Weak hadronic decays: $K \to 2\pi$ and $K \to 3\pi$

Thomas J. Devlin and Jean O. Dickey*

Department of Physics, Rutgers, The State University, Busch Campus, Piscataway, New Jersey 08854

We present a phenomenological review of the weak hadronic decay processes $K\mu 2\pi$ and $K \to 3\pi$, primarily the *CP*-conserving transitions. We have paid particular attention to consistency between different experimental measurements of the same quantity. We have performed a least-squares analysis of the data to determine transition amplitudes to various final states, and the dependence on final-state energy variables. Tests of isotopic spin selection rules have been made by applying various constraints to the least-squares analysis. Amplitudes for pure isotopic spin transitions to specific final states are presented. Current algebra predictions are remarkably well satisfied for the pure $\Delta I = 1/2$ amplitude.

CONTENTS

I. INTRODUCTION

Over the past two decades, a considerable amount of experimental work has been devoted to studies of K meson decay processes. Much of this is relevant to the purely hadronic processes $K \to 2\pi$ and $K \to 3\pi$. Almost every conceivable type of measurement has been attempted, some often repeated. Now a new spectroscopy, with a new set of decay modes and selection rules, is upon us, and is likely to absorb our attention in the coming decades. Before K meson decays become an archaic study and fade from memory, it seems useful to attempt a critical review of the experimental and phenomenological situation.

Much of the crucial labor in a study like this one has already been done by the Particle Data Group (PDG, 1976). They have assembled a comprehensive list of experimental results with careful attention given to selecting directly measured quantities. Further, they have adopted a consistent set of variables for the final-state energy spectra. When necessary, they have transformed the originally published data to conform to these variables. As a starting point for this review, we have adopted their variables virtually unchanged, and their data set with the additions, deletions, and substitutions discussed below.

Our major concern is with the weak hadronic decays, $K \to 2\pi$ and $K \to 3\pi$. We consider the transition rates, the branching ratios, and the energy dependence in the Dalitz plots. Strong experimental constraints on the hadronic transition rates can be derived from the lifetimes (total rates) and the leptonic and semileptonic rates. Therefore those processes have been included in our analysis. The minor modes (branching fraction $< 10^{-3}$) have been summed from the PDG values. We account for $K_L^0 \to 2\pi$ explicitly as the largest of the minor modes, but otherwise *CP* conservation is assumed. Also, we make no distinction between charge conjugate processes.

Our goals are as follows: To perform a general least-squares analysis on all the data relevant to these decay modes, parametrized in terms of transitions amplitudes with appropriate energy dependence. To examine the experimental results for consistency and, where inconsistencies exist, to search for the cause and to test the stability of the fitted solutions against deletion of various pieces of data. To test isotopic spin selection rules. To determine the minimal parameter set required to describe the data adequately, and especially, to test the need for quadratic terms in the energy dependence of the $K \to 3\pi$ transition amplitudes. And, finally, to comment on the usefulness of further experimental studies in this area.

II. AMPLITUDES AND OBSERVABLES OF HADRONIC K DECAYS

A. $K \to 2\pi$

There are three distinct two-pion decay modes for the K meson, $K^{\pm} \to \pi^{\pm}\pi^0$, $K_s^0 \to \pi^+\pi^-$, and $K_s^0 \to \pi^0\pi^0$. Because of Bose statistics and angular momentum conservation, the final states can occur only with $I=0$ or $I=2$. Four parameters are needed to describe all the physical observables (see Appendix A). They are three amplitudes: $a_{1/2}$, $a_{3/2}$, and $a_{5/2}$, and $\delta_2 - \delta_0$, the phases difference between the $I=2$ and $I=0$ s-wave $\pi\pi$ states. The

*Present address: Physics Department, California Institute of Technology, Pasadena, California 91125.

subscript on the amplitudes represents the change in isotopic spin between initial and final states.

B. $K\to 3\pi$

Four distinct three-pion decay modes exist for the K meson. They are $K^\pm\to\pi^\pm\pi^\pm\pi^\mp$, $K^\pm\to\pi^0\pi^0\pi^\pm$, $K_L^0\to\pi^+\pi^-\pi^0$, and $K_L^0\to\pi^0\pi^0\pi^0$. The amplitudes are expanded in terms of the final-state energy variables X and Y, where

$$Y=(s_3-s_0)/m_{\pi^+}^2 \text{ and } X=(s_2-s_1)/m_{\pi^+}^2 ,$$

which are similar, but not identical, to the Dalitz variables. The Lorentz invariants are

$$s_i=(\bar{p}_k-\bar{p}_i)^2 \quad s_0=\sum_{i=1}^{3} s_i/3 .$$

Here $\bar{p}_k$ and $\bar{p}_i$ are four-vectors for the K and the ith pion with index 3 referring to the odd pion. If we adopt a series expansion of the amplitudes in terms of X and Y, the number of parameters is unlimited. In practice, we feel that the available experimental data do not justify expansion beyond a small quadratic term. Further, the processes seem dominated by a $\Delta I=1/2$ transition to the $I=1$ final state. Only this amplitude will be expanded to second order. As we shall see, the fits show no evidence for $I=3$ in the final state. Therefore we retain only the constant term in the corresponding amplitude. All remaining transition amplitudes will be cut off at the linear term. (In principle, there is no linear term in the $I=3$ amplitude, and no constant term in the $I=2$ amplitude.) The parameters required are m_{11}, m_{13}, m_{35}, m_{37}, a'_{11}, a'_{13}, a'_{23}, a'_{25}, b', c', $\delta_M-\delta_1$, $\delta_2-\delta_1$, and $\delta_3-\delta_1$ (see Appendix A). Eight of the parameters have two subscripts; the first indicates the isospin of the final state and the second is twice the change in isospin between initial and final states. The parameters m are the constants in the expansion. The parameters a' are coefficients of the terms linear in Y. The parameters b' and c' are coefficients of the symmetric and mixed-symmetry quadratic terms, $(Y^2+X^2/3)$ and $(Y^2-X^2/3)$, respectively. Finally, $\delta_M-\delta_1$, $\delta_2-\delta_1$, and $\delta_3-\delta_1$ are the phase differences between the s-wave three-pion states, where the subscripts 1, M, 2, and 3 indicate the isospin eigenstates: symmetric $I=1$, mixed symmetry $I=1$, $I=2$ and $I=3$, respectively. The relationships between these parameters and the physical observables are given in Appendix A.

Two $K_S^0\to 3\pi$ modes should exist, but have not yet been observed. The best experimental upper limits are: $(K_S^0\to\pi^+\pi^-\pi^0)/(K_L^0\to\pi^+\pi^-\pi^0)<0.12$ (Metcalf *et al.*, 1972) and $(K_S^0\to 3\pi^0)/(K_L^0\to 3\pi^0)<0.28$ (Gjesdal *et al.*, 1974). Both branching ratios are expected to be of order 10^{-3}, dominated by CP-violating transitions. We shall not include these two modes in our study.

III. DATA AND PARAMETERS

Sixty-six measurable quantities were considered in the fitting program (Appendix B). They are listed in Table I. Experimental measurements of fifty of these (1–3, 10–24, 26–33, 36–43, 47–50, 52–55, 57, 58, 60–63, 65, 66) are available and were used as input data for the fits. The values of all sixty-six were computed from the fits.

Twenty-seven parameters were used in the fitting functions. They are listed in Table II. (We shall see in Sec. IV that not all these parameters are needed to describe the data.) In some cases, e.g., leptonic and semileptonic decay rates, the parameter selected was one of the physically measurable quantities. Since our primary interest here is in CP-allowed hadronic decays, we avoided more detailed parametrization of other processes.

A. Overall selection of data

In selecting data for this study, we have followed the guidance of the PDG (1976) wherever possible. The measurements consisted of 186 pieces of data from 126 published experimental papers. Most of these are listed in PDG 1976. Recent results not included in that review are:

(1) Aronson *et al.* (1976), K_S^0 mean lifetime,
(2) Bertrand *et al.* (1976), energy dependence for $K^+\to\pi^0\pi^0\pi^+$,
(3) Everhart *et al.* (1976), the ratio $(K_S^0\to\pi^+\pi^-)/(K_S^0\to 2\pi^0)$,
(4) Rey *et al.* (1976), the ratio $(K_2^0\to 2\pi^0)/(K_S^0\to 2\pi^0)$, (This includes measurements from Cence *et al.*, 1969. The earlier paper was deleted from our analysis.)
(5) Weissenberg *et al.* (1976), ratios of leptonic and semileptonic partial widths for K^+.
(6) Cho *et al.* (1977), the ratio $(K_L^0\to\pi^+\pi^-\pi^0)/(K_L^0\to \text{charged})$, and energy dependence for $K_L^0\to\pi^+\pi^-\pi^0$.
(7) Peach *et al.* (1977), Energy dependence for $K_L^0\to\pi^+\pi^-\pi^0$.

Some aspects of the data selection which required special attention are discussed in the following sections.

B. K_S^0 lifetime

Before 1972 the PDG value for the K_S^0 mean life was $(0.862\pm0.006)\times10^{-10}$ sec. Then three high-precision experiments [Skjeggestad *et al.* (1972), Geweniger *et al.* (1974), and Carithers et al. (1975)] found values compatible with each other which averaged to $(0.8930\pm0.0023)\times10^{-10}$ sec. The origin of the difference is not known. Since the newer experiments are thought to be superior, the PDG has chosen to average them separately from the older experiments. They quote the newer value in the Stable Particle Table. We have followed this approach, discarding the pre-1972 measurements. We have added the new measurement of Aronson *et al.* (1976).

C. Energy dependence in three-pion final states

The PDG listings give the linear term, g, for the final-state energy spectrum of the processes $K^\pm\to\pi^\pm\pi^\pm\pi^\mp$, $K^\pm\to\pi^0\pi^0\pi^\pm$, and $K_L^0\to\pi^+\pi^-\pi^0$ for all experi-

TABLE I. Measurable quantities and results of fits with $\Delta I \leq \frac{3}{2}$ constraint.

	Measurable quantity	PDG Symbol	Units	Fit LF3A[c]	Fit QF3A[c] (where different)
	$K^\pm$ Quantities				
1	$\tau^\pm$ $K^\pm$ Lifetime	T	10^{-8} sec	1.2373 ± 0.0013	
2	$\Gamma^\pm_{\mu 2}$ Rate $K^\pm \to \mu^\pm \nu$	W1	10^6 sec^{-1}	51.03 ± 0.13	
3	$\Gamma_{\pm\pm\mp}$ Rate $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$	W2	10^6 sec^{-1}	4.52 ± 0.02	
4	$\Gamma_{00\pm}$ Rate $K^\pm \to \pi^0\pi^0\pi^\pm$		10^6 sec^{-1}	1.335 ± 0.006	
5	$\Gamma_{\pm 0}$ Rate $K^\pm \to \pi^\pm\pi^0$		10^6 sec^{-1}	16.90 ± 0.11	
6	$\Gamma^\pm_{\mu 3}$ Rate $K^\pm \to \pi^0\mu^\pm\nu$		10^6 sec^{-1}	2.59 ± 0.04	
7	$\Gamma^\pm_{e3}$ Rate $K^\pm \to \pi^0 e^\pm \nu$		10^6 sec^{-1}	3.90 ± 0.04	
8	$\Gamma^\pm_{MM}$ Rate $K^\pm \to$ Minor modes		10^6 sec^{-1}	0.536[a]	
9	Ratio $\Gamma^\pm_{\mu 2}/\Gamma^\pm_{total}$	R1	None	0.6317 ± 0.0015	
10	Ratio $\Gamma_{\pm 0}/\Gamma^\pm_{total}$	R2	None	0.2090 ± 0.0013	
11	Ratio $\Gamma_{\pm\pm\mp}/\Gamma^\pm_{total}$	R3	None	0.0560 ± 0.0003	
12	Ratio $\Gamma_{00\pm}/\Gamma^\pm_{total}$	R4	None	0.01651 ± 0.00008	
13	Ratio $\Gamma^\pm_{\mu 3}/\Gamma^\pm_{total}$	R5	None	0.0320 ± 0.0005	
14	Ratio $\Gamma^\pm_{e3}/\Gamma^\pm_{total}$	R6	None	0.0482 ± 0.0005	
15	Ratio $(\Gamma_{\pm 0} + \Gamma^\pm_{\mu 3})/\Gamma^\pm_{total}$	R7	None	0.2410 ± 0.0013	
16	Ratio $\Gamma^\pm_{MM}/\Gamma^\pm_{total}$		None	0.00664[a]	
17	Ratio $\Gamma_{\pm 0}/\Gamma_{\pm\pm\mp}$	R17	None	3.74 ± 0.03	
18	Ratio $\Gamma_{00\pm}/\Gamma_{\pm\pm\mp}$	R18	None	0.2952 ± 0.0001	
19	Ratio $\Gamma^\pm_{\mu 3}/\Gamma_{\pm\pm\mp}$	R19	None	0.572 ± 0.009	
20	Ratio $\Gamma^\pm_{e3}/\Gamma_{\pm\pm\mp}$	R20	None	0.862 ± 0.009	
21	Ratio $\Gamma^\pm_{e3}/(\Gamma_{\pm 0} + \Gamma^\mp_{\mu 2})$	R23	None	0.0574 ± 0.0006	
22	Ratio $\Gamma_{\pm 0}/\Gamma^\pm_{\mu 2}$	R24	None	0.331 ± 0.003	
23	Ratio $\Gamma^\pm_{e3}/\Gamma^\pm_{\mu 2}$	R25	None	0.0764 ± 0.0009	
24	Ratio $\Gamma^\pm_{\mu 3}/\Gamma^\pm_{\mu 2}$	R26	None	0.0506 ± 0.0009	
25	Ratio $\Gamma^\pm_{\mu 2}/\Gamma_{\pm\pm\mp}$	R27	None	11.29 ± 0.07	
26	Ratio $\Gamma^\pm_{\mu 3}/\Gamma^\pm_{e3}$	R29	None	0.663 ± 0.010	
27	$g_{\pm\pm\mp}$ Coefficient of Y for $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$	GT$^\pm$	None	−0.2106 ± 0.0023	−0.2128 ± 0.0023
28	$h_{\pm\pm\mp}$ Coefficient of Y^2 for $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$		None	0.01109 ± 0.00024[b]	0.0177 ± 0.0025
29	$k_{\pm\pm\mp}$ Coefficient of X^2 for $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$		None	⋯	−0.0071 ± 0.0009
30	$g_{00\pm}$ Coefficient of Y for $K^\pm \to \pi^0\pi^0\pi^\pm$	GTP	None	0.550 ± 0.013	0.603 ± 0.023
31	$h_{00\pm}$ Coefficient of Y^2 for $K^\pm \to \pi^0\pi^0\pi^\pm$		None	0.076 ± 0.004[b]	0.056 ± 0.008
32	$k_{00\pm}$ Coefficient of X^2 for $K^\pm \to \pi^0\pi^0\pi^\pm$		None	⋯	0.0067 ± 0.0012
	K_S Quantities				
33	τ_S K_S Lifetime	T	10^{-10} sec	0.8923 ± 0.0022	
34	Γ^S_{+-} Rate $K_S \to \pi^+\pi^-$		10^{10} sec^{-1}	0.770 ± 0.003	
35	Γ^S_{00} Rate $K_S \to \pi^0\pi^0$		10^{10} sec^{-1}	0.351 ± 0.002	
36	Ratio $\Gamma^S_{+-}/\Gamma^S_{total}$	R1	None	0.687 ± 0.002	
37	Ratio $\Gamma^S_{00}/\Gamma^S_{total}$	R2	None	0.313 ± 0.002	
38	Ratio $\Gamma^S_{+-}/\Gamma^S_{00}$	R3	None	2.19 ± 0.02	
	K_L Quantities				
39	τ_L K_L Lifetime	T	10^{-8} sec	5.18 ± 0.04	5.19 ± 0.04
40	Γ^L_{000} Rate $K_L \to \pi^0\pi^0\pi^0$	W1	10^6 sec^{-1}	3.85 ± 0.04	3.82 ± 0.04
41	Γ^L_{+-0} Rate $K_L \to \pi^+\pi^-\pi^0$	W2	10^6 sec^{-1}	2.41 ± 0.03	2.42 ± 0.03
42	Γ^L_{e3} Rate $K_L \to \pi e \nu$	W3	10^6 sec^{-1}	7.48 ± 0.10	7.50 ± 0.10
43	$\Gamma^L_{\mu 3}$ Rate $K_L \to \pi\mu\nu$	W6	10^6 sec^{-1}	5.23 ± 0.08	
44	Γ^L_{+-} Rate $K_L \to \pi^+\pi^-$		10^6 sec^{-1}	0.0394 ± 0.0011	

TABLE I. (*Continued*)

	Measurable quantity	PDG Symbol	Units	Fit LF3A [c]	Fit QF3A [c] (where different)
45	Γ^L_{00} Rate $K_L\to\pi^0\pi^0$		10^6 sec^{-1}	0.017 ± 0.002	
46	Γ^L_{MM} Rate $K_L\to$ Minor modes		10^6 sec^{-1}	0.2615 [a]	
47	Γ^L_{CH} Rate $K_L\to$ Charged modes	W4	10^6 sec^{-1}	15.2 ± 0.1	
48	Γ^L_{LEPT} Rate $K_L\to$ Leptonic modes	W5	10^6 sec^{-1}	12.7 ± 0.1	
49	Ratio $\Gamma^L_{000}/\Gamma^L_{CH}$	R1	None	0.254 ± 0.003	0.251 ± 0.003
50	Ratio $\Gamma^L_{+-0}/\Gamma^L_{CH}$	R2	None	0.159 ± 0.002	
51	Ratio $\Gamma^L_{\mu 3}/\Gamma^L_{CH}$	R3	None	0.345 ± 0.005	
52	Ratio $\Gamma^L_{e3}/\Gamma^L_{CH}$	R4	None	0.494 ± 0.005	
53	Ratio $\Gamma^L_{e3}/\Gamma^L_{LEPT}$	R5	None	0.589 ± 0.006	
54	Ratio $\Gamma^L_{+-}/\Gamma^L_{CH}$	R9	10^{-3}	2.60 ± 0.07	
55	Ratio $\Gamma^L_{\mu 3}/\Gamma^L_{e3}$	R10	None	0.699 ± 0.016	0.697 ± 0.016
56	Ratio $\Gamma^L_{00}/\Gamma^L_{total}$	R17	10^{-3}	0.87 ± 0.11	
57	Ratio $\Gamma^L_{000}/\Gamma^L_{+-0}$	R18	None	1.599 ± 0.002	1.579 ± 0.003
58	Ratio $\Gamma^L_{00}/\Gamma^L_{000}$	R19	10^{-3}	4.4 ± 0.6	
59	Ratio $\Gamma^L_{+-}/\Gamma^L_{LEPT}$	R20	10^{-3}	3.10 ± 0.08	
60	Ratio $\Gamma^L_{MM}/\Gamma^L_{total}$		None	0.0136 [a]	
61	g_{+-0} Coefficient of Y for $K_L\to\pi^+\pi^-\pi^0$	GTO	None	0.627 ± 0.005	0.677 ± 0.008
62	h_{+-0} Coefficient of Y^2 for $K_L\to\pi^+\pi^-\pi^0$		None	0.098 ± 0.002 [b]	0.075 ± 0.005
63	k_{+-0} Coefficient of X^2 for $K_L\to\pi^+\pi^-\pi^0$		None	...	0.0076 ± 0.0014
64	$h_{000}=3k_{000}$ Coefficient of $(Y^2+X^2/3)$ $K_L\to\pi^0\pi^0\pi^0$		None	...	−0.0083 ± 0.0024
	Additional Quantities				
65	Ratio $\Gamma^S_{MM}/\Gamma^S_{total}$		10^{-3}	0.0020 [a]	
66	$\delta_2-\delta_0$ Phase difference s-wave π-π states		Radians	−0.79 ± 0.08	

[a] Fixed at this value because of fixed parameter (see text Sec. IV).
[b] The quadratic term arises from the square of the linear term in the amplitude, $h=g^2/4$.
[c] Uncertainities are those which emerge from the fit. No scale factors have been applied as is the case in PDG, 1976. See text, Appendix B.

ments. We have excluded from our analysis those with less than one thousand events. The number of measurements is large enough that a few deviant results do not affect the fitted parameters significantly. Therefore we have not excluded any further experiments from the linear fits.

Although the quadratic energy dependence is discussed in PDG, 1976, a comprehensive listing of the experimental results is not given. We have considered data from the following experiments: $K^\pm\to\pi^\pm\pi^\pm\pi^\mp$ Mast *et al.*, 1969; Ford *et al.*, 1972; Hoffmaster *et al.*, 1972. $K^\pm\to\pi^0\pi^0\pi^\pm$ Davison *et al.*, 1969; Aubert *et al.*, 1972; Sheaff, 1975; Smith *et al.*, 1975; Bertrand *et al.*, 1976. $K^0_L\to\pi^+\pi^-\pi^0$ Hopkins *et al.*, 1967; Basile *et al.*, 1968; Albrow *et al.*, 1970; Smith *et al.*, 1970; Bisi and Ferrero, 1974; Messner, 1974; Messner *et al.*, 1974; Slone, 1974; Buchanan *et al.*, 1975; Cho *et al.*, 1977; Peach *et al.*, 1977. For Ford *et al.*, (1972), we used the combined K^+ and K^- result. Some experimenters performed separate linear and quadratic fits to their data. Our analysis does the same thing, and the appropriate fits from the original papers have been used in each case.

The number of measurements of quadratic terms is fairly small, and the experiments are vulnerable to systematic errors. This is particularly true of $K^0_L\to\pi^+\pi^-\pi^0$. (See the discussion in Peach *et al.*, 1977.) The fits of Basile *et al.* (1968) to g and h for $K^0_L\to\pi^+\pi^-\pi^0$ were deleted from our analysis. The linear term g is in substantial disagreement with a number of newer and more precise experiments, and we doubt the reliability of the quadratic term. There is substantial disagreement about the value of h_{+-0} measured in the three most precise experiments, Albrow *et al.* (1970), Bisi and Ferrero (1974), and Messner (1974). This last paper included radiative corrections while the others did not. The reported semileptonic backgrounds were 17%, 0.5%, and 9%, respectively. We have tried a number of fits to data sets with and without various specific experiments. This will be discussed at length in Sec. IV.A. We find the data of Messner (1974) to be more consistent with the overall fit than the other two experiments. We have excluded the data of Albrow *et al.* (1970) and Bisi and Ferrero (1974) from the quadratic fits.

TABLE II. Fitted parameters.[a]

	Parameter	Units	Fit LF7A	Fit LF3A	Fit QF3A
	Leptonic and semileptonic rates				
1	$\Gamma^L_{e3}(K^0_L \to \pi e \nu)$	10^6 sec^{-1}	7.41 ±0.11	7.48 ±0.10	7.51 ±0.10
2	$\Gamma^L_{\mu 3}(K^0_L \to \pi \mu \nu)$	10^6 sec^{-1}	5.22 ±0.08	5.23 ±0.08	5.22 ±0.09
3	$\Gamma^{\pm}_{e3}(K^{\pm} \to \pi e \nu)$	10^6 sec^{-1}	3.90 ±0.04	3.90 ±0.04	3.91 ±0.04
4	$\Gamma^{\pm}_{\mu 3}(K^{\pm} \to \pi \mu \nu)$	10^6 sec^{-1}	2.58 ±0.04	2.59 ±0.04	2.59 ±0.04
5	$\Gamma^{\pm}_{\mu 2}(K^{\pm} \to \mu \nu)$	10^6 sec^{-1}	51.02 ±0.13	51.06 ±0.13	51.03 ±0.13
	$K \to 3\pi$ Amplitudes (see Appendix A)				
6	m_{11} (Constant term)	10^{-8}	91.75 ±0.29	91.25 ±0.21	91.46 ±0.24
7	m_{13} (Constant term)	10^{-8}	3.42 ±0.24	3.51 ±0.17	3.57 ±0.18
8	m_{35} (Constant term)	10^{-8}	−0.86 ±0.32	0	0
9	m_{37} (Constant term)	10^{-8}	−0.08 ±0.28	0	0
10	a'_{11} (Linear term)	10^{-8}	24.30 ±0.26	24.15 ±0.24	25.83 ±0.41
11	a'_{13} (Linear term)	10^{-8}	−0.99 ±0.15	−1.14 ±0.15	−1.24 ±0.24
12	a'_{23} (Linear term)	10^{-8}	3.36 ±0.37	3.06 ±0.33	4.35 ±0.62
13	a'_{25} (Linear term)	10^{-8}	0	0	0
14	b' (Quadratic term)	10^{-8}	0	0	−0.37 ±0.11
15	c' (Quadratic term)	10^{-8}	0	0	−1.25 ±0.12
	Phase-shift differences				
16	$\delta_{M1}(3\pi)$	Radians	0	0	0
17	$\delta_{21}(3\pi)$	Radians	0	0	0
18	$\delta_{31}(3\pi)$	Radians	0	0	0
19	$\delta_2 - \delta_0(2\pi)$	Radians	−0.79 ±0.09	−0.79 ±0.08	−0.79 ±0.08
	$K \to 2\pi$ Amplitudes (see Appendix A)				
20	$a_{1/2}$	keV	0.4687 ± 0.0006	0.4687 ± 0.0006	0.4687 ± 0.0006
21	$a_{3/2}$	keV	0.0210 ± 0.0001	0.0210 ± 0.0001	0.0210 ± 0.0001
22	$a_{5/2}$	keV	0	0	0
	CP-violating rates				
23	$\Gamma^L_{+-}(K^0_L \to \pi^+\pi^-)$	10^6 sec^{-1}	0.0391 ± 0.0011	0.0394 ± 0.0011	0.0395 ± 0.0011
24	$\Gamma^L_{00}(K^0_L \to \pi^0\pi^0)$	10^6 sec^{-1}	0.0174 ± 0.0023	0.0168 ± 0.0023	0.0167 ± 0.0022
	Remaining minor modes				
25	$\Gamma^{\pm}_{MM}(K^{\pm} \to$ minor modes)	10^6 sec^{-1}	0.536	0.536	0.536
26	$\Gamma^L_{MM}(K^0_L \to$ minor modes)	10^6 sec^{-1}	0.262	0.262	0.262
27	$\Gamma^S_{MM}(K^0_S \to$ minor modes)	10^6 sec^{-1}	22.40	22.40	22.40

[a] Parameters with no quoted uncertainties were fixed at the values given in the table during the fitting procedure.

D. Two-pion and three-pion phase shifts

The Particle Data Group (1976) does not list the S-wave $\pi\pi$ phase-shift differences between the $I=2$ and $I=0$ final states $\delta_2 - \delta_0$. We have used as input data for our analysis the value $\delta_2 - \delta_0 = (-41.4° \pm 8.1°)$ which is a weighted average of results from Baton *et al.* (1970), Baubillier *et al.* (1972), Cohen *et al.* (1973), Colton *et al.* (1971), Grayer *et al.* (1973), Katz *et al.* (1969), Losty *et al.* (1974), Maratek *et al.* (1968), Morgan and Pišút (1970), Protopopescu *et al.* (1972), Sander *et al.* (1972), Skuja (1972), Sonderegger and Bonamy (1969), Villet *et al.* (1973), and Walker *et al.* (1967).

No corresponding results are available for $\delta_n - \delta_1$, $\delta_2 - \delta_1$ and $\delta_3 - \delta_1$ in the 3π final state. The $K \to 3\pi$ data are not sufficient to constrain these parameters. (See Sec. III.G.) Therefore we have fixed the values of $\delta_n - \delta_1$, $\delta_2 - \delta_1$, and $\delta_3 - \delta_1$ at zero.

E. Correlations

A number of experiments quote values for several numbers of interest from the same experimental data. For example, Chiang *et al.* (1972) quote seven branching fractions from a large sample of K^+ decays. Statistical and possible systematic correlations exist among such results. Whenever the original reference provides the correlation matrix, we have used it. In cases where it was not quoted, we have attempted to determine it, either from inquiries to the experimenters, from evidence within the paper, or from comparison with similar experiments. Our fits with and without correlations establish that the results are insensitive to the correlation terms. Therefore we have not exerted extraordinary efforts to determine them precisely.

The correlation matrix becomes singular in cases where experimenters have used the constraint that measured branching fractions must sum to 1.00. This difficulty is eliminated by deleting one of the branching fractions and the appropriate row and column of the matrix from the input data set. No information is lost because the fitting program has the same constraints built in. The choice of which branching fraction to eliminate is arbitrary. We selected $(K^{\pm} \to \mu^{\pm}\nu)/(K^{\pm} \to \text{ALL})$ and $(K_L^0 \to \pi^{\pm}\mu^{\mp}\nu)/(K_L^0 \to \text{ALL CHARGED})$ in cases where the problem arose.

F. Input data sets

Several data sets were used as inputs to the fitting program. Data set LD included experimental measurements of lifetimes, decay rates, branching ratios, $\pi\pi$ phase-shift differences, and linear energy dependence in 3π final states. Only linear fits were included in this set. We did not include linear terms from quadratic fits. The second data set, QD, was like LD except that all linear results were deleted, and we added the results of quadratic fits by experimenters. In order to study some inconsistencies in quadratic terms, we formed several additional data sets, QDA, QDB, QDC, and QDD, the characteristics of which are discussed in Sec. IV.

G. Parameter sets

The fitting program provided facilities for fixing any parameter at a specified value. We defined a number of parameter sets to test various hypotheses. Their characteristics, along with results, will be tabulated in Sec. IV. For example, the $\Delta I = 1/2$ rule can be tested by setting to zero all amplitudes and associated phases which violate this rule, as is done in parameter sets LF1 and QF1. The notation is coded so that LF (QF) stands for linear (quadratic) fitting function for the $K \to 3\pi$ energy spectra. The number, $n = 1$, 3, 5, or 7, stands for a $\Delta I \geq n/2$ selection rule. Other sets used for specific purposes are displayed in the table.

Some of the parameters are redundant. This is obvious in the case of the slope parameters for $K \to 3\pi$. Three experimental measurements are available to constrain six parameters: a'_{11}, a'_{13}, a'_{23}, a'_{25}, $\delta_n - \delta_1$, and $\delta_2 - \delta_1$. When a fit is attempted in which more than three of these parameters are allowed to vary, pathologies develop in the matrix of second derivatives of χ^2 (or its inverse, the variance matrix), and the fitting program, understandably, has convergence problems. Another less obvious example is the set $a_{3/2}$ and $a_{5/2}$. Since $\delta_2 - \delta_1$ is not precisely known, the fit is poorly constrained and has difficulties.

Another feature is that of quadratic ambiguities. For example, solutions exist with $\Delta I = 3/2$ terms dominant and $\Delta I = 1/2$ terms small. We have chosen to ignore such solutions.

IV. RESULTS

A. Consistency of input data

For each piece of input data y, we examined the contribution to χ^2, $\chi_i^2 = (y_i - F_i)^2/\sigma_i^2$. We have plotted this information in terms of the distribution of $P(\chi_i^2)$ in Fig.

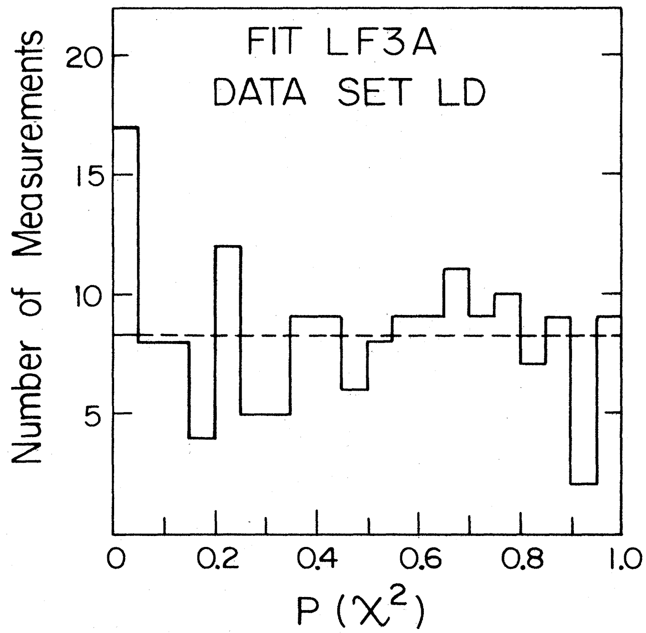

FIG. 1. The distribution of $P(\chi_i^2)$ for 166 measurements in linear data set LD, fit with parameter set LF3A. In each case, $\chi_i^2 = |Y_i - F_i|^2/\sigma_i^2$ is the square of the deviation of the measurement Y_i from the fitted value F_i in units of the reported standard deviation σ_i. The dashed line is the distribution expected if the errors are Gaussian.

1 for fit LF3A to data set LD. The observed distribution is close to the expected distribution. The lowest bin has an excess of about eight data points out of 166 in the whole histogram. The experiments with very low $P(\chi_i^2)$ are scattered throughout the data, showing no concentration on a particular type of measurement. We have tried excluding these experiments and have observed no significant change in the fitted parameters. We have also tried excluding all experiments reported prior to 1970. The value of $\chi^2/\text{DF} = 96.6/56 = 1.72$ is higher because of smaller uncertainties on the more recent measurements, but the resulting parameters are not significantly different from the fit with all the data.

When the same procedures were carried out for fit QF3A to data set QD, the situation was somewhat different. The overall distribution of $P(\chi_i^2)$ is similar to that in Fig. 1, but, as mentioned in Sec. III.C, there is a concentration of large deviations in measurements of h_{+-0}, the coefficient of Y^2 in the reaction $K_L^0 \to \pi^+\pi^-\pi^0$. The three most precise measurements, Albrow et al. (1970), Bisi and Ferrero (1974), and Messner (1974), contribute a combined total of 73 to χ^2. Figure 2 displays all the measurements of this quantity. While Albrow *et al.* (1970) and Bisi and Ferrero (1974), are in agreement, they are clearly inconsistent with Messner (1974). The other, less precise, measurements tend to favor Messner (1974). The values of the quadratic coefficients are also influenced by the $K^{\pm}$ decay modes through our assumption of $\Delta I = 1/2$ for those terms.

In order to study the inconsistencies, we tried a number of different input data sets, QDA, QDB, QDC, and QDD. When data on the $K^{\pm} \to \pi^{\pm}\pi^{\pm}\pi^{\mp}$ energy dependence

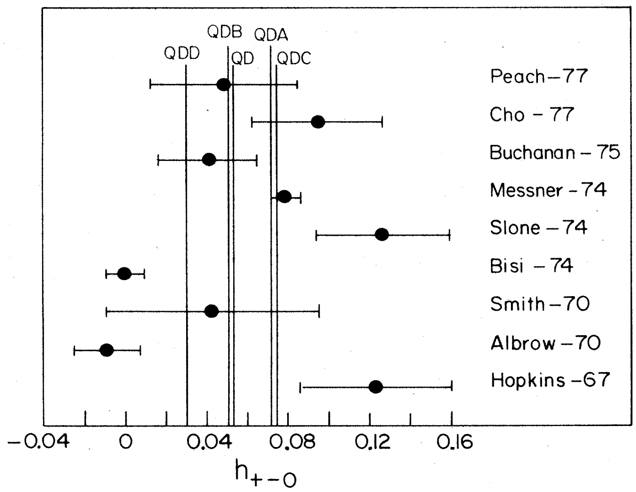

FIG. 2. Measurements of h_{+-0}, the coefficient of Y^2 in $K_L^0 \to \pi^+\pi^-\pi^0$. The vertical lines are the values resulting from fits to various data sets described in the text and in Table III. QD contains all the data. QDA excludes all $K_L^0 \to \pi^+\pi^-\pi^0$ spectrum measurements. QDB does the same for $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$. QDC excludes only Albrow *et al.* (1970) and Bisi and Ferrero (1974). QDD excludes all the $K_L^0 \to \pi^+\pi^-\pi^0$ spectrum measurements except those two. QDC was the data set finally adopted for use in this study.

was entirely excluded (data set QDB), there was little improvement in χ^2. When the corresponding data for $K_L^0 \to \pi^+\pi^-\pi^0$ was excluded instead (data set QDA), χ^2 improved more significantly, and the fitted coefficient was consistent with the results of Messner, 1974. Data set QDC, with only Albrow *et al.* (1970) and Bisi and Ferrero (1974) excluded, shows the most dramatic improvement. Table III shows the changes in χ^2 and Fig. 2 shows the values of h_{+-0} resulting from the various fits used to study this problem. The final quadratic data set adopted for further studies was QDC.

It might be argued that the addition of a $\Delta I = 3/2$ contribution to the quadratic term in the amplitude could change the conclusion drawn above. However, it should be noted from the results of the linear fits in Table I, that the dominant contribution to the Y^2 dependence comes from the square of the linear term, and relatively little comes from the $\Delta I = 1/2$ quadratic term in the amplitude. In order to reconcile the results of Albrow *et al.* (1970) and Bisi and Ferrero (1974) with the $K^\pm \to 3\pi$ data, the $\Delta I = 3/2$ term would have to be larger than the $\Delta I = 1/2$ term. While such an explanation cannot be excluded, we regard it as unlikely.

B. Isotopic spin selection rules

Table IV lists the values of χ^2/DF for data set LD with various parameter sets designed to test isotopic spin selection rules. The procedure consisted in adding each successive parameter to the list varied by the program, and examining the change in χ^2. Large improvements were observed for every amplitude with $\Delta I = 3/2$. Allowing m_{35} to vary resulted in a small improvement in χ^2, but at a level easily ascribed to systematic errors. The remaining parameters produced no improvements in χ^2. A similar study was made with the quadratic data set, QDC, with very similar results and identical conclusions.

The results of this procedure are consistent with the five tests performed in Appendix I of PDG, 1976.

C. Linear fits

The fitted parameter sets LF3A and LF7A are given in Table II. The various functions computed from the parameters of LF3A are given in Table I. The set LF7A is shown only to indicate the level at which $\Delta I = 5/2$, $7/2$ contributions are excluded. The amplitudes m_{35}, m_{37}, and $a_{5/2}$ are consistent with zero at a level of 1% of the leading term in both $K \to 2\pi$ and $K \to 3\pi$. The linear term, a'_{25}, is redundant with a'_{23} and it has been set to zero.

D. Quadratic fits

The parameters of the quadratic fit, QF3A, to the data set QDC are shown in Table II. The various functions computed from the parameters are given in Table I. With the exception of the $K \to 3\pi$ energy spectra, the results differ little from the linear fit.

Most of the Y^2 term in the spectrum can be explained by the square of the linear term in the amplitude. This can be seen by comparing the various values for h_{123} in Table I. However, Ford *et al.* (1972) and Messner (1974) report the presence of X^2 terms in the spectrum with four to five standard deviation significance. In our formulation, with the final state interaction phases con-

TABLE III. Study of inconsistencies in measurements of $K \to 3\pi$ energy dependence.

Data[a] set	Characteristics	χ^2/DF	Compared with	$\Delta\chi^2/\Delta DF$
QD	Experimental data as discussed in Sec. III. All quadratic fits to $K \to 3\pi$ energy dependence included	324.87/169		
QDA	Delete all measurements of $\pi^+\pi^-\pi^0$ energy dependence	214.89/149	QD	109.98/20 = 5.5
QDB	Delete all measurements of $\pi^\pm\pi^\pm\pi^\mp$ energy dependence	302.34/160	QD	22.53/9 = 2.5
QDC	Delete Albrow *et al.* (1970) and Bisi and Ferrero (1974)	245.54/105	QD	79.33/4 = 19.8
QDD	Delete all $\pi^+\pi^-\pi^0$ energy dependence *except* Albrow *et al.* (1970) and Bisi and Ferrero (1974)	242.74/153	QD	82.13/16 = 5.1

[a] The parameter set used for these studies was QF3A.

TABLE IV. Tests of isotopic spin selection rules with data set LD.

Parameter set	χ^2/DF	$\Delta\chi^2$	Compared with	Selection rule $K\to 2\pi$	Selection rule $K\to 3\pi$	Parameters allowed to vary[a] and comments
LF1	20 104/159			$\Delta I=\frac{1}{2}$	$\Delta I=\frac{1}{2}$	All consistent with $\Delta I=\frac{1}{2}$
LF13	1 408/158	18 696	LF1	$\Delta I\leq\frac{3}{2}$	$\Delta I=\frac{1}{2}$	LF1 + $a_{3/2}$ Test for $\Delta I=\frac{3}{2}$ in $K\to 2\pi$
LF3E[b]	628/157	780	LF13	$\Delta I\leq\frac{3}{2}$	$\Delta I\leq\frac{3}{2}$	LF13 + m_{13} Test for $\Delta I=\frac{3}{2}$ in $I=1$ amplitude
LF3D	322/156	306	LF3E	$\Delta I\leq\frac{3}{2}$	$\Delta I\leq\frac{3}{2}$	LF3E + a_{13} Test for $\Delta I=\frac{3}{2}$ in $I=1$ slope
LF3C	303/156	325	LF3E	$\Delta I\leq\frac{3}{2}$	$\Delta I\leq\frac{3}{2}$	LF3E + a_{23} Test for $\Delta=\frac{3}{2}$ in $I=2$ slope
LF3A	240.6/155	63	LF3C	$\Delta I\leq\frac{3}{2}$	$\Delta I\leq\frac{3}{2}$	LF3E + a_{13} + a_{23} Test for $\Delta I=\frac{3}{2}$ in both slopes
LF5A[b]	233.1/154	7.5	LF3A	$\Delta I\leq\frac{3}{2}$	$\Delta I=\frac{5}{2}$	LF3A + m_{35} Test for $\Delta I=\frac{5}{2}$ in $I=3$ amplitude
LF5B	233.1/154	0	LF5A	$\Delta I\leq\frac{3}{2}$	$\Delta I=\frac{5}{2}$	LF5A + a_{25} Test for $\Delta I=\frac{5}{2}$ in $I=2$ slope
						Pathological fit. a_{25} and a_{23} redundant
LF5D	233.1/154	0	LF5A	$\Delta I\leq\frac{3}{2}$	$\Delta I=\frac{5}{2}$	LF5A + $a_{25}-a_{23}$ confirms above
LF5C	232.7/153	0.4	LF5A	None	$\Delta I=\frac{5}{2}$	LF5A + $a_{5/2}$ Test for $\Delta I=\frac{5}{2}$ in $K\to 2\pi$
LF7A[b]	233.1/153	0.02	LF5A	$\Delta I\leq\frac{3}{2}$	None	LF5A + m_{37} Test for $\Delta I=\frac{7}{2}$ in $I=3$ amplitude

[a] We have fixed $\delta_M-\delta_1=\delta_2-\delta_1=\delta_3-\delta_1=0$ for all these fits. The two-pion phase shift, $\delta_2-\delta_0$ was allowed to vary, constrained by independent data.
[b] In several cases, quadratic ambiguities exist. We have ignored solutions with $m_{13}\gg m_{11}$, $a_{5/2}\gg a_{3/2}$, and $m_{37}\gg m_{35}$.

stant, this requires quadratic terms in the amplitude.

In order to test the need for b' and c', we tried fitting the quadratic data set with the combinations shown in Table V. It is clear that c' is needed. The change in the fit resulting from the b' term is smaller, and may not be significant. We have chosen to allow it to vary in the results presented in Tables I and II.

E. Results: $K\to 2\pi$

The fitted and fixed parameters for $K\to 2\pi$ amplitudes are listed as items 19–22 in Table II. These can be combined to form the transitions among various physical charge states according to the equations in Appendix A. The dominant term is the $\Delta I=\frac{1}{2}$ amplitude $a_{1/2}=(0.4687\pm0.0006)$ keV. There is a small but significant $\Delta I=3/2$ contribution, $a_{3/2}=(0.021\pm0.0001)$ keV. The phase-shift difference is $(\delta_2-\delta_0)=(-0.79\pm0.08)$ radians, determined largely by hadron scattering results (see Sec. III.D). The $\Delta I=5/2$ amplitude is not well determined because of the uncertainty in $\delta_2-\delta_0$. We have assumed $a_{5/2}=0$.

F. Results: $K\to 3\pi$

The fitted and fixed parameters for the $K\to 3\pi$ amplitudes are listed as items 6–18 in Table II. These can be combined to form the transition amplitudes among various physical charge states according to the equations in the text of Appendix A, and in Tables VII and VIII.

It is useful to write down the explicit energy-dependent amplitudes for $K\to 3\pi$ in the pure isospin transitions (Table VI). The results from fit LF3A are:

$$f_3'=f_{3S}'+f_{3M}'=(91.25+24.15\,Y)\times10^{-8}\quad(I=1,\ \Delta I=1/2)\tag{4.1a}$$

$$f_3''=f_{3S}''+f_{3M}''=(3.51-1.14\,Y)\times10^{-8}\quad(I=1,\ \Delta I=3/2)\tag{4.1b}$$

$$g_3''=(3.06\,Y)\times10^{-8}\quad(I=2,\ \Delta I=3/2)\tag{4.1c}$$

With quadratic terms included, the amplitudes for the $K\to 3\pi$ transitions become:

$$f_3'=[91.46+25.83\,Y-0.37\,(Y^2+X^2/3)-1.25\,(Y^2+X^2/3)]\times10^{-8}\quad(I=1,\ \Delta I=1/2)\tag{4.2a}$$

$$f_3''=(3.57-1.24\,Y)\times10^{-8}\quad(I=1,\ \Delta I=3/2)\tag{4.2b}$$

$$g_3''=(4.35\,Y)\times10^{-8}\quad(I=2,\ \Delta I=3/2)\tag{4.2c}$$

G. Soft-pion calculations

A valuable set of constraints among the parameters of this study can be obtained from current algebra relations between $K\to 2\pi$ and $K\to 3\pi$ decay modes. (For more detailed discussions see, for example, Callan and Treiman, 1966; Hara and Nambu, 1966; Treiman *et al.*, 1972; Marshak *et al.*, 1969. Figure 3 illustrates

TABLE V. Tests for quadratic energy dependence in $K\to 3\pi$ with data set QDC.

Parameter set	χ^2/DF	$\Delta\chi^2$	Compared with	Parameters allowed to vary
QF3L	341.88/167	···		Identical to LF3A, $b'=c'=0$
QF3QP	338.16/166	3.72	QF3L	QF3L + b'
QF3QM	253.72/166	88.16	QF3L	QF3L + c'
QF3A	245.54/165	8.18	QF3QM	QF3L + b' + c'

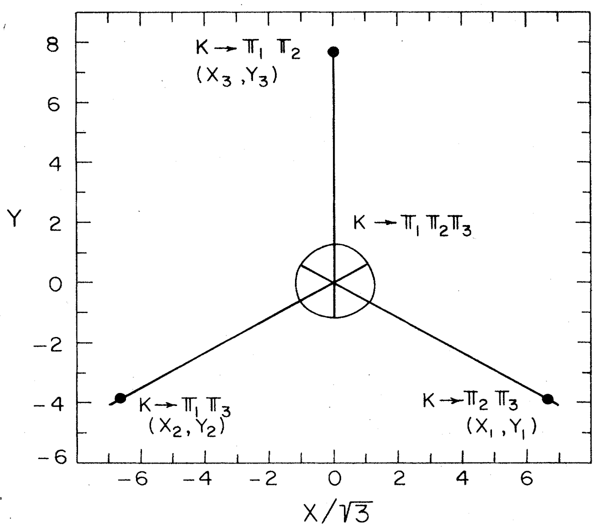

FIG. 3. A Dalitz plot for $K \to 3\pi$ showing the three possible soft-pion extrapolations to $K \to 2\pi$ processes.

the kinematic quantities involved. The Dalitz plot for the three-pion system is shown in the center. If all components of the four-momentum of one of the pions are set to zero, then $s_i = 0$ and $m_i = 0$. This can be done for any of the three pions, and the X-Y values for this are shown in the figure. If the amplitude for the three-pion decay is analytically continued outside the boundaries of the Dalitz plot, then it can be related to the two-pion amplitude. When a charged pion is taken off shell, the resulting two-pion process is unphysical. Depending on the details of the current commutation relations, it can then be argued that the amplitude should be zero or related to a physical two-pion process through isospin considerations. Off-mass-shell corrections are expected to be small.

The relationship involves the pion decay constant f_π, which can be determined from $\pi^\pm \to \mu^\pm \nu$ to be $f_\pi = 135$ MeV. It can also be calculated from the Goldberger–Treiman relation (Goldberger and Treiman, 1958) to be $f_\pi = 122$ MeV. Some authors define f_π differently, a factor of $2^{1/2}$ smaller than the convention we use here.

A convenient example is the process $K^+ \to \pi^+\pi^+\pi^-$. Under the assumption that the $\Delta I = 1/2$ rule is true, we should observe

$$A_{++-}(X_1, Y_1) = A_{+-}/(2^{1/2} f_\pi) , \tag{4.3a}$$

$$A_{++-}(X_3, Y_3) = 0 . \tag{4.3b}$$

The experimental results can be represented by the relationships of Appendix A with the parameters of Table II. For the linear fits, this yields

$$A_{++-} = (189.52 - 19.95\ Y) \times 10^{-8}$$

$$|A_{+-}| = 0.3913 \times 10^{-3}\ \text{MeV}.$$

These are plotted in Fig. 4. The agreement is very crude and rather disappointing. If the results of the quadratic fit, QF3A, are used, the picture changes, but the overall agreement is at least as bad as the linear case.

Several authors (for example, Bouchiat and Meyer, 1967; Holstein, 1969) have refined this comparison by including $\Delta I = 3/2$ effects.

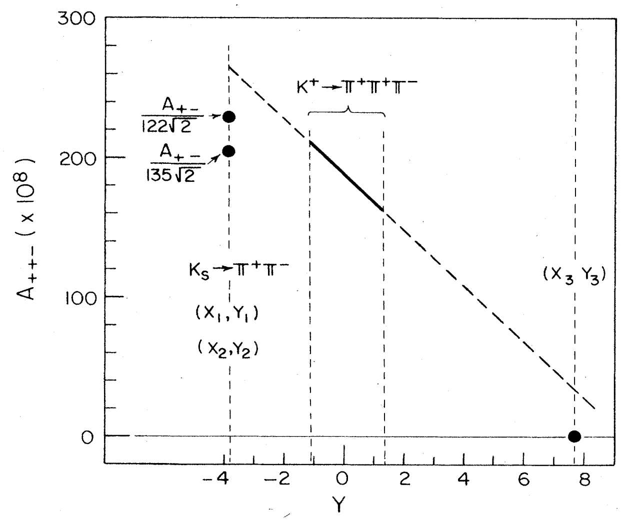

FIG. 4. A plot to the Y dependence (linear fit) of $K^+ \to \pi^+\pi^+\pi^-$ with extrapolations to the soft-pion limits. According to the current algebra predictions, assuming $\Delta I = \frac{1}{2}$, the amplitude should extrapolate to zero at $Y = Y_3$, and to values calculated from Eq. (4.3a) at $Y = Y_1 = Y_2$. The latter is shown for two different values of f_π.

Our objectives are better served by considering the amplitude, f_3' of Zemach (1964) (see Table VI) for the pure $\Delta I = \frac{1}{2}$ transition. In the soft-pion limits, the amplitude should satisfy the relations:

$$f_3'(X_3, Y_3) = a_{1/2}/(2^{1/2} f_\pi) , \tag{4.4a}$$

$$f_3'(X_1, Y_1) = 0. \tag{4.4b}$$

The experimental results can be represented by Eq. 4.1a, and by $a_{1/2} = (0.4687 \pm 0.0006) \times 10^{-3}$ MeV. These are plotted in Fig. 5. Equation 4.4b is satisfied quite

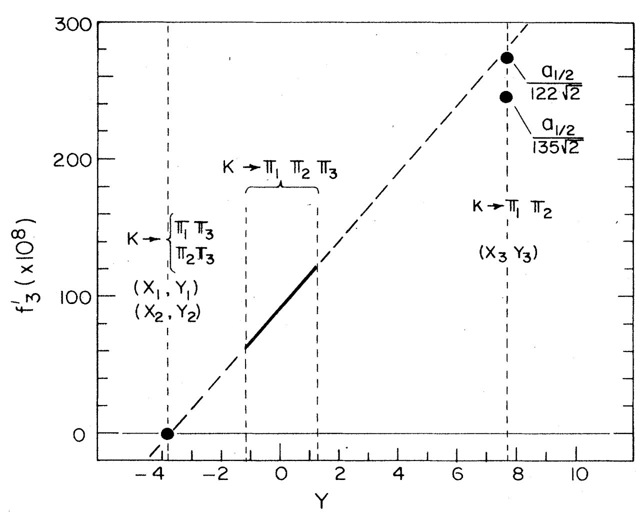

FIG. 5. A plot of the Y dependence (linear fit) for the pure $\Delta I = \frac{1}{2}$ transition $K \to \pi_1\pi_2\pi_3$ with extrapolations to the soft-pion limits. The extrapolated amplitude is very nearly zero at $Y = Y_1 = Y_2$, and is in excellent agreement with the $\Delta I = \frac{1}{2}$ part of the $K \to 2\pi$ amplitude if the Goldberger–Treiman relation is used to calculate f_π.

TABLE VI. Expansion of the Zemach amplitudes for $K\to 3\pi$.

I	ΔI	Expansion of $K\to 3\pi$ amplitude[a]
$1(S)$	$\frac{1}{2}$	$f'_{3S}=m_{11}+b'(Y^2+X^2/3)$
$1(M)$	$\frac{1}{2}$	$f'_{3M}=a'_{11}Y+c'(Y^2-X^2/3)$
$1(S)$	$\frac{3}{2}$	$f''_{3S}=m_{13}$
$1(M)$	$\frac{3}{2}$	$f''_{3M}=a'_{13}Y$
2	$\frac{3}{2}$	$g''_3=a'_{23}Y$
2	$\frac{5}{2}$	$g'_3=a'_{25}Y$
3	$\frac{5}{2}$	$g'_e=m_{35}$
3	$\frac{7}{2}$	$g''_e=m_{37}$

[a]The amplitudes, g, of Zemach (1964) are quoted only in this table, Sec. IV. C, and Sec. IV. D. They should not be confused with the slope parameters, g, of the Particle Data Group.

closely. If the Goldberger–Treiman value of f_π is used, the consistency with Eq. 4.4a is at the 3% level! Using an alternative approach, it is possible to compute f_π from the experimental data. The result is

$$f_\pi=(119\pm 3)\ \text{MeV}\ ,$$

where the uncertainty has been expanded to reflect the slight inconsistency in the extrapolation. This is clearly in excellent agreement with the Goldberger–Treiman relation.

If the results of the quadratic fit (Eq. 4.2a) are used, the good agreement of Fig. 5 is destroyed.

H. Comments on the quadratic energy dependence for $K\to 3\pi$

The very close agreement between the linear term in the $K\to 3\pi$ energy spectrum and the current algebra prediction supports the view that this term arises "from the structure of the weak interaction itself, and that the final-state interaction among the pions is relatively unimportant" (Marshak *et al.*, 1969). From what we have seen, this interpretation cannot be extended to the quadratic terms. A number of explanations for this are possible:

1. The apparent good agreement in Fig. 5 may be accidental, and the soft-pion predictions should not be expected to be this good.
2. The experimental results for the quadratic terms may be incorrect.
3. The electromagnetic corrections to the raw spectra may be inadequate.
4. The quadratic energy dependence may be real, but it may result from strong final-state interactions, i.e., variations in the phases which we assumed to be constant.

While we are biased against the first possibility, we see no reliable basis for choosing amongst these alternatives.

V. SUMMARY AND FINAL REMARKS

With the exceptions noted in Sec. III, there seems to be reasonable consistency amongst a large number of experimental measurements. The value of χ^2 is about 1.5 per data point, which argues that the average systematic inconsistencies are of the same order as the quoted uncertainties. The data definitely require the presence of $\Delta I=\frac{3}{2}$ amplitudes in both $K\to 2\pi$ and $K\to 3\pi$ at the level of 3%–4% of the $\Delta I=\frac{1}{2}$ amplitude. No need exists for amplitudes with $\Delta I=\frac{5}{2},\frac{7}{2}$ at the level of 1% of the dominant amplitude. The linear term in the $K\to 3\pi$ energy spectrum also requires a $\Delta I=\frac{3}{2}$ contribution to both final states, $I=1,2$, at the level of a few percent. The linear fit to the $K\to 3\pi$ amplitude for the pure $\Delta I=\frac{1}{2}$ transition is in agreement with the current algebra prediction within about 3%. The experimental data seem to require the presence of quadratic terms in the $K\to 3\pi$ energy spectrum. The quadratic fits do not agree with the current algebra predictions as well as the linear fits.

It is always perilous to state that some particular subject of basic research is complete, and that no further work is needed. We shall carefully avoid making such a statement about the subject of this review. Nevertheless, some remarks (which may be obvious to workers in this area) are in order. The principal features of $K\to 2\pi$ and $K\to 3\pi$ are well established and measured. They fit into a consistent picture within the framework of the weak interactions. The smallness (or largeness) of the $\Delta I=\frac{3}{2}$ contribution may still be poorly understood, but it is no longer an experimental problem. The $\Delta I=\frac{5}{2},\frac{7}{2}$ terms seem to be excluded at a level which is satisfactory for the present. Nonlinear terms in the $K\to 3\pi$ spectrum are not large, and if they exist, they can probably be explained in a way which does not disturb the overall consistency. Unless some new theoretical insight arrives to enlighten us, there are no "burning questions" left.

The experiments have been expensive in time, equipment, and money. The recent, and more precise, experiments have data samples up to several millions of events, and the analyses alone have been difficult, costly, and time-consuming. Further, these measurements are subject to important corrections such as ambiguous events, background subtractions, and detector acceptance. We question, on general grounds, whether or not these experiments have reached some practical limit of precision with present techniques. Therefore, we suggest that proposals for future experimental efforts, even by-product experiments, should demonstrate convincingly that significant improvements in precision will be made, free of systematic error, and that these improvements will address significant theoretical issues.

VI. ACKNOWLEDGMENTS

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APPENDIX A. TRANSITION AMPLITUDES AND PHYSICAL OBSERVABLES

The relationships between the physical observables and the $K\to 2\pi$ and $K\to 3\pi$ transition amplitudes are discussed in this appendix.

1. $K\to 2\pi$

We use the formalism and notation of Marshak, Riazuddin, and Ryan (1969). The isospin structure of the normalized s-wave two-pion states is given by

$$|\pi^+\pi^0\rangle = |I=2, I_Z=1\rangle\ ,$$
$$|\pi^+\pi^-\rangle = (\tfrac{2}{3})^{1/2}|0,0\rangle + (\tfrac{1}{3})^{1/2}|2,0\rangle\ ,$$
$$|\pi^0\pi^0\rangle = (\tfrac{1}{3})^{1/2}|0,0\rangle - (\tfrac{2}{3})^{1/2}|2,0\rangle\ .$$

Next consider transition amplitudes between initial-state kaons and final states of pure isotopic spin.

$$\langle I=n|H|K^0\rangle = a_n e^{i\delta_n}$$
$$\langle I=n|H|\bar{K}^0\rangle = -a_n^* e^{i\delta_n}$$
$$\langle I=2|H|K^+\rangle = a_2^+ e^{i\delta_2}\ .$$

Here $n=0, 2$ and δ_n is the $\pi\pi$ phase shift for $I=n$. CPT invariance has been assumed. The resulting relations are:

$$a(K^0\to\pi^+\pi^-) = (\tfrac{2}{3})^{1/2}a_0 e^{i\delta_0} + (\tfrac{1}{3})^{1/2}a_2 e^{i\delta_2}\ ,$$
$$a(\bar{K}^0\to\pi^+\pi^-) = -(\tfrac{2}{3})^{1/2}a_0^* e^{i\delta_0} - (\tfrac{1}{3})^{1/2}a_2^* e^{i\delta_2}\ ,$$
$$a(K^0\to\pi^0\pi^0) = (\tfrac{1}{3})^{1/2}a_0 e^{i\delta_0} - (\tfrac{2}{3})^{1/2}a_2 e^{i\delta_2}\ ,$$
$$a(\bar{K}^0\to\pi^0\pi^0) = -(\tfrac{1}{3})^{1/2}a_0^* e^{i\delta_0} + (\tfrac{2}{3})^{1/2}a_2^* e^{i\delta_2}\ ,$$
$$a(K^+\to\pi^+\pi^0) = a_2^+ e^{i\delta_2}\ .$$

Using the Wigner–Eckart theorem, a_0, a_2, and a_2^+ amplitudes can be expressed in terms of the reduced amplitudes (for isospin transitions $\Delta I=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$) as follows:

$$a_0 = (\tfrac{1}{2})^{1/2}a_{1/2}\ ,$$
$$a_2 = (\tfrac{1}{2})^{1/2}(a_{3/2}+a_{5/2})\ ,$$
$$a_2^+ = (\tfrac{1}{3})^{1/2}(\tfrac{3}{2}a_{3/2} - a_{5/2})\ .$$

Assuming CP invariance and using $|K_S^0\rangle = (\frac{1}{2})^{1/2}(|K^0\rangle - |\bar{K}^0\rangle)$, we find the squared transition amplitudes

$$|A_{+-}|^2 = |(\tfrac{2}{3})^{1/2}a_{1/2} + (\tfrac{1}{3})^{1/2}(a_{3/2}+a_{5/2})e^{i(\delta_2-\delta_0)}|^2\ ,$$
$$|A_{00}|^2 = |(\tfrac{1}{3})^{1/2}a_{1/2} - (\tfrac{2}{3})^{1/2}(a_{3/2}+a_{5/2})e^{i(\delta_2-\delta_0)}|^2\ ,$$
$$|A_{+0}|^2 = \tfrac{1}{3}[\tfrac{3}{2}a_{3/2} - a_{5/2}]^2\ .$$

The partial rate, Γ_{12} in sec^{-1}, for $K\to\pi_1\pi_2$ is

$$\hbar\Gamma_{12} = \int C_{12}|A_{12}|^2\delta^4(\bar{p}_K - \bar{p}_1 - \bar{p}_2)d^3p_1 d^3p_2/(32\pi^2 M_K e_1 e_2)\ ,$$
$$\Gamma_{12} = |A_{12}|^2 C_{12}\phi_{12}/(8\pi^2 M_K \hbar)\ ,$$

where C_{12} is a Coulomb correction, and ϕ_{12} is the Lorentz-invariant phase-space volume

$$\phi_{12} = \int \delta^4(\bar{p}_K - \bar{p}_1 - \bar{p}_2)\frac{d^3p_1 d^3p_2}{4e_1 e_2} = \pi p/M_K\ .$$

Here, p is the momentum of either pion in the rest system of the K. For the three processes under consideration

$$\phi_{+-} = 1.3005\ ,$$
$$\phi_{00} = 1.3197\ ,$$
$$\phi_{+0} = 1.3055\ .$$

Final-state Coulomb interactions will modify the effective matrix element. Schiff (1955) and Dalitz (1956) suggest a factor $C_{12}=n/(e^n-1)$ where $n=2\pi q_1 q_2\alpha/v_{12}$, $\alpha=1/137$, v_{12} is the relative velocity of the pions and $q_i=0, \pm 1$ is the pion charge. This yields

$$C_{+-} = 1.0235$$
$$C_{00} = C_{+0} = 1.000\ .$$

Other calculations of a radiative correction factor, $1+\Delta_{RC}$, have been done (Abbud *et al.*, 1967; Belavin and Narodetsky, 1968; Nachtmann, 1970). The results vary, even in the sign of Δ_{RC}, but all yield $|\Delta_{RC}|$ less than a few percent. Since the uncertainties in $C_{12}\phi_{12}$ are of the same order as the differences, we have assumed $C_{+-}\phi_{+-} = C_{00}\phi_{00} = C_{+0}\phi_{+0} = 1.31$, the same for all three processes, with an uncertainty of about 2%.

2. $K\to 3\pi$

Zemach (1964) has studied the symmetries to be expected in the amplitudes and energy dependence of various three-pion final states. Given the restrictions discussed in Sec. II.B, we expand the Zemach amplitudes as shown in Table VI. We have separated the amplitudes for the symmetric $I=1$ final state from those of the mixed-symmetry $I=1$ state. Three phase differences arise: that between the two $I=1$ states, $\delta_{M1}=\delta_M-\delta_1$, and the differences between the $I=2$ ($I=3$) and symmetric $I=1$ states, $\delta_{21}=\delta_2-\delta_1$ ($\delta_{31}=\delta_3-\delta_1$). It is then possible to write the transition amplitudes for the various processes as

$$A_{++-} = \langle ++-|T|K^+\rangle = 2(m_{11}+m_{13}) + (m_{35}+m_{37})e^{i\delta_{31}} + [-(a'_{11}+a'_{13})e^{i\delta_{M1}} + (a'_{23}+a'_{25})e^{i\delta_{21}}]Y + 2b'(Y^2+X^2/3) + c'(Y^2-X^2/3)e^{i\delta_{M1}}$$

$$A_{00+} = \langle 00+|T|K^+\rangle = (m_{11}+m_{13}) - 2(m_{35}+m_{37})e^{i\delta_{31}} + [(a'_{11}+a'_{13})e^{i\delta_{M1}} + (a'_{23}+a'_{25})e^{i\delta_{21}}]Y + b'(X^2/3+Y^2/3) + c'(Y^2-X^2/3)e^{i\delta_{M1}}\ ,$$

$$A_{000} = \langle 000|T|K_L^0\rangle = -3(m_{11}-2m_{13}) + (3m_{35}-4m_{37})e^{i\delta_{31}} - 3b'(Y^2+X^2/3)\ ,$$

$$A_{+-0} = \langle +-0|T|K_L^0\rangle = -(m_{11}-2m_{13}) - (\tfrac{1}{2})(3m_{35}-4m_{37})e^{i\delta_{31}} - (a'_{11}-2a'_{13})Ye^{i\delta_{M1}} - b'(Y^2+X^2/3) - c'(Y^2-X^2/3)e^{i\delta_{M1}}\ .$$

In order to emphasize the dominance of $\Delta I=\frac{1}{2}$ terms and to express the amplitudes in a manner more closely related to the experimental data, we factor out leading terms. The resulting amplitudes are of the form

$$A_{123} = \langle 123|T|K\rangle = B_{123}M_{123}[1 + a_{123}Y + b_{123}(Y^2+X^2/3) + c_{123}(Y^2-X^2/3)]\ .$$

For individual processes the relationships are given in Table VII. With appropriate factors for identical bosons, the amplitudes can then be squared to give the probability densities

$$p_{++-}(X,Y) = 2|M_{++-}|^2\{1 + 2\,\mathrm{Re}a_{++-}Y + |a_{++-}|^2Y^2 + 2\,\mathrm{Re}b_{++-}(Y^2+X^2/3) + 2\,\mathrm{Re}c_{++-}(Y^2-X^2/3)\}\ ,$$

$$p_{00+}(X,Y) = (\tfrac{1}{2})|M_{00+}|^2\{1 + 2\,\mathrm{Re}a_{00+}Y + |a_{00+}|^2Y^2 + 2\,\mathrm{Re}b_{00+}(Y^2+X^2/3) + 2\,\mathrm{Re}c_{00+}(Y^2-X^2/3)\}\ ,$$

TABLE VII. Amplitude coefficients for $K\to 3\pi$.

$$M_{++-}=m_{11}[1+m_{13}/m_{11}-2e^{i\delta_{13}}(m_{35}+m_{37})/m_{11}]$$
$$M_{00+}=m_{11}[1+m_{13}/m_{11}-2e^{i\delta_{13}}(m_{35}+m_{37})/m_{11}]$$
$$M_{000}=m_{11}[1-2m_{13}/m_{11}-e^{i\delta_{13}}(m_{35}-\tfrac{4}{3}m_{37})/m_{11}]$$
$$M_{+-0}=m_{11}[1-2m_{13}/m_{11}+\tfrac{1}{2}e^{i\delta_{13}}(m_{35}-\tfrac{4}{3}m_{37})/m_{11}]$$
$$a_{++-}=-\frac{a'_{11}e^{i\delta_{M1}}}{2M_{++-}}[1+a'_{13}/a'_{11}-(a'_{23}+a'_{25})e^{i\delta_{12}}/a'_{11}]$$
$$a_{00+}=\frac{a'_{11}e^{i\delta_{M1}}}{M_{00+}}[1+a'_{13}/a'_{11}+(a'_{23}+a'_{25})e^{i\delta_{12}}/a'_{11}]$$
$$a_{000}=0$$
$$a_{+-0}=a'_{11}e^{i\delta_{M1}}(1-2a'_{13}/a'_{11})/M_{+-0}$$
$$b_{++-}=b'/M_{++-}\qquad c_{++-}=-c'e^{i\delta_{M1}}/2M_{++-}$$
$$b_{00+}=b'/M_{00+}\qquad c_{00+}=c'e^{i\delta_{M1}}/M_{00+}$$
$$b_{000}=b'/M_{000}\qquad c_{000}=0$$
$$b_{+-0}=b'/M_{+-0}\qquad c_{+-0}=c'e^{i\delta_{M1}}/M_{+-0}$$

$$p_{000}(X,Y)=(\tfrac{3}{2})|M_{000}|^2\{1+2\,\mathrm{Re}b_{000}(Y^2+X^2/3)\}$$
$$p_{+-0}(X,Y)=|M_{+-0}|^2\{1+2\,\mathrm{Re}a_{+-0}Y+|a_{+-0}|^2Y^2+2\,\mathrm{Re}b_{+-0}(Y^2+X^2/3)+2\,\mathrm{Re}c_{+-0}(Y^2-X^2/3)\}\,.$$

The experimental data are expressed in terms of the PDG parameters

$$p_{123}(X,Y)\sim(1+g_{123}Y+h_{123}Y^2+k_{123}X^2)\,.$$

Table VIII relates these to the parameters in our expansion of the amplitudes.

The partial decay rates can be calculated as an integral over phase space with Coulomb corrections:

$$\hbar\Gamma_{123}=\frac{(2\pi)^{-5}}{2M_K}\int p_{123}(X,Y)c_{12}(X,Y)c_{23}(X,Y)c_{31}(X,Y)\times\delta^4(\bar{p}_K-\sum_i\bar{p}_i)\frac{d^3p_1d^3p_2d^3p_3}{8e_1e_2e_3}\,.$$

The C_{ij} are the Coulomb terms (as discussed in the two-pion case) for each pair of pions. Mass differences among the mesons give rise to substantial differences in the phase-space volumes and in the effect of the slope parameters g on the overall transition rates. The original formulation by Dalitz (1956) included nonrelativistic phase-space calculations and an estimate of the Coulomb corrections. Various refinements have been added since then: The relativistic calculation by Trilling (1965); slope corrections (Devlin and Barshay, 1967; Devlin, 1968; Kenny, 1968), and finally a full calculation of all these effects by Mast *et al.* (1969).

In order to incorporate these effects into a fitting program it is useful to perform the integral explicitly for each term in the expansion

$$\Gamma_{123}=p_{123}(0,0)[I_0+g_{123}I_1+h_{123}I_2+k_{123}I_3]/(64\pi^3\hbar M_K)\,,$$

where

$$I_m=\int H_m(X,Y)c_{12}(X,Y)c_{23}(X,Y)c_{31}(X,Y)de_1de_2\,,$$

and $H_0=1$, $H_1=Y$, $H_2=Y^2$, and $H_3=X^2$. It is also useful to calculate $I_\pm=I_2\pm I_3/3$. Table IX gives these integrals for each of the processes.

APPENDIX B. FITTING PROCEDURE

The least-squares fit was achieved by an iterative procedure which minimized the value of

$$\chi^2=\sum_{ijk}(y_{ik}-F_i)(E^k_{ij})^{-1}(y_{jk}-F_j)\,,$$

where

$$F_i=F_i(26\text{ parameters})$$

is the fitted value of the ith observable, y_{ik} is the measured value of the ith observable measured in the kth experiment. E^k_{ij} is the variance matrix for the kth experiment. It was assumed that there are no correlations between experiments.

The program computed the values of the parameters, the variance matrix G_{lm}, and the fitted values of the selected observable quantities F_i. It also used the variance matrix to compute the uncertainties in these quantities, ΔF_i. The computed F's included not only those corresponding to the input data, but also a number of others potentially measurable. It was possible to fix any of the parameters, and appropriate adjustments to the variance matrix were made.

We have assumed the distribution of errors to be purely Gaussian, even though the presence of systematic errors makes it clear that this is not strictly true. We have not scaled the errors, nor have we applied a long-tailed error distribution as suggested in PDG, 1976. The central values and uncertainties of fitted parameters can be changed slightly by such procedures, but not significantly for our purposes. Nevertheless, the presence of systematic errors does violate our assumption, and this is reflected in the high values of χ^2, e.g., $\chi^2/DF=240/160=1.5$. However, in the distribution of $P(\chi^2_i)$ shown in Fig. 1, it can be seen that these effects are not terribly bad.

Under such conditions, the standard χ^2 tests of significance cannot be used to test a specific hypothesis. We have adopted an alternate procedure for testing any given hypothesis, namely, performing the constrained

TABLE VIII. Probability density coefficients used by the PDG written in terms of the $K\to 3\pi$ transition amplitudes.

Final state	g	h	k
$++-$	$2\,\mathrm{Re}a_{++-}$	$\lvert a_{++-}\rvert^2+2\,\mathrm{Re}(b_{++-}+c_{++-})$	$(\frac{2}{3})\mathrm{Re}(b_{++-}-c_{++-})$
$00+$	$2\,\mathrm{Re}a_{00+}$	$\lvert a_{00+}\rvert^2+2\,\mathrm{Re}(b_{00+}+c_{00+})$	$(\frac{2}{3})\mathrm{Re}(b_{00+}-c_{00+})$
000	0	$2\,\mathrm{Re}b_{000}$	$(\frac{2}{3})\mathrm{Re}b_{000}$
$+-0$	$2\,\mathrm{Re}a_{+-0}$	$\lvert a_{+-0}\rvert^2+2\,\mathrm{Re}(b_{+-0}+c_{+-0})$	$(\frac{2}{3})\mathrm{Re}(b_{+-0}-c_{+-0})$

TABLE IX. Integrals over phase space of terms in the expansion of $K \to 3\pi$ amplitudes.[a]

	$K^\pm \to \pi^\pm\pi^\pm\pi^\mp$	$K^\pm \to \pi^0\pi^0\pi^\pm$	$K_2^0 \to \pi^+\pi^-\pi^0$	$K_2^0 \to \pi^0\pi^0\pi^0$
I_0 (GeV^2)	1.621×10^{-3}	1.955×10^{-3}	1.993×10^{-3}	2.353×10^{-3}
I_1 (GeV^2)	0.017×10^{-3}	-0.193×10^{-3}	0.188×10^{-3}	0.0
I_2 (GeV^2)	0.600×10^{-3}	0.908×10^{-3}	0.968×10^{-3}	1.336×10^{-3}
I_3 (GeV^2)	1.821×10^{-3}	2.769×10^{-3}	2.720×10^{-3}	4.003×10^{-3}
I_+ (GeV^2)	1.206×10^{-3}	1.831×10^{-3}	1.875×10^{-3}	2.670×10^{-3}
I_- (GeV^2)	-0.008×10^{-3}	-0.015×10^{-3}	0.061×10^{-3}	0.0

[a] Errors due to numerical integration are of order 0.002×10^{-3} GeV^2.

fit with and without the hypothesis and judging the results on the basis of the change in χ^2. Fortunately, in the fits considered in this paper, this naive procedure works. The comparisons split rather nicely into two groups: those with $\Delta\chi^2/\Delta DF \gg 10$, and those with $\Delta\chi^2/\Delta DF < 10$. Thus fairly clear conclusions can be drawn.

For individual quantities, where experiments disagree, PDG, 1976 scales the statistical uncertainties by a factor $S = \sqrt{\chi^2/(N-1)}$, where N is the number of experiments. We have not done this, and some uncertainties will appear differently. For example, PDG, 1976 lists the $K^\pm$ lifetime as $(1.2371 \pm 0.0026) \times 10^{-8}$ sec$(S = 1.9)$, whereas our fit yields $(1.2373 \pm 0.0013) \times 10^{-8}$ sec. The apparent discrepancy is simply a matter of interpretation. When a particular result is determined almost entirely by precise experiments which disagree, as is the case for the $K^\pm$ lifetime, then the procedure of PDG, 1976 is probably safest. When the constrained fit yields uncertainties substantially better than the raw data, then the reader must exercise judgment about which result is most trustworthy.

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