which do not vanish in the leading order, namely F_1^{NS}, F_2^{NS}, and F_3^{NS}, the generalizations of Eqs. (2.69a) and (1.69b) are given as follows:

$$M_k^{\mathrm{NS}}(n,Q^2)=\delta_{\mathrm{NS}}^k A_n^{\mathrm{NS}}(Q_0^2)\left[1+\frac{R_{k,n}^{\mathrm{NS}}(Q^2)}{\beta_0\ln(Q^2/\Lambda^2)}-\frac{R_{k,n}^{\mathrm{NS}}(Q_0^2)}{\beta_0\ln(Q_0^2/\Lambda^2)}\right]\times\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d_{\mathrm{NS}}^n} k=1,2,3 \tag{2.90a}$$

[11]Numerical values of higher-order corrections to F_2^{ep} considered in this paper are *wrong* and should be ignored.

and

$$M_k^{\mathrm{NS}}(n,Q^2)=\delta_{\mathrm{NS}}^k\bar{A}_n^{\mathrm{NS}}\left[1+\frac{R_{k,n}^{\mathrm{NS}}(Q^2)}{\beta_0\ln(Q^2/\Lambda^2)}\right]\times[\ln(Q^2/\Lambda^2)]^{-d_{\mathrm{NS}}^n}\ k=1,2,3\,, \tag{2.90b}$$

where $\bar{A}_n^{\mathrm{NS}}$ are (independent of Q_0^2) constants which are related to $A_n^{\mathrm{NS}}(Q_0^2)$ by Eq. (7.22). Furthermore

$$R_{k,n}^{\mathrm{NS}}(Q^2)=R_{k,n}^{\mathrm{NS}}-(\beta_1/\beta_0)d_{\mathrm{NS}}^n\ln\ln(Q^2/\Lambda^2)\,, \tag{2.91}$$

where

$$R_{k,n}^{\mathrm{NS}}=B_{k,n}^{\mathrm{NS}}+\frac{\gamma_{\mathrm{NS}}^{(1),n}}{2\beta_0}-\frac{\gamma_{\mathrm{NS}}^{(0),n}}{2\beta_0^2}\beta_1,\ k=1,2,3\,. \tag{2.92}$$

The parameters $B_{k,n}^{\mathrm{NS}}$ and $\gamma_{\mathrm{NS}}^{(1),n}$ have been defined in Eqs. (2.67) and (2.68), respectively, and d_{NS}^n is given by Eq. (2.70). For the longitudinal structure function we have

$$\int_0^1 dx\,x^{n-2}F_L^{\mathrm{NS}}(x,Q^2)=A_n^{\mathrm{NS}}(Q_0^2)\delta_{\mathrm{NS}}^L\times\frac{B_{L,n}^{\mathrm{NS}}}{\beta_0\ln(Q^2/\Lambda^2)}\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d_{\mathrm{NS}}^n}, \tag{2.93}$$

where $B_{L,n}^{\mathrm{NS}}$ is defined in Eq. (2.67). Because the longitudinal structure function vanishes in the leading order it follows that in the order considered $A_n^{\mathrm{NS}}(Q_0^2)$ is the same as in Eq. (2.69a). Furthermore in obtaining (2.93) the leading-order formula for $\bar{g}^2(Q^2)$, (2.50), should be used. It turns out that $\delta_{\mathrm{NS}}^{(2)}=\delta_{\mathrm{NS}}^L$ and therefore Eq. (2.93) can be written as follows (Zee, Wilczek, and Treiman, 1974):

$$\int_0^1 dx\,x^{n-2}F_L^{\mathrm{NS}}(x,Q^2)=\frac{B_{L,n}^{\mathrm{NS}}}{\beta_0\ln(Q^2/\Lambda^2)}\int_0^1 dx\,x^{n-2}F_2^{\mathrm{NS}}(x,Q^2)|_{\mathrm{LO}}\,, \tag{2.94}$$

where "LO" indicates that the moments of $F_2^{\mathrm{NS}}(x,Q^2)$ are given by the leading-order expression (2.69).

Finally we give formulas for $B_{k,n}^{\mathrm{NS}}$ and numerical results for $\gamma_{\mathrm{NS}}^{(1),n}$. The parameters $B_{L,n}^{\mathrm{NS}}$ have been calculated by Zee, Wilczek, and Treiman (1974) and are given as follows:

$$B_{L,n}^{\mathrm{NS}}=\tfrac{16}{3}[1/(n+1)]\,. \tag{2.95}$$

$B_{1,n}^{\mathrm{NS}}$, $B_{2,n}^{\mathrm{NS}}$, and $B_{3,n}^{\mathrm{NS}}$ have been calculated by Bardeen, Buras, Duke, and Muta (1978) and recalculated by Floratos, Ross, and Sachrajda (1979).[12,13] They are given as follows:

$$B_{2,n}^{\mathrm{NS}}=\frac{4}{3}\left\{3\sum_{j=1}^{n}\frac{1}{j}-4\sum_{j=1}^{n}\frac{1}{j^2}-\frac{2}{n(n+1)}\sum_{j=1}^{n}\frac{1}{j}+4\sum_{s=1}^{n}\frac{1}{s}\sum_{j=1}^{s}\frac{1}{j}+\frac{3}{n}+\frac{4}{(n+1)}+\frac{2}{n^2}-9\right\}+\frac{1}{2}\gamma_{\mathrm{NS}}^{0,n}(\ln 4\pi-\gamma_E)\,, \tag{2.96}$$

$$B_{1,n}^{\mathrm{NS}}=B_{2,n}^{\mathrm{NS}}-B_{L,n}^{\mathrm{NS}} \tag{2.97}$$

[12]The results of these two papers have been recently confirmed by Altarelli, Ellis, and Martinelli (1979a) and Harada, Kaneko, and Sakai (1979).

[13]These calculations have been done in the minimal subtraction scheme of 't Hooft (1973) and are the only existing calculations which can be combined with the two-loop anomalous dimensions calculated by Floratos, Ross, and Sachrajda (1977, 1979). See discussions below.

TABLE III. Coefficients of $g^4/(16\pi^2)$ in the anomalous dimensions $\gamma_{\rm NS}^{(1),n}$, $\gamma_{\psi\psi}^{(1),n}$, $\gamma_{\psi G}^{(1),n}$, $\gamma_{G\psi}^{(1),n}$, and $\gamma_{GG}^{(1),n}$ as given in 't Hooft's scheme for $f=3$ and $f=4$. This table has been calculated on the basis of the results of Floratos, Ross, and Sachrajda (1977, 1979).

	$\gamma_{\rm NS}^{(1),n}$		$\gamma_{\psi\psi}^{(1),n}$		$\gamma_{\psi G}^{(1),n}$		$\gamma_{G\psi}^{(1),n}$		$\gamma_{GG}^{(1),n}$	
n	3	4	3	4	3	4	3	4	3	4
2	77.70	71.37	65.84	55.56	−45.25	−60.34	−65.84	−55.56	45.25	60.34
4	133.25	120.14	132.6	119.28	7.75	10.34	−28.64	−27.40	178.9	151.61
6	164.26	147.00	164.1	146.82	16.56	22.08	−18.46	−18.28	242.9	201.94
8	186.68	166.39	186.6	166.34	19.47	25.96	−13.94	−14.08	287.6	238.16
10	204.5	181.78	204.4	181.74	20.44	27.25	−11.40	−11.67	323.1	267.48
12	219.3	194.63	219.3	194.58	20.63	27.51	−9.78	−10.11	353.1	292.44
14	232.1	205.7	232.1	205.7	20.46	27.29	−8.65	−9.00	379.0	314.2
16	243.3	215.4	243.3	215.4	20.11	26.82	−7.81	−8.17	402.1	333.7
18	253.3	224.1	253.3	224.1	19.68	26.25	−7.16	−7.52	422.8	351.2
20	262.3	231.9	262.3	231.9	19.22	25.63	−6.64	−7.00	441.6	367.3

and

$$B_{3,n}^{\rm NS}=B_{2,n}^{\rm NS}-\frac{4}{3}\,\frac{4n+2}{n(n+1)}\,. \tag{2.98}$$

The constant γ_E is the Euler–Masheroni constant $\gamma_E=0.5772\ldots$. We shall comment on the terms $(\ln 4\pi-\gamma_E)$ in Sec. II.C.2.e.

The two-loop anomalous dimensions $\gamma_{\rm NS}^{(1),n}$ have been calculated by Floratos, Ross, and Sachrajda (1977). We give only their numerical values in Table III since the corresponding analytic expressions in the original paper are rather complicated. Relatively simple analytic expressions for $\gamma_{\rm NS}^{(1),n}$ can be found in the paper by Gonzalez-Arroyo, Lopez, and Yndurain (1979b).

c. Corrections to parton model sum rules

It follows from Eqs. (2.95) to (2.98) that the sum rules (2.41) to (2.43) are no longer satisfied. The Adler sum rule (2.40), which expresses charge conservation, is, however, still respected. We have (Bardeen, Buras, Duke, and Muta, 1978; Altarelli, Ellis, and Martinelli, 1978)[14]

$$\int_0^1 dx\,[F_3^{\bar\nu p}+F_3^{\nu p}]=6\left[1-\frac{12}{(33-2f)\ln(Q^2/\Lambda^2)}\right] \tag{2.99}$$

and

$$\int_0^1 dx\,[F_1^{\bar\nu p}-F_1^{\nu p}]=1-\frac{8}{(33-2f)\ln(Q^2/\Lambda^2)}\,. \tag{2.100}$$

The violation of Callan–Gross relation (2.43) is expressed by Eq. (2.94).

d. Singlet structure functions

The expressions for the moments of the singlet structure functions with higher-order corrections included were first found by Floratos, Ross, and Sachrajda (1979). Here we present the equivalent but simpler expressions of Bardeen and Buras (1979b).

For the singlet structure functions which do not vanish in the leading order, namely F_1^s and F_2^s, the generalizations of Eqs. (2.74a) and (2.74b) are as follows[15]:

$$\begin{aligned}M_k^s(n,Q^2)={}&\delta_\psi^k A_n^+(Q_0^2)\left[1+\frac{R_{k,n}^+(Q^2)}{\beta_0\ln(Q^2/\Lambda^2)}-\frac{R_{k,n}^+(Q_0^2)}{\beta_0\ln(Q_0^2/\Lambda^2)}+f_{+-}(Q^2,Q_0^2)R_n^{+-}\right]\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d_+^n}\\&+\delta_\psi^k A_n^-(Q_0^2)\left[1+\frac{R_{k,n}^-(Q^2)}{\beta_0\ln(Q^2/\Lambda^2)}-\frac{R_{k,n}^-(Q_0^2)}{\beta_0\ln(Q_0^2/\Lambda^2)}+f_{-+}(Q^2,Q_0^2)R_n^{-+}\right]\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d_-^n}\quad k=1,2\end{aligned} \tag{2.101a}$$

and[16]

$$M_k^s(n,Q^2)=\delta_\psi^k\bar A_n^+\left[1+\frac{R_{k,n}^+(Q^2)}{\beta_0\ln(Q^2/\Lambda^2)}\right][\ln(Q^2/\Lambda^2)]^{-d_+^n}+\delta_\psi^k\bar A_n^-\left[1+\frac{R_{k,n}^-(Q^2)}{\beta_0\ln(Q^2/\Lambda^2)}\right][\ln(Q^2/\Lambda^2)]^{-d_-^n}\quad k=1,2\,, \tag{2.101b}$$

where

$$f_{\mp\pm}(Q^2,Q_0^2)=\frac{1}{\beta_0\ln(Q^2/\Lambda^2)}\left[\frac{\ln(Q_0^2/\Lambda^2)}{\ln(Q^2/\Lambda^2)}\right]^{d_\pm^n-d_\mp^n-1}-\frac{1}{\beta_0\ln(Q_0^2/\Lambda^2)}\,, \tag{2.102}$$

and

$$R_n^{\mp\pm}=\frac{\gamma_{\mp\pm}^{(1),n}}{2\beta_0+\lambda_\mp^n-\lambda_\pm^n}\,, \tag{2.103}$$

with $\gamma_{\mp\pm}^{(1),n}$ given by Eqs. (2.110) and (2.111). Furthermore

$$R_{k,n}^\mp(Q^2)=R_{k,n}^\mp-(\beta_1/\beta_0)d_\mp^n\ln\ln(Q^2/\Lambda^2) \tag{2.104}$$

and $A_n^\pm(Q_0^2)$ and $\bar A_n^\pm$ are constants to be determined from

[14]The first calculations of QCD corrections to the sum rules (2.41) and (2.43) have been done by Calvo (1977). There are, however, discrepancies between his results and results presented here.

[15]As discussed in Sec. VII.E the structure function F_3 does not depend on gluon contributions and Eq. (2.90) is therefore the full result.

[16]As discussed in Sec. VIII care must be taken when continuing Eq. (2.101b) to noninteger values of n.

experiment.

The parameters $R^{\pm}_{k,n}$ are given as follows:

$$R^{-}_{k,n}=B^{-}_{k,n}+\frac{\gamma^{(1),n}_{--}}{2\beta_0}-\frac{\lambda^n_-\beta_1}{2\beta_0^2}-\frac{\gamma^{(1),n}_{-+}}{2\beta_0+\lambda^n_- -\lambda^n_+} \tag{2.105}$$

$$k=1,2$$

$$R^{+}_{k,n}=B^{+}_{k,n}+\frac{\gamma^{(1),n}_{++}}{2\beta_0}-\frac{\lambda^n_+\beta_1}{2\beta_0^2}-\frac{\gamma^{(1),n}_{+-}}{2\beta_0+\lambda^n_+ -\lambda^n_-} \tag{2.106}$$

where

$$B^{-}_{k,n}=B^{\psi}_{k,n}+\frac{(\lambda^n_- -\gamma^{0,n}_{\psi\psi})}{\gamma^{0,n}_{\psi G}}B^{G}_{k,n} \tag{2.107}$$

and $\qquad k=1,2$

$$B^{+}_{k,n}=B^{\psi}_{k,n}+\frac{(\lambda^n_+ -\gamma^{0,n}_{\psi\psi})}{\gamma^{0,n}_{\psi G}}B^{G}_{k,n}\,. \tag{2.108}$$

$B^{\psi}_{k,n}$ and $B^{G}_{k,n}$ are defined in Eqs. (2.75) and (2.76), respectively. Furthermore

$$\gamma^{(1),n}_{--}=\frac{1}{(\lambda^n_- -\lambda^n_+)}\frac{1}{\gamma^{0,n}_{\psi G}}[\mathfrak{D}^n_1(\gamma^{0,n}_{\psi\psi}-\lambda^n_+)+\mathfrak{D}^n_2\gamma^{0,n}_{\psi G}]\,, \tag{2.109}$$

$$\gamma^{(1),n}_{-+}=\frac{1}{(\lambda^n_- -\lambda^n_+)}\frac{1}{\gamma^{0,n}_{\psi G}}[\mathfrak{D}^n_1(\gamma^{0,n}_{\psi\psi}-\lambda^n_-)+\mathfrak{D}^n_2\gamma^{0,n}_{\psi G}]\,, \tag{2.110}$$

$$\gamma^{(1),n}_{+-}=\frac{1}{(\lambda^n_- -\lambda^n_+)}\frac{1}{\gamma^{0,n}_{\psi G}}[\mathfrak{D}^n_3(\gamma^{0,n}_{\psi\psi}-\lambda^n_+)+\mathfrak{D}^n_4\gamma^{0,n}_{\psi G}]\,, \tag{2.111}$$

$$\gamma^{(1),n}_{++}=\frac{1}{(\lambda^n_- -\lambda^n_+)}\frac{1}{\gamma^{0,n}_{\psi G}}[\mathfrak{D}^n_3(\gamma^{0,n}_{\psi\psi}-\lambda^n_-)+\mathfrak{D}^n_4\gamma^{0,n}_{\psi G}]\,, \tag{2.112}$$

where

$$\mathfrak{D}^n_1=\gamma^{0,n}_{\psi G}\gamma^{(1),n}_{\psi\psi}+(\lambda^n_- -\gamma^{0,n}_{\psi\psi})\gamma^{(1),n}_{\psi G}\,, \tag{2.113}$$

$$\mathfrak{D}^n_2=\gamma^{0,n}_{\psi G}\gamma^{(1),n}_{G\psi}+(\lambda^n_- -\gamma^{0,n}_{\psi\psi})\gamma^{(1),n}_{GG}\,, \tag{2.114}$$

$$\mathfrak{D}^n_3=-\gamma^{0,n}_{\psi G}\gamma^{(1),n}_{\psi\psi}+(\gamma^{0,n}_{\psi\psi}-\lambda^n_+)\gamma^{(1),n}_{\psi G}\,, \tag{2.115}$$

$$\mathfrak{D}^n_4=-\gamma^{0,n}_{\psi G}\gamma^{(1),n}_{G\psi}+(\gamma^{0,n}_{\psi\psi}-\lambda^n_+)\gamma^{(1),n}_{GG}\,, \tag{2.116}$$

and $\gamma^{(1),n}_{ij}$ are defined in Eq. (2.73). Finally $d^n_{\pm}$ are given by Eq. (2.77). For the longitudinal structure function we have

$$\int_0^1 dx\,x^{n-2}F^{\,s}_L(x,Q^2)=A^-_n(Q_0^2)\delta^L_\psi\frac{B^-_{L,n}}{\beta_0\ln(Q^2/\Lambda^2)}\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d^n_-}+A^+_n(Q_0^2)\delta^L_\psi\frac{B^+_{L,n}}{\beta_0\ln(Q^2/\Lambda^2)}\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d^n_+}, \tag{2.117}$$

where $A^{\mp}_n(Q_0^2)$ are the same constants which enter Eq. (2.74) and $B^{\mp}_{L,n}$ are given by formulas (2.107) and (2.108) with $k=L$.

Now we give the formulas for $B^{\psi}_{k,n}$ and $B^{G}_{k,n}$ and numerical results for the two-loop anomalous dimensions $\gamma^{(1),n}_{\psi\psi}$, $\gamma^{(1),n}_{\psi G}$, $\gamma^{(1)}_{G\psi}$, and $\gamma^{(1)}_{GG}$. We have (Bardeen, Buras, Duke, and Muta, 1978)

$$B^{\psi}_{k,n}=B^{\rm NS}_{k,n}\qquad k=1,2,L\,, \tag{2.118}$$

where $B^{\rm NS}_{k,n}$ are given by Eqs. (2.98)–(2.100). Next (Walsh and Zerwas, 1973; Kingsley, 1973; Hinchliffe and Llewellyn-Smith, 1977a)

$$B^{G}_{L,n}=8f/(n+1)(n+2)\,, \tag{2.119}$$

where f is the number of flavors. The parameters $B^{G}_{2,n}$ and $B^{G}_{1,n}$ have been calculated by Bardeen, Buras, Duke, and Muta (1978) and Floratos, Ross, and Sachrajda (1979).[12,13] They are given as follows:

$$B^{G}_{2,n}=2f\left[\frac{4}{n+1}-\frac{4}{n+2}+\frac{1}{n^2}-\frac{n^2+n+2}{n(n+1)(n+2)}\left(1+\sum_{j=1}^{n}\frac{1}{j}\right)\right]+\tfrac{1}{2}\gamma^{0,n}_{\psi G}(\ln 4\pi-\gamma_E)\,, \tag{2.120}$$

$$B^{G}_{1,n}=B^{G}_{2,n}-B^{G}_{L,n}\,. \tag{2.121}$$

Finally the elements of the two-loop anomalous dimension matrix $\gamma^{(1),n}_{ij}$ have been calculated by Floratos, Ross, and Sachrajda (1979) and are collected in Table III.

This completes the presentation of the formal expressions needed for phenomenological study of higher-order corrections. Parton model formulas for higher-order corrections are discussed in Sec. II.F.

e. Miscellaneous remarks

We want to make a few explanatory remarks about the formulas above. Derivations and detailed discussions are included in Secs. VII and VIII.

As pointed out by Floratos, Ross, and Sachrajda (1977), the parameters $B^{j}_{k,n}$[17] and the two-loop anomalous dimensions $\gamma^{(1),n}_{\rm NS}$ and $\gamma^{(1),n}_{ij}$ are renormalization prescription dependent and generally gauge dependent. In other words they depend on how one renormalizes the divergent amplitudes used to calculate these quantities. Any physical quantity cannot, of course, depend on the renormalization scheme, and the renormalization prescription dependences of $B^{j}_{k,n}$ and of two-loop anomalous dimensions cancel in the expressions for physical quantities, i.e., in formulas (2.92), (2.105), and (2.106).[18] However, in order for the cancellation to occur, $B^{j}_{k,n}$, $\gamma^{(1),n}_{\rm NS}$, and $\gamma^{(1),n}_{ij}$ have to be calculated in the same scheme. All the expressions for $B^{j}_{k,n}$ and the two-loop anomalous dimensions listed above correspond to 't Hooft's minimal subtraction scheme ('t Hooft, 1973). A nice property of this scheme is that $B^{j}_{k,n}$, $\gamma^{(1),n}_{\rm NS}$, and $\gamma^{(1),n}_{ij}$ calculated in this scheme are gauge independent (Caswell and Wilczek, 1974). Calculations of $B^{j}_{k,n}$ in other schemes have been made by Kingsley (1973), Calvo (1977), De Rujula, Georgi, and Politzer (1977a), Altarelli, Ellis, and Martinelli (1978), and Kubar-André and Paige (1979).[19]

The terms which involve $(\ln 4\pi-\gamma_E)$ can be absorbed by redefining the scale parameter Λ (Bardeen, Buras, Duke, and Muta, 1978). Generally this can be done

[17] Except for $k=L$. Also corrections to various sum rules are, in this order in g^2, automatically renormalization prescription independent.

[18] This cancellation of renormalization prescription dependence is a particular example of a general theorem of Stueckelberg and Peterman (1953).

[19] The results of these calculations should not be combined with the results for $\gamma^{(1),n}_{\rm NS}$ and $\gamma^{(1),n}_{ij}$ of Floratos, Ross, and Sachrajda (1977, 1979).

with any term in $R^i_{k,n}$ proportional to $\gamma^{0,n}_{NS}$ in the case of nonsinglet structure functions and with any term proportional to $\lambda^n_\pm$ in the case of singlet structure functions. This freedom of defining Λ is related to the freedom of defining the effective coupling constant when solving the renormalization group equations. Therefore a numerical value of Λ extracted from experiment on the basis of higher-order formulas only has a meaning if the definition of the effective coupling constant is given at the same time. The same comment applies to numerical values of parameters $R^i_{k,n}$.[20] We shall give specific examples in Sec. VII.

D. Mass corrections

So far in this review we have discussed only the scaling violations due to QCD effects. Certainly at low values of $Q^2\sim 0$ (few GeV^2) target mass, heavy quark mass effects, and higher twist corrections will not be negligible. Here we shall indicate how the formulas presented above are modified in the presence of masses. We shall comment on higher-twist contributions later on.

In the last two years there has been a lot of progress in the understanding of mass corrections in the framework of QCD but we think the question is not completely solved. Neglecting for a moment the warnings which will be made subsequently, the modifications due to mass effects in the formulas above are as follows. We shall discuss here only target mass corrections and mass corrections in the case of heavy quark production off light quarks in ν and $\bar\nu$ scattering. More general cases are discussed by Georgi and Politzer (1976) and Barbieri, Ellis, Gaillard, and Ross (1976a, b).

1. Target mass corrections

In Sec. II.C we have seen that asymptotic freedom predictions are particularly simple when given for the moments (Cornwall and Norton, 1968)

$$M(n,Q^2)=\int_0^1 dx\,x^{n-2}F(x,Q^2)\,, \tag{2.122}$$

since to each given moment $(n-2)$ and for large values of Q^2 only a small number of operators of a given spin-n (so-called twist-2 operators) contribute.

For lower values of Q^2 the mass effects are non-negligible and this is no longer true. In fact there are infinitely many operators of leading twist and different spin which contribute to the nth moment in the presence of masses. It has been demonstrated by Nachtmann (1973), however, that one can redefine the moments (2.122) in such a way that to the $(n-2)$ moment only operators of spin-n will contribute as in the massless case.

The Nachtmann moments take the following form:

$$M_i(n,Q^2)=\int_0^1 dx\,\frac{\xi^{n+1}}{x^k}K_i(n,x,Q^2)F_i(x,Q^2)\,, \tag{2.123}$$

where $k=2$ for $i=3$ and $k=3$ for $i=2$. n is even for electromagnetic structure functions. For $\nu,\bar\nu$ structure functions we have

$$F_2^{\nu+\bar\nu},\ F_3^{\nu-\bar\nu}\quad n\ \text{even}\,,$$
and
$$F_2^{\nu-\bar\nu},\ F_3^{\nu+\bar\nu}\quad n\ \text{odd}\,. \tag{2.124}$$

Furthermore

$$K_2(n,x,Q^2)=\frac{n^2+2n+3+3(n+1)(1+4m^2x^2/Q^2)^{1/2}+n(n+2)(4m^2x^2/Q^2)}{(n+2)(n+3)} \tag{2.125}$$

and

$$K_3(n,x,Q^2)=\frac{1+(n+1)(1+4m^2x^2/Q^2)^{1/2}}{(n+2)}\,. \tag{2.126}$$

Here

$$\xi=\frac{2x}{[1+(1+4m^2x^2/Q^2)^{1/2}]}\,, \tag{2.127}$$

and m is the mass of the target. For $m^2/Q^2\to 0$ Eq. (2.123) reduces to (2.122). In Eq. (2.124), $F_i(x,Q^2)$ are the experimentally measured structure functions and for $M_i(n,Q^2)$ we can take the asymptotic freedom predictions as calculated for the massless case. In other words the functions $K_i(n,x,Q^2)$ are supposed to take care of target mass effects present in the data so that in evaluating the lhs of Eq. (2.123) by means of the formulas of Sec. II.C we do not need to think about target mass effects at all. For the derivation of the formulas (2.123)–(2.127) we refer the interested reader to the papers by Nachtmann (1973, 1974) and Wandzura (1977). Similar formulas have also been discussed by Baluni and Eichten (1976a, b).

One can also apply target mass effects directly to the structure functions as has been done by Georgi and Politzer (1976), De Rujula, Georgi, and Politzer (1977a, b), and by Barbieri, Ellis, Gaillard, and Ross (1976a, b). If in the massless case

$$\nu W_2(x,Q^2)\equiv \mathfrak{F}(x,Q^2)\,, \tag{2.128}$$

then with target mass corrections included

$$\begin{aligned}\nu W_2(x,Q^2)=&\frac{1}{(1+4xm^2/Q^2)^{3/2}}\frac{x^2}{\xi^2}\mathfrak{F}(\xi,Q^2)+6\frac{m^2}{Q^2}\frac{x^3}{(1+4x^2m^2/Q^2)^2}\int_\xi^1 d\xi'\,\frac{\mathfrak{F}(\xi',Q^2)}{\xi'^2}\\&+12\frac{m^4}{Q^4}\frac{x^4}{(1+4x^2m^2/Q^2)^{5/2}}\int_\xi^1 d\xi'\int_{\xi'}^1 d\xi''\,\frac{\mathfrak{F}(\xi'',Q^2)}{\xi''^2}\,.\end{aligned} \tag{2.129}$$

[20]The numerical values for higher-order parameters in Tables I and II correspond to the so-called $\overline{\rm MS}$ scheme for the effective coupling constant (see Sec. VII). In this scheme the terms $(\ln 4\pi-\gamma_E)$ in Eqs. (2.96) and (2.120) are dropped.

Similar expressions exist for $W_1(x,Q^2)$ and $\nu W_3(x,Q^2)$ which can be found in the original papers.

The mass effects introduced through Eq. (2.129) are supposed to be equivalent to those represented by the expressions (2.123). This is only true formally (Buras and Duke, 1978; Bitar, Johnson, and Tung, 1979). In practical applications in which one has to take into account that for finite Q^2 $\xi<1$, formulas (2.129) and (2.123) lead to different results. In particular, Nachtmann moments lead to the decrease of the parameter Λ relative to the massless case (Bossetti *et al.*, 1978) whereas the expressions like (2.129) lead to its increase (Buras, Floratos, Sachrajda, and Ross, 1977; Fox, 1977). Critical discussions of the two approaches above can be found in the papers by Barbieri, Ellis, Gaillard, and Ross (1976a,b), Ellis, Petronzio, and Parisi (1976), Gross, Treiman, and Wilczek (1976), and Bitar, Johnson, and Tung (1979). It follows from these discussions that one has to be very careful in using equations like (2 129) for $\xi\to 0$ or $\xi\sim 1$ [for earlier discussions on related points see Broadhurst (1975), Schnitzer (1971), and Dash (1972)]. Bitar, Johnson, and Tung (1979) and Johnson and Tung (1979) have suggested how the problems of the formulas above for $\xi\to 1$ can be overcome. It is, however, not clear whether the problems above can be solved in perturbation theory in a unique way.

2. Heavy quark mass corrections

If a heavy quark with mass m_f is produced off a light quark in ν or $\bar\nu$ scattering the standard parton model formulas are modified. The modifications in question have been formulated and calculated by Georgi and Politzer (1976) and subsequently rederived in various ways by Barbieri *et al.* (1976a,b). They have also been discussed in detail by Barnett (1976) and by Kaplan and Martin (1976). The procedure is as follows. For a light to heavy quark transition one replaces the light quark distribution as follows:

$$xq_L(x,Q^2)\to \tilde\xi q_L(\tilde\xi,Q^2)\,, \tag{2.130}$$

where

$$\tilde\xi=(Q^2+m_f^2)/2\nu=x+m_f^2/2mEy \tag{2.131}$$

and m is the proton mass. In addition the corresponding y distribution is modified according to the following rule:

$$\begin{bmatrix}1\\(1-y)^2\end{bmatrix}\to\left\{(1-y)+\frac{x}{\tilde\xi}\left[\frac{y^2}{2}\pm\left(y-\frac{y^2}{2}\right)\right]\right\}\theta(1-\tilde\xi)\,. \tag{2.132}$$

The kinematical bounds are

$$x\le 1-m_f^2/2mEy\le 1-m_f^2/2mE\,, \tag{2.133}$$

$$y\ge m_f^2/2mE(1-x)\ge m_f^2/2mE\,, \tag{2.134}$$

from which it follows that the effect of a new quark is seen first at high y and small x values. In addition since $\tilde\xi>x$ and $q_L(\tilde\xi,Q^2)$ is a decreasing function of $\tilde\xi$ the heavy quark contribution is suppressed at all values of x relative to the contribution of the light-to-light quark transition. Furthermore the y distributions are distorted relative to the massless case. In summary, the full power of the new contributions due to heavy quark production is expected to be seen only at energies well above the threshold. For a detailed description of these effects we refer the reader to papers by Barnett (1976), Kaplan and Martin (1976), and Albright and Shrock (1977).[21]

We have dealt here only with mass effects due to light quark to heavy quark transitions. The treatment of light or heavy quark production off heavy quarks is more complicated and can be found in the papers by Georgi and Politzer (1976) and Barbieri, Ellis, Gaillard, and Ross (1976a,b).

E. Structure of common asymptotic freedom phenomenology

In order to help the reader to use the formulas of this section in phenomenological applications we present here the structure of common asymptotic freedom phenomenology in the form of a recipe. This procedure should be regarded only as a first try in testing the theory. More fancy ways of confronting asymptotic freedom predictions with the data are discussed in Secs. VI, VII, and VIII.

1. Leading order

In the leading order of asymptotic freedom all parton model sum rules and relations are satisfied. Therefore all known parton model formulas (see Sec. II.B) for deep-inelastic processes are still valid except that now the parton distributions depend on Q^2.

An idealized version of a procedure for testing asymptotic freedom predictions would be then as follows:

Step 1. Write parton model formulas for all experimentally measured deep-inelastic processes as ep, μp, νN, $\bar\nu N$ and in particular consider the quantities $\langle x\rangle$, $\langle y\rangle$, νW_2, νW_3, $d\sigma/dxdy$, and also the moments of structure functions.

Step 2. Collect all existing deep-inelastic data.

Step 3. Choose a value of $Q^2=Q_0^2$ for which perturbation theory makes sense (say $Q_0^2\approx 2\ \mathrm{GeV}^2$, preferably higher) and for which the data are good enough so that all quark distributions can be found at this value of Q^2 for as broad a range in x as possible.

Step 4. Find $q_i(x,Q_0^2)$.

Step 5. Choose a gluon distribution $G(x,Q_0^2)$. The shape of the gluon distribution [e.g., the power of $(1-x)$] is not specified directly by the data. However, the second moment $n=2$ is fixed by energy–momentum conservation once the quark distributions are known.

Step 6. Choose Λ. A good starting point is $\Lambda=0.5$ GeV.

Step 7. (i) In the moment version (a) calculate $\langle\Delta_{ij}(Q_0^2)\rangle_n$, $\langle\Sigma(Q_0^2)\rangle_n$, $\langle G(Q_0^2)\rangle_n$; (b) calculate $\langle\Delta_{ij}(Q^2)\rangle_n$ and $\langle\Sigma(Q^2)\rangle_n$ using Eqs. (2.84)–(2.85); (c) calculate the moments of structure functions using formulas of Sec. II.B; (d) try to include target mass corrections using

[21]For further discussions of mass effects and heavy quark contributions to deep-inelastic scattering we refer the reader to the papers by Witten (1976) and Close, Scott, and Sivers (1976).

Nachtmann moments, keeping in mind warnings of Sec. II.D.

(ii) In the local version (a) calculate $\Delta_{ij}(x, Q_0^2)$, $\Sigma(x, Q_0^2)$, and $G(x, Q_0^2)$ and use them as boundary conditions to the integrodifferential Eqs. (2.52)–(2.54); (b) solve these equations either numerically or by means of approximate methods discussed in Sec. V; (c) insert the results for $\Delta_{ij}(x, Q^2)$, $\Sigma(x, Q^2)$ in the standard parton model formulas (Sec. II.B) and calculate various cross sections and structure functions.

Step 8. Compare the results of *Step 7* with the data. If the agreement is bad, change a few FORTRAN cards in Steps 5 and 6 and possible in Step 4.

Step 9. Check whether fits with various values of Q_0^2 give results compatible with each other. This last step can be omitted if the formal Eqs. (2.69b) and (2.74b) are used.

A few comments are necessary. In calculating the $\nu, \bar{\nu}$ total cross sections or x and y distributions as functions of energy, one has to integrate over Q^2 essentially in the range from 0 to 2 ME. Since the effective coupling constant $\bar{g}^2(Q^2)$ is large for $Q^2 < 1-2$ GeV^2 and consequently perturbative methods are inapplicable, one has to make assumptions about the Q^2 dependence of the quark distributions for the low values of Q^2. Presumably the best method is to use the data itself in this Q^2 range (Fox, 1978a). Another way to circumvent this problem is to make cuts in Q^2 in the data and consequently calculate the experimental as well as theoretical cross sections without including the low range of Q^2 where perturbation theory does not make sense (Buras, 1977).

2. Higher orders

The phenomenology of higher-order asymptotic freedom corrections has only reached an early stage of its development, and, consequently, we make only a few comments here.

One can either directly compare formulas (2.90a, b) and (2.101a, b) with the data or devise methods by means of which the effects of higher-order corrections on the leading-order predictions are most clearly seen. A typical example of such a method is the Λ_n scheme proposed by Bace (1978) and developed by Bardeen *et al.* (1978). We shall discuss this scheme and its various versions in Sec. VII and turn now to a parton model formulation of asymptotic freedom beyond the leading order.

F. Parton model formulas for higher-order corrections

We first recall that in the leading order of asymptotic freedom the formulas for the Q^2 development of deep-inelastic structure functions can be found by means of two simple rules.

Rule 1. Write a given structure function or its moments in terms of parton distributions using the standard parton model formulas of Sec. II.B, e.g.,

$$F_2^{ep}(x, Q^2) = \tfrac{5}{18} x\Sigma(x, Q^2) + \tfrac{1}{6} x\Delta^{ep}(x, Q^2) \tag{2.135a}$$

or

$$M_2^{ep}(n, Q^2) = \tfrac{5}{18} \langle\Sigma(Q^2)\rangle_n + \tfrac{1}{6}\langle\Delta(Q^2)\rangle_n^{ep}. \tag{2.135b}$$

Rule 2. Find the Q^2 evolution of the parton distributions or their moments by using Eqs. (2.52)–(2.54) or Eqs. (2.84)–(2.86), respectively.

Here we want to present a generalization of these two rules which includes next-to-leading order corrections.

As we shall discuss in more detail in Sec. VIII, there is no unique way to define parton distributions beyond the leading order of asymptotic freedom, and various definitions are possible. Here we shall present one (Baulieu and Kounnas, 1978; Kodaira and Uematsu, 1978) which is particularly useful in connection with so-called perturbative QCD (Politzer, 1977a) on which we shall comment in Sec. IX.

Let us illustrate the new Rules 1′ and 2′ by considering the moments $M_2^{ep}(n, Q^2)$ of Eq. (2.135b).

Rule 1′

$$M_2^{ep}(n, Q^2) = \frac{5}{18}\left[\langle\Sigma(Q^2)\rangle_n\left(1 + \frac{\bar{g}^2(Q^2)}{16\pi^2}\bar{B}^{\psi}_{2,n}\right) + \frac{\bar{g}^2(Q^2)}{16\pi^2}\bar{B}^{G}_{2,n}\langle G(Q^2)\rangle_n\right] + \frac{1}{6}\left[\langle\Delta(Q^2)\rangle_n^{ep}\left(1 + \frac{\bar{g}^2(Q^2)}{16\pi^2}\bar{B}^{\rm NS}_{2,n}\right)\right], \tag{2.136}$$

where $\bar{B}^{\rm NS}_{2,n}$, $\bar{B}^{\psi}_{2,n}$, and $\bar{B}^{G}_{2,n}$ are obtained from Eqs. (2.96), (2.118), and (2.120), respectively, by removing there terms $(\ln 4\pi - \gamma_E)$ (see Sec. VII). The factors which multiply various parton distributions are simply the Wilson coefficient functions $C^{\psi}_{2,n}(1, \bar{g}^2)$, $C^{G}_{2,n}(1, \bar{g}^2)$, and $C^{\rm NS}_{2,n}(1, \bar{g}^2)$. For any other structure function, one just replaces the index "2" by "k" and changes charge factors and nonsinglet quark distributions in accordance with the rules of the standard parton model. For F_3, $\langle\Sigma(Q^2)\rangle_n$ and $\langle G(Q^2)\rangle_n$ are absent. The contributions of the two-loop β function and the two-loop anomalous dimension matrix are contained in the definition of the parton distributions.

We have found explicit expressions for the Q^2 evolution of the parton distributions in Eq. (2.136) which read as follows:

Rule 2′

$$\langle\Delta_{ij}(Q^2)\rangle_n = \langle\Delta_{ij}(Q_0^2)\rangle_n\left[\frac{\bar{g}^2(Q^2)}{\bar{g}^2(Q_0^2)}\right]^{d^n_{\rm NS}} H^n_{\rm NS}(Q^2, Q_0^2) \tag{2.137}$$

$$\langle\Sigma(Q^2)\rangle_n = [(1-\alpha_n)\langle\Sigma(Q_0^2)\rangle_n - \tilde{\alpha}_n\langle G(Q_0^2)\rangle_n]\left[\frac{\bar{g}^2(Q^2)}{\bar{g}^2(Q_0^2)}\right]^{d^n_+} H^n_{+\psi}(Q^2, Q_0^2) + [\alpha_n\langle\Sigma(Q_0^2)\rangle_n + \tilde{\alpha}_n\langle G(Q_0^2)\rangle_n]\left[\frac{\bar{g}^2(Q^2)}{\bar{g}^2(Q_0^2)}\right]^{d^n_-} H^n_{-\psi}(Q^2, Q_0^2) \tag{2.138}$$

$$\langle G(Q^2)\rangle_n = [\alpha_n\langle G(Q_0^2)\rangle_n - \varepsilon_n\langle\Sigma(Q_0^2)\rangle_n]\left[\frac{\bar{g}^2(Q^2)}{\bar{g}^2(Q_0^2)}\right]^{d^n_+} H^n_{+G}(Q^2, Q_0^2) + [(1-\alpha_n)\langle G(Q_0^2)\rangle_n + \varepsilon_n\langle\Sigma(Q_0^2)\rangle_n]\left[\frac{\bar{g}^2(Q^2)}{\bar{g}^2(Q_0^2)}\right]^{d^n_-} H^n_{-G}(Q^2, Q_0^2), \tag{2.139}$$

where α_n, $\tilde{\alpha}_n$, and ε_n are given in Eq. (2.87), and d^n_{NS} and $d^n_\pm$ have been defined in Eqs. (2.70) and (2.77), respectively. Notice similarities with the leading-order expressions (2.84)–(2.86). The factors H^n_i which include higher-order corrections are given as follows:

$$H^{NS}_n(Q^2, Q_0^2) = 1 + \frac{[\bar{g}^2(Q^2) - \bar{g}^2(Q_0^2)]}{16\pi^2} Z^n_{NS} \tag{2.140}$$

$$H^n_{\pm i} = 1 + \frac{[\bar{g}^2(Q^2) - \bar{g}^2(Q_0^2)]}{16\pi^2} Z^n_\pm + \left\{ \frac{\bar{g}^2(Q_0^2)}{16\pi^2} \left[\frac{\bar{g}^2(Q^2)}{\bar{g}^2(Q_0^2)} \right]^{d^n_\mp - d^n_\pm} - \frac{\bar{g}^2(Q^2)}{16\pi^2} \right\} K^i_{\pm\mp} \quad i = \psi, G, \tag{2.141}$$

where

$$K^i_{\pm\mp} = \begin{cases} \dfrac{\gamma^{(1),n}_{\pm\mp}}{2\beta_0 + \lambda^n_\pm - \lambda^n_\mp} & i = \psi \\[2ex] \dfrac{\gamma^{(1),n}_{\pm\mp}}{2\beta_0 + \lambda^n_\pm - \lambda^n_\mp} \dfrac{\gamma^{0,n}_{\psi\psi} - \lambda^n_\mp}{\gamma^{0,n}_{\psi\psi} - \lambda^n_\pm} & i = G \end{cases} \tag{2.142}$$

and

$$Z^n_\pm = \frac{\gamma^{(1),n}_{\pm\pm}}{2\beta_0} - \frac{\lambda^n_\pm \beta_1}{2\beta_0^2}, \tag{2.143}$$

$$Z^n_{NS} = \frac{\gamma^{(1),n}_{NS}}{2\beta_0} - \frac{\gamma^{(0),n}_{NS}}{2\beta_0^2} \beta_1. \tag{2.144}$$

Finally $\gamma^{(1),n}_{ij}$ are defined in Eqs. (2.109)–(2.112) and the Q^2 evolution of $\bar{g}^2(Q^2)$ which enters Eqs. (2.137)–(2.139) is given in Eq. (2.88). On the other hand, in the order considered, the leading-order formula (2.50) for $\bar{g}^2(Q^2)$ should be used in Eqs. (2.136), (2.140), and (2.141). The derivation and properties of Eqs. (2.138)–(2.139) are discussed in Sec. VIII. The numerical values of the parameters which enter Eqs. (2.137)–(2.144) can be found in Table II.

We would like to remark that the parton distributions as given in Eqs. (2.136)–(2.139) are renormalization-prescription dependent, i.e., they depend on how various operators in the Wilson operator product expansion are renormalized. This renormalization prescription dependence is canceled, however, by that of the B^i's which enter Eq. (2.136). Since one can define parton distributions in many ways anyhow, we think that one should not worry about this renormalization prescription dependence of parton distributions discussed here. For different definitions of parton distributions we refer the reader to the papers by Altarelli, Ellis, and Martinelli (1978) and Floratos, Ross, and Sachrajda (1979). In particular Floratos *et al.* present explicit expressions for the Q^2 evolution of their parton distributions.

G. Longitudinal structure functions

Finally we quote the formula for the longitudinal structure function which we shall derive in Sec. VIII. We have

$$F_L(x, Q^2) = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)} \int_x^1 \frac{dy}{y} \frac{x^2}{y^2} \left[\frac{16}{3} F_2(y, Q^2) + \delta^{(2)}_\psi 8f \left(1 - \frac{x}{y}\right) y G(y, Q^2) \right], \tag{2.145}$$

where $F_2(y, Q^2)$ and $G(y, Q^2)$ are the full structure functions F_2 (singlet + nonsinglet) and the gluon distribution, respectively. The leading-order formulas for the Q^2 evolution of $F_2(y, Q^2)$ and $G(y, Q^2)$ should be used in Eq. (2.144). For $f=4$, $\delta^{(2)}_\psi = 5/18$ for ep scattering and $\delta^{(2)}_\psi = 1$ for ν and $\bar{\nu}$ scattering. The generalizations to an arbitrary number of flavors are straightforward.

III. QUANTUM CHROMODYNAMICS AND TOOLS TO STUDY IT

In this section we shall collect certain information about quantum chromodynamics (QCD). We shall also present the tools necessary to extract QCD predictions for deep-inelastic scattering. We shall discuss regularization, renormalization, operator product expansion, and renormalization group equations. Our discussion is mainly devoted to those readers who would like to gain enough information about these topics to be able to calculate such quantities as the β function, the anomalous dimensions of quark and gluon fields, and the anomalous dimensions of various operators relevant to the discussion of scaling violations. Therefore in our presentation we shall try only to give the reader a feeling for what is going on—very often by showing examples. We refer the interested reader to the textbooks (Bogolubov and Shirkov, 1959; Bjorken and Drell, 1965; De Rafael, 1977) and to various papers, reviews, and lectures (Zimmerman, 1971; Abers and Lee, 1973; Politzer, 1974; Coleman, 1971; Gross and Wilczek, 1973b, 1974; Abarbanel, 1974; Gross, 1976; Crewther, 1976; Ellis, 1976; Lautrup, 1976; Taylor, 1976; Marciano and Pagels, 1978) where the material of this section is presented in great depth. This section may be omitted by the experts, without loss of continuity.

A. Lagrangian and Feynman rules

Quantum chromodynamics is an $SU(3)_c$ color gauge theory which describes the interactions between quarks and gluons. Quarks are arranged in color triplets and come in f flavors. The QCD Lagrangian is given by

$$\mathcal{L} = -\tfrac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + i\bar{\psi}_\alpha (\gamma^\mu \mathfrak{D}^\mu_{\alpha\beta} + im\,\delta_{\alpha\beta}) \psi_\beta, \tag{3.1}$$

where $a = 1, \ldots 8$, $\alpha, \beta = 1, 2, 3$ and the summation over repeated indices and over flavors is understood. Furthermore

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f^{abc} G^b_\mu G^c_\nu \tag{3.2}$$

is the field strength and

$$\mathfrak{D}^\mu_{\alpha\beta} = \delta_{\alpha\beta} \partial_\mu - ig \lambda^a_{\alpha\beta} G^a_\mu \tag{3.3}$$

is the covariant derivative. ψ_α and G^a_μ are fermion and gluon fields, respectively. Finally g is the strong interaction coupling constant. The matrices λ^a obey the commutation relations

$$[\lambda^a, \lambda^b] = i f^{abc} \lambda^c , \tag{3.4}$$

with f^{abc} being the structure constants of $SU(3)_c$. It should be remarked that in order to specify the theory completely one must add a gauge fixing term to $\mathcal{L}$, which for commonly used gauges (covariant gauges) has the form $-\mathrm{Tr}[(\partial^\mu G_\mu)^2]/\alpha$, where α is a gauge parameter. In these gauges one must add Fadeev–Popov ghost interactions to the Lagrangian. In the axial gauge ($G_3^a = 0$) there are no ghosts but the calculations of Feynman diagrams are generally more complicated than in covariant gauges. From (3.1), by means of standard techniques, (e.g., 't Hooft and Veltman, 1973; Gross and Wilczek, 1973b) one can derive Feynman rules which are shown in Fig. 4.

The new features relative to quantum electrodynamics (QED) are (i) SU(3) group factors, (ii) existence of the triple and quartic gluon couplings, and (iii) existence of fictitious ghost couplings. Otherwise the calculations of QCD Feynman diagrams are very similar to the corresponding QED calculations. The relevant group factors are defined as follows:

$$\begin{aligned} &\delta_{ij} C_2(R) = \lambda^a_{ik}\lambda^a_{kj} ,\\ &\delta_{ab} C_2(G) = f^{acd} f^{bcd} ,\\ &\delta_{ab} T(R) = f \lambda^a_{jk}\lambda^b_{kj} . \end{aligned} \tag{3.5}$$

For an SU(3) gauge theory with f flavors, as discussed here,

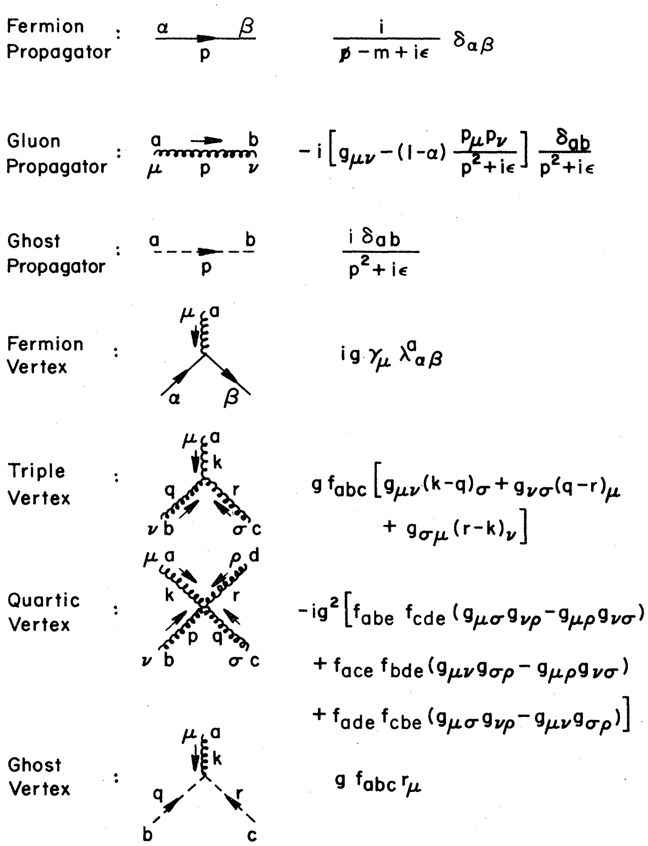


FIG. 4. Feynman rules for quantum chromodynamics.

$$C_2(R) = \tfrac{4}{3},\ C(G) = 3 , \text{ and } T(R) = \tfrac{1}{2} f . \tag{3.6}$$

To make the phenomenological applications easier, in all expressions presented in this review, the values of the group theory factors as given by Eq. (3.6) have been explicitly used.

B. Renormalization and renormalization group equations

As in QED many calculations of QCD Feynman diagrams with one or more loops lead to divergent results, and renormalizations of various quantities which enter the calculations (vertices, propagators) are necessary in order to obtain finite physical answers. There exists a whole machinery for extracting finite physical answers from perturbation theory, called the renormalization program. It consists of two steps: (i) regularization and (ii) renormalization.

1. Dimensional regularization

The first step in any renormalization program is to identify the singularities of a given Feynman diagram and extract them in such a way that renormalization can be easily performed. One can achieve this is many ways, but a particularly elegant and simple method is the dimensional regularization procedure of 't Hooft and Veltman (1972)[22] in which Feynman diagrams are evaluated in $D = 4 - \varepsilon$ dimensions and singularities are extracted as poles: $1/\varepsilon, 1/\varepsilon^2$, etc.

Let us illustrate this method with a simple example. Consider the one-loop diagram of Fig. 5(a). It represents the virtual gluon correction to the quark propagator. Here we treat gluons and quarks as massless particles, and, in order to avoid mass singularities, we take external quark momentum slightly off-shell ($p^2 < 0$). Using the Feynman rules of Fig. 4, we first obtain (in the Feynman gauge)

$$\sum^{(2)}(\not p) = r g^2 \int \frac{d^D k}{(2\pi)^D} \frac{\gamma_\nu \not k \gamma_\nu}{k^2 (k-p)^2} , \tag{3.7}$$

$$= r g^2 (\varepsilon - 2) \int \frac{d^D k}{(2\pi)^D} \frac{\not k}{k^2 (k-p)^2} , \tag{3.8}$$

where we have used formula (A18) to reduce the Dirac algebra in D dimensions and we have put all group and i factors in one symbol r. Conventionally we have denoted the whole result by $\sum^{(2)}(\not p)$. Next using the Feynman parametrization (A13) for the denominators and integrating over k by means of the formula (A1), we obtain

$$\sum^{(2)}(\not p) = \not p \sum^{(2)}(p^2) , \tag{3.9}$$

where

$$\sum^{(2)}(p^2) = i r \frac{g^2}{(4\pi)^{D/2}} \frac{\Gamma(\varepsilon/2)}{(-p^2)^{\varepsilon/2}} (\varepsilon - 2) B\left(2 - \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right), \tag{3.10}$$

and $\Gamma(\varepsilon/2)$ and $B(2 - \varepsilon/2, 1 - \varepsilon/2)$ are Euler-gamma and

[22] For a review see Leibrandt (1974) and references therein. Basic formulas of dimensional regularization are collected in Appendix A. Useful calculations can be found in the paper by Marciano (1975).

Euler-beta functions, respectively. The singularity has been nicely isolated in the gamma function $\Gamma(\varepsilon/2)$. It is instructive to expand Eq. (3.10) around the singular point $\varepsilon \approx 0$. Before doing this let us recall that in $D \neq 4$ dimensions, g^2 is not dimensionless although it is dimensionless in four dimensions. It is convenient to work with a dimensionless coupling in D dimensions, and so we make in (3.10) the replacement

$$g^2 \to g^2 \bar{\mu}^{\varepsilon}, \tag{3.11}$$

where $\bar{\mu}$ is an arbitrary parameter with the dimensions of mass and g^2 on the rhs. of Eq. (3.11) is dimensionless. Now expanding Eq. (3.10) around $\varepsilon = 0$ by utilizing formula (A11), we obtain after neglecting terms $O(\varepsilon)$

$$\sum^{(2)}(p^2) = -i\not{r}\frac{g^2}{16\pi^2}\left[\frac{2}{\varepsilon} - \ln\frac{-p^2}{\bar{\mu}^2} + 1 + \ln 4\pi - \gamma_E\right], \tag{3.12}$$

where γ_E is the Euler–Mascheroni constant which we already encountered in Sec. II. We have thus extracted the singularity as a $1/\varepsilon$ pole and have obtained a well-defined finite part.

2. Renormalization

In order to illustrate the general idea of renormalization, consider the QCD Lagrangian (3.1) continued to $D = 4 - \varepsilon$ dimensions. For simplicity we put fermion masses to zero ($m_\alpha = 0$) and we work with the dimensionless coupling constant g by making the replacement (3.11) in (3.1). We denote the resulting Lagrangian by

$$\mathcal{L}_\varepsilon(G^a_\mu, \psi_\alpha, g^2\bar{\mu}^\varepsilon). \tag{3.13}$$

Using this Lagrangian we shall encounter singularities in the Feynman diagrams which will appear as poles in ε. A specific example has been shown above. To remove these divergences we add to $\mathcal{L}_\varepsilon$ counter terms. We can write the resulting, renormalized, Lagrangian $\mathcal{L}_R$ in terms of the bare fields $(G^a_\mu)^0, \psi^0_\alpha$, and the bare coupling g_0 as follows:

$$\mathcal{L}_R = \mathcal{L}_\varepsilon(G^a_\mu, \psi_\alpha, g^2\bar{\mu}^\varepsilon) + \mathcal{L}(\varepsilon)_{\substack{\text{counter}\\ \text{terms}}} = \mathcal{L}_\varepsilon[(G^a_\mu)^0, \psi^0_\alpha, g_0^2]. \tag{3.14}$$

The bare and renormalized quantities are related to each other by the following equations:

$$(G^a_\mu)^0 = Z_G^{1/2} G^a_\mu, \tag{3.15}$$

$$\psi^0_\alpha = Z_\psi^{1/2}\psi_\alpha, \tag{3.16}$$

$$g_0 = \mu^{\varepsilon/2} g Z_g, \tag{3.17}$$

where Z_G, Z_ψ, and Z_g are renormalization constants, which diverge for $\varepsilon \to 0$. Z_G, Z_ψ, and Z_g are known once the counter terms are specified. Before going into detail let us get a feeling for the renormalization procedure. One can look upon it in two obviously equivalent ways.

(a) The calculation of a Feynman diagram is performed in terms of the bare parameters (e.g., g_0) and the resulting divergent expression is rendered finite by rewriting it in terms of the renormalized parameters by utilizing relations (3.15)–(3.17). One can imagine that all singularities have been absorbed by introducing the renormalized quantities. In practical applications it is often more convenient to deal exclusively with renormalized parameters and proceed as follows:

(b) The calculation of a Feynman diagram is performed in terms of the renormalized parameters (e.g., g), as in the example of Sec. III.B.1, and the resulting divergent expression is rendered finite by subtracting the singularities in one way or another (see below). Once the *subtraction scheme* has been specified, the renormalization constants Z_g, Z_G, Z_ψ can be found.

Let us illustrate the idea of a subtraction scheme by renormalizing the expression (3.12).

3. Two subtraction schemes

a. Subtraction at $p^2 = -\mu^2$

In this subtraction scheme the renormalized vertices and propagators are found by specifying their values at particular values in momentum space. Let us consider the following example.

The unrenormalized inverse fermion propagator $S^{-1}(p)$ resulting from Figs. 5(a) and 5(b) is

$$S^{-1}(p) = \not{p} + \not{p}\sum^{(2)}(p^2), \tag{3.18}$$

with $\sum^{(2)}(p^2)$ given by Eq. (3.12). We can require that the renormalized inverse fermion propagator

$$S_R^{-1}(p) = \not{p} + \not{p}\sum^{(2)}_{(R)}(p^2/\mu^2), \tag{3.19}$$

at $p^2 = -\mu^2$ satisfies

$$S_R^{-1}(p)\,|_{p^2=-\mu^2} = \not{p}, \tag{3.20}$$

where μ is an arbitrary momentum. This is achieved if

$$\sum^{(2)}_{(R)}(p^2/\mu^2) = \sum^{(2)}(p^2) - \sum^{(2)}(-\mu^2). \tag{3.21}$$

Using (3.12) we obtain

$$\sum^{(2)}_{(R)}(p^2/\mu^2) = i\not{r}(g^2/16\pi^2)\ln(-p^2/\mu^2). \tag{3.22}$$

Next writing

$$S_R^{-1}(p) = Z_\psi S^{-1}(p) \tag{3.23}$$

we obtain from (3.18), (3.19), (3.12), and (3.22)

$$Z_\psi = 1 + i\not{r}(g^2/16\pi^2)[2/\varepsilon - \ln(\mu^2/\bar{\mu}^2) + 1 + \ln 4\pi - \gamma_E]. \tag{3.24}$$

b. 't Hooft's minimal subtraction scheme

The method presented above is not the only way to render $S^{-1}(p)$ finite. By putting condition (3.20) we

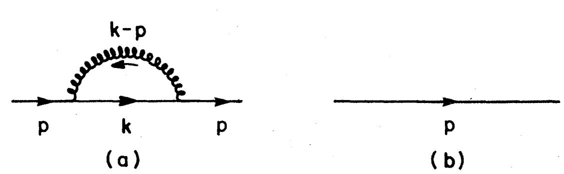


FIG. 5. g^2 and g^0 order contributions to quark self-energy.

have subtracted from $\sum^{(2)}(p^2)$ not only the singularity $1/\varepsilon$ but also the finite terms. The subtraction procedure in which one subtracts only the singularity $1/\varepsilon$ is called the minimal subtraction scheme (MS) and has been proposed by 't Hooft (1973). Applying this procedure to Eq. (3.12) we obtain

$$\sum\nolimits_{(R)}^{(2)}\left(\frac{p^2}{\bar{\mu}^2}\right)\Big|_{\text{MS}} = ir\frac{g^2}{16\pi^2}\left[\ln\frac{-p^2}{\bar{\mu}^2} - 1 - \ln 4\pi + \gamma_E\right] \tag{3.25}$$

and

$$Z_\psi|_{\text{MS}} = 1 + ir(g^2/16\pi^2)(2/\varepsilon). \tag{3.26}$$

Notice that the expression for the renormalization constant Z_ψ is very simple in 't Hooft's scheme. Generally in the scheme in question any renormalization constant is just a series of powers of $1/\varepsilon$ which greatly facilitates the calculations of various renormalization group parameters; in particular the calculation of two-loop anomalous dimensions (for a discussion see Sec. 3 of Ross, 1979). Notice that $\sum_R^{(2)}$ is different in the two renormalization schemes considered. This does not bother us, however, because $\sum_R^{(2)}$ is not a physical quantity. $\sum_R^{(2)}$ is, in fact, generally only one element of a physical expression. There are other elements in a given physical expression which also depend on renormalization procedure. If all these elements are put together, their renormalization prescription dependences cancel each other as required for a physical quantity (Stuekelberg and Peterman, 1953). Specific examples of such situations, relevant for deep-inelastic scattering, will be discussed in Sec. VII.

Our discussion of 't Hooft's minimal subtraction scheme was very superficial. Full exposition of this elegant subtraction procedure can be found in the papers by 't Hooft (1973), Collins and Macfarlane (1974), and Gross (1976). In particular, Collins and Macfarlane discuss a whole class of subtraction schemes which differ from the one presented here by a different continuation of the renormalized coupling constant to D dimensions. In general, instead of (3.11) one can have

$$g^2 \to g^2\bar{\mu}^\varepsilon f(\varepsilon, g^2), \tag{3.27a}$$

where $f(\varepsilon, g^2)$ is any function of ε which has the property $f(0, g^2) = 1$. Each different continuation will lead to different finite parts in Eq. (3.25). As we shall see in Sec. VII this type of ambiguity in the finite parts can be absorbed in the definition of the effective coupling constant. In particular the $\overline{\text{MS}}$ scheme of Sec. VII in which the terms involving $(\ln 4\pi - \gamma_E)$ are not present in Eqs. (2.96) and (2.120) corresponds to

$$f(\varepsilon, g^2) = 1 - \varepsilon/2\,(\ln 4\pi - \gamma_E). \tag{3.27b}$$

4. Renormalization group equations

So far we have shown how to regularize the renormalize Feynman amplitudes. In the process we have introduced an arbitrary normalization scale μ. In the $p^2 = \mu^2$ subtraction scheme μ^2 stands for the spacelike momentum where we specify the values of particular Green's functions. In the 't Hooft's scheme the $\mu = \bar{\mu}$ in the renormalized expression appeared through the continuation of the renormalized coupling constant to $D = 4 - \varepsilon$ dimensions. In both schemes the value of μ is arbitrary. Certainly the final result of a renormalization procedure cannot depend on the value of μ and any change in μ can be compensated by the change in g and the scales of the fields. This is most elegantly expressed by renormalization group equations (Stueckelberg and Peterman, 1953; Gell-Mann and Low, 1954; Bogolubov and Shirkov, 1959; Callan, 1970, 1972; Symanzik, 1970; Wilson, 1969), which we shall discuss now very briefly.

It is convenient to work with the amputated, renormalized, one-particle irreducible, proper vertex functions $\Gamma_R^{(N_\psi, N_G)}(p_j, g, \varepsilon)$ which are (suppressing arguments) defined as follows:

$$\Gamma_R^{(N_\psi, N_G)} = \frac{G_R^{(N_\psi, N_G)}}{\prod^{N_F} G_R^{(2,0)} \prod^{N_G} G_R^{(0,2)}}. \tag{3.28}$$

The renormalized Green functions $G_R^{(N_\psi, N_G)}$ are given by

$$G_R^{(N_\psi, N_G)} = \langle 0|T(\psi_1 \cdots \psi_{N_\psi} G_1 \cdots G_{N_G}|0\rangle, \tag{3.29}$$

and N_ψ and N_G stand for the number of external fermion legs and gluon legs, respectively. In this notation

$$S_R^{-1} \equiv \Gamma_R^{(2,0)}. \tag{3.30}$$

Similar expressions exist for the unrenormalized proper vertex functions $\Gamma_U^{(N_\psi, N_G)}(p_j, g_0, \varepsilon)$ with all renormalized parameters and fields replaced by the corresponding bare quantities. Because of (3.15) and (3.16) the proper vertex functions $\Gamma_U^{(N_\psi, N_G)}$ and $\Gamma_R^{(N_\psi, N_G)}$ are related to each other by

$$\Gamma_R^{(N_\psi, N_G)}(p_j, g, \varepsilon, \mu) = Z_\psi^{N_\psi/2} Z_G^{N_G/2} \Gamma_U^{(N_\psi, N_G)}(p_j, g_0, \varepsilon), \tag{3.31}$$

and the limit

$$\lim_{\varepsilon \to 0} \Gamma_R^{(N_\psi, N_G)}(p_j, g, \varepsilon, \mu) = \Gamma_R^{(N_\psi, N_G)}(p_j, g, \mu), \tag{3.32}$$

exists.

Since $\Gamma_U^{(N_\psi, N_G)}$ does not depend on μ, i.e.,

$$\frac{d}{d\mu}\Gamma_U^{(N_\psi, N_G)} = 0, \tag{3.33}$$

we obtain from (3.33) and (3.31) the following renormalization group equations

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - N_\psi\gamma_\psi(g) - N_G\gamma_G(g)\right]\Gamma_R^{(N_\psi, N_G)} = 0, \tag{3.34}$$

where

$$\gamma_\psi(g) = \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_\psi, \tag{3.35}$$

$$\gamma_G(g) = \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_G, \tag{3.36}$$

and

$$\beta(g) = \mu\frac{\partial g}{\partial\mu}. \tag{3.37}$$

The renormalization group functions $\gamma_\psi(g)$, $\gamma_G(g)$, and $\beta(g)$ depend only on g. $\gamma_\psi(g)$ and $\gamma_G(g)$ are called anomalous dimensions of the fermion field and gluon field, respectively. $\beta(g)$ is the well-known function which governs the Q^2 evolution of the effective coupling constant, as we shall discuss in more detail below. It should be remarked that $\beta(g)$, $\gamma_\psi(g)$, and $\gamma_G(g)$ depend only on the given theory and not on the particular Green's function considered.

We would like to remark that generally the renormalization group equations (3.34) also involve the derivatives with respect to the masses and the gauge parameter α. In order to simplify the presentation we neglect these derivatives here. For a careful discussion of the renormalization of the gauge parameter we refer the reader to Sec. 5.4 of the review by Gross (1976).

Equation (3.34) when combined with the standard dimensional analysis can be used to relate the vertex functions evaluated at momenta p_i to the same vertex functions evaluated at rescaled momenta $e^t p_i$. In applications to deep-inelastic scattering $t = \ln Q^2/\mu^2$. One obtains [see, e.g., Gross (1976)]

$$\Gamma_R^{(N_\psi, N_G)}(e^t p_i, g) = \Gamma_R^{(N_\psi, N_G)}[p_i, \bar{g}(t)] \times \exp\left[\tilde{D}t - \int_g^{\bar{g}(t)} dg' \, \frac{N_\psi \gamma_\psi(g') + N_G \gamma_G(g')}{\beta(g')}\right], \tag{3.38}$$

where $\bar{g}(t)$ is an effective coupling constant which satisfies the following equation

$$\frac{d\bar{g}^2}{dt} = \bar{g}\beta(\bar{g}) \, ; \; \bar{g}(t=0) = g \tag{3.39}$$

and $\tilde{D}$ is the physical dimension of Γ_R.

We observe that once $\gamma_\psi(g)$, $\gamma_G(g)$, and $\beta(g)$ are known, and $\Gamma_R^{(N_\psi, N_G)}$ is calculated at momenta corresponding to a *single value of t*, say $t=0$, then Eq. (3.38) gives us $\Gamma_R^{(N_\psi, N_G)}$ at any rescaled momenta $e^t p_i$ with $t \neq 0$. We shall see that equations like (3.38) will be at the basis of discussions of scaling violations as predicted in QCD.

5. Calculations of renormalization group functions

We can obtain $\gamma_\psi(g)$ in g^2 order by inserting Z_ψ of Eq. (3.26) into (3.35) with the result $(r = iC_2(R) = i4/3)$

$$\gamma_\psi(g) = \mu \frac{\partial}{\partial\mu} \ln Z_\psi = \frac{4}{3}\frac{g^2}{16\pi^2} \equiv \gamma_\psi^0 \frac{g^2}{16\pi^2}, \tag{3.40}$$

which corresponds to the Feynman gauge $(\alpha = 1)$. In an arbitrary covariant gauge α the result of Eq. (3.40) is to be multiplied by α. Alternatively we can calculate $\gamma_\psi(g)$ by using renormalization group Eq. (3.34). We insert $S_R^{-1} \equiv \Gamma_R^{(2,0)}$ as given by Eqs. (3.19) and (3.25) into renormalization group equations and compare the powers of g^2. Since the g^2 expansion of $\beta(g)$ begins with g^3, $\beta(g)$ can be dropped on the rhs of Eq. (3.34) if we are interested in $\gamma(g)$ in order g^2. Therefore we obtain first

$$\left[\mu\frac{\partial}{\partial\mu} - 2\gamma_\psi(g)\right]\Gamma_R^{(2,0)} = 0 \, , \tag{3.41}$$

and consequently (in g^2 order)

$$\gamma_\psi(g) = \frac{1}{2}\mu\frac{\partial}{\partial\mu}\sum\nolimits_{(R)}^{(2)}\left(\frac{p^2}{\mu^2}\right), \tag{3.42}$$

which by Eq. (3.22) or (3.25) leads to the previous result (3.40).

Similarly one can calculate the anomalous dimension of the gluon field, $\gamma_G(g)$, by considering the diagrams of Fig. 6, with the result (Politzer, 1974; Gross and Wilczek, 1973)

$$\gamma_G(g) = -\frac{g^2}{16\pi^2}\left[\left(\frac{13}{6} - \frac{\alpha}{2}\right)3 - \frac{2}{3}f\right] \equiv \gamma_G^0 \frac{g^2}{16\pi^2}, \tag{3.43}$$

where α is the gauge parameter.

In order to evaluate $\beta(g)$ to order g^3 one considers either the diagrams of Fig. 7(a) or the diagrams of Fig. 7(b). If the resulting renormalized vertex functions are $\Gamma_R^{(0,3)}$ and $\Gamma_R^{(2,1)}$, respectively, then the equations for $\beta(g)$ which follow from renormalization group Eq. (3.34) are

$$\beta(g) = -\frac{g^3}{16\pi^2}\left[\mu\frac{\partial}{\partial\mu} f_R^{(0,3)}\left(\frac{p^2}{\mu^2}\right) + 3\gamma_G^0\right], \tag{3.44a}$$

$$\beta(g) = -\frac{g^3}{16\pi^2}\left[\mu\frac{\partial}{\partial\mu} f_R^{(2,1)}\left(\frac{p^2}{\mu^2}\right) + 2\gamma_\psi^0 + \gamma_G^0\right], \tag{3.44b}$$

where $f^{(0,3)}(p^2/\mu^2)$ and $f^{(2,1)}(p^2/\mu^2)$ are defined as follows:

$$\Gamma_R^{(0,3)} = g + (g^3/16\pi^2) f_R^{(0,3)}(p^2/\mu^2) \tag{3.45a}$$

and

$$\Gamma_R^{(2,1)} = g + (g^3/16\pi^2) f_R^{(2,1)}(p^2/\mu^2) \, . \tag{3.45b}$$

Evaluating $f_R^{(0,3)}$ and $f_R^{(2,1)}$ (Politzer, 1973; Gross and Wilczek, 1973) and using (3.40) and (3.43) one obtains from both (3.44a) and (3.44b)

$$\beta(g) = -\beta_0(g^3/16\pi^2) \, , \tag{3.46}$$

where

$$\beta_0 = 11 - \tfrac{2}{3}f \, . \tag{3.47}$$

The fact that the coefficient of g^3 is negative (for $f \leq 16$) has a very important consequence—*asymptotic freedom*. Inserting (3.46) into (3.39) and choosing $t = \ln Q^2/\mu^2$ as an example, we obtain

$$\frac{\bar{g}^2(Q^2)}{16\pi^2} = \frac{\bar{g}^2(\mu^2)/16\pi^2}{1 + [\bar{g}^2(\mu^2)/16\pi^2]\beta_0 \ln(Q^2/\mu^2)} = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}, \tag{3.48}$$

where we have introduced the parameter Λ which is related to μ^2 and $\bar{g}^2(\mu^2) = g^2$ by Eq. (2.51). For $Q^2 \to \infty$ this coupling constant which measures the effective interaction between quarks and gluons at momentum scale Q decreases to zero. This is what we mean by asymp-

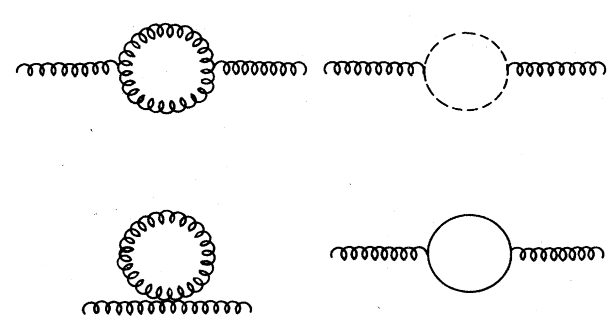


FIG. 6. Lowest-order corrections to gluon self-energy.

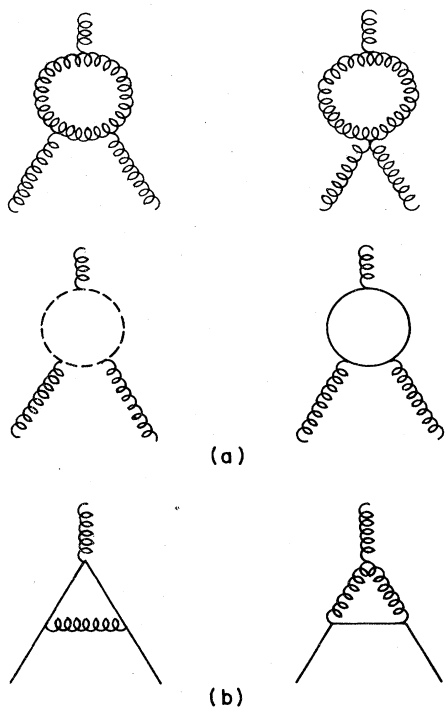


FIG. 7. Lowest-order corrections to (a) triple gluon vertex and (b) fermion–gluon vertex.

totic freedom. As we shall see in subsequent sections the smallness of $\bar g^2(Q^2)$ for sufficiently large Q^2 will allow us to calculate many quantities in perturbation theory in $\bar g^2(Q^2)$. This spectacular property is a special property of non-Abelian gauge theories.[23]

So far we have discussed renormalization, renormalization group equations, and asymptotic freedom in general. Before showing how these ideas can be used in the study of deep-inelastic scattering, we have to discuss operator product expansion.

C. Operator product expansion

The basic object in any discussion of deep-inelastic scattering is the spin averaged amplitude $T_{\mu\nu}$ for the forward scattering of a current J_μ off a hadronic state $|p\rangle$. Here J_μ stands either for the electromagnetic current ($ep, \mu p$ scattering) or a weak current ($\nu, \bar\nu$ scattering). We first introduce the operator

$$\tilde T_{\mu\nu}=i\int d^4z e^{iqz}T[J_\mu(z)J_\nu^+(0)]\,, \tag{3.49}$$

which is related to $T_{\mu\nu}$ by

$$T_{\mu\nu}(Q^2,\nu)=\langle p|\tilde T_{\mu\nu}|p\rangle_{\text{spin averaged}}\,. \tag{3.50}$$

Next the amplitude $T_{\mu\nu}(Q^2,\nu)$ can be decomposed into invariant amplitudes as follows:

$$T_{\mu\nu}(Q^2,\nu)=e_{\mu\nu}T_L(Q^2,\nu)+d_{\mu\nu}T_2(Q^2,\nu) - i\varepsilon_{\mu\nu\alpha\beta}(p_\alpha q_\beta/\nu)T_3(Q^2,\nu)\,, \tag{3.51}$$

with $\nu=pq$ and $Q^2=-q^2$. The tensors $e_{\mu\nu}$ and $d_{\mu\nu}$ are defined in Eqs. (2.2) and (2.3).

Following Wilson (1969) we can expand the product of currents, which enters Eq. (3.49), as a sum of products of local operators $O_i^{\mu_1\cdots\mu_n}$ of definite spin n times certain coefficient functions C_n^i. The index i stands for the type of operator and will be specified below. In what follows we shall only consider so-called twist (twist ≡ dimension-spin) two operators which give the dominant contributions to the moments of the structure functions in the Bjorken limit. The higher-twist operators are suppressed relative to twist-two operators by powers in Q^2. At not too large values of Q^2 they are expected (De Rujula, Georgi, and Politzer, 1977a, b; Gottlieb, 1978) to be of some importance only for x close to 1.

The operator product expansion for $\tilde T_{\mu\nu}$ is as follows:

$$\tilde T_{\mu\nu}=\sum_{n,i}\Big[-(g_{\mu\mu_1}g_{\nu\mu_2}q^2-g_{\mu\mu_1}q_\nu q_{\mu_2}-q_\mu q_{\mu_1}g_{\nu\mu_2}+g_{\mu\nu}q_{\mu_1}q_{\mu_2})C^i_{2,n}\left(\frac{Q^2}{\mu^2},g^2\right) + \left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)q_{\mu_1}q_{\mu_2}C^i_{L,n}\left(\frac{Q^2}{\mu^2},g^2\right) - i\varepsilon_{\mu\nu\alpha\beta}g_{\alpha\mu_1}q_\beta q_{\mu_2}C^i_{3,n}\left(\frac{Q^2}{\mu^2},g^2\right)\Big]q_{\mu_3}\cdots q_{\mu_n}\left(\frac{2}{Q^2}\right)^n O_i^{\mu_1\cdots\mu_n}\,. \tag{3.52}$$

Writing next

$$\langle p|O_i^{\mu_1\cdots\mu_n}|p\rangle=A_n^i(\mu^2)p_{\mu_1}\cdots p_{\mu_n}-\text{traces}\,, \tag{3.53}$$

and combining Eqs. (3.50)–(3.53) we obtain

$$T_{\mu\nu}(Q^2,\nu)=\sum_{i,n}\frac{1}{x^n}\Big[e_{\mu\nu}C^i_{L,n}\left(\frac{Q^2}{\mu^2},g^2\right)+d_{\mu\nu}C^i_{2,n}\left(\frac{Q^2}{\mu^2},g^2\right) - i\varepsilon_{\mu\nu\alpha\beta}\frac{p_\alpha q_\beta}{\nu}C^i_{3,n}\left(\frac{Q^2}{\mu^2},g^2\right)\Big]A_n^i(\mu^2) \tag{3.54}$$

where x is the Bjorken variable $(Q^2/2\nu)$. In writing (3.54) we have dropped the trace terms of Eq. (3.53). This is justified if target mass corrections (see Sec. II.D) can be neglected. The arguments of the coefficient functions indicate that they will be calculated in perturbation theory in g^2. The sum on the rhs of Eq. (3.54) runs over spin-n, twist-2 operators such as

$$O^{\mu_1\cdots\mu_n}_{NS,\kappa}=S(\bar\psi_\alpha\lambda^\kappa_{\alpha\beta}\gamma^{\mu_1}\mathfrak{D}^{\mu_2}\cdots\mathfrak{D}^{\mu_n}\psi_\beta-\text{traces})\,, \tag{3.55}$$

$$O^{\mu_1\cdots\mu_n}_{\psi}=S(\bar\psi_\alpha\gamma^{\mu_1}\mathfrak{D}^{\mu_2}\cdots\mathfrak{D}^{\mu_n}\psi_\alpha-\text{traces})\,, \tag{3.56}$$

$$O^{\mu_1\cdots\mu_n}_{G}=S(G^{\mu_1\nu}\mathfrak{D}^{\mu_2}\cdots\mathfrak{D}^{\mu_{n-1}}G^{\mu_n\,\nu}-\text{traces})\,, \tag{3.57}$$

where S denotes symmetrization over all Lorentz indices. Since we shall deal only with foward spin-averaged matrix elements we do not consider operators with

[23]Prior to the discovery of asymptotic freedom it has been argued that approximate Bjorken scaling requires an asymptotically free theory (Callan and Gross, 1973) and it has been shown that only non-Abelian gauge theories can be asymptotically free (Coleman and Gross, 1973). One exception to this is a $\lambda\phi^4$ theory with $\lambda<0$ (Symanzik, 1973), but this theory is rejected on the ground that its spectrum is unbounded from below.

negative parity. $O_{\rm NS}^{\mu_1\cdots\mu_n}$ are the fermion nonsinglet (under physical symmetries) operators, whereas $O_\psi^{\mu_1\cdots\mu_n}$ and $O_G^{\mu_1\cdots\mu_n}$ are singlet fermion and gluon opertors, respectively. The index κ distinguishes between various nonsinglet operators. Since the Q^2 dependence of the Wilson coefficient functions, corresponding to various nonsinglet operators, is common, we shall in the following drop the index κ.

Using dispersion relations between deep-inelastic structure functions of Eq. (2.1) and the invariant amplitudes of Eq. (3.51), and taking into account (3.54), one can express the moments of the structure functions in terms of the Wilson coefficient functions and the hadronic matrix elements of various local operators. One obtains (Christ, Hasslacher, and Mueller, 1972)

$$\int_0^1 dx\, x^{n-2} F_k(x,Q^2) = \sum_i A_n^i(\mu^2) C_{k,n}^i\left(\frac{Q^2}{\mu^2}, g^2\right) \quad k=2, L \tag{3.58}$$

and

$$\int_0^1 dx\, x^{n-1} F_3(x,Q^2) = \sum_i A_n^i(\mu^2) C_{3,n}^i\left(\frac{Q^2}{\mu^2}, g^2\right). \tag{3.59}$$

Notice that by taking moments the operator product expansion has been projected on a given spin; the $(n-2)$ moment of the structure function depends only on operators of spin n. Since there are at most three types of leading operators of a given spin [Eqs. (3.55)–(3.57)] the theoretical analysis of QCD predictions for deep-inelastic scattering is most conveniently done in terms of the moments of various structure functions rather than in terms of structure functions themselves.

A few final remarks about operator product expansion (OPE) are necessary. The matrix elements A_n^i depend on the target $|p\rangle$, and are uncalculable in perturbation theory if the target is a composite object as for instance the proton. The coefficient functions on the other hand *do not* depend on the target since they are determined by the expansion (3.52). They can be calculated in perturbation theory. In fact what OPE does for us is to separate perturbatively calculable pieces (coefficient functions) in the expression for the moments of structure functions from nonperturbative pieces—matrix elements of local operators. A brief discussion on this factorization in the framework of the perturbative QCD can be found in Sec. IX.

We shall now show that the Q^2 dependence of the coefficient function is governed by the renormalization group equations similar to Eq. (3.34).

D. Renormalization group equations for Wilson coefficient functions

We begin the discussion with the coefficient functions of the nonsinglet operators of Eq. (3.55). Suppressing for simplicity all the arguments, we write the nonsinglet part of the operator product expansion of two currents symbolically as follows:

$$JJ|_{\rm NS} = \sum_n C_n^{\rm NS} O_{\rm NS}^n. \tag{3.60}$$

Sandwiching Eq. (3.60) between nonsinglet (quark) states we obtain

$$\langle {\rm NS}|JJ|{\rm NS}\rangle = \sum_n C_n^{\rm NS} O_{\rm NS,NS}^n, \tag{3.61}$$

where

$$\langle {\rm NS}|O_{\rm NS}^n|{\rm NS}\rangle \equiv O_{\rm NS,NS}^n. \tag{3.62}$$

In order to derive renormalization group equations for $C_n^{\rm NS}$, we have to find first renormalization group equations for $\langle {\rm NS}|JJ|{\rm NS}\rangle$ and $O_{\rm NS,NS}^n$.

For $\langle {\rm NS}|JJ|{\rm NS}\rangle$ we have similar to Eq. (3.34)

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - 2\gamma_\psi(g)\right]\langle {\rm NS}|JJ|{\rm NS}\rangle = 0, \tag{3.63}$$

where $\gamma_\psi(g)$ is the anomalous dimension of the quark field [see Eq. (3.40)]. Due to current conservation, the anomalous dimension of the current J is zero.

Next we define the wave-function renormalization $Z_{\rm NS}^n$ of the operator $O_{\rm NS}^n$ by

$$O_{\rm NS}^n = \frac{O_{\rm NS}^{0,n}}{Z_{\rm NS}^n} \tag{3.64}$$

where $O_{\rm NS}^{0,n}$ is the bare operator.

For the matrix element $O_{\rm NS,NS}^n$ we therefore have

$$O_{\rm NS,NS}^n = \frac{Z_\psi}{Z_{\rm NS}^n} O_{\rm NS,NS}^{0,n}, \tag{3.65}$$

with Z_ψ defined by Eq. (3.16).

Repeating the steps which led us from Eq. (3.31) to Eq. (3.34) we obtain

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma_{\rm NS}^n(g) - 2\gamma_\psi(g)\right] O_{\rm NS,NS}^n = 0, \tag{3.66}$$

where

$$\gamma_{\rm NS}^n(g) = \mu\frac{\partial}{\partial\mu}\ln Z_{\rm NS}^n, \tag{3.67}$$

is the anomalous dimension of the operator $O_{\rm NS}^n$. Next combining Eqs. (3.61), (3.63), and (3.66) and taking into account that the tensor structure in the expansion (3.60) is different for different n [see Eq. (3.52)], we finally obtain for each n

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma_{\rm NS}^n(g)\right] C_n^{\rm NS}\left(\frac{Q^2}{\mu^2}, g^2\right) = 0. \tag{3.68}$$

Notice that we have now written explicitly the arguments of the coefficient functions.

The case of singlet operators O_ψ^n and O_G^n is more complicated because these operators mix under renormalization and Eq. (3.64) is replaced by

$$O_a^n = \sum_b (Z^{n^{-1}})_{ab} O_b^{0,n} \quad a,b=\psi,G. \tag{3.69}$$

Here $(Z^{n^{-1}})$ is a 2×2 matrix. Consequently, Eq. (3.65) is generalized to

$$O_{a,c}^n = \sum_b Z_c (Z^{n^{-1}})_{ab} O_{b,c}^{0,n} \quad a,b,c=\psi,G, \tag{3.70}$$

where

$$O_{a,c}^n \equiv \langle c|O_a^n|c\rangle \tag{3.71}$$

and Z_c stands either for Z_ψ or Z_G, which are defined in Eqs. (3.16) and (3.17), respectively. Therefore we have

$$\sum_b \left[\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-2\gamma_c(g)\right)\delta_{ab}+\gamma^n_{ab}(g)\right]O^N_{b,c}=0\,, \quad (3.72)$$

where γ^n_{ab} are the elements of the 2×2 anomalous dimension matrix and are defined by

$$\gamma^n_{ab}=\left(\mu\frac{\partial}{\partial\mu}\ln Z\right)_{ab}. \quad (3.73)$$

Equations (3.61) and (3.63) are generalized to the singlet case as follows:

$$\langle c|JJ|c\rangle=\sum_{n,b} C^b_n O^n_{b,c}\,, \quad (3.74)$$

and

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-2\gamma^n_c(g)\right]\sum_b C^b_n O^n_{b,c}=0\,. \quad (3.75)$$

Therefore combining Eqs. (3.72) and (3.75) we finally obtain

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right]C^a_n\left(\frac{Q^2}{\mu^2},g^2\right)=\sum_b \gamma^n_{ba}C^b_n\left(\frac{Q^2}{\mu^2},g^2\right). \quad (3.76)$$

Equations (3.66) and (3.75) are the basic results of this section. We shall discuss the solutions of these equations in Secs. IV, VII, and VII, and now we turn to the calculation of the anomalous dimensions γ^n_{NS} and γ^n_{ab}.

E. Calculations of anomalous dimensions of local operators

We first write the perturbative expansions of the matrix elements $O^n_{\mathrm{NS,NS}}$ and $O^n_{b,c}$, dropping p^2 independent terms, as follows:

$$O^n_{\mathrm{NS,NS}}=1+\frac{g^2}{16\pi^2}\frac{r^n_{\mathrm{NS}}}{2}\ln\frac{-p^2}{\mu^2}+O(g^4)\,, \quad (3.77)$$

and

$$O^n_{b,c}=\delta_{bc}+\frac{g^2}{16\pi^2}\frac{r^n_{bc}}{2}\ln\frac{-p^2}{\mu^2}+O(g^4)\,, \quad (3.78)$$

where r^n_{NS} and r^n_{bc} are calculable numbers and p^2 is the spacelike momentum of the quark or gluon states (NS,c) between which the operators are sandwiched.

Furthermore, the perturbative expansions of $\gamma^n_{\mathrm{NS}}(g)$ and $\gamma^n_{ab}(g)$ are

$$\gamma^n_{\mathrm{NS}}(g)=\gamma^{0,n}_{\mathrm{NS}}\frac{g^2}{16\pi^2}+O(g^4) \quad (3.79)$$

and

$$\gamma^n_{ab}(g)=\gamma^{0,n}_{ab}\frac{g^2}{16\pi^2}+O(g^4)\,. \quad (3.80)$$

Inserting Eqs. (3.77) through (3.80) into Eqs. (3.66) and (3.72) and choosing 4 different combinations of a and c in Eq. (3.72), we obtain the following relations between the coefficients of $\ln(-p^2/\mu^2)$ in Eqs. (3.77) and (3.78) and the anomalous dimensions of the operators

$$\gamma^{0,n}_{\mathrm{NS}}=r^n_{\mathrm{NS}}+2\gamma^0_\psi\,, \quad (3.81)$$

$$\gamma^{0,n}_{ab}=r^n_{ab}+2\gamma^0_a\delta_{ab} \quad a,b=\psi,G\,. \quad (3.82)$$

Here γ^0_a stands for either γ^0_ψ or γ^0_G which are defined in Eqs. (3.40) and (3.43), respectively.

It follows from (3.81) and (3.82) that in order to find $\gamma^{0,n}_{\mathrm{NS}}$ and $\gamma^{0,n}_{ab}$, we have to calculate the matrix elements of local operators sandwiched between quark and gluon states and pick out the coefficients of $\ln(-p^2)$. For diagonal elements $\gamma^{0,n}_{\psi\psi}$, $\gamma^{0,n}_{\mathrm{NS}}$, and $\gamma^{0,n}_{GG}$ we have to add, in addition, twice the anomalous dimensions of the quark and gluon fields which we have already calculated earlier.

The diagrams which enter the calculation of $\gamma^{0,n}_{\psi\psi}$ or $\gamma^{0,n}_{\mathrm{NS}}$ are shown in Fig. 8. The diagrams which enter the calculation of the whole anomalous dimension matrix in order g^2 are shown in Fig. 10. The virtual gluon corrections on the external lines need not be calculated if $2\gamma^0_\psi$ or $2\gamma^0_G$ is added explicitly as in Eqs. (3.81) and (3.82). On the other hand, if these diagrams are included in the calculations, the anomalous dimensions γ^0_ψ and γ^0_G should be dropped in Eqs. (3.81) and (3.82).

In order to evaluate the diagrams of Figs. 8 and 10 we have to extend the list of Feynman rules of Fig. 4 by the rules for the vertices "×" which represent operator insertions into a two-point function. Simple rules for the vertices in question have been found by (Gross and Wilczek, 1974) and are shown in Fig. 9. The Δ_μ appearing there is an arbitrary four vector with the property

$$\Delta^2=0\,. \quad (3.83)$$

The derivation of these rules can be found in the appendix of the paper by Gross and Wilczek. Additional rules necessary for the calculation of the anomalous dimensions in order g^4 can be found in the paper by Floratos, Ross, and Sachrajda (1979). Here we shall only indicate how to reproduce formula (2.79a) for $\gamma^{0,n}_{\psi\psi}$. We begin with the diagram of Fig. 8(b). We work in the Feynman gauge[24] and obtain first

$$I_b=h\int\frac{d^Dk}{(2\pi)^D}\frac{\gamma_\nu \not{k}\not{\Delta}\not{k}\gamma_\nu}{(k^2)^2(k-p)^2}(\Delta k)^{n-1}\,,$$
$$=h(\varepsilon-2)\int\frac{d^Dk}{(2\pi)^D}\frac{\not{k}\not{\Delta}\not{k}}{(k^2)^2(k-p)^2}(\Delta k)^{n-1}\,, \quad (3.84)$$

where we have used formula (A.20) to reduce the Dirac

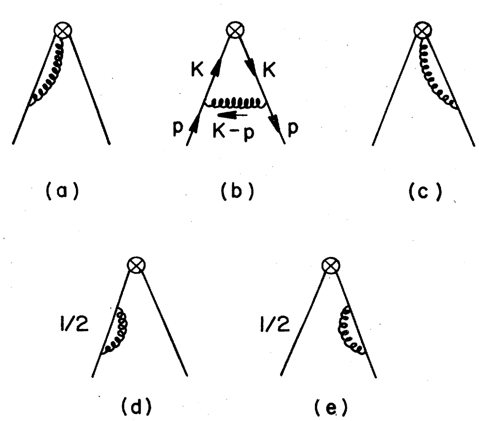
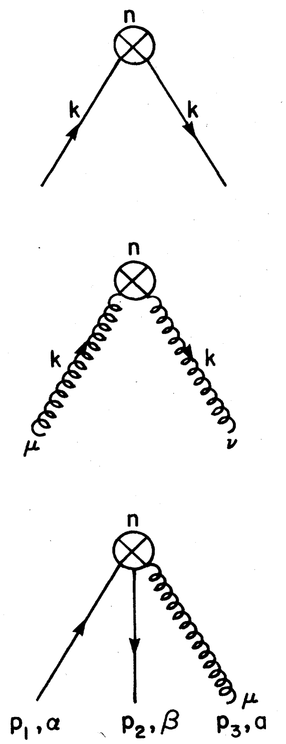


FIG. 8. Diagrams entering the calculation of $\gamma^{0,n}_{\psi\psi}$ or $\gamma^{0,n}_{\mathrm{NS}}$.

[24] Anomalous dimensions of local operators considered in this review are gauge independent in order g^2.



$$= \not\Delta\,(\Delta\cdot k)^{n-1}$$

$$= g_{\mu\nu}(\Delta\cdot k)^{n} + k^2\Delta_\mu\Delta_\nu(\Delta\cdot k)^{n-2} - k_\mu\Delta_\nu(\Delta\cdot k)^{n-1} - k_\nu\Delta_\mu(\Delta\cdot k)^{n-1}$$

$$= g\,\lambda^a_{\alpha\beta}\;\Delta_\mu\not\Delta\sum_{j=0}^{n-2}(\Delta\cdot p_1)^j(\Delta\cdot p_2)^{n-2-j}$$

FIG. 9. Feynman rules for the operator insertions.

algebra in D dimensions and we have put all group and i factors in one symbol h. Now as the reader may convince herself (himself), if we are interested only in the coefficients of $\ln p^2$, we can put $\varepsilon=0$ at all places where this substitution does not lead to a singularity. From this it follows (see below) that, equivalently, anomalous dimensions can be found by calculating the coefficients of the divergent parts $1/\varepsilon$. On the other hand, if we are interested in the calculations of the so-called constant pieces [e.g., $1+\ln 4\pi-\gamma_E$ in Eq. (3.12)] we have to keep all ε factors different from zero until the calculation is finished. In particular the "1" in Eq. (3.12) comes from the product of ε in (Eq. 3.8) and the divergence $1/\varepsilon$ in $\Gamma(\varepsilon/2)$.

Using the formulas of Appendix A, Eqs. (3.11), (3.83), and $h=-iC_2(R)$, with $C_2(R)$ given by (3.6), we obtain

$$I_b=\frac{g^2}{16\pi^2}\left[\frac{2C_2(R)}{n(n+1)}\right]\left[\frac{2}{\varepsilon}-\ln\frac{-p^2}{\mu^2}\right]\not\Delta(\Delta p)^{n-1}+\text{const}\,. \quad (3.85)$$

The factor $(\Delta p)^{n-1}$ in Eq. (3.85) arises in the following way. After using the Feynman parametrization of Eq. (A.13) one makes a change from k to $\hat{k}$:

$$\hat{k}=k-p(1-x)\,,$$

where x is the Feynman parameter. Therefore one obtains

$$(\Delta k)^{n-1}=(1-x)^{n-1}(\Delta p)^{n-1}+(n-1)(1-x)^{n-2}(\Delta p)^{n-2}\Delta\hat{k}+\cdots\,. \quad (3.86)$$

The terms which involve more than one factor $\Delta\hat{k}$ can be dropped in Eq. (3.86) because they lead after $\hat{k}$ integration to Δ^2, which is zero. An additional factor Δp in the second term in Eq. (3.86) is obtained after $\hat{k}$ integration, when Eq. (3.86) is inserted into (3.84).

Adding zeroes order contribution $\{[1]\not\Delta(\Delta p)^{n-1}\}$ to I_b, and comparing the result with Eq. (3.78) and taking into account (3.81), we obtain the following contribution of diagram 8(b) to $\gamma^{0,n}_{\psi\psi}$:

$$-4C_2(R)/n(n+1)\,. \quad (3.87)$$

Notice that result (3.87) could also be read off the coefficient of $1/\varepsilon$. This is particularly useful in the calculation of two-loop anomalous dimensions as discussed in detail by Floratos *et al.* (1977, 1979).

The diagrams 8(a) and 8(c) give the following contribution to $\gamma^{0,n}_{\psi\psi}$:

$$8C_2(R)\sum_{j=2}^{n}\frac{1}{j}\,, \quad (3.88)$$

and when the result for $2\gamma^0_\psi=2C_2(R)$ is added to Eqs. (3.87) and (3.88), we obtain

$$\gamma^{0,n}_{\psi\psi}=2C_2(R)\left[1-\frac{2}{n(n+1)}+4\sum_{j=2}^{n}\frac{1}{j}\right], \quad (3.89)$$

which by Eq. (3.6) for $C_2(R)$ agrees with (2.79a). The calculation of the remaining elements of the anomalous dimension matrix proceeds in a similar way.

IV. Q^2 DEPENDENCE OF THE MOMENTS OF STRUCTURE FUNCTIONS IN ASYMPTOTICALLY FREE GAUGE THEORIES

A. Preliminaries

In this section we shall find the Q^2 dependence of deep-inelastic structure functions as predicted by asymptotic freedom in the leading order. The basic formulas of this section [(4.17), (4.18), (4.41)] express the moments of structure functions in terms of unknown (Q^2 independent) matrix elements of certain operators times their coefficient functions with explicit Q^2 dependence. In the next section we shall cast formulas [(4.17), (4.18), (4.41)] into the standard parton model expressions with Q^2 dependent quark distributions. The reason for a careful discussion of structure functions in terms of Wilson coefficient functions first rather than immediately in terms of parton distributions is that beyond the leading order the definition of parton distributions is not unambiguous and the language developed in this section is more appropriate. This section is slightly formal but we invite the reader to go through it carefully since the techniques presented here will be at the basis of Secs. VII and VIII.

As we saw in Sec. III, the basic tools necessary to study QCD implications for deep-inelastic scattering are the Wilson operator product expansion and the renormalization group equations. The operator product expansion allowed us to systematically identify the dominant contributions to the moments of the structure functions at large Q^2 and to express them in terms of a sum of products of (perturbatively) calculable coefficient functions and (by present methods) uncalculable matrix elements of certain operators between hadronic states. The Q^2 dependence of the coefficient functions could then be found by means of renormalization group equations. Explicitly we have

$$\int_0^1 dx\, x^{n-2} F_L(x, Q^2) = \sum_i A_n^i(\mu^2) C_{L,n}^i\left(\frac{Q^2}{\mu^2}, g^2\right), \tag{4.1}$$

$$\int_0^1 dx\, x^{n-2} F_2(x, Q^2) = \sum_i A_n^i(\mu^2) C_{2,n}^i\left(\frac{Q^2}{\mu^2}, g^2\right), \tag{4.2}$$

and

$$\int_0^1 dx\, x^{n-1} F_3(x, Q^2) = A_n^{\rm NS}(\mu^2) C_{3,n}^{\rm NS}\left(\frac{Q^2}{\mu^2}, g^2\right), \tag{4.3}$$

where the sum runs over spin-n, twist-2 operators such as the fermion nonsinglet operator $O_{\rm NS}^n$ and the singlet fermion and gluon operators O_ψ^n and O_G^n, respectively. The $A_n^i(\mu^2)$, which are independent of Q^2, are the reduced hadronic matrix elements of the operators in question. They are defined in Eq. (2.63). We have shown explicitly that $A_n^i(\mu^2)$ depend on μ^2, the subtraction point (see discussion in Sec. III). Notice that to the moments of F_3 only one type of operators contributes. This is explained in Sec. VII.E.

We would like to recall that in our notation $O_{\rm NS}^n$ stands for any linear combination of nonsinglet operators which differ from each other by λ^κ in Eq. (3.55). Therefore as emphasized after Eq. (2.64) $A_n^{\rm NS}(\mu^2)$ *depend generally on the process and the structure function considered.* This dependence is discussed in Appendix B.

We shall now find the explicit Q^2 dependence of the coefficient functions $C_{k,n}^i(Q^2/\mu^2, g^2)$ as given in the leading order of asymptotic freedom. To this end it is convenient to decompose any structure function into a sum of singlet and nonsinglet contributions as follows:

$$F_k(x, Q^2) = F_k^{\rm NS}(x, Q^2) + F_k^s(x, Q^2) \quad k = L, 2, 3\,. \tag{4.4}$$

Particular examples of such decomposition in the framework of the simple parton model are given in Sec. II.B. The moments of the functions $F_k^{\rm NS}$ and F_k^s are given as follows:

$$\int_0^1 dx\, x^{n-2} F_k^{\rm NS}(x, Q^2) = A_n^{\rm NS}(\mu^2) C_{k,n}^{\rm NS}\left(\frac{Q^2}{\mu^2}, g^2\right) \quad k = L, 2\,, \tag{4.5}$$

$$\int_0^1 dx\, x^{n-1} F_3(x, Q^2) = A_n^{\rm NS}(\mu^2) C_{3,n}^{\rm NS}\left(\frac{Q^2}{\mu^2}, g^2\right), \tag{4.6}$$

and

$$\int_0^1 dx\, x^{n-2} F_k^s(x, Q^2) = A_n^\psi(\mu^2) C_{k,n}^\psi\left(\frac{Q^2}{\mu^2}, g^2\right) + A_n^G(\mu^2) C_{k,n}^G\left(\frac{Q^2}{\mu^2}, g^2\right) \quad k = 2, L\,. \tag{4.7}$$

B. Nonsinglet structure functions

The Q^2 dependence of $C_{k,n}^{\rm NS}(Q^2/\mu^2, g^2)$ is governed by the following renormalization group equations:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma_{\rm NS}^n(g)\right] C_{k,n}^{\rm NS}\left(\frac{Q^2}{\mu^2}, g^2\right) = 0\,, \tag{4.8}$$

where $\gamma_{\rm NS}^n(g)$ is the anomalous dimension of the nonsinglet operator $O_{\rm NS}^n$ and $\beta(g)$ is the renormalization group function which governs the Q^2 dependence of the effective coupling constant

$$\frac{d\bar g^2}{dt} = \bar g\,\beta(\bar g); \ \bar g(t=0) = g\,. \tag{4.9}$$

Here $t = \ln Q^2/\mu^2$ and g is the renormalized strong interaction coupling constant.

The solution of Eq. (4.8) is given as follows:

$$C_{k,n}^{\rm NS}\left(\frac{Q^2}{\mu^2}, g^2\right) = C_{k,n}^{\rm NS}(1, \bar g^2)\exp\left[-\int_{\bar g(\mu^2)}^{\bar g(Q^2)} dg'\,\frac{\gamma_{\rm NS}^n(g')}{\beta(g')}\right]. \tag{4.10}$$

To proceed further one has to calculate $C_{k,n}^{\rm NS}(1, \bar g^2)$, $\gamma_{\rm NS}^n(g)$, and $\beta(g)$ in perturbation theory. In the leading order it is enough to calculate one-loop contributions to $\gamma_{\rm NS}^n(g)$ and $\beta(g)$ using the methods of Sec. II, and take the zero-loop (parton model) values for $C_{k,n}^{\rm NS}(1, \bar g^2)$. Thus in the leading order we have

$$\gamma_{\rm NS}^n(g) = \gamma_{\rm NS}^{(0),n}(g^2/16\pi^2)\,, \tag{4.11}$$

$$\beta(g) = -\beta_0(g^2/16\pi^2)\,, \tag{4.12}$$

and

$$C_{k,n}^{\rm NS}(1, \bar g^2) = \begin{cases} \delta_{\rm NS}^{(k)} & k = 2, 3 \\ 0 & k = L \end{cases}, \tag{4.13}$$

where $\delta_{\rm NS}^k$ are constants which depend on weak and electromagnetic charges (see Appendix B), and β_0 and $\gamma_{\rm NS}^{(0),n}$ are given in Eqs. (2.49) and (2.71), respectively. Inserting Eqs. (4.11) to (4.13) into (4.10) and performing the integral we obtain

$$C_{k,n}^{\rm NS}\left(\frac{Q^2}{\mu^2}, g^2\right) = \begin{cases} \delta_{\rm NS}^{(k)}\left[\dfrac{\bar g^2(Q^2)}{\bar g^2(\mu^2)}\right]^{d_{\rm NS}^n} & k = 2, 3 \\ 0 & k = L \end{cases}, \tag{4.14}$$

where

$$d^n_{NS}=\gamma^{(0),n}_{NS}/2\beta_0 . \tag{4.15}$$

Now from Eqs. (4.9) and (4.12) we have

$$\bar{g}^2(Q^2)=16\pi^2/\beta_0\ln(Q^2/\Lambda^2), \tag{4.16}$$

where the scale parameter Λ is related to μ and $\bar{g}^2(\mu^2)=g^2$ by Eq. (2.51). Combining Eqs. (4.5), (4.6), (4.14), and (4.16) we finally obtain

$$\int_0^1 dx\, x^{n-2}F_2^{NS}(x,Q^2)=\delta^{(2)}_{NS}A_n^{NS}(Q_0^2)\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d^n_{NS}}, \tag{4.17a}$$

$$\int_0^1 dx\, x^{n-1}F_3(x,Q^2)=\delta^{(3)}_{NS}A_n^{NS}(Q_0^2)\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d^n_{NS}}, \tag{4.18a}$$

and $F_L^{NS}=0$. In order to unify notation we have put $\mu^2=Q_0^2$ in Eqs. (4.17a) and (4.18a).

Equations (4.17a) and (4.18a) can be used directly in phenomenological applications. $\delta^{(k)}_{NS}$ can be taken from Appendix B and $A_n^{NS}(Q_0^2)$ can be found from the data by measuring the moments of structure functions at $Q^2=Q_0^2$. Once $A_n^{NS}(Q_0^2)$ are known, Eqs. (4.17a) and (4.18a) give the moments of the structure functions at any (sufficiently large) value of Q^2 in terms of one free parameter Λ.

The value of Q_0^2 in Eqs. (4.17a) and (4.18a) is arbitrary as required by the renormalization group and the predictions for the moments in question should be independent of it. However, by picking out one particular value of Q_0^2 in order to determine $A_n^{NS}(Q_0^2)$ one gives this value specific significance. For consistency one should find $A_n^{NS}(Q_0^2)$ from the data by choosing various values of Q_0^2 and check whether expressions (4.17a) and (4.18a) with various values of Q_0^2 give results compatible with each other. In order to simplify this procedure and at the same time to impose the independence of the phenomenological fit of Q_0^2, it is convenient to get rid of Q_0^2 by writing Eqs. (4.17a) and (4.18a) as follows:

$$\int_0^1 dx\, x^{n-2}F_2^{NS}(x,Q^2)=\delta^{(2)}_{NS}A_n^{NS}\left[\ln\frac{Q^2}{\Lambda^2}\right]^{-d^n_{NS}} \tag{4.17b}$$

and

$$\int_0^1 dx\, x^{n-1}F_3^{NS}(x,Q_0^2)=\delta^{(3)}_{NS}A_n^{NS}\left[\ln\frac{Q^2}{\Lambda^2}\right]^{-d^n_{NS}}. \tag{4.18b}$$

Here A_n^{NS} are constants (independent of Q_0^2) and are related to $A_n^{NS}(Q_0^2)$ by the following equation:

$$A_n^{NS}(Q_0^2)=A_n^{NS}\left[\ln\frac{Q_0^2}{\Lambda^2}\right]^{-d^n_{NS}}. \tag{4.19}$$

Numerical values for d^n_{NS} can be found in Table I.

C. Singlet structure functions

The Q^2 dependence of $C^\psi_{k,n}(Q^2/\mu^2,g^2)$ and $C^G_{k,n}(Q^2/\mu^2,g^2)$ is governed by the following two coupled renormalization group equations

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right]C^i_{k,n}\left(\frac{Q^2}{\mu^2},g^2\right)=\sum_j\gamma^n_{ji}(g)C^j_{k,n}\left(\frac{Q^2}{\mu^2},g^2\right)\qquad i,j=\psi,G, \tag{4.20}$$

where $\gamma^n_{ij}(g)$ are the elements of the anomalous dimension matrix and $\beta(g)$ is the same function as in Eq. (4.12). These equations are more complicated than Eq. (4.8) due to the mixing between singlet operators as we discussed in Sec. III. In other words, under renormalization O_ψ transforms into a linear combination of O_ψ and O_G and the same happens with O_G. This mixing has a very intuitive interpretation which we shall present in Sec. V.

In what follows it will be convenient to work with matrix notation and introduce the column vector

$$\mathbf{C}_{k,n}\left(\frac{Q^2}{\mu^2},g^2\right)=\begin{bmatrix}C^\psi_{k,n}(Q^2/\mu^2,g^2)\\ C^G_{k,n}(Q^2/\mu^2,g^2)\end{bmatrix}\quad k=2,L \tag{4.21}$$

and the matrix[25]

$$\hat{\gamma}^n(g)=\begin{bmatrix}\gamma^n_{\psi\psi}(g) & \gamma^n_{G\psi}(g)\\ \gamma^n_{\psi G}(g) & \gamma^n_{GG}(g)\end{bmatrix}. \tag{4.22}$$

Then the solution of Eq. (4.20) can be written as follows:

$$\mathbf{C}_{k,n}\left(\frac{Q^2}{\mu^2},g^2\right)=\left[T_g\exp\int_{\bar{g}(Q^2)}^{g}dg'\frac{\hat{\gamma}^n(g')}{\beta(g')}\right]\mathbf{C}_{k,n}(1,\bar{g}^2). \tag{4.23}$$

The T ordering is necessary since $[\hat{\gamma}(g_1),\hat{\gamma}(g_2)]\neq 0$ and is defined as follows:

$$T_g\exp\int_{\bar{g}}^{g}dg'\frac{\hat{\gamma}^n(g')}{\beta(g')}=1+\int_{\bar{g}}^{g}dg'\frac{\hat{\gamma}^n(g')}{\beta(g')}+\int_{\bar{g}}^{g}dg'\int_{\bar{g}}^{g'}dg''\frac{\hat{\gamma}(g')}{\beta(g')}\frac{\hat{\gamma}(g'')}{\beta(g'')}+\cdots. \tag{4.24}$$

To proceed further one has to calculate $\mathbf{C}_{k,n}(1,\bar{g}^2)$ and $\hat{\gamma}^n(g)$ in perturbation theory. In the *leading order* it is enough to calculate one-loop contributions to $\hat{\gamma}^n(g)$ and take zero-loop (parton model) values for $\mathbf{C}_{k,n}(1,\bar{g}^2)$. Explicitly

$$\hat{\gamma}^n(g)=\hat{\gamma}^{(0),n}(g^2/16\pi^2), \tag{4.25}$$

$$C_{2,n}(1,\bar{g}^2)=\begin{bmatrix}\delta^{(2)}_\psi\\ 0\end{bmatrix}, \tag{4.26}$$

and

$$\mathbf{C}_{L,n}(1,\bar{g}^2)=\begin{bmatrix}0\\ 0\end{bmatrix}. \tag{4.27}$$

Notice that $C^G_{k,n}(1,\bar{g}^2)$ vanishes to this order. This does not mean, however, that $C^G_{k,n}(Q^2/\mu^2,g^2)$ is zero for $Q^2\neq\mu^2$, as one can check by inserting (4.26) into Eq. (4.23).

In what follows it will be useful to choose the basis in which $\hat{\gamma}^{(0),n}$ is diagonal. We introduce the matrix $\hat{U}$ which diagonalizes $\hat{\gamma}^{(0),n}$ by

$$\hat{U}^{-1}\hat{\gamma}^{0,n}\hat{U}=\begin{bmatrix}\lambda^n_- & 0\\ 0 & \lambda^n_+\end{bmatrix}, \tag{4.28}$$

where $\lambda^n_\pm$ are the eigenvalues of $\hat{\gamma}^{0,n}$ and are given in Eq. (2.78). It should be remarked that the matrix $\hat{U}$ is not

[25]Notice that we work with a transposed matrix.

defined uniquely by Eq. (4.28). In fact any matrix $\hat{U}'$ which is related to $\hat{U}$ by

$$\hat{U}'=\hat{U}\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}, \tag{4.29}$$

where a and b are arbitrary real, finite numbers, satisfies Eq. (4.28). Of course the final expression for the moments of the singlet structure functions does not depend on which a and b we take. Here we choose

$$\hat{U}=\begin{bmatrix}\gamma_{\psi\psi}^{(0),n}-\lambda_+^n & \gamma_{\psi\psi}^{(0),n}-\lambda_-^n\\ \gamma_{\psi G}^{(0),n} & \gamma_{\psi G}^{(0),n}\end{bmatrix}\frac{1}{(\lambda_-^n-\lambda_+^n)}, \tag{4.30}$$

and consequently

$$\hat{U}^{-1}=\begin{bmatrix}\gamma_{\psi G}^{(0),n} & \lambda_-^n-\gamma_{\psi\psi}^{(0),n}\\ -\gamma_{\psi G}^{(0),n} & \gamma_{\psi\psi}^{(0),n}-\lambda_+^n\end{bmatrix}\frac{1}{\gamma_{\psi G}^{(0),n}}. \tag{4.31}$$

The elements $\gamma_{ij}^{(0),n}$ have been calculated by Georgi and Politzer (1974) and Gross and Wilczek (1974) and are given in Eq. (2.79).

Notice next that if we introduce a row vector

$$\mathbf{A}_n(\mu^2)=[A_n^{\psi}(\mu^2),A_n^{G}(\mu^2)], \tag{4.32}$$

we can rewrite Eq. (4.7) as follows[26]:

$$\begin{aligned}\int_0^1 dx\,x^{n-2}F_2^s(x,Q^2)&=\mathbf{A}_n(\mu^2)\mathbf{C}_{2,n}\left(\frac{Q^2}{\mu^2},g^2\right)\\ &=\mathbf{A}_n(\mu^2)\hat{U}\hat{U}^{-1}\mathbf{C}_{2,n}\left(\frac{Q^2}{\mu^2},g^2\right) \qquad (4.33)\\ &=\tilde{A}_n^-(\mu^2)C_{2,n}^-\left(\frac{Q^2}{\mu^2},g^2\right)\\ &\quad+\tilde{A}_n^+(\mu^2)C_{2,n}^+\left(\frac{Q^2}{\mu^2},g^2\right), \qquad (4.34)\end{aligned}$$

where

$$[\tilde{A}_n^-(\mu^2),\tilde{A}_n^+(\mu^2)]=\mathbf{A}_n(\mu^2)\hat{U} \tag{4.35}$$

and

$$\begin{bmatrix}C_{2,n}^-(Q^2/\mu^2,g^2)\\ C_{2,n}^+(Q^2/\mu^2,g^2)\end{bmatrix}=\hat{U}^{-1}\mathbf{C}_{2,n}\left(\frac{Q^2}{\mu^2},g^2\right). \tag{4.36}$$

Using Eq. (4.36) for $Q^2=\mu^2$ and taking into account Eqs. (4.26) and (4.31) we obtain (leading order)

$$\begin{bmatrix}C_{2,n}^-(1,\bar{g}^2)\\ C_{2,n}^+(1,\bar{g}^2)\end{bmatrix}=\delta_\psi^{(2)}\begin{bmatrix}1\\ -1\end{bmatrix}. \tag{4.37}$$

We next write

$$\begin{aligned}\hat{U}^{-1}\mathbf{C}_{k,n}\left(\frac{Q^2}{\mu^2},g^2\right)&=\hat{U}^{-1}\left[T_g\exp\int_{\bar{g}(Q^2)}^{g}dg'\frac{\hat{\gamma}^n(g')}{\beta(g')}\right]\\ &\quad\times\hat{U}\hat{U}^{-1}\mathbf{C}_{k,n}(1,\bar{g}^2). \qquad (4.38)\end{aligned}$$

The T ordering is irrelevant to the order considered and we first obtain, using (4.11), (4.12), (4.16), and (4.28)

$$\hat{U}^{-1}\left[T_g\exp\int_{\bar{g}(Q^2)}^{g}dg'\frac{\hat{\gamma}^n(g')}{\beta(g')}\right]\hat{U}=\begin{bmatrix}\left[\dfrac{\ln(Q^2/\Lambda^2)}{\ln(\mu^2/\Lambda^2)}\right]^{-d_-^n} & 0\\ 0 & \left[\dfrac{\ln(Q^2/\Lambda^2)}{\ln(\mu^2/\Lambda^2)}\right]^{-d_+^n}\end{bmatrix}, \tag{4.39}$$

where

$$d_\pm^n=\lambda_\pm^n/2\beta_0. \tag{4.40}$$

Combining Eqs. (4.34) and (4.37)–(4.39) we finally obtain the generalization of Eq. (4.17a),

$$\int_0^1 dx\,x^{n-2}F_2^s(x,Q^2)=\delta_\psi^{(2)}A_n^-(Q_0^2)\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d_-^n}+\delta_\psi^{(2)}A_n^+(Q_0^2)\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d_+^n} \tag{4.41a}$$

and $F_L^s=0$. Here we have put $\mu^2=Q_0^2$ and defined

$$A_n^\mp(Q_0^2)=\pm\tilde{A}^\mp(Q_0^2). \tag{4.42}$$

Equation (4.18a) applies for F_3.

Equations (4.41) and (4.42) with $\tilde{A}^i(Q_0^2)$ given by (4.35) are very useful in relating the formal approach developed here to the intuitive approach of Sec. V. However, when the formal approach is used without any reference to the parton distributions, it is convenient to repeat the steps which led us in the nonsinglet case from Eq. (4.17a) to Eq. (4.17b) and write

$$\int_0^1 dx\,x^{n-2}F_2^s(x,Q^2)=\delta_\psi^{(2)}A_n^-[\ln(Q^2/\Lambda^2)]^{-d_-^n}+\delta_\psi^{(2)}A_n^+[\ln(Q^2/\Lambda^2)]^{-d_+^n}, \tag{4.41b}$$

where the constants (independent of Q_0^2) $A_n^\mp$ are related to $A_n^\mp(Q_0^2)$ as follows:

$$A_n^\mp(Q_0^2)=A_n^\mp[\ln(Q_0^2/\Lambda^2)]^{-d_\mp^n}. \tag{4.43}$$

Numerical values for $d_\pm^n$ can be found in Table I.

Equation (4.41b) can be used directly in phenomenological applications. $\delta_\psi^{(2)}$ can be taken from Appendix B. Then Eq. (4.41b) describes the Q^2 evolution of the moments of F_2^s in terms of two sets of unknown numbers $A_n^\mp$ and the scale parameter Λ. $A_n^\mp$ and Λ are to be found by comparing Eq. (4.41b) with the data.

As can be seen in Table I

$$d_+^n\gtrsim d_-^n+1\,,\quad 4\leqslant n\leqslant 14 \tag{4.44}$$

and

$$d_+^n\gtrsim d_-^n+2\,,\quad n\geqslant 14\,. \tag{4.45}$$

In addition

$$d_-^n\approx d_{\rm NS}^n\,,\quad n\geqslant 4\,. \tag{4.46}$$

Therefore for $n\geqslant 4$ and for sufficiently large Q^2 the second term in Eq. (4.41) can be dropped and consequently the Q^2 dependence of the singlet structure function is essentially the same as that of the nonsinglet structure functions. This could be spoiled by large values of A_n^+ but experimentally this is not the case. In formal terms Eq. (4.46) expresses the fact that the mixing of gluon and fermion operators of high spin-n is

[26]Since F_L vanishes in the leading order we discuss here only F_2, i.e., $k=2$. F_L is discussed in Secs. VII and VIII.

very weak. We also observe that because of the inequality (4.44) the $\bar{g}^2$ corrections to the "−" term in Eq. (4.41) are for $n>4$ as important as the leading-order contribution to the "+" term. We shall later discuss it in more detail.

V. Q^2 DEPENDENCE OF PARTON DISTRIBUTIONS IN THE LEADING ORDER

The discussion of the Q^2 dependence of the structure functions as presented in Sec. IV was rather formal and it is useful to develop a more intuitive picture. In fact, this can be done at least in the leading order. The result is simple and was already announced in Sec. II: all well-known parton model expressions remain unchanged except that now parton distributions depend on Q^2. Thus if we only find the Q^2 dependence of parton distributions predicted by asymptotic freedom, we can study QCD effects in deep-inelastic scattering by means of the standard parton model formulas. The aim of this section is to present equations which determine the Q^2 dependence of parton distributions, solve them, and show that they are equivalent to the formalism developed in Sec. IV.

A. Intuitive picture and integrodifferential equations

We begin with the intuitive picture of Kogut and Susskind (1974). Imagine the photon, Z^0 boson, or a W boson to be a microscope by means of which we probe the inner structure of the proton or generally of a hadron. Increasing Q^2 while holding x fixed is equivalent to increasing the power of the microscope or looking at shorter and shorter distances. By scanning the proton at fixed Q^2 and $0\leq x\leq 1$, we obtain the picture of the proton at this particular value of Q^2. According to the simple parton model of Sec. II.B, the picture of the proton does not depend on how strong a microscope we use. The pictures at different values of Q^2 are the same. This is not the case in QCD. By increasing the power of our microscope from Q_1^2 to $Q_2^2>Q_1^2$, we can resolve a quark with momentum fraction x into a quark with $x'<x$ and a gluon with $x''\simeq x-x'$ as illustrated in Fig. 3(a). Similarly a gluon with momentum fraction x can be resolved into a quark–antiquark pair as illustrated in Fig. 3(b). There exists also the process of Fig. 3(c) which can be interpreted as resolving a gluon into a gluon pair. Since a gluon couples neither to γ nor to Z^0 or $W^\pm$, this does not happen directly. However, the existence of this process affects the probability of finding a quark in a gluon since a gluon can either fragment into a quark–antiquark pair or into a pair of gluons and the sum of these two probabilities (plus the probability that the gluon does not fragment at all) is just unity. In summary the picture of the proton or, equivalently, parton distributions, depends on Q^2.

The Q^2 dependence is different for different parton distributions. Intuitively, we see that valence quarks can effectively emit only gluons. They cannot be produced effectively in the Fig. 3(b) process because this would lead to baryon number nonconservation. As a result, valence quarks lose their momentum in favor of gluons and consequently [through the process of Fig. 3(b)] in favor of the sea. In the simple language developed above, by increasing Q^2 we cannot find a valence quark in a gluon or in a sea quark but only in a valence quark itself. Of course this picture is oversimplified since it assumes one can make a clear distinction between valence and sea quarks as in Eqs. (2.12) and (2.13).

The behavior of the sea distribution with increasing Q^2 is different. Here both processes [Figs. 3(a) and 3(b)] can contribute. On one hand, gluon bremsstrahlung leads to a shift of the sea distribution to smaller values of x. On the other hand the process of Fig. 3(b) increases the amount of sea at all (mostly at small) values of x. As we shall see below, because of the coexistence of the two processes instead of just one as in the case of valence quarks, the asymptotic freedom equations for the Q^2 evolution of the sea distribution are more complicated than for the valence quark distribution. For the same reason also, equations for the Q^2 development of the gluon distribution are rather complicated. In the formal language of the previous section the complex Q^2 dependence of the gluon and sea distributions is due to the mixing between gluon and singlet fermion operators. On the other hand, the simple behavior of the valence quark distributions is due to the fact that the corresponding nonsinglet operators do not mix under renormalization with the singlet operators.

It is obvious from the picture developed above that the Q^2 dependence of the parton distributions will be determined by the variation (with Q^2) of the probability of finding a parton i inside the parent parton j with the fraction of the parent momentum, $z=x_i/x_j$.

Adopting the notation of Altarelli and Parisi (1977) we write the variation of the probability in question as

$$[\alpha_s(Q^2)/2\pi\, P_{ij}(z)\,, \tag{5.1}$$

where $\alpha_s(Q^2)=\bar{g}^2(Q^2)/4\pi$ and i,j stand for q and G. Then the equations which determine the Q^2 dependence of the parton distributions are given as follows:

$$\frac{dV(x,t)}{dt}=\frac{\alpha(Q^2)}{2\pi}\int_x^1\frac{dy}{y}\,V(y,t)P_{qq}\left(\frac{x}{y}\right) \tag{5.2}$$

$$\frac{dq_i(x,t)}{dt}=\frac{\alpha(Q^2)}{2\pi}\times\int_x^1\frac{dy}{y}\left[q_i(y,t)P_{qq}\left(\frac{x}{y}\right)+G(y,t)P_{qG}\left(\frac{x}{y}\right)\right]\qquad i=1,\ldots f \tag{5.3}$$

$$\frac{dG(x,t)}{dt}=\frac{\alpha(Q^2)}{2\pi}\int_x^1\frac{dy}{y}\left[\Sigma(y,t)P_{Gq}\left(\frac{x}{y}\right)+G(y,t)P_{GG}\left(\frac{x}{y}\right)\right]\,. \tag{5.4}$$

Here $t=\ln(Q^2/\mu^2)$, $V(x,t)$ is the valence quark distribution and

$$\Sigma(x,t)\equiv\sum_i[q_i(x,t)+\bar{q}_i(x,t)]\,,$$

where the sum is over all flavors. The equation for $\bar{q}_i(x,t)$ is obtained from (5.2) by replacing $q_i(x,t)$ by $\bar{q}_i(x,t)$. From Eq. (5.3) it is a trivial matter to obtain Eqs. (2.52) and (2.53) of Sec. II.

Equations (5.2)–(5.4) have been obtained in QCD by Altarelli and Parisi (1977) and Dokshitser, Dyakonov, and Troyan (1978). Similar equations in the context of other theories have been discussed previously by Gribov and Lipatov (1972) and Kogut and Susskind (1974). The structure of Eqs. (5.2)–(5.4) is easy to understand in terms of the intuitive picture discussed above. The quark distribution at the value x is determined by the quark (gluon) distribution in the range $x \leq y \leq 1$ and the probability for the $q(y)[G(y)] \to q(x)$ transition which is given by $P_{qq}(x/y)[P_{qG}(x/y)]$. Similar comments apply to Eq. (5.4).

Notice that the functions P_{qq}, P_{qG}, and P_{Gq} do not depend on flavor. Strictly speaking this is only true for massless quarks.

The "splitting" functions $P_{ij}(z)$ can be calculated in QCD by considering the vertices of Fig. 3. We refer the reader to the paper by Altarelli and Parisi (1977) for details. The result of this calculation is summarized in Eqs. (2.56)–(2.59) of Sec. II.

Here we shall only discuss certain properties of $P_{ij}(z)$. To this end it is useful, following Dokshitser, Dyakonov, and Troyan (1978), to factor out group theory factors from $P_{ij}(z)$ and introduce the functions $V_{ij}(z)$ as follows:

$$P_{qq}(z) = \tfrac{4}{3} V_{qq}(z), \quad z<1 \tag{5.5}$$

$$P_{qG}(z) = V_{qG}(z), \tag{5.6}$$

$$P_{Gq}(z) = \tfrac{4}{3} V_{Gq}(z), \tag{5.7}$$

$$P_{GG}(z) = 3V_{GG}(z). \quad z<1 \tag{5.8}$$

The functions $P_{ij}(z)$ and $V_{ij}(z)$ satisfy certain relations and sum rules which we shall list now:

(i) Charge conservation:

$$\int_0^1 dz\, P_{qq}(z) = 0. \tag{5.9}$$

(ii) Total momentum conservation:

$$\int_0^1 dz\, z[P_{qq}(z) + P_{Gq}(z)] = 0,$$

$$\int_0^1 dz\, z[2fP_{qG}(z) + P_{GG}(z)] = 0. \tag{5.10}$$

(iii) Momentum conservation at the vertices of Fig. 3:

$$V_{qq}(z) = V_{Gq}(1-z),$$
$$V_{qG}(z) = V_{qG}(1-z),$$
$$V_{GG}(z) = V_{GG}(1-z). \tag{5.11}$$

The relations above are obvious. There exist in addition two other relations (Dokshitser, 1977), which are very interesting although not completely clear:

(iv) The crossing relation:

$$V_{qG}(z) = z\, V_{Gq}(1/z). \tag{5.12}$$

(v) Quark–gluon symmetry:

$$V_{qq}(z) + V_{Gq}(z) = V_{qG}(z) + V_{GG}(z). \tag{5.13}$$

The crossing relation (5.12) leads to the well-known relation between the deep-inelastic and e^+e^- structure functions (Drell, Levy, and Yan, 1969; Gribov and Lipatov, 1972; Lipatov, 1975; Bukhvostov, Lipatov, and Popov, 1975). Equation (5.13) could be interpreted as the equality of total probabilities for finding quark and gluon in a quark, and quark and gluon in a gluon. It is possible, however, that the relation (5.13) is just accidental.

The important consequence of relations (5.9)–(5.13) is that it is enough to know one function $P_{ij}(z)$ in order to determine the remaining three splitting functions. As we shall see below this implies that in order to find the whole one-loop anomalous dimension matrix as given by Eq. (2.79) it is enough to calculate only one of its elements! This does not turn out to be true for the two-loop anomalous dimension matrix (see Sec. VIII).

B. Asymptotic freedom equations for the moments of parton distributions

Here we shall show that the integrodifferential equations (2.52)–(2.54) are equivalent to the equations for the moments of parton distributions as given by Eqs. (2.84)–(2.86).

We first quote the well-known convolution theorem for Mellin transforms which says that if

$$H_1(x) = \int_x^1 \frac{dy}{y} H_2(y) H_3\left(\frac{x}{y}\right), \tag{5.14}$$

where $H_i(x)$ are some functions, then

$$M_n^{(1)} = M_n^{(2)} M_n^{(3)}, \tag{5.15}$$

where

$$M_n^{(i)} = \int_0^1 dx\, x^{n-1} H_i(x) \quad i = 1,2,3. \tag{5.16}$$

We next notice (Altarelli and Parisi, 1977) the relations between the moments of the splitting functions $P_{ij}(z)$ and the elements of the anomalous dimension matrix $\hat{\gamma}^{0,n}$[Eq. (2.79)]

$$\int_0^1 dz\, z^{n-1} P_{qq}(z) = -\frac{\gamma^{0,n}_{\psi\psi}}{4}, \tag{5.17}$$

$$\int_0^1 dz\, z^{n-1} P_{qG}(z) = -\frac{\gamma^{0,n}_{\psi G}}{8f}, \tag{5.18}$$

$$\int_0^1 dz\, z^{n-1} P_{Gq}(z) = -\frac{\gamma^{0,n}_{G\psi}}{4}, \tag{5.19}$$

and

$$\int_0^1 dz\, z^{n-1} P_{GG}(z) = -\frac{\gamma^{0,n}_{GG}}{4}. \tag{5.20}$$

Conventionally we have kept different notations for the indices of P_{ij} functions and the indices of the elements of the anomalous dimension matrix. ψ in the formal approach stands for q in the intuitive approach. G is the same in both approaches. The one-to-one correspondence between diagrams needed for the calculation of $\hat{\gamma}^{0,n}$ and the vertices needed for a similar calculation of the P_{ij} functions is illustrated in Fig. 10.

Finally recall that in the one-loop approximation to the β function

$$\alpha(Q^2) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda^2)}, \tag{5.21}$$

we can write

$$\frac{2\pi}{\alpha(Q^2)} \frac{d}{dt} = \frac{\beta_0}{2} \frac{d}{d\bar{s}} \tag{5.22}$$

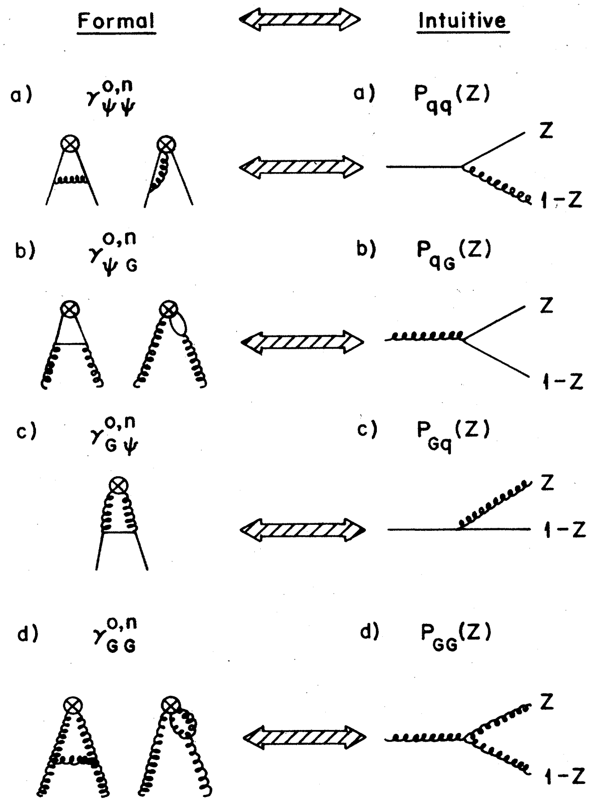


FIG. 10. Formal approach versus intuitive approach.

where

$$\tilde{s} \equiv \ln\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right], \tag{5.23}$$

with Q_0^2 being some reference value of Q^2.

Applying the convolution theorem to Eqs. (2.52)–(2.54) and using Eqs. (5.17)–(5.20) and (5.22) we obtain differential equations for the moments of parton distributions which can be trivially integrated to give the promised Eqs. (2.84)–(2.86).

C. Equivalence of the intuitive and the formal approach

In Sec. V.B we have demonstrated how the moment Eqs. (2.84)–(2.86) can be obtained from the integro-differential Eqs. (2.52)–(2.54). Here we shall show that Eqs. (2.84)–(2.86) can also be derived from the formal approach of Sec. IV.

As we have discussed in Sec. II.B, any parton model formula for an arbitrary structure function can be written as a sum of singlet [$\Sigma(x)$] and nonsinglet [$\Delta(x)$] combinations of quark distributions weighted by the appropriate weak and electromagnetic charges. The latter are represented in the formal approach by the constants $\delta_\psi^{(k)}$ and $\delta_{\rm NS}^{(k)}$. Therefore writing generally

$$F_2^{\rm NS}(x, Q^2) = \delta_{\rm NS}^{(2)} x\Delta(x, Q^2), \tag{5.24}$$

and inserting it into Eq. (4.17a), we obtain

$$\langle\Delta(Q^2)\rangle_n \equiv \int_0^1 dx\, x^{n-1}\Delta(x, Q^2) = A_n^{\rm NS}(Q_0^2)\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d^n_{\rm NS}}, \tag{5.25}$$

and consequently

$$A_n^{\rm NS}(Q_0^2) = \langle\Delta(Q_0^2)\rangle_n . \tag{5.26}$$

Therefore Eq. (5.25) is clearly identical to Eq. (2.84). Thus the matrix elements of local operators normalized at $\mu^2 \equiv Q_0^2$ are interpreted as the moments of quark distributions at $Q^2 = Q_0^2$. The relation (5.26) between moments of quark distributions and the matrix elements of local operators has been anticipated a long time ago in the context of the parton model and light cone algebra (Jaffe, 1972).

The case of singlet structure functions is slightly more complicated. Writing

$$F_2^s(x, Q^2) = \delta_\psi^{(2)} x\Sigma(x, Q^2), \tag{5.27}$$

and inserting it into Eq. (4.41a) we obtain first

$$\langle\Sigma(Q^2)\rangle_n \equiv \int_0^1 dx\, x^{n-1}\Sigma(x, Q^2) = A_n^-(Q_0^2)\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d^n_-} + A_n^+(Q_0^2)\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d^n_+}, \tag{5.28}$$

where by Eqs. (4.30), (4.35), and (4.42)

$$A_n^-(Q_0^2) = A_n^\psi(Q_0^2)\alpha_n + A_n^G(Q_0^2)\tilde{\alpha}_n \tag{5.29}$$

and

$$A_n^+(Q_0^2) = A_n^\psi(Q_0^2)(1-\alpha_n) - A_n^G(Q_0^2)\tilde{\alpha}_n . \tag{5.30}$$

The parameters α_n and $\tilde{\alpha}_n$ are defined in Eq. (2.87). It follows from Eqs. (5.28) to (5.30) that

$$A_n^\psi(Q_0^2) = \langle\Sigma(Q_0^2)\rangle_n . \tag{5.31}$$

Therefore if we take

$$A_n^G(Q_0^2) = \langle G(Q_0^2)\rangle_n , \tag{5.32}$$

the formal Eq. (5.28) is identical to the moment Eq. (2.85).

As pointed out by Floratos, Ross, and Sachrajda (1979), Eq. (2.86) for the moments $\langle G(Q^2)\rangle_n$ can be directly obtained from Eq. (2.85). To this end we find $\langle G(Q_0^2)\rangle_n$ from Eq. (2.85) and make the relabeling, $Q_0^2 \leftrightarrow Q^2$, with the result

$$\langle G(Q^2)\rangle_n = \frac{\langle\Sigma(Q^2)\rangle_n\{(1-\alpha_n)\exp[d_+^n\tilde{s}] + \alpha_n\exp[d_-^n\tilde{s}]\} - \langle\Sigma(Q_0^2)\rangle_n}{\tilde{\alpha}_n\{\exp[d_+^n\tilde{s}] - \exp[d_-^n\tilde{s}]\}} . \tag{5.33}$$

Using next Eq. (2.85) on the rhs of Eq. (5.33), we are led to Eq. (2.86).

D. Properties of parton distributions

The Q^2 dependence of parton distributions, as predicted by asymptotic freedom, can be obtained by integrating Eqs. (2.52)–(2.54) or equivalently (5.2)–(5.3). Before doing this we shall first list basic properties of the Q^2 evolution of parton distributions. These properties can be obtained most directly from the moment Eqs. (2.84)–(2.86).

a. The momentum fraction carried by valence quarks, $\langle V\rangle_2$, and $\langle x\rangle_v$ of the valence quark distribution decrease with increasing Q^2. This is consistent with the intuitive picture developed in Sec. V.A. Explicitly from Eqs.

(2.49a), (2.84), and (2.79a) we have

$$\langle V(Q^2)\rangle_2 = \langle V(Q_0^2)\rangle_2 \exp\{-[64/3(33-2f]\bar{s}\} \tag{5.34}$$

and

$$\langle x(Q^2)\rangle = \langle x(Q_0^2)\rangle \exp\{-[12/(33-2f)]\bar{s}\}\,, \tag{5.35}$$

where $\bar{s}$ is given by Eq. (5.23) and $\langle x(Q^2)\rangle$ is defined as follows:

$$\langle x(Q^2)\rangle \equiv \frac{\langle V(Q^2)\rangle_3}{\langle V(Q^2)\rangle_2}\,. \tag{5.36}$$

Notice that the rate of decrease of $\langle V(Q^2)\rangle_2$ and of $\langle x\rangle_v$ increases with the number of flavors, which is not difficult to understand in the intuitive picture of Sec. V.A.

b. The momentum fraction carried by gluons, $\langle G\rangle_2$ *and sea,* $\langle S\rangle_2$, *increases with increasing* Q^2. Since energy–momentum is conserved the momentum lost by valence quarks must be carried by gluons and sea quarks. In the Altarelli–Parisi Eqs. (5.2)–(5.3) energy–momentum conservation is ensured by the sum rules (5.10), which by Eqs. (5.17)–(5.20) are equivalent to

$$\gamma^{0,(2)}_{\psi\psi} + \gamma^{0,(2)}_{G\psi} = 0\,, \tag{5.37}$$

$$\gamma^{0,(2)}_{GG} + \gamma^{0,(2)}_{\psi G} = 0\,, \tag{5.38}$$

and consequently we have by Eq. (2.78)

$$\lambda_-^{(2)} = 0\,. \tag{5.39}$$

In more formal terms this just expresses the fact that the anomalous dimension of the energy-momentum tensor is zero. In Eqs. (5.37)–(5.39) "(2)" stands for $n=2$.

From Eqs. (2.87) and (2.79) we have for f flavors

$$\alpha_2 = \tilde{\alpha}_2 = 3f/(16+3f) \tag{5.40}$$

and

$$\epsilon_2 = 1-\alpha_2 = 16/(16+3f) \tag{5.41}$$

and using (2.85) and (2.86), we verify momentum conservation

$$\langle \Sigma(Q^2)\rangle_2 + \langle G(Q^2)\rangle_2 = \langle \Sigma(Q_0^2)\rangle_2 + \langle G(Q_0^2)\rangle_2\,. \tag{5.42}$$

Normalizing the total momentum of the hadron to 1 we obtain the following asymptotic predictions

$$\langle \Sigma(Q^2)\rangle_2 \xrightarrow[Q^2\to\infty]{} 3f/(16+3f)\,, \tag{5.43}$$

$$\langle G(Q^2)\rangle_2 \xrightarrow[Q^2\to\infty]{} 16/(16+3f)\,. \tag{5.44}$$

Because $\langle V(\infty)\rangle_2 = 0$ we also have $\langle \Sigma(\infty)\rangle_2 = \langle S(\infty)\rangle_2$. For instance, for $f=4$, asymptotically 43% of the proton (or other target) momentum will be carried by the sea and the remaining 57% by gluons. For $f=6$, asymptotically 47% of momentum is carried by gluons and the remaining 53% by the sea. Notice that the asymptotic fraction of momentum carried by quarks increases with the number of flavors. Similarly the asymptotic fraction of momentum carried by gluons increases with the number of colors since for a gauge group SU(N) the 16 in Eqs. (5.40) and (5.41) is replaced by $2(N^2-1)$. It should be remarked that these asymptotic predictions do not depend on the target. On the other hand, at moderate values of Q^2 the momentum decomposition in the proton is, for instance, different from that in the pion. For a recent discussion of these questions we refer the reader to the paper by Brodsky and Gunion (1979).

At low values of Q^2, roughly 45% of proton momentum is carried by gluons, 7% by the sea, and the remaining 48% by valence quarks. We expect therefore, on the basis of predictions (5.43) and (5.44), a rapid increase with Q^2 of the amount of the sea, and a very slow (fast) increase (decrease) of the momentum carried by gluons (valence quarks). *Thus, effectively, valence quarks lose their momentum almost entirely in favor of the sea.* This is confirmed by explicit calculations.

c. The average values of x, $\langle x\rangle$, *of the sea and gluon distributions decrease with increasing* Q^2. Although the momentum carried by the sea and gluons increases, their $\langle x\rangle$ values decrease as one can verify by means of Eqs. (2.84)–(2.86). This is obvious if we recall the intuitive picture at the beginning of this section or notice (Nachtmann, 1973) that $d_\pm^n$, as given in Table I, increase monotonically with n and are positive for $n>2$. Similar comments apply to higher moments of $\langle x\rangle$. Consequently we expect a *decrease* (as in the case of valence quarks) of the sea and gluon distributions at large values of x and *increase* (due to property b) of the distributions in question at small x values. This behavior has profound consequences for the Q^2 development of the deep-inelastic structure functions.

d. The flavor symmetry breaking in the sea decreases with increasing Q^2. Equation (2.84) implies that

$$\langle \bar{c}(Q^2)\rangle_n - \langle \bar{d}(Q^2)\rangle_n = [\langle \bar{c}(Q_0^2)\rangle_n - \langle \bar{d}(Q_0^2)\rangle_n]\exp[-d^n_{\rm NS}\bar{s}]\,, \tag{5.45}$$

and similarly for any pair of different quark distributions. Thus asymptotically all different quark distributions will be equal. Strictly speaking, Eq. (5.45) is only approximate because it does not take care of thresholds effects. Consequently, for the values of Q^2 not much bigger than the (mass)^2 of the relevant heavy quark, Eq. (5.45) overestimates the rate of approach to the flavor symmetry limit.

e. Generation of heavy quarks in the sea. It follows from Eq. (5.45) that if we set the charm contribution equal to zero at some value of $Q^2=Q_0^2$, then for $Q^2>Q_0^2$ the distribution in question will be different from zero. This is due to the $q\bar{q}$ creation of Fig. 3(b). Of course, due to neglect of mass effects, Eq. (5.45) overestimates the rate of generation of heavy quarks in the sea.

f. The Q^2 *evolution of the sea distribution depends on the shape of the gluon distribution.* Since the sea is produced in the process of Fig. 3(b), its Q^2 evolution depends on the shape of the gluon distribution. From the intuitive picture developed at the beginning of this section, it is clear that the steeper the gluon distribution, the stronger the increase of the sea at small values of x. Similarly a broad gluon distribution would lead to a non-negligible generation of sea quarks at intermediate values of x, say $x\approx 0.3$. Since the shape of the gluon distribution is rather poorly determined experimentally, in practical applications the parameters of the gluon distribution are very often kept free and are varied to get the best fit to the data. Of course, this freedom is limited to one value of $Q^2=Q_0^2$ and to the moments $n>2$, since the momentum carried by gluons is known due to momentum conservation (Eq. 5.42) once the momentum carried by quarks is determined.

This completes the listing of the main properties of the Q^2 evolution of parton distributions as predicted by asymptotic freedom. We shall see below that the knowledge of these properties greatly simplifies the discussion of QCD effects in deep-inelastic scattering.

So far our discussion was rather qualitative. Choosing certain quark and gluon distributions at $Q=Q_0^2$ we can either integrate numerically Eqs. (2.52)–(2.54) or invert numerically moment Eqs. (2.84)–(2.86) to find the distributions in question for $Q^2 \neq Q_0^2$. The result of such a calculation is presented in Fig. 11. All the properties discussed above are clearly seen.

E. Approximate solutions of asymptotic freedom equations

For practical applications it is often convenient to have analytic expressions for Q^2 dependent parton distributions which to a good accuracy represent the numerical solutions of Eqs. (2.52)–(2.54) or Eqs. (2.84)–(2.86). We shall here present the method for obtaining such analytic expressions proposed by Buras (1977) and Buras and Gaemers (1978). We shall also refer to other methods which can be found in the literature.

Let us parametrize the solutions to Eqs. (2.52)–(2.54) or Eqs. (2.84)–(2.86) by analytic expressions as follows:

$$xV(x,Q^2)=\frac{3}{B[\eta_1(s),1+\eta_2(s)]}\,x^{\eta_1(\bar{s})}(1-x)^{\eta_2(\bar{s})}\,, \tag{5.46}$$

for valence quark distribution,

$$xS(x,Q^2)=A_s(\bar{s})(1-x)^{\eta_s(\bar{s})}\,, \tag{5.47}$$

for any sea distribution, and

$$xG(x,Q^2)=A_G(\bar{s})(1-x)^{\eta_G(\bar{s})}\,, \tag{5.48}$$

for the gluon distribution.

Here $\bar{s}$ is given by Eq. (5.23) and $B[\eta_1(\bar{s}),1+\eta_2(\bar{s})]$ is Euler's beta function. Its appearance is necessary if we want to satisfy the known sum rule

$$\int_0^1 dx\, V(x,Q^2)=3\,. \tag{5.49}$$

Notice that the parametrizations in Eqs. (5.46)–(5.48) are simple generalizations of the standard parametrizations used in the simple parton model (e.g., Field and Feynman, 1977; Barger and Phillips, 1974). The functions $\eta_i(\bar{s})$ and $A_i(\bar{s})$ can be found as follows. At some $Q^2=Q_0^2$, or, equivalently, $\bar{s}=0$, $\eta_i(0)$, and $A_i(0)$ are taken from the data. This allows us to calculate the moments $\langle V(Q_0^2)\rangle_n$, $\langle\Sigma(Q_0^2)\rangle_n$, and $\langle G(Q_0^2)\rangle_n$, and consequently by Eqs. (2.84)–(2.86) we obtain $\langle V(Q^2)\rangle_n$, $\langle\Sigma(Q^2)\rangle_n$, and $\langle G(Q^2)\rangle_n$ for any (not too small) value of Q^2 for which the equations in question apply. The functions $\eta_1(\bar{s})$ and $\eta_2(\bar{s})$ which describe the evolution of $V(x,Q^2)$ are then found by assuming

$$\eta_i(\bar{s})=\eta_i(0)+\eta_i'\bar{s}\quad i=1,2\,, \tag{5.50}$$

and determining the constant slopes η_i' by fitting the moments of the analytic expression (5.46) to the asymptotic freedom prediction for $\langle V(Q^2)\rangle_n$ which we have just obtained. One obtains for instance (for four flavors, $f=4$)

$$\begin{aligned}\eta_1(\bar{s})&=0.70-0.176\bar{s}\,,\\ \eta_2(\bar{s})&=2.60+0.8\bar{s}\,,\end{aligned} \tag{5.51}$$

where the input values 0.7 and 2.6 correspond to $Q_0^2=1.8\ \text{GeV}^2$ and have been chosen on the basis of SLAC data (Riordan *et al.*, 1975; Bodek *et al.*, 1979). The formula (5.46) with $\eta_i(Q^2)$ given by (5.51) is a good representation of the asymptotic freedom Eq. (2.84) for $0.02\leq x\leq 0.8$ and $0\leq\bar{s}\leq 1.6$. This range of $\bar{s}$ is larger than that explored by present experiments and experiments to be performed in the near future.

Similar analytic expressions can be found for $d_v(x,Q^2)$ and $u_v(x,Q^2)$ separately. One obtains, for in-

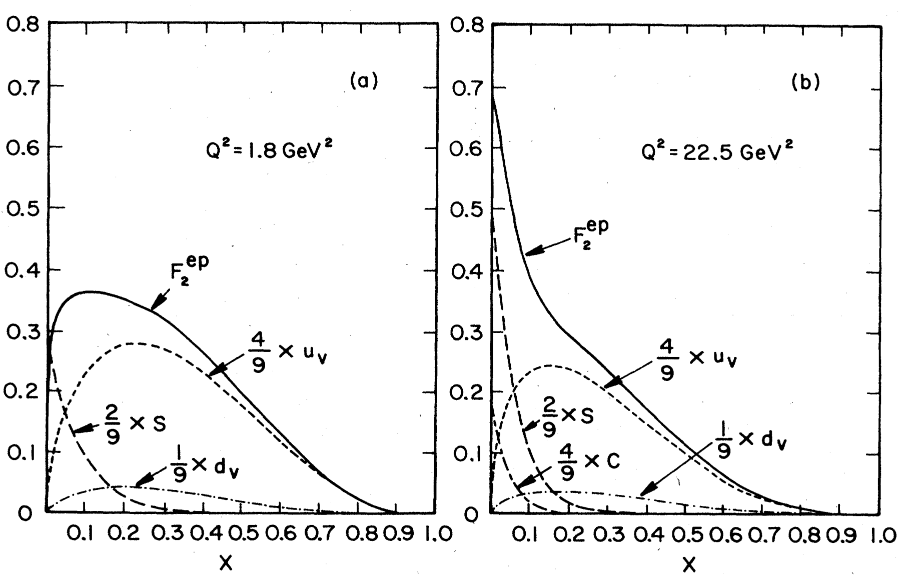


FIG. 11. F_2^{ep} as a function of x for (a) $Q^2=1.8$ GeV^2 and (b) $Q^2=22.5$ GeV^2 together with contributions from u_v, d_v, noncharmed sea, and charmed sea (Buras and Gaemers, 1978).

stance,

$$xd_v(x,Q^2)=\frac{1}{B[\eta_3(\bar{s}),1+\eta_4(\bar{s})]}x^{\eta_3(\bar{s})}(1-x)^{\eta_4(\bar{s})}, \qquad (5.52)$$

where (for $f=4$)

$$\eta_3(\bar{s})=0.85-0.24\bar{s},$$

$$\eta_4(\bar{s})=3.35+0.816\bar{s}. \qquad (5.53)$$

The input values 0.85 and 3.35 again correspond to $Q_0^2=1.8$ GeV^2 and have been chosen on the basis of SLAC data. As we shall see in Sec. VI, parametrizations (5.46) and (5.53) fit well the data. Needless to say the method just outlined can be easily generalized, if required by the data at Q_0^2, to any linear combination such as $\sum_{i,j}A_i x^{\eta_i}(1-x)^{\eta_j}$.

The method just discussed is less powerful in reproducing the Q^2 dependence of the sea and gluon distributions. This is due to the fact that the corresponding asymptotic freedom equations [(2.85) and (2.86)] are very complicated. However, for a limited range of x, $0.02\leq x\leq 0.3$ and $0\leq\bar{s}\leq 1.6$, it is enough to use the moments $n=2$ and $n=3$ of Eqs. (2.85) and (2.86) in order to find $xS(x,Q^2)$ and $xG(x,Q^2)$. For instance

$$A_s(\bar{s})=\langle S(Q^2)\rangle_2\left(\frac{1}{\langle x\rangle_s}-1\right), \qquad (5.54)$$

$$\eta_s(\bar{s})=1/\langle x\rangle_s-2, \qquad (5.55)$$

where

$$\langle x\rangle_s=\langle S(Q^2)\rangle_3/\langle S(Q^2)\rangle_2, \qquad (5.56)$$

and $\langle S(Q^2)\rangle_2$ and $\langle S(Q^2)\rangle_3$ are given by Eqs. (2.85). A similar formula exists for gluon distribution which, however, turns out to be only a fair representation of asymptotic freedom due to a very rapid increase with Q^2 of the gluon distribution at very small values of x, as predicted by Eq. (2.86). This rapid increase cannot be reproduced well by a simple formula like (5.48). Fortunately in the leading order for deep-inelastic processes one has to deal only with valence quark distributions and sea distributions. In addition, for $x>0.3$ where the formula (5.47) is not applicable, the sea distribution is very small, and all deep-inelastic formulas are governed for this range of x by the valence quark distribution. Therefore the method just outlined is useful for deep-inelastic phenomenology. For further details we refer the reader to Buras and Gaemers (1978).

For applications to other than deep-inelastic proceses, as for instance the Drell–Yan process, one needs asymptotic freedom expressions for the sea distributions which are valid for $x>0.3$. Such expressions turn out to be very complicated. They can be found in papers by Owens and Reya (1978), and Kato, Shimizu and Yamamoto (1979).

There exist in the literature other methods for obtaining analytic expressions for the Q^2-dependent parton distributions (Glück and Reya, 1977*b*; Parisi and Sourlas, 1979; De Grand, 1979), which the interested reader may consult. Simple numerical inversion methods can be found in the papers by Fox (1977), Yndurain (1978), Martin (1979), Furmanski and Pokorski, (1979b). The first inversion of the moment equations for nonsinglet structure functions by means of Mellin transform techniques is due to Parisi (1973), Gross (1974), and DeRujula *et al.* (1974). First numerical integration of integrodifferential Eqs. (2.52)–(2.54) has been done by Cabibbo and Petronzio (1978). For recent refinements in the use of Eqs. (2.52)–(2.54) see the papers by Baulieu and Kounnas (1979), and Gonzâlez–Arroyo, Lopez and Yndurain (1979a, b, c).

VI. SHORT REVIEW OF ASYMPTOTIC FREEDOM PHENOMENOLOGY

Using the procedure of Sec. II.E modified appropriately by the mass corrections of Sec. II.D it is a straightforward matter to obtain asymptotic freedom predictions for various quantities of interest and to confront them with the experimental data.

There have been many phenomenological papers in the recent past, and it is not a purpose of this section either to review them in detail or to present the best comparison of asymptotic freedom with the data. Instead we shall try to present the pattern of scaling violations and its size as predicted by QCD and as seen in the data. We shall do this quite systematically. For each quantity we shall first give qualitative predictions based on the properties of parton distributions which we have listed in Sec. V.D. We shall then give quantitative estimates based on the procedure of Sec. II.E, and we shall subsequently confront them with the existing data. In all cases we shall refer to various papers where details on the experimental data and their detailed comparison with asymptotic freedom predictions can be found. The analysis of this section is based on *leading-order* predictions only.[27]

A. Electroproduction and muon scattering

1. Structure functions

According to asymptotic freedom, with increasing Q^2 one expects a decrease of the structure functions at large values of x and an increase at small values of x. The increase at small x values is due to the sea component, whereas the decrease at large values of x is caused mainly by the decrease of the valence component. This qualitative behavior is certainly consistent with the ep, μp, ed, and μFe data (Watanabe *et al.*, 1975; Riordan *et al.*, 1975; Taylor, 1975; Atwood *et al.*, 1976; Anderson *et al.*, 1977; Gordon *et al.*, 1979; Bodek *et al.*, 1979). These data show a definite decrease of the structure function $F_2(x,Q^2)$ for $x>0.25$ and a Q^2-independent behavior for $0.15<x<0.25$. For $x<0.15$ the data for ep and ed scattering are poor and nothing definite can be said. The scaling violations in question increase with increasing x in accordance with the increase of the anomalous dimensions with increasing n. Asymptotic freedom fits, with or without mass effects, give good agreement with the data above. The parameter Λ is found to be in the range $0.3<\Lambda<0.5$

[27]Phenomenological studies of scaling violations without reference to asymptotic freedom can be found in the papers by Karliner and Sullivan (1978), Perkins, Schreiner, and Scott (1977) and Kirk (1978).

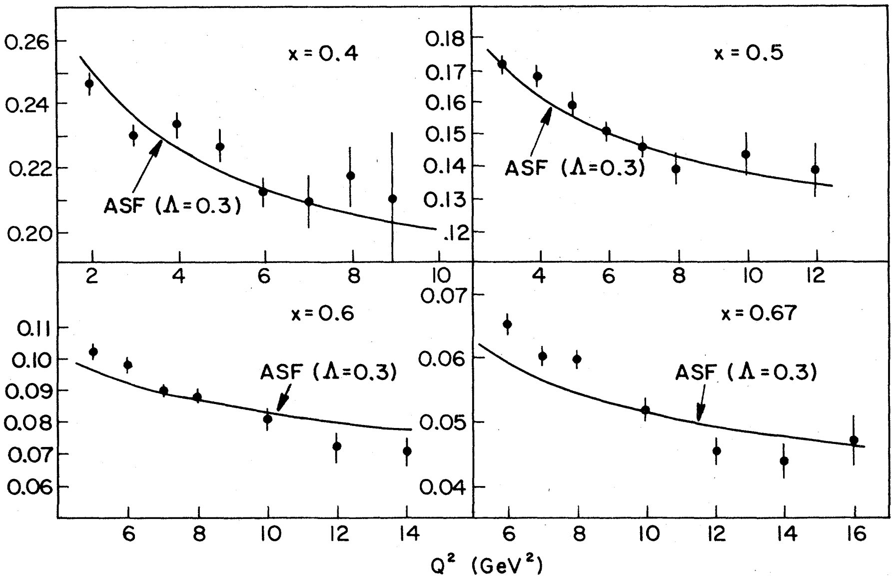


FIG. 12. The Q^2 behavior of F_2^{ep} for various values of x, compared with the SLAC data of Riordan *et al*. (1975). The solid line corresponds to parametrizations of Eqs. (5.46), (5.51), (5.52), and (5.53). The sea contribution is negligible at these values of x.

GeV. The details of such asymptotic freedom fits can be found in the papers by Parisi and Petronzio (1976), DeRujula, Georgi, and Politzer (1977a), Glück and Reya (1977a), Buras and Gaemers (1978), Fox (1977), Tung (1978), Kogut and Shigimitsu (1977a), and Johnson and Tung (1977a,b). We show a typical asymptotic freedom fit in Fig. 12. The best data for structure functions at small values of x come from μp scattering (Anderson *et al*., 1977; Gordan *et al*., 1979).[28] For $x < 0.15$, and especially for $x < 0.10$, a definite increase with Q^2 of $F_2^{\mu p}$ is observed. As shown in Fig. 13, the agreement of asymptotic freedom with the data is again good with a value of Λ consistent with that obtained from ep and ed scattering. We should like to remark that the increase at small values of x is expected to be caused by the increase of both the noncharmed sea as well as of the charmed sea component of the proton.

The data above extend over the range of Q^2 up to 60 GeV^2 with the majority of experimental points below $Q^2 = 30$ GeV^2. Recently results from a μFe experiment have been reported for Q^2 up to 150 GeV^2 (Ball *et al*., 1979). The scaling violations observed in this experiment agree well with asymptotic freedom predictions for $Q^2 < 20$ GeV^2 but disagree with it for larger values of

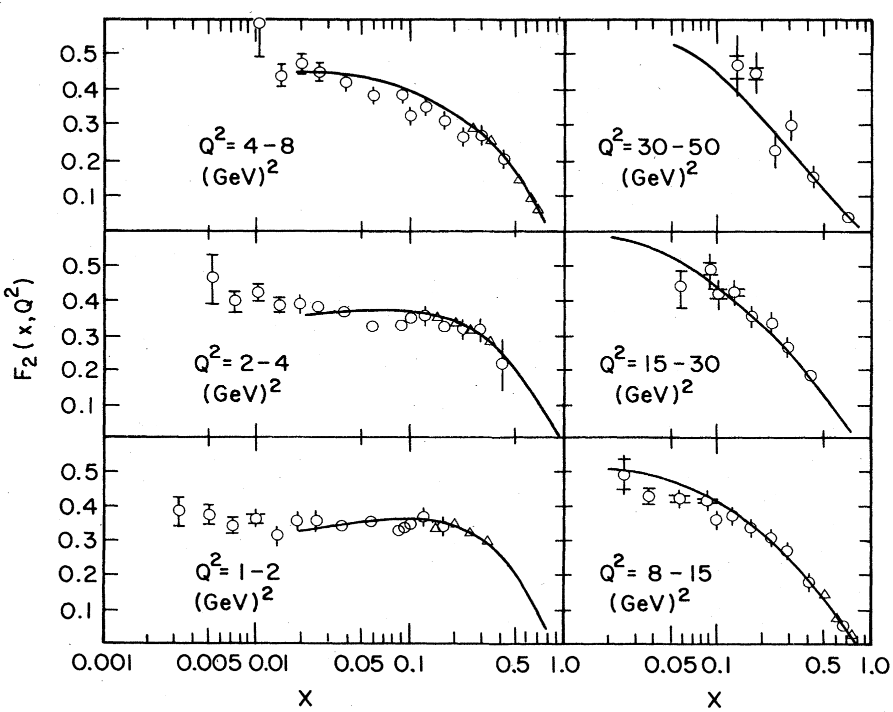


FIG. 13. Comparison of asymptotic freedom predictions with the μp data of Gordon *et al*. (1979) (open circles). For comparison the ep data of Riordan *et al*. (1975) (triangles) are also shown. The curves correspond to the parametrizations of Figs. 12 and 15 with $\Lambda = 0.4$ GeV.

[28]For an excellent review of the deep-inelastic muon data we refer the reader to the paper by Francis and Kirk (1979).

Q^2. In fact a decrease of the structure functions for $0.15 < x < 0.25$ and for $Q^2 < 20$ GeV^2 is followed by an increase for $Q^2 > 20$ GeV^2. Whether these data cause a problem for QCD remains to be seen. The effect is much too strong to be explained by new flavor production with the conventional charge assignment. It is of great interest to see whether the new μ experiments at CERN and at Fermilab will confirm the finding of Ball *et al.*

2. Moment analysis

Anderson, Matis, and Myrianthopoulos (1978) have made a comparison of the asymptotic freedom predictions for the moments of $F_2(x, Q^2)$ with the experimentally extracted Nachtmann moments as defined by Eq. (2.124). Their analysis includes ep, ed, and μp data. The agreement of QCD with the data is impressive as can be seen in Fig. 14. The preferred value of the parameter Λ turns out to be 0.66 ± 0.08 GeV, a slightly higher value than that obtained from the direct analysis of the structure functions.

B. ν and $\bar{\nu}$ deep-inelastic scattering (charged currents)

Asymptotic freedom predictions for ν and $\bar{\nu}$ deep-inelastic scattering can be obtained by means of the parton model formulas of Sec. II and the Q^2 dependent parton distributions of Sec. V.

1. Total cross sections

In the simple parton model the total cross sections σ^{ν}/E_{ν} and $\sigma^{\bar{\nu}}/E_{\bar{\nu}}$ are independent of energy except for possible threshold effects due to heavy quark production. In addition $\sigma^{\bar{\nu}}/\sigma^{\nu} = 1/3$ in the absence of sea quarks and $\sigma^{\bar{\nu}}/\sigma^{\nu} \approx 0.40$ if the sea carries 5%–10% of the momentum of the nucleon as observed at Gargamelle ($E \approx 5$ GeV)(Eichten *et al.*, 1973; Deden *et al.*, 1975). In the presence of asymptotic freedom effects σ_{ν}, $\sigma_{\bar{\nu}}$ and also $\sigma^{\bar{\nu}}/\sigma^{\nu}$ depend on energy. This energy dependence arises as follows. With increasing energy, a larger range of Q^2 is explored and, consequently, the valence and sea contributions to any of the cross sections are effectively decreased and increased, respectively. The sea contribution to ν and $\bar{\nu}$ cross sections is roughly the same except for the difference in charm production. On the other hand, the valence quark contribution is roughly 3 times larger in the ν cross section. Consequently in the range of energies explored by present experiments, the energy dependence of σ_{ν} is expected to be governed by the decrease of the valence quark distribution, i.e., σ_{ν}/E is expected to fall (Hinchliffe and Llewellyn-Smith, 1977b) with increasing energy. For $\bar{\nu}$ scattering the decrease of the valence quark contribution is roughly compensated by the increase of the sea. Thus $\sigma_{\bar{\nu}}/E$ is expected to be roughly constant at moderate energies. At higher energies, where charm production is at full strength, $\sigma_{\bar{\nu}}/E$ is expected to rise slowly. Asymptotically it should approach σ_{ν}/E. In summary, in the range of energies explored by present experiments, one expects a decrease of σ_{ν}/E and a constant behavior followed by an increase for $\sigma_{\bar{\nu}}/E$. Consequently the ratio $\sigma_{\bar{\nu}}/\sigma_{\nu}$ is expected to increase. These expectations (Altarelli, Petronzio, and Parisi, 1976; Barnett, Georgi, and Politzer, 1976; Buras, 1977; Glück and Reya, 1977a; Barnett and Mar-

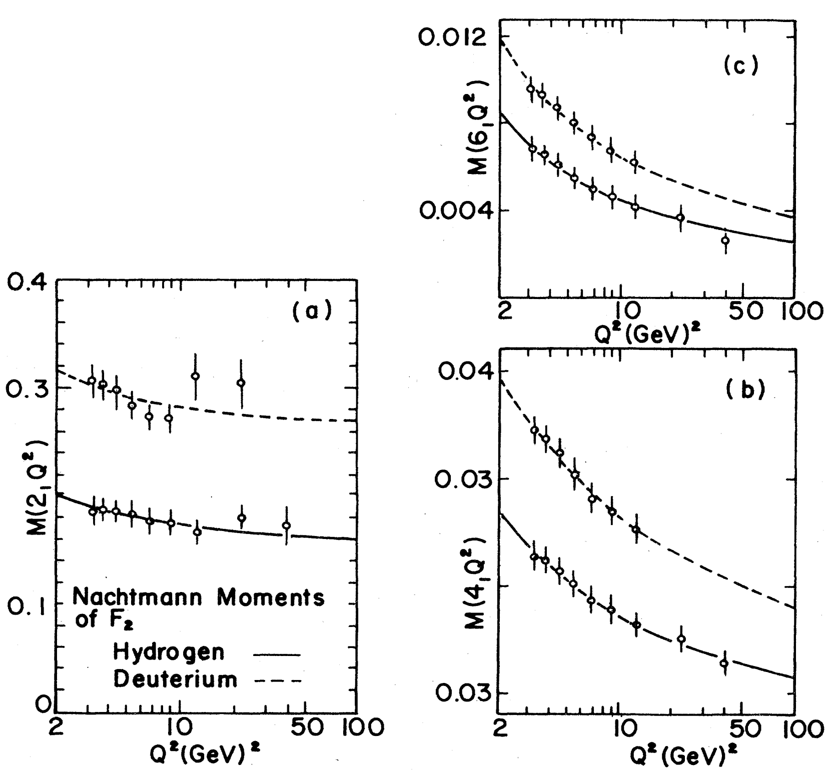


FIG. 14. Comparison of the QCD predictions for the moments of F_2 with the ep, ed, μp, and μd data (Anderson, Matis, and Myrianthopoulos, 1978).

tin, 1977; Hinchliffe and Llewellyn-Smith, 1977b; Buras and Gaemers, 1978; Barger and Phillips, 1978; Fox, 1978; Avilez *et al.*, 1977; Graham *et al.*, 1977; Roy *et al.*, 1977) are confirmed by the recent high-energy experiments BEBC (Bosseti *et al.*, 1977), CALT (Barish *et al.*, 1977b, 1978), Serpukhov (Asratyan, 1978), and CDHS (de Groot *et al.*, 1979a) when combined with low-energy data as is shown in Figs. 15 and 16. We should like to remark, however, that above $E=40$ GeV the changes in the cross sections are very weak both in the theory and experiment, and to a good approximation the total cross sections in the range $40<E<200$ GeV can be represented by a simple parton model formula with Q^2-independent quark distributions, with the amount of the sea (valence) larger (smaller) than that observed at Gargamelle.

The very slow change of the total cross sections with energy is easily understood. Integrating over x amounts to summing up the parts of the structure functions which increase and decrease with Q^2 and which partly compensate each other, leading to a small effect. The same phenomenon happens in the case of $\langle y\rangle$ on which we now comment briefly.

2. $\langle y\rangle$

In the simple parton model for the strong interactions and for the structure of weak interactions given by the Weinberg–Salam model the distributions $1/E\,d\sigma_\nu/dy$ and $1/E\,d\sigma_{\bar\nu}/dy$ are, in the absence of the sea quarks in the nucleon, flat and $\sim(1-y)^2$, respectively. Consequently $\langle y\rangle_\nu=0.5$ and $\langle y\rangle_{\bar\nu}=0.25$. If the sea carries 5%–10% momentum of the nucleon as observed at Gargamelle the distributions $1/E\,d\sigma_\nu/dy$ and $1/E\,d\sigma_{\bar\nu}/dy$ have additional small $(1-y)^2$ and small flat components respectively which lead to $\langle y\rangle_\nu=0.48$ and $\langle y\rangle_{\bar\nu}\approx 0.3$. If strong interactions as described by QCD are switched on, the sea component increases and the valence component decreases with energy. Consequently $\langle y\rangle_\nu$ and $\langle y\rangle_{\bar\nu}$ are expected to decrease and increase with energy, respectively. These expectations are confirmed by the recent high-energy experiments, CITFR (Barish *et al.*, 1977a, 1978) and CDHS (De Groot, 1979a) as shown in Fig. 17. The rate of change is, as in the case of total cross sections, very slow. In particular the CDHS group hardly sees any dependence. At $E\approx 200$ GeV, $\langle y\rangle_\nu\approx 0.46$, and $\langle y\rangle_{\bar\nu}\approx 0.34$. Asymptotically we expect

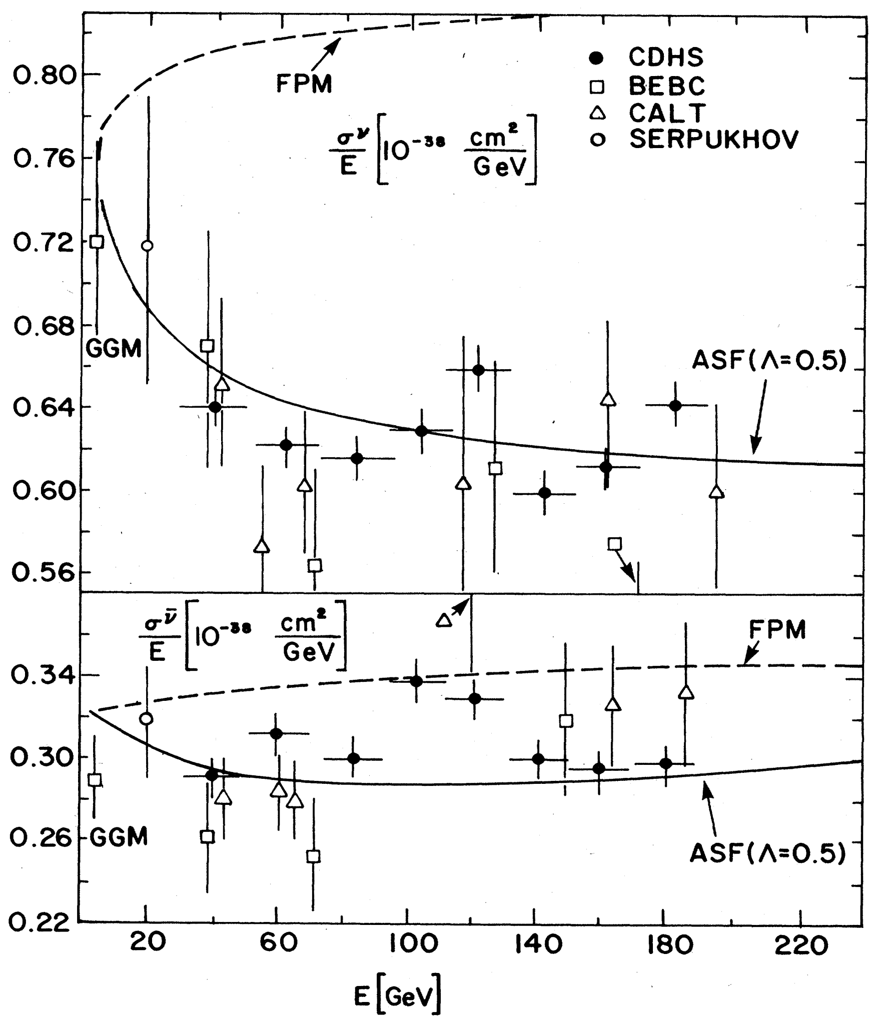


FIG. 15. The absolute cross sections σ^ν/E and $\sigma^{\bar\nu}/E$ as functions of energy, compared with the high-energy data. The calculated curves are for the free parton model (FPM) and the leading order of asymptotic freedom (ASF). The parametrizations of the valence quarks are as in Fig. 12. The input parametrizations for the remaining distributions are: $xS(x,Q_0^2)=0.99(1-x)^8$, $xC(x,Q_0^2)=0$, and $xG(x,Q_0^2)=2.41(1-x)^5$ at $Q_0^2=1.8$ GeV^2. The increase of the cross sections in the parton model is due to charm production.

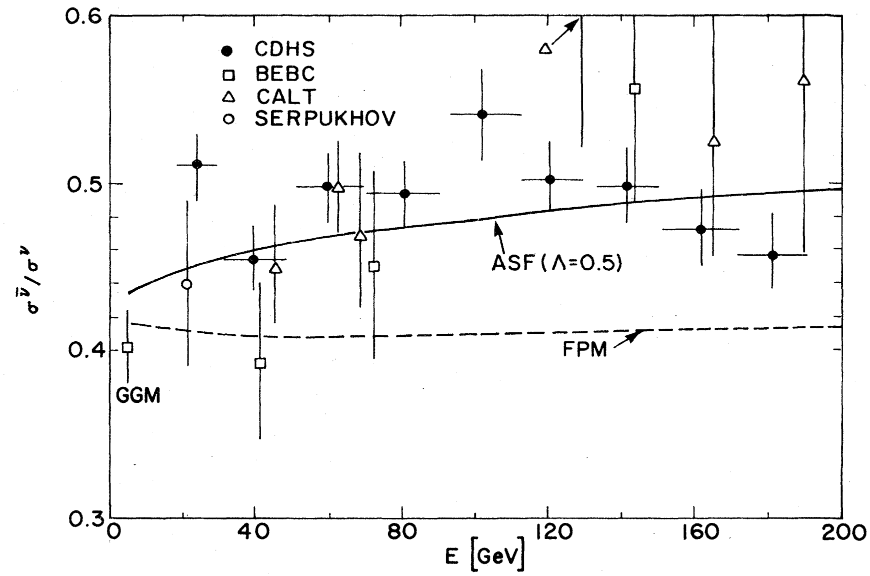


FIG. 16. The ratio $\sigma^{\bar{\nu}}/\sigma^{\nu}$ as a function of energy, compared with the high-energy data. The curves correspond to the parametrizations of Fig. 15.

$\langle y\rangle_\nu = \langle y\rangle_{\bar{\nu}} \approx 0.44$. Therefore if nothing but asymptotic freedom effects are present at higher energies one should observe a detectable increase of $\langle y\rangle_{\bar{\nu}}$, but almost constant $\langle y\rangle_\nu$. These slow changes with energy of $\langle y\rangle_{\bar{\nu}}$ and $\langle y\rangle_\nu$, as well as of $1/E\, d\sigma_\nu/dy$ and $1/E\, d\sigma_{\bar{\nu}}/dy$, predicted by asymptotic freedom are very fortunate because they will not mask the changes in y distributions due to W boson propagator. The latter effect (for $m_W \approx 80$ GeV) is much stronger in the range $500 < E < 10^4$ GeV than asymptotic freedom effects. We refer the interested reader to a paper by Halprin (1979)[29] where a detailed study of the W boson propagator effects in y distributions can be found.

3. $\langle x\rangle, \langle xy\rangle, \langle x^n\rangle$

More useful quantities to test asymptotic freedom ideas than those discussed in B.1 and B.2 are the aver-

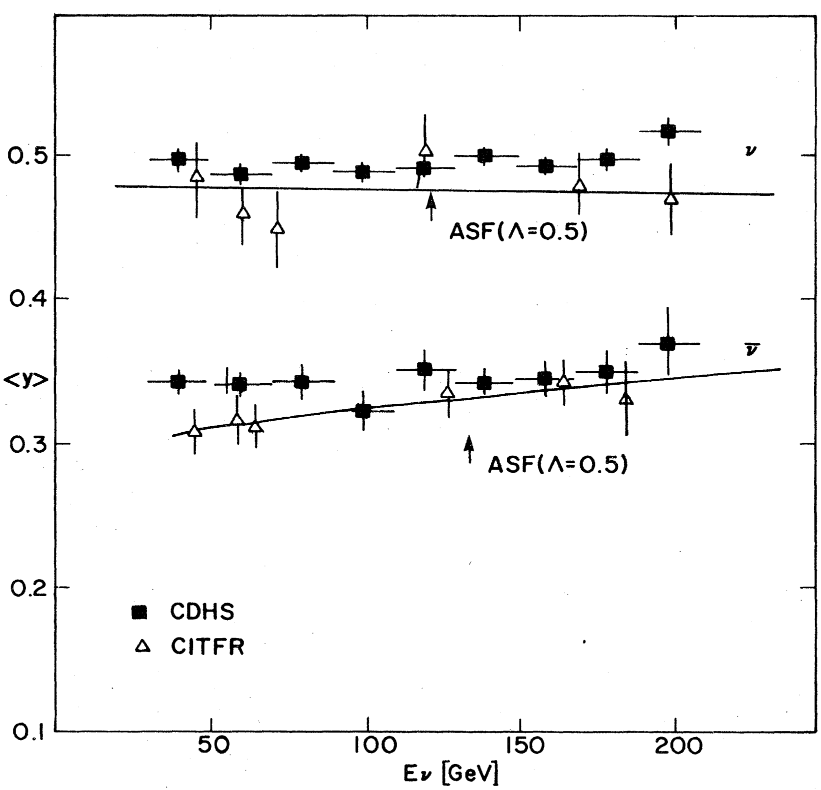


FIG. 17. $\langle y\rangle_\nu$ and $\langle y\rangle_{\bar{\nu}}$ as functions of energy. The curves correspond to asymptotic freedom parametrizations used in Fig. 15.

[29]For recent calculations of asymptotic freedom and W boson propagator effects at very high energies (1–100 TeV) see the papers by Halprin and Oakes (1979) and Oakes and Tung (1979).

ages $\langle x\rangle$ and $\langle xy\rangle$, or, more generally, the moments

$$\langle x^n\rangle=\frac{1}{\sigma}\int dx\, x^n\frac{d\sigma}{dx}\,, \tag{6.1}$$

$$\langle x^m y^n\rangle=\frac{1}{\sigma}\int\int dx\, dy\, x^m y^n\frac{d\sigma}{dx dy}\,. \tag{6.2}$$

On the basis of the properties of the Q^2 dependence of parton distributions listed in Sec. V.D and the formulas of Sec. II.B, both $\langle x^n\rangle$ and $\langle x^m y^n\rangle$ should decrease with increasing energy. As an example we show in Fig. 18 the data for $\langle Q^2/E\rangle=2M\langle xy\rangle$ which exhibit the expected energy dependence. The solid curve in the figure corresponds to a typical asymptotic freedom fit with $\Lambda=0.5$ GeV. The data are from GGM (Eichten *et al.*, 1973), BEBC (Bosetti *et al.*, 1977), FNAL (Berge *et al.*, 1976), SKAT (Baranov *et al.*, 1978), CDHS (De Groot *et al.*, 1979a), and IHEP–ITEP (Asratyan *et al.*, 1978).

4. $\int dx\, F_i(x,E)$ and $\int dx\, x^n F_i(x,E_h)$

If asymptotic freedom or any renormalizable field theory ideas are correct, then the structure functions F_i *do not* depend directly on the incoming energy E but on Q^2. This is obvious if we recall the intuitive picture of Sec. V.A. Therefore it is not very convenient to test asymptotic freedom ideas by measuring the integrals $\int dxF_i(x,E)$. Experimentally they are extracted by using simple parton model formulas and assuming factorization in x and y. This last assumption is not true in QCD. The only way to compare the integrals of the structure functions in question as presented, for instance, in the papers by Bosetti *et al.* (1977) and Barish *et al.* (1978) is to relate them to the total cross sections σ_ν and $\sigma_{\bar\nu}$ and calculate the latter using asymptotic freedom formulas as discussed previously. Such an exercise is performed in Fig. 19 where we have used the relations

$$\int_0^1 F_2(x,E)\,dx=\frac{3\pi}{4G^2M}\,\frac{\sigma_\nu+\sigma_{\bar\nu}}{E}\,, \tag{6.3}$$

$$\int_0^1 xF_3(x,E)\,dx=\frac{3\pi}{2G^2M}\,\frac{\sigma_\nu-\sigma_{\bar\nu}}{E}\,. \tag{6.4}$$

The integral in Eq. (6.3) measures essentially the fraction of the proton momentum carried by quarks and antiquarks. This fraction decreases very slowly since some of the momentum is effectively transferred to gluons. The decrease of the integral in Eq. (6.4) is mainly due to the decrease of the valence component of the nucleon. Asymptotically the integral of F_3 is expected to be zero.

More sensitive tests of asymptotic freedom can be made by measuring the moments

$$\int_0^1 dx\, x^n F_i(x,E_h)\,, \tag{6.5}$$

where E_h is the hadronic energy which is related to Q^2 by

$$Q^2=2x\nu=2xM(E_h-M)\approx 2xME_h\,. \tag{6.6}$$

In order to calculate the E_h dependence of the moments in Eq. (6.5) in the framework of asymptotic freedom one computes first $F_i(x,E_h)$ from $F_i(x,Q^2)$ by using Eq. (6.6). Subsequently the moments of Eq. (6.5) are calculated by a straightforward integration. It should be kept in mind, however, that for finite fixed energy E_h the lower limit of integration corresponds to $Q^2=0$, for which perturbative calculations do not make sense. If we take $Q^2_{\min}=2$ GeV^2 to be the minimal value of Q^2 for which perturbative calculations are reliable, then for $E_h=5$, 20, and 200 GeV the corresponding minimal values of x are 0.10, 0.025, and 0.0025. Therefore only for $E_h>20$ GeV can we reliably estimate the $n=0$ moment of Eq. (6.5) in perturbation theory. The situation is better for $n>0$ because these moments receive only a very small contribution from very small x regions. In Fig. 20, which we took from the paper by de Groot *et al.* (1979a), we show the average values

$$\bar{x}_{F_2}=\int_0^1 xF_2(x,E_h)\,dx\Big/\int_0^1 F_2(x,E_h)\,dx \tag{6.7}$$

and

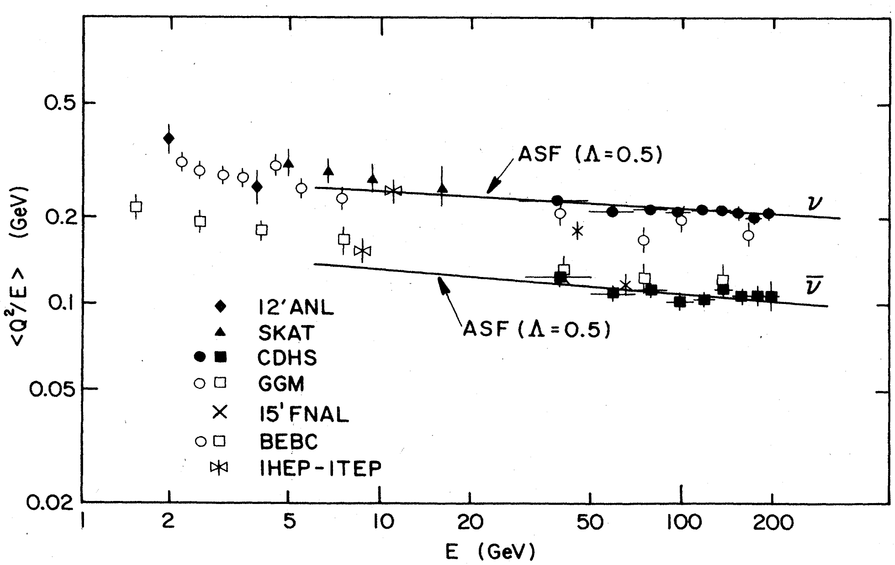


FIG. 18. $\langle Q^2/E\rangle$ as functions of energy. The curves correspond to asymptotic freedom parametrizations used in Fig. 15. The collection of data points is from Tittel (1979).

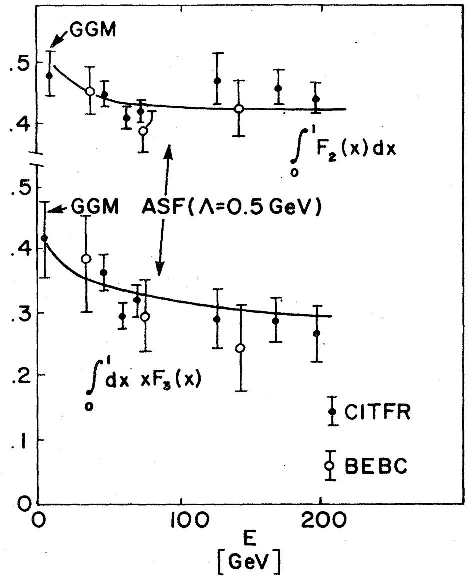


FIG. 19. Integrals $\int_0^1 F_2(x)dx$ and $\int_0^1 dx\ xF_3(x)$ as functions of energy. The curves have been obtained on the basis of Eqs. (6.3) and (6.4) with the values of σ^ν and $\sigma^{\bar{\nu}}$ as in Fig. 15.

$$\bar{x}_{F_3} = \int_0^1 x^2 F_3(x, E_h)\,dx \Big/ \int_0^1 xF_3(x, E_h)\,dx\ , \tag{6.8}$$

as functions of E_h. The solid curve is the asymptotic freedom prediction with $\Lambda = 0.47$ GeV.

5. x distributions

From the properties of parton distributions as discussed in Sec. V.D and the formulas of Sec. II.B, it immediately follows that the distribution $(1/E)(1/\sigma)d\sigma/dx$ in both ν and $\bar{\nu}$ processes should increase and decrease with energy at small and large x values, respectively. These trends are in agreement with the existing high-energy data. We refer the interested reader to the paper by Fox (1978) where a detailed comparison of asymptotic freedom predictions with the experimentally measured x distributions has been made. Also very recent data for x distributions (Benvenuti *et al.*, 1979) exhibit the expected pattern of scaling violations. The x distributions deserve, certainly, further experimental studies since among the quantities which directly depend on energy and not on Q^2 the quantities in question are expected to show the largest asymptotic freedom effects.

6. ν and $\bar{\nu}$ structure functions

So far we have discussed only the energy dependence of various quantities which can be measured in ν and $\bar{\nu}$ induced processes. We concluded that, except for the x distributions and the moments (6.5), the measurements of the energy dependence are not very sensitive tests of asymptotic freedom ideas. Certainly the best way to compare the theory with the data is to consider the structure functions as functions of Q^2. The Q^2 dependence of $F_2^{\nu,\bar{\nu}}$ is expected to be similar to that of

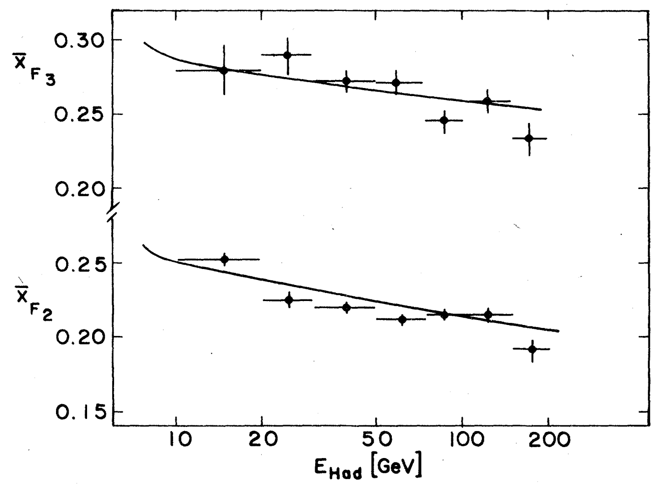


FIG. 20. $\bar{x}_{F_3}$ and $\bar{x}_{F_2}$ as functions of energy. The curves correspond to an asymptotic freedom fit with $\Lambda = 0.47$ GeV. The figure is from de Groot *et al.* (1979a).

F_2^{ep}, i.e., they should increase at small values of x and decrease at large values of x. On the other hand, the Q^2 dependence of F_3 should be similar to that of the valence quark distribution. These expectations are confirmed by the recent high-energy data obtained at CERN (Bosseti *et al.*, 1978; de Groot *et al.*, 1979a,b). As an example we show in Fig. 21 the Q^2 dependence of F_2^ν as measured by de Groot *et al.* The solid lines correspond to an asymptotic freedom fit with $\Lambda = 0.47$ GeV.

7. Moment analyses of BEBC and CDHS

One of the predictions of QCD is the n dependence for the anomalous dimensions of various operators. This n dependence can be tested indirectly by comparing the scaling violations, as predicted by the theory, with the experimental data. Such a test is not ideal because one has to make assumptions about the structure functions or parton distributions (in particular about the gluon distribution) at some value of $Q^2 = Q_0^2$ and for the whole range of x. Also there is one free parameter, Λ. It would be useful to have a direct way of experimentally "measuring" the n dependence of the anomalous dimensions in experiment. This is, in fact, possible for the nonsinglet anomalous dimensions, as has been suggested and measured by the BEBC group (Bosseti *et al.*, 1978). Recently a similar analysis has also been carried out by the CDHS group (de Groot *et al.*, 1979c). Consider the moments of the structure function F_3 which in QCD are given as follows:

$$M_n(Q^2) = M_n(Q_0^2)\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-\gamma_{NS}^{0,n}/2\beta_0}. \tag{6.9}$$

Consequently it follows that

$$\ln M_n(Q^2) = \ln M_n(Q_0^2) - \frac{\gamma_{NS}^{0,n}}{2\beta_0}\ln\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]. \tag{6.10}$$

Therefore, if we plot $\ln M_{n_1}(Q^2)$ for a given $n = n_1$ vs $\ln M_{n_2}(Q^2)$ with $n_2 \neq n_1$, we should obtain a straight line with a slope given by the ratio

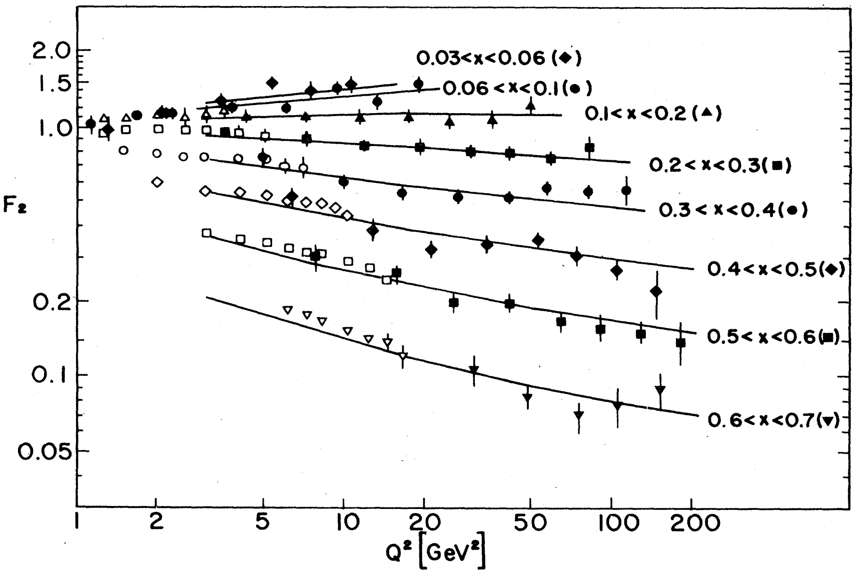


FIG. 21. $F_2(x, Q^2)$ as measured by the CDHS group (de Groot *et al.*, 1979a) (black points) compared with the leading-order asymptotic freedom predictions for $\Lambda = 0.47$ GeV. The open points are from SLAC. The figure is from de Groot *et al.* (1979a).

$$\frac{\gamma_{NS}^{0,n_1}}{\gamma_{NS}^{0,n_2}} = \frac{1-2/n_1(1+n_1)+4\sum_{j=2}^{n_1} 1/j}{1-2/n_2(n_2+1)+4\sum_{j=2}^{n} 1/j} . \quad (6.11)$$

Notice that this ratio is independent of the gauge group as well as the number of flavors. It is also independent of Λ. It should be remarked that formula (6.11) expresses the vector character of the gluons and is true in any theory in which strong interactions are mediated by vector particles. In the case of theories with scalar gluons the sums in Eq. (6.11) should be dropped (Christ, Hasslacher, and Mueller, 1972). The combined data from Gargamelle and BEBC (Bosetti *et al.*, 1978) and the CDHS data (de Groot *et al.*, 1979c) exhibit straight lines for the plots in question as can be seen in Fig. 22. The extracted slopes are compared with the predictions of vector and scalar theories in Fig. 23. The following observations can be made on the basis of these results:

(i) Gargamelle-BEBC results agree very well with the formula (6.11) and disagree with the predictions of the scalar theory (Ellis, 1978).

(ii) The ratios of the ordinary moments (Cornwall and Norton, 1969) extracted by the CDHS group favor the vector theory whereas the ratios for the Nachtmann moments lie almost exactly between predictions of vector and scalar gluon theories.

(iii) The predictions of the scalar theory are systematically below all the data considered.[30]

We may therefore conclude that the results above give some support to the belief that the mediators of strong interactions are spin-1 particles. Recently Abbott and Barnett (1979)[31] reanalyzed the data of BEBC and CDHS and investigated how the plots in Figs. 22 and 23 depend on the cuts in Q^2 and how they could be affected by higher-twist contributions. Their analysis weakened somewhat the conclusion made above. We refer the reader to this interesting paper for details.

It should be remarked that although the plots of Figs. 22 and 23 may help to distinguish between vector and scalar theories, only ratios of anomalous dimensions are "measured" in this way. By taking ratios, some of the predictions of the theory, namely the size of the anomalous dimensions, are lost. Furthermore, the anomalous dimensions of nonsinglet operators which we discussed here represent only a part of the theory in which singlet operators are also present. Therefore to test the theory more critically and in particular to

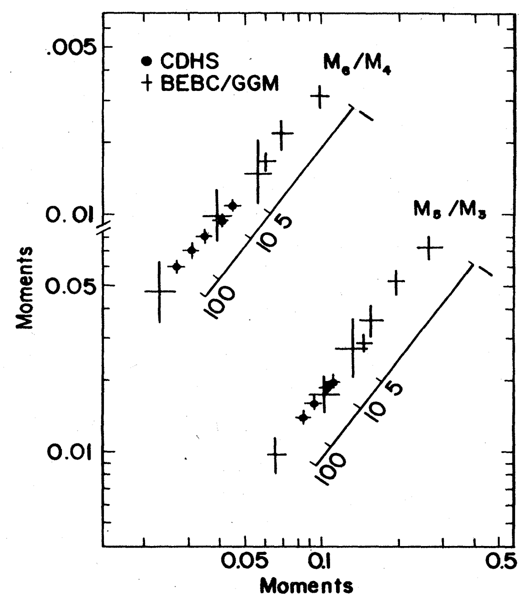


FIG. 22. Log M_i vs log M_j plots as obtained by CDHS and BEBC/GGM. The figure is from de Groot *et al.* (1979a).

[30]It should be emphasized that the predictions of scalar theories as presented here are based on an unproven assumption that perturbative calculations are reliable for these theories. See discussion in Sec. VI.C.

[31]See also Abbott (1979).

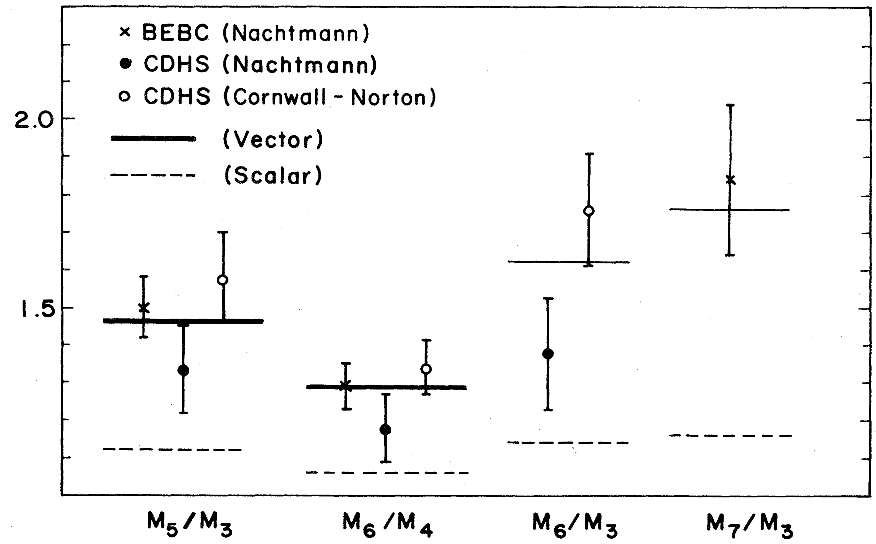


FIG. 23. The measured slopes as defined by the plots of Fig. 22. The straight horizontal lines are the predictions of vector gluon and scalar gluon theories.

distinguish it from other vector theories, it is necessary to study the full Q^2 evolution of structure functions with both singlet and nonsinglet contributions taken into account, as we did in the previous subsections. Discussion of the predictions of other field theories will be presented below.

Finally we would like to comment on a paper by Harari (1979) who derived "bounds" on the slopes in Fig. 23, assuming that $xF_3(x,Q^2)$ behaves as

$$xF_3(x,Q^2)=3\frac{x^{a_1(Q^2)}(1-x)^{a_2(Q^2)}}{B[a_1(Q^2),1+a_2(Q^2)]}, \tag{6.12}$$

where $a_1(Q^2)$ and $a_2(Q^2)$ are slowly varying functions of Q^2, decreasing and increasing, respectively. $B[a_1(Q^2),1+a_2(Q^2)]$ is the Euler's beta function needed to ensure the Gross–Llewellyn-Smith relation. The bounds obtained in this way are rather stringent, and the results of the CDHS group and the BEBC collaboration cover the entire range allowed by these bounds. Therefore Harari concluded that the data in question cannot be considered as evidence for the validity of QCD. It should, however, be remembered that the form (6.12) is exactly the form of Eq. (5.46) which turned out to be a good representation of QCD. Furthermore, the assumption that the functional form of $xF_3(x,Q^2)$ will not be changed with Q^2 is a strong assumption which is approximately satisfied by QCD, but will in general not be true in an arbitrary theory. In fact it is not difficult to violate Harari's bounds by choosing arbitrarily the n dependence of the anomalous dimensions $\gamma_{NS}^{0,n}$ in Eqs. (6.9)–(6.11). Next, inverting Eq. (6.9), one finds that the functional form assumed at one value of Q^2 cannot in general be retained with varying Q^2. For instance, scalar gluon theories violate Harari's bounds. Therefore, although Harari's analysis is interesting in itself, we do not agree with Harari's conclusion and think that the BEBC and CDHS data do give support to QCD. We do agree with him, however, as we stated above, that a better test of the theory can be made by studying the full Q^2 evolution of the structure functions or their moments.

8. Comments on neutral current processes

In QCD, scaling violations are also expected in the neutral current processes (Barnett and Martin, 1977; Buras and Gaemers, 1977; Barger and Phillips, 1978; Hinchliffe and Llewellyn-Smith, 1977c). In particular, the well-known plot R_ν vs $R_{\bar\nu}$ is expected to change with energy. The present data on neutral currents are, however, not precise enough to make any QCD analysis meaningful.

C. Comments on fixed point theories

Until now our discussion of scaling violations concentrated on asymptotically free gauge theories. Here we shall comment on theories in which the effective coupling constant approaches for $Q^2\to\infty$ a constant value $g^*\neq 0$ [so-called fixed point at which $\beta(g^*)=0$]. If g^* is small then we may hope to calculate predictions of these theories in perturbation theory in g^*. It should be emphasized however that the structure of fixed point theories is not well known. In particular, we do not know whether a fixed point with a small value of g^* exists. Therefore if we assume g^* to be small, use perturbation theory and show that the result disagrees with the data, we still cannot claim that we have ruled out the theory in question. It could, for instance, happen that g^* was large in fact and the true prediction of the theory obtained by nonperturbative methods was consistent with the data.

Nevertheless it is interesting to see what happens if perturbation theory is used. Since the ratios of anomalous dimensions in scalar gluon theories (obtained in perturbation theory) are systematically below BEBC and CDHS data, we shall discuss here only Abelian vector theories. These theories have been studied extensively by Glück and Reya (1976, 1977a, 1979) and we shall only recall the most important points of their analysis.

For the moments of nonsinglet structure functions we have

$$M^{\mathrm{NS}}(n,Q^2)=M^{\mathrm{NS}}(n,Q_0^2)\exp\left(-\int_{1/2\ln Q_0^2}^{1/2\ln Q^2} dt\,\gamma_{\mathrm{NS}}^n\right)$$
$$=M^{\mathrm{NS}}(n,Q_0^2)[Q^2/Q_0^2]^{-\gamma^{*,n}/2}, \qquad (6.13)$$

where $\gamma_{\mathrm{NS}}^{*,n}$ is determined by its fixed point value

$$\gamma_{\mathrm{NS}}^{*,n}=\frac{g^{*2}}{16\pi}\,\frac{8}{3}\left[1-\frac{2}{n(n+1)}+4\sum_{j=2}^{n}\frac{1}{j}\right]+O(g^{*}\ ). \qquad (6.14)$$

The n dependence of $\gamma_{\mathrm{NS}}^{*n}$ is exactly the same as in QCD. The Q^2 dependence in Eq. (6.13) is different but as noticed first by Tung (1975) and Llewellyn-Smith (1975), if g^* is properly chosen, Eq. (6.13) can mimic the corresponding QCD prediction in the range of Q^2 available in present experiments. In particular Reya (1979) and Abbott and Barnett (1979) find that Abelian vector theories agree well with CDHS and BEBC data for the moments of F_3 if $g^{*2}/16\pi^2\approx 0.04$. Therefore if g^* is small and the leading-order formulas (6.13) and (6.14) are used, the fixed point vector gluon theories cannot be at present distinguished from QCD on the basis of scaling violations observed in nonsinglet structure functions. It has been pointed out however by Glück and Reya that such a distinction can be made on the basis of the singlet structure functions. They propose to look at the second moment of $F_2^{\nu N}(x,Q^2)$, which in the parton language measures the fraction of the proton momentum carried by the quarks. In experiment the moment in question is roughly equal to 0.5 at low values of Q^2 and decreases very slowly with increasing Q^2. It turns out that only QCD and so-called "fixed point QCD" (FP-QCD, QCD with the vanishing triple gluon vertex) agree with this behavior. All other theories considered by Glück and Reya predict an increase of the second moment with Q^2 [as pointed out by Abbott and Barnett (1979), higher-twist effects could invalidate these results].

Glück and Reya (1979) also investigated the full x and Q^2 dependence of $F_2^{\nu N}(x,Q^2)$ and concluded that further discrimination between QCD and the FP-QCD cannot be made on the basis of the present data on scaling violations.

Once again we would like to emphasize that when judging these results one should keep in mind that the predictions of fixed point theories as discussed here cannot be treated on the same footing as the QCD predictions. The reason is that whereas we can believe in the results of perturbative calculations in QCD, there is no reason that such calculations are justified for fixed point theories.

D. Critical summary

As we have seen in this section, asymptotic freedom (leading-order) predictions agree well with the scaling violations observed in $ep, ed, \mu p, \nu N$, and $\bar{\nu}N$ deep-inelastic scattering. We should, however, be very careful in judging these results. The reasons are as follows:

(a) The mass corrections, which enter any asymptotic freedom analysis, are, on the one hand, non-negligible at low values of Q^2 and, on the other hand, as discussed in Sec. II, not completely understood. Presumably the problem of target mass corrections cannot be completely solved within perturbation theory. The effects of heavy quarks can, however, be discussed in the framework of perturbation theory except for the region close to various thresholds, where nonperturbative effects are probably important. Since the ultimate QCD predictions depend on the treatment of mass and threshold effects, further study of the effects in question is very desirable. One way to circumvent partially the problem of mass correction is to make QCD comparison with the data for $(4x^2m^2/Q^2)\ll 1$, where at least target mass effects are expected to be small.

(b) The effects of higher-twist operators, which we have not included in our analysis, may turn out to be of some importance at low values of $Q^2\approx 0(5\ \mathrm{GeV}^2)$ (for a recent analysis see Abbott and Barnett, 1979).

(c) There is the question whether the use of leading-order predictions at low values of Q^2 is justified in view of the existence of calculable higher-order corrections. We shall try to answer this question in the next two sections.

VII. HIGHER-ORDER ASYMPTOTIC FREEDOM CORRECTIONS TO DEEP-INELASTIC SCATTERING (NONSINGLET CASE)

A. Preliminaries

In the last three sections we have discussed leading-order predictions of asymptotic freedom for deep-inelastic processes. We have seen that these predictions are in good agreement with all experimental data with the value of the scale parameter Λ in the range from 0.3 to 0.7 GeV. However, at $Q^2\approx$ few GeV^2 the leading asymptotic behavior cannot be the whole story and it is of interest and of importance to ask whether higher-order corrections in the effective coupling constant $\bar{g}^2(Q^2)$ modify these results. In this and the next section we shall discuss these corrections in great detail. We shall see that these corrections are different for different structure functions and consequently various parton model relations and current algebra sum rules, which were true in the leading order, are no longer satisfied. The experimental verification of the violations of these sum rules is very important although a difficult task.

There is still another reason why higher-order calculations are important. This is the fact that without them the value of Λ *cannot* be extracted from experiment in a theoretically meaningful way (Bace, 1978). To see this consider the moments of a nonsinglet structure function as given by the leading-order expression and, to simplify the argument, take the appropriate roots

$$\tilde{M}_n(Q^2)\equiv[M_n(Q^2)]^{-2\beta_0/\gamma_{\mathrm{NS}}^{0,n}}=\tilde{A}_n\ln(Q^2/\Lambda_{\mathrm{LO}}^2). \qquad (7.1)$$

where $\tilde{A}_n$ are Q^2-independent numbers. If the experimentally measured moments are

$$\tilde{A}_n\ln(Q^2/0.49), \qquad (7.2)$$

then (in units of GeV)

$$\Lambda_{\mathrm{LO}}=0.7\,.$$

Now let us introduce next-to-leading order corrections to the leading-order formula (7.1) and write it as

$$\tilde{M}_n(Q^2)=\tilde{A}_n\ln(Q^2/\Lambda^2)+R_n. \quad (7.3)$$

If $R_n=\delta\tilde{A}_n$, where δ is an n independent number, then we can rewrite (7.3) as

$$\tilde{M}_n(Q^2)=\tilde{A}_n\ln(Q^2/\Lambda'^2), \quad (7.4)$$

with

$$\Lambda'^2=\Lambda^2 e^{-\delta} \quad (7.5)$$

Now we have various options. We can work with Eq. (7.4) and say that we have absorbed all higher-order corrections by redefining the parameter Λ. In that case $\Lambda'=\Lambda_{\rm LO}$. We can also work with expression (7.3) but in this case

$$\Lambda^2=e^{\delta}\Lambda_{\rm LO}^2. \quad (7.6)$$

In practice R_n is not proportional to $\tilde{A}_n$ but one can always redefine Λ (Bardeen, Buras, Duke, and Muta, 1978) by using the equality

$$\tilde{A}_n\ln(Q^2/\Lambda^2)+R_n=\tilde{A}_n\ln(Q^2/\Lambda'^2)+[R_n-\delta\tilde{A}_n]. \quad (7.7)$$

The freedom in defining Λ, as discussed here, is related to the freedom which we have in defining the effective coupling constant $\bar{g}^2(Q^2)$ when solving renormalization group equations. (For more details, see Sec. VII F.) All these examples show clearly that one cannot discuss numerical values of Λ in a theoretically meaningful way without calculating higher-order corrections and without specifying the definition of the effective coupling constant.

Once a definition of $\bar{g}^2(Q^2)$ is made and is used in calculations of higher-order corrections in various processes it is possible to make a meaningful comparison of higher-order corrections to various processes. We shall see that these corrections are generally different for different processes. This teaches us that it is in principle unjustified to use the same value of Λ in the leading-order expressions for different processes. On the other hand, once higher-order corrections are included in the analysis and $\bar{g}^2(Q^2)$ is properly defined in a universal way, it is justified to use the same value of Λ in different processes. We shall discuss all these questions in greater detail and with specific examples, but first we have to calculate the higher-order corrections. As we shall see there are many subtle points related to higher-order calculations which one does not encounter in the leading order. These are, for instance, various gauge dependences and renormalization prescription dependences of separate elements of the higher-order formulas. We shall deal with all these questions in detail.

B. Wilson coefficient functions of nonsinglet operators to order $\bar{g}^2$

We begin the discussion of higher-order corrections with the nonsinglet structure functions (e.g., $F_2^{ep}-F_2^{en}$, $F_2^{\bar{\nu}p}-F_2^{\nu p}$, F_3, etc.), which we generally denote by $F_k^{\rm NS}(x,Q^2)$. In quantum chromodynamics the moments of $F_k^{\rm NS}(x,Q^2)$ are given as follows:

$$M_k^{\rm NS}(n,Q^2)=\int_0^1 dx\, x^{n-2}F_k^{\rm NS}(x,Q^2)=A_n^{\rm NS}(\mu^2)C_{k,n}^{\rm NS}\left(\frac{Q^2}{\mu^2},g^2\right) \quad k=L,2 \quad (7.8)$$

$$M_3^{\rm NS}(n,Q^2)=\int_0^1 dx\, x^{n-1}F_3(x,Q^2)=A_n^{\rm NS}(\mu^2)C_{3,n}^{\rm NS}\left(\frac{Q^2}{\mu^2},g^2\right). \quad (7.9)$$

Here $C_{k,n}^{\rm NS}(Q^2/\mu^2,g^2)$ are the Wilson coefficient functions of the nonsinglet operators and $A_n^{\rm NS}(\mu^2)$ are the corresponding reduced hadronic matrix elements. The $A_n^{\rm NS}$'s are uncalculable by present methods and, as discussed in the previous sections, must be taken from experiment. The coefficient functions $C_{k,n}^{\rm NS}(Q^2/\mu^2,g^2)$ are, on the other hand, calculable in perturbation theory. They satisfy the renormalization group equations (4.8) which have the following solution:

$$C_{k,n}^{\rm NS}\left(\frac{Q^2}{\mu^2},g^2\right)=C_{k,n}^{\rm NS}(1,\bar{g}^2)\exp\left(-\int_{\bar{g}(\mu^2)}^{\bar{g}(Q^2)}dg'\frac{\gamma_{\rm NS}^n(g')}{\beta(g')}\right), \quad (7.10)$$

where $\gamma_{\rm NS}^n(g)$ is the anomalous dimension of the nonsinglet operator $O_{\rm NS}^n$ and $\bar{g}^2(Q^2)$ is the effective coupling constant. $\bar{g}^2(Q^2)$ satisfies the equation

$$\frac{d\bar{g}^2}{dt}=\bar{g}\beta(\bar{g})\,;\ \bar{g}(t=0)=g. \quad (7.11)$$

Here $t=\ln Q^2/\mu^2$ and g is the renormalized strong interaction coupling constant. In order to find explicit expressions for the leading and next-to-the-leading contributions to $C_{k,n}^{\rm NS}(Q^2/\mu^2,g^2)$ we expand $\gamma_{\rm NS}^n(\bar{g})$, $\beta(\bar{g})$, and $C_{k,n}^{\rm NS}(1,\bar{g}^2)$ in powers of $\bar{g}$:

$$\gamma_{\rm NS}^n(\bar{g})=\gamma_{\rm NS}^{(0),n}\frac{\bar{g}^2}{16\pi^2}+\gamma_{\rm NS}^{(1),n}\left(\frac{\bar{g}^2}{16\pi^2}\right)^2, \quad (7.12)$$

$$\beta(\bar{g})=-\beta_0\frac{\bar{g}^3}{16\pi^2}-\beta_1\frac{\bar{g}^5}{(16\pi^2)^2}-\cdots, \quad (7.13)$$

and (through order $\bar{g}^2$)

$$C_{k,n}^{\rm NS}(1,\bar{g}^2)=\begin{cases}\delta_{\rm NS}^k[1+(\bar{g}^2/16\pi^2)B_{k,n}^{\rm NS}] & k=2,3\\ \delta_{\rm NS}^L[0+(\bar{g}^2/16\pi^2)B_{L,n}^{\rm NS}] & k=L\end{cases}. \quad (7.14)$$

Here $\delta_{\rm NS}^k$ are constants which depend on weak and electromagnetic charges.

Inserting Eqs. (7.12)–(7.14) into Eq. (7.10), expanding in $\bar{g}^2(Q^2)$ and inserting the result into Eqs. (7.8) and (7.9), we obtain after putting $\mu^2=Q_0^2$

$$M_k^{\rm NS}(n,Q^2)=\delta_{\rm NS}^k A_n^{\rm NS}(Q_0^2)\left[1+\frac{[\bar{g}^2(Q^2)-\bar{g}^2(Q_0^2)]}{16\pi^2}R_{k,n}^{\rm NS}\right]\left[\frac{\bar{g}^2(Q^2)}{\bar{g}^2(Q_0^2)}\right]^{d_{\rm NS}^n}, \quad k=2,3 \quad (7.15)$$

where

$$d_{\rm NS}^n=\frac{\gamma_{\rm NS}^{0,n}}{2\beta_0}, \quad (7.16)$$

$$R_{k,n}^{\rm NS}=B_{k,n}^{\rm NS}+\gamma_{\rm NS}^{(1),n}/2\beta_0-(\gamma_{\rm NS}^{(0),n}/2\beta_0^2)\beta_1 \quad (7.17)$$

and $\bar{g}^2(Q^2)$ is to be calculated by means of Eq. (7.11) with the β function given by Eq. (7.13). For the longitudinal structure function we obtain

$$M_L^{\rm NS}(n,Q^2)=\delta_{\rm NS}^L A_n^{\rm NS}(Q_0^2)\frac{B_{L,n}^{\rm NS}}{\beta_0\ln(Q^2/\Lambda^2)}\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d_{\rm NS}^n}. \quad (7.18a)$$

Because the longitudinal structure function vanishes in the leading order it is sufficient to use here the leading order formula for $\bar{g}^2(Q^2)$, i.e., Eq. (2.50).

In the phenomenological applications it is often convenient to insert into Eq. (7.15) the explicit expression for $\bar{g}^2(Q^2)$, which is given as follows:

$$\frac{\bar{g}^2(Q^2)}{16\pi^2}=\frac{1}{\beta_0\ln(Q^2/\Lambda^2)}-\frac{\beta_1}{\beta_0^3}\frac{\ln\ln(Q^2/\Lambda^2)}{\ln^2(Q^2/\Lambda^2)}+O\left[\frac{1}{\ln^3(Q^2/\Lambda^2)}\right]. \quad (7.19)$$

Here Λ has been arbitrarily chosen so that there are no further terms of order $1/(\ln^2 Q^2/\Lambda^2)$. Clearly this choice of Λ is not unique and one could use other definitions for Λ which lead to additional terms of order $1/(\ln^2 Q^2/\Lambda^2)$ in Eq. (7.19). In this review, however, we shall only use the functional form of $\bar{g}^2(Q^2)$ as given in Eq. (7.19). Λ, μ^2, and $\bar{g}(\mu^2)$ are related to each other by Eq. (2.89).

Inserting (7.19) into (7.15) we obtain the following generalization of the leading-order formulas (4.17a) and (4.18a):

$$M_k^{\rm NS}(n,Q^2)=\delta_{\rm NS}^k A_n^{\rm NS}(Q_0^2)\left[1+\frac{R_{k,n}^{\rm NS}(Q^2)}{\beta_0\ln(Q^2/\Lambda^2)}-\frac{R_{k,n}^{\rm NS}(Q_0^2)}{\beta_0\ln(Q_0^2/\Lambda^2)}\right]\times\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d_{\rm NS}^n}, \quad k=2,3 \quad (7.20a)$$

where

$$R_{k,n}^{\rm NS}(Q^2)=R_{k,n}^{\rm NS}-(\beta_1/2\beta_0^2)\gamma_{\rm NS}^{0,n}\ln\ln(Q^2/\Lambda^2)\,, \quad (7.21)$$

with $R_{k,n}^{\rm NS}$ given by Eq. (7.17).

The value of Q_0^2 in Eq. (7.20a) is arbitrary as required by the renormalization group equations and the predictions for $M_k^{\rm NS}(n,Q^2)$ are independent of it. Therefore it is convenient to get rid of Q_0^2 by writing Eq. (7.20a) as follows:

$$M_k^{\rm NS}(n,Q^2)=\delta_{\rm NS}^k\bar{A}_n^{\rm NS}\left[1+\frac{R_{k,n}^{\rm NS}(Q^2)}{\beta_0\ln(Q^2/\Lambda^2)}\right][\ln(Q^2/\Lambda^2)]^{-d_{\rm NS}^n} \quad k=2,3\,. \quad (7.20b)$$

Here $\bar{A}_n^{\rm NS}$ are constants (independent of Q_0^2), which are related to $A_n^{\rm NS}(Q_0^2)$ by the following equation:

$$A_n^{\rm NS}(Q_0^2)=\bar{A}_n^{\rm NS}\left[1+\frac{R_{k,n}^{\rm NS}(Q_0^2)}{\beta_0\ln(Q_0^2/\Lambda^2)}\right][\ln(Q_0^2/\Lambda^2)]^{-d_{\rm NS}^n}. \quad (7.22)$$

Notice that Eqs. (7.20b) and (7.22) are straightforward generalizations of the leading-order formulas (4.17b), (4.18b), and (4.19).

Similarly we can write (7.18a) as

$$M_L^{\rm NS}(n,Q^2)=\delta_{\rm NS}^L A_n^{\rm NS}\frac{B_{L,n}^{\rm NS}}{\beta_0\ln(Q^2/\Lambda^2)}[\ln(Q^2/\Lambda^2)]^{-d_{\rm NS}^n}, \quad (7.18b)$$

where $A_n^{\rm NS}$ is defined by Eq. (4.19).

It shoudl be remarked that theoretically $\bar{A}_n^{\rm NS}=A_n^{\rm NS}$. This is because for sufficiently large values of Q^2, for which higher-order corrections are small, Eq. (7.20b) and the corresponding leading-order formulas (4.17b) and (4.18b) should coincide. In phenomenological applications, however, $A_n^{\rm NS}$ and $\bar{A}_n^{\rm NS}$, being uncalculable in perturbation theory, are regarded as free parameters and are found by fitting the formulae in question with the data. Since formulas (7.20b), (4.17b), and (4.18b) have different structures, fits to the same data will lead to different numerical values for $A_n^{\rm NS}$ and $\bar{A}_n^{\rm NS}$. Therefore we use different notation for $A_n^{\rm NS}$ and $\bar{A}_n^{\rm NS}$.

We see that, in order to find the next-to-the-leading order corrections, one has to calculate two-loop contributions to $\gamma_{\rm NS}^n(g)$ and $\beta(g)$ and one-loop corrections to $C_{k,n}^{\rm NS}(1,\bar{g}^2)$. The two-loop contribution to the β function, i.e., parameter β_1, has been calculated by Caswell (1974) and Jones (1974) and is given for an $SU(3)_c$ gauge theory with f flavors by

$$\beta_1=102-(38/3)f\,. \quad (7.23)$$

It should be remarked that β_1 as well as $\gamma_{\rm NS}^{0,n}$ and β_0 are renormalization prescription- and gauge-independent.

The parameters $B_{k,n}^{\rm NS}$ for electromagnetic processes and ν scattering have been calculated by Calvo (1977), and for electromagnetic processes by de Rujula, Georgi, and Politzer (1977a). The results obtained in these two papers disagree with each other. The reason for the disagreement between these two calculations is that they have been performed in two different renormalization schemes. Calvo has used renormalization on the mass shell, whereas de Rujula, Georgi, and Politzer made subtractions at $p^2=-\mu^2$ (see Sec. III). In fact, as has been pointed out by Floratos, Ross, and Sachrajda (1977), the parameters $B_{k,n}^{\rm NS}$ are renormalization prescription dependent. Of course the moments of the structure functions cannot depend on the renormalization schemes used, and it can be shown (see Sec. VII D) that the renormalization prescription dependence of the parameters $B_{k,n}^{\rm NS}$ is canceled by that of the two-loop parameters $\gamma_{\rm NS}^{(1),n}$. In other words the quantity

$$B_{k,n}^{\rm NS}+\gamma_{\rm NS}^{(1),n}/2\beta_0\,, \quad (7.24)$$

which enters the formula (7.17) is renormalization prescription *independent*. This means that the calculations of higher-order corrections can be performed in any renormalization scheme but care must be taken that both quantities are calculated in the same scheme. This implies that without doing explicit calculations one cannot *a priori* neglect any of the two quantities $B_{k,n}^{\rm NS}$ and $\gamma_{\rm NS}^{(1),n}/2\beta_0$ in any higher-order formulas. The reason is that in some schemes the two-loop contributions are dominant in the sum (7.24), whereas in other schemes $B_{n,k}^{\rm NS}$ are most important.

The full calculation of the sum of Eq. (7.24) has been performed in the literature only in the 't Hooft's minimal subtraction scheme. The parameters $B_{k,n}^{\rm NS}$ have been calculated by Bardeen, Buras, Duke, and Muta (1978) and recalculated by Floratos, Ross, and Sachrajda (1979). The latter authors have also calculated the two-loop anomalous dimensions $\gamma_{\rm NS}^{(1),n}$ (Floratos, Ross, and Sachrajda, 1977).

We shall now outline the procedure for the calculation of the parameters $B_{k,n}^{\rm NS}$ and subsequently prove the renormalization prescription independence of the sum in Eq. (7.24).

C. Procedure for the calculation of $B_{k,n}^{\rm NS}$

We first notice that in order to find $B_{k,n}^{\rm NS}$ as defined in Eq. (7.14), it is sufficient to calculate $C_{k,n}^{\rm NS}(Q^2/\mu^2,g^2)$ in perturbation theory to order g^2 and put $Q^2=\mu^2$. This is obvious from Eqs. (7.10) and (7.11). In order to calculate $C_{k,n}^{\rm NS}(Q^2/\mu^2,\,g^2)$ in perturbation theory we proceed as follows. We write first the lhs of Eq. (3.54) as

$$T_{\mu\nu}(Q^2,\nu)=\sum_n \frac{1}{x^n}[e_{\mu\nu}T_{L,n}(Q^2/p^2,g^2)+d_{\mu\nu}T_{2,n}(Q^2/p^2,g^2) - i\epsilon_{\mu\nu\alpha\beta}(p_\alpha q_\beta/\nu)T_{3,n}(Q^2/p^2,g^2)], \quad (7.25)$$

where p^2 is the target momentum squared and we have indicated on the rhs of Eq. (7.25) that $T_{L,n}$, $T_{2,n}$, and $T_{3,n}$ will be calculated in perturbation theory. The tensors $e_{\mu\nu}$ and $d_{\mu\nu}$ are defined in Eqs. (2.2) and (2.3), respectively. Restricting the discussion to the nonsinglet contributions, we obtain by comparing (3.54) and (7.25) the following relation for each n separately

$$T^{\rm NS}_{k,n}\left(\frac{Q^2}{p^2},g^2\right)=C^{\rm NS}_{k,n}\left(\frac{Q^2}{\mu^2},g^2\right)A^{\rm NS}_n\left(\frac{p^2}{\mu^2},g^2\right) \quad k=L,2,3\,. \quad (7.26)$$

We observe that in order to find $C^{\rm NS}_{k,n}(Q^2/\mu^2,g^2)$ we generally have to calculate both $T^{\rm NS}_{k,n}$ and $A^{\rm NS}_n$. We have mentioned before that the matrix elements of local operators between hadronic states are incalculable in perturbation theory. Fortunately, the coefficient functions of operators do not depend on the states between which the operators are sandwiched, and therefore in the problem under investigation we can choose any state for which perturbative calculations can be performed. Consequently, in Eq. (7.26) $T^{\rm NS}_{k,n}$ are to be found from the virtual Compton scattering off quarks and $A^{\rm NS}_n$ stands for the matrix element of the spin n nonsinglet operator between quark states. In order to avoid mass singularities in what follows, we shall keep the external quarks at spacelike momenta $p^2<0$.

Next we expand the elements of Eq. (7.26) in a perturbation series as follows:

$$T^{\rm NS}_{k,n}(Q^2/p^2,g^2)=h_k+(g^2/16\pi^2)[-\tfrac{1}{2}\gamma^{0,n}_{\rm NS}\ln(Q^2/-p^2)+T^{(2),\rm NS}_{k,n}]\,, \quad (7.27)$$

$$C^{\rm NS}_{k,n}(Q^2/\mu^2,g^2)=h_k+(g^2/16\pi^2)[-\tfrac{1}{2}\gamma^{0,n}_{\rm NS}\ln(Q^2/\mu^2)+B^{\rm NS}_{k,n}]\,, \quad (7.28)$$

$$A^{\rm NS}_n(p^2/\mu^2,g^2)=1+(g^2/16\pi^2)[\tfrac{1}{2}\gamma^{0,n}_{\rm NS}\ln(-p^2/\mu^2)+A^{(2),\rm NS}_n]\,, \quad (7.29)$$

where

$$h_k=\begin{cases}1 & k=2,3\\ 0 & k=L\end{cases} \quad (7.30)$$

and the coefficients of the logarithms are fixed by the renormalization group equations which $C^{\rm NS}_{k,n}$ and $A^{\rm NS}_n$ have to satisfy (see Sec. III). In order to simplify the notation we have dropped the overall factors $\delta^k_{\rm NS}$.

Inserting Eqs. (7.27)–(7.29) into (7.26) and comparing the coefficients of g^2 we obtain

$$B^{\rm NS}_{k,n}=\begin{cases}T^{(2),\rm NS}_{k,n}-A^{(2),\rm NS}_n & k=2,3\\ T^{(2),\rm NS}_{L,n} & k=L\end{cases}\,. \quad (7.31)$$

We shall comment on the details of the calculation of $T^{(2),\rm NS}_{k,n}$, $A^{(2),\rm NS}_n$, and $B^{\rm NS}_{k,n}$ in Sec. VII E; we turn now to a discussion of the renormalization prescription dependence of $B^{\rm NS}_{k,n}$ and of its cancellation by the renormalization scheme dependence of the two-loop anomalous dimensions $\gamma^{(1),n}_{\rm NS}$.

D. Renormalization prescription independence of higher-order corrections

Here we follow the proof of Floratos, Ross, and Sachrajda (1977).

Consider two renormalization schemes a and b in which the matrix elements of the operator $O^n_{\rm NS}$ calculated to order g^2 are normalized differently as follows (we drop the index NS):

$$A^{(a)}_n(p^2/\mu^2,g^2)=A^0_n[1+(g^2/16\pi^2)\tfrac{1}{2}\gamma^{0,n}_{\rm NS}\ln(-p^2/\mu^2)] \quad (7.32)$$

and

$$A^{(b)}_n(p^2/\mu^2,g^2)=A^0_n[1+(g^2/16\pi^2)(\tfrac{1}{2}\gamma^{0,n}_{\rm NS}\ln(-p^2/\mu^2)+r_n)]\,. \quad (7.33)$$

Here A^0_n are the zero-loop matrix elements which are obviously the same in both schemes. In scheme a, at $p^2=-\mu^2$ the renormalized matrix element $A^{(a)}_n$ is equal to the zero-loop matrix element. This is a very common renormalization scheme. In scheme b we have

$$A^{(b)}_n(-1,g^2)=A^0_n[1+(g^2/16\pi^2)r_n]\,, \quad (7.34)$$

where r_n are nonzero numbers specific to a given renormalization scheme. In particular, 't Hooft's minimal subtraction scheme ('t Hooft, 1973) falls into the class of b schemes.

Since $T^{(2)}_{k,n}$ are independent of renormalization scheme (the virtual Compton amplitude is finite and no renormalization is required) we have from Eqs. (7.31)–(7.33)

$$B^{(a)}_{k,n}=\begin{cases}B^{(b)}_{k,n}-r_n & k=2,3\\ B^{(b)}_{L,n} & k=L\end{cases}\,. \quad (7.35)$$

Thus, $B_{2,n}$ and $B_{3,n}$ are renormalization prescription dependent, whereas $B_{L,n}$ is independent of renormalization scheme.

Recall next that

$$O^{0,n}=Z_iO^n \quad i=a,b\,, \quad (7.36)$$

where $O^{0,n}$ is the bare operator. From (7.32) and (7.33) we have therefore the following relation between the renormalization constants Z_a and Z_b:

$$Z_a=(1+(g^2/16\pi^2)r_n)Z_b\,. \quad (7.37)$$

Since the anomalous dimension of the operator O^n is defined by

$$\gamma^n_i=\mu\frac{\partial}{\partial\mu}\log Z_i\,\Big|_{\text{Bare quantities fixed}} \quad i=a,b \quad (7.38)$$

we obtain from (7.37), (7.38) and the definition of the β function, e.g., Eq. (3.37), the following relation between γ^n_a and γ^n_b:

$$\gamma^n_b=\gamma^n_a+2r_n\beta_0[g^4/(16\pi^2)^2] \quad (7.39)$$

or equivalently

$$\gamma^{(1),n}_b=\gamma^{(1),n}_a+2r_n\beta_0\,. \quad (7.40)$$

Equations (7.35) and (7.40) taken together lead to

$$B^{(a)}_{k,n}+\gamma^{(1),n}_a/2\beta_0=B^{(b)}_{k,n}+\gamma^{(1),n}_b/2\beta_0 \quad k=2,3\,, \quad (7.41)$$

i.e., the combination (7.24) is independent of the renormalization scheme.[32]

[32]Generalization of this proof to all orders in $\bar{g}^2(Q^2)$ has been discussed by Moshe (1978) and Schellekens (1979). For a very nice discussion of this topic see Peterman (1979).

Just before Eq. (7.35) we stated that $T^{(2)}_{k,n}$ are independent of the renormalization scheme. On the other hand it is obvious that $T^{(2)}_{k,n}$ depend on the assumptions about the "quark target" used to extract the coefficient function. In the discussion above we concentrated on a class of calculations in which the "quark target" is massless with spacelike momentum $p^2<0$. One could equally well consider massive quarks with $p^2=0$. In that case $T^{(2)}_{k,n}$ would be different from the case considered here. Also $A^{(2)}_n$ in a given renormalization scheme depends on the target (the state between which O_n is sandwiched). The dependences of $T^{(2)}_{k,n}$ and $A^{(2)}_n$ on the "quark target" cancel, however, in the calculation of $B_{k,n}$, as should be the case. We shall illustrate this with an example in Sec. VIII. It is important to keep in mind the points discussed in this section when comparing various calculations in the literature.

E. Results for $\gamma^{(1),n}_{\mathrm{NS}}$ and $B^{\mathrm{NS}}_{k,n}$

As we have stressed several times, one has to make sure that $\gamma^{(1),n}_{\mathrm{NS}}$ and $B^{\mathrm{NS}}_{k,n}$ are calculated in the same renormalization scheme in order that a physical answer for the moments of structure functions is obtained. The calculation of the two-loop anomalous dimensions $\gamma^{(1),n}_{\mathrm{NS}}$ is much more involved than that of $B^{\mathrm{NS}}_{k,n}$ and therefore it is useful to choose a renormalization scheme in which the calculation of $\gamma^{(1),n}_{\mathrm{NS}}$ is simplest. The minimal subtraction scheme of 't Hooft, which we have discussed in Sec. III, turns out to be a convenient scheme for this purpose.

1. Two-loop anomalous dimensions $\gamma^{(1),n}_{\mathrm{NS}}$

It has been shown by Floratos, Ross, and Sachrajda (1977) that in 't Hooft's renormalization scheme the two-loop anomalous dimensions can be simply obtained from the coefficient of the $1/\varepsilon$ pole in the quark matrix element of the nonsinglet operator calculated to order g^4, plus twice the two-loop anomalous dimensions of the quark field. The latter anomalous dimension is also obtained in 't Hooft's scheme by calculating to order g^4 the coefficient of $1/\varepsilon$ pole in the quark self-energy. Typical diagrams for these two calculations are shown in Figs. 24 and 25. In the whole there are about 30 two-loop diagrams which one has to calculate in order to obtain $\gamma^{(1),\mathrm{NS}}_{\mathrm{NS}}$. All these diagrams have been calculated in the paper by Floratos, Ross, and Sachrajda where the interested reader can find the details of the calcula-

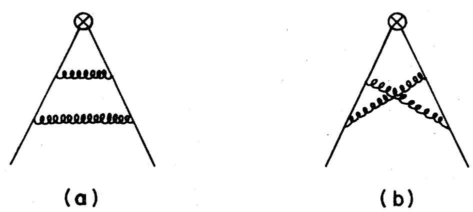


FIG. 24. Examples of diagrams which enter the calculation of $\gamma^{(1),n}_{\mathrm{NS}}$.

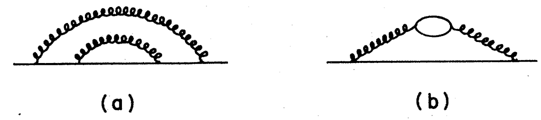


FIG. 25. Examples of g^4 order contributions to quark self-energy.

tion.[33] The analytic expressions for $\gamma_n^{(1),\mathrm{NS}}$ as given in the original paper is very complicated. A simpler formula for $\gamma^{(1),n}_{\mathrm{NS}}$ can be found in the paper by Gonzalez-Arroyo *et al.* (1979b). The numerical values for $\gamma_n^{(1),\mathrm{NS}}$ are given in Table III. We would like to remark that $\gamma^{(1),n}_{\mathrm{NS}}$ in the minimal subtraction scheme are gauge independent (Caswell and Wilczek, 1974).

2. $B^{\mathrm{NS}}_{k,n}$ in 't Hooft's scheme (electromagnetic currents)

The first calculation of $B^{\mathrm{NS}}_{k,n}$ in the minimal subtraction scheme has been done by Bardeen, Buras, Duke, and Muta (1978). Contrary to the calculation of the two-loop anomalous dimensions, the calculation of $B^{\mathrm{NS}}_{k,n}$ in the scheme in question is generally more complicated than in other schemes. The reason is that in 't Hooft's scheme the g^2 corrections to the matrix elements of local operators $A_n^{(2),\mathrm{NS}}$ are nonzero and must be explicitly calculated in addition to the g^2 corrections to the virtual Compton amplitude.

The calculation of the virtual Compton amplitude for scattering off quarks in g^2 order involves the diagrams of Fig. 26. The diagrams contributing in g^2 order to the matrix elements of nonsinglet operator between

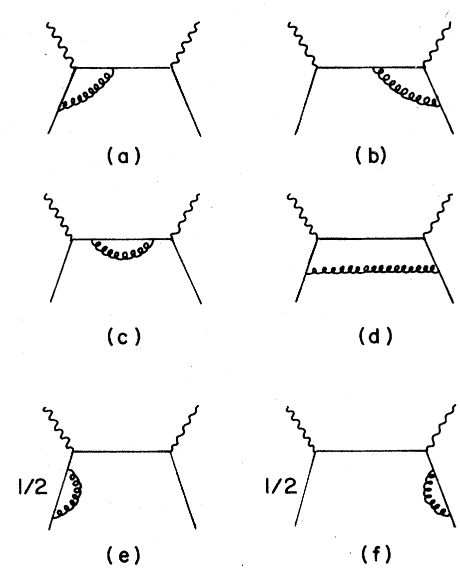


FIG. 26. Diagrams entering the calculation of $T^{(2),\mathrm{NS}}_{k,n}$ of Eq. (7.31).

[33]It turns out that $\gamma^{(1),n}_{\mathrm{NS}}=\gamma^{\alpha}_n+(-1)^n\gamma^{\beta}_n$, where γ^{α}_n and γ^{β}_n may be analytically continued in n. Because of the factor $(-1)^n$ the even and odd values of $\gamma^{(1),n}_{\mathrm{NS}}$ (see Footnote 9) must be (in the process of inversion) analytically continued to $\gamma^{\alpha}_n+\gamma^{\beta}_n$ and $\gamma^{\alpha}_n-\gamma^{\beta}_n$, respectively. The fact that $\gamma^{\beta}_n\neq 0$ can be related to flavor symmetry breaking in antiquark distributions (Ross and Sachrajda, 1979).

quark states are shown in Fig. 8.[34] Explicit expressions for $T_{k,n}^{(2),\mathrm{NS}}$ and $A_n^{(2),\mathrm{NS}}$ can be found in the original paper. Here we only remark that although $T_{k,n}^{(2),\mathrm{NS}}$ and $A_n^{(2),\mathrm{NS}}$ are separately *gauge dependent*, the resulting expression for $B_{2,n}^{\mathrm{NS}}$ is *gauge independent* as expected in the minimal subtraction scheme. We have

$$B_{2,n}^{\mathrm{NS}}=\frac{4}{3}\Bigg\{3\sum_{j=1}^{n}\frac{1}{j}-4\sum_{j=1}^{n}\frac{1}{j^2}-\frac{2}{n(n+1)}\sum_{j=1}^{n}\frac{1}{j}\qquad n,\ \text{even}$$
$$+4\sum_{s=1}^{n}\frac{1}{s}\sum_{j=1}^{s}\frac{1}{j}+\frac{3}{n}+\frac{4}{(n+1)}+\frac{2}{n^2}-9\Bigg\}+\tfrac{1}{2}\gamma_{\mathrm{NS}}^{0,n}(\ln 4\pi-\gamma_E) \tag{7.42}$$

and

$$B_{L,n}^{\mathrm{NS}}=\frac{16}{3}[1/(n+1)],\quad n\ \text{even}. \tag{7.43}$$

The last result for $B_{L,n}^{\mathrm{NS}}$, which is renormalization prescription independent, has been previously obtained by other authors (Zee, Wilczek, and Treiman, 1974; Kingsley, 1973; Walsh and Zerwas, 1973). Recently the calculation of $B_{2,n}^{\mathrm{NS}}$ in the minimal subtraction scheme has been repeated by Floratos, Ross, and Sachrajda (1979), who have reproduced the results of Eq. (7.42).

For the calculations of $B_{2,n}^{\mathrm{NS}}$ in renormalization schemes different from that considered here we refer the reader to the papers by Calvo (1977); De Rujula, Georgi, and Politzer (1977a); Altarelli, Ellis, and Martinelli (1978); Kubar-André and Paige (1979); and Abad and Humpert (1978).

3. $B_{k,n}^{\mathrm{NS}}$ in 't Hooft's scheme (weak currents)

In the evaluation of g^2 corrections to ν and $\bar{\nu}$ deep-inelastic scattering it is convenient to consider certain combinations of $\nu,\bar{\nu}$ structure functions which have simple properties under crossing. These are

$$F_2^{\bar{\nu}p}-F_2^{\nu p}, \tag{7.44}$$

$$F_2^{\bar{\nu}p}+F_2^{\nu p}, \tag{7.45}$$

$$F_3^{\bar{\nu}p}-F_3^{\nu p}, \tag{7.46}$$

and

$$F_3^{\bar{\nu}p}+F_3^{\nu p}. \tag{7.47}$$

The remaining structure functions for scattering off neutron or nuclear targets can be directly obtained from (7.44)–(7.47) using charge symmetry. For instance,

$$F_2^{\bar{\nu}p}-F_2^{\nu p}=F_2^{\nu n}-F_2^{\nu p}=F_2^{\bar{\nu}p}-F_2^{\bar{\nu}n}. \tag{7.48}$$

In order to calculate g^2 corrections to F_2, one considers again the diagrams of Figs. 8 and 26, except that now the diagrams with both vector currents replaced by axial-vector currents also contribute. The combinations (7.44) and (7.45) correspond to subtracting and adding crossed diagrams of Fig. 26, respectively.

The structure function F_3 corresponds to the vector–axial-vector interference and therefore the diagrams contributing to it are obtained from Fig. 26 by replacing one of the vector currents by an axial vector current. Again the calculation of the g^2 corrections to the combinations (7.46) and (7.47) corresponds to subtracting and adding crossed diagrams respectively.

By inspecting the diagrams directly or by considering the decomposition (3.51) and taking into account known properties of various structure functions under the transformations $\mu\to\nu$, $x\leftrightarrow -x$ one can easily find whether even or odd spin operators contribute to each of the combinations (7.44–7.47). It turns out (Bailin, Love, and Nanopoulos, 1974; Politzer, 1974) that to $F_2^{\bar{\nu}-\nu}$ and $F_3^{\bar{\nu}+\nu}$ only odd spin and to $F_2^{\bar{\nu}+\nu}$ and $F_3^{\bar{\nu}-\nu}$ only even spin operators contribute.

Finally we have to determine which combinations are independent of gluon operators and therefore satisfy simple renormalization group equations as given in Eq. (4.8). The combinations (7.44) and (7.46) transform obviously as nonsinglets under flavor symmetry and therefore satisfy equations like (4.8). $F_2^{\bar{\nu}p}+F_2^{\nu p}$ is a singlet combination as discussed in Sec.II. Therefore because of mixing between gluon and fermion singlet operators this combination will satisfy more complicated renormalization group equations, which we shall discuss in Sec.VIII. On the other hand, $F_3^{\bar{\nu}+\nu}$ still satisfies Eq. (4.8) in spite of having contributions from singlet fermion operators. This is because the gluon operators of odd spin transform differently under charge conjugation than the corresponding singlet fermion operators and therefore there is no mixing.

In the minimal subtraction scheme the results for the parameters $B_{k,n}$ relevant for ν, $\bar{\nu}$ scattering are as follows (Bardeen, Buras, Duke, and Muta, 1978):

$$B_{2,n}^{\bar{\nu}-\nu}=B_{2,n}^{\mathrm{NS}},\quad n\ \text{odd} \tag{7.49}$$

$$B_{L,n}^{\bar{\nu}-\nu}=B_{L,n}^{\mathrm{NS}},\quad n\ \text{odd} \tag{7.50}$$

$$B_{3,n}^{\bar{\nu}\pm\nu}=B_{2,n}^{\mathrm{NS}}-\frac{4}{3}\frac{4n+2}{n(n+1)}\qquad \begin{matrix} n\ \text{odd}\\ n\ \text{even}\end{matrix}, \tag{7.51}$$

where $B_{2,n}^{\mathrm{NS}}$ and $B_{L,n}^{\mathrm{NS}}$ are given by (7.42) and (7.43), respectively. Result (7.50) has been previously obtained by Zee, Wilczek, and Trieman (1974). For the calculation of $B_{2,n}^{\bar{\nu}-\nu}$ and $B_{3,n}^{\bar{\nu}\pm\nu}$ in different renormalization schemes from those considered here we refer the reader to the papers by Calvo (1977), and Altarelli, Ellis, and Martinelli (1978).

4. Corrections to sum rules and parton model relations

It is well known that in the leading order of asymptotic freedom parton model relations and sum rules are satisfied. The $\bar{g}^2$ corrections discussed in this section can generally introduce violations of the sum rules and relations in question. Notice in particular that the $\bar{g}^2$ corrections to the Q^2 dependence of $F_3(x,Q^2)$ differ from the corresponding corrections for $F_2(x,Q^2)$.

Evaluating formulas (7.49)–(7.51) for $n=1$ and recalling that $\gamma_{\mathrm{NS}}^{n}=0$ for $n=1$ due to current conservation one obtains corrections to the Gross–Llewellyn-Smith sum rule (Gross and Llewellyn-Smith, 1969) and the Bjorken sum rule (Bjorken, 1967) as shown in Eqs. (2.99) and (2.100), respectively. In Fig. 27 we have plotted predictions of Eqs. (2.99) and (2.100) versus Q^2/Λ^2. We ob-

[34] We recall that we have calculated the diagrams of Fig. 8 in Sec. III in order to find $\gamma_{\mathrm{NS}}^{0,n}$, the coefficient of $\ln(-p^2/\mu^2)$ in Eq. (7.29). This time we are interested in the constant pieces $A_n^{(2),\mathrm{NS}}$.

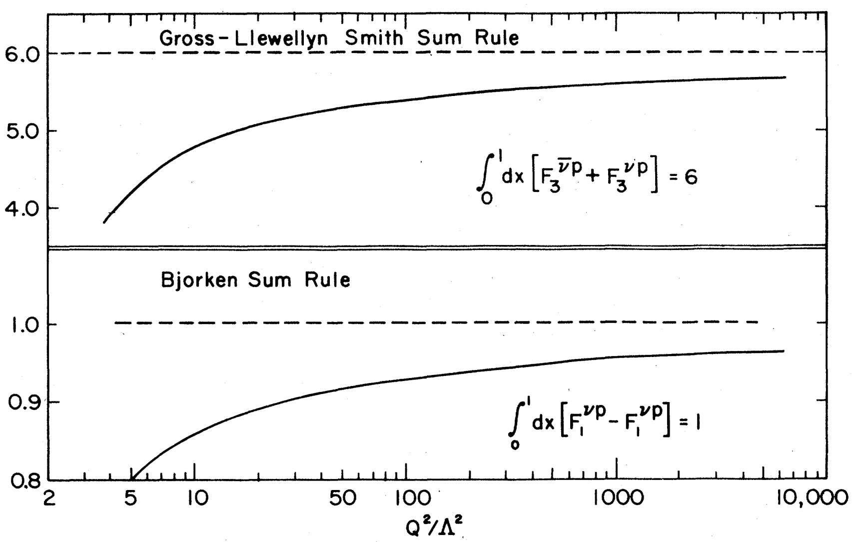


FIG. 27. Order $\bar g^2$ deviations from the Gross–Llewellyn-Smith and Bjorken sum rules. The dashed lines (---) are parton model predictions. The solid (—) lines follow from Eqs. (2.99) and (2.100). The figure is from Bardeen *et al.* (1978).

serve that the deviations from the two sum rules in question are predicted to be non-negligible and accurate measurements should detect them.

We defer the discussion of QCD corrections to the Callan–Gross relation (Eq. 2.43) to Sec. VIII.

F. Phenomenology of the order $\bar g^2$ corrections (nonsinglet case)

In this section we shall compare the formula

$$M_k^{NS}(n,Q^2)=\delta^k_{NS}\bar A_n^{NS}\left[1+\frac{R^{NS}_{k,n}(Q^2)}{\beta_0\ln(Q^2/\Lambda^2)}\right]\left[\ln(Q^2/\Lambda^2)^{-d^n_{NS}}\right], \tag{7.52}$$

with experiment (Bardeen, Buras, Duke, and Muta, 1978). The only free parameters in Eq. (7.52) are the constants $\bar A_n^{NS}$ and the scale parameter Λ. These parameters are to be found by fitting the formula (7.52) to the data for as large a range of Q^2 as possible.

As we have discussed in Sec. VII.A, there is freedom in defining the effective coupling constant and, correspondingly, the parameter Λ. The Λ which enters Eq. (7.52) corresponds to $R^{NS}_{k,n}$ given by Eq. (7.17) with $B^{NS}_{k,n}$ calculated in 't Hooft's minimal scheme and to the following form of $\bar g^2(Q^2)$:

$$\frac{\bar g^2(Q^2)}{16\pi^2}=\frac{1}{\beta_0\ln(Q^2/\Lambda^2)}-\frac{\beta_1}{\beta_0^3}\frac{\ln\ln(Q^2/\Lambda^2)}{\ln^2(Q^2/\Lambda^2)}+O\left[\frac{1}{\ln^3(Q^2/\Lambda^2)}\right]. \tag{7.53}$$

Clearly this definition of Λ is not unique, and we shall now discuss other possible definitions.

The effect of the redefinition of Λ is equivalent through order $\bar g^2(Q^2)$ to the shift of $R^{NS}_{k,n}(Q^2)$ by a constant amount proportional to the one-loop anomalous dimension $\gamma^{0,n}_{NS}$. In fact rescaling Λ in Eq. (7.52) to Λ' by

$$\Lambda=\kappa\Lambda', \tag{7.54}$$

where κ is a constant, and dropping terms of order $\bar g^4(Q^2)$ generated by this rescaling, one obtains

$$M_k^{NS}(n,Q^2)=\delta^k_{NS}\bar A_n^{NS}\left[1+\frac{1}{\beta_0\ln(Q^2/\Lambda'^2)}R'^{NS}_{k,n}(Q^2)\right]\left[\ln\frac{Q^2}{\Lambda'^2}\right]^{-d^n_{NS}}, \tag{7.55}$$

where

$$R'^{NS}_{k,n}(Q^2)=R^{NS}_{k,n}(Q^2)+\gamma^{0,n}_{NS}\ln\kappa. \tag{7.56}$$

The Λ' thus corresponds to the $\bar g^2$ corrections given by Eq. (7.56) and $\bar g^2(Q^2)$ having the form of Eq. (7.53) with Λ replaced by Λ'.

It should be remarked that Eqs. (7.52) and (7.55) are equivalent representations of next-to-the-leading order corrections. On the other hand they correspond to different estimates of the higher-order terms $O[\bar g^4(Q^2)]$ not included in the analysis. This is obvious, since in going from Eq. (7.52) to (7.55) we drop terms of $O[\bar g^4(Q^2)]$. It should be remarked that since the n dependences in Eqs. (7.52) and (7.55) are different from each other so will be the free parameters $\bar A_n^{NS}$ and Λ extracted in both cases. However, if the estimates of terms of $O(\bar g^4)$ by Eqs. (7.52) and (7.55) are not very different from each other, the equations in question should give equally good fits to experimental data. To illustrate this we have compared two different schemes for Λ with data of BEBC for the moments of F_3^ν (Bardeen *et al.*, 1978).

The first scheme we call the minimal subtraction scheme, and we denote the corresponding value of Λ by Λ_{MS}. This scheme is defined by Eq. (7.52) with Λ replaced by Λ_{MS}. The second scheme is defined by choosing in Eq. (7.54)

$$\kappa=\exp[-\tfrac{1}{2}(\ln 4\pi-\gamma_E)]. \tag{7.57}$$

We shall denote the corresponding Λ by $\Lambda_{\overline{MS}}$. Effectively [see (7.55)] the $\overline{MS}$ scheme is represented as the MS scheme by Eq. (7.52), but with Λ replaced by $\Lambda_{\overline{MS}}$ and $R^{NS}_{k,n}(Q^2)$ replaced by $\bar R^{NS}_{k,n}(Q^2)$, which is given as follows:

$$\bar R^{NS}_{k,n}(Q^2)=R^{NS}_{k,n}(Q^2)-\tfrac{1}{2}\gamma^{0,n}_{NS}(\ln 4\pi-\gamma_E). \tag{7.58}$$

Recalling Eqs. (7.17) and (7.42) we observe that the $\overline{MS}$ scheme does not involve the terms $(\ln 4\pi-\gamma_E)$. The

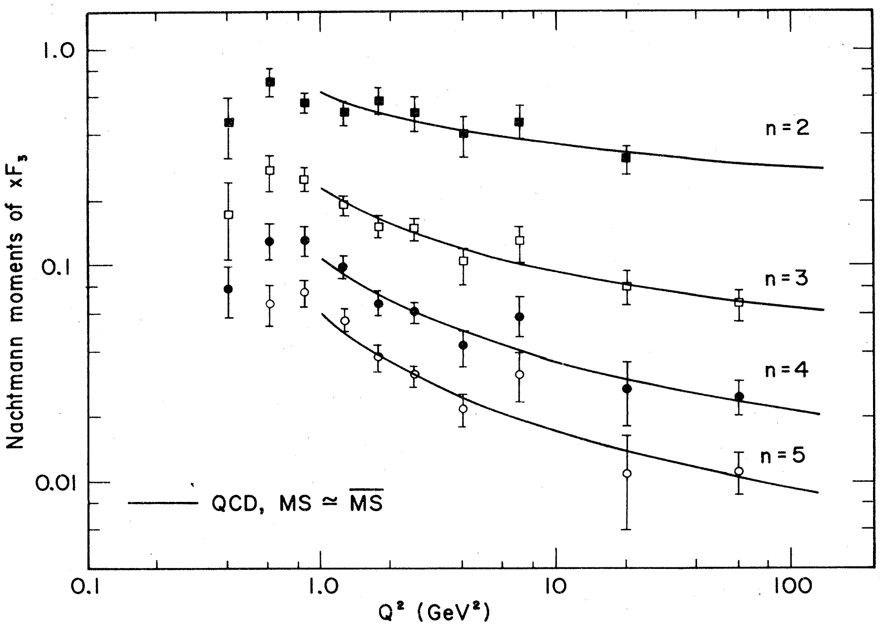


FIG. 28. Nachtmann moments of $xF_3(x, Q^2)$ vs Q^2. The data are from Bossetti *et al.* (1978). The solid lines represent the MS and $\overline{\text{MS}}$ schemes.

comparison of these two schemes for Λ with the BEBC data leads to the following values for Λ_{MS} and $\Lambda_{\overline{\text{MS}}}$:

$$\Lambda_{\text{MS}} = 0.40 \text{ GeV},$$
$$\Lambda_{\overline{\text{MS}}} = 0.52 \text{ GeV}. \tag{7.59}$$

In obtaining these values, Nachtmann moments of Eq. (2.123) have been used. The fits to the data are indistinguishable from each other and are represented in Fig. 28 by a single curve. The leading-order prediction (not shown in the figure)

$$M_3^{\text{NS}}(n, Q^2)\big|_{\text{LO}} = \delta_{\text{NS}}^{(3)} A_n^{\text{NS}} [\ln(Q^2/\Lambda_{\text{LO}}^2)]^{-d_{\text{NS}}^n}, \tag{7.60}$$

with

$$\Lambda_{\text{LO}} = 0.73 \text{ GeV}. \tag{7.61}$$

follows very closely the solid line of Fig. 28. We shall discuss below more sensitive ways of comparing higher-order predictions with the leading-order results.

We recall that in fitting the data both Λ's and A_n's have been treated as free parameters. Therefore the fitted values of the uncalculable matrix elements A_n's are different for different schemes. The similarity of the LO, MS, and $\overline{\text{MS}}$ fits simply indicates that it is possible for the A_n's and Λ in each case to conspire to mask the combined n and Q^2 dependence of the order $\bar{g}^2$ corrections. Of course the similarity of the fits of higher-order corrections (MS and $\overline{\text{MS}}$ schemes) and of leading-order predictions is, strictly speaking, only true over some not too large range in Q^2, as is the case for the presently available data.

It is instructive to calculate the term

$$1 + \frac{R_{k,n}^{\text{NS}}(Q^2)}{\beta_0 \ln(Q^2/\Lambda^2)} \tag{7.62}$$

in Eq. (7.52), which is equal to unity in the leading order. The numerical values for the quantity (7.62) are given for the MS and $\overline{\text{MS}}$ schemes in Table IV. We observe in accordance with earlier expectations (Gross, 1974; Gross, Wilczek, and Treiman, 1976) that, for large values of n and for not too large values of Q^2, higher-order corrections are large and perturbative calculations cannot be trusted. This behavior is mainly due to the constants $B_{k,n}^{\text{NS}}$ which for $k \neq L$ grow like $(\ln n)^2$. We observe also that the terms (7.62) are much smaller in the $\overline{\text{MS}}$ scheme as compared to the MS scheme. The MOM scheme of Table IV is discussed in Sec. VII.H.

As we already discussed above some part of the effects due to higher-order corrections in the second term in Eq. (7.52) can be absorbed over a not too large range of Q^2 in the (incalculable by present methods) hadronic matrix elements of local operators. This makes the phenomenological study of higher-order corrections in deep-inelastic processes complicated. We shall see in Sec. IX that the situation is much better in photon–photon scattering where the leading-order expression and the next-to-the-leading order predictions

TABLE IV. The values of the quantity $1 + [R_{2,n}^{\text{NS}}(Q^2)/\beta_0 \ln(Q^2/\Lambda^2)]$ as a function of n and Q^2 in various schemes: $\overline{\text{MS}}$ ($\bar{\Lambda} = 0.5$ GeV), MOM ($\Lambda_{\text{MOM}} = 0.85$ GeV), and MS ($\Lambda = 0.4$ GeV).

n	Scheme	Q^2[GeV^2] 5	10	50	200
	$\overline{\text{MS}}$	0.97	0.96	0.95	0.95
2	MOM	0.68	0.73	0.80	0.83
	MS	1.20	1.15	1.09	1.07
	$\overline{\text{MS}}$	1.10	1.05	0.99	0.97
4	MOM	0.63	0.65	0.71	0.75
	MS	1.53	1.42	1.26	1.20
	$\overline{\text{MS}}$	1.24	1.15	1.05	1.01
6	MOM	0.69	0.68	0.71	0.74
	MS	1.79	1.62	1.40	1.30
	$\overline{\text{MS}}$	1.37	1.25	1.11	1.06
8	MOM	0.79	0.74	0.73	0.75
	MS	2.00	1.79	1.52	1.39

are free of the incalculable matrix elements of local operators.

The values of Λ as given in Eqs. (7.59) and (7.61) have been obtained for $f=4$ and without taking quark mass effects into account. At low values of Q^2 (few GeV^2), the massless approximation is probably justified for the light quarks, but not justified for the charm quark contributions. The effect of heavy quark masses (in the β function) for the extraction of the value of Λ in the analysis with higher-order corrections included has been recently studied by Abbott and Barnett (1979), who find the values of Λ, which are smaller by roughly 20% than those in Eqs. (7.59) and (7.61). We think that heavy quark mass effects deserve further study.

G. Λ_n schemes

As we discussed at the beginning of this section, if the coefficient of $1/[\beta_0 \ln(Q^2/\Lambda^2)]$ in Eq. (7.52) were independent of Q^2 and had exactly the same n dependence as $\gamma_{\mathrm{NS}}^{0,n}$, then all $\bar{g}^2$ corrections could be absorbed in the parameter Λ, and the higher-order formula would look like the leading-order expression. Conversely, we could say that the leading-order formula assumes that the next-to-leading order corrections have the same n dependence as the $\gamma_{\mathrm{NS}}^{0,n}$. Therefore it is of interest to see whether the next-to-leading order corrections, which we have calculated in this section, exhibit a nontrivial n dependence different from $\gamma_{\mathrm{NS}}^{0,n}$. To this end it is useful to cast Eq. (7.52) into a different form (Bace, 1978; Bardeen *et al.*, 1978). One can, for instance (Bardeen *et al.*, 1978), perform the integral in Eq. (7.10) exactly using γ_{NS}^n and $\beta(g)$ of Eqs. (7.12) and (7.13), respectively, and define an n-dependent Λ_n as follows[35]:

$$\Lambda_n^{(k)} = \bar{\Lambda} \exp[\bar{B}_{k,n}^{\mathrm{NS}}/\gamma_{\mathrm{NS}}^{0,n}] \,. \tag{7.63}$$

This leads to

$$M_k^{\mathrm{NS}}(n,Q^2) = \delta_{\mathrm{NS}}^k \bar{A}_n^{\mathrm{NS}} \left[1 + \frac{\beta_1}{\beta_0^2 \ln(Q^2/\Lambda_n^{(k)2})}\right]^{\gamma^{(1),n}/2\beta_1 - d_{\mathrm{NS}}^n} \times \left[\frac{\bar{g}_n^2}{16\pi^2}\right]^{d_{\mathrm{NS}}^n}, \tag{7.64}$$

where $\bar{g}_n^2$ satisfies the following equation:

$$\frac{16\pi^2}{\beta_0 \bar{g}_n^2} - \frac{\beta_1}{\beta_0^2} \ln\left(\frac{16\pi^2}{\beta_0 \bar{g}_n^2} + \frac{\beta_1}{\beta_0^2}\right) = \ln \frac{Q^2}{\Lambda_n^{(k)2}} \,. \tag{7.65}$$

It turns out that the factor involving $\gamma_{\mathrm{NS}}^{(1),n}$ in Eq. (7.64) is always very near unity in the region of interest, hence Eq. (7.64) has essentially the same form as the leading order Eq. (7.60). Therefore the difference between the leading-order and the higher-order corrections, so far as the n dependence is concerned, resides almost entirely in the scale $\Lambda_n^{(k)}$. Notice that if $\bar{B}_{k,n}^{\mathrm{NS}}$ were proportional to $\gamma_{\mathrm{NS}}^{0,n}$, $\Lambda_n^{(k)}$ would be independent of n. There is one weak point in the Λ_n scheme discussed above. Although $M_k^{\mathrm{NS}}(n,Q^2)$ is renormalization-prescription independent, Λ_n as defined in Eq. (7.63) depends on the renormalization scheme through $\bar{B}_{k,n}^{\mathrm{NS}}$. Similarly, the third factor in Eq. (7.64) depends on the renormalization scheme through $\gamma_{\mathrm{NS}}^{(1),n}$. Fortunately, in 't Hooft's scheme the $\bar{B}_{k,n}^{\mathrm{NS}}$ give the dominant contribution to the higher-order corrections and the third factor in Eq. (7.64) is close to unity. Therefore, in the 't Hooft's scheme discussed here, the $\Lambda_n^{(k)}$ as defined in Eq. (7.63) is a useful quantity for testing the n dependence of the higher-order corrections. It should be kept in mind, however, that one can find renormalization schemes in which the main n dependence of higher-order corrections resides in $\gamma_{\mathrm{NS}}^{(1),n}$; in this case $\Lambda_n^{(k)}$ of Eq. (7.63) would be useless for testing the higher-order corrections.

In order to compare the n dependence of $\Lambda_n^{(k)}$ with the data, one fits $M_k^{\mathrm{NS}}(n,Q^2)$ as given by Eq. (7.64) to the data for each n separately and extracts in this way the experimental values for Λ_n. It turns out (Bardeen *et al.*, 1978) that for $n<5$ the n dependence predicted by formula (7.63) is in fair agreement with the BEBC data for F_3 (Bosetti *et al.*, 1978). For higher values of n the BEBC data do not agree with Eq. (7.63). Recently the analysis in question has been repeated by (Duke and Roberts, 1979a) who also took into account the CDHS data for F_3 (de Groot *et al.*, 1979a) and Fermilab (Gordon *et al.*, 1978) and SLAC data for F_2^{p-n}. A similar analysis has been carried out in a slightly different way by Anderson *et al.* (1979). It follows from the analysis of Duke and Roberts (see Fig. 29)[36] that there is remarkable agreement of formula (7.63) with the data for F_2^{p-n} and agreement for low n with the CDHS data for F_3. We may conclude that there *are* indications in the data for the n dependence of Λ_n as predicted by QCD.

A comparison of the higher-order prediction (7.52) and the leading-order prediction (7.60) can be done in a simpler way than discussed above, at the price of introducing a weak Q^2 dependence into Λ_n. The method discussed below is very similar to that proposed by Bace (1978).

Equation (7.52) can be written as follows:

$$M_k^{\mathrm{NS}}(n,Q^2) = \delta_{\mathrm{NS}}^k \bar{A}_n^{\mathrm{NS}} \left[\ln \frac{Q^2}{\Lambda_n^{(k)2}(Q^2)}\right]^{-d_{\mathrm{NS}}^n}, \tag{7.66}$$

where

$$\Lambda_n^{(k)}(Q^2) = \bar{\Lambda} \exp\left[\frac{\bar{R}_{k,n}^{\mathrm{NS}}(Q^2)}{\gamma_{\mathrm{NS}}^{0,n}}\right]. \tag{7.67}$$

Now the difference between the leading-order and higher-order corrections resides *totally* in $\Lambda_{(n)}^{(K)}(Q^2)$. Next using Eqs. (7.17) and (7.21) we can write

$$\Lambda_n^{(k)}(Q^2) = \bar{\Lambda} Z_n(Q^2, \bar{\Lambda}) \,, \tag{7.68}$$

where

$$Z_n(Q^2,\bar{\Lambda}) = \left[\ln \frac{Q^2}{\bar{\Lambda}^2}\right]^{-\beta_1/2\beta_0^2} \exp\left[\frac{\gamma_{\mathrm{NS}}^{(1),n}}{2\beta_0 \gamma_{\mathrm{NS}}^{0,n}} - \frac{\beta_1}{2\beta_0^2}\right] \exp\left[\frac{\bar{B}_{k,n}^{\mathrm{NS}}}{\gamma_{\mathrm{NS}}^{0,n}}\right]. \tag{7.69}$$

Notice that the first factor on the rhs of Eq. (7.69) represents the Q^2 evolution of $\Lambda_n^{(k)}(Q^2)$. The second factor,

[35]The parameters $\bar{B}_{k,n}^{\mathrm{NS}}$ correspond to the $\overline{\mathrm{MS}}$ scheme of Eq. (7.58) and are obtained from Eq. (7.42) by dropping there $(\ln 4\pi - \gamma_E)$ terms. Notice that the n dependence of $\Lambda_n^{(k)}$ is independent of the definition of $\bar{g}^2(Q^2)$.

[36]Similar conclusions have been reached by Anderson *et al.* (1979).

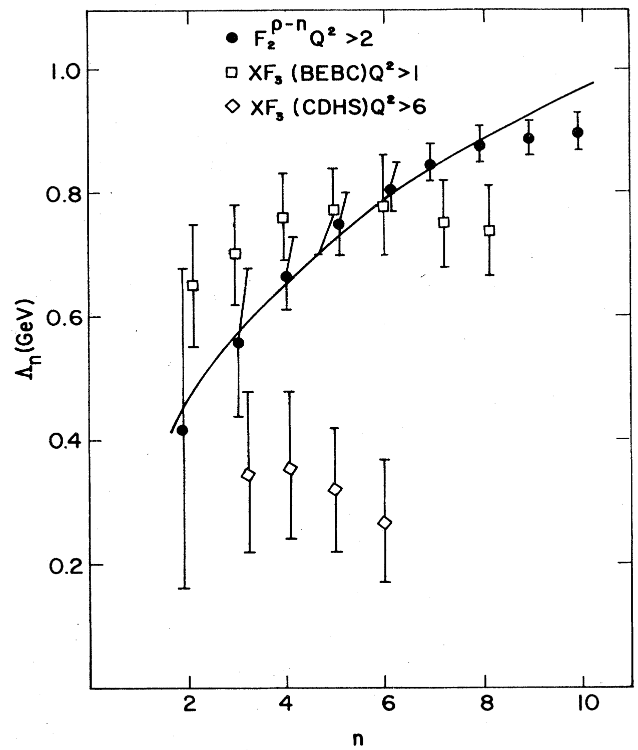


FIG. 29. Experimental Λ_n values obtained by Duke and Roberts (1979a) using the data of BEBC (open box), CDHS (open diamond), and the entire SLAC data. The solid curve corresponds to $\Lambda_n^{(2)}$. Strictly speaking the data for F_3 should be compared with $\Lambda_n^{(3)}$ (see Eq. 7.63). For a discussion see Para and Sachrajda (1979).

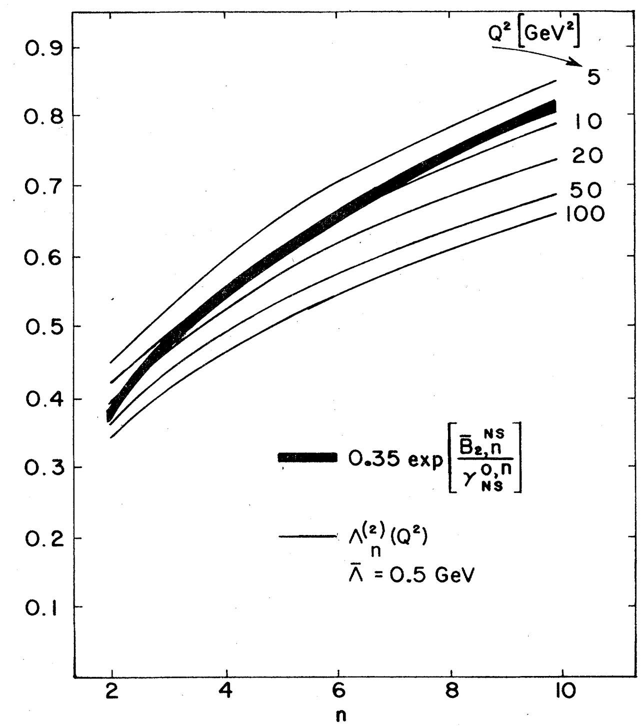


FIG. 30. The effective $\Lambda_n^{(2)}(Q^2)$ (——) as defined by Eqs. (7.68) and (7.69) as functions of n for various values of Q^2 and $\bar{\Lambda}=0.5$ GeV. For comparison the $\Lambda_n^{(2)}$ as defined by Eq. (7.63) is plotted as functions of n for $\bar{\Lambda}=0.35$ GeV.

on the other hand, introduces additional n dependence as compared with the $\Lambda_n^{(k)}$ scheme of Eq. (7.63). Since this additional n dependence (a 15% decrease from $n=2$ to 10) is very weak, the n dependence of $\Lambda_n^{(k)}(Q^2)$ is essentially the same as that of $\Lambda_n^{(k)}$, which changes roughly by a factor 2 over the range from $n=2$ to 10. The Q^2 dependence of the first factor in Eq. (7.69) is such that for $f=4$ and $\bar{\Lambda}=0.3$ and $\bar{\Lambda}=0.5$, at $Q^2=100$ GeV^2, $Z_n(Q^2,\bar{\Lambda})$ is suppressed by the factors 1.23 and 1.29, respectively.

In summary, the $\Lambda_n^{(n)}(Q^2)$ scheme as defined by Eq. (7.66) is (except for the overall factor) quite similar to the $\Lambda_n^{(k)}$ scheme of Eq. (7.64), but very accurate experiments should detect the Q^2 dependence as predicted by Eq. (7.69). In Fig. 30 we have shown $\Lambda_n^{(2)}(Q^2)$ for $\bar{\Lambda}=0.5$ and $f=4$ as functions of n and Q^2. For comparison we plot the last factor in Eq. (7.70) which represents the $\Lambda_n^{(k)}$ scheme. It should be remarked that $\Lambda_n^{(k)}(Q^2)$ in Eq. (7.67) is renormalization-prescription *independent*.

Finally we would like to remark that Anderson *et al.* (1979) have extracted $\Lambda_n^{(2)}$ using the formula (7.66) and neglecting the Q^2 dependence of $\Lambda_n^{(2)}(Q^2)$. Their results agree very well with Eq. (7.63). In order to test the Q^2 dependence of $\Lambda_n^{(2)}(Q^2)$ as given by (7.67) one should repeat Anderson's analysis in various ranges of Q^2, e.g., 2–5, 5–10, 10–30 GeV^2, etc. The prediction of the theory is that for a fixed n value a slow decrease of $\Lambda_n^{(2)}$ with increasing Q^2 should be observed. The present data are, however, not accurate enough to detect this Q^2 dependence. Notice that the Q^2 dependence in question is entirely due to the two-loop contributions to the β function.

H. Other definitions of $\bar{g}^2(Q^2)$

In the previous subsections we have discussed the MS and $\overline{\text{MS}}$ schemes for the effective coupling constant. The corresponding parameters $R_{2,n}^{\text{NS}}$ and $\bar{R}_{2,n}^{\text{NS}}$ are related to each other by Eq. (7.58) and the numerical values of the quantities of Eq. (7.62) for these two schemes are collected in Table IV. We observe that in the expansion in the inverse powers of logarithms the next-to-the-leading order corrections to $M_2^{\text{NS}}(n,Q^2)$ are smaller in the $\overline{\text{MS}}$ scheme than in the MS scheme. Generally one can introduce other definitions of $\bar{g}^2(Q^2)$ for which the parameters R are related to the corresponding parameters of the MS and $\overline{\text{MS}}$ scheme as follows:

$$\begin{aligned} R_{k,n}^{\text{NS}}\big|_a &= R_{k,n}^{\text{NS}} - \tfrac{1}{2}\gamma_{\text{NS}}^{0,n} a \\ &= \bar{R}_{k,n}^{\text{NS}} - \tfrac{1}{2}\gamma_{\text{NS}}^{0,n}(a-1.95)\,. \end{aligned} \tag{7.70}$$

Here a is a constant, which distinguishes between various schemes and $1.95 = \ln 4\pi - \gamma_E$. In particular Barbieri *et al.* (1979) and Celemaster and Gonsalves (1979) have discussed $\bar{g}^2(Q^2)$ as defined by momentum space subtraction. The $\bar{g}^2(Q^2)$ so defined is gauge dependent but the gauge dependence is very weak. Celemaster and Gonsavles have used the Landau gauge for which $a=3.5$. The case discussed by Barbieri *et al.* corresponds to $a=3.6$. We observe that in both cases the parameters R are smaller than in the MS and $\overline{\text{MS}}$ schemes. One could conclude from this that the next-to-the-leading order corrections in the schemes based on momentum subtraction are smaller than in the MS and $\overline{\text{MS}}$ schemes. However, in order to be able to draw any conclusion one has to determine first from experiment the values of Λ for momentum subtraction schemes. We have performed the following exercise. We took $\bar{\Lambda}=0.50$ GeV extracted from the BEBC data and $\bar{\Lambda}=0.30$ GeV a value more relevant for the CDHS data and calculated the moments $M_2^{\text{NS}}(n,Q^2)$ in the $\overline{\text{MS}}$ scheme for these two cases. Next we found the values of Λ_{MOM} (momentum subtraction) by fitting the moments $M_2^{\text{NS}}(n,Q^2)$ calculated in the momentum subtraction scheme with $a=3.5$ to the moments $M_2^{\text{NS}}(n,Q^2)$ calculated in the $\overline{\text{MS}}$ scheme. For $\bar{\Lambda}=0.50$ and 0.30 GeV the corresponding values of Λ_{MOM} turn out to be $\Lambda_{\text{MOM}}=0.85$ GeV and $\Lambda_{\text{MOM}}=0.55$ GeV. The effective coupling constants in the three schemes considered for $\Lambda=0.40$ GeV, $\bar{\Lambda}=0.50$ GeV, and $\Lambda_{\text{MOM}}=0.85$ GeV are plotted in Fig. 31. We observe the following inequalities, $\bar{g}^2(Q^2)|_{\text{MOM}} > \bar{g}^2(Q^2)|_{\overline{\text{MS}}} > \bar{g}^2(Q^2)|_{\text{MS}}$, which correspond to $R_{2,n}^{\text{NS}} > \bar{R}_{2,n}^{\text{NS}} > R_{2,n}^{\text{NS}}|_{\text{MOM}}$. Furthermore we observe that in all cases considered the effective coupling constant is smaller than that given by the leading order expression. The numerical values of the quantities of Eq. (7.62) for the three schemes are shown in Table IV. We conclude that in the expansion in the inverse powers of logarithms the next-to-the-leading order corrections to $M_2^{\text{NS}}(n,Q^2)$ calculated in the momentum subtraction scheme with $a=3.5$ are larger than those in the $\overline{\text{MS}}$ scheme but smaller than in the MS scheme.

In spite of this analysis we cannot say which of the schemes considered leads to the best convergence of the perturbative series. In order to be able to answer this question one would have to calculate higher orders in $\bar{g}^2(Q^2)$ not included in the analysis. Needless to say the n dependence of $\Lambda_n(Q^2)$ or Λ_n discussed in the previous subsection is independent of the definition of $\bar{g}^2(Q^2)$.

VIII. HIGHER-ORDER ASYMPTOTIC FREEDOM CORRECTIONS TO DEEP-INELASTIC SCATTERING (SINGLET CASE)

A. Preliminaries

In the last section we have discussed next-to-leading asympototic freedom corrections to the moments of the nonsinglet contributions to the deep-inelastic structure functions. In this section we shall extend the analysis to the singlet contributions. Such an analysis requires the calculation of the two-loop anomalous dimension matrix and of the one-loop corrections to the fermion singlet and gluon Wilson coefficient functions. As in the nonsinglet case, one has to take care that all these quantities are calculated in the same renormalization scheme. In the minimal subtraction scheme one-loop corrections to the fermion singlet and gluon Wilson coefficient functions have been calculated by Bardeen, Buras, Duke, and Muta (1978) and by Floratos, Ross, and Sachrajda (1979). The latter authors have also computed the two-loop anomalous dimension matrix. The study of the next-to-the-leading corrections in the singlet sector is complicated by the mixing of gluon and fermion singlet operators. The problem of mixing for the next-to-the-leading order corrections was first solved by Floratos, Ross, and Sachrajda (1979). Here we shall present the equivalent, but slightly simpler, approach of Bardeen and Buras (1979b).[37] Most of our

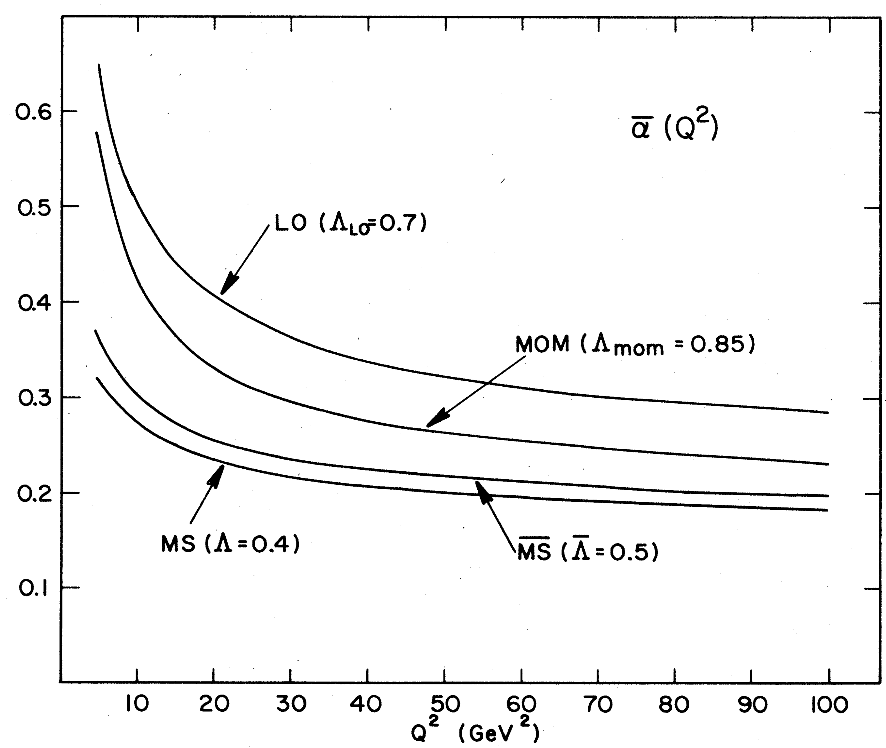


FIG. 31. The effective coupling constants $\bar{\alpha}(Q^2)$ as extracted from the BEBC data for the leading order (L.O.), MS scheme, $\overline{\text{MS}}$ scheme, and momentum subtraction scheme MOM.

[37]The material of Secs. VIII.B and VIII.D has been drawn from collaboration with W. A. Bardeen.

discussion will be rather formal and only at the end of this section shall we present a parton model formulation of asympotic freedom beyond the leading order.

B. Moments of the singlet structure functions

We shall now derive Eqs. (2.101)–(2.117) of Sec. II. We begin with $F_2(x, Q^2)$. In the formal approach of Sec. IV the moments of the singlet part of F_2 are given as follows:

$$M_2^S(n, Q^2) \equiv \int_0^1 dx\, x^{n-2} F_2^s(x, Q^2) = A_n^\psi(\mu^2) C_{2,n}^\psi\left(\frac{Q^2}{\mu^2}, g^2\right) + A_n^G(\mu^2) C_{2,n}^G\left(\frac{Q^2}{\mu^2}, g^2\right), \tag{8.1}$$

with all symbols defined in Sec. IV.A.

The Q^2 dependence of the coefficient functions $C_{2,n}^\psi(Q^2/\mu^2, g^2)$ and $C_{2,n}^G(Q^2/\mu^2, g^2)$ is governed by the renormalization group Eq. (4.20) which has the following solution:

$$\mathbf{C}_{2,n}\left(\frac{Q^2}{\mu^2}, g^2\right) = \left[T_g \exp \int_{\bar{g}(Q^2)}^{g} dg' \frac{\hat{\gamma}^n(g')}{\beta(g')}\right] \mathbf{C}_{2,n}(1, \bar{g}^2), \tag{8.2}$$

with $\bar{g}(\mu^2) = g$. As in Sec. IV.C we have introduced here the column vector

$$\mathbf{C}_{2,n}(Q^2/\mu^2, g^2) = \begin{bmatrix} C_{2,n}^\psi(Q^2/\mu^2, g^2) \\ C_{2,n}^G(Q^2/\mu^2, g^2) \end{bmatrix}. \tag{8.3}$$

The 2×2 anomalous dimension matrix $\hat{\gamma}^n(g)$ is shown explicitly in Eq. (4.22). It has the following perturbative expansion

$$\hat{\gamma}^n(g) = \hat{\gamma}^{0,n} \frac{g^2}{16\pi^2} + \hat{\gamma}^{(1),n} \frac{g^4}{(16\pi^2)^2} + \cdots. \tag{8.4}$$

We shall now express the T ordered exponential in Eq. (8.2) in terms of the coefficients in the expansions (7.13) and (8.4). Denote first

$$\hat{W}(g, \bar{g}) = T_g \exp \int_{\bar{g}(Q^2)}^{g} dg' \frac{\hat{\gamma}^n(g')}{\beta(g')}, \tag{8.5}$$

and write it as

$$\hat{W}(g, \bar{g}) = \hat{V}(g)\left\{\exp\left(\frac{\hat{\gamma}^{0,n}}{\beta_0} \ln \frac{\bar{g}}{g}\right)\right\} \hat{V}^{-1}(\bar{g}), \tag{8.6}$$

where $\hat{V}(g)$ is a 2×2 matrix which we shall now find. To simplify the notation we shall drop the index n in Eqs. (8.7)–(8.11).

Differentiating both sides of Eqs. (8.5) and (8.6) with respect to g we obtain, respectively,

$$\partial_g \hat{W}(g, \bar{g}) = \frac{\hat{\gamma}(g)}{\beta(g)} \hat{V}(g) \exp\left[\frac{\hat{\gamma}^0}{\beta_0} \ln \frac{\bar{g}}{g}\right] \hat{V}^{-1}(\bar{g}) \tag{8.7}$$

and

$$\partial_g \hat{W}(g, \bar{g}) = \left[\partial_g \hat{V}(g) - \hat{V}(g) \frac{\hat{\gamma}^0}{\beta_0} \frac{1}{g}\right] \exp\left[\frac{\hat{\gamma}^0}{\beta_0} \ln \frac{\bar{g}}{g}\right] \hat{V}^{-1}(\bar{g}). \tag{8.8}$$

Equating the rhs of these two equations we obtain after some manipulations the following differential equation for $\hat{V}(g)$:

$$\partial_g \hat{V}(g) + \frac{1}{g}\left[\frac{\hat{\gamma}^0}{\beta_0}, \hat{V}(g)\right] = \left(\frac{\hat{\gamma}(g)}{\beta(g)} + \frac{\hat{\gamma}^0}{\beta_0 g}\right) \hat{V}(g). \tag{8.9}$$

Writing next

$$\hat{V}(g) = \hat{1} + (g^2/16\pi^2)\hat{V}_2, \tag{8.10}$$

where $\hat{1}$ is a unit matrix, we obtain the following algebraic equation for $\hat{V}_2$:

$$2\hat{V}_2 + \left[\frac{\hat{\gamma}^0}{\beta_0}, \hat{V}_2\right] = -\frac{\hat{\gamma}^{(1)}}{\beta_0} + \frac{\hat{\gamma}^0 \beta_1}{\beta_0^2}. \tag{8.11}$$

In order to solve Eq. (8.11) it is useful to choose the basis in which $\hat{\gamma}^{0,n}$ is diagonal. As in Sec. IV.C we introduce the matrix U which diagonalizes $\hat{\gamma}^{0,n}$ by

$$\hat{U}^{-1}\hat{\gamma}^{0,n}\hat{U} = \begin{bmatrix} \lambda_-^n & 0 \\ 0 & \lambda_+^n \end{bmatrix}. \tag{8.12}$$

Explicit expression for a $\hat{U}$ matrix which does this job is given in Eq. (4.30).

In the basis in which $\hat{\gamma}^{0,n}$ is diagonal, the matrices $\hat{\gamma}^{(1),n}$ and $\hat{V}_2$ are given as follows:

$$\hat{U}^{-1}\hat{\gamma}^{(1),n}\hat{U} = \begin{bmatrix} \gamma_{--}^{(1),n} & \gamma_{-+}^{(1),n} \\ \gamma_{+-}^{(1),n} & \gamma_{++}^{(1),n} \end{bmatrix} \tag{8.13}$$

and

$$\hat{U}^{-1}\hat{V}_2\hat{U} = \begin{bmatrix} V_2^{--} & V_2^{-+} \\ V_2^{+-} & V_2^{++} \end{bmatrix}. \tag{8.14}$$

Diagonalizing Eq. (8.11) and using Eqs. (8.12)–(8.14) we obtain

$$V_2^{--} = -\frac{\gamma_{--}^{(1),n}}{2\beta_0} + \frac{\lambda_-^n \beta_1}{2\beta_0^2}, \tag{8.15}$$

$$V_2^{++} = -\frac{\gamma_{++}^{(1),n}}{2\beta_0} + \frac{\lambda_+^n \beta_1}{2\beta_0^2}, \tag{8.16}$$

$$V_2^{-+} = -\frac{\gamma_{-+}^{(1),n}}{2\beta_0 + \lambda_-^n - \lambda_+^n}, \tag{8.17}$$

$$V_2^{+-} = -\frac{\gamma_{+-}^{(1),n}}{2\beta_0 + \lambda_+^n - \lambda_-^n}. \tag{8.18}$$

Explicit expressions for $\gamma_{--}^{(1),n}$, $\gamma_{++}^{(1),n}$, $\gamma_{-+}^{(1),n}$, and $\gamma_{+-}^{(1),n}$ are given in Eqs. (2.109) to (2.112). As discussed in Sec. IV.C the $\hat{U}$ matrix which diagonalizes $\hat{\gamma}^{0,n}$ is not uniquely defined by Eq. (8.12), and any other matrix related to $\hat{U}$ by Eq. (4.29) will also satisfy Eq. (8.12). It follows from this that only the diagonal elements $\gamma_{--}^{(1),n}$, $\gamma_{++}^{(1),n}$, and the product $\gamma_{+-}^{(1),n}\cdot\gamma_{-+}^{(1),n}$ are independent of our choice of $\hat{U}$. Equations (2.109)–(2.112) correspond to $\hat{U}$ given by Eq. (4.30).

To proceed further we introduce a column vector

$$\begin{bmatrix} C_{2,n}^-(Q^2/\mu^2, g^2) \\ C_{2,n}^+(Q^2/\mu^2, g^2) \end{bmatrix} = \hat{U}^{-1}\mathbf{C}_{2,n}(Q^2/\mu^2, g^2). \tag{8.19}$$

The components $C_{2,n}^\mp(Q^2/\mu^2, g^2)$ are easily obtained by first writing

$$\hat{U}^{-1}\mathbf{C}_{2,n}(Q^2/\mu^2, g^2) = \hat{U}^{-1}\hat{W}(g, \bar{g})\hat{U}\hat{U}^{-1}\mathbf{C}_{2,n}(1, \bar{g}^2), \tag{8.20}$$

and then using Eqs. (8.6) and (8.12)–(8.14). The result is

$$\begin{bmatrix} C^-_{2,n}(Q^2/\mu^2, g^2) \\ C^+_{2,n}(Q^2/\mu^2, g^2)\end{bmatrix} = [\hat{1} + (g^2/16\pi^2)\hat{U}^{-1}\hat{V}_2\hat{U}]\begin{bmatrix}\tilde{C}^-_{2,n}(Q^2/\mu^2, g^2) \\ \tilde{C}^+_{2,n}(Q^2/\mu^2, g^2)\end{bmatrix}, \tag{8.21}$$

where $\tilde{C}^{\pm}_{2,n}(Q^2/\mu^2, g^2)$ are given as follows:

$$\tilde{C}^-_{2,n}\left(\frac{Q^2}{\mu^2}, g^2\right) = C^-_{2,n}(1, \bar{g}^2)\left(1 - \frac{\bar{g}^2(Q^2)}{16\pi^2}V^{--}_2\right)\left[\frac{\bar{g}^2}{g^2}\right]^{d^n_-} - C^+_{2,n}(1, \bar{g}^2)\left(\frac{\bar{g}^2(Q^2)}{16\pi^2}\right)V^{-+}_2\left[\frac{\bar{g}^2}{g^2}\right]^{d^n_-} \tag{8.22}$$

and

$$\tilde{C}^+_{2,n}\left(\frac{Q^2}{\mu^2}, g^2\right) = C^+_{2,n}(1, \bar{g}^2)\left(1 - \frac{\bar{g}^2}{16\pi^2}V^{++}_2\right)\left[\frac{\bar{g}^2}{g^2}\right]^{d^n_+} - C^-_{2,n}(1, \bar{g}^2)\left(\frac{\bar{g}^2}{16\pi^2}\right)V^{+-}_2\left[\frac{\bar{g}^2}{g^2}\right]^{d^n_+}. \tag{8.23}$$

Here $C^{\pm}_{2,n}(1, \bar{g}^2)$ are obtained by putting $Q^2 = \mu^2$ in Eq. (8.19). Furthermore

$$d^n_{\pm} = \lambda^n_{\pm}/2\beta_0. \tag{8.24}$$

We next expand $C^{\psi}_{2,n}(1, \bar{g}^2)$ and $C^G_{2,n}(1, \bar{g}^2)$ in a power series in $\bar{g}^2$ as follows:

$$C^{\psi}_{2,n}(1, \bar{g}^2) = \delta^{(2)}_{\psi}[1 + (\bar{g}^2/16\pi^2)B^{\psi}_{2,n}] \tag{8.25}$$

and

$$C^G_{2,n}(1, \bar{g}^2) = \delta^{(2)}_{\psi}(\bar{g}^2/16\pi^2)B^G_{2,n}, \tag{8.26}$$

where $\delta^{(2)}_{\psi}$ depend on weak and electromagnetic charges. Defining also the perturbative expansion for $C^{\mp}_{2,n}(1, \bar{g}^2)$ by

$$C^{\mp}_{2,n}(1, \bar{g}^2) = C^{\mp,0}_{2,n}[1 + \bar{g}^2/16\pi^2)B^{\mp}_{2,n}], \tag{8.27}$$

we obtain from Eqs. (8.19), (8.25), (8.26), and (4.31)

$$C^{-,0}_{2,n} = -C^{+,0}_{2,n} = \delta^{(2)}_{\psi} \tag{8.28}$$

and the formulas (2.107) and (2.108) for $B^{\mp}_{2,n}$. Equation (8.27) is the generalization of the leading-order expression (4.37). With all the formulas above at hand, we could now perform matrix multiplication in Eq. (8.21), and we would reproduce the Eq. (2.13) in the paper by Floratos, Ross, and Sachrajda (1979). In order to obtain slightly simpler equations we proceed in a different way.

We first write Eq. (8.1) as

$$M^s_2(n, Q^2) = \mathbf{A}_n(\mu^2)\mathbf{C}_{2,n}(Q^2/\mu^2, g^2), \tag{8.29}$$

where $\mathbf{A}_n(\mu^2)$ is a two-component row vector given by

$$\mathbf{A}_n(\mu^2) = [A^{\psi}_n(\mu^2), A^G_n(\mu^2)]. \tag{8.30}$$

Defining next

$$[\tilde{A}^-(\mu^2), \tilde{A}^+(\mu^2)] = \mathbf{A}_n(\mu^2)\hat{U}[\hat{1} + (g^2/16\pi^2)\hat{U}^{-1}\hat{V}_2\hat{U}], \tag{8.31}$$

we obtain

$$M^s_2(n, Q^2) = \tilde{A}^-_n(\mu^2)\tilde{C}^-_{2,n}(Q^2/\mu^2, g^2) + \tilde{A}^+_n(\mu^2)\tilde{C}^+_{2,n}(Q^2/\mu^2, g^2), \tag{8.32}$$

with $\tilde{C}^{\mp}_{2,n}(Q^2/\mu^2, g^2)$ given by Eqs. (8.22) and (8.23).

Next using Eq. (7.19) for $\bar{g}^2(Q^2)$, we obtain the final expression for the moments of $F_2(x, Q^2)$

$$M^s_2(n, Q^2) = \delta^{(2)}_{\psi}\bar{A}^-_n\left[1 + \frac{R^-_{2,n}(Q^2)}{\beta_0 \ln(Q^2/\Lambda^2)}\right]\left[\ln\frac{Q^2}{\Lambda^2}\right]^{-d^n_-} + \delta^{(2)}_{\psi}\bar{A}^+_n\left[1 + \frac{R^+_{2,n}(Q^2)}{\beta_0 \ln Q^2/\Lambda^2}\right]\left[\ln\frac{Q^2}{\Lambda^2}\right]^{-d^n_+}, \tag{8.33}$$

where

$$R^{\mp}_{2,n}(Q^2) = R^{\mp}_{2,n} - (\lambda^n_{\mp}\beta_1/2\beta_0^2)\ln\ln(Q^2/\Lambda^2) \tag{8.34}$$

and the constants $R^{\mp}_{2,n}$ are given as follows:

$$R^{\mp}_{2,n} = B^{\mp}_{2,n} + \frac{\gamma^{(1),n}_{\mp\mp}}{2\beta_0} - \frac{\lambda^n_{\mp}\beta_1}{2\beta_0^2} - \frac{\gamma^{(1),n}_{\mp\pm}}{2\beta_0 + \lambda^n_{\mp} - \lambda^n_{\pm}}. \tag{8.35}$$

$\bar{A}^{\mp}_n$ are constants which are related to $\tilde{A}^{\mp}_n(\mu^2)$ by

$$\bar{A}^{\mp}_n = \pm\tilde{A}^{\mp}(\mu^2)[\bar{g}^2(\mu^2)]^{-d^n_{\mp}}. \tag{8.36}$$

$\bar{A}^{\mp}$ are *independent* of μ^2. Equation (8.33) is the generalization of the leading order formula (4.41b). It should be remarked that $\bar{A}^{\mp}_n$ and $R^{\mp}_{2,n}$ are independent of the choice of the matrix U.

The matrix U of Eq. (4.30) differs from that used in the paper of Floratos, Ross, and Sachrajda (1979). However, as the interested reader may ascertain, $\gamma^{(1),n}_{--}$, $\gamma^{(1),n}_{++}$, and $\gamma^{(1),n}_{-+}\gamma^{(1),n}_{+-}$ are the same as in the paper in question.

Let us briefly discuss Eq. (8.33). It is probably the simplest possible representation of the next-to-leading order corrections for the singlet structure functions. We would like to emphasize two important features of Eq. (8.33):

(i) no reference is made to a special value of $Q^2 = Q_0^2$, and (ii) no reference is made to the parton distributions. The property (i) has already been discussed in Sec. IV in connection with the leading-order expressions. It is required by renormalization group equations and incorporating it in $M^s_2(n, Q^2)$ simplifies phenomenological applications. In particular the Step 9 of the procedure of Sec. II can be omitted if Eq. (8.33) is used.

Concerning (ii) we would like to recall that the parton distributions cannot be uniquely defined beyond the leading order of asymptotic freedom (see Sec. VIII.E for details). Many definitions are possible, which differ from each other by next-to-leading order corrections. Therefore the study of higher-order effects on the Q^2 evolution of quark and gluon distributions does not make much sense because the result of such a study is not a prediction of the theory but depends sensitively on one's definition of parton distributions. Furthermore equations for the Q^2 evolution of parton distributions are much more complicated than Eq. (8.33). Therefore we think that the simplest and most straightforward tests of higher-order corrections can be performed directly by means of Eq. (8.33) and (7.20b) without any reference to parton distributions. Still, with a given definition of parton distributions, the parton language may be useful in comparing asymptotic freedom predictions in various processes such as deep-inelastic scattering, the Drell–Yan process, etc. Therefore at the end of this section we shall discuss a parton model formulation of higher-order corrections.

After having discussed some attractive features of Eq. (8.33) we should mention a possible limitation in the use of it. We observe that the last term in the expression for $R^-_{2,n}$ in Eq. (8.35) is singular when $d^n_+ = d^n_- + 1$.

While this singularity does not appear for physical values of n and f, it can lead to anomalously large higher-order corrections to the "$-$" contributions and an apparent breakdown of perturbation theory.

The singularity in $R_{2,n}^-$ is, of course, spurious and must be canceled by other terms in Eq. (8.33). At this stage it should be recalled that $\bar{A}_n^\mp$ are rather complicated functions [see Eqs. (8.31) and (8.36)] of the matrix elements of singlet fermion and gluon operators and of various renormalization group parameters which we lumped together in order to obtain a simple expression and get rid of $\mu^2 = Q_0^2$ dependence. In doing this we have generated singularities in $R_{2,n}^-$ and $\bar{A}_n^+$ for noninteger f and n which cancel each other in the full expression. In fact, on the basis of Eqs. (8.31) and (8.36), $\bar{A}_n^+$ can be written as follows:

$$\bar{A}_n^+ = A_n^+(Q_0^2) + \frac{R_n^{-+}}{\beta_0 \ln(Q_0^2/\Lambda^2)} \bar{A}_n^- \left[\ln\frac{Q_0^2}{\Lambda^2}\right]^{d_+^n - d_-^n} + O\left[\frac{1}{\ln^2(Q^2/\Lambda^2)}\right], \tag{8.37}$$

where $A_n^+(Q_0^2)$ is nonsingular and the singularity is in R_n^{-+}, which is given as follows:

$$R_n^{-+} = \frac{\gamma_{-+}^{(1),n}}{2\beta_0 + \lambda_-^n - \lambda_+^n}; \tag{8.38}$$

Q_0^2 is an arbitrary scale. Inserting Eq. (8.37) into (8.33) we convince ourselves that the singularity in R_n^- is indeed canceled by that in R_n^{-+}.

From a detailed numerical study of the singularities in $R_{2,n}^-$ (Bardeen and Buras, 1979b) it follows that only in the case of $n=2$ and then only for $f=5$ and 6 is the separation of the singular part as shown in Eq. (8.37) necessary.[38] For all other physical values of n and f, the existence of a nearby singularity does not disturb the validity of the perturbative nature of the corrections and Eq. (8.33) can be safely used.

For completeness, however, we would like to mention that one can derive an equation for $M_2^s(n, Q^2)$ which, although it is slightly more complicated than Eq. (8.33) and involves explicitly Q_0^2, can be easily continued to noninteger values of f and n. This is the Eq. (2.101a), which can be easily derived by performing matrix multiplication in Eq. (8.21), instead of absorbing the first factor on the r.h.s. of Eq. (8.21) into matrix elements of local operators as was done in the present derivation.

So far we have discussed only $F_2^s(x, Q^2)$. The formula for the singlet part of the longitudinal structure function can be derived in a similar way by replacing the Eqs. (8.25) and (8.26) by

$$C_{L,n}^\psi(1, \bar{g}^2) = \delta_\psi^L(\bar{g}^2/16\pi^2) B_{L,n}^\psi \tag{8.39}$$

and

$$C_{L,n}^G(1, \bar{g}^2) = \delta_\psi^L(\bar{g}^2/16\pi^2) B_{L,n}^G. \tag{8.40}$$

The result is

$$M_L^s(n, Q^2) = A_n^- \delta_\psi^L \frac{B_{L,n}^-}{\beta_0 \ln(Q^2/\Lambda^2)} \left[\ln\frac{Q^2}{\Lambda^2}\right]^{-d_-^n} + A_n^+ \delta_\psi^L \frac{B_{L,n}^+}{\beta_0 \ln(Q^2/\Lambda^2)} \left[\ln\frac{Q^2}{\Lambda^2}\right]^{-d_+^n}, \tag{8.41}$$

where $A_n^\mp$ are given by Eq. (4.43) and $B_{L,n}^\mp$ are obtained from formulas (2.107) and (2.108) for $k=L$. Longitudinal structure function is discussed in more detail in Sec. VIII.F.

This completes the derivation of the basic asymptotic freedom formulas for the moments of singlet structure functions with next-to-leading-order corrections taken into account. We shall now discuss the calculations of various parameters which enter the formulas (8.33) and (8.41).

C. Results for $\hat{\gamma}^{(1),n}$, $B_{k,n}^\psi$ and $B_{k,n}^G$

As in the case of nonsinglet contributions, one has to make sure that $\hat{\gamma}^{(1),n}$, $B_{k,n}^\psi$ and $B_{k,n}^G$ are calculated in the same renormalization scheme. Below we give results of calculations of these quantities in the minimal subtraction scheme. A nice feature of this scheme is that $\hat{\gamma}_n^{(1)}$, $B_{k,n}^\psi$, and $B_{k,n}^G$ in this scheme are gauge independent.

1. Two-loop anomalous dimension matrix

The two-loop anomalous dimension matrix, $\hat{\gamma}^{(1),n}$, has been calculated by Floratos, Ross, and Sachrajda (1979). One has to calculate quark and gluon matrix elements of the fermion singlet and gluon operators to order g^4 and the anomalous dimensions of the gluon and fermion field to the same order. The details of this calculation are given in the original paper. Typical diagrams are shown in Figs. 32 and 33. In the whole there are about 100 two-loop diagrams which one has to calculate in order to obtain $\hat{\gamma}^{(1),n}$. The analytic expressions for the elements of $\hat{\gamma}^{(1),n}$ are very long and the authors quote the numerical values of the coefficients of various group theoretical factors. On the basis of this information we have calculated the elements of $\hat{\gamma}^{(1),n}$ which are collected in Table III. We only make a few remarks related to Table III.

(a) The nondiagonal elements $\gamma_{\psi G}^{(1),n}$ and $\gamma_{G\psi}^{(1),n}$ differ by sign from those of Floratos *et al.* as we use the same normalizations of operators as Gross and Wilczek (1974). Notice that, consistent with these normalizations, the

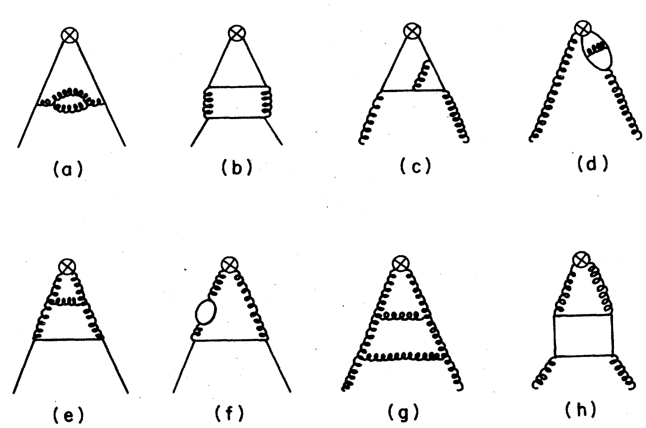


FIG. 32. Examples of the diagrams which enter the calculation of the two-loop anomalous dimension matrix $\gamma_{\psi\psi}^{(1),n}(a,b)$, $\gamma_{\psi G}^{(1),n}(c,d)$, $\gamma_{G\psi}^{(1),n}(e,f)$, and $\gamma_{GG}^{(1),n}(g,h)$.

[38] For $n=2$, 4, 6, 8, and 10 the position of the singularity is at $f=5.583$, 3.788, 1.627, 0.142, and -0.988, respectively. The corresponding residue in $R_{2,n}^-$ is equal to -15.43, 1.36, 0.2, 0.007, and -0.02.

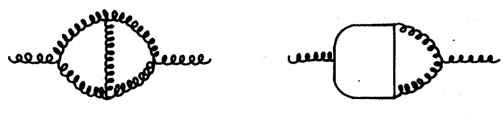


FIG. 33. Examples of the diagrams which enter the calculation of the anomalous dimension of the gluon field in order g^4.

nondiagonal one-loop elements $\gamma_{\psi G}^{0,n}$ and $\gamma_{G\psi}^{0,n}$ as given in Eq. (2.79) also differ by a minus sign from those of Floratos *et al.*

(b) For $n=2$ we have the following relations:

$$\gamma_{\psi\psi}^{(1)}+\gamma_{G\psi}^{(1)}=0\,,\qquad \gamma_{GG}^{(1)}+\gamma_{\psi G}^{(1)}=0\,. \tag{8.42}$$

This is the generalization of Eqs. (5.38) and (5.39) and corresponds to the vanishing of the anomalous dimension of the energy momentum tensor.

(c) $\gamma_{\psi\psi}^{(1),n}$, contrary to the one-loop case, differs for low values of n from $\gamma_{\rm NS}^{(1),n}$ due to the appearance of various diagrams which contribute to $\gamma_{\psi\psi}^{(1)}$ but do not contribute to $\gamma_{\rm NS}^{(1)}$. An example of such a diagram is presented in Fig. 32b. For $n\geq 4$ however $\gamma_{\rm NS}^{(1),n}\approx\gamma_{\psi\psi}^{(1),n}$.

2. $B_{k,n}^{\psi}$ and $B_{k,n}^{G}$ in 't Hooft's scheme

In order to calculate $B_{k,n}^{\psi}$ and $B_{k,n}^{G}$ as defined by Eqs. (8.25), (8.26), (8.39), and (8.40), consider the two forward Compton amplitudes

$$T_{\mu\nu}^{\psi}(Q^2,\nu)=i\int d^4z e^{iqz}\langle\psi;p\,|T[J_\mu(z)J_\nu(0)]\,|\psi;p\rangle \tag{8.43}$$

and

$$T_{\mu\nu}^{G}(Q^2,\nu)=i\int d^4z e^{iqz}\langle G;p\,|T[J_\mu(z)J_\nu(0)]\,|G;p\rangle\,, \tag{8.44}$$

where $\langle\psi;p\,|$ and $\langle G;p\,|$ stand for fermion singlet and gluon states, respectively. As in the calculation of $B_{k,n}^{\rm NS}$ we choose these states to be massless with spacelike momenta $p^2<0$.

The Compton amplitudes above have a Lorentz decomposition as in Eq. (7.25), and employing the operator product expansion we obtain the following generalizations of Eq. (7.26) for each n separately:

$$T_{k,n}^{\psi}(Q^2/p^2,g^2)=C_{k,n}^{\psi}(Q^2/\mu^2,g^2)A_{n\psi}^{\psi}(p^2/\mu^2,g^2)+C_{k,n}^{G}(Q^2/\mu^2,g^2)A_{n\psi}^{G}(p^2/\mu^2,g^2)\quad k=1,2,L\,, \tag{8.45}$$

$$T_{k,n}^{G}(Q^2/p^2,g^2)=C_{k,n}^{\psi}(Q^2/\mu^2,g^2)A_{nG}^{\psi}(p^2/\mu^2,g^2)+C_{k,n}^{G}(Q^2/\mu^2,g^2)A_{nG}^{G}(p^2/\mu^2,g^2)\quad k=1,2,L\,. \tag{8.46}$$

The matrix elements $A_{nj}^{i}(p^2/\mu^2,g^2)$ are defined as follows:

$$\langle p;j\,|O_i^{\mu_1\cdots\mu_n}|p;j\rangle\equiv A_{nj}^{i}(p^2/\mu^2,g^2)p_{\mu_1}\cdots p_{\mu_n}+\text{trace terms}\,. \tag{8.47}$$

We next expand the elements of Eqs. (8.45) and (8.46) in a perturbation series as follows:

$$T_{k,n}^{\psi}(Q^2/\mu^2,g^2)=h_k+(g^2/16\pi^2)[-\tfrac{1}{2}\gamma_{\psi\psi}^{0,n}\ln(Q^2/-p^2)+T_{k,n}^{(2),\psi}] \tag{8.48}$$

$$T_{k,n}^{G}(Q^2/\mu^2,g^2)=(g^2/16\pi^2)[-\tfrac{1}{2}\gamma_{\psi G}^{0,n}\ln(Q^2/-p^2)+T_{k,n}^{(2),G}] \tag{8.49}$$

$$C_{k,n}^{\psi}(Q^2/\mu^2,g^2)=h_k+(g^2/16\pi^2)[-\tfrac{1}{2}\gamma_{\psi\psi}^{0,n}\ln(Q^2/\mu^2)+B_{k,n}^{\psi}] \tag{8.50}$$

$$C_{k,n}^{G}(Q^2/\mu^2,g^2)=(g^2/16\pi^2)[-\tfrac{1}{2}\gamma_{\psi G}^{0,n}\ln(Q^2/\mu^2)+B_{k,n}^{G}] \tag{8.51}$$

and

$$A_{nj}^{i}(p^2/\mu^2,g^2)=\delta_{ij}+(g^2/16\pi^2)[\tfrac{1}{2}\gamma_{ij}^{0,n}\ln(-p^2/\mu^2)+A_{nj}^{(2)i}]\,, \tag{8.52}$$

where $i,j=\psi,G$. Here

$$h_k=\begin{cases}1 & k=1,2\\ 0 & k=L\end{cases}\,, \tag{8.53}$$

and the coefficients of the logarithms are fixed by the renormalization group equations which the quantities appearing on the lhs of Eqs. (8.48)–(8.52) have to satisfy. In order to simplify notation we have dropped the overall factors δ_ψ^k.

Inserting Eqs. (8.48)–(8.52) into (8.45) and (8.46) and comparing the coefficients of g^2 we obtain

$$B_{k,n}^{\psi}=\begin{cases}T_{k,n}^{(2),\psi}-A_{n\psi}^{(2),\psi} & k=1,2\\ T_{L,n}^{(2),\psi} & k=L\end{cases} \tag{8.54}$$

and

$$B_{k,n}^{G}=\begin{cases}T_{k,n}^{(2),G}-A_{nG}^{(2),\psi} & k=1,2\\ T_{L,n}^{(2),G} & k=L\end{cases}\,. \tag{8.55}$$

Since to the order considered $T_{k\psi}^{\psi}$ and $A_{n\psi}^{\psi}$ are equal to the corresponding quantities for nonsinglet operators, Eq. (8.54) is equivalent to (7.31), and we obtain

$$B_{k,n}^{\psi}=B_{k,n}^{\rm NS}\qquad n=1,2,L\,. \tag{8.56}$$

On the other hand, Eq. (8.55) tells us that in order to calculate $B_{2,n}^{G}$ we have to find the forward Compton amplitude for a photon scattering off a gluon and subtract from it the matrix element of the fermion singlet operator between gluon states. In the case of the longitudinal structure function, $B_{L,n}^{G}$, only the forward Compton amplitude has to be calculated. The diagrams which one has to calculate in order to extract $B_{k,n}^{G}$ are shown in Figs. 34 and 10b.[39] We only quote the final results and refer the interested reader to the original papers by Bardeen *et al.* (1978) and Floratos *et al.* (1979) for details. The results for $B_{k,n}^{G}$ are

$$B_{L,n}^{G}=8f/(n+1)(n+2)\,, \tag{8.57}$$

$$B_{2,n}^{G}=2f\left[\frac{4}{n+1}-\frac{4}{n+2}+\frac{1}{n^2}-\frac{n^2+n+2}{n(n+1)(n+2)}\left(1+\sum_{j=1}^{n}\frac{1}{j}\right)\right]+\tfrac{1}{2}\gamma_{\psi G}^{0,n}(\ln 4\pi-\gamma_E)\,. \tag{8.58}$$

[39]Notice that the diagrams of Fig. 10(b) enter also the calculation of $\gamma_{\psi G}^{0,n}$, in which case one is interested only in the coefficient of $\ln(-p^2/\mu^2)$ [see Eq. (8.52)]. This time we are interested in the constant pieces $A_{nG}^{(2)\psi}$.

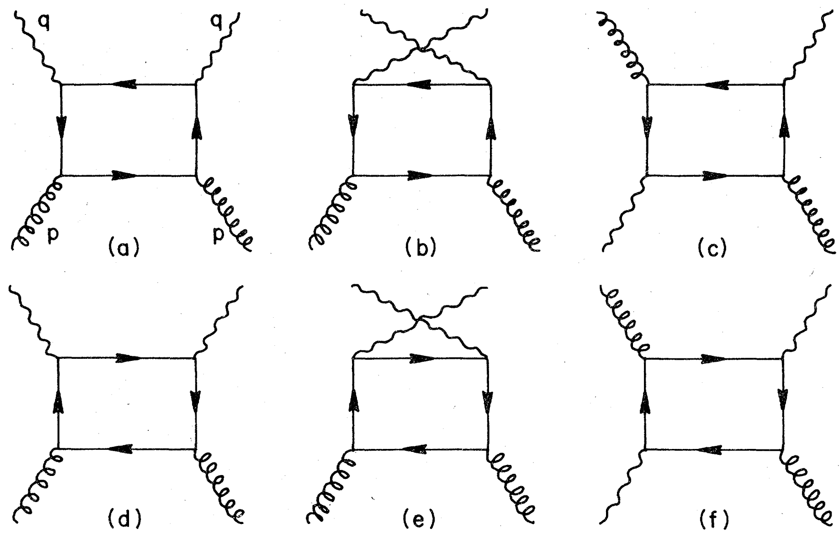


FIG. 34. Diagrams entering the calculation of $T^{(2),G}_{k,n}$ of Eq. (8.55).

Notice the appearance of the terms $(\ln 4\pi - \gamma_E)$ which, as we shall show below, can be absorbed in the redefinition of the scale parameter Λ. $B^G_{L,n}$, which is renormalization prescription independent, has been previously calculated by other authors (Kingsley, 1973; Walsh and Zerwas, 1973; Hinchliffe and Llewellyn-Smith, 1977a). For the calculations of $B^G_{2,n}$ in different renormalization schemes from that considered here we refer the reader to the papers by Kingsley (1973), Witten (1976), Calvo (1976), Llewellyn-Smith and Hinchliffe (1977a), Altarelli, Ellis, and Martinelli (1978), Kubar-André and Paige (1979), Hill and Ross (1979), Sheiman (1979), and Abad and Humpert (1978).

3. Comparison of various calculations

In Sec. VII.D we stated that

(i) $T^{(2),i}_{k,n}$, as defined by Eqs. (7.27), (8.48), and (8.49), are *independent* of the renormalization scheme but are *dependent* on the assumptions about the "quark target" or "gluon target" used to extract the coefficient functions;

(ii) $B^i_{k,n}$, as defined by Eqs. (7.28), (8.50). and (8.51), are *independent* of the target but are *dependent* on the renormalization scheme;

(iii) $A^{(2)i}_{nj}$, as defined in Eqs. (7.29) and (8.52), are *dependent* on both target and renormalization scheme.

Keeping these properties in mind it is not very difficult to compare various calculations existing in the literature performed in different renormalization schemes and corresponding to various different targets. We shall illustrate this by an example (Bardeen *et al.*, 1978).

In the papers by Kingsley (1973), Witten (1976), Hinchliffe and Llewellyn-Smith (1977a), Kubar-André and Paige (1979), and Hill and Ross (1979) the parameters $T^{(2),G}_{2,n}$ have been calculated by considering massive quarks in the fermion loop of Fig. 15 and by putting $p^2=0$ for the external gluons. On the other hand, in the papers by Bardeen *et al.*, (1978), Altarelli *et al.*, (1978), and Floratos *et al.* (1979), $T^{(2),G}_{2,n}$ have been calculated by considering massless quarks and putting $p^2<0$ for the external gluons. The resulting $T^{(2),G}_{2,n}$ are different in the two cases. To check the compatibility of these two calculations we proceed as follows.

We calculate A^{ψ}_{nG} in the minimal subtraction scheme with $p^2=0$ and $m\neq 0$, where m is the quark mass, with the result (we drop the terms which vanish as $m\to 0$)

$$A^{\psi}_{nG}(\mu^2/m^2, g^2)=(g^2/16\pi^2)\{-\tfrac{1}{2}\gamma^{0,n}_{\psi G}[\ln(4\pi\mu^2/m^2)-\gamma_E]\}. \tag{8.59}$$

On the other hand we have in the same renormalization scheme (for any target)

$$C^G_{2,n}(Q^2/\mu^2, g^2)=(g^2/16\pi^2)[-\tfrac{1}{2}\gamma^{0,n}_{\psi G}\ln Q^2/\mu^2+B^G_{2,n}], \tag{8.60}$$

where $B^G_{2,n}$ is given by Eq. (8.58). Inserting (8.59) and (8.60) into (8.46) and replacing there p^2 by $-m^2$ we obtain

$$T^G_{2,n}\left(\frac{Q^2}{m^2}, g\right)=\frac{g^2}{16\pi^2}\left[-\frac{1}{2}\gamma^{0,n}_{\psi G}\left(\ln\frac{Q^2}{m^2}-1-\sum_{j=1}^{n}\frac{1}{j}\right)+\frac{4}{n+1}-\frac{4}{n+2}+\frac{1}{n^2}\right], \tag{8.61}$$

or, equivalently, (in the notation of the papers which use $p^2=0$ and $m^2\neq 0$)

$$F^G_2(x, Q^2)=\frac{g^2}{16\pi^2}2fx\left[(1-2x+2x^2)\ln\frac{Q^2(1-x)}{m^2x}-1+8x(1-x)\right]. \tag{8.62}$$

This result agress with that of Witten (1976) (if one corrects his expression by a factor 4), Kingsley (1973), Kubar-André and Paige (1978), Hill and Ross (1978), and Llewellyn-Smith and Hinchliffe (1977a) (if we replace 6 by 8 in the last term of their expression). A similar exercise can be performed for $T^{\psi}_{2,n}(Q^2/p^2, g^2)$ (Muta, 1979).

4. Discussion of the $\ln 4\pi - \gamma_E$ terms

The quantities $B^G_{2,n}$ and $B^{\psi}_{2,n}$ have terms which include the factors $(\ln 4\pi - \gamma_E)$. It should be possible to absorb these terms by redefining the parameter Λ as in Eqs. (7.54) and (7.57). To check this we insert Eqs. (8.56) and (8.58) into (2.107) and (2.108) and find

$$B_n^{\mp}=\overline{B}_n^{\mp}+\tfrac{1}{2}\lambda_{\mp}^{n}(\ln 4\pi-\gamma_E)\,, \tag{8.63}$$

where $\overline{B}_n^{\mp}$ are free of terms involving $(\ln 4\pi-\gamma_E)$. Inserting formula (8.63) into (8.34) and (8.35) and subsequently into (8.33), we convince ourselves that, in fact, the $(\ln 4\pi-\gamma_E)$ terms can be absorbed by redefining the parameter Λ as in Eqs. (7.54) and (7.57).

Absorbing the $(\ln 4\pi-\gamma_E)$ terms into the parameter Λ corresponds to the $\overline{\mathrm{MS}}$ scheme of Sec. VII.F. For this scheme in analogy with Eq. (7.58) the functions $R_{2,n}^{\mp}(Q^2)$ in Eq. (8.33) are replaced by $\overline{R}_{2,n}^{\mp}(Q^2)$ which are given as follows:

$$\overline{R}_{2,n}^{\mp}(Q^2)=R_{2,n}^{\mp}(Q^2)-\tfrac{1}{2}\lambda_{\mp}^{n}(\ln 4\pi-\gamma_E)\,. \tag{8.64}$$

Equivalently $\overline{R}_{2,n}^{\mp}(Q^2)$ is obtained from $R_{2,n}^{\mp}(Q^2)$ by replacing in (8.35) $B_{2,n}^{\mp}$ by $\overline{B}_{2,n}^{\mp}$ of Eq. (8.63). Similarly the Q^2 independent part of $\overline{R}_{2,n}^{\pm}(Q^2)$ is denoted by $\overline{R}_n^{\pm}$.

D. Numerical estimates

The numerical values of the parameters $\overline{R}_{2,n}^{\pm}$ and $d_{\pm}^{n}$ are given in Table I. First we notice that for sufficiently large values of Q^2 and for $n\geq 4$ the next-to-the-leading order corrections to the "$-$" operator are at least as important as the leading contributions to the "+" operator. This is due to the fact that

$$d_{+}^{n}>d_{-}^{n}+1 \text{ for } n\geq 4\,. \tag{8.65}$$

Therefore for $n\geq 4$ the next-to-the-leading order corrections to $\overline{A}^{-}$ should be treated on the same footing as the leading-order contributions to the $\overline{A}^{+}$ operator. Furthermore for $n>8$ the former contributions dominate over the latter ones. Similarly the next-to-the-leading order corrections to the "+" operator are for $n>4$ and large Q^2 only as important as $1/\ln^2(Q^2/\Lambda^2)$ corrections to the "$-$" operator. We further notice that for $n>4$ $\overline{R}^{-}\approx\overline{R}^{\mathrm{NS}}$ and $d_{-}^{n}\approx d_{\mathrm{NS}}^{n}$, which results from the small mixing between quark and gluon operators for large n and the identification of the "$-$" operator with the singlet quark operator. In addition in the framework of the parton model one expects $\overline{A}_n^{-}$ to be much larger than $\overline{A}_n^{+}$, which is confirmed by the data.[40] Thus one expects that for $n>4$ the singlet structure function will behave essentially the same as the nonsinglet structure function for typical hadronic targets.

In terms of the effective coupling constant

$$\overline{\alpha}(Q^2)\equiv\frac{\overline{g}^2(Q^2)}{4\pi}=\frac{4\pi}{\beta_0\ln(Q^2/\overline{\Lambda}^2)}-4\pi\frac{\beta_1}{\beta_0^3}\frac{\ln\ln(Q^2/\overline{\Lambda}^2)}{\ln^2(Q^2/\overline{\Lambda}^2)}\,, \tag{8.66}$$

the formulas (7.20b) and (8.33) can be written as follows:

$$M_2^{\mathrm{NS}}(n,Q^2)=\delta_{\mathrm{NS}}^{(2)}\overline{A}_n^{\mathrm{NS}}[\overline{\alpha}(Q^2)]^{d_{\mathrm{NS}}^n}I_{2,n}^{\mathrm{NS}}[\overline{\alpha}(Q^2)] \tag{8.67}$$

$$M_2^{S}(n,Q^2)=\delta_{\psi}^{(2)}\overline{A}_n^{-}[\overline{\alpha}(Q^2)]^{d_-^n}I_{2,n}^{-}[\overline{\alpha}(Q^2)]+\delta_{\psi}^{(2)}\overline{A}_n^{+}[\overline{\alpha}(Q^2)]^{d_+^n}I_{2,n}^{+}[\overline{\alpha}(Q^2)]\,, \tag{8.68}$$

where

$$I_{2,n}^{i}(\overline{\alpha})=1+(\overline{\alpha}/4\pi)\overline{R}_{2,n}^{i}\quad i=\mathrm{NS},+,-\,. \tag{8.69}$$

The quantities $I_{2,n}^{i}(\overline{\alpha})$ are plotted in Fig. 35 as functions of $\overline{\alpha}$. The figure is presented mainly for illustration since the actual size of $I_{2,n}^{i}(\overline{\alpha})$ depends on the definition of Λ or, equivalently, of $\overline{\alpha}(Q^2)$. The curves in the figure correspond to the $\overline{\mathrm{MS}}$ scheme for which $0.2<\overline{\alpha}(Q^2)<0.5$ for $2<Q^2<100$ GeV^2 (see Sec. VII).

We note the difference between $\overline{g}^2$ corrections to "NS" and "$-$" components for low values of n which is due to mixing between quark and gluon operators. Furthermore the $\overline{g}^2$ corrections to the "+" contribution are generally larger than to the "$-$" and NS contributions. This, however, does not spoil the perturbative expansion for the full singlet structure function due to the smallness of $\overline{A}_n^{+}$ for large n and due to the large values of d_{+}^{n} as discussed above.

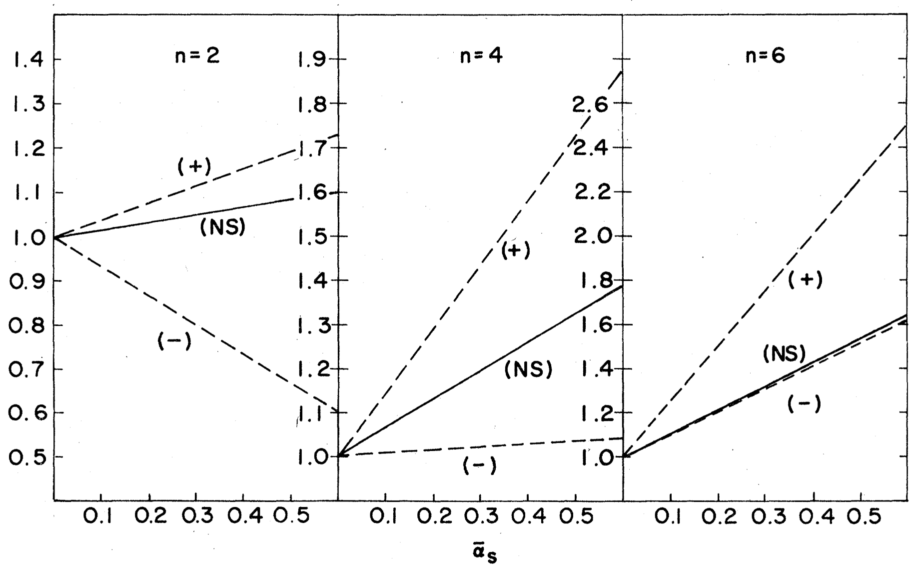


FIG. 35. Size of the explicit second-order corrections $I_{2,n}^{i}(\overline{\alpha})$ in the $\overline{\mathrm{MS}}$ scheme for $f=4$.

[40] See Appendix B.

E. Parton model and higher-order corrections

So far our discussion of higher-order corrections was very formal. Although the most straightforward tests of higher-order corrections can be presumably done by using Eqs. (8.33) and (7.20b), it is sometimes useful to express these equations in terms of parton distributions. As we already remarked before, this cannot be done in a unique way (Kodaira and Uematsu, 1978; Altarelli, Ellis and Martinelli, 1978), and the functional form of the resulting "higher-order parton model formulas" depends on the definition of the parton distributions.

In order to illustrate this point, consider the moments of a nonsinglet structure function which, in the leading order, is expressed through the moments of a nonsinglet quark distribution $\Delta(x,Q^2)$ as follows:

$$M_k^{\rm NS}(n,Q^2)=\delta_{\rm NS}^{(k)}A_n^{\rm NS}(Q_0^2)\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d_{\rm NS}^n}\equiv\delta_{\rm NS}^{(k)}\langle\Delta(Q^2)\rangle_n\,, \tag{8.70}$$

with $\delta_{\rm NS}^{(k)}$ being a charge factor; e.g., $\delta_{\rm NS}^{(k)}=1/6$ for F_2^{ep}.

The Q^2 evolution of $\langle\Delta(Q^2)\rangle_n$ is given by

$$\langle\Delta(Q^2)\rangle_n=\langle\Delta(Q_0^2)\rangle_n\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d_{\rm NS}^n}, \tag{8.71}$$

where $\langle\Delta(Q_0^2)\rangle_n\equiv A_n^{\rm NS}(Q_0^2)$.

If next-to-leading order corrections are taken into account we have first

$$M_k^{\rm NS}(n,Q^2)=\delta_{\rm NS}^{(k)}A_n^{\rm NS}(Q_0^2)\left[1+\frac{[\bar g^2(Q^2)-\bar g^2(Q_0^2)]}{16\pi^2}\left(\frac{\gamma_{\rm NS}^{(1),n}}{2\beta_0}-\frac{\gamma_{\rm NS}^{(0),n}\beta_1}{2\beta_0^2}\right)+\frac{\bar g^2(Q^2)}{16\pi^2}B_{k,n}^{\rm NS}\right]\left[\frac{\bar g^2(Q^2)}{\bar g^2(Q_0^2)}\right]^{d_{\rm NS}^n}. \tag{8.72}$$

Next Eq. (8.72) can be expressed in terms of parton distributions and this can be done in many ways. Here we shall discuss only two examples:

(a) $$M_2^{\rm NS}(n,Q^2)=\delta_{\rm NS}^{(2)}\langle\Delta(Q^2)\rangle_n^{(a)} \tag{8.73a}$$

and

(b) $$M_k^{\rm NS}(n,Q^2)=\delta_{\rm NS}^{(k)}\langle\Delta(Q^2)\rangle_n^{(b)}\{1+[\bar g^2(Q^2)/16\pi^2]B_{k,n}^{\rm NS}\}\equiv\langle\Delta(Q^2)\rangle_n^{(b)}C_{k,n}^{\rm NS}(1,\bar g^2)\,, \tag{8.74}$$

where $\langle\Delta(Q^2)\rangle_n^{(a)}$ is just defined by Eqs. (8.72) and (8.73a) and $\langle\Delta(Q^2)\rangle_n^{(b)}$ in the order considered reads as follows:

$$\langle\Delta(Q^2)\rangle_n^{(b)}=\langle\Delta(Q_0^2)\rangle_n^{(b)}\left[1+\frac{[\bar g^2(Q^2)-\bar g^2(Q_0^2)]}{16\pi^2}\left(\frac{\gamma_{\rm NS}^{(1),n}}{2\beta_0}-\frac{\gamma_{\rm NS}^{(0),n}\beta_1}{2\beta_0^2}\right)\right]\left[\frac{\bar g^2(Q^2)}{\bar g^2(Q_0^2)}\right]^{d_{\rm NS}^n}. \tag{8.75}$$

In the first example the structure function is totally expressed in terms of a quark distribution which has the Q^2 dependence different from that given by the leading-order formula (8.71). It should be remarked, however, that the inclusion of *all* higher-order corrections in the definition of parton distributions can only be done for *one* structure function, e.g., F_2 as in Eq. (8.73a). The reason is that the parameters $B_{k,n}$ depend on the structure function considered, and if the parton distribution is defined by using F_2, the formulas for the remaining structure functions (F_1, F_3) will involve explicit higher-order corrections in addition to the parton distributions in question.

In the second example the term which depends on the structure function has been factored out. Writing Eq. (8.74) as

$$M_k^{\rm NS}(n,Q^2)=\langle\Delta(Q^2)\rangle_n\sigma_{k,n}^{\rm NS}(Q^2)\,, \tag{8.76}$$

and using the convolution theorem of Eqs. (5.14)–(5.16) we obtain

$$F_k^{\rm NS}(x,Q^2)=\int_x^1 dy\,\sigma_k^{\rm NS}(x/y,Q^2)\Delta^{(b)}(y,Q^2)\,. \tag{8.77}$$

The factor $\sigma_k^{\rm NS}(x,Q^2)$ can be interpreted as an elementary cross section for scattering of a current off a quark with an effective Q^2 dependent distribution $\Delta^{(b)}(y,Q^2)$. This interpretation can be extended to other processes such as Drell-Yan, large $p_\perp$ processes, etc., with σ different for different processes. Therefore in this formulation all measurable cross sections are expressed as a convolution of *universal* quark distributions and *process-dependent* (also structure-function-dependent) elementary parton cross sections. This picture is at the basis of the perturbative QCD to be discussed briefly in Sec. IX. It should be remarked, however, that the parton distributions and the elementary parton cross sections defined in this way are separately renormalization-prescription dependent and generally gauge dependent. These renormalization prescription and gauge dependences cancel in the final expression if the same gauge and renormalization scheme are used in the calculations of $\Delta^{(b)}(x,Q^2)$ and $\sigma(x,Q^2)$. Since one can define parton distributions in many ways anyhow, one should not worry about this renormalization prescription dependence of parton distributions discussed here. The only important thing is to *define the parton distributions consistently in the same renormalization scheme for all structure functions and all processes.*

Equations (8.73a) and (8.74) can be generalized to the singlet structure functions. In the case of definition (a) one has (Altarelli, Ellis, and Martinelli, 1978; Floratos, Ross, and Sachrajda, 1979)

$$M_2^s(n,Q^2)=\delta_\psi^{(2)}\langle\Sigma(Q^2)\rangle_n^{(a)} \tag{8.73b}$$

where $\langle\Sigma(Q^2)\rangle_n^{(a)}$ has rather complicated dependence on Q^2. Explicit expressions for $\langle\Sigma(Q^2)\rangle_n^{(a)}$ can be found in Floratos *et al.* (1979). Notice that in this example all higher-order corrections (including those from gluon distribution) have been absorbed in the definition of the singlet quark distribution.

Here we shall discuss in detail only the singlet analog of Eq. (8.74) (Baulieu and Kounnas, 1978; Kodaira and Uematsu, 1978; Ellis, Georgi, Machacek, Politzer, and Ross, 1979). For the discussion of the distributions of Eq. (8.73b) we refer the reader to the papers by

Altarelli, Ellis, and Martinelli (1978) and Floratos, Ross, and Sachrajda (1979). Equation (8.75) is just the nonsinglet formula (2.137). In order to derive Eqs. (2.138) and (2.139) we proceed as follows:

Using Eqs. (8.29) and (8.2) we first obtain

$$M_2^s(n,Q^2) = \mathbf{A}_n(\mu^2)\mathbf{C}_{2,n}(Q^2/\mu^2, g^2)\,,$$
$$= \mathbf{A}_n(\mu^2)\, W(g,\bar{g})\mathbf{C}_{2,n}(1,\bar{g}^2) \tag{8.78}$$

with $\mathbf{A}_n(\mu^2)$ and $\hat{W}(g,\bar{g})$ defined by Eqs. (8.30) and (8.5), respectively. Next in analogy with Eq. (8.74) the moments $\langle\Sigma(Q^2)\rangle_n$ and $\langle G(Q^2)\rangle_n$ are defined as follows (we drop the index b in what follows):

$$[\langle\Sigma(Q^2)\rangle_n, \langle G(Q^2)\rangle_n] \equiv \mathbf{A}_n(\mu^2)\hat{W}(g,\bar{g}) \tag{8.79}$$

and consequently using Eq. (8.3) for $Q^2=\mu^2$ we obtain

$$M_2^s(n,Q^2) = \langle\Sigma(Q^2)\rangle_n C_{2,n}^{\psi}(1,\bar{g}^2) + \langle G(Q^2)\rangle_n C_{2,n}^{G}(1,\bar{g}^2)\,. \tag{8.80}$$

Notice that since $\hat{W}(g,\bar{g})=1$ for $Q^2=\mu^2$, the parton distributions defined by (8.79) are just the matrix elements of various local operators renormalized at $\mu^2=Q^2$. It is now a simple matter to obtain Eqs. (2.138) and (2.139). In order to use some of the results of Sec. VIII.B we write (putting $\mu^2=Q_0^2$) Eq. (8.79) as follows:

$$[\langle\Sigma(Q^2)\rangle_n, \langle G(Q^2)\rangle_n] = [\langle\Sigma(Q_0^2)\rangle_n, \langle G(Q_0^2)\rangle_n] \times \hat{U}\hat{U}^{-1}\hat{W}(g,\bar{g})\hat{U}\hat{U}^{-1} \tag{8.81}$$

and use

$$\hat{U}^{-1}\hat{W}(g,\bar{g})\hat{U} = \left(\hat{1} + \frac{\bar{g}^2(Q_0^2)}{16\pi^2}\hat{U}^{-1}\hat{V}_2\hat{U}\right)\begin{bmatrix} \left[\dfrac{\bar{g}^2(Q^2)}{\bar{g}^2(Q_0^2)}\right]^{d_-^n} & 0 \\ 0 & \left[\dfrac{\bar{g}^2(Q^2)}{\bar{g}^2(Q_0^2)}\right]^{d_+^n}\end{bmatrix} \times\left(1 - \frac{\bar{g}^2(Q^2)}{16\pi^2}\hat{U}^{-1}\hat{V}_2\hat{U}\right), \tag{8.82}$$

where the matrix $\hat{U}^{-1}\hat{V}_2\hat{U}$ is given by Eqs. (8.14)–(8.18). Inserting (8.82) into (8.81) and using Eqs. (4.30) and (4.31) for $\hat{U}$ and $\hat{U}^{-1}$, we are led to the formulas (2.138) and (2.139) which describe the Q^2 evolution of $\langle\Sigma(Q^2)\rangle_n$ and $\langle G(Q^2)\rangle_n$. Numerical values of the parameters which enter these formulas are collected for the $\overline{\mathrm{MS}}$ scheme of Sec. VII in Table II. In addition, $\bar{g}^2/16\pi^2 \sim 0.03$ for $Q^2 \sim 5$ GeV^2 and therefore the next-to-leading order corrections in Eqs. (2.138) and (2.139) are relatively small. Consequently the Q^2 evolution of the parton distributions defined by Eq. (8.79), with $\bar{g}^2$ corrections calculated in the $\overline{\mathrm{MS}}$ scheme, should not be very different from the Q^2 dependence predicted by the leading-order expression of Sec. V. It would be interesting in the future to invert Eqs. (2.138) and (2.139) to obtain $G(x,Q^2)$ and $\Sigma(x,Q^2)$. The large value of the parameter K_{-+}^G for $n=4$ and $f=4$ is related to the previously discussed singularity at $d_+^n = d_-^n + 1$ which appears for $f \approx 3.8$. The singularity in K_{-+}^G is, however, canceled by the factor multiplying K_{-+}^G in Eq. (2.141), and the resulting correction is small. We would like also to remark that due to properties (5.38), (5.39), and (8.42), Eqs. (2.138) and (2.139) satisfy energy momentum conservation.

Although the Q^2 dependence of the quark and gluon distributions defined in the $\overline{\mathrm{MS}}$ scheme does not differ very much from the leading-order predictions, the input distributions for quark and gluon distributions which have to be taken from the data will differ considerably at low Q^2 and large x from those used in the leading-order phenomenology. The reason is that the parameters $\bar{B}_{2,n}^{\mathrm{NS}}$ and $\bar{B}_{2,n}^{\psi}$ are large for large values of n.

Finally, as in the case of the nonsinglet structure functions, we can invert Eq. (8.80) to obtain

$$F_2^s(x,Q^2) = \int_x^1 dy\,[\sigma_2^{\psi}(x/y,Q^2)\Sigma(y,Q^2) + \sigma_2^G(x/y,Q^2)G(x,Q^2)]\,, \tag{8.83}$$

where $\sigma_2^{\psi}(x/y,Q^2)$ is, except for the charge factor, equal to $\sigma_2^{\mathrm{NS}}(x/y,Q^2)$, and $\sigma_2^G(x/y,Q^2)$ can be interpreted as an elementary cross section for scattering of a current (γ, W, Z) off a gluon with an effective Q^2-dependent distribution $G(y,Q^2)$.

So far nobody has done a detailed comparison of the higher-order corrections to the singlet structure functions with the experimental data, but such a comparison will be available in the near future (Field and Ross, 1979; Duke and Roberts, 1979b).[41] See note added in proof. Some applications of these definitions of the parton distributions [Eqs. (8.73a, b), (8.74), and (8.80)] in semi-inclusive processes are discussed by Altarelli *et al.* (1979b) and Buras (1979).

F. More about the Callan-Gross relation

In this section we shall express the longitudinal structure function $F_L(x,Q^2)$ in terms of quark and gluon distributions. Combining Eqs. (7.18a) and (8.41) and utilizing Eq. (4.43), we first obtain the following expression for the moments of $F_L(x,Q^2)$:

$$\begin{aligned} M_L(n,Q^2) &= M_L^{\mathrm{NS}}(n,Q^2) + M_L^s(n,Q^2) \\ &= \delta_{\mathrm{NS}}^L A_n^{\mathrm{NS}}(Q_0^2)\frac{B_{L,n}^{\mathrm{NS}}}{\beta_0\ln(Q^2/\Lambda^2)}\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d_{\mathrm{NS}}^n} \\ &+ \delta_{\psi}^L A_n^-(Q_0^2)\frac{B_{L,n}^-}{\beta_0\ln(Q^2/\Lambda^2)}\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d_-^n} \\ &+ \delta_{\psi}^L A_n^+(Q_0^2)\frac{B_{L,n}^+}{\beta_0\ln(Q^2/\Lambda^2)}\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]^{-d_+^n}, \end{aligned} \tag{8.84}$$

where

$$B_{L,n}^{\mp} = B_{L,n}^{\psi} + [(\lambda_{\mp}^n - \gamma_{\psi\psi}^{0,n})/\gamma_{\psi G}^{0,n}]B_{L,n}^G\,, \tag{8.85}$$

and $B_{L,n}^{\psi}$ and $B_{L,n}^G$ are given by Eqs. (2.98) and (2.120), respectively.

Inserting Eq. (8.85) into (8.84) and using Eqs. (5.29)–(5.32) we obtain

$$M_L(n,Q^2) = \frac{B_{L,n}^{\psi}}{\beta_0\ln(Q^2/\Lambda^2)}M_2(n,Q^2) + \frac{B_{L,n}^G}{\beta_0\ln(Q^2/\Lambda^2)}\delta_{\psi}^{(2)}\langle G(Q^2)\rangle_n \tag{8.86}$$

where the Q^2 dependence of $M_2(n,Q^2)$ and of $\langle G(Q^2)\rangle_n$ is given in the order considered by the *leading-order formulas* of Secs. IV and V. As the reader may convince

[41]While completing this review we received a paper by Anderson *et al.* (1979) where a comparison of higher-order corrections with the measured moments of $F_2^{\mu p}$ and $F_2^{\mu d}$ has been made. The agreement with the data is good with the value of $\Lambda_{\overline{\mathrm{MS}}} = 0.459 \pm 0.111$. This is consistent with the value obtained from the analysis of nonsinglet structure functions [Eq. (7.59)].

himself Eq. (8.86) can also be derived from Eqs. (8.77) and (8.83) by replacing there the indices k or 2 by the index L.

Finally using the convolution theorem of Eqs. (5.14)–(5.16) and the explicit expressions for $B^{\psi}_{L,n}$ and $B^{G}_{L,n}$, as given in Eqs. (7.43), (8.56), and (8.57), we obtain

$$F_L(x,Q^2)=\frac{1}{\beta_0\ln(Q^2/\Lambda^2)}\int_x^1\frac{dy}{y}\frac{x^2}{y^2}\times\left[\frac{16}{3}F_2(y,Q^2)+\delta_{\psi}^{(2)}8f\left(1-\frac{x}{y}\right)yG(y,Q^2)\right]. \tag{8.87}$$

In the case of four flavors, $\delta_{\psi}^{(2)}=5/18$ for ep scattering and $\delta_{\psi}^{(2)}=1$ for $\nu,\bar{\nu}$ scattering. Equation (8.87) with $\delta_{\psi}^{(2)}=5/18$ agrees with Eq. (127) of Altarelli (1978), and for $\delta_{\psi}^{(2)}=1$ with Eq. (24) of Kodaira and Uematsu (1978) if we take into account that the definition of $F_L(x,Q^2)$ in the latter paper differs from our definition by a factor x.

It should be emphasized that the parameter Λ which enters Eq. (8.87) *need not be the same* as that obtained from the phenomenological applications of the leading-order or next-to-leading order expressions for the Q^2 evolution of the structure functions F_1, F_2, and F_3 which do not vanish in the leading order. In fact the value of Λ in Eq. (8.87) cannot be meaningfully determined from experiment unless $1/(\ln^2 Q^2/\Lambda^2)$ corrections to F_L are computed. Therefore in comparing (8.87) with experiment we are free to choose Λ to be different from that extracted from phenomenological applications of asymptotic freedom equations for other structure functions. We do not think this point was realized previously in the literature.

Experimentally the formula (8.87) is consistent with the ep data (Riordan *et al.*, 1975) for $x<0.4$ (Hinchliffe and Llewellyn-Smith, 1977a) but disagrees with these data for large x (De Rujula *et al.*, 1977a). The predictions of the theory for large x lie systematically below the data. This is also confirmed by recent analyses (Bodek *et al.*, 1979 and Mestayer, 1978). The disagreement between theoretical predictions and the data for F_L might not be a problem for QCD, however, and could be due to our neglect of higher-twist operators,[42] nonperturbative effects, etc., which are present in QCD but are difficult to calculate. In particular it has been suggested by Schmidt and Blankenbecler (1977) that the diquark systems in the proton could be responsible for the observed large values of F_L at large x. Recent phenomenological applications of this idea can be found in the paper by Abbott, Berger, Blankenbecler, and Kane (1979). Certainly the longitudinal structure functions deserve further study.

IX. ASYMPTOTIC FREEDOM BEYOND DEEP-INELASTIC SCATTERING

A. Preliminaries

So far our discussion of asymptotic freedom effects concentrated on deep-inelastic scattering. In the past year there has been a lot of progress in understanding the structure of asymptotic freedom in other than deep-inelastic processes. For completeness we shall briefly review here some of the results of these studies. We shall only discuss basic ideas and present results of various calculations without confronting them with the data.

Historically, the study of asymptotic freedom in the inclusive deep-inelastic scattering began in the framework of the formal approach of Sec. IV and only in the last two years have calculations been made in the intuitive approach of Sec. V and, in particular, in the framework of so-called perturbative QCD, which we have not discussed so far. In the case of semi-inclusive processes, as for instance massive μ-pair production, or processes in which hadron momenta are measured in the final state, progress proceeded in the reverse order. Most of the calculations were first done in the framework of perturbative QCD and only recently have studies been made to develop a technique similar to the powerful methods of operator product expansion and renormalization group equations.

As we discussed in detail in this review, the operator product expansion (OPE) allows us to identify systematically the dominant contributions to the moments of the structure functions at large Q^2 and to express them in terms of a sum of products of (perturbatively) calculable coefficient functions and (by present methods) incalculable matrix elements of certain operators taken between hadronic states. The Q^2 dependence of the coefficient functions can then be found by means of renormalization group equations. In other words the OPE assures the *factorization* of nonperturbative pieces (matrix elements of local operators) from perturbatively calculable pieces (coefficient functions). We would like to stress that this factorization is true to all orders in the renormalized coupling constant g^2 and in all logarithms of Q^2 (leading, next-to-the-leading, etc.). Such a proof has been missing for semi-inclusive processes and the strategy (Politzer, 1977a, b; Sachrajda, 1978a, b) has been to calculate these processes in perturbation theory in g^2 and to show that the nonperturbative pieces (mass singularities) can be factored out and absorbed in the (by present methods) uncalculable wave functions of the incoming and the outgoing hadrons. There have been very many papers on this subject, and it is impossible to quote all of them here. An incomplete list of theoretical papers involved with the question of factorization includes the works by Politzer (1977a, b), Sachrajda (1978a, b), Dokshitser, Dyakonov, and Troyan (1978), Llewellyn-Smith (1978b), Mueller (1974, 1978), Libby and Sterman (1978), Kazama and Yao (1978, 1979), Amati, Petronzio, and Veneziano (1978a, b), Ellis, Georgi, Machacek, Politzer, and Ross (1978a, 1979), Gupta and Mueller (1979), Kripfganz (1979), and Frazer and Gunion (1979a, b). References to phenomenological studies of semi-inclusive processes can be found, for instance, in the reviews by Field (1979), Hwa (1978), Berger (1979), and Halzen (1979).

In the theoretical papers above one can find demonstrations of all order proofs of factorization (Ellis *et al.*, 1978a, 1979; Amati *et al.*, 1978a, b; Libby and

[42]See Nanopoulos and Ross (1975). For a recent analysis see Abbott, Atwood, and Barnett (1979).

Sterman, 1978; Mueller, 1978) and a formulation of the whole study of semi-inclusive processes similar to the formal approach of Sec. IV (Gupta and Mueller, 1979).

Before we proceed with our review let us illustrate the idea of factorization in the framework of perturbative QCD with the example of deep-inelastic scattering (Politzer, 1977a).

B. Factorization and perturbative QCD

We consider the diagrams of Fig. 36 contributing to the photon–quark scattering to order g^2,[43] and as in the calculations of Sec. VII we take the incoming (massless) quarks slightly off-shell ($p^2 < 0$). The result is most simply expressed in terms of moments in x (Bjorken variable), and we obtain

$$M_n = A_n^0\{1 + (g^2/16\pi^2)[-\tfrac{1}{2}\gamma_{\rm NS}^{0,n}\ln(Q^2/-p^2) + r_n]\}, \qquad (9.1)$$

where A_n^0 stands for the empty ("bare") blob in Fig. 36 and, as in Eq. (7.27), $r_n = T_{2,n}^{(2),\rm NS}$.

The result (9.1) is not very useful for various reasons:

(i) It is singular for $p^2 \to 0$.

(ii) The constants r_n depend on our assumptions about the target, i.e., the incoming quark. We could, for instance, perform the calculation with $p^2 = 0$ but keep the quark mass $m \neq 0$. In that case we would obtain

$$M_n = A_n^0\{1 + (g^2/16\pi^2)[-\tfrac{1}{2}\gamma_{\rm NS}^{0,n}\ln(Q^2/m^2) + \tilde{r}_n]\}, \qquad (9.2)$$

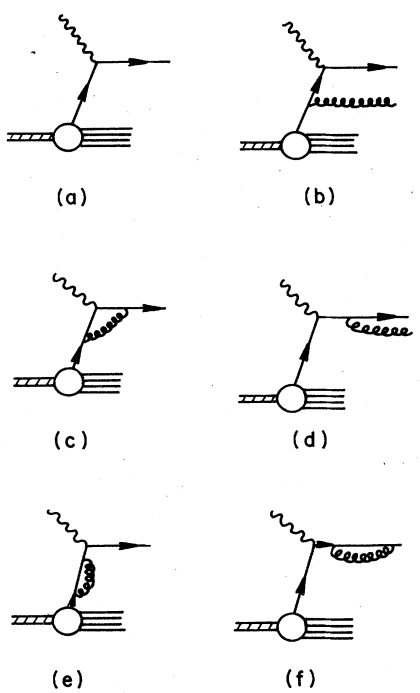


FIG. 36. Diagrams contributing to photon-quark scattering to order g^2. The empty ("bare") blob stands for the "bare" (Q^2-independent) quark distribution in the proton. denotes gluon.

[43]Strictly speaking the diagrams of Fig. 36 represent the contribution of the photon–quark scattering to the photon–proton cross section. In order to calculate the full photon–proton cross section, the contribution of the photon gluon scattering has also to be considered (see Sec. VIII).

where $\tilde{r}_n \neq r_n$. In general r_n or $\tilde{r}_n$ are also gauge dependent. This is exactly the same problem which we discussed in Sec. VIII, but we mention it here again because it also enters the calculations of Drell-Yan and other semi-inclusive processes as we shall see below. Of course we know already how this problem is solved (see Sec. VIII), and we shall leave it for a moment.

The problem (i) is solved by first rewriting Eq. (9.1) as follows (we drop constant terms):

$$M_n = A_n^0\left(1 + \frac{g^2}{16\pi^2}\frac{\gamma_{\rm NS}^{0,n}}{2}\ln\frac{-p^2}{\mu^2}\right)\left(1 - \frac{g^2}{16\pi^2}\frac{\gamma_{\rm NS}^{0,n}}{2}\ln\frac{Q^2}{\mu^2}\right), \qquad (9.3)$$

and then absorbing the p^2 dependent singular factor into A_n^0. We obtain therefore

$$M_n = A_n(\mu^2)[1 - (g^2/16\pi^2)(\gamma_{\rm NS}^{0,n}/2)\ln(Q^2/\mu^2)], \qquad (9.4)$$

where μ is an arbitrary scale which we introduced to protect the Q^2-dependent factor from the singularity at $p^2 = 0$.

In the parton language we can interpret $A_n(\mu^2)$ as the moments of a parton distribution at $Q^2 = \mu^2$. In the formal language this is just the matrix element of Eq. (7.29). The second factor on the rhs of Eq. (9.4) describes how the moments M_n behave for $Q^2 \neq \mu^2$. Notice that this factor is free of any singularity for $p^2 \to 0$. In formal terms it is just the coefficient function calculated to first order in g^2 (keeping only the leading logarithm). This factorization of singular (nonperturbative) terms from well-behaved terms can be proven to all orders in perturbation theory in g^2 (see references above). When all orders in g^2 are summed and in each order only leading logarithms are kept, then the leading-order corrections of asymptotic freedom discussed in the previous sections are obtained. Summing next-to-leading logarithms to all orders in g^2, one obtains the next-to-leading order corrections of asymptotic freedom of Secs. VII and VIII, and so on.

C. Lessons from deep-inelastic scattering

In our presentation of asymptotic freedom effects in processes other than deep-inelastic scattering we shall discuss both the leading and next-to-leading order corrections, and it is useful to summarize the lessons which we gained from the study of deep-inelastic scattering. They are as follows:

(a) In the leading order there is no reason that the numerical values of the scale parameter Λ should be the same for different processes.

(b) If next-to-leading order effects are taken into account and the effective coupling constant defined universally for various processes, then it is justified to use the same value of Λ in different processes {here we tacitly assume that still higher-order corrections $O[g^4(Q^2)]$ are small}.

(c) The definition of parton distributions is not unambiguous beyond the leading order. Therefore in comparing the parton distributions extracted from various processes one has to make sure that the definition of parton distributions is common to all processes.

(d) One has to make sure that all renormalization prescription-dependent parts of a physical expression are calculated in the same scheme, so that at the end a renormalization prescription-independent answer is obtained.

(e) One has to check that the final answer for the coefficient functions does not depend on the (in principle arbitrary) assumptions about the gluon and quark target used in perturbative calculations to extract the coefficient functions.

We shall keep these lessons in mind while discussing various processes. We begin our presentation with e^+e^- annihilation.

D. $e^+e^- \rightarrow$ hadrons

This process has already been studied in the framework of asymptotically free gauge theories a long time ago (Appelquist and Georgi, 1973; Zee, 1973). Therefore we only quote the result and discuss it briefly.

Consider the ratio

$$R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}. \tag{9.5}$$

In the simple parton model and in QCD one obtains for $Q^2 \equiv E^2_{\text{c.m.}} \rightarrow \infty$

$$R_\infty = 3\sum_i e_i^2, \tag{9.6}$$

where 3 is the number of colors, e_i are the charges of the quarks, and the sum runs over the flavors. The fact that R approaches a constant value is a consequence of the lack of renormalization of the conserved electromagnetic current. For finite values of Q^2 there are calculable asymptotic freedom corrections to Eq. (9.6), and the formula for R reads as follows:

$$R = R_\infty\left[1 + \frac{\bar{g}^2(Q^2)}{16\pi^2}b_1 + \frac{\bar{g}^4(Q^2)}{(16\pi^2)^2}b_2 + \cdots\right]. \tag{9.7}$$

The coefficient b_1 has been calculated by Jost and Luttinger (1950), Appelquist and Georgi (1973), and Zee (1973) and is given as follows:

$$b_1 = 4. \tag{9.8}$$

Neglecting for the moment $\bar{g}^4(Q^2)$ corrections and using the leading-order expression for $\bar{g}^2(Q^2)$ (Eq. 2.50) one obtains

$$R = R_\infty\left[1 + \frac{b_1}{\beta_0 \ln(Q^2/\Lambda^2)}\right]. \tag{9.9}$$

b_1 is positive and, therefore, R_∞ will be approached from above. In Fig. 37 R is plotted as a function of $\sqrt{Q^2}$ for $f=4$ and two values of Λ. The contribution of the heavy lepton ($\Delta R = 1$) has been added there. The experimental values of R range at $\sqrt{Q^2} \approx 5$ GeV from 4.5 to 5.0 (see review by Feldman, 1979). The curves in Fig. 37 are shown only to illustrate the size of the second term in Eq. (9.9). A careful comparison of Eq. (9.9) with the data involves smearing over the resonances and inclusion of threshold effects. We refer the interested reader to the papers by Poggio, Quinn, and Weinberg (1976), Moorhouse, Pennington, and Ross (1977), Barbieri and Gatto (1977), and Shankar (1977) for details. Recalling lesson (a) there is no reason why the value for Λ extracted from the data on the basis of Eq. (9.9) should be the same as that obtained from the leading-order analysis of deep-inelastic scattering. A meaningful comparison of QCD effects in e^+e^- annihilation with those expected in deep-inelastic scattering can therefore only be made once the $\bar{g}^4$ corrections are taken into account as in Eq. (9.7). Using Eq. (2.88) for $\bar{g}^2(Q^2)$ calculated to two loops, we obtain from (9.7)

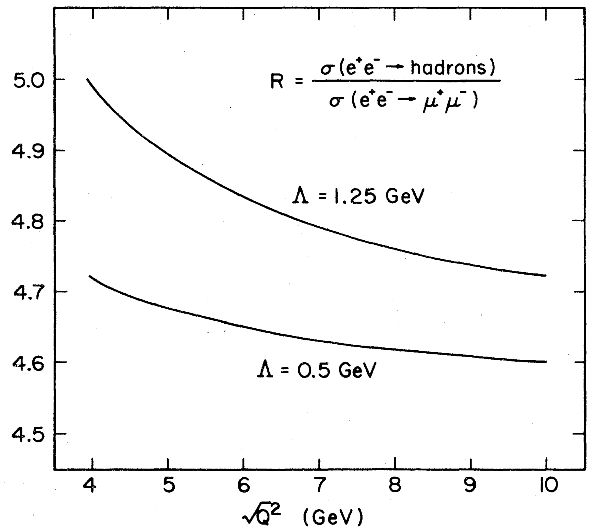


FIG. 37. The ratio R as given by Eq. (9.9) as a function of $\sqrt{Q^2}$ for two values of Λ. The heavy lepton contribution ($\Delta R = 1$) has been added. The parton model prediction is $R = 4.3$.

$$R = R_\infty\left[1 + \frac{b_1}{\beta_0 \ln Q^2/\Lambda^2} - b_1\frac{\beta_1}{\beta_0^3}\frac{\ln\ln(Q^2/\Lambda^2)}{\ln^2(Q^2/\Lambda^2)} + \frac{b_2}{\beta_0^2\ln^2(Q^2/\Lambda^2)} + \cdots\right]. \tag{9.10}$$

Of course the exact value for b_2 in the expansion in Eq. (9.10) depends on the definition of $\bar{g}^2(Q^2)$ or, equivalently, on the definition of Λ. Rescaling Λ to Λ' with

$$\Lambda = \kappa\Lambda', \tag{9.11}$$

changes the last term in Eq. (9.10) to

$$\frac{b_2 + \beta_0 b_1 \ln\kappa^2}{\beta_0^2 \ln^2(Q^2/\Lambda'^2)}, \tag{9.12}$$

with Λ replaced by Λ' in the first two terms. The calculation of b_2 has been recently done by Dine and Sapirstein (1979), and Chetyrkin, Kataev and Trachov (1979) with the result $b_2 = 89.3$, 24.3, and -27.2 for MS, $\overline{\text{MS}}$, and MOM schemes, respectively. Since $\bar{g}^2/16\pi^2$ is 0 (0.03) for $Q^2 \approx 30$ GeV^2 we observe that $\bar{g}^4$ corrections to R are relatively small. These results are being now checked by Ross, Terrano and Wolfram (1979) and Celemaster and Gonsalves (1979).

E. Photon-photon collisions

It is well known that photon–photon inelastic collisions in e^+e^- storage rings become an increasingly important source of hadrons as the center-of-mass energy is raised (Brodsky, Kinoshita, and Terazawa, 1971; Terazawa, 1973; Budnev *et al.*, 1975). Whereas the e^+e^- annihilation cross section decreases quadratically with energy, the cross section for $e^+e^- \rightarrow e^+e^- +$ hadrons increases logarithmically with energy. The dominant contribution to the latter cross section arises from the annihilation of two nearly on-shell photons emitted at small angles to the beam. Here we shall study the case in which one of the virtual photons is very far off shell (large Q^2) and the other one is close to the mass shell (small p^2) as shown in Fig. 38.

The subprocess

$$\gamma+\gamma \rightarrow \text{hadrons}, \tag{9.13}$$

can be viewed as deep-inelastic scattering on a photon target. The corresponding virtual Compton amplitude is shown in Fig. 38(b), and as in the standard deep-inelastic scattering one can introduce structure functions as F_2^γ, this time photon structure functions. In the early days process (9.13) was studied in the framework of the vector dominance model (VDM) and predictions similar to that for standard deep-inelastic scattering have been obtained, i.e., Bjorken scaling in the simple parton model and logarithmic scaling violations in the framework of asymptotically free gauge theories (Ahmed and Ross, 1975a). It turns out, however, that in addition to the VDM contributions there are contributions to photon–photon scattering in which the photon behaves as a pointlike particle (Walsh and Zerwas, 1973; Kingsley, 1973). In the parton model these contributions are represented by the box diagram of Fig. 38(c). The box diagram contribution cannot only be exactly calculated, but at large values of Q^2 increases as $\ln Q^2$ and dominates over the (incalculable in perturbation theory) vector dominance terms. The latter are suppressed by powers of $\ln Q^2$ as in the standard deep-inelastic scattering. Neglecting VDM contributions, the parton model result for the photon structure function $F_2^\gamma(x,Q^2)$ for

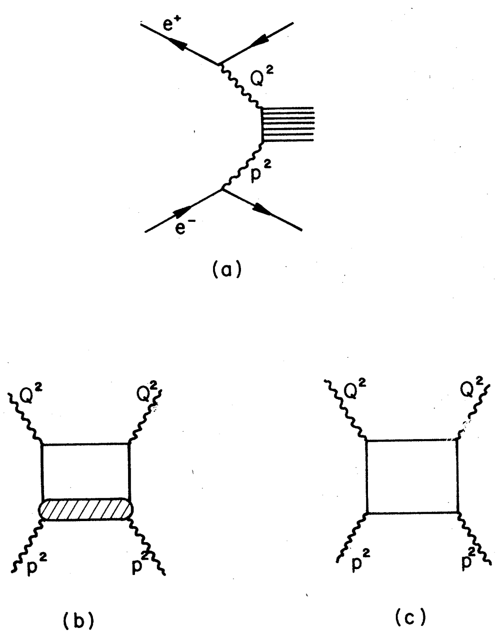


FIG. 38. The process $e^+e^- \rightarrow \text{hadron}+e^+e^-$: (a) The dominant two-photon contribution, (b) vector dominance contribution to the photon–photon scattering, (c) Contributions to photon–photon scattering in which the photon behaves like a pointlike particle (parton model diagram).

large Q^2 has the form

$$F_2^\gamma(x,Q^2)\,|_{\rm PM}=\alpha^2 p(x)\ln Q^2\,, \tag{9.14}$$

where α is the electromagnetic coupling constant and $p(x)$ is given as follows:

$$p(x)=8\,\delta_\gamma x P_{qG}(x)\,. \tag{9.15}$$

Here $P_{qG}(x)$ is the familiar splitting function of Eq. (2.57) which expresses the probability of finding a quark in a gluon (now photon). Furthermore

$$\delta_\gamma=3\sum_i e_i^4 \tag{9.16}$$

where the sum runs over the flavors and e_i are the quark charges. "3" stands for the number of colors.

It has been pointed out by Witten (1977) that there are asymptotic freedom corrections to the parton model result of Eqs. (9.14) and (9.15). Witten calculated these corrections in the leading order of asymptotic freedom and his calculation has recently been extended to the next-to-leading order (Bardeen and Buras, 1979a). All these corrections are independent of the unknown matrix elements of local operators in constrast to the case of standard deep-inelastic scattering, and can be exactly calculated. Moreover, at sufficiently large values of Q^2 these exactly calculable terms are more important than the VDM contributions. Therefore the process under consideration is, from a theoretical point of view, an excellent place to study properties of higher-order corrections. The two calculations above have been done using operator product expansion and renormalization group methods. Witten's result has been rederived last year by Llewellyn-Smith (1978c) and recently by Frazer and Gunion (1979b) in the framework of perturbative QCD. Furthermore, the process in question has been studied in the intuitive approach of Sec. V (De Witt, Jones, Sullivan, Willen, and Wyld, 1979; Brodsky, De Grand, Gunion, and Weis, 1978, 1979).[44] In what follows we shall present in more detail the formal approach to photon-photon scattering.

The formal approach to photon–photon scattering has been first discussed by Witten (1977). The moments of the photon structure function $F_2^\gamma(x,Q^2)$ are given as follows

$$\int_0^1 dx x^{n-2}F_2^\gamma(x,Q^2)=\sum_i C_{2,n}^i(Q^2/\mu^2,g^2,\alpha)\langle\gamma\,|O_i^n|\,\gamma\rangle\,, \tag{9.17}$$

where $\alpha=e^2/4\pi$ is the electromagnetic coupling constant. The sum on the rhs of Eq. (9.16) runs over spin-n, twist-2 operators such as the fermion nonsinglet operator $O_{\rm NS}$, singlet fermion and gluon operators, O_ψ and O_G, and the photon operator O_γ. The latter operator, which is not present in the analysis of the deep-inelastic scattering off hadronic targets, is the analog of the gluon operator O_G with the non-Abelian field strength tensor $G_{\alpha\beta}$ replaced by the electromagnetic tensor $F_{\alpha\beta}$ [see Eq. (3.57)]. As noted by Witten, O_γ must be included in the analysis of photon–photon scattering. The

[44]See also Koller, Walsh, and Zerwas (1978) and Kajantie (1979). Gunion and Jones (1979) have discussed the parameter $\tilde{\alpha}_n$ in the intuitive approach.

reason is that, although the Wilson coefficients C_n^γ are $O(\alpha)$, the matrix elements $\langle\gamma|O_\gamma^n|\gamma\rangle$ are $O(1)$. Therefore the photon contribution in Eq. (9.17) is of the same order in α as the contributions of quark and gluon operators. The latter have Wilson coefficients $O(1)$ but matrix elements in photon states $O(\alpha)$. We want to evaluate Eq. (9.17) to lowest order in α but to all order in g. In lowest order in α

$$C_{2,n}^i(Q^2/\mu^2, g^2, \alpha) = C_{2,n}^i(Q^2/\mu^2, g^2) \quad i = \psi, G, NS \tag{9.18}$$

where the functions on the rhs of Eq. (9.18) are the familiar Wilson coefficients which we discussed in Secs. II–VIII. (Recall that the coefficient functions do not depend on the target.) Therefore the first three terms involving hadronic operators will be suppressed by powers of logarithms except for $n=2$, due to the vanishing of the anomalous dimension of the hadronic energy momentum tensor. Moreover, the matrix elements of hadronic operators in photon states cannot be calculated in perturbation theory. In the language used above the three terms in Eq. (9.17) involving hadronic operators belong to vector dominance contributions. We are, in fact, mainly interested in the coefficient function of the photon operator, whose matrix element between photon states is known,

$$\langle\gamma|O_\gamma|\gamma\rangle = 1\,. \tag{9.19}$$

$C_{2,n}^\gamma(Q^2/\mu^2, g^2, \alpha)$ satisfy the following renormalization group equations:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)C_{2,n}^i\left(\frac{Q^2}{\mu^2}, g^2, \alpha\right) = \sum_j \gamma_{ji} C_{2,n}^j\left(\frac{Q^2}{\mu^2}, g^2, \alpha\right)$$

$$i = \psi, NS, G, \gamma\,, \tag{9.20}$$

where γ_{ji} are the elements of the 4×4 anomalous dimension matrix, which has in lowest order in α the following structure:

$$\gamma_n(g^2, \alpha) = \begin{bmatrix} \hat{\gamma}_n^H(g^2) & 0 \\ \mathbf{K}_n(g^2, \alpha) & 0 \end{bmatrix}. \tag{9.21}$$

Here $\hat{\gamma}_n^H(g^2)$ is the standard hadronic anomalous dimension matrix of Eq. (4.22) extended by one column and one row to include the anomalous dimension of the nonsinglet operator. $\mathbf{K}_n(g^2, \alpha)$ is a three-component vector

$$\mathbf{K}_n(g^2, \alpha) = [K_\psi^n(g^2, \alpha), K_G^n(g^2, \alpha), K_{\mathrm{NS}}^n(g^2, \alpha)]\,, \tag{9.22}$$

which represents the mixing between the photon operator and the remaining three operators. The components K_j^n can be calculated in perturbation theory and have the following expansion:

$$K_j^n(g^2, \alpha) = -\frac{e^2}{16\pi^2}K_j^{0,n} - \frac{e^2 g^2}{(16\pi^2)^2}K_j^{(1),n} \quad j = \psi, NS \tag{9.23}$$

and

$$K_G^n(g^2, \alpha) = -\frac{e^2 g^2}{(16\pi^2)^2}K_G^{(1),n}\,. \tag{9.24}$$

Examples of diagrams necessary for the calculation of the coefficients $K_j^{0,n}$, $K_j^{(1),n}$ and $K_G^{(1),n}$ are shown in Fig. 39.

Because of the mixing between the photon operators and hadronic operators the coefficient function

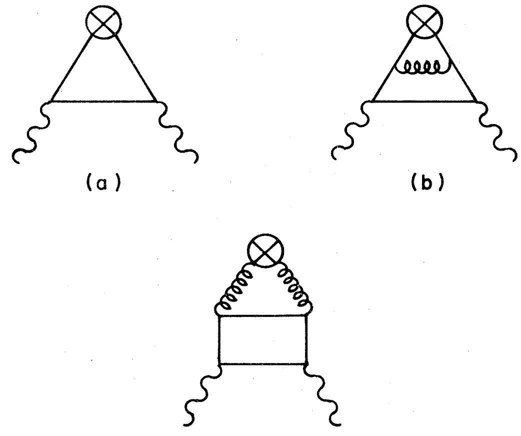


FIG. 39. Typical diagrams contributing to the mixing between hadronic operators and the photon operator (a) $K_\psi^{0,n}$, (b) $K_\psi^{(1),n}$, (c) $K_G^{(1),n}$.

$C_{2,n}^\gamma(Q^2/\mu^2, g^2, \alpha)$ depends on the matrix $\hat{\gamma}_n^H(g^2)$ and the coefficients $C_{2,n}^G(1,\bar{g}^2)$, $C_{2,n}^{\mathrm{NS}}(1,\bar{g}^2)$, and $C_{2,n}^\psi(1,\bar{g}^2)$, in addition to the mixing anomalous dimensions K_j^n and the coefficient function $C_{2,n}^\gamma(1, g^2, \alpha)$. The resulting expression for the moments of $F_2^\gamma(x, Q^2)$ is as follows:

$$\int_0^1 dx\, x^{n-2} F_2^\gamma(x, Q^2) = \alpha^2\left[a_n \ln\frac{Q^2}{\Lambda^2} + \tilde{a}_n \ln\ln\frac{Q^2}{\Lambda^2} + \bar{b}_n + O\left(\frac{1}{\ln(Q^2/\Lambda^2)}\right)\right], \tag{9.25}$$

where the terms $O[1/(\ln Q^2/\bar{\Lambda}^2)]$ include the VDM contributions.

The constants a_n have been calculated by Witten, (1977) and the parameters $\tilde{a}_n$ and $\bar{b}_n$ by Bardeen and Buras (1979a). The a_n depend on one-loop anomalous dimensions and one-loop β functions while the $\tilde{a}_n$ depend, in addition, on the two-loop contributions to the β functions. Finally, the $\bar{b}_n$ depend on the two-loop anomalous dimensions and the one-loop contributions to the Wilson coefficient functions, in addition to the renormalization group parameters on which $\tilde{a}_n$ and a_n depend. Analytic expressions for a_n, $\tilde{a}_n$, and $\bar{b}_n$ and their detailed derivations can be found in the original papers. The numerical values for these parameters are collected in Table V. The numerical values of $\bar{b}_n$ depend on the definitions of the scale parameter $\bar{\Lambda}$. The $\bar{b}_n$ in Table V are for the $\overline{\mathrm{MS}}$ of Sec. VII. Notice that we do not give the values of b_2, which involves the perturbatively uncalculable photon matrix element of the hadronic energy momentum tensor.

Remembering lesson (b) of Sec. IX.C we take for $\bar{\Lambda}$ the value of 0.5 GeV which we have extracted from deep-inelastic scattering in Sec. VII using the same definition for Λ. Equation (9.25) is plotted in Fig. 40 for various values of n on Q^2. In Fig. 40 L.O. stands for the first term in Eq. (9.26) and PM stands for the parton model result of Eq. (9.15). We conclude that asymptotic freedom effects suppress the photon-structure function at large values of n or equivalently large x and that this suppression is enhanced by higher-order corrections as compared to the leading-order re-

TABLE V. Numerical values of the parameters a_n, $\tilde{a}_n$, and $\bar{b}_n$ which enter Eq. (9.25) for $f=3$ and 4. p_n are the moments of the $p(x)$ which enter Eq. (9.14). The table is from Bardeen and Buras (1979a).

	a_n		$\tilde{a}_n$		$\bar{b}_n$		p_n	
n	3	4	3	4	3	4	3	4
2	0.660	1.245	0.353	0.529			0.889	1.679
4	0.276	0.504	0.218	0.373	−0.604	−1.028	0.489	0.924
6	0.175	0.317	0.138	0.235	−0.418	−0.716	0.349	0.660
8	0.127	0.230	0.100	0.170	−0.327	−0.561	0.274	0.518
10	0.0989	0.179	0.0781	0.132	−0.269	−0.463	0.226	0.427
12	0.0806	0.146	0.0637	0.108	−0.228	−0.394	0.193	0.364
14	0.0678	0.122	0.0536	0.0904	−0.198	−0.343	0.168	0.318
16	0.0584	0.105	0.0461	0.0777	−0.175	−0.303	0.149	0.282
18	0.0511	0.0919	0.0404	0.0680	−0.157	−0.271	0.134	0.253
20	0.0453	0.0815	0.0358	0.0603	−0.142	−0.245	0.122	0.230

sult. A similar result for the higher-order corrections to the moments of $F_2^\gamma(x,Q^2)$ is obtained if $\bar{g}^2(Q^2)$ is defined by momentum subtraction. In that case the parameters $\bar{b}_n$ are replaced (Celemaster and Gonsalves, 1979) by

$$b_n^{\mathrm{MOM}} = \bar{b}_n + a_n[1.54] \tag{9.26}$$

and $\bar{\Lambda}=0.5$ GeV by $\Lambda_{\mathrm{MOM}}=0.85$ GeV as extracted in this scheme from deep-inelastic scattering (see Sec. VII.H). More elaborate comparison of higher-order corrections with the leading-order result and the higher-order corrections in deep-inelastic scattering can be found in the paper by Bardeen and Buras (1979a).

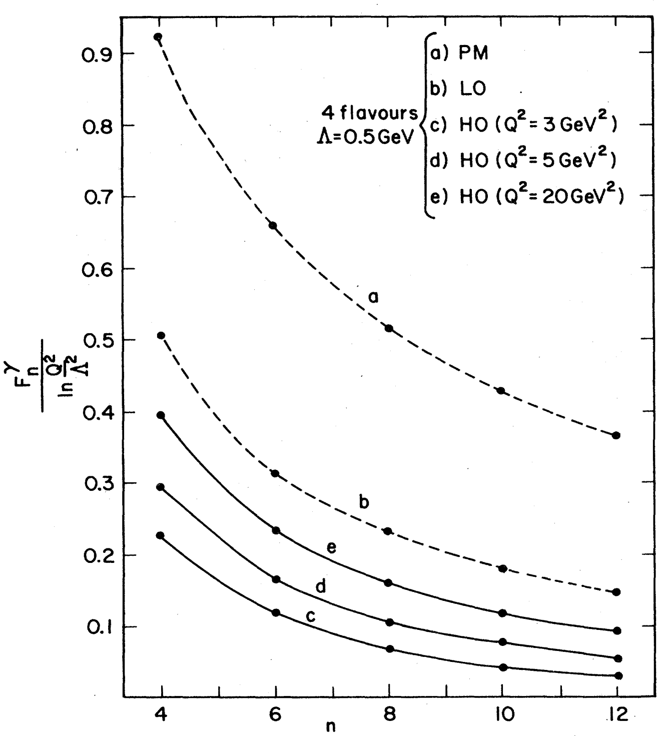


FIG. 40. Moments of the photon structure function in units of α^2 as predicted by the parton model (a), asymptotic freedom in the leading order (b), and asymptotic freedom with higher-order corrections (c, d, e). For comparison the same value of Λ for all cases has been chosen.

For recent reviews of the photon–photon physics in connection with QCD ideas we refer the interested reader to the papers by Llewellyn-Smith (1978c), Brodsky *et al.* (1978), Brodsky (1978), Kajantie (1979), Koller, Walsh, and Zerwas (1979), and Hill and Ross (1979). In particular Hill and Ross discuss heavy quark mass effects in photon–photon scattering which turn out to be important.

F. Semi-inclusive processes in QCD

1. Preliminaries

We shall now turn to the presentation of the basic structure of QCD formulas for semi-inclusive processes. To this end it will be useful to introduce certain notation which we shall illustrate with the familiar deep-inelastic scattering. Consider a photon of momentum q which scatters off a parton of momentum p. If P is the momentum of the hadron to which the parton in question belongs then we can introduce the following variables

$$x = Q^2/2Pq\,, \tag{9.27}$$

$$\tilde{x} = Q^2/2pq\,, \tag{9.28}$$

and ξ, the fraction of the hadron momentum carried by the struck parton.

The deep-inelastic photon–hadron cross section can then be written in QCD as follows[45]:

$$\sigma_H(x,Q^2) = \sum_j \int d\tilde{x} \int d\xi\, \delta(x-\tilde{x}\xi)\, \sigma_p^j(\tilde{x},Q^2)[\xi f_j^H(\xi,Q^2)]$$

$$= \sum_j \int_x^1 \frac{d\xi}{\xi}\, \sigma_p^j\!\left(\frac{x}{\xi},Q^2\right)[\xi f_j^H(\xi,Q^2)]\,. \tag{9.29}$$

Here σ_p^j is the photon–parton cross section and f_j^H is the distribution of partons of type j in the hadron. The sum runs over all types of partons, i.e., quarks and gluons. Equation (9.29) just represents Eqs. (8.77) and (8.83). Equation (9.29) is illustrated in Fig. 41.

In the leading order of asymptotic freedom

[45]See, for instance, Ellis, Georgi, Machacek, Politzer, and Ross (1979) except that we denote differential cross sections as $d\sigma/dx$ by $\sigma(x)$, $d\sigma/(dxdz)$ by $\sigma(x,z)$, etc. For simplicity, and following these authors, we consider only the cross sections which are projected out by contracting the indices of the virtual photon with the tensor $-g_{\mu\nu}$.

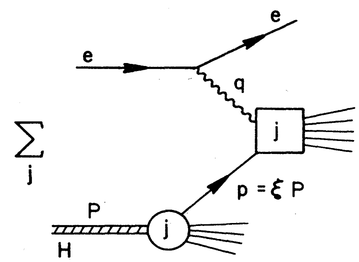


FIG. 41. Illustration of the rhs of Eq. (9.29). The sum runs over quarks and gluons.

$$\sigma_p^j(\tilde{x}, Q^2) = \begin{cases} e_j^2 \delta(1-\tilde{x}) & j = q, \bar{q} \\ 0 & j = G \end{cases} \tag{9.30}$$

and

$$\xi f_j^H(\xi, Q^2) \equiv \xi q_j(\xi, Q^2) \text{ or } \xi \bar{q}_j(\xi, Q^2), \tag{9.31}$$

with Q^2 dependence given by Eqs. (2.52)–(2.54). Inserting Eqs. (9.30) and (9.31) into (9.29) we obtain the standard result

$$\sigma_H(x, Q^2) = \sum_j e_j^2 [x q_j(x, Q^2) + x \bar{q}_j(x, Q^2)], \tag{9.32}$$

which is also true in the simple parton model if Q^2 dependence is neglected. If next-to-the-leading order corrections are taken into account the following things happen:

(i) there are $\bar{g}^2(Q^2)$ corrections to the photon–quark cross sections of Eq. (9.30);

(ii) the Q^2 dependence of quark distributions is modified [see Eqs. (2.137)–(2.139)];

(iii) the photon–gluon cross section, which is of order $\bar{g}^2(Q^2)$, also enters the final formula for $\sigma_H(x, Q^2)$. As discussed extensively in Sec. VIII the points (i)–(iii) are related to each other. For instance the explicit $\bar{g}^2(Q^2)$ corrections to various parton cross sections depend on the definition of parton distributions beyond the leading order.

In what follows we shall briefly discuss QCD formulas for semi-inclusive processes, which will turn out to have a structure similar to that of Eq. (9.29).

2. Fragmentation functions

In addition to quark distributions, extensively discussed in previous sections, important quantities in the study of semi-inclusive processes are the fragmentation functions (Feynman, 1972; Field and Feynman, 1977) which describe how a parton decays into a final hadron. The best process (at least from a theoretical point of view) to study these functions is the semi-inclusive e^+e^- annihilation in which a single hadron is detected in the final state:

$$e^+e^- \to h(P) + \text{anything}. \tag{9.33}$$

This process is shown in Fig. 42.

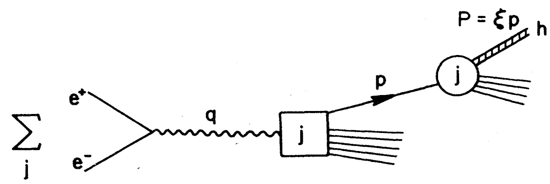


FIG. 42. Illustration of the rhs of Eq. (9.36). The sum runs over quarks and gluons.

The relevant variables are, in analogy with Eqs. (9.27) and (9.28),

$$z = 2Pq/Q^2 \tag{9.34}$$

and

$$\tilde{z} = 2pq/Q^2. \tag{9.35}$$

One also introduces ξ, which this time measures the fraction of the parton momentum carried by the hadron in the final state.

The cross section for process (9.33) can be written in QCD (in units of 3 $[4\pi\alpha_E^2/3Q^2]$), as follows (Georgi and Politzer, 1978b):

$$\begin{aligned}\bar{\sigma}_h(z, Q^2) &= \sum_j \int d\tilde{z} \int d\xi\, \delta(z - \tilde{z}\xi) \bar{\sigma}_p^j(\tilde{z}, Q^2) \\ &\quad \times [\xi D_j^h(\xi, Q^2)] \\ &= \sum_j \int_z^1 \frac{d\xi}{\xi} \bar{\sigma}_p^j\left(\frac{z}{\xi}, Q^2\right) [\xi D_j^h(\xi, Q^2)].\end{aligned} \tag{9.36}$$

Here $\bar{\sigma}_p^j$ is the cross section for the production of the parton j and $D_j^h(\xi, Q^2)$ is the fragmentation function which measures the probability for a parton j to decay into a hadron h carrying the fraction ξ of the parton momentum. Let us recall that in the simple parton model (PM)

$$\bar{\sigma}_p^j = \begin{cases} e_j^2 \delta(1-\tilde{z}) & j = q, \bar{q} \\ 0 & j = G \end{cases} \tag{9.37}$$

and the fragmentation functions do not depend on Q^2. Consequently one obtains

$$\bar{\sigma}_h(z)\big|_{\rm PM} = \sum_j e_j^2 \left[z D_{q_j}^h(z) + z D_{\bar{q}_j}^h(z) \right]. \tag{9.38}$$

In QCD the fragmentation functions acquire a Q^2 dependence which has been studied by various authors (Georgi and Politzer, 1978b; Sachrajda, 1978b; Dokshitser, Dyakanov and Troyan, 1978; Owens, 1978; Uematsu, 1978; Meuller, 1978; Ellis *et al.*, 1978).[46] We quote only the results of these studies and refer the reader to the papers above for details. In the leading order of asymptotic freedom, or equivalently by summing the leading logarithms to all orders in g^2,[47] formula (9.38) is unchanged except that the fragmentation functions depend on Q^2. The following integrodifferential equations analogous to Eqs. (2.52)–(2.54) determine the Q^2 evolution of the fragmentation functions

[46] Previous studies of the fragmentation functions in the context of QCD can be found in Callan and Goldberger (1975) and Mueller (1974). Semi-inclusive deep-inelastic scattering has been discussed previously by many authors, in particular by Georgi and Politzer (1978a) and by Mendez (1978).

[47] Some of the order g^2 corrections to $e^+e^- \to h(P)$ + anything are shown in Fig. 44.

$$\frac{dD^h_{\Delta ij}}{dt}(z,t)=\frac{\bar{\alpha}(Q^2)}{2\pi}\int_z^1\frac{dy}{y}D^h_{\Delta ij}(y,t)P_{qq}\left(\frac{z}{y}\right) \tag{9.39a}$$

$$\frac{dD^h_{\Sigma}}{dt}(z,t)=\frac{\bar{\alpha}(Q^2)}{2\pi}\int_z^1\frac{dy}{y}\left[D^h_{\Sigma}(y,t)P_{qq}\left(\frac{z}{y}\right)+2fD^h_G(y,t)P_{Gq}\left(\frac{z}{y}\right)\right] \tag{9.39b}$$

$$\frac{dD^h_{G}}{dt}(z,t)=\frac{\bar{\alpha}(Q^2)}{2\pi}\int_z^1\frac{dy}{y}\left[D^h_{\Sigma}(y,t)P_{qG}\left(\frac{z}{y}\right)+D^h_G(y,t)P_{GG}\left(\frac{z}{y}\right)\right], \tag{9.39c}$$

where $t=\ln Q^2/\mu^2$. Here

$$D^h_{\Delta ij}(z,t)\equiv D^h_{q_i}(z,t)-D^h_{q_j}(z,t), \tag{9.40}$$

is a nonsinglet fragmentation function and

$$D^h_{\Sigma}(z,t)\equiv\sum_i\left[D^h_{q_i}(z,t)+D^h_{\bar{q}_i}(z,t)\right], \tag{9.41}$$

is the singlet fragmentation function. Furthermore, D^h_G measures the probability for a gluon to decay into a hadron h carrying a fraction z of gluon momentum. The functions P_{ij} are exactly the "splitting functions" of Eqs. (2.56)–(2.59). Notice however that P_{Gq} and P_{qG} have been interchanged relative to Eqs. (2.53) and (2.54). The structure of Eqs. (9.39) can be easily understood (see Fig. 43). The process of obtaining hadrons from a given quark can proceed in three ways. The quark can fragment directly into hadrons or fragment into them after emission of a gluon. These two processes correspond to the first term in Eq. (9.39b). The second term in Eq. (9.39b) corresponds to the situation in which the quark emits a gluon, which subsequently fragments into hadrons. Similarly one can interpret Eq. (9.39c).

From Eqs. (9.39) it is a simple matter to derive the equations for the moments of the fragmentation functions, which are defined analogously to the moments of quark distributions, e.g.,

$$\langle D^h_{\Sigma}(Q^2)\rangle_n\equiv\int_0^1 dz\,z^{n-1}D^h_{\Sigma}(z,Q^2). \tag{9.42}$$

The moment equations for the fragmentation functions are obtained from Eqs. (2.84)–(2.86) by making there the following replacements:

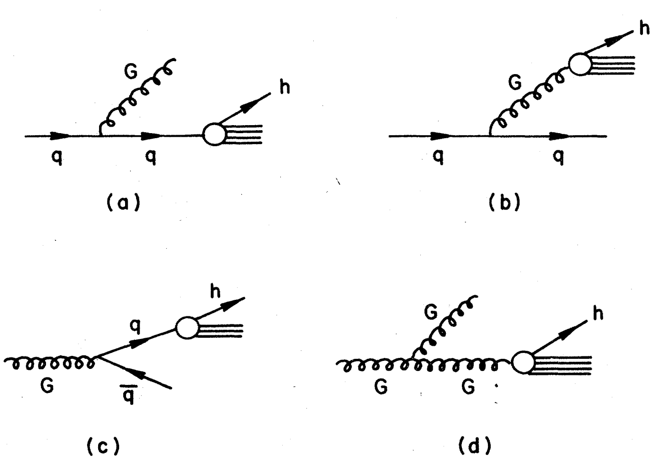


FIG. 43. Basic processes responsible for the Q^2 evolution of the fragmentation functions.

$$\tilde{\alpha}_n\to\frac{2f\gamma^{0,n}_{G\psi}}{\lambda^n_- -\lambda^n_+}=2f\varepsilon_n \tag{9.43}$$

and

$$\varepsilon_n\to\frac{1}{2f}\frac{\gamma^{0,n}_{\psi G}}{\lambda^n_- -\lambda^n_+}=\frac{\tilde{\alpha}_n}{2f}. \tag{9.44}$$

α_n remains unchanged and the anomalous dimension matrix is, as before, given by Eq. (2.79). One can check that due to the properties (5.10) of the splitting functions the momentum sum rule

$$\sum_h\int_0^1 zD^h_{q_i}(z,Q^2)=1, \tag{9.45}$$

and an analogous sum rule for the gluon fragmentation function are satisfied.

In order to solve Eqs. (9.39) the values of the fragmentation functions at some value of $Q^2=Q_0^2$ are needed. As in the case of quark distributions, they have to be taken from the data. Once they are given one can find fragmentation functions at other values of Q^2 by solving (9.39) numerically. Such an exercise can be found in the paper by Field (1979). The pattern of scaling violations in fragmentation functions is predicted to be very similar to that found in quark distributions in spite of the interchange of the "nondiagonal" splitting functions P_{qG}, P_{Gq} or equivalently $\gamma^{0,n}_{\psi G}$ and $\gamma^{0,n}_{G\psi}$. This is not surprising since for $n>2$ mixing between quark and gluons is very weak and the interchange of the functions in question irrelevant. It should be of course kept in mind that, although the patterns of scaling violations in fragmentation functions and parton distributions are very similar, the boundary conditions to Eqs. (2.53)–(2.54) and (9.39) as determined from the data are different and so are the functional forms of the Q^2 dependent parton distributions and fragmentation functions. Before presenting the structure of next-to-leading order QCD corrections to process (9.33) let us briefly discuss the question of factorization of mass singularities.

Consider the diagrams of Fig. 44 which contribute in order g^2 to the cross section $\tilde{\sigma}_h(z,Q^2)$.[48] As in the example of Sec. IX.B, the quarks are assumed to be massless and slightly off-shell. For the moments of $\tilde{\sigma}_h(z,Q^2)$ defined by

$$\tilde{\sigma}^n_h(Q^2)=\int_0^1 dz\,z^{n-2}\tilde{\sigma}_h(z,Q^2), \tag{9.46}$$

we obtain

$$\tilde{\sigma}^n_h(Q^2)=V^0_n\{1+(g^2/16\pi^2)[-\tfrac{1}{2}\gamma^{0,n}_{NS}\ln(Q^2/-p^2)+u_n]\} \tag{9.47}$$

where V^0_n stands for the empty ("bare") blob in Fig. 44, $\gamma^{0,n}_{NS}$ is the standard nonsinglet anomalous dimension, and u_n are constant numbers analogous to the r_n's of Eq. (9.1). As in Eq. (9.1) also here there is a mass singularity for $p^2\to 0$ and u_n depends on the assumption about the quarks. For $m^2\neq 0$ and $p^2=0$ a different u_n would be obtained. Equation (9.47) can be rewritten as follows

[48] In this example we do not discuss the g^2 corrections which arise from gluon production and its subsequent decay into hadrons.

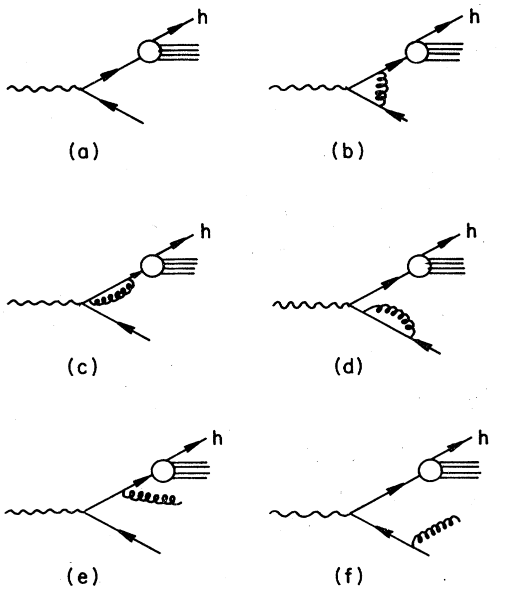


FIG. 44. Diagrams contributing to quark production in $e^+e^- \to h$ + anything to order g^2. The empty ("bare") blob stands for the "bare" Q^2 independent quark fragmentation functions. denotes gluons.

$$\tilde{\sigma}_n^h(Q^2) = V_n^0\left[1+\frac{g^2}{16\pi^2}\left(\frac{1}{2}\gamma_{\rm NS}^{0,n}\ln\frac{-p^2}{\mu^2}+u_n^{(1)}\right)\right]\times\left[1+\frac{g^2}{16\pi^2}\left(-\frac{1}{2}\gamma_{\rm NS}^{0,n}\ln\frac{Q^2}{\mu^2}+u_n^{(2)}\right)\right], \tag{9.48}$$

where μ^2 is a scale and

$$u_n^{(1)}+u_n^{(2)}=u_n. \tag{9.49}$$

Combining the first two factors on the rhs of Eq. (9.48) we obtain

$$\tilde{\sigma}_n^h(Q^2) = V_n(\mu^2)\left[1+\frac{g^2}{16\pi^2}\left(-\frac{1}{2}\gamma_{\rm NS}^{0,n}\ln\frac{Q^2}{\mu^2}+u_n^{(2)}\right)\right]. \tag{9.50}$$

In the parton language we can interpret $V_n(\mu^2)$ as the moments of the fragmentation function at $Q^2=\mu^2$. In the formal language of Mueller (1978), $V_n(\mu^2)$ is an analog of the matrix element of the local operator and is called the *timelike cut vertex*. In the same language the matrix elements of local operators are called *spacelike cut vertices*. The second factor on the rhs of Eq. (9.50), which is free of any singularity as $p^2\to 0$, is the analog of the coefficient function. This time a coefficient function of the timelike cut vertex in an expansion similar to the operator product expansion.

This factorization of singular (nonperturbative) terms from well-behaved terms which can be calculated in perturbation theory can be proved to all orders in g^2 and in all logarithms (Ellis *et al.*, 1978, 1979; Amati *et al.*, 1978a, b; Mueller, 1978).

Notice that through Eq. (9.49) $u_n^{(2)}$ depends on $u_n^{(1)}$ or equivalently on the normalization of the cut vertex at $p^2=\mu^2$. Different renormalization schemes will therefore lead to different values of $u_n^{(2)}$. As in the case of deep-inelastic scattering, also here this renormalization prescription dependence will be canceled by that of the two-loop anomalous dimensions of the cut vertices when the full Q^2 evolution of the fragmentation function to all orders in g^2 and in the first two orders in $\bar{g}^2$ is calculated. We observe, therefore, that the study of the next-to-leading order corrections to the Q^2 evolution of the fragmentation functions proceeds in an analogous way to that for quark distributions. The structure of the formal and intuitive formulas (beyond the leading-order approximation) for the process (9.33) is very similar to that presented in the previous sections for deep-inelastic scattering. Questions of definitions of fragmentation functions, of the definition of $\bar{g}^2(Q^2)$, and of the cancellation of renormalization-prescription dependences also arise here. Consequently also the comments (i)–(iii) made after Eq. (9.32) also apply to $\tilde{\sigma}_n(z,Q^2)$ as given in Eq. (9.36).

Equation (9.50) can be rewritten as follows:

$$\tilde{\sigma}_n^h = \langle D(Q^2)\rangle_n[1+(g^2/16\pi^2)u_n^{(2)}], \tag{9.51}$$

where

$$\langle D(Q^2)\rangle_n \equiv V_n(\mu^2)[1-(g^2/16\pi^2)(\gamma_{\rm NS}^{0,n}/2)\ln(Q^2/\mu^2)], \tag{9.52}$$

are the moments of the fragmentation functions and

$$[1+(g^2/16\pi^2)u_n^{(2)}] \equiv \tilde{\sigma}_p^n, \tag{9.53}$$

are the moments of the cross section for quark production calculated to order g^2. When all orders in g^2 and the two first orders in $\bar{g}^2(Q^2)$ are taken into account, $\langle D(Q^2)\rangle_n$ acquires the full Q^2 dependence with two-loop anomalous dimensions of the cut vertices and the two-loop β function included. Furthermore, in Eq. (9.53) g^2 is replaced by $\bar{g}^2(Q^2)$. We recall once more that, although $\tilde{\sigma}_n^h$ is unambiguous, the separation of $\tilde{\sigma}_n^h$ into the fragmentation function and the cross section for parton production is arbitrary beyond the leading order.

The full study of $\bar{g}^2(Q^2)$ corrections to the process $e^+e^-\to h$ + anything has not yet been discussed in the literature. In particular, it is not known whether the two-loop anomalous dimensions for timelike cut vertices are the same as those for spacelike cut vertices, which are given in Table III.

3. Drell-Yan and semi-inclusive deep-inelastic scattering

In the simple parton model, parton distributions and parton fragmentation functions are the building blocks of any expression for inclusive and semi-inclusive processes. These building blocks do not depend on the process, although in different processes they enter in different well-defined ways. Thus if we can extract all parton distributions from deep-inelastic processes and fragmentation functions from e^+e^- annihilation, then the cross sections for other processes such as the Drell-Yan process (Drell and Yan, 1971), semi-inclusive deep-inelastic scattering, etc., can be predicted.

We have seen that, in QCD, parton distributions and fragmentation functions acquire a Q^2 dependence, and it is of interest to ask whether the QCD predictions for semi-inclusive processes amount to using these Q^2-dependent functions in the parton model formulas for the processes in question. This has been studied by many authors during the last year, in particular by Politzer (1977a), Sachrajda, (1978a,b), Dokshitser, Dyakanov, and Troyan (1978), Llewllyn-Smith (1978b), Amati *et al.* (1978a, b), Ellis, Georgi, Machacek, Politzer, and Ross (1978, 1979), Gupta and Mueller (1979), and Buras (1979). In what follows we shall present the formulas for the two processes

$$eH \to e + h + \text{anything}\,, \tag{9.54}$$

and

$$H_1H_2 \to \mu^+\mu^- + \text{anything}\,, \tag{9.55}$$

in the leading order and next-to-leading order of asymptotic freedom. Subsequently, we shall briefly discuss the basic features of these formulas which are characteristic for all QCD expressions for semi-inclusive cross sections.

The cross sections for the processes (9.54) and (9.55) are given in QCD as follows:

$$\begin{aligned}\tilde\sigma_{Hh}(x,z,Q^2) &= \sum_{j,k}\int d\bar z\, d\bar x\, d\xi_1\, d\xi_2\,\delta(x-\bar x\xi_1)\,\delta(z-\bar z\xi_2)\,.\\ &\quad \tilde\sigma_P^{jk}(\bar x,\bar z,Q^2)[\xi_1 f_j^H(\xi_1,Q^2)][\xi_2 D_k^h(\xi_2,Q^2)]\\ &= \sum_{j,k}\int_x^1\frac{d\xi_1}{\xi_1}\int_z^1\frac{d\xi_2}{\xi_2}\tilde\sigma_P^{j,k}\left(\frac{x}{\xi_1},\frac{z}{\xi_2},Q^2\right)\\ &\quad [\xi_1 f_j^H(\xi_1,Q^2)][\xi_2 D_k^h(\xi_2,Q^2)]\,,\end{aligned} \tag{9.56}$$

for process (9.54) and

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha_E^2}{3Q^2}\int\frac{dx_1}{x_1}\int\frac{dx_2}{x_2}\sigma_{H_1H_2}(x_1,x_2,x_{12},Q^2) \tag{9.57}$$

with

$$\begin{aligned}\sigma_{H_1H_2}(x_1,x_2,x_{12},Q^2) &= \sum_{j,k}\int d\bar x_1\, d\bar x_2\, d\xi_1\, d\xi_2\,\delta(x_1-\bar x_1\xi_1)\,\delta(x_2-\bar x_2\xi_2)\\ &\quad \sigma_p^{jk}(\bar x_1,\bar x_2,\bar x_{12},Q^2)[\xi_1 f_j^{H_1}(\xi_1,Q^2)][\xi_2 f_k^{H_2}(\xi_2,Q^2)]\\ &= \sum_{j,k}\int_{x_1}^1\frac{d\xi_1}{\xi_1}\int_{x_2}^1\frac{d\xi_2}{\xi_2}\sigma_P^{jk}\left(\frac{x_1}{\xi_1},\frac{x_2}{\xi_2},\frac{\tau}{x_1x_2},Q^2\right)[\xi_1 f_j^{H_1}(\xi_1,Q^2)][\xi_2 f_k^{H_2}(\xi_2,Q^2)]\end{aligned} \tag{9.58}$$

for process (9.55). In Eq. (9.57) α_E is the electromagnetic coupling constant. The processes are shown schematically in Figs. 45 and 46. Variables $x_i, \bar x_i, \xi_i$ in Eqs. (9.57) and (9.58) are obvious generalizations of the variables of Eqs. (9.27) and (9.28). The new variable x_{12} is given as follows

$$x_{12} = \frac{2(P_1\cdot q)(P_2\cdot q)}{Q^2(P_1\cdot P_2)}\,,$$

and $\bar x_{12}$ is obtained from x_{12} by replacing P_i by p_i. Furthermore $\tau = Q^2/s$, where $s=(P_1+P_2)^2$.

In Eq. (9.56) $\tilde\sigma_P^{jk}$ stands for the photon–parton j cross-section with the parton k in the final state. The parton j belongs to the incoming hadron H and its distribution is given by $\xi_1 f_j^H(\xi_1,Q^2)$ with ξ_1 being the momentum fraction of H carried by parton j. The parton k, on the other hand, fragments into the hadron h, and this process is described by the fragmentation function $\xi_2 D_k^h(\xi_2,Q^2)$. The sums in Eq. (9.56) run over all types of partons i.e., quarks and gluons. Similar comments apply to Eq. (9.58) with σ_P^{jk} being the cross section for parton j-parton k scattering or annihilation with a $\mu^+\mu^-$ pair in the final state.

Formulas (9.56) and (9.58) are obtained by summing various QCD diagrams to all orders in g^2. Keeping leading logarithms in each order corresponds to the leading order in $\bar g^2(Q^2)$. Summing next-to-leading logarithms corresponds to next-to-leading order in $\bar g^2(Q^2)$ and so on. As in deep-inelastic scattering and semi-inclusive e^+e^- annihilation, one encounters mass singularities which must be factored out and absorbed in the incalculable (in perturbation theory) wave functions of the incoming and outgoing hadrons: parton distributions and fragmentation functions, or in more formal language spacelike and timelike cut vertices. The structure of mass singularities (anomalous dimensions) turns out to be the same for incoming hadrons as in deep-inelastic scattering and for outgoing hadrons as in e^+e^- annihilation. Therefore the parton distributions and parton fragmentation functions *can* be defined universally independent of the process considered. We write "can" because, due to ambiguities in the definition of parton distributions and parton fragmentation functions beyond the leading order of asymptotic freedom, one could in principle define parton distributions in a different way for different processes. This of course would not be a very useful thing to do.

Let us discuss Eqs. (9.56) and (9.58) in slightly more detail. In the leading order of asymptotic freedom

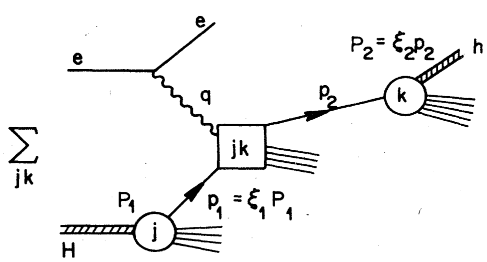


FIG. 45. Illustration of the rhs of Eq. (9.56). The sums run over quarks and gluons.

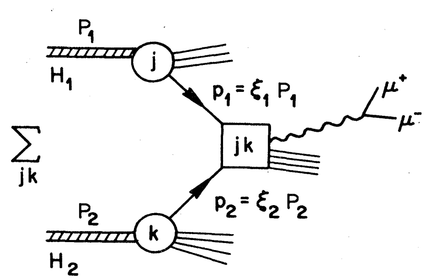


FIG. 46. Illustration of the rhs of Eq. (9.57). The sums run over quarks and gluons.

$$\tilde{\sigma}_P^{jk}(\tilde{x},\tilde{z},Q^2)=\delta_{jk}e_j^2\delta(1-\tilde{z})\delta(1-\tilde{x}) \tag{9.59}$$

and

$$\sigma_P^{jk}(\tilde{x}_1,\tilde{x}_2,\tilde{x}_{12},Q^2)=\begin{cases}\frac{1}{3}e_j^2\delta(1-\tilde{x}_1)\delta(1-\tilde{x}_2)\delta(1/\tilde{x}_{12}-\tilde{x}_1-\tilde{x}_2+\tilde{x}_1\tilde{x}_2) & \text{for } \begin{matrix} j=q \\ k=\bar{q}\end{matrix}\ , \\ 0 & \text{otherwise}\end{cases} \tag{9.60}$$

where 1/3 is the color factor. Therefore, inserting Eqs. (9.59) and (9.60) into Eqs. (9.56) and (9.58), we obtain

$$\tilde{\sigma}_{Hh}(x,z,Q^2)=\sum_j e_j^2[xf_j^H(x,Q^2)][zD_j^h(z,Q^2)] \tag{9.61}$$

and

$$\sigma_{H_1H_2}(x_1,x_2,Q^2)=\frac{1}{3}\sum_{j,\bar{j}}e_j^2[x_1f_j^{H_1}(x_1,Q^2)]\times[x_2f_{\bar{j}}^{H_2}(x_2,Q^2)]\delta\left(1-\frac{\tau}{x_1x_2}\right), \tag{9.62}$$

where j runs over all flavors, $\bar{j}$ denotes antiquarks, and $f_j^H(x,Q^2)[f_{\bar{j}}^H(x,Q^2)]$ are just the quark (antiquark) distributions of Sec. V. $f_j^H(x,Q^2)$ and $D_j^h(z,Q^2)$ satisfy Eqs. (2.53), (2.54), and (9.39), respectively. There is no explicit gluon contribution to the cross sections $\tilde{\sigma}_{Hh}$ and $\sigma_{H_1H_2}$ to this order in $\bar{g}^2(Q^2)$. Gluons, however, contribute indirectly in this order through the scaling violations in the quark distributions and quark fragmentation functions. The formulas (9.61) and (9.62) are, except for the Q^2 dependence, exactly the same as in the simple parton model. Notice in particular the factorization between the x and z dependence for $\tilde{\sigma}_{Hh}$ and between the x_1 and x_2 dependence for $\sigma_{H_1H_2}$ (except for the δ function).

If next-to-leading order corrections are taken into account the Q^2 dependence of parton distributions and parton fragmentation is modified, and there are $\bar{g}^2(Q^2)$ corrections to the parton cross sections of Eqs. (9.59) and (9.60). In addition there are explicit contributions involving gluons. For instance, there is an explicit contribution of quark-gluon scattering to the μ-pair production and an explicit appearance of the gluon fragmentation function in the semi-inclusive deep-inelastic cross sections. Furthermore the factorization property shown in Eqs. (9.61) and (9.62) is broken through the $\bar{g}^2(Q^2)$ corrections to the parton cross sections.

For explicit calculations of next-to-leading order corrections to μ-pair production, we refer the reader to the interesting papers by Altarelli, Ellis, and Martinelli (1978, 1979a), Kubar-André and Paige (1979), Harada, Kaneko, and Sakai (1979), Contogouris and Kripfganz (1979b), and Abad and Humpert (1979). As discussed in particular by Altarelli *et al.* (1979a) and Kubar-André and Paige (1979) the $\bar{g}^2(Q^2)$ corrections to the $q\bar{q}$ annihilation are very large. Unfortunately the authors of these two papers used in their calculations the leading-order predictions for the Q^2 evolution of the quark distributions whereas, consistently to this order, one should include the next-to-leading order corrections to the quark distributions in addition to the $\bar{g}^2(Q^2)$ corrections to the parton cross sections. If the parton distributions are defined as in Eq. (2.136) the next-to-leading order corrections to their Q^2 evolution are small. However, the definitions of quark distributions (beyond the leading order) in the two papers above differ from ours and it would be interesting to check how the author's conclusions about the size of the $\bar{g}^2(Q^2)$ corrections are changed when the Q^2 dependence of parton distributions is properly taken into account.

In summary, since the $\bar{g}^2(Q^2)$ corrections to the elementary parton cross sections depend on the definition of parton distribution (or fragmentation functions) beyond the leading-order approximation, both $\bar{g}^2(Q^2)$ corrections to the parton cross sections and to the parton distributions (fragmentation functions) must be consistently included in a phenomenological analysis. Only then can a physical answer independent of a particular definition of parton distributions (fragmentation functions) be obtained.

The explicit calculations of $\bar{g}^2(Q^2)$ corrections to semi-inclusive deep-inelastic scattering have been done by Sakai (1979), Altarelli *et al.* (1979b), and Baier and Fey (1979), who find breakdown of factorization between z and x at the 10% to 20% level for intermediate z and x values and larger breakdown of factorization for higher z and x values. The comparisons of these predictions with the data are now in progress. Altarelli *et al.* (1979b) have also calculated $\bar{g}^2(Q^2)$ corrections to $e^+e^-\to h_1+h_2+$anything, which turn out to be large only at the kinematical boundaries. Furthermore the qq contribution [order $\bar{g}^4(Q^2)$] to massive μ-pair production has been calculated by Contogouris and Kripfganz (1979a) and Schellekens and Van Neerven (1979). This contribution turns out to be small in the presently accessible kinematic range of τ.

G. Miscellaneous remarks

There are quite a few applications of perturbative QCD which we have not discussed in this review. These include jets, large $p_\perp$ processes, $p_\perp$ distributions in massive μ-pair production, etc. These topics have been nicely discussed for instance in the papers by Ellis, Gaillard, and Ross (1976), Sterman and Weinberg (1977), Farhi (1977), Georgi and Machacek (1977), Cutler and Sivers (1977), Combridge, Kripfganz, and Ranft (1977), Floratos (1978), Furmanski (1978, 1979), Ellis (1978b), Brodsky (1978), Field (1978, 1979), Llewellyn-Smith (1978b), Sachrajda (1978c), Dokshitser, Dyakonov, and Troyan (1978a), Berger (1979), Hwa (1978), Halzen (1979), Veneziano (1979), Politzer (1979), Brown (1979), De Rujula, Ellis, Floratos, and Gaillard (1978), Einhorn and Weeks (1978), Fox and Wolfram (1979), Koller and Walsh (1978), Shizuya and Tye (1979), Fritzsch and Streng (1978), Altarelli (1978b), Furmanski and Pokorski (1979a), Konishi, Ukawa, and Veneziano (1978), Contogouris, Gaskell, and Papadopoulos (1978), Basham, Brown, Ellis, and Love (1978, 1979), and De Grand, Ng, and Tye (1977), where the interested reader may find further references.

The study of nonperturbative effects in the inclusive and semi-inclusive processes can be found in the papers by Andrei and Gross (1978), Appelquist and Shankar (1978), Baulieu *et al.* (1978), Ellis, Gaillard, and Zakrzewski (1979), Carlitz and Lee (1978), and Shifman, Vainshtein, and Zaharov (1979).

X. SUMMARY

In this review we have presented in detail asymptotic freedom predictions for inclusive deep-inelastic scattering. We have also briefly discussed the structure of QCD formulas for other inclusive and semi-inclusive processes such as massive μ-pair production, semi-inclusive deep-inelastic scattering, e^+e^- annihilation, and $\gamma\gamma$ scattering. We have presented confrontations of asymptotic freedom predictions with the deep-inelastic data, and we may conclude that asymptotic freedom survives these confrontations very well, with the possible exception of the longitudinal structure function where the situation is still unclear. The disagreement between theoretical predictions and the data for F_L might not be a problem for QCD, however, and could be due to our neglect of higher-twist operators, nonperturbative effects, etc., which are present in QCD but are difficult to calculate.

We have devoted a considerable part of this review to a discussion of higher-order corrections, the study of which began only two years ago. We have seen that the structure of QCD formulas with higher-order corrections taken into account is fairly complicated and involves many features not encountered in the leading order. These new features include:

(i) Gauge and renormalization-prescription dependences of separate elements of the physical expressions.

(ii) Well-defined dependence of the functional form of the explicit higher-order corrections on the definition of $\bar{g}^2(Q^2)$ or, equivalently, on Λ.

(iii) Freedom in the definition of parton distributions and parton fragmentation functions beyond the leading-order approximation.

These features have to be kept in mind when carrying out calculations to make sure that various parts of the higher-order calculations are compatible with each other. Only then can a physical result be obtained which is independent of gauge, renormalization scheme, particular definition of $\bar{g}^2(Q^2)$, and particular definition of the parton distributions.

Although the structure of higher-order corrections to the Q^2 dependence of parton distributions and fragmentation functions is fairly complicated, the formulas for inclusive and semi-inclusive processes expressed in terms of these effective Q^2-dependent functions are simple and have intuitive interpretations similar to that of the standard parton model.

We have seen that the higher-order corrections are quite large and, moreover, that there are some indications for their presence in the deep-inelastic scattering data. This is most clearly seen in the n dependence of the parameter Λ extracted from the data on the basis of the leading-order formulas. This n dependence agrees well with that obtained from higher-order calculations.

We think it is important to calculate higher-order QCD corrections for other than deep-inelastic processes. This has been already done for massive μ-pair production, photon–photon scattering, and e^+e^- annihilation. In the near future results for the higher-order corrections to fragmentation functions and large $p_\perp$ processes should be available. At this point we would like to re-emphasize that without the higher-order calculations, a meaningful, detailed comparison of QCD effects in various processes cannot be made. This again shows the importance of the calculations in question.

Besides higher-order corrections there are other effects which deserve further study. These are target mass effects, heavy quark mass effects, higher twist operator effects and nonperturbative effects.

In spite of the fact that there is still much to be done, both theoretically and phenomenologically, we believe that a lot of progress has been made in the past few years in the calculations of QCD predictions and in their confrontation with newer and more statistically significant data.

Note added in proof: Here we would like to list a few papers which appeared after the completion of our review. Various phenomenological aspects of higher order corrections have been discussed in the papers by Para and Sachrajda (1979), Pennington and Ross (1979) and Moshe (1979). The effect of higher order corrections on the x and Q^2 dependence of structure functions has been studied by Duke and Roberts (1979b), Gonzalez–Arroyo, Lopez and Yndurain (1979b, c) and Bialas and Buras (1979). Haruyama and Kanazawa (1979) have done an analysis of higher order effects in the moments of deep-inelastic structure functions. References to recent papers related to Sec. IX of this review can be found in the paper by Ellis (1979).

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APPENDIX A: BASIC FORMULAS OF THE DIMENSIONAL REGULARIZATION

1. D-dimensional integrals

$$\int \frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2-M^2+i\varepsilon)^N} = i\frac{(-1)^N}{(4\pi)^{D/2}}\frac{\Gamma(N-D/2)}{\Gamma(N)}\frac{1}{(M^2-i\varepsilon)^{N-D/2}}, \tag{A1}$$

$$\int \frac{d^Dk}{(2\pi)^D}\frac{k^2}{(k^2-M^2+i\varepsilon)^N} = \frac{i}{2}\frac{(-1)^{N-1}}{(4\pi)^{D/2}}\frac{\Gamma(N-1-D/2)}{\Gamma(N)}\frac{D}{(M^2-i\varepsilon)^{N-1-D/2}}, \tag{A2}$$

$$\int \frac{d^Dk}{(2\pi)^D}\frac{k_\mu k_\nu}{(k^2-M^2+i\varepsilon)^N} = \frac{i}{2}\frac{(-1)^{N-1}}{(4\pi)^{D/2}}\frac{\Gamma(N-1-D/2)}{\Gamma(N)}\frac{g_{\mu\nu}}{(M^2-i\ )^{N-1-D/2}}, \tag{A3}$$

$$\int \frac{d^Dk}{(2\pi)^D}k_\mu k_\nu f(k^2) = \frac{1}{D}g_{\mu\nu}\int\frac{d^Dk}{(2\pi)^D}k^2 f(k^2), \tag{A4}$$

where $f(k^2)$ is a function of k^2 and M^2 is a parameter. Integrals with odd number of k's in the numerator are zero.

2. Expansions of Euler-gamma and Euler-beta functions

$$\Gamma(N-\varepsilon/2) = \Gamma(N)(1-\varepsilon/2\psi(N)) + O(\varepsilon^2), \tag{A5}$$

where

$$\psi(N) = S_{N-1} - \gamma_E \tag{A6}$$

and

$$S_N = \sum_{j=1}^{N}\frac{1}{j}. \tag{A7}$$

Here $\varepsilon = 4-D$ and $\gamma_E = 0.5772\ldots$ Since

$$\Gamma(A) = \Gamma(1+A)/A \tag{A8}$$

and

$$B(A_1,A_2) = \frac{\Gamma(A_1)\Gamma(A_2)}{\Gamma(A_1+A_2)}, \tag{A9}$$

we have for instance

$$\Gamma(\varepsilon/2) = 2/\varepsilon - \gamma_E + O(\varepsilon), \tag{A10}$$

$$B(N-\varepsilon/2, 1-\varepsilon/2) = \frac{1}{N}\left[1+\varepsilon S_N - \frac{\varepsilon}{2}S_{N-1}\right] + O(\varepsilon^2) \tag{A11}$$

$$B(N-\varepsilon/2, 2-\varepsilon/2) = \frac{1}{N(N+1)}\left[1-\frac{\varepsilon}{2}S_{N-1} - \frac{\varepsilon}{2} + \varepsilon S_{N+1}\right] + O(\varepsilon^2). \tag{A12}$$

3. Feynman parametrization

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 dx\frac{x^{\alpha-1}(1-x)^{\beta-1}}{[ax+b(1-x)]^{\alpha+\beta}}. \tag{A13}$$

Generalization of (A13) to more factors can be found in 't Hooft and Veltman (1972).

4. Dirac algebra in D dimensions

$$g_{\mu\mu} = D, \tag{A14}$$

$$\{\gamma^\mu,\gamma^\nu\} = 2g_{\mu\nu}\hat{1}, \tag{A15}$$

$$\gamma^\mu\gamma^\mu = D\hat{1}, \tag{A16}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g_{\mu\nu}, \tag{A17}$$

$$\gamma^\mu \not{p}\gamma^\mu = (2-D)\not{p}, \tag{A18}$$

$$\gamma^\mu\not{p}_1\not{p}_2\gamma^\mu = 4p_1p_2 + (D-4)\not{p}_1\not{p}_2, \tag{A19}$$

$$\gamma^\mu\not{p}_1\not{p}_2\not{p}_3\gamma^\mu = -2\not{p}_3\not{p}_2\not{p}_1 - (D-4)\not{p}_1\not{p}_2\not{p}_3. \tag{A20}$$

APPENDIX B: PARTON DISTRIBUTIONS AND MATRIX ELEMENTS OF LOCAL OPERATORS. CHARGE FACTORS

In the course of our review we have denoted the matrix elements of any nonsinglet operator by A_n^{NS}. A_n^{NS} depends in fact on the process and the structure function considered. This dependence can be read from the parton model formulas of Sec. II. We give now a few examples. If $A^N(x)$ is defined by

$$A_n^{\mathrm{NS}} = \int_0^1 dx x^{n-1}A^{\mathrm{NS}}(x), \tag{B1}$$

then

$$A^{\mathrm{NS}}(x) = \begin{cases} \Delta^{ep}(x) & \text{for } F_2^{ep} \\ \Delta^{eN}(x) & \text{for } F_2^{eN}, F_2^{\nu,\bar\nu}|_{\mathrm{NC}} \\ V(x)\mp\Delta^{eN}(x) & \text{for } F_3^{\nu,\bar\nu}|_{\mathrm{CC}} \\ V(x) & \text{for } F_3^{\nu,\bar\nu}|_{\mathrm{NC}} \end{cases}, \tag{B2}$$

where $\Delta^{eN}(x)$ and $\Delta^{ep}(x)$ are defined in Eqs. (2.22) and (2.23), respectively.

Next we give examples of the charge factors $\delta_i^{(k)}$ which appeared in our formulas. For the *nonsinglet* charge factors, $\delta_{\mathrm{NS}}^{(k)}$ we have

$$\delta_{\mathrm{NS}}^{(k)} = \begin{cases} \frac{1}{6} & \text{for } F_2^{ep}, F_2^{eN} \\ 1 & \text{for } F_3^{\nu,\bar\nu}|_{\mathrm{CC}} \\ \delta_1^2+\delta_3^2-\delta_2^2-\delta_4^2 & \text{for } F_2^{\nu,\bar\nu}|_{\mathrm{NC}} \\ \delta_1^2+\delta_2^2-\delta_3^2-\delta_4^2 & \text{for } F_3^{\nu,\bar\nu}|_{\mathrm{NC}} \end{cases}. \tag{B3}$$

For the *singlet* charge factors, $\delta_\psi^{(k)}$, we have:

$$\delta_\psi^{(k)} = \begin{cases} \frac{5}{18} & \text{for } F_2^{ep}, F_2^{eN} \\ 1 & \text{for } F_2^{\nu,\bar\nu}|_{\mathrm{CC}} \\ \delta_1^2+\delta_2^2+\delta_3^2+\delta_4^2 & \text{for } F_2^{\nu,\bar\nu}|_{\mathrm{NC}} \end{cases}. \tag{B4}$$

The parameters δ_i are given in Eq. (2.27). The formulas above are for the case of four flavors but it is a simple matter to generalize them to any number of flavors.

The relations between $A_n^{\mp}(Q_0^2)$ and parton distributions are given in Eqs. (5.29) and (5.30). Combining these equations with Table II and the expectation $A_n^\psi(Q_0^2) > A_n^G(Q_0^2)$ for $n \geq 4$ we observe that $A_n^-(Q_0^2) \gg A_n^+(Q_0^2)$ for $n \geq 4$.

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