Physical interpretation of inverse scattering formalism applied to self-induced transparency*

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The inverse scattering method (ISM) as applied to self-induced transparency (SIT) is reviewed. The linear scattering equations (Zakharov-Shabat equations) that are the basis of the inverse scattering method applied to SIT are physically interpreted. They in turn are solved by establishing an analogy with the equations are shown to follow from application of the principle of causality of the parametric amplifier analog.

CONTENTS

INTRODUCTION

The inverse scattering method (ISM) is one of the very few general methods of solution of a class of nonlinear differential equations in one spatial dimension and time (Gardner *et al.*, 1967; Zakharov and Shabat, 1972; Whitham, 1974; Ablowitz *et al.*, 1974a; Lamb, 1973). In the application of the ISM a linear scattering problem of quantum-mechanical nature is associated with the nonlinear differential equation (Whitham, 1974). The sought-for solution of the nonlinear differential equation at the initial time $t=0$ plays the role of the scattering potential or "well" of the linear scattering problem. The initial conditions of the problem to be solved prescribe the transmission and reflection coefficients of the linear scattering problem from which the scattering "well" may be determined by standard techniques of (inverse) scattering theory. The evolution in time of the scattering problem, which may take several forms (Ablowitz *et al.*, 1974a,b), then prescribes the evolution in time of the solution of the nonlinear equation—or, alternately, the form of the nonlinear differential equation associated with this particular scattering problem (see Fig. 1).

In 1973 Lamb showed (Lamb, 1973) how one may associate with the equations of self-induced transparency (SIT) one of the standard equations of the ISM, the Zakharov-Shabat equations (Zakharov and Shabat, 1972). He proceeded through a set of variable transformations with no apparent physical interpretation. In fact, one of the intriguing unsolved problems of the ISM is the development of a procedure to find the scattering problem associated with a particular differential equation.

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In many cases the scattering problem of quantum-mechanical character will have no physical interpretation. It is to be expected, however, that in those cases in which the physics underlying the nonlinear differential equation is based on quantum mechanics the associated scattering problem must have a direct physical meaning. In the case of the Josephson transmission line and SIT, this has been pointed out by McLaughlin and Corones (McLaughlin and Corones, 1974). The physical interpretation of the scattering problem obtained after Lamb had reduced the SIT problem to one amenable to the inverse scattering method is but one example of various developments presented by different authors in the course of time. The present author attempted to gain an understanding of the ISM by uncovering physical interpretations for the mathematical steps. Even though most of the specifics have appeared in the literature (Gardner *et al.*, 1967; Zakharov and Shabat, 1972; Whitham, 1974; Ablowitz *et al.*, 1974a,b; Lamb, 1973; McLaughlin and Corones, 1974; Ablowitz *et al.*, 1973; Faddeyev, 1962), this paper may serve as a guide for the physically inclined to an important mathematical method.

The Zakharov–Shabat equations written as differential equations in normalized time τ with the electric field $\mathcal{E}(\tau, z)$ as the scattering "well" define the linear scattering problem associated with SIT. The spatial variable z plays the role of a parameter. In Sec. I we show the direct connection between the Zakharov–Shabat equations and the equations of the two-level systems interacting with the electric field in SIT.

The nature of the solutions to the Zakharov–Shabat equations may be anticipated by recognizing their sim-

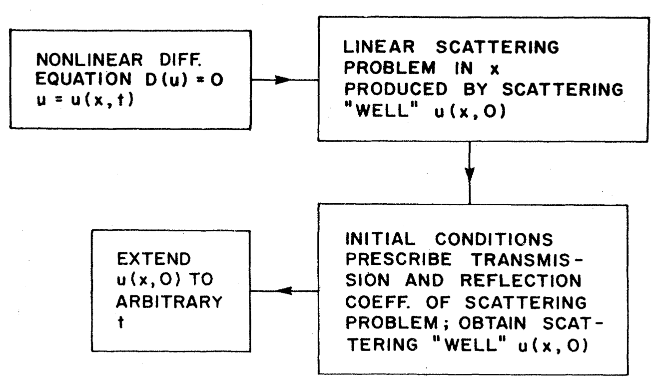

FIG. 1. Flow chart.

ilarity with a well-known physical problem: the parametric interaction of two waves via a nonlinear medium excited by a pump wave (Yariv, 1976). This is done in Sec. II. In particular, it is known that "unstable" solutions growing in time are encountered in parametric interactions. This fact may be used to predict the location of the eigenvalues of the Zakharov–Shabat equations in the complex plane.

The inverse scattering problem is solved and the shape of the scattering well ($\mathcal{E}(\tau, z)$) is obtained in terms of the socalled Jost functions which satisfy the Marchenko equation. In Sec. III we show the "physical" significance of the Jost function, and in Sec. IV we show that the Marchenko equation is the result of Laplace transformation of the Zakharov–Shabat equations (i.e., the parametric interaction) and application of causality.

In Sec. V we show that the assumption of independence of z of the eigenvalues of the Zakharov–Shabat equations leads to a form of the Maxwell–Bloch equations of SIT. In Sec. VI we solve the Marchenko equation in the same way as Lamb has done, filling in some of the omitted steps.

I. THE ZAKHAROV-SHABAT EQUATIONS AS THE EQUATIONS OF THE TWO-LEVEL SYSTEMS

The Zakharov–Shabat equations are central to the inverse scattering method applied to self-induced transparency. In this section we review briefly the equations of a two-level system excited by an E field and show that the resulting equations are equivalent to the Zakharov–Shabat equations (Zakharov and Shabat, 1972) arrived at by Lamb (Lamb, 1973).

In the slow envelope approximation, the wave equation for the electric field envelope $\overline{E}(x,t)$ of a plane wave propagating in the x direction is in mks units (compare Lamb, 1973)

$$\frac{\partial \overline{E}}{\partial x}+\frac{1}{c}\frac{\partial \overline{E}}{\partial t}=i\frac{\omega}{2c}\frac{\overline{P}}{\varepsilon}, \tag{1.1}$$

where c is the speed of light, the dielectric constant, ω the "carrier" frequency, and $\overline{P}$ is the polarization of the medium. $\overline{P}$ and $\overline{E}$ are parallel to each other and transverse to the x direction. The polarization of the medium is obtained from the analysis of two-level systems with a distribution of energy-level spacings. Denote the amplitude of the wave function of the upper level (1) by a_1, that of the lower level (2) by a_2. One may write down two differential equations for the amplitudes a_1 and a_2 as coupled by the $\overline{E}$ field (Vuylsteke, 1960). Factoring out the natural time dependences and retaining only the slowly time-varying portions of the variables, one has

$$\dot{a}_1=(i/\hbar)\overline{E}^*\cdot\overline{p}_{12}\exp i\delta t\, a_2, \tag{1.2}$$

$$\dot{a}_2=\frac{i}{\hbar}\overline{E}\cdot\overline{p}_{12}^*\exp -i\delta t\, a_1, \tag{1.3}$$

where $\delta=\omega-\omega_{21}$, $\overline{p}_{12}$ is the matrix element between the two levels and $\hbar\omega_{21}$ is the energy separation of the levels. If we define

$$a_1\exp-(i\delta/2)t\equiv v_1^*, \tag{1.4}$$

$$a_2\exp+(i\delta/2)t\equiv v_2^*, \tag{1.5}$$

we obtain

$$\frac{dv_1}{dt}-i\frac{\delta}{2}v_1=\frac{\overline{E}\cdot\overline{p}_{21}}{i\hbar}v_2 \tag{1.6}$$

$$\frac{dv_2}{dt}+i\frac{\delta}{2}v_2=\frac{\overline{E}^*\cdot\overline{p}_{21}^*}{i\hbar}v_1. \tag{1.7}$$

These are already the Zakharov–Shabat equations, except for a normalization. The density matrix ρ is the statistical average of the products of the amplitudes a_1, a_2 or v_1, v_2 and their complex conjugates. Since we are dealing here with a pure state (no collisions) no statistical average need be performed.

$$\rho_{ij}=v_i^*v_j. \tag{1.8}$$

The positive frequency portion of the polarization $\overline{P}$ is given by

$$\overline{P}=\langle N\overline{p}_{12}\rho_{21}\rangle=N\overline{p}_{12}\langle v_1v_2^*\rangle \tag{1.9}$$

where N is the particle density and the brackets indicate an average over all two-level system. One obtains from (1.1) and (1.11) by dot-multiplication of both sides by $\overline{p}_{21}/i\hbar$

$$\frac{c}{\Omega^2}\frac{\partial}{\partial x}\left(\frac{(\overline{p}_{21}\cdot\overline{E})}{i\hbar}\right)+\frac{\partial}{\Omega^2\partial t}\left(\frac{\overline{p}_{21}\cdot\overline{E})}{i\hbar}\right)=\langle v_1v_2^*\rangle \tag{1.10}$$

where

$$\Omega^2\equiv\frac{\omega N|\overline{p}_{12}|^2}{2\hbar\epsilon}. \tag{1.11}$$

The equation for the electric field (1.10) completes the system of equations; the solution of the Zakharov–Shabat equations appears directly as a drive in the equation of the field. The system of equations is nonlinear in that the drive is nonlinear in v_1, v_2^*.

Through the use of the normalized variables

$$\mathcal{E}\equiv 2\overline{p}_{21}\cdot\overline{E}/i\hbar\Omega, \quad \zeta\equiv-\delta/2\Omega,$$

$$\tau=\Omega[t-(x/c)], \quad z=\Omega x/c,$$

Eqs. (1.6), (1.7), and (1.10) assume the form

$$\partial\mathcal{E}/\partial z=\langle 2v_1v_2^*\rangle \tag{1.12}$$

$$\partial v_1/\partial\tau+i\zeta v_1=\tfrac{1}{2}\mathcal{E}v_2 \tag{1.13}$$

$$\partial v_2/\partial\tau-i\zeta v_2=-\tfrac{1}{2}\mathcal{E}^*v_1. \tag{1.14}$$

These equations are already in one of the standard forms of inverse scattering theory. McLaughlin and Corones (1974) have pointed out the relation between the linear (Zakharov–Shabat) problem and the quantum-mechanical equations of the Josephson junction. They also touched on the problem of SIT without making the connection of the v's with the wave function amplitudes.

For later reference, and to make connection with the Bloch equations, we also list the differential equations for the density matrix elements (1.8). They follow directly from Eqs. (1.6) and (1.7)

$$\frac{d\rho_{12}}{dt}+i\delta\rho_{12}=\frac{\overline{E}^*\cdot\overline{p}_{12}}{i\hbar}(\rho_{11}-\rho_{22}), \tag{1.15}$$

$$\frac{d}{dt}(\rho_{11}-\rho_{22})=-2\left(\frac{\overline{E}^*\cdot\overline{p}_{12}}{i\hbar}\rho_{21}-\frac{\overline{E}\cdot p_{12}^*}{i\hbar}\rho_{12}\right). \tag{1.16}$$

After introduction of the variable τ and the definitions

$$\lambda\equiv 2\rho_{21},$$

$$N \equiv \rho_{22} - \rho_{11} \, ,$$

one obtains the normalized Bloch equations

$$\partial\lambda/\partial\tau + 2i\zeta\lambda = \mathcal{E}N \, , \tag{1.17}$$

$$\partial N/\partial\tau = -\tfrac{1}{2}(\mathcal{E}^*\lambda + \mathcal{E}\lambda^*) \, . \tag{1.18}$$

Lamb (1973) used the Bloch equations, and the field equation

$$\partial\mathcal{E}/\partial z = \langle\lambda\rangle \tag{1.19}$$

as the defining equation of SIT. Lamb had to go through a series of transformations to derive the Zakharov–Shabat equations. Our way of deriving the equations shows, much more simply, that the Bloch equations are implied by the Zakharov–Shabat equations.

II. THE SCATTERING PROBLEM

We have shown that the nonlinear self-induced transparency equations are cast naturally in terms of a set of linear differential equations for the amplitudes of the wave functions of the upper and lower levels coupled by the electric field. These were the equations of Zakharov and Shabat (1972) central to their formulation of the inverse scattering problem for the nonlinear Schroedinger equation and derived by Lamb (1973) from the density matrix equations of the two-level system by a set of variable transformations. In this section we shall elaborate on the significance of the Zakharov-Shabat equations.

We consider them to be a set of equations of mode coupling in *space*, treating τ as if it were the distance coordinate, and ζ as if it were the propagation constant (ζ is real by definition); the amplitudes v_1 and v_2 are then wave amplitudes. The function $\mathcal{E}(\tau)$ plays the role of the coupling coefficient. In the absence of an $\mathcal{E}$ field,

$$v_1 \propto \exp - i\zeta\tau \tag{2.1}$$

and

$$v_2 \propto \exp + i\zeta\tau \, . \tag{2.2}$$

The wave v_1 propagates in the $-\tau$ direction, and the wave v_2 in the $+\tau$ direction [assuming the physicist's definition of phase delay as represented by the factor $\exp i\zeta\tau$ with $\zeta > 0$]. The amplitudes v_1 and v_2 are functions of ζ and τ. We shall consider later the Fourier transform

$$\int_{-\infty}^{\infty} d\zeta \exp(-i\zeta y) v_1(\zeta, \tau) = V_1(y, \tau) \, . \tag{2.3}$$

If ζ is taken to be a propagation constant, $\zeta = \omega/u$ with u the phase velocity of the uncoupled wave, and ω the frequency, then y may be interpreted as a time variable ($y = ut$). This further interpretation endows the waves v_1 and v_2 with dispersion-free propagation at group velocity u, in the absence of $\mathcal{E}$. The original self-induced transparency problem involving interactions of electromagnetic *pulses* with the nonlinear medium requires that $\mathcal{E}(\tau)$ has to vanish at $|\tau| \to \infty$. Hence Eqs. (1.13) and (1.14) describe coupling of waves in an interaction region extending from $-\infty < \tau < +\infty$, with vanishing interaction in the limit $|\tau| \to \infty$.

In the region where $\mathcal{E} \neq 0$, the forward and backward

	POWER FLOW	ENERGY	GROUP VELOCITY
v_1	→	−	←
v_2	→	+	→

FIG. 2. Schematic of power flow and energy.

waves are coupled. The coupling is lossless in the sense that (for real ζ)

$$d/d\tau(|v_1(\zeta, \tau)|^2 + |v_2(\zeta, \tau)|^2) = 0 \tag{2.4}$$

as can be demonstrated easily from Eqs. (1.13) and (1.14). In Eq. (2.4) waves (1) and (2) may be assigned powers $|v_1(\zeta, \tau)|^2$ and $|v_2(\zeta, \tau)|^2$. According to Eq. (2.4), both waves carry power in the same direction—say the $+\tau$ direction. Because they have oppositely directed group velocities, their energies must be of opposite sign (Pierce, 1974). (See Fig. 2.)

The concept of negative small signal energy is widely used in plasma physics (Sturrock, 1961). Negative energy commonly occurs in energy conservation principles derived from the linearized equations of motion of a nonlinear system which contains an energy "reservoir" [such as the kinetic energy of a moving plasma or an electron beam (Pierce, 1974)]. Excitation of a wave [usually a so-called slow wave (Sturrock, 1961)] may lower the overall energy of the system, a fact that manifests itself in terms of a negative energy attributed to the wave. The energy is quadratic in the excitation amplitude of the wave. If a negative-energy wave is coupled to a positive-energy wave, both wave amplitudes may grow. The growth of positive energy is balanced by the growth of the negative energy, net small signal energy is conserved. One example of such a system is the backward-wave oscillator (Kleen, 1958).

More familiar may be the example of parametric interaction (Yariv, 1976) of two waves of frequencies ω_1 and ω_2 with a pump wave of frequency ω_p, so that $\omega_1 + \omega_2 = \omega_p$. The phase-matching condition of the (collinear) propagation vectors $\bar{k}_1$ and $\bar{k}_2$ is then $\bar{k}_p = \bar{k}_1 + \bar{k}_2$. In the steady state, when phase matching is not realized, one may define

$$\zeta \equiv \tfrac{1}{2}[(k_1 + k_2) - k_p] \, .$$

The equations of parametric coupling between the two waves as a function of the spatial coordinate (τ) are of the form of Eqs. (1.13) and (1.14) when the "fast" spatial variations of the waves are removed and only the slowly varying variations of "envelopes" are considered. The energy densities must be reinterpreted as photon number densities, the power flows as photon number flows, and energy conservation as photon number conservation. To be more specific, in a parametric process of the type where a pump photon (ω_p) produces a "signal" photon at frequency ω_1, and an "idler" photon at frequency ω_2, the number of signal photons generated either spontaneously, or by induced emission, must be equal to the number of idler photons. The wave amplitudes, v_1 and

v_2, may be so normalized that $|v_1|^2$ and $|v_2|^2$ are proportional to the flux of photons in the interacting waves (1) and (2). Then

$$\frac{\partial}{\partial\tau}|v_1|^2 \pm \frac{\partial}{\partial\tau}|v_2|^2 = 0$$

may be interpreted as the Manley–Rowe conservation relations applied to this parametric process (Manley and Rowe, 1959; Weiss, 1957). A parametric instability occurs if the group velocities of waves (1) and (2) are oppositely directed, as indicated by the plus sign in the above equation.

The waves v_1 and v_2 have further properties somewhat analogous to lossless coupling of electromagnetic waves that obey reciprocity relations. Indeed, from Eqs. (1.13) and (1.14) it is easily shown that, given a solution (f is treated as a column matrix of components f_1 and f_2)

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

then

$$\bar{f} \equiv \begin{bmatrix} f_2^* \\ -f_1^* \end{bmatrix}$$

is also a solution for the same ζ (if ζ is real). Further, these two solutions are physically different. Indeed, let f describe the coupling of wave v_2 to v_1 via $\mathcal{E}(\tau)$ with boundary conditions as indicated schematically in Fig. 3(a). The solution $\bar{f}$ is the one shown in Fig. 3(b). The function $\bar{f}$ in relation to f is like the time-reversed solution of electromagnetic waves used to demonstrate reciprocity.

One may generalize Eq. (2.4) to show conservation of "cross power," i.e., prove the conservation law

$$d/d\tau(f_1g_1^* + f_2g_2^*) = 0\,, \tag{2.5}$$

where f and g are any two solutions of Eqs. (1.13) and (1.14). Using the property that

$$\bar{g} = \begin{bmatrix} g_2^* \\ -g_1^* \end{bmatrix}$$

is a solution, if g is one, Eq. (2.5) becomes

$$d/d\tau(f_1g_2 - f_2g_1) = 0\,. \tag{2.6}$$

This is known as conservation of the Wronskian (Gardner *et al.*, 1967).

Thus far we have studied general properties of the scattering problem. One may use Eqs. (1.13) and (1.14) to find solutions v_1 and v_2 for given $\mathcal{E}$. More relevant to the solution of the self-induced transparency problem is the inverse scattering problem: given v_1 and v_2, what coupling function $\mathcal{E}(\tau)$ produces this particular v_1 and v_2. Then by picking solutions for v_1 and v_2 one may find the $\mathcal{E}$ field that is described by them.

The SIT problem calls for a very special kind of solution $v_1(\zeta,\tau)$, $v_2(\zeta,\tau)$. Indeed, ζ is the parameter describing the detuning of the two level systems from the carrier frequency ω_0. If there is to be no loss, $|v_1(\zeta,\tau)| = 1$, $|v_2(\zeta,\tau)| = 0$ for $\tau \to \pm\infty$; i.e., every two-level system has to start from the ground state before the arrival of the pulse and must return into the ground state after passage of the pulse. This requirement in turn calls for a scattering well $\mathcal{E}(\tau)$ which produces no reflection ($v_2 = 0$) for an incident "wave" $v_1(\zeta,\tau)$ for any ζ! There are wells that are capable of doing this. The Schroedinger equation for a secant hyperbolic well has a continuum of eigenstates that are traveling waves outside the well and experience only a phase shift as they pass through the well (Morse and Feshbach, 1953). The solutions of the Marchenko equation—to be derived in a novel way in Sec. IV—prove the existence of such wells in general.

III. THE JOST FUNCTION

We shall concentrate on a particular solution $f(\zeta,\tau)$ whose limit for $\tau \to -\infty$ is

$$\lim_{\tau\to-\infty} f(\zeta,\tau) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i\zeta\tau}\,. \tag{3.1}$$

In the schematic representation of Fig. 3, the solution is shown in Fig. 4. Here $f(\zeta,\tau)$ represents an experiment in which a wave is incident from $\tau \to +\infty$, partly transmitted and partly coupled to the (reflected) wave v_2. The complete solutions may be written

$$f(\zeta,\tau) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i\zeta\tau} + \int_{-\infty}^{\tau} ds\, A(\tau,s)e^{-i\zeta s}\,, \tag{3.2}$$

where

$$A(\tau,s) = \begin{Bmatrix} A_1(\tau,s) \\ A_2(\tau,s) \end{Bmatrix}$$

is the so-called Jost function, independent of ζ. If we define $A(\tau,s) = 0$ for $s > \tau$, then the upper limit in Eq. (3.2) may be extended to infinity.

We can show that Eq. (3.2) is general and, further, that it has a simple physical interpretation. For this purpose we Fourier transform Eq. (3.2) as indicated by Eq. (2.3)

$$\frac{1}{2\pi}\int d\zeta\, e^{-i\zeta y} f(\zeta,\tau) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta(y+\tau) + A(\tau,-y)\,. \tag{3.3}$$

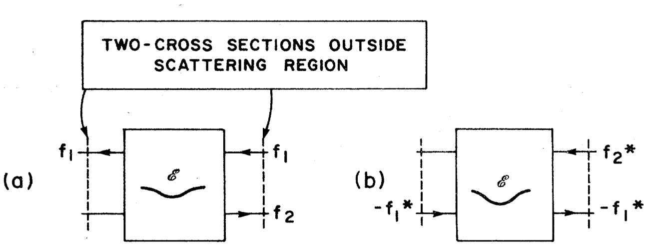

FIG. 3. "Experiments" defining f and $\bar{f}$.

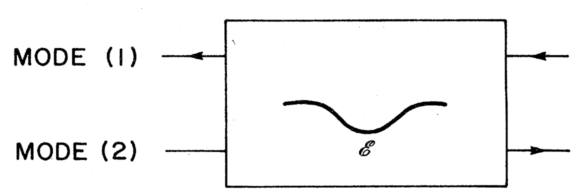

FIG. 4. Coupling of modes.

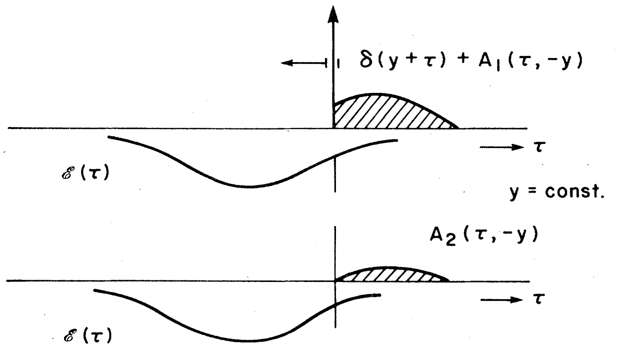

FIG. 5. The coupling via $\mathcal{E}(\tau)$ as expressed by $f(\zeta,\ \tau)$ in the space (τ)–time (y) domain.

This is the space (τ)–time (y) representation of the coupling of modes experiment in Fig. 4. A unit impulse in mode (1) is incident from the right upon the coupling region, partly transmitted and partly coupled to mode (2). Here $A(\tau,-y)$ contains the spreading and scattering information. It is a general function of the space-time variables with the sole restriction that $A(\tau,-y)=0$ for $-y>\tau$, or $y+\tau<0$. This restriction may be interpreted as one of causality, namely that the "response" $A(\tau,-y)$ does not appear at the position τ until $y+\tau>0$, i.e., the time is advanced enough for the impulse to have passed. The process is sketched in Fig. 5.

The Jost function contains information about the scattering well $\mathcal{E}(\tau)$. This is obvious on physical grounds. Mathematically the information is extracted as follows. Consider Eq. (1.13) and Eq. (1.14) in the limit $\zeta\to\infty$. Then, with mode (1) incident from the right, $v_1(\zeta,\tau)$ is to lowest order in $1/\zeta$ the uncoupled v_1 so that

$$v_1=f_1\simeq e^{-i\zeta\tau}\,. \tag{3.4}$$

The amplitude v_2 is obtained from Eq. (1.14). Set

$$f_2=v_2=V_2(\tau)e^{+i\zeta\tau}\,, \tag{3.5}$$

then

$$(d/d\tau)V_2=-\tfrac{1}{2}\mathcal{E}^*(\tau)e^{-2i\zeta\tau}$$

to lowest order in $1/\zeta$. Integration gives (remember the mode v_2 vanishes at $\tau=-\infty$)

$$\begin{aligned}V_2&=-\frac{1}{2}\int_{-\infty}^{\tau}\mathcal{E}^*(s)e^{-2i\zeta s}ds\cong-\frac{1}{2}\mathcal{E}^*(\tau)\int_{-\infty}^{\tau}e^{-2i\zeta s}ds\\&=\frac{1}{2}\frac{\mathcal{E}^*(\tau)}{2i\zeta}e^{-2i\zeta\tau}\end{aligned} \tag{3.6}$$

to lowest order in $1/\zeta$. On the other hand, Eq. (3.2) evaluated to lowest order in $1/\zeta$ gives

$$f_2(\zeta,\tau)\simeq\frac{A_2(\tau,\tau)}{-i\zeta}e^{-i\zeta\tau}\,. \tag{3.7}$$

Comparing Eq. (3.7) with Eqs. (3.6) and (3.5), we find

$$\mathcal{E}^*=-4A_2(\tau,\tau)\,. \tag{3.8}$$

This relation is important in that it gives the coupling "well" $\mathcal{E}(\tau)$ for given information on the coupled mode solution, the Jost function A_2. The evaluation of the scattering well reduces to the evaluation of the Jost function for given scattering information. In the next section we show that the Jost function obeys the Marchenko equation.

Note that the Jost function does not contain the parameter ζ, because it describes the response in the Fourier transform space of ζ (in "time" y). In the spirit of Fourier (or Laplace) transforms, the functional dependence of $f(\zeta,\tau)$ is extended into the entire complex ζ plane. Hence the physical interpretation of ζ as the detuning parameter has been abandoned, at least for the purpose of obtaining a suitable Jost function. The detuning parameter is reintroduced at the end of the analysis after inverse Fourier transforming $A(\tau,-y)$ according to Eq. (3.2) and interpreting $f(\zeta,\tau)$ for real ζ only.

IV. THE MARCHENKO EQUATION

We have pointed out that the function $f(\zeta,\tau)$ defined in Eq. (3.2) represents a coupling-of-modes experiment in which one mode (wave) is incident from the right of the interaction region, partly reflected and partly transmitted. Here $\bar{f}$ is the experiment "run backwards"; $\bar{f}$ is also a solution of Eqs. (1.13) and (1.14). The two solutions f and $\bar{f}$ are linearly independent because they describe different "physical events." Any other solution may be represented as a linear superposition of f and $\bar{f}$. In particular, consider the solution g which, asymptotically, is

$$\lim_{\tau\to\infty}g(\zeta,\tau)=\begin{bmatrix}0\\1\end{bmatrix}e^{i\zeta\tau}\,. \tag{4.1}$$

Figure 6 shows the experiment represented by g. A mode (wave) is incident from the left upon the interaction region and partly coupled to the backward mode (wave), partly transmitted. Because f and $\bar{f}$ describe the system completely, $g(\zeta,\tau)$, must be expressible as a linear superposition of f and $\bar{f}$ [1] and

$$g(\zeta,\tau)=a(\zeta)f(\zeta,\tau)+b(\zeta)\bar{f}(\zeta,\tau)\,. \tag{4.2}$$

In the limit $\tau\to-\infty$, $f(\zeta,\tau)$ describes the reflected wave

$$\lim_{\tau\to-\infty}f=\begin{bmatrix}1\\0\end{bmatrix}e^{-i\zeta\tau}\,, \tag{4.3}$$

and $\bar{f}$ the incident wave

$$\lim_{\tau\to-\infty}\bar{f}=\begin{bmatrix}0\\-1\end{bmatrix}e^{i\zeta\tau}\,. \tag{4.4}$$

The expression

$$\frac{1}{b(\zeta)}g(\zeta,\tau)=\frac{a(\zeta)}{b(\zeta)}f(\zeta,\tau)+\bar{f}(\zeta,\tau) \tag{4.5}$$

evaluated in the limit $|\tau|\to\infty$ contains the conventional

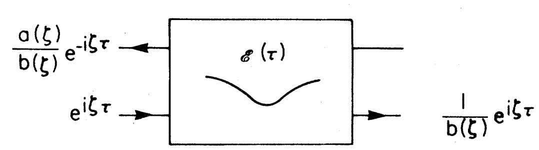

FIG. 6. The "experiment" described by Eq. (4.5).

[1] The a and b coefficients are chosen in the notation of Lamb (1973), which differs from that of Zakharov and Shabat (1972).

reflection coefficient $[a(\xi)/b(\xi)]e^{-2i\xi\tau}$ on the left of the interaction region, and the transmitted wave

$$[1/b(\xi)]e^{i\xi\tau}$$

on the right. Note that power conservation assures, for real ξ.

$$|a|^2+|b|^2=1 \tag{4.6}$$

and thus

$$|1/b|^2-|a/b|^2=1. \tag{4.7}$$

This is a variant of the conservation relation between transmission coefficient T and reflection coefficient Γ of a conventional transmission line

$$T^2+\Gamma^2=1\,. \tag{4.8}$$

In the present case the powers of both waves are positive, hence the difference in sign of the contribution of the reflection coefficient. The "steady-state" (real frequency) scattering solution is completely described by the functions $1/b(\xi)$ and $a(\xi)/b(\xi)$ for real ξ. The interaction region may give rise, however, to solutions growing or decaying in time (the Fourier transform variable y). This is due to the coupling of a positive-energy wave with a negative-energy wave as explained in Sec. II. Solutions growing in "time" are contained in the analytic continuation of Eq. (4.5) into the upper half of the *complex* ξ plane. In doing so, we interpret $\bar{f}(\xi,\tau)$ as complex conjugated for *real* ξ. When ξ is made complex, it appears as ξ in $\bar{f}$, not as ξ^*. Here $b(\xi)$ analytically continued may have zeros in the upper half-plane. When this happens at $\xi=\xi_k$

$$g(\xi_k,\tau)=a(\xi_k)f(\xi_k,\tau) \tag{4.9}$$

and g is not linearly independent of f. Because g decays with $\tau\to+\infty$, and f with $\tau\to-\infty$, we are faced with a "trapped" solution which decays spatially away in both directions from the coupling region. This is an "eigenmode" of the system described by $\mathcal{E}(\tau)$. In the parametric interaction example this would be the "parametrically" unstable solution.

Every solution growing in time is paired with a solution decaying in time, $\xi\to\xi^*$. Indeed, taking the solutions $v_1(\xi,\tau)$ and $v_2(\xi,\tau)$ of Eqs. (1.13) and (1.14), one may construct a new solution by the replacement $v_1^*(\xi^*,\tau)\to v_2(\xi,\tau)$, $v_2^*(\xi^*,\tau)\to -v_1(\xi,\tau)$, $\xi^*\to\xi$.

From the preceding discussion we gather that (4.5) describes the steady-state coupling for real ξ and that analytic continuation into the upper half-plane may uncover eigensolutions pertaining to zeros of $b(\xi)$ at $\xi=\xi_k$. The complex conjugate values of ξ_k also lead to eigensolutions which are decaying with time.

Because Eq. (4.5) describes solutions that may be growing in time (y), the Fourier transform (2.3) must be replaced by a Laplace transform. In Laplace transforming (4.5) we pick a contour in the upper half-plane of ξ at $\mathrm{Im}\,\xi=\kappa$

$$\frac{1}{2\pi}\int_{i\kappa-\infty}^{i\kappa+\infty} d\xi e^{-i\xi y}g(\xi,\tau)/b(\xi)$$

$$=\frac{1}{2\pi}\int_{i\kappa-\infty}^{i\kappa+\infty} d\xi\, e^{-i\xi y}\left[\frac{a(\xi)}{b(\xi)}f(\xi,\tau)+\bar{f}(\xi,\tau)\right] \tag{4.10}$$

The contour may be deformed to coincide with the real ξ axis. In this case integrations around the poles of $a(\xi)/b(\xi)$ have to be added. It is easily confirmed that Eq. (4.10) reduces to

$$\frac{1}{2\pi}\int_{i\kappa-\infty}^{i\kappa+\infty} d\xi\, e^{-i\xi y}g(\xi,\tau)/b(\xi)$$

$$=\begin{bmatrix}0\\-1\end{bmatrix}\delta(y-\tau)+\bar{A}(\tau,y)$$

$$+\begin{bmatrix}1\\0\end{bmatrix}F(y+\tau)+\int_{-\infty}^{\infty}A(\tau,s)F(s+y)ds\,, \tag{4.11}$$

where $F(y)$ incorporates the scattering information,

$$F(y)\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi\frac{a(\xi)}{b(\xi)}e^{-i\xi y}+\sum_k C_k e^{-i\xi_k y}\,, \tag{4.12}$$

with

$$C_k\equiv -i\frac{a(\xi_k)}{b'(\xi_k)}\,,$$

and $(\xi-\xi_k)b'(\xi_k)$ is the leading term in the expansion of $b(\xi)$ around ξ_k. Note that the C_k's are independent of τ, a fact that will be exploited later. The term $\delta(y-\tau)$ is the "incident" impulse producing the response. Clearly, there cannot be any response until the impulse has arrived. Therefore, for $y-\tau<0$, or $y<\tau$

$$\begin{bmatrix}A_2^*(\tau,y)\\-A_1^*(\tau,y)\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}F(y+\tau)$$

$$+\int_{-\infty}^{\tau}A(\tau,s)F(s+y)ds=0\,, \tag{4.13}$$

where we have replaced the upper limit in the integral by τ, because $A(\tau,s)=0$ for $s>\tau$. Equation (4.13) is the pair of the desired Marchenko equations of the inverse scattering problem. Given the "reflection coefficient" $a(\xi)/b(\xi)$ for real ξ and the excitation of the eigensolutions C_k, one may construct $F(y)$. The A's may be found from Eq. (4.13). Finally, the scattering potential is determined from Eq. (3.8).

We have followed here a simple intuitive approach due to Balanis (1972) in order to arrive at the Marchenko equation. Zakharov and Shabat (1972) followed a more formal approach.[2] Ablowitz *et al.* (1974b) presented a derivation very similar to the one here, with no emphasis on the contour deformation in the complex ξ plane.

The result—Eq. (4.13) with Eq. (4.12)—has interesting physical implications. In (4.12) the scattering information is expressed in terms of the residues C_k at the poles ξ_k of the scattering function a/b, and the value of $a(\xi)/b(\xi)$ on the real ξ axis. If one chooses values for the residues, and sets $a(\xi)/b(\xi)=0$ for real ξ, one postulates the existence of a scattering "well" that produces no reflection for an incident "wave" $v_1\propto\exp-i\xi\tau$ $(\tau\to+\infty)$ for all values of real ξ. In other words, one postulates the existence of a reflection-free "well." If Eq. (4.13) has a solution under this assumption (in Sec. VI one such solution is found) one finds $\mathcal{E}(\tau)$ from Eq. (3.8) and

[2]There is an error in their definition of c_k.

one has proven the existence of such a scattering "well."

The parameter ζ has a physical meaning in SIT; it is the detuning parameter. The complex eigenvalue ζ_k does not have this meaning—it is only a parameter characteristic of the potential $\mathcal{E}(\tau)$.

V. INVARIANCE OF ζ WITH z

In the preceding sections we have reviewed the solution of the Zakharov–Shabat equations and given physical interpretations to the mathematical steps. We recognized that SIT implies the existence of a reflection free "well" $\mathcal{E}(\tau)$ so that the lower-level amplitude $v_1(\zeta,\tau)$ starts with unity magnitude at $\tau=+\infty$ and returns to unity magnitude at $\tau\to-\infty$. Such wells exist and one of them is the well known secant hyperbolic to be reviewed in Sec. VI. It is not obvious, however, that such reflection-free wells $\mathcal{E}(\tau)$ are consistent with the propagation equation (1.12), which has not been used as yet. Reflection-free wells are characterized by the zeros ζ_k of the function $b(\zeta)$. Consistency with the propagation equation (1.12) is proven when it is shown that ζ_k is independent of distance z. Ablowitz *et al.* (1974a) provided a general proof. The proof will be presented here in a slightly modified form.

The Zakharov–Shabat equations were shown to be the normalized form of the two-level system equations. The parameter ζ was the detuning parameter, obviously real. When the Laplace transform was taken in Eq. (4.10), the ζ parameter was extended analytically into the complex plane. No physical connection can be made between the responses $f_1(\zeta,\tau)$ and $f_2(\zeta,\tau)$ for complex ζ and the two-level system, for which the normalization $|v_1|^2+|v_2|^2=1$ holds, a normalization not maintained for complex ζ. The complex ζ has a new significance not contained in the two-level system equations or the Bloch equations. A reinterpretation of these equations in terms of the complex ζ parameter is in order.

The solutions of the Zakharov–Shabat equations for any complex ζ define the well $\mathcal{E}(\tau)$. The same holds for the solutions of the Marchenko equations written in terms of the Laplace transform variable y of ζ. Because $\mathcal{E}(\tau)$ is also a function of another variable (the spatial variable z in the case of SIT), the solution of the Marchenko equations contains z as a parameter. The dependence upon the spatial parameter is a simple one if y (or ζ) can be proven to be independent of z. In particular, in the case of the reflection-free well, where the solutions are functions of the ζ_k's and C_k's, only the C_k's may depend on z. We now proceed to prove the invariance of ζ with respect to z. Note that the Bloch equations were written originally in terms of the real detuning parameter δ/Ω for which we used the symbol -2ζ. When ζ is made complex, by analytic continuation of the solutions of the Zakharov Shabat equations, it lost its meaning of a detuning parameter. Henceforth we use the symbol ξ for the detuning parameter, which is real by definition. We introduce the complex ζ into the Bloch equations by Hilbert transforms of the functions $N(\xi,z,\tau)$, $\lambda(\xi,z,\tau)$, and $\lambda^*(\xi,z,\tau)$ with respect to the line-shape function $g(\xi)(\int_{-\infty}^{\infty}g(\xi)d\xi=1)$. We define

$$A(\zeta,z,\tau)\equiv\frac{i}{4}\int_{-\infty}^{\infty}\frac{N(\xi,z,\tau)g(\xi)}{\zeta-\xi}d\xi\,, \tag{5.1}$$

$$B(\zeta,z,\tau)\equiv\frac{i}{4}\int_{-\infty}^{\infty}\frac{\lambda(\xi,z,\tau)g(\xi)}{\zeta-\xi}d\xi\,, \tag{5.2}$$

$$C(\zeta,z,\tau)\equiv\frac{i}{4}\int_{-\infty}^{\infty}\frac{\lambda^*(\xi,z,\tau)g(\xi)}{\zeta-\xi}d\xi\,. \tag{5.3}$$

When ζ approaches the real axis, the integration paths must be interpreted as properly indented.

By transformation of Eqs. (1.17) and (1.18) and the complex conjugate of (1.18) we find that A, B, and C obey the equations

$$\frac{\partial A}{\partial\tau}=-\left(\tfrac{1}{2}\mathcal{E}C+\tfrac{1}{2}\mathcal{E}^*B\right), \tag{5.4}$$

$$\frac{\partial B}{\partial\tau}+2i\zeta B+\tfrac{1}{2}\langle\lambda\rangle=\mathcal{E}A\,, \tag{5.5}$$

$$\frac{\partial C}{\partial\tau}-2i\zeta C-\tfrac{1}{2}\langle\lambda^*\rangle=\mathcal{E}^*A\,. \tag{5.6}$$

The field equation may be written explicitly in terms of the line-shape function $g(\xi)$

$$\frac{\partial\mathcal{E}}{\partial z}=\langle\lambda\rangle=\int_{-\infty}^{\infty}d\xi\,g(\xi)\,\lambda(\xi)\,. \tag{5.7}$$

These equations contain the same information as the Bloch equations, but now they are written in terms of the generalized complex parameter ζ. From here one may proceed analogously to Ablowitz *et al.* One assumes that the evolution of v_1 and v_2 with the variable z proceeds according to the equations

$$\partial v_1/\partial z=A(\zeta,z,\tau)v_1-B(\zeta,z,\tau)v_2\,, \tag{5.8}$$

$$\partial v_2/\partial z=-C(\zeta,z,\tau)v_1+D(\zeta,z,\tau)v_2\,. \tag{5.9}$$

In order that Eqs. (5.8) and (5.9) be consistent with Eqs. (1.13) and (1.14), cross derivatives $\partial^2/\partial\tau\,\partial z$ must be equal to $\partial^2/\partial z\,\partial\tau$. From the requirement that ζ be independent of z one finds

$$A=-D\,, \tag{5.10}$$

where an integration constant has been set equal to zero, and

$$\partial A/\partial\tau=-\left(\tfrac{1}{2}\mathcal{E}C+\tfrac{1}{2}\mathcal{E}^*B\right), \tag{5.11}$$

$$\partial B/\partial\tau+2i\zeta B=-\tfrac{1}{2}\mathcal{E}_z+A\mathcal{E}\,, \tag{5.12}$$

$$\partial C/\partial\tau-2i\zeta C=\tfrac{1}{2}\mathcal{E}_z^*+A\mathcal{E}^*\,. \tag{5.13}$$

With Eq. (5.7), these are the transformed Bloch equations (5.4) through (5.6). Hence conservation of ζ is proved.

VI. THE ε FIELD OF SIT

From the preceding, one gathers the generality of the ISM for the solution of the SIT problem. Any given scattering function (4.12), $F(y,C_k)$, leads to the (linear) Marchenko equation (4.13). From the solution of this equation one finds $A_2(\tau,\tau)$ which yields the electric field $\mathcal{E}(\tau)$ at $z=0$. $\mathcal{E}(\tau,z)$ at arbitrary z is obtained from Eq. (1.12) by integration with respect to z. In the process, the ζ_k's are kept independent of z as shown in the preceding section.

Useful solutions can be obtained by *assuming* simple forms of the scattering function $F(y)$. Suppose we assume $b(\xi)$ has only one zero so that

$$F(y)=C_1 e^{-i\xi_1 y}, \tag{6.1}$$

where

$$C_1 \equiv -i\ a(\xi_1)/b'(\xi_1)$$

and

$$a(\xi)/b(\xi)\to 0 \text{ for real } \xi.$$

The "continuum" contribution to $F(y)$ of Eq. (4.12) is then zero. The coordinate z of the original nonlinear differential equation can appear in $F(y)$ as a parameter, i.e., C_1 is a function of z.

Next we solve the Marchenko equations by separation of variables. We set

$$A_1(\tau,y)=T_1(\tau)\ Y_1(y), \tag{6.2}$$

$$A_2^*(\tau,y)=T_2(\tau)\ Y_2(y). \tag{6.3}$$

When these expressions are introduced in Eq. (4.13) we obtain the two equations

$$T_2(\tau)\ Y_2(y)+C_1 \exp -i\xi_1(\tau+y)+T_1(\tau) \times\int_{-\infty}^{\tau} Y_1(s)\, C_1 \exp -i\xi_1(s+y)\,ds=0 \tag{6.4}$$

$$-T_1(\tau)\,Y_1(y)+C_1^*\,T_2(\tau)\int_{-\infty}^{\tau} Y_2(s)\exp i\xi_1^*(s+y)\,ds=0. \tag{6.5}$$

We divide Eqs. (6.4) and (6.5) by $\exp -i\xi_1 y$ and Eq. (6.5) by $\exp i\xi_1^* y$, and obtain sums of functions of τ alone except for the first term in each of the two expressions. Hence these must be independent of y, culminating in the relations

$$Y_2(y)=\exp -i\xi_1 y, \tag{6.6}$$

$$Y_1(y)=\exp i\xi_1^* y. \tag{6.7}$$

The constant multipliers are absorbed in $T_1(\tau)$ and $T_2(\tau)$. The remaining equations for $T_1(\tau)$ and $T_2(\tau)$ are $(\xi_1=\alpha+i\beta)$

$$T_2(\tau)+(C_1/2\beta)T_1(\tau)\exp 2\beta\tau=-C_1\exp -i(\alpha+i\beta)\tau, \tag{6.8}$$

$$T_1(\tau)-(C_1^*/2\beta)T_2(\tau)\exp 2\beta\tau=0 \tag{6.9}$$

with the solutions

$$T_1(\tau)=-\left|\frac{C_1}{2\beta}\right|^2\left(\exp -i\alpha\tau\exp 3\beta\tau/1+\left|\frac{C_1}{2\beta}\right|^2\exp 4\beta\tau\right), \tag{6.10}$$

$$T_2(\tau)=-C_1\left(\exp -i(\alpha+i\beta)\tau/1+\left|\frac{C_1}{2\beta}\right|^2\exp 4\beta\tau\right). \tag{6.11}$$

The $\mathcal{E}$ field is obtained from Eq. (3.8) which, in combination with Eqs. (6.3), (6.6), and (6.11), gives

$$\mathcal{E}=-4T_2(\tau)\,Y_2(\tau)=4$$

$$=4\left(C_1\exp -2i(\alpha+i\beta)\tau/1+\left|\frac{C_1}{2\beta}\right|^2\exp 4\beta\tau\right). \tag{6.12}$$

Through C_1, which is a function of z, $\mathcal{E}$ is also a function of z. A differential equation for $C_1(z)$ is obtained by substituting $\mathcal{E}(\tau)$ or Eq. (6.12) into Eq. (1.12) and evaluating $f_1(\xi,\tau)$ and $f_2(\xi,\tau)$ in terms of C_1. A great simplification is achieved by using the fact that C_1 is independent of τ and thus the differential equation for $C_1(z)$ cannot depend upon τ. In the limit $\tau\to-\infty$ all expressions are particularly simple. From Eq. (6.12),

$$\lim_{\tau\to-\infty}\mathcal{E}=4C_1\exp -2i(\alpha+i\beta)\tau. \tag{6.13}$$

From Eq. (3.1)

$$\lim_{\tau\to-\infty} f_1(\xi,\tau)=e^{-i\xi\tau}. \tag{6.14}$$

Here $f_2^*(\xi,\tau)$ is obtained from Eq. (1.14), using Eqs. (6.13) and (6.14)

$$\lim_{\tau\to-\infty} f_2^*(\xi,\tau)=-\tfrac{1}{2}e^{-i\xi\tau}\int_{-\infty}^{\tau}\mathcal{E}\,e^{2i\xi\tau}\,d\tau$$

$$=C_1 e^{-2i(\alpha+i\beta)\tau}\frac{i}{\xi-(\alpha+i\beta)}e^{i\xi\tau}. \tag{6.15}$$

Introducing Eqs. (6.15) and (6.14) into Eq. (1.12), one obtains

$$\frac{dC_1}{dz}=\frac{i}{2}\left\langle\frac{1}{\xi-(\alpha+i\beta)}\right\rangle C_1 \tag{6.16}$$

with the solution

$$C_1(z)=C_1(0)\exp\frac{iz}{2}\left\langle\frac{1}{\xi-(\alpha+i\beta)}\right\rangle. \tag{6.17}$$

By an arbitrary adjustment of the initial position of the pulse, which picks $C_1(0)$, one finds for $\mathcal{E}(\tau,z)$

$$\mathcal{E}(\tau,z)=4\beta\exp -2i\alpha(\tau-\tau_1)\operatorname{sech}2\beta(\tau-\tau_0), \tag{6.18}$$

where

$$\tau_0=\frac{1}{4}\left\langle\frac{1}{(\xi-\alpha)^2+\beta^2}\right\rangle z$$

$$\tau_1=\frac{1}{4}\left\langle\frac{(\xi-\alpha)/\alpha}{(\xi-\alpha)^2+\beta^2}\right\rangle z.$$

VII. CONCLUSIONS

We have shown that the scattering equations of the ISM applied to SIT are identical with the basic equations for the wave function amplitudes of the two-level system. With the time variable interpreted as a spatial variable, we have established an analogy of the scattering equations with the backward-wave oscillator or the parametric oscillator equations. The insights gained in the vast literature on this subject aid the scattering analysis. The Marchenko equation was obtained by a simple Laplace transform of the scattering equations and use of the causality condition.

The analytic continuation of the scattering problem into the complex ξ plane called for an identification of ξ distinct from the detuning parameter. To introduce this parameter into the Bloch equations, a Hilbert-like transform was performed on them. Independence of ξ from the coordinate z was then shown to lead to the modified Bloch equations and the field equation (1.19).

Whereas most of the results in this paper are available in the literature, the introduction of physical analogies in the mathematical derivations may help to build up an intuitive grasp of the ISM.

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