Galaxy correlations and cosmology

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Correlations in the distribution of galaxies provide some important clues about the structure and evolution of the Universe on scales larger than individual galaxies. In recent years much effort has been devoted to estimating and interpreting galaxy correlations. This is a review of these efforts. It is meant to provide both an introductory overview of the subject and a critical assessment of some recent developments.

CONTENTS

I. INTRODUCTORY REMARKS

Galaxy clustering has been a subject of considerable interest and importance in astronomy for the better part of a century. It exhibits a variety of forms, is related to a variety of astrophysical problems, and has been described in a variety of ways. In recent years, much effort has been devoted to developing a description of galaxy clustering in the language of probability theory and to interpreting it within the framework of cosmological theory. In this approach clustering is expressed in terms of correlations in the distribution of galaxies, and recent efforts have centered on both estimating and explaining these correlations. In many respects this approach has turned out to be a fruitful one and some of its methods and results have already been accepted into the canon of astronomical knowledge. Much of the credit for this must go to P. J. E. Peebles and his associates at Princeton.

For methods and materials, workers in this field have drawn liberally from several branches of astronomy, mathematics, and physics, and the technical literature on the subject has now become rather formidable. The somewhat abstract nature of the subject and its bearing on several fundamental topics, such as the mean mass density of the Universe and the spectrum of primordial fluctuations, has also given rise to some stimulating controversy. My purpose in writing this review is therefore twofold: first, it is to give readers not familiar with the correlation approach an introductory overview, and second, it is to offer a critical assessment of some recent work in this field.[1] Comments on some aspects of the subject at the semipopular level have been given by Davis (1976) and by Groth *et al.* (1977).

This article is not meant to be a comprehensive review of the general subject of galaxy clustering. The correlation approach tends to focus attention on an important but nevertheless restricted subset of problems within this vast subject and is perhaps best thought of as being complementary to other approaches. For reviews which emphasize other aspects of galaxy clustering the reader is referred to the articles by Abell (1975) and Bahcall (1977). Even within the statistical approach this review is not meant to be comprehensive. Prior to the recent interest in correlation functions, various statistical methods have been used to describe and interpret galaxy clustering. However, since most of this work has already been reviewed by de Vaucouleurs (1971) and Layzer (1975) and since it has had surprisingly little impact on the present subject matter, it will be referred to only occasionally in what follows. This imbalance is not intended to be a reflection of the importance of the earlier work, but rather of the relatively self-contained nature of the recent work.

The remainder of this article is organized into two fairly distinct parts. The first part, consisting of Secs. II and III, is meant to be a relatively informal overview of the correlation approach. Here the basic definitions, assumptions, and most widely accepted empirical results are introduced, and the simplest and most concrete theoretical ideas for interpreting these results are discussed. This part is largely self-contained but often sacrifices completeness in the interests of simplicity. The second part, consisting of Sec. IV, treats several problems of current research interest, including some of their more technical and controversial aspects. Topics include the problem of estimating correlations in very deep samples and recent work on the nonlinear development of clustering, especially within the framework of N-body simulations and kinetic theory. The article is concluded in Sec. V with some attempts at assessing the present status of the subject and its prospects for future development.

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[1]The final literature search for this article ended in July, 1978.

II. BASIC EMPIRICAL RESULTS

The distribution of galaxies is remarkable for its diversity of form. Anyone who has even casually inspected high Galactic latitude prints from the Palomar Sky Survey must have been struck by this fact. There are isolated galaxies, pairs, groups, and clusters with as many as several hundred members. There are also configurations intermediate between these simple categories and configurations best described as combinations of them. The first aim of the correlation approach to galaxy clustering is to supplement various aspects of this subjective impression with some simple, well-defined, and relatively objective measures of clustering.

A. Definitions, assumptions, and methods

The basic element of the correlation approach is the *spatial pair correlation function*,[2] which is usually denoted by ξ. It is defined such that

$$\delta p(r) \equiv n^2[1+\xi(r)]\delta v_1 \delta v_2 \tag{1}$$

is the joint probability of finding galaxies in the elemental volumes δv_1 and δv_2 separated by a distance r, where n is the mean space density of galaxies. If galaxies were distributed in a uniformly random or Poisson manner the joint probability of finding any two galaxies at any two points would be independent of their separation and equal to $n^2\delta v_1 \delta v_2$. The fact that galaxies are not distributed in a Poisson manner results in a nonzero value of ξ and the variation of this function with separation r then tells us something about the strength of pairwise clustering on various scales. Before proceeding, two remarks about the definition of ξ are in order. First, it has been defined in terms of the "probability" δp, as is now common practice. It is clear, however, that δp is a true probability only for infinitesimal δv_1 and δv_2 because it does not have unit norm even when $\xi(r)$ vanishes for all r. Nevertheless, this does not affect the interpretation of ξ as an "excess probability." Second, a statistical version of the cosmological principle—the assumption that the Universe is homogeneous and isotropic in the large—is already implicit in Eq. (1) since ξ has been written as a function only of the magnitude of the relative separation between galaxies. In the language of probability theory, the distribution of galaxies is assumed to represent a "stationary random process."[3]

Since only a limited amount of information about the positions of galaxies in space is available, most estimates of ξ have been derived through a subsidiary function which is usually denoted by w. This *angular pair correlation function* is defined in direct analogy with its spatial counterpart;

$$\delta p(\theta) \equiv \mathfrak{N}^2[1+w(\theta)]\delta\sigma_1 \delta\sigma_2 \tag{2}$$

is the joint probability of finding two galaxies in the elemental solid angles $\delta\sigma_1$ and $\delta\sigma_2$ separated by an angle θ, where $\mathfrak{N}$ is the mean angular (or surface) density of galaxies in the sample under consideration. Estimates of w can be made directly from counts in a sample for which the positions of individual galaxies on the sky are available using the formula[4]

$$2\delta N_p(\theta) \simeq N\mathfrak{N}[1+w(\theta)]2\pi\theta\delta\theta\,. \tag{3}$$

Here $\delta N_p(\theta)$ is the total number of pairs with angular separations lying in the interval $(\theta-\frac{1}{2}\delta\theta,\ \theta+\frac{1}{2}\delta\theta)$, N is the total number of sample galaxies, and the approximate equality emphasizes the fact that a statistical estimate of w is being made. When the sample consists of counts of galaxies in cells, a slightly different approach is required. A basic assumption of the correlation approach is that although ξ is not directly measurable, its character is universal in the sense that estimates of it from large but different samples will agree to within statistical uncertainties (sampling errors). Even apart from statistical uncertainties, estimates of w will, however, vary from one sample to another depending on the sample depths and other factors.

The basic relation between the angular and spatial pair correlation functions is a linear integral equation which was first derived by Limber (1953). The derivation of *Limber's equation* from the basic definitions (1) and (2) above is straightforward and, in the "narrow angle approximation," takes the form[5]

$$w(\theta)=\int_0^\infty dx x^4\phi^2(x)\int_{-\infty}^{+\infty} dy\,\xi[(x^2\theta^2+y^2)^{1/2}]\Big/\left[\int_0^\infty dx x^2\phi(x)\right]^2. \tag{4}$$

Here $\phi(x)$ is the sample selection function, defined as the mean number of sample galaxies per unit volume of space at a distance x from Earth. This equation is analogous to the relation between the spatial and projected densities of a spherical star cluster, except that, through ϕ, it includes the effects of galaxies sampled at different distances. It can also be inverted in closed form (Fall and Tremaine, 1977; Parry, 1977). For a sample which is magnitude-limited at the apparent magnitude m_0, the selection function is readily expressed in terms of the *integral luminosity function* for galaxies $\Phi(M)$ (the mean density of galaxies brighter than absolute magnitude M):

$$\phi(x)=\Phi(M=m_0-5\log(x/\mathrm{Mpc})-25)\,. \tag{5}$$

Since Φ is in general a rather broad function [cf. Eq. (11) below], any features in ξ, when convolved with ϕ^2 in Eq. (4), will be suppressed or even lost in estimates of w.

[2]Statisticians often refer to this function and related quantities as "covariance functions" and often take expressions like Eq. (23) below as the basic definition. In statistical physics the term "correlation function" is in general use and will be adopted throughout this article.

[3]The word "stationary" is unfortunate in this context as it refers to the spatial rather than the temporal distribution.

[4]This formula follows directly from the definition of w when allowance is made for overcounting in pairs. It is important to recognize that estimates of w made using Eq. (3) will have integrals over the sample area which are very nearly zero even if the underlying distribution has a spatial pair correlation function ξ which is everywhere positive.

[5]This approximation, introduced by Totsuji and Kihara (1969), is valid when $|(d/dr)\ln\phi(r)| \ll |(d/dr)\ln\xi(r)|$ as will be the case in practice.

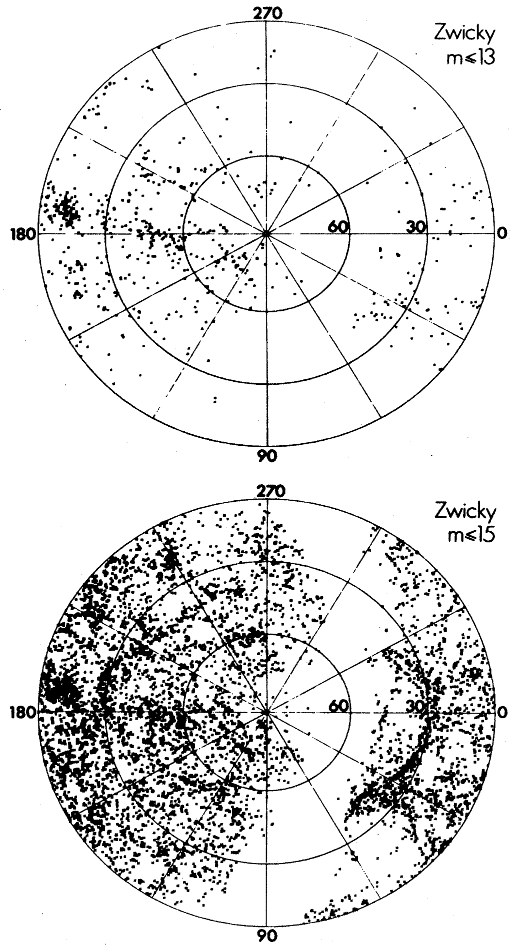

FIG. 1. Distribution of galaxies on the north celestial hemisphere: top, galaxies in the catalog of Zwicky *et al.* with $m \leq 13$; bottom, same but with $m \leq 15$. Note how clustering appears to smear out with increasing sample depth; cf., the scaling relation (7) and Fig. 2. The numbers indicate celestial coordinates α (right ascension) and δ (declination) in degrees. The large blank areas are due to obscuration in the galactic plane. The Local Supercluster is at $\alpha \simeq 190°$, $\delta \simeq 13°$ and the Perseus-Pisces chain is at $\alpha \simeq 20°$, $\delta \simeq 35°$. (Reproduced with the kind permission of N. A. Sharp.)

Two simple but important results follow directly from Limber's equation. The first is the *power-law solution*

$$w(\theta) = A\theta^{1-\gamma}, \quad \xi(r) = Br^{-\gamma}, \tag{6a}$$

$$B = A\Gamma(\tfrac{1}{2}\gamma)\left[\int_0^\infty dx x^2 \phi(x)\right]^2 \Big/ \left[\sqrt{\pi}\,\Gamma(\tfrac{1}{2}\gamma - \tfrac{1}{2}) \int_0^\infty dx x^{5-\gamma}\phi^2(x)\right], \tag{6b}$$

where γ, A, and B are constants and Γ denotes the usual gamma function. Although the relation between the amplitudes A and B of these two power law functions depends on the sample selection procedure, their indices $\gamma - 1$ and γ always differ by unity because a distribution in space has been projected onto the sky. The second simple consequence of Limber's equation is a *scaling relation* (Peebles, 1973) which is useful in comparing estimates of w from different samples. If the selection functions for a collection of samples are similar and differ only in their characteristic depths D^*, then the selection functions for these samples $\phi(x; D^*)$ must depend on x and D^* only through the dimensionless combination x/D^*. A straightforward change of variables in Eq. (4) then gives

$$w(\theta) = (1/D^*)F(\theta D^*), \tag{7}$$

where F is a function which is determined by ξ but which depends on θ only through θD^*. This relation simply means that on a given spatial scale θD^*, the apparent strength of clustering decreases inversely with the sample depth D^* because the number of uncorrelated intervening galaxies along the line of sight is proportional to D^*. (Fig. 1 illustrates this point in another way.)

As a practical example, suppose one has estimates of w from several samples which are limited at various magnitudes m_0, and that these estimates all show similar power-law form (same index γ). Then according to Eqs. (6a) and (7), the amplitudes of the angular correlation estimates should scale as

$$A = BC(D^*)^{-\gamma}, \quad D^* \propto \mathrm{dex}(\tfrac{1}{5}m_0), \tag{8}$$

where C is a constant that depends on γ and the shape of the luminosity function through Eqs. (5) and (6b). This scaling relation has proved useful in assessing the reliability of correlation estimates and the assumption that ξ is universal. More complete discussions of the method outlined above and its relation to other methods, particularly those involving power spectra, have been given by Peebles (1973) and by Layzer (1975). For a more detailed treatment of Limber's equation and its properties, the reader is referred to Fall and Tremaine (1977). For a discussion of some of the practical problems which can arise when estimating w, the reader is referred to Sharp (1979).

B. Pair correlation estimates

From time to time various statistical measures of galaxy clustering closely related to or equivalent to the correlation functions defined above, have been estimated (Neyman, Scott, and Shane, 1953, 1956; Limber, 1954; Rubin, 1954; Karachentsev, 1966; Kiang, 1967; Kiang and Saslaw, 1969). The present discussion, however, will begin with the work of Totsuji and Kihara (1969). They used essentially the procedure outlined above and the galaxy counts made by Shane and Wirtanen (1954, 1967) from the Lick survey plates (Shane, 1976). This sample has an average magnitude limit of roughly 19^m (or perhaps slightly less), includes nearly a million galaxies (!), and has a mean density of about 1.5×10^5 galaxies per steradian. The original Lick data consist of counts in $10' \times 10'$ cells but they were consolidated into $1° \times 1°$ cells for the published version of the catalog. Totsuji and Kihara found that their estimate of w was fitted best by the power-law model and

$$\gamma_L \simeq 1.8, \quad A_L \simeq 8 \times 10^{-2}(\mathrm{deg})^{0.8}, \tag{9}$$

(L for Lick) over a moderate range of angular separations. Using a Gaussian differential luminosity function $d\Phi(M)/dM$, they then obtained a power-law estimate of ξ with amplitude $B_L \simeq 16\,(\mathrm{Mpc}/h)^{1.8}$. (Here and throughout h denotes Hubble's constant H in units of $100\ \mathrm{km\,s^{-1}Mpc^{-1}}$; it is now thought to have a value somewhere in the approximate range $0.5 \lesssim h \lesssim 1.0$.) Although Totsuji and Kihara's (1969) paper is rather brief and was largely neglected until recently, it is remarkable for what is perhaps the basic empirical result in this field, namely the power-law form of ξ with index 1.8. Unaware of this paper, Peebles and Hauser (1974) repeated the analysis of the $1^\circ \times 1^\circ$ Lick counts in somewhat greater detail, but with essentially the same result as the angular correlation estimate (9).

Peebles and Hauser also analyzed counts from the catalog of Zwicky *et al.* (1961-1968) and gave a more thorough treatment of the spatial correlation function. Their estimates of w from the Zwicky catalog also show power-law form and can be fitted with

$$\gamma_Z \simeq 1.8, \quad A_Z \simeq 0.7(\mathrm{deg})^{0.8}, \tag{10}$$

(Z for Zwicky) when the sample is limited at $15\overset{m}{.}0$ and corrected for Galactic absorption (with a cosecant model). In order to estimate ξ, Peebles and Hauser evaluated the integrals of Eq. (6b) with a luminosity function of the form

$$\Phi(M) \propto \begin{cases} \mathrm{dex}\ [\beta(M - M^*)] & (M > M^*) \\ \mathrm{dex}\ [\alpha(M - M^*)] & (M^* > M > M_0) \\ 0 & (M < M_0)\,. \end{cases} \tag{11}$$

Apart from the cutoff at the bright end, this function is of the form introduced by Abell (1962) for galaxies in rich clusters. Peebles and Hauser adopted the parameters

$$\alpha = 0.75\,, \quad \beta = 0.25\,, \quad M_0 = M^* - 2.67\,, \tag{12a}$$

$$M^*_{pg} = -18.6 + 5\log h\,, \tag{12b}$$

and then computed the coefficient C in the scaling relation (8) using the definition

$$D^* \equiv \mathrm{dex}[\tfrac{1}{5}(m_0 - M^*) - 5]\,\mathrm{Mpc} \tag{13}$$

for the characteristic sample depth. The results of the integration[6] are $C = 1.83$ for $\gamma = 1.7$, $C = 1.58$ for $\gamma = 2.0$ and, by interpolation

$$C = 1.75 = 42\ (\mathrm{deg})^{0.8}\,, \quad (\gamma = 1.8)\,. \tag{14}$$

With the magnitude limit of $15\overset{m}{.}0$ for the Zwicky sample, Eqs. (12b) and (13) give the characteristic depth

$$D_Z^* \simeq 50\,h^{-1}\,\mathrm{Mpc}\,. \tag{15}$$

An estimate of the amplitude of ξ for this sample is thus

$$B_Z \simeq 20\,(\mathrm{Mpc}/h)^{1.8}\,, \tag{16}$$

by Eqs. (8), (10), and (14).

In principle, an estimate of the depth of the Lick sample could be made in the same way, but since its limiting magnitude is not known to very high accuracy another method is required. The method adopted by Peebles and Hauser begins with their estimate [Eq. (15)] for the Zwicky sample depth and then scales this depth by the ratio of sample densities

$$D_L^* \simeq D_Z^*(\mathfrak{N}_L/\mathfrak{N}_Z)^{1/3} \simeq 220\,h^{-1}\,\mathrm{Mpc}\,. \tag{17}$$

In conjunction with the estimates of w for the Lick sample [Eq. (9)] and the scaling relation [Eq. (8)], this method, however, leads to an estimate of the amplitude B_L of ξ which is some 50% larger than B_Z. This discrepancy probably gives some indication of the internal consistency of the method. Peebles (1974a) has also analyzed the 15th-magnitude Zwicky sample at smaller angular separations than was done in the original analysis and has made correlation estimates from this sample with a 13th-magnitude limit. The results, along with those from the Lick sample, are shown in scaled form in Fig. 2. Although the log-log nature of this plot can be somewhat deceptive, it does indicate that the galaxy pair correlation function has convincing power-law form over a fairly broad range of scales and that estimates from different samples do scale roughly as expected.[7]

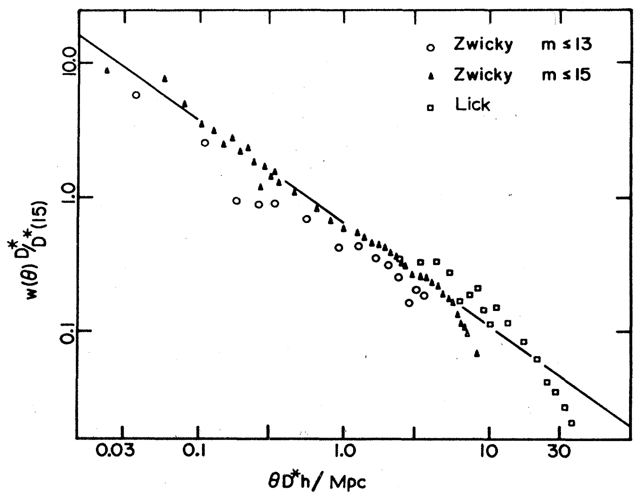

FIG. 2. Scaled correlation estimates from the Zwicky and Lick catalogs (limited to the region of sky with $b \geq 40°$, $\delta \geq 0°$). The counts have been corrected for galactic obscuration and, in the case of the $m \leq 13$ Zwicky sample, galaxies within 14° of the Virgo cluster center ($\alpha = 187°$, $\delta = 13\overset{\circ}{.}5$) have not been included. The solid line is $w(\theta) = 0.70\ (\mathrm{deg}/\theta)^{0.77}$ and the assumed sample depths are $D_Z^*(13) = 20\ h^{-1}$ Mpc, $D_Z^*(15) = 50\ h^{-1}$ Mpc, and $D_L^* = 220\ h^{-1}$ Mpc. The dependence of these estimates on sample depth is in reasonable agreement with the scaling relation (7) and indicates that the estimates have been made from a "fair sample" of galaxies. (From Peebles, 1974a, with kind permission.)

The scaling of these correlation estimates indicates that the samples from which they were derived are at least approximately representative of a universal underlying distribution.[8] It also indicates that the estimates of ξ are very largely the result of real galaxy clustering

[6] C is $2GH/E^2$ in Peebles and Hauser's notation.

[7] For a different opinion, see Wesson (1976).

[8] Following Hubble (1936), Peebles (1973) has called this the "Fair Sample Hypothesis."

in space and are not due to patchy Galactic obscuration because obscuration would affect estimates of $w(\theta)$ on fixed angular scales θ, whereas clustering in space affects estimates of $w(\theta)$ on fixed spatial scales θD^*. As the solid line in Fig. 2 indicates, the best correlation estimates probably come from the 15th-magnitude Zwicky sample. With the luminosity function (11) above, this gives [cf. Eq. (16)]

$$\xi(r) \simeq (r_0/r)^\gamma , \quad (0.1\,h^{-1}\,\mathrm{Mpc} \lesssim r \lesssim 10\,h^{-1}\,\mathrm{Mpc}) , \tag{18}$$

$$\gamma = 1.8 \pm 0.1 , \quad r_0 = 5.3(1.5)^{\pm 1} h^{-1}\,\mathrm{Mpc} .$$

The uncertainties indicated above are entirely subjective and are meant to reflect three separate effects. (i) Scatter in the data: error estimates of pair correlation data depend on higher-order correlations and are therefore difficult to make. (ii) Uncertainties in the luminosity function: the amplitude of ξ depends on the shape of the luminosity function, particularly at its bright end (cf. Peebles and Hauser, 1974, Table 3). (iii) Uncertainties in the magnitude scales: a change of $0^m.5$ (for example) in either M^* or m_0 would result in a 26% change in D^* and hence in r_0.

Figure 2 and Eqs. (18) indicate in a succinct way that pairwise galaxy clustering is a long-range phenomenon with no obvious features on scales ranging almost from those of individual galaxies up to about $10\,h^{-1}$ Mpc (and perhaps even more). Recently, Groth and Peebles (1977) have done a high-resolution analysis of the Lick counts using Seldner *et al.*'s (1977) reduction of the original $10' \times 10'$ counts. The new analysis is more complicated than the previous one by Peebles and Hauser (1974) and includes statistical corrections for plate-to-plate variations in the limiting magnitude and corrections for the overlap of different fields. The results are essentially the same as Eq. (18) (with $r_0 \simeq 4.7\,h^{-1}$ Mpc), except that there is some indication that beyond about $2^\circ.5$ the estimates of w fall below an extrapolated power law. The corresponding spatial scale of the feature is $r_b \simeq 2r_0$, which is where the amplitude of the spatial correlation function is $\xi_b \simeq 0.3$ (independent of r_0 and D^*). For reasons that will be explained in Sec. III.C, such a feature may have interesting cosmological consequences. In view of the large number of corrections made in the data analysis, however, it is difficult to assess the significance of the feature Groth and Peebles claim to have found.[9]

All of the correlation estimates discussed up to this point have used only two-dimensional information about the positions of galaxies. Redshifts, however, give some information about the positions of galaxies in the third dimension through Hubble's relation $v = Hx$ (v and x being, respectively, the radial velocity and distance). If all galaxies had perfect Hubble motion, with no random or peculiar velocities, one could estimate ξ directly (without having first to estimate w). Of course, galaxies do have peculiar velocities and this tends to smooth out the power-law behavior on scales smaller than a few megaparsecs when ξ is estimated in this way. On larger scales, however, the method should give a reliable estimate of ξ which is free from uncertainties in the luminosity function, and on smaller scales it should give information about the peculiar velocities of galaxies. The only difficulty is in obtaining redshifts for a sample of galaxies which is large enough to be representative in the sense discussed above. Davis, Geller, and Huchra (1978) have recently applied the method to a 13th-magnitude sample (essentially, the Shapley-Ames list with about 800 redshifts). Their correlation estimates have an amplitude of about $r_0 \simeq 3\,h^{-1}$ Mpc, considerably smaller than those from estimates of w, and their estimate of the rms peculiar velocity is about 300 km s^{-1}, a fairly typical value. Since the sample is dominated by the Virgo cluster and surrounding supercluster (cf., Fig. 1), Davis *et al.* regard these results as tentative. Within the near future, enough redshifts should be available for considerably more reliable estimates.

[9]In this connection, it is worth noting that shortly before the feature was found, it was predicted by Davis and Peebles (1977) and that it is not present in the "unsmoothed" correlation estimates of Groth and Peebles (1977; cf. their Figs. 2 and 3).

C. Higher-order and cross correlations

From the previous section it should be clear that an enormous amount of information about the distribution of galaxies has been neglected in obtaining pair correlation estimates. Additional information is contained in the higher-order correlation functions, such as the *spatial triplet correlation function*, which is usually denoted by ζ (and is often referred to as the "three-point function"). This function is defined such that

$$\delta p \equiv n^3[1 + \xi(r_{12}) + \xi(r_{23}) + \xi(r_{31}) + \zeta(r_{12}, r_{23}, r_{31})]\delta v_1 \delta v_2 \delta v_3 \tag{19}$$

is the joint probability of finding galaxies in the three elemental volumes δv_1, δv_2, δv_3 separated by the distances r_{12}, r_{23}, and r_{31}. It must be a symmetric function of these arguments. The middle three terms on the right hand side of Eq. (19) account for the clustering in triples from uncorrelated singles and correlated pairs, while the last term ζ accounts for purely triple clustering and is therefore referred to as the "irreducible" triplet correlation function. A corresponding angular function, usually denoted by z, can also be defined in direct analogy with ζ and can be estimated from galaxy counts by a procedure analogous to that described above for estimating w. The triplet functions ζ and z are related by a straightforward generalization of Limber's equation, which also has power-law solutions and a simple scaling property. For details, the reader is referred to Peebles and Groth (1975).

They have used this procedure to estimate ζ from the 15th-magnitude Zwicky sample and the $10' \times 10'$ Lick counts (Groth and Peebles, 1977) and find that the data can be fitted by a function of the form

$$\zeta(r_{12}, r_{23}, r_{31}) \simeq Q[\xi(r_{12})\xi(r_{23}) + \xi(r_{23})\xi(r_{31}) + \xi(r_{31})\xi(r_{12})] , \tag{20}$$

with $Q = 1.3 \pm 0.2$. This interesting result has played an important role in theoretical work in this subject (Sec. III.A). Uncertainties in it are somewhat larger and the range of scales over which it applies are somewhat smaller than those for ξ. Fry and Peebles (1978) have recently estimated fourth-order correlation functions for the $10' \times 10'$ Lick counts and have found them to be consistent (in a rough way) with a straightforward generalization of Eq. (20). In principle, one could estimate even higher-order correlation functions, but in practice the required effort increases rapidly with the order, and in any case the amplitudes would be so low that the estimates would be dominated by sampling errors.

As a complementary method for exploiting some additional information about galaxy clustering within the correlation approach, cross-correlation techniques have also proved useful. In direct analogy with the other correlation functions, one can define a *spatial cross-correlation function* ξ_{ab} for objects of type a and objects of type b;

$$\delta p(r) \equiv n_a n_b [1 + \xi_{ab}(r)] \delta v_a \delta v_b \tag{21}$$

is the joint probability of finding objects of types a and b, respectively, in the elemental volumes δv_a and δv_b separated by the distance r when the mean densities are n_a and n_b. (Note that ξ_{ab} and ξ_{ba} are equal by this definition and that ξ is the "autocorrelation function" ξ_{gg} where g refers to individual galaxies.) As a hypothetical example, one might cross-correlate x-ray counts with galaxy clusters in order to study the extent of hot gas in clusters. The methods for estimating ξ_{ab} and its angular analog w_{ab} are much the same as those described above except that the selection functions ϕ_a and ϕ_b enter Limber's equation with the product $\phi_a(x)\phi_b(x)$ replacing $\phi^2(x)$ in the numerator of (4) and $\int_0^\infty dx x^2 \phi_a(x) \times \int_0^\infty dy y^2 \phi_b(y)$ replacing the entire denominator.

Using the $10' \times 10'$ Lick counts, Seldner and Peebles (1977a) have repeated Peebles' (1974b) original cross-correlation analysis of Abell (1958) cluster centers and galaxies in the $1° \times 1°$ Lick cells. The analysis is complicated by the fact that Abell's definition of a cluster is distance dependent, but when averages are taken over distance and richness classes, the results are consistent with a power-law model for ξ_{gc} with $\gamma_{gc} \simeq 2.4$ and $B_{gc} \simeq 170\ (\mathrm{Mpc}/h)^{2.4}$ over the range $0.5\,h^{-1}$ Mpc to about $15\,h^{-1}$ Mpc. There are uncertainties in the results, but they do seem to be in good agreement with the cluster density profile

$$\nu(r) \propto r^{-\alpha}, \quad \alpha = 2.3 \pm 0.2\,, \quad (2\,h^{-1}\ \mathrm{Mpc} \lesssim r \lesssim 15\,h^{-1}\ \mathrm{Mpc}) \tag{22}$$

found by Chincarini and Rood (1976) to fit the distribution of Zwicky galaxies in the Coma cluster field. This should not be surprising because on scales small enough that the clustering of clusters (superclustering) can be ignored, ξ_{gc} is essentially the (dimensionless) density run of a typical rich cluster with the "background" taken out. Together with estimates of the correlation between the positions of Abell clusters themselves (Hauser and Peebles, 1973), Peebles (1974b) has suggested that estimates of ξ_{gc} are consistent with a picture in which clusters and galaxies are grouped in great "clouds" or "superclusters" with about two rich clusters per cloud and about 25% of all galaxies as members of some cloud. [This represents a change of view from the earlier conclusion of Yu and Peebles (1969); see also de Vaucouleurs (1971) and references therein.]

Several other applications of cross-correlation techniques are worth mentioning here. These include Davis and Geller's (1976) cross-correlation analysis of galaxies of different morphological types in the Uppsala catalog (Nilson, 1973), Sharp, Jones, and Jones' (1978) cross-correlation analysis of DDO (dwarf) galaxies and galaxies in the Zwicky catalog, and Seldner and Peebles (1978) cross-correlation analysis of 4C radio sources and galaxies in the Lick survey. It is important to recognize that the interpretation of these results in terms of ξ_{ab} is quite sensitive to the assumed luminosity functions for the different types of objects a and b through the selection functions ϕ_a and ϕ_b. If the luminosity functions are known to high precision, ξ_{ab} can be reliably estimated from w_{ab}.[10] Conversely, if the clustering properties of the different objects, and therefore ξ_{ab}, are known (or assumed), some information about the luminosity functions can be inferred from estimates of w_{ab}.

D. Power spectra and related measures

In what follows, it will sometimes be helpful to think of the distribution of galaxies as representing a fluctuating density field. In this case, an alternative expression for the spatial pair correlation function ξ in terms of the local density of galaxies $n(\mathbf{x})$ is useful:

$$\xi(r) = \langle \Delta(\mathbf{x})\Delta(\mathbf{x}+\mathbf{r})\rangle - \delta(r)/n\,, \tag{23a}$$

$$\Delta(\mathbf{x}) \equiv n(\mathbf{x})/n - 1\,, \quad n \equiv \langle n(\mathbf{x})\rangle\,, \quad n(\mathbf{x}) = \sum_{\text{galaxies}} \delta(\mathbf{x} - \mathbf{x}_i)\,. \tag{23b}$$

Here $\Delta(\mathbf{x})$ is the density contrast at $\mathbf{x}$, δ denotes the (three-dimensional) delta function, the averages are over large volumes, and n (without a position argument) is the mean density of galaxies (as before). This expression can be shown to be equivalent to the definition (1) in terms of probabilities (e.g., Ichimaru, 1973; Layzer, 1975). The second term on the right hand side of Eq. (23a) accounts for the self-correlation of discrete objects and must be interpreted as vanishing in the continuum limit. Similarly, correlation functions of order s can be expressed as averages over s factors of Δ, each with a different position argument.

A useful measure of clustering closely related to ξ is the *power spectrum* or structure factor $S(k)$. It is defined as $\langle |\Delta_{\mathbf{k}}|^2\rangle$ (up to proportionality) where $\Delta_{\mathbf{k}}$ is the Fourier transform of $\Delta(\mathbf{x})$; thus

$$S(k) = 1 + P(k)\,, \tag{24a}$$

$$P(k) = 4\pi n \int_0^\infty dr r^2 \xi(r) \sin(kr)/(kr)\,. \tag{24b}$$

Sometimes, $P(k)$ is also called the power spectrum

[10] Note that not all cross-correlation functions can have power-law form with different indices because, for example, $n_g \xi_{ag} = n_b \xi_{ab} + n_a \xi_{aa}$ where a and b refer to galaxies g of different types.

because it is equal to $S(k)$ in continuum. In an analogous way, one can define an angular power spectrum u_l in terms of averages over spherical harmonic transforms of density fluctuations on the sky (Peebles, 1973). This in turn can be related to the angular correlation function w by an integral transform, which, in the narrow angle approximation, reduces to the usual two-dimensional Fourier transform:

$$u_l = 2\pi\mathfrak{N}\int_0^\infty d\theta\,\theta w(\theta)J_0(l\theta)\,, \tag{25}$$

where J_0 denotes the Bessel function of order zero. The power spectra $P(k)$ and u_l are related by an equation which is completely equivalent to Limber's equation (4) and which also has a simple scaling property; $u_l(D^*)$ is a function of the single variable l/D^*. (This fact follows directly from Eqs. (7), (25) and $\mathfrak{N}\propto D^{*3}$.) For the power-law model (6), one has

$$P(k) = 2\pi^2 nBk^{\gamma-3}/\Gamma(\gamma-1)\sin[\pi(\gamma-1)/2]\,,\quad (1<\gamma<3) \tag{26a}$$

$$u_l = \pi\mathfrak{N}A(l/2)^{\gamma-3}\Gamma(\tfrac{3}{2}-\tfrac{1}{2}\gamma)/\Gamma(\gamma/2-\tfrac{1}{2}),\quad (\tfrac{3}{2}<\gamma<3)\,. \tag{26b}$$

In principle, the power spectra contain no new information about pairwise clustering. Nevertheless, estimates of $P(k)$ and u_l are not in general equal to the Fourier transforms of estimates of $\xi(r)$ and $w(\theta)$ with a finite amount of data; so in practice the power-spectrum approach is complementary to the correlation approach. Indeed, because of the convolution property of power spectra, it is often more convenient to remove any scale-dependent selection effects (e.g., variations in plate sensitivity) in the transform domain. Peebles (1973) has developed an elaborate procedure for estimating power spectra in the distributions of extragalactic objects, and he and Hauser have applied it to the Zwicky, Lick, and Abell catalogs in conjunction with their correlation estimates (Hauser and Peebles, 1973; Peebles and Hauser, 1974; Peebles, 1974b). For alternative and more direct approaches to the problem of estimating power spectra, the reader is referred to Webster (1976a).

Two other statistical measures of clustering are of interest here. The first is $\Xi(r)$, the expected number of galaxies, in excess of the Poisson number, within a distance r of a randomly chosen galaxy:

$$\Xi(r) = 4\pi n\int_0^r ds\,s^2\xi(s)\,. \tag{27}$$

The second is $\sigma^2(r)$, the variance in $N(r)/\langle N(r)\rangle$, where $N(r)$ is the number of galaxies in a randomly placed volume of radius r [given by an integral over $n(\mathbf{x})$] and $\langle N(r)\rangle$ is the average number (given by $\frac{4}{3}\pi nr^3$). On the length scale r, $\sigma(r)$ is a good measure of the "typical" density contrast and from Eq. (23) it follows that this quantity is given by the expressions

$$\sigma^2(r)\langle N(r)\rangle^2 \equiv \langle[N(r)-\langle N(r)\rangle]^2\rangle \tag{28a}$$

$$= \langle N(r)\rangle + n^2\int_{r>|\mathbf{x}|} d^3\mathbf{x}\int_{r>|\mathbf{y}|} d^3\mathbf{y}\,\xi(|\mathbf{x}-\mathbf{y}|)\,. \tag{28b}$$

The first term of Eq. (28b) is due to the usual $\sqrt{N}$ fluctuations in a Poisson distribution and results from the delta-function term in Eq. (23a). For the present distribution of galaxies, the first terms of Eqs. (24a) and (28b) are small on most of the scales where ξ has been estimated (Fall, 1978). Neglecting these "discreteness terms," one has the following power-law relations

$$S(k) \simeq 18(r_0/\lambda)^3(kr_0)^{\gamma-3}\,, \tag{29a}$$

$$\Xi(r) \simeq 10(r_0/\lambda)^3(r/r_0)^{3-\gamma}\,, \tag{29b}$$

$$\sigma(r) \simeq 1.4(r_0/r)^{\gamma/2}\,, \tag{29c}$$

where λ denotes the mean intergalaxy separation $n^{-1/3}$, and the coefficients have been evaluated specifically for the case $\gamma\simeq 1.8$.

These relations have some useful consequences. From Eq. (29c) it follows that typical fluctuations on scales of about 1.4 r_0 have unit amplitude. A randomly placed sphere of this radius contains, on average, about $12(r_0/\lambda)^3$ galaxies. According to Eq. (29b), a sphere of the same radius, centered on a galaxy, typically contains another $15(r_0/\lambda)^3$ galaxies, for a total of $27(r_0/\lambda)^3$. In order to estimate this number and the coefficients of Eqs. (29a) and (29b), an estimate of the mean density n is required. Because of the large number of faint galaxies, this is in general a difficult quantity to define [cf. Eqs. (11) and (12) above]. However, most of the galaxies counted in a magnitude-limited sample have absolute magnitudes near the shoulder of the luminosity function at M^*. Thus, for many purposes, a reasonable definition of n is $\mathfrak{L}/L^*$ where $\mathfrak{L}$ is the mean luminosity density and L^* is the luminosity corresponding to M^*. Recent estimates of these quantities give $n\simeq 0.02\,h^3$ Mpc^{-3} and hence $\lambda\simeq 3.6\,h^{-1}$ Mpc (Felten, 1977). With this estimate and Eq. (18), the number of bright galaxies in typical fluctuations of unit amplitude is of order 80. This number, however, depends sensitively on the values of λ and r_0.

Finally, a few additional remarks are in order concerning the distribution of bright galaxies as inferred from spots of light on photographic plates. In dynamical arguments, it is usually assumed that this distribution reflects the underlying *mass distribution* and, in particular, that the number density $n(\mathbf{x})$ is proportional to the mass density $\rho(\mathbf{x})$. To the extent that the mass and light distributions are proportional, ξ and ζ are also two- and three-point correlation functions for mass density fluctuations. This is a natural assumption to make and is one that will be made throughout the remainder of this article, but one should remember that the evidence for it is not terribly compelling. (It consists mainly of the fact that the virial mass-to-light ratio of groups and clusters varies by about five or less for thousandfold variations in mass; see Sec. III.D). In this case, the average mass contained within a sphere of radius r, centered on a randomly chosen galaxy, is

$$M(r) = 4\pi\rho\int_0^r ds\,s^2[1+\xi(s)] = 4.3\times 10^{14}m_\odot \times\Omega h^{-1}(r_0h/5.3\text{ Mpc})^3(r/r_0)^{3-\gamma}[1+0.4(r/r_0)^\gamma]\,, \tag{30}$$

where $m_\odot$ is the mass of the sun, ρ is the mean mass density, and Ω is the cosmological density parameter [defined by Eq. (38) below]. This is a reasonable measure of the mass of a fluctuation of characteristic dimension r. The mass of a typical fluctuation of unit amplitude ($r \simeq 1.4 r_0$) is therefore

$$M_1 \simeq 10^{15} m_\odot \Omega h^{-1} (r_0 h/5.3\ \text{Mpc})^3 . \quad (31)$$

This is a useful datum for theoretical discussion, but its exact value depends sensitively on r_0.

III. SIMPLE THEORETICAL IDEAS

The distribution of galaxies poses an important and, as it turns out, surprisingly formidable challenge to theorists: How did the present distribution come about and what does it tell us about the Universe in the past? Of course, these questions are not new to cosmologists. On the basis of some simple order-of-magnitude arguments it has usually been assumed that some form of gravitational instability in the context of a hot big-bang cosmological model has done the job. The recent studies of galaxy clustering, particularly those involving correlation functions, however, have stimulated a rethinking of the problem at a much more detailed level and in a somewhat different language.

A. Connection with traditional descriptions

Before considering evolutionary and dynamical problems it will be useful to make at least a rough connection between the foregoing description of galaxy clustering in terms of correlation functions and more traditional descriptions.[11] This is also worthwhile because the correlation approach may at first seem somewhat foreign to one's intuitive notions of clustering. Three additional measures of galaxy clustering are of use here. (i) *Internal density profile*: counts of galaxies in the fields of several rich clusters are consistent with a power-law density profile of the form

$$\nu(r) \propto r^{-\alpha} , \quad (32)$$

with index α in the range 2.1 to 2.5 [cf. Eq. (22) above]. It seems reasonable to suppose that this relation applies in some average sense to smaller clusters and groups. (ii) *Membership-size relation*: Carpenter (1938) first noticed that clusters and groups follow a number of members, m, versus size, l, relation of power-law form

$$m(l) \propto l^{3-\theta} . \quad (33)$$

de Vaucouleurs (1971) has coined the term "thinning factor" for the index θ and advocates the value 1.7 for it.[12] (iii) *Membership spectrum*: this is closely related to the (differential) luminosity function and mass spectrum of groups and is often referred to as the "multiplicity function." It has been studied by Holmberg (1940) and, more recently, by Gott and Turner (1977b). They have found that, in their recent group catalog, the number of groups $\eta(m)\delta m$ with membership in the interval $(m, m+, \delta m)$ can be fitted by a power law

$$\eta(m) \propto m^{-\beta} , \quad (34)$$

with index $\beta \simeq 2.3$ over the range $1 \lesssim m \lesssim 300$.

It is remarkable that all three of these measures of clustering have no preferred scales and can be approximated by power laws, even though they refer to three different aspects of the distribution of galaxies. One may therefore wonder to what extent the power-law forms of the pair and triplet correlation functions, ξ and ζ, reflect these different aspects of clustering. Because each measure gives only a partial description of clustering, it is not possible to make exact comparisons. Some impression of their relative importance, however, can be gained by considering the following extreme models for clustering. Model I (Isolated clustering): all galaxies are in clusters and all clusters are alike; each cluster has a power-law density profile with index $\alpha \simeq 2.4$ and the cluster centers are randomly distributed (no subclustering or superclustering). Model H (Hierarchical clustering): all galaxies are members of some aggregate (group or cluster); each aggregate is a member of a larger one and contains smaller ones (subclustering and superclustering is nested); the mean density of aggregates of different sizes follows Carpenter's relation with $\theta \simeq 1.8$. Model I is, of course, rather naive and can already be ruled out on the basis of Carpenter's relation and the multiplicity function. Even model H is not completely realistic because it is easy to find groups which cannot be assigned to a larger cluster and clusters which show few signs of subclustering.[13] Nevertheless, it is instructive to compare the estimates of ξ and ζ on these two simple models.

Peebles and Groth (1975) have given the following arguments to suggest that the estimates of ξ and ζ are more consistent with model H than with model I. In the case of model I, $\xi(r)$ is proportional to the density profile overlap $\int d^3\mathbf{x}\nu(\mathbf{x})\nu(\mathbf{x}+\mathbf{r})$, or to $r^{3-2\alpha}$, for small r ($r \lesssim r_0$), (Peebles, 1974a). Thus the correlation and density profile indices are related by the equation $\gamma = 2\alpha - 3$, which is in good agreement with $\gamma \simeq 1.8$ and $\alpha \simeq 2.4$. On this model, however, $\zeta(r,r,r)$ is proportional to $\int d^3\mathbf{x}\nu(\mathbf{x})\nu^2(\mathbf{x}+\mathbf{r})$, or to $r^{3-3\alpha}$, for small r. Comparison with the Peebles–Groth expression (20) for ζ in terms of ξ then gives $2\gamma = 3\alpha - 3$, which is not compatible with $\gamma \simeq 1.8$ and $\alpha \simeq 2.4$. With model H, on the other hand, the correlation index γ and the thinning factor θ are equal, in reasonable agreement with de Vaucouleur's estimate $\theta \simeq 1.7$. This is because with each galaxy at the top of the hierarchy, the probability of finding another one at a distance r is proportional to the density on that scale and $\xi(r)$ is therefore proportional to $m(r)/r^3$ (for $r \lesssim r_0$). Similarly, the probability of finding a

[11]In keeping with the remarks at the end of the previous section and the crude nature of the arguments to follow, the terms "mass," "membership," and "luminosity" will be used more or less interchangeably throughout this subsection.

[12]Carpenter suggested Eq. (33) with $\theta \simeq 1.5$ as an upper envelope for the membership-size relation, but a power-law fit through his data is better with a somewhat larger value of θ. In view of point (i) above, the notion of a cluster "size" is very subjective, and Carpenter's relation should be regarded accordingly.

[13]The hierarchy envisaged here is essentially the same on small scales ($l \lesssim r_0$) as that advocated by de Vaucouleurs (1971) but differs on larger scales in that here the mean density is assumed to have a well-defined finite value n. (See also Mandelbrot, 1975, 1977.)

triplet with the separations (r, r, r) is proportional to the square of the density on that scale so $\zeta(r, r, r)$ is proportional to $m^2(r)/r^6$ (for $r \lesssim r_0$). Evidently, these arguments suggest that hierarchical clustering is an important contributor to the forms of the lower-order correlation functions. However, as Shanks (1979) has recently emphasized, the Peebles–Groth arguments are not as convincing as they seem and there may be statistics better than the correlation functions for testing different models of clustering.

It is also interesting to compare the pair correlation estimates with the multiplicity function on the hierarchical model. This may be done as follows (McClelland and Silk, 1977b). An aggregate with m members will have $\frac{1}{2}m(m-1)$ pairs, so the number of pairs contributed by aggregates with membership in the interval $(m, m+\delta m)$ is roughly $\frac{1}{2}m^2\eta(m)\delta m$, and this is proportional to $l^{(2-\beta)(3-\theta)}l^{(2-\theta)}\delta l$ by Carpenter's relation [Eq. (33)]. On model H, it is also proportional to $\xi(l)l^2\delta l$ (for $l \lesssim r_0$), so the indices θ, β, and γ are related by the equation $\gamma = 3(\theta+\beta) - \theta\beta - 6$. With the specific values $\theta \simeq 1.7$ and $\beta \simeq 2.3$, this gives $\gamma \simeq 2.1$, which is probably satisfactory considering the crude nature of the argument.[14] The relations between correlation functions of various orders and traditional measures of clustering have been considered in some detail by Peebles (1974f), McClelland and Silk (1977b, 1978), Soneira and Peebles (1977, 1978), Bhavsar (1978), and White (1979).

Correlation studies have also had a bearing on several related issues: specifically, the existence of "holes" round clusters, the existence of a "field" population of galaxies, and the existence of "chains" of galaxies. The fact that all of the correlation estimates discussed in Sec. II are positive, at least on scales up to about $15\,h^{-1}$ Mpc, suggests that, in a statistical sense, there are not prominent holes round clusters[15] and that the density profiles of clusters merge smoothly into the background (Peebles, 1974a). Correlation studies also suggest that it is difficult, if not impossible, to isolate a genuine field population which does not participate in the general pattern of clustering (Turner and Gott, 1975; Fall *et al.*, 1976; Soneira and Peebles, 1977). Finally, estimates of the triplet correlation function ζ suggest that the importance of chains is minimal on small scales ($l \lesssim r_0$). One way to see this is as follows. Consider a triplet (1, 2, 3) with separations $r_{12} = r$, $r_{23} = r(\frac{1}{4} + \lambda^2 - \lambda\mu)^{1/2}$, and $r_{31} = r(\frac{1}{4} + \lambda^2 + \lambda\mu)^{1/2}$, where μ is the cosine of the angle between the line joining 1 to 2 and the line joining 3 to the midpoint of (1,2), and λ is the ratio of the lengths of the two lines. With this notation and the Peebles–Groth form [Eq. (20)] for ζ, one has

$$\zeta(r_{12}, r_{23}, r_{31}) \simeq \tfrac{2}{3}\zeta(r, r, r)\begin{cases}\lambda^{-\gamma} & (\lambda \gg 1)\\ \text{const} & (\lambda \ll 1)\end{cases}, \tag{35}$$

which is independent of μ, and hence the orientation of the triplet, for both large and small displacements of the third galaxy from the other two. On scales larger than about $10\,h^{-1}$ Mpc, this argument does not apply. Indeed, some very long chains do appear in the maps of Selder *et al.* (1977), but their significance is difficult to estimate quantitatively (Groth and Peebles, 1977).

B. Linear evolution of clustering

The description of galaxy clustering in terms of correlation functions suggests a gravitational origin for the structure of matter on scales larger than those of individual galaxies, especially in view of the long-range, scale-free nature of gravitational attraction. One of the goals of the correlation approach has been to make this connection more precise within the framework of the "standard big-bang" cosmological model. This model (with $\Lambda = 0$) will be adopted throughout the remainder of the article; that is, it will be assumed that the Universe is homogeneous and isotropic in the large, that redshifts are cosmological, and that the 3° background radiation has an extragalactic nature and is the relic of a dense "fireball" phase of the Universe.[16] (For overviews of the standard model, the reader is referred to the books by Peebles, 1971; Sciama, 1971; and Weinberg, 1972.) Moreover, for reasons that will become clear later, the "recombination epoch," when the fireball plasma became transparent, will be taken as the starting point. This, of course, leaves out of consideration the many interesting pre-recombination processes which have traditionally played a role in attempts to understand the origin of structure in the Universe.

In homogeneous cosmological models, the large-scale evolution is completely specified by the *cosmological scale parameter* $a(t)$, which may be thought of as the average proper distance between particles at epochs specified by the proper time t. After recombination, matter and radiation evolve separately, with matter the only dynamically significant component.[17] In this case, the Newtonian approximation applies and *Friedmann's equations* for the scale parameter take the familiar forms

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 + \frac{k}{a^2} = \frac{8}{3}\pi G\rho\,, \quad \rho \propto a^{-3}\,, \tag{36}$$

where G is the gravitational constant, $\rho(t)$ is the mean mass density, and k is a constant, which in the relativistic formulation determines the curvature of the model (negative for $k<0$, positive for $k>0$). Depending on the context, it is sometimes convenient to specify evolution in terms of the redshift z, which is related to a and therefore to t by the expression

[14] Since the Gott–Turner procedure for identifying groups involves a minimum surface density criterion, it is likely that the effective value of θ for their sample is somewhat smaller than 1.7 (i.e., $m/l^2 \propto$ surface density $\simeq$ const). This results in a smaller value of γ by the above arguments; e.g., $\gamma \simeq 1.6$ with $\theta \simeq 1.0$.

[15] All of the remarks of this paragraph apply in a statistical sense; it is easy to find individual exceptions.

[16] These assumptions are made here to keep the discussion concrete. Much of what follows, however, could easily be adapted to other models (cf. Layzer, 1975).

[17] Unless the intergalactic medium received a substantial input of heat at recent epochs, recombination took place at the redshift $z_r \simeq 1500$ and the energy densities of matter and radiation were equal at the redshift $z_e \simeq 4.3 \times 10^4 \Omega h^2$. Thus, if $\Omega h^2 \gtrsim 3.5 \times 10^{-2}$ is satisfied, recombination occurred after the epoch of equal energy densities.

$$a(t) \equiv a(z) \propto 1/(1+z)\,, \tag{37}$$

with $z=0$ corresponding to the present epoch and $z \simeq 1500$ corresponding to the epoch of recombination (on the standard model; cf. footnote 17).

The character of the solutions of Eq. (36) is determined by the *cosmological density parameter*

$$\Omega \equiv \rho_0/\rho_{\rm crit} = 8\pi G\rho_0/3H^2\,, \tag{38}$$

where the subscript 0 refers to the present epoch ($z=0$, $t=t_0$), H is Hubble's constant [the present value of $d(\ln a)/dt$], and $\rho_{\rm crit}$ is the critical density (1.9×10^{-29} h^2 gm cm^{-3}) required to close the Universe. If Ω is greater than unity, the Universe will eventually stop expanding and begin to contract; otherwise it will expand forever. At early times, the solutions of Eq. (36) all have the *Einstein-de Sitter* ($k=0$) form, $a(t) \propto t^{2/3}$. The solutions for the open model ($k<0$, $\Omega<1$) may be approximated by the expressions

$$a(t) \propto \begin{Bmatrix} t^{2/3} & (z \gtrsim z_f) \\ t & (z \lesssim z_f) \end{Bmatrix}, \quad z_f \simeq \Omega^{-1}-1\,, \tag{39}$$

with the later evolution corresponding to undecelerated expansion. The solution for the closed model ($k>0$, $\Omega>1$) is a cycloid. The values of Ω which are of current interest lie in the range 0.02 to 1, corresponding, respectively, to the density associated with the luminous parts of galaxies and the closure density. The value $\Omega \simeq 0.1$ is becoming fairly standard but is subject to uncertainty (see Sec. III.D, Gott *et al.*, 1974, and Gunn, 1978).

The behavior of inhomogeneities in the "background universe" described above depends on whether they are "linear" or "nonlinear," that is, whether the density contrast $\delta\rho/\rho$ associated with them is smaller or larger than unity. The behavior of small-amplitude perturbations is relatively straightforward and may be obtained by linearizing the basic fluid-dynamical equations (for conservation of mass, energy, and momentum) about the Friedmann equations (36) (see, for example, Harrison, 1967; Weinberg, 1972). Neglecting pressure gradients, the result is

$$\frac{d^2\Delta}{dt^2} + \frac{2}{a}\frac{da}{dt}\frac{d\Delta}{dt} = 4\pi G\rho\Delta\,, \quad v_\parallel \propto a\frac{d\Delta}{dt}\,, \tag{40}$$

where Δ is $\delta\rho/\rho$ and $v_\parallel$ is the compressional component of the proper peculiar velocity associated with the perturbation. These equations apply to perturbations on mass scales larger than the Jeans mass, which is of order $10^6\, m_\odot$ just after recombination and decreases as $a^{-3/2}$ from then on. They have the approximate solutions for $\Omega<1$ [cf. Eq. (39) above]

$$\Delta^{(+)}(t) \propto \begin{Bmatrix} t^{2/3} & (z \gtrsim z_f) \\ \text{const} & (z \lesssim z_f) \end{Bmatrix}, \quad v_\parallel^{(+)} \propto \begin{Bmatrix} t^{1/3} & (z \gtrsim z_f) \\ 0 & (z \lesssim z_f) \end{Bmatrix}, \tag{41a}$$

$$\Delta^{(-)}(t) \propto \begin{Bmatrix} t^{-1} & (z \gtrsim z_f) \\ t^{-1} & (z \lesssim z_f) \end{Bmatrix}, \quad v_\parallel^{(-)}(t) \propto \begin{Bmatrix} t^{-4/3} & (z \gtrsim z_f) \\ t^{-1} & (z \lesssim z_f) \end{Bmatrix}. \tag{41b}$$

The growing mode (+) "freezes out" at a redshift z_f of about $\Omega^{-1}-1$ when the universe begins undecelerated expansion. After that, only nonlinear perturbations continue to grow. The decaying mode (−) is usually neglected because, even if present at recombination, it soon becomes negligible.[18] In this case, it follows from the fact that $\xi(r)$ is $\langle\Delta(\mathbf{x})\Delta(\mathbf{x}+\mathbf{r})\rangle$ (in the continuum limit) and the fact that linear modes behave independently, that the linear part of ξ is proportional to its recombination form at any later time; in fact

$$\xi(r,z) \simeq \xi_r(ra_r/a(z)) \begin{Bmatrix} (a(z)/a_r)^2 & (z \gtrsim z_f) \\ \text{const} & (z \lesssim z_f) \end{Bmatrix} \; (\xi \lesssim 1)\,, \tag{42}$$

where the subscript r refers to the recombination epoch and the argument r denotes proper separation.

It is now common practice to assume a power-law model for the spectrum of fluctuations at recombination, over at least some range of scales

$$S_r(k) \propto k^n\,, \quad (-3<n\leqslant 4)\,, \tag{43}$$

where n is an index to be determined (and not to be confused with the mean number density of galaxies). According to Eqs. (24) and (42), the present linear part of ξ has the form (Peebles, 1974d; Bonometto and Lucchin, 1978a)

$$\xi(r) \propto r^{-(n+3)}\text{sgn}(-n)\,, \quad (r \gtrsim r_0)\,, \tag{44}$$

so long as Eq. (43) applies on mass scales larger than M_1 [given by Eq. (31) above]. The exact range of scales over which Eq. (44) applies may depend somewhat on Ω but, in any case, it is not likely that present estimates of ξ are reliable on large enough scales to determine unambiguously the exponent n from this linear relation. When large samples of galaxies with measured redshifts become available, it may then be possible to estimate n directly from the linear part of ξ.

C. Nonlinear evolution of clustering

One might reasonably expect that the nonlinear development of inhomogeneities is not so simple because it should depend not only on the cosmological model and the spectrum of fluctuations at recombination but also on the specific physical processes involved in the formation and clustering of galaxies. In one picture (see, for example, Peebles, 1974c), inhomogeneities on all mass scales larger than some "seed mass" ($\sim 10^7$–$10^9 m_\odot$) managed to survive the fireball[19] (e.g., the fluctuations had an isothermal component). Fluctuations on all larger scales then grew by purely gravitational clustering to form structures in the approximate order: subgalactic structures, galaxies, cores of rich clusters,

[18] In addition, there is another decaying mode associated with the rotational component $v_\perp$ of peculiar velocities: $v_\perp \propto a^{-1}$.

[19] The notion of a "seed mass" can be made somewhat more precise; from Eqs. (28), (41), and (43), it follows that density fluctuations on the mass scale M had amplitude $(1500\Omega)^{-1}$ $(M_1/M)^{1/2+n/6}$ at recombination (where M_1 is the current nonlinear mass scale). Extrapolating to smaller scales and using Eq. (31), one finds that the nonlinear mass scale at recombination was about $6\times10^8 m_\odot$ for $\Omega=1.0$, $n=0$ and about $3\times10^7 m_\odot$ for $\Omega=0.1$, $n=-1$.

groups, etc. In another picture (see, for example, Doroshkevich, Sunyaev, and Zel'dovich, 1974), inhomogeneities on all mass scales smaller than those of present clusters (10^{13}–$10^{14} m_{\odot}$) were either absent initially or were erased in the fireball (e.g., adiabatic fluctuations damped by photon viscosity on mass scales smaller than $M_D \simeq 10^{12}\Omega^{-5/4} h^{-5/2} m_{\odot}$). These large-scale perturbations then condensed out fairly recently ($z \simeq 3-5$), collapsed into flat structures ("pancakes"), shocked while in the gaseous state and fragmented into substructures which later became small clusters and individual galaxies.[20] Intermediate pictures, embodying some features of each of these two extremes, are also possible (e.g., White and Rees, 1978).

In any picture, one of the fundamental problems has been to explain the characteristic masses and sizes of galaxies. Recently, a number of authors (Binney, 1977; Rees and Ostriker, 1977; Silk, 1977) have noticed that the condition that a protogalactic gas cloud be able to cool and fragment into stars on its free-fall time-scale leads naturally to just such masses and sizes ($m_g \simeq 10^{12} m_{\odot}$, $r_g \simeq 75$ Kpc). This is not the place for an exhaustive discussion of the basic pictures and the detailed models which have been constructed within them; the reader is referred to Jones (1976), Gott (1977), and Rees (1978) for recent reviews of galaxy formation which emphasize the connection with galaxy clustering. A persistent question in recent work on galaxy clustering concerns the degree to which the present nonlinear pattern of galaxy clustering reflects the distribution of matter at earlier times. Is the present distribution the inevitable result of relaxation or dissipation or does it uniquely reflect the distribution at recombination? Clearly, the answer to this question depends to some extent on which of the general pictures mentioned above is correct.

Peebles (1974d) has suggested that in a picture involving purely gravitational clustering, the present nonlinear pattern of galaxy clustering should reflect the distribution of matter at recombination, and he has given a simple *scaling argument* for relating the two distributions in the case $\Omega \simeq 1$. In slightly modified form, the argument runs as follows (Fall, 1978). The initial power spectrum is assumed to have the power-law form [Eq. (43)] with exponent n over all mass scales larger than that of individual galaxies. According to Eq. (28) above, density perturbations then have typical amplitudes which vary with scale l as

$$\sigma_r(l) \propto l^{-(3+n)/2}, \quad (-3 < n \leq 4). \tag{45}$$

A perturbation of size l_r at the recombination epoch z_r grows roughly at the linear rate until it reaches a maximum size l_m at redshift z_m and then condenses out of the general expansion (see Fig. 3). At that time its mean internal density is a fixed factor α times the mean cosmological density[21]: $\alpha \simeq 5.5$. Thus the characteristic size, amplitude, and internal density of the perturbation at z_m are related by the expressions

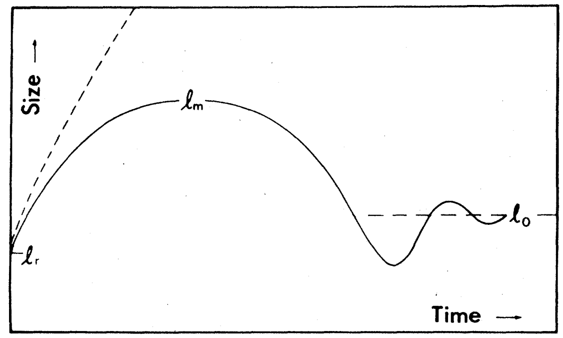

FIG. 3. Model behavior of a condensing inhomogeneity assumed in the scaling arguments of Sec. III. C. The dashed line at the upper left indicates the behavior of the cosmological scale factor a, and the horizontal dashed line indicates the final equilibrium size of the condensed aggregate ($l_0 \simeq \frac{1}{2} l_m$).

$$l_m \simeq l_r a_m, \quad \alpha \simeq a_m \sigma_r(l_r), \quad \rho_m \simeq \alpha \rho_r a_m^{-3}, \tag{46}$$

where a_m is the expansion parameter $(1+z_r)/(1+z_m)$ and ρ_r is the mean cosmological density at z_r. Assuming that the final size of the resulting aggregate l_0 is approximately $\frac{1}{2} l_m$ (by the virial theorem), these equations lead to a characteristic density-size relation $\rho_0(l_0) \propto l_0^{-\theta}$ in the form of Carpenter's relation[22] with $\theta \simeq (9+3n)/(5+n)$. With the Peebles–Groth (1975) arguments for a hierarchical pattern of clustering, this gives

$$\xi(r) \simeq (r_0/r)^{\gamma}, \quad \gamma \simeq (9+3n)/(5+n), \quad (r \lesssim r_0)(\Omega \simeq 1) \tag{47}$$

for the nonlinear part of ξ. Since empirical estimates of ξ have $\gamma \simeq 1.8$, this argument suggests $n \simeq 0$, a "white noise" spectrum at recombination for $\Omega \simeq 1$. Assuming that the resulting hierarchy is stable, the nonlinear growth law for ξ is

$$\xi(r, z) \simeq (1+z)^{-3}(r_0/r)^{\gamma}, \quad (\xi \gtrsim 1)(\Omega \simeq 1), \tag{48}$$

because $\rho\xi(r)$, the typical density within a condensed aggregate of size r, is constant.

The scaling arguments in their simplest form neglect several potentially important effects. First, spherical perturbations on most of the scales over which ξ has been estimated could not yet have reached their equilibrium sizes (l_0) as was assumed in the derivation of Eq. (47). Indeed, the density contrast of a spherical perturbation which has just reached equilibrium is of order 10^2 or more, corresponding to scales of order $10^{-1} r_0$ or less (Gott and Rees, 1975). Second, two-body and collective relaxation within condensed aggregates and the disruptive collisions of subunits within them might be expected to influence the small-scale form of ξ and its growth rate (Press and Lightman,

[20]In the picture of Doroshkevich *et al.*, one might have expected to find a prominent feature in ξ on some nonlinear scale set by the transition between gas-dynamical and gravitational effects. The fact that one is not observed may therefore be difficult to reconcile with this picture. (See, however, Doroshkevich and Shandarin, 1978).

[21]The value $\alpha \simeq 5.5$ follows by considering a uniform density spherical region; it evolves according to the Freidmann equations (36) with a higher density than that of the background (see, for example, Gunn and Gott, 1972; Field, 1975).

[22]A relation equivalent to this one for θ has been derived in a different way by Press and Schechter [1974; cf. their Eq. (27)].

1978). It is therefore somewhat surprising that the N-body experiments designed to simulate gravitational clustering agree reasonably well with Eq. (47) and the notion of a stable hierarchy over most nonlinear scales (Aarseth, Gott, and Turner, 1978; Fall, 1978). In part this may be due to the tendency for perturbations to develop into nonuniform and nonspherical shapes and thus to smear out much of the distinction between expanding, collapsing, and stationary aggregates (Icke, 1973). However, for reasons that will be discussed in Sec. IV. B, the interpretation of the N-body experiments is, in many cases, difficult and Eq. (47) has been well-tested only for the case $n=0$.

The relations (47) and (48) should hold on sufficiently small scales for any value of Ω. Over some range of scales, the form of ξ should depend on the cosmological model because perturbations on scales near r_0 would have stopped growing some time ago if Ω is small. Until recently, there have been two distinct views of how ξ should reflect this process. In the first view (Peebles, 1974d, Davis, Groth, and Peebles, 1977), the pair correlation function is expected to have a prominent bend at $\xi_b \simeq 0.3\Omega^{-3}$ with Eq. (44) holding at smaller ξ and Eq. (47) holding at larger ξ. Assuming the bend in Groth and Peebles' (1977) analysis of the $10' \times 10'$ Lick counts is real, one has $\Omega \simeq 1$ and $n \simeq 0$. In the second view (Gott and Rees, 1975), lower values of Ω are expected to steepen ξ to an approximate power law over most of the scales on which it has been estimated. In this case, one requires more long-range correlation in the matter distribution at recombination for smaller values of Ω. This implies negative values of n with the exact value depending sensitively on Ω. Recent N-body experiments suggest that the second view may be more nearly correct (Efstathiou, 1979, and Figs. 5 and 6 below), perhaps because of smearing effects. But once again the interpretation is not straightforward and is deferred to Sec.IV. B. Finally, it is worth noting that some efforts have been made to connect the recombination spectra inferred from these arguments with pre-recombination processes (Zel'dovich, 1972; Gott and Rees, 1975; Eichler, 1977; Jones, 1977; Liang, 1979). There is, however, scope for further work along these lines.

D. Estimates of the mean mass density

Dynamical methods for estimating Ω have traditionally been based on the *virial theorem*, applied either to individual galaxies or to groups and clusters. In the case of aggregates of galaxies, one estimates the total kinetic energy from the radial velocities of member galaxies and the total potential energy from their positions on the sky, both averaged in ways meant to account for projection effects (e.g., Burbidge and Burbidge, 1975). Applying the virial theorem to the aggregates then gives an estimate of the mean mass-to-light ratio $\langle m/L \rangle$ of their members, including any material between the galaxies. An estimate of Ω then follows from the expression

$$\Omega \simeq 3.6 \times 10^{-4} h^{-2} (\mathcal{L}/10^8 L_\odot \mathrm{Mpc}^{-3}) \langle m/L \rangle_\odot, \tag{49}$$

where $\mathcal{L}$ is the mean luminosity density. Recent estimates typically give $\mathcal{L} \simeq (1-2) \times 10^8\ h\ L_\odot \mathrm{Mpc}^{-3}$ (Felten, 1977) and $\langle m/L \rangle_\odot \simeq (100-300)h$ (Gott and Turner, 1977a), so the corresponding value of Ω is of order 0.1, independent of Hubble's constant. The major disadvantage of this method arises from the problem of assigning galaxies to a parent group. Geller and Peebles (1973) have developed a statistical version of the method which averages over groups and thereby reduces the membership problem somewhat. Another disadvantage of these methods is that the virial theorem can be applied only to the parts of groups and clusters which have reached a state of dynamical equilibrium. To overcome this problem, several methods for estimating Ω from the decelerated expansion in the outer parts of clusters have been developed, but the application of the methods to real data has not yet given unambiguous results (Sandage, Tammann, and Hardy, 1972; Silk, 1974; Peebles, 1976c).

Basically, there are two new dynamical methods for estimating Ω from correlation information and, in principle, they are free of some of the difficulties mentioned above. The simplest method (Fall, 1975; Peebles, 1976b) balances clustering against the peculiar (non-Hubble) motions of galaxies. It does this by associating with each of these effects a corresponding energy (per unit mass)

$$W \equiv \tfrac{1}{2}\rho \int d^3\mathbf{r}(-G/r)\xi(r), \quad T \equiv \tfrac{1}{2}\langle v^2 \rangle, \tag{50}$$

where the integral is over a large comoving volume, $\mathbf{v}$ is the peculiar velocity of a galaxy, and the average is over all galaxies. The basic relation between T and W is a *cosmic energy equation*

$$\frac{d}{dt}(T+W) + \frac{1}{a}\frac{da}{dt}(2T+W) = 0, \tag{51}$$

which guarantees that the total energy of an expanding system is conserved[23] (Irvine, 1961, 1965; Layzer, 1963; Dmitriev and Zeldovich, 1964). An additional equation and some boundary conditions are required to derive an exact relation between T and W, but an entirely satisfactory expression can be obtained as follows. Neglecting the decaying modes, Eqs. (41a) and (42) above indicate that at early times both T and W grow as $t^{2/3}$. In this case $T = -\frac{2}{3}W$ according to Eq. (51). In the other extreme, where perturbations have condensed into bound aggregates which have reached dynamical equilibrium, the result is $T = -\frac{1}{2}W$ (the ordinary virial theorem). Throughout the clustering process, one may therefore expect the relation

$$T \simeq -\kappa W, \quad \tfrac{1}{2} \lesssim \kappa \lesssim \tfrac{2}{3}, \tag{52}$$

to hold[24] (except at very early times if dynamically un-

[23]This equation applies to a system with fluctuations of arbitrary amplitudes so long as their characteristic length-scale is much smaller than the horizon [or, so long as $|\xi(r)| \to 0$, at least as fast as r^{-2} for large r]. It can be derived from either the fluid-dynamical equations or the BBGKY equations (Sec. IV. C).

[24]This equation is sometimes referred to as a "cosmic virial theorem," but the use of this term is to be discouraged because Eq. (52) applies even when most of the contribution to T and W is from scales on which perturbations have not reached a state of virial equilibrium. Perhaps "cosmic energy condition" is a better name for it.

supported peculiar motions are present). Cosmological N-body experiments indicate that Eq. (52) is satisfied for a fairly wide range of initial conditions and values of Ω (Fall, 1976a; Aarseth *et al.*, 1978).

An estimate of Ω now follows directly from estimates of ξ using Eqs. (38), (50), and (52). For the power-law model [Eq. (18)], the result (with $\gamma < 2$) is

$$\Omega \simeq \tfrac{2}{3}(2-\gamma)(R/r_0)^\gamma \langle v^2\rangle/\kappa(RH)^2, \tag{53}$$

where R is an effective cutoff scale beyond which $|\xi(r)|$ decreases more rapidly than r^{-2} [corresponding to $n > -1$; cf. Eq. (44) above]. For example, R is about $2r_0$ if the Groth–Peebles (1977) bend at $9h^{-1}$ Mpc is real. Note, however, that this estimate of Ω is not sensitive to the exact value of R; it changes by a factor of only four for a thousandfold change in R (with $\gamma = 1.8$). Also, the above estimate of Ω is not sensitive to the shape of ξ because ξ enters Eq. (53) only through the integral quantity W (Fall and Tremaine, 1977). Thus the major uncertainties are in r_0 and in $\langle v^2\rangle$. With $r_0 \simeq 5.3\ h^{-1}$ Mpc, $\kappa = \frac{1}{2}$, $R = 2r_0$, and $\langle v^2\rangle^{\frac{1}{2}} = 300$ km s^{-1}, one has $\Omega \simeq 0.07$, in reasonable agreement with traditional methods and the Deuterium arguments (Wagoner, 1973). With the same parameter values, except $r_0 \simeq 3.0$ (Davis *et al.*, 1978) or $\langle v^2\rangle^{\frac{1}{2}} \simeq 600$ km s^{-1} (Smoot *et al.*, 1977) one has, respectively, $\Omega \simeq 0.2$ or $\Omega \simeq 0.3$. Note that this method is also independent of Hubble's constant because H enters all of the distance estimates which occur in Eq. (53) in the same way.

As Peebles (1976b) has emphasized, the quantity $\langle v^2\rangle$ required for the estimates of Ω given above is somewhat difficult to measure because in practice one only observes the relative motions of galaxies and not their absolute motions with respect to a universal comoving frame. Thus one can imagine that, even though the small-scale relative velocities of galaxies are small, the quantity $\langle v^2\rangle$ may be large because of large-scale matter currents. However, as the following argument shows, this criticism only applies for certain values of the initial spectrum index n. The typical peculiar velocity induced by perturbations on the scale l should vary with l roughly as

$$\langle v^2(l)\rangle \simeq \begin{Bmatrix} Gm(l)/l & (l \lesssim r_0) \\ [g(l)/H]^2 & (l \gtrsim r_0) \end{Bmatrix}, \quad g(l) \propto l\sigma(l), \tag{54}$$

since the rms peculiar acceleration $g(l)$ is proportional to l^{-2} times the mass enhancement of a perturbation, which is proportional to $l^3\sigma(l)$. Thus, according to Eqs. (33) and (45), one has

$$\langle v^2(l)\rangle^{\frac{1}{2}} \propto \begin{cases} l^{(2-\gamma)/2} & (l \lesssim r_0) \\ l^{-(1+n)/2} & (l \gtrsim r_0) \end{cases}. \tag{55}$$

With $\gamma \simeq 1.8$, the small l variation of $\langle v^2(l)\rangle^{\frac{1}{2}}$ is very weak but the large l variation of $\langle v^2(l)\rangle^{\frac{1}{2}}$ depends on n. For $n > -1$, the major contribution to (55) is from small scales and it is safe to estimate $\langle v^2\rangle$, and hence Ω, from the relative velocities of galaxies separated by a few Mpc. Otherwise, the method does not apply.

Peebles (1976a, b) has developed another dynamical method for estimating Ω with correlation information. This method is also a statistical method but differs from the previous one in that the required velocities are the relative velocities $\mathbf{v}_{12}(r)$ of pairs of galaxies separated by the distance r, and are therefore directly measurable. Like traditional methods, Peebles' method requires the assumption of dynamical equilibrium on the scales to which it can be applied, and he has given it the name *cosmic virial theorem*. The derivation will not be given here but the result is

$$\Omega \simeq \frac{4}{9}\pi \langle v_{12}^2(r)\rangle I(r)/H^2,$$
$$I(r) = \xi(r)\left[\int_r^\infty dx\, x^{-1}\int d^3\mathbf{y}(\mathbf{x}\cdot\mathbf{y}/y^3)\zeta(x, y, |\mathbf{x}-\mathbf{y}|)\right]^{-1}, \tag{56}$$

where ζ is the triplet correlation function. For the values $\gamma = 1.8$, $r_0 = 5.3h^{-1}$ Mpc, and the Peebles–Groth (1975) form for ζ, the function $I(r)$ has the approximate value $1.2\times10^{-3}h^2$ Mpc^{-2} at $r = 1h^{-1}$ Mpc. Thus, with $\langle v_{12}^2(h^{-1}\text{ Mpc})\rangle \simeq 3\ (300\text{ km s}^{-1})^2$, the result is $\Omega \simeq 0.05$, which is in good agreement with the other methods. A more complete discussion of these methods and their relation to other methods, particularly that of Geller and Peebles (1973), can be found elsewhere (Fall, 1976a; Peebles, 1976b; Davis *et al.*, 1978; Geller and Davis, 1978). Finally, Seldner and Peebles (1977b) have recently proposed a method using cluster-galaxy cross-correlation estimates in a fairly straightforward application of the virial theorem. The method seems promising, but the application to existing data is subject to several important uncertainties.[25]

IV. FURTHER DEVELOPMENTS

With a few notable exceptions, the material covered to this point is fairly widely recognized as the best established part of the subject. Much of what follows concerns developments which are so recent that they have not yet gained general acceptance. They are included here because the general methods are certain to play a role in future developments even if some of the specific results are found to be in need of modification. Some of the material in this section, however, is quite standard and has been included only because of its close relation to recent work.

A. Deep samples and related problems

In the gravitational instability picture described above, galaxy clustering evolves with time and the possibility exists that we may actually be able to "observe" this evolution by studying the correlation of galaxies in very deep samples. The problem, of course, is a difficult one and no conclusive results have yet been obtained, but several attempts have shown that it is an interesting and important problem for future research.

[25]The preferred value of Seldner and Peebles is $\Omega \simeq 0.7$. This method, however, is sensitive to the assumed run of velocity dispersion with the radial distance from cluster centers; little is known about this at present.

In addition to the evolution of the pair correlation function ξ, the simple scaling relation (7) above cannot be expected to hold for deep samples because: (i) proper lengths and the angles they subtend are not linearly related (curvature effects), (ii) the spectral energy distributions of galaxies are not the same in the emitted and observed wavebands (K corrections), (iii) the luminosities of galaxies have almost certainly changed during the look-back times (evolutionary corrections). Curvature effects can be included simply by generalizing Limber's equation to include them. Similarly, *K corrections* can be included in a straightforward way but, empirically, they are still somewhat uncertain in the relevant wavebands (e.g., Pence, 1976). *Evolutionary corrections* can either be computed theoretically (e.g., Tinsley, 1976; Gunn, 1978; Sunyaev *et al.*, 1978) or determined empirically (e.g., Ellis, Fong, and Phillipps, 1977; Butcher and Oemler, 1978), but at present they are not understood very well. Finally, if and when these three effects are accounted for, the comparison of deep samples with shallow samples will give information about the evolution of ξ.

In the narrow-angle approximation, the *relativistic generalization of Limber's equation* (4) takes the form

$$w(\theta)=\frac{\int_0^\infty dz f^4(z)g^2(z)\psi^2(z)(1+z)^6\int_{-\infty}^{+\infty}dy\,\xi\{[f^2(z)\theta^2+g^2(z)y^2]^{1/2},z\}}{[\int_0^\infty dz f^2(z)g(z)\psi(z)(1+z)^3]^2} \tag{57}$$

(Dodd *et al.*, 1976; Fall, 1976b; Dautcourt, 1977b; Groth and Peebles, 1977; Phillipps *et al.*, 1978). Here $\psi(z)$ is the number of sample galaxies per unit proper volume at the redshift z, $f(z)$ is the angular diameter distance at z, and $g(z)$ is the derivative of proper distance with respect to z

$$f(z)=\frac{c\{q_0z+(q_0-1)[(2q_0z+1)^{1/2}-1]\}}{Hq_0^2(1+z)^2}, \tag{58a}$$

$$g(z)=c[H(1+z)^2(2q_0z+1)^{1/2}]^{-1}, \tag{58b}$$

where c is the speed of light, H is Hubble's constant (present value), and q_0 is the present value of the deceleration parameter (equal to $\frac{1}{2}\Omega$ in the cosmological models considered here, $\Lambda=0$). The arguments of $\xi(r,z)$ are the redshift z and the proper separation r of galaxies at the corresponding epoch. This equation reduces to the nonrelativistic version of Limber's equation when $\psi(z)$ decreases rapidly to zero for $z>z^*$ where z^* is small ($z^*\ll 1$) and, in some cases, can be inverted (Bonometto and Lucchin, 1978b). For a fairly detailed treatment of Eq. (57) and related equations the reader is referred to Dautcourt (1977b).

In order to make further progress, it is necessary to make some assumptions about the evolution of ξ and the selection function ψ. A useful one-parameter model for correlation evolution is

$$\xi(r,z)=(r_0/r)^\gamma(1+z)^{-(3+\epsilon)},\quad (\xi\gtrsim 1, z\lesssim 1), \tag{59}$$

where ϵ is an evolutionary parameter to be determined.[26] In this case, the angular correlation function also has power-law form, $w(\theta)=A\theta^{1-\gamma}$, and Eq. (57) can be put into a form[27] analogous to Eq. (6)

$$\frac{A}{B}=\sqrt{\pi}S\Gamma\left(\frac{\gamma}{2}-\frac{1}{2}\right)\Big/\Gamma(\tfrac{1}{2}\gamma), \tag{60a}$$

$$S=\int_0^\infty dz f^{5-\gamma}(z)g(z)\psi^2(z)(1+z)^{3-\epsilon}\Big/\left[\int_0^\infty dz f^2(z)g(z)\psi(z)(1+z)^3\right]^2, \tag{60b}$$

where, as before, B is r_0^γ. If the selection procedure and evolutionary and K corrections are known (or assumed), the selection function ψ for the sample can be constructed, and from it the "scaling factor" S can be computed using Eq. (60b). In principle, one can then determine the evolutionary parameter ϵ from the angular correlation estimates for a deep sample (D) and a shallow sample, such as the Zwicky sample (Z), through the relation[28]

$$A_Z/A_D=S_Z/S_D. \tag{61}$$

The left-hand side of this expression is empirical and the right-hand side depends on ϵ; it is independent of r_0, is virtually independent of q_0 for $z^*\lesssim 1$, and de-

[26]The case $\epsilon=0$ corresponds to stable clustering [cf. Eq. (48) above] and $\epsilon=\gamma-3$ and $\epsilon=3-\gamma$ correspond, respectively, to clusters expanding and collapsing at the same rate as the Universe expands.

[27]The scaling parameter S in Eq. (60) is identical to the one used by Phillipps *et al.* (1978). The parameter ϵ (Groth and Peebles, 1977) is $\eta-3$ in the notation of Phillipps *et al.* and is $s-3$ in the notation of Dautcourt.

[28]This assumes that the estimated correlation indices γ_1 and γ_2 are equal; if they are not equal comparison can still be made but a slightly different approach is required.

pends on H only through the time-scale for luminosity evolution [cf. Eq. (64) below].

For deep samples, it is necessary to include separately the contributions to ψ from spiral galaxies (s) and elliptical galaxies (e) because galaxies of different types have different spectral energy distributions, different evolutionary histories, and different image characteristics; thus

$$\psi(z) = \psi_s(z) + \psi_e(z) . \tag{62}$$

If a deep sample is limited at the apparent magnitude m_0, the selection functions ψ_s and ψ_e can be related to the present integral luminosity functions Φ_s and Φ_e as follows:

$$\psi_i(z) = \Phi_i(M_i(z)) \quad (i = s, e) , \tag{63a}$$

$$M_i(z) = m_0 - 5 \log[d(z)/\mathrm{Mpc}] - 25 - K_i(z) - E_i(z) , \tag{63b}$$

where K_i and E_i denote K corrections and evolutionary corrections in the relevant wavebands for galaxies of different types ($i = s, e$), and $d(z) = (1+z)^2 f(z)$ is the luminosity distance corresponding to z. For a sample deeper than about $m_0 \simeq 20$, the assumption that the sample is magnitude-limited is likely to be a bad one. This is because the images of galaxies in such a sample are so small and so faint that the way in which they are recognized must also be included in ψ. If the data are taken from plates which have been scanned by an automated device, this can be done in an objective way.

Evolutionary corrections and K corrections become important for z larger than about 0.1 and, in models with galaxies brighter in the past, these effects tend to cancel each other. In the most naive model for galactic evolution, galaxies begin their lives with a single burst of star formation at a redshift of order 10 or less and then become continuously fainter as their stars evolve off the main sequence and onto the giant branch of the color-magnitude diagram. In this case, evolutionary corrections take the simple form

$$\begin{aligned} E(z) &\simeq (1.3 - 0.3x)\ln(t/t_0) \\ &\simeq -(1.3 - 0.3x)\ln[(1+z)(1+\Omega z)^{1/2}] , \quad (\Omega z \lesssim 1) \end{aligned} \tag{64}$$

where x is the logarithmic slope of the initial stellar mass function and t_0 is the age of the Universe. ($t_0 \simeq 1/H \simeq 10^{10}$ yr and $x \simeq 1.35$ in the solar neighborhood.) This model is thought to be a reasonable one for the recent ($z \lesssim 1$) evolution of ellipticals because they show no signs of recent star formation. The evolution of spirals, however, is more difficult to model because vigorous star formation is currently taking place. Nevertheless, fairly sophisticated models for both kinds of evolution have been constructed. (See, for example, Tinsley, 1976, 1977, and references therein.) Like other cosmological tests, the scaling relation (61) is sensitive to luminosity evolution. Thus, although our theoretical understanding of luminosity evolution is uncertain, some information about it can be obtained empirically from the number-magnitude and number-angular diameter relations in a way that is independent of correlation estimates. In principle, this information can then be fed into the scaling relation through ψ in order to estimate correlation evolution.

To date, the only serious attempt to carry through a program like the one outlined above is that of the Durham group (Ellis *et al.*, 1977; Phillipps *et al.*, 1978). They have analyzed machine-scanned Schmidt plates taken in two colors (R and J) at a high Galactic latitude. The sample consists of about 4000 galaxies, most of which have redshifts between about 0.1 and 0.4. A moderate amount of luminosity evolution is consistent with their results but an unexpectedly large amount of clustering evolution is required to scale the deep sample correlation estimates with those from the Zwicky sample ($\epsilon \simeq 6$). If correct, the importance of these results to cosmology is very great indeed. However, because of the many difficult steps in the procedure required to obtain them, the results of Phillipps *et al.* must be considered as tentative.[29] Other deep samples have been analyzed in less ambitious ways. The one that has received the most attention is the $6° \times 6°$ Jagellonian field, which has a limit somewhere near 20^m (Rudnicki *et al.*, 1973). It has been analyzed by Peebles (1975), by Dautcourt (1977b), and by Groth and Peebles (1977), but each with a different result.[30] Perhaps this is another indication of how difficult and uncertain this kind of work is.

A closely related problem concerns small-scale fluctuations in the brightness of the night sky. Theoretically, this is identical to the problem discussed above because fluctuations in the cosmic background radiation at optical frequencies are thought to be due to the clustering of unresolved galaxies (Gunn, 1965; Dautcourt, 1977a). Observationally, however, a somewhat different approach is required (Shectman, 1973, 1974). Using power-spectrum techniques, Shectman has measured intensity fluctuations on scales of the order of a few arcminutes on deep Schmidt plates. Unfortunately, the interpretation of his results is no less difficult than the interpretation of the results from deep samples with resolved sources. Another related problem concerns the clustering of radio sources. At the large redshifts of these objects, their distribution on angular scales of a few degrees gives information about the homogeneity of the Universe on spatial scales of order 10^2 Mpc. Webster and others have analyzed the distribution of radio sources in several catalogs using power-spectrum techniques (Webster, 1976b, 1977; Webster and Pearson, 1977; Fanti *et al.*, 1978). As might be expected from the extrapolation of the small-scale clustering of galaxies, their results are consistent with a uniform distribution on these scales. On smaller scales, Seldner and Peebles (1978) have found marginally signifi-

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[29]The machine-scanned data of Phillipps *et al.* (1978) are from a different area of the same plate as the eye-scanned data of Dodd *et al.* (1976). With the Dodd *et al.* sample, correlation scaling is nearly satisfactory (with $\epsilon \simeq 0$), but the analysis is not thought to be as reliable as that with the Phillipps *et al.* sample.

[30]In the present notation, the assumptions of Groth and Peebles are $E_s = E_e = 0$, $K_s = K_e = 3.0z$, $m_0 = 20^m\!.3$. Those of Dautcourt are $E_s = E_e = 0$, $K_s = K_e = 2.6z$, $m_0 = 20^m\!.0$. Both studies assume a luminosity function of the form of Eqs. (11) and (12) above. The results of Groth and Peebles are consistent with $\epsilon \simeq 0$, whereas those of Dautcourt require $\epsilon \simeq 3$.

cant evidence for both the self-correlation of radio sources and the cross correlation of radio sources with galaxies in the Lick sample. These results seem to indicate that the luminosity function for radio sources is broader than was previously thought and that the fraction of radio galaxies in clusters is higher than the fraction of radio-quiet galaxies in clusters.

B. Cosmological *N*-body experiments

Computer experiments with expanding N-body systems have been done by several groups (Haggerty and Janin, 1974; Press and Schechter, 1974; Miyoshi and Kihara, 1975; Peebles and Groth, 1976; Aarseth, Gott, and Turner, 1978; Fall, 1978; Efstathiou, 1979). Particular attention is directed to the forthcoming series of cosmological N-body experiments by Aarseth *et al.* (1979). The appeal of the experimental method is obvious; it is relatively free of the many approximations which are made in the simple models for the nonlinear evolution of clustering. In principle, the method is simple. One puts down a distribution of discrete masses within a spherical boundary that expands according to the Friedmann equations and then one integrates numerically the Newtonian equations of motion for the N particles. Particles are specularly reflected from the boundary in order to mimic the effects of particles whose peculiar motions would have brought them into the system from outside. The pair correlation function ξ, or any other measure of clustering, is then computed at various time intervals. In practice, one requires an integration code, such as the one developed by Aarseth, which minimizes the buildup of numerical errors during the calculation. In this case, one can test some aspects of the clustering process described in Sec.III.C. The results of two 1000-body experiments are shown in Figs. 4–6.

The N-body method does, however, have its limitations. Because of the relatively small number of particles that can be handled by current integration codes ($N \simeq 10^3$), experiments designed to simulate the clustering of galaxies must "break into" the clustering process at times corresponding roughly to the epoch of galaxy formation ($z \simeq 10$). This is despite the fact that structure in the real Universe has already undergone considerable prior development. It is difficult to put down a distribution of particles at the start of an N-body experiment which does not have white noise form on small scales. This is because of the discreteness effects mentioned in Sec.II.D. They almost guarantee that the initial power spectrum $S(k)$ will be nearly constant on scales smaller than the mean interparticle separation even though real galaxy clustering, developing from subgalactic seed masses, would not be expected to have this feature. By the scaling arguments of Sec.III.C, this leads to a final N-body pair correlation function with index $\gamma \simeq 1.8$ on the corresponding mass scales (about four particles). On scales comparable with the size of the system, the clustering process may be influenced in an artificial way by the expanding boundary. With $N \simeq 10^3$, this leaves only a small range of mass scales (roughly 4 to 400 particles) and an even smaller range of spatial scales (less than a decade) over which

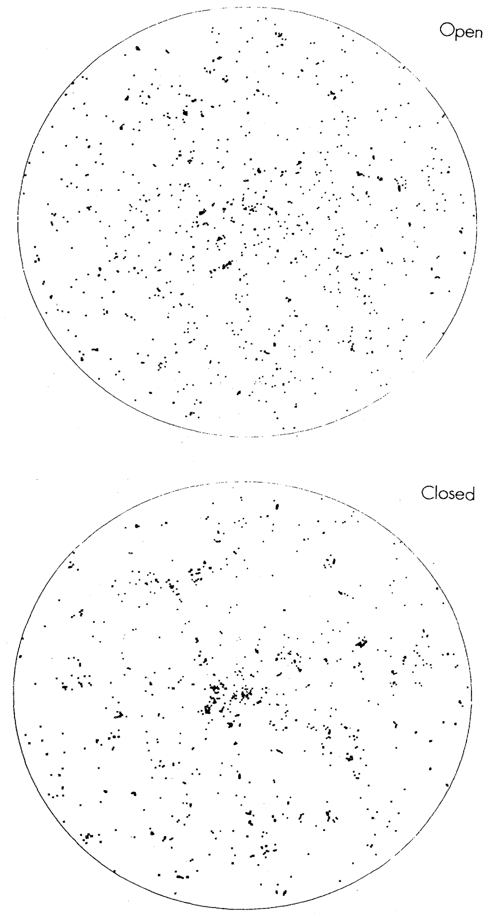

FIG. 4. Results of two cosmological 1000-body experiments with identical initial distributions (Poisson) but with different (final) density parameters: top, $\Omega = 0.26$; bottom, $\Omega = 1.0$. Crosses mark the positions of particles, projected onto the plane of the page, after each system had expanded by a factor of 9.0 from its initial size. Large-scale clustering has ceased to grow in the low-density experiment. (From data which were kindly supplied by S. J. Aarseth.)

to test for the effects of initial conditions and cosmological parameters. Except for the Poisson ($n = 0$) case, it is not a simple matter to guarantee that the initial spectrum has power-law form, as assumed in the scaling relation (47), over this range of scales.[31] In addition, the results of individual experiments tend to be noisy and experiments with statistically similar initial conditions often have slightly different distributions at later times.

At the time of writing (July 1978), the conclusions that can be drawn from the N-body experiments would seem to be the following.[32] (i) In an $n = 0$, $\Omega = 1$ model, ξ develops power-law form with index 1.9 over most

[31] For this reason, it is to be hoped that the published results of future N-body experiments will be accompanied by discussion and plots of the initial power spectra (at least, for the non-Poisson cases).

[32] Unfortunately, the results and conclusions of Aarseth *et al.* (1979) are not yet available for comparison with previous work.

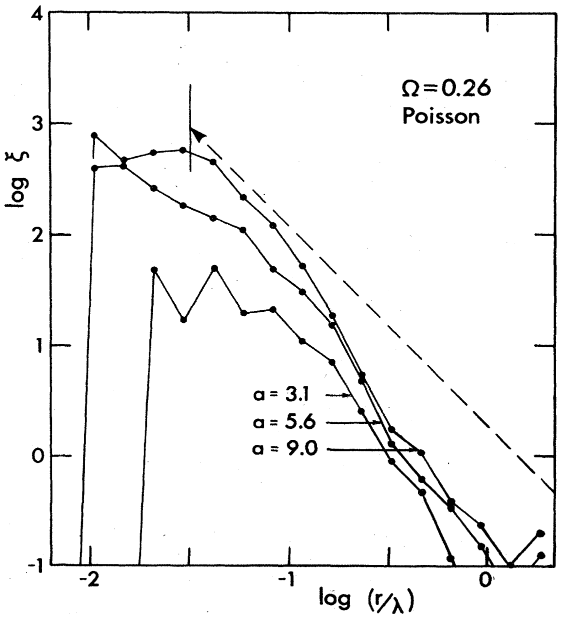

FIG. 5. Correlation estimates for the low density N-body experiment of Fig. 4 (open). The expansion parameter at which the estimates were made is denoted by a and the instantaneous mean separation of particles $n^{-1/3}$ is denoted by λ. The dashed line is a power law with $\gamma = 1.8$ and $r_0 = 1.45\lambda$, corresponding to empirical estimates (Sec. II and Fig. 2). A power-law fit to the estimates at $a = 9.0$ gives $\gamma \simeq 2.4$ (for $0.03 \lesssim r/\lambda \lesssim 1$). The behavior of the correlation function in this experiment illustrates the steepening effect discussed in Sec. IV. B and is to be compared with Fig. 6.

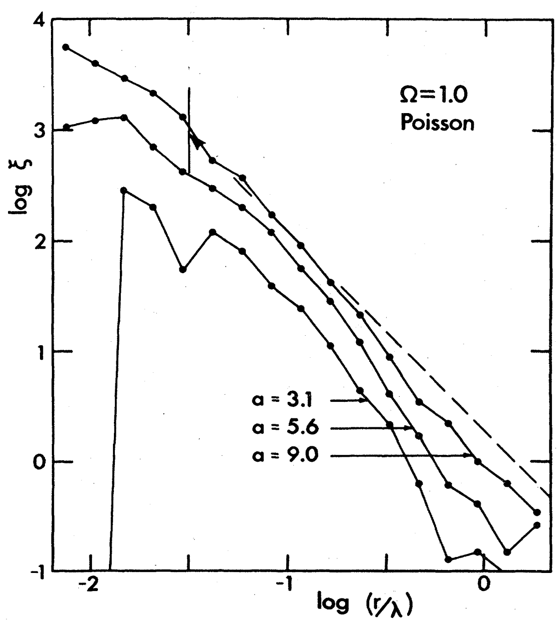

FIG. 6. Correlation estimates for the high density N-body experiment of Fig. 4 (closed). The notation and position of the dashed line are the same as in Fig. 5. A power-law fit to the estimates at $a = 9.0$ gives $\gamma \simeq 1.9$ (for $0.03 \lesssim r/\lambda \lesssim 1$). The behavior of the correlation function in this experiment is reasonably consistent with the simple scaling argument of Sec. III. C and is to be compared with Fig. 5.

nonlinear scales (Aarseth *et al.*, 1978; Fall, 1978; Efstathiou, 1979). This is in reasonable agreement with Peebles' (1974d) scaling relation (47). (ii) The later, nonlinear development of ξ depends somewhat on the initial distribution of particles but, because of discreteness effects, a test of the scaling relation for nonzero values of n will be difficult to make with current integration codes (Fall, 1978). (iii) The evolution of ξ depends on the density parameter in the sense that lower values of Ω result in steeper correlation functions (Efstathiou, 1979; Figs. 5 and 6). For $\Omega < 1$, ξ has reasonably convincing power-law form over a wide range of scales, possibly with a weak bend; but the Ω dependence has been tested only for $n = 0$. (iv) The evolution of ξ is consistent with the notion of a stable hierarchy on small scales and is not sensitive to the internal evolution of condensed aggregates. This last conclusion is based on the comparison of ordinary N-body experiments with some special ones in which the motion of particles within aggregates was stopped just after condensation (Fall, 1978). It is also based on several non-expanding N-body systems in which a hierarchical distribution of particles was found to be fairly stable as measured by ξ (Peebles, 1978). Relaxation and disruption effects are certainly important in the cores of rich clusters (White, 1976) but these experiments indicate that ξ is not sensitive to them, probably because so few particles are involved. (For different opinions see Aarseth *et al.*, 1978, and Press and Lightman, 1978).

The behavior of ξ in low-density N-body experiments suggests a simple way to modify the scaling relation for $\Omega < 1$. The argument is similar to the one given by Gott and Rees (1975) and runs as follows. Up to the time at which the Universe begins free expansion ($z_f \simeq \Omega^{-1} - 1$), the nonlinear part of the correlation function $\xi(r, z_f)$ should have the $\Omega = 1$ power-law form $(r_{0f}/r)^{\gamma_f}$ with index γ_f given approximately by Eq. (47). At later times, and on nonlinear scales larger than r_v (say), ξ has the approximate power-law form $(r_0/r)^{\gamma_e}$ with γ_e increasing as Ω decreases (cf. Fig. 5). Now r_0 and r_{0f} are related by the expression $r_0 \simeq \Omega^{-1} r_{0f}$ because the scale of perturbations with unit amplitude simply expands as $(1+z_f)/(1+z)$ after z_f. By assumption, clustering on the scale r_v is stable so $(r_0/r_v)^{\gamma_e}$ is approximately equal to $\Omega^{-3}(r_{0f}/r_v)^{\gamma_f}$. Combining these relations gives

$$\xi(r) \simeq (r_0/r)^{\gamma_e}, \quad \gamma_e \simeq \frac{-3\log\Omega + \log\xi_v}{-\log\Omega + \gamma_f^{-1}\log\xi_v}, \quad (1 \lesssim \xi \lesssim \Omega^{-3}) \tag{65}$$

Here $\xi_v \equiv \xi(r_v, z_f)$ is the amplitude of perturbations when they first reach a stable equilibrium state and must be considered as a parameter to be determined from the N-body experiments. With Table I of Efstathiou (1979) one finds that the best fitting value of $\log\xi_v$ is somewhere between 2.0 and 2.3. Using Eq. (47) for γ_f one then finds values of n between -1.5 and -2.0 for the case $\gamma_e \simeq 1.8$, and $\Omega \simeq 0.1$ [cf. Eq. (16) of Gott and Rees, 1975]. Although qualitatively reasonable, these arguments are obviously rather crude and should not be taken too seriously until they have been tested more fully.

Most of the cosmological N-body experiments done to date have ignored the effects of galaxy collisions and mergers. If galaxies have large halos, as most astrophysicists now believe (e.g., Ostriker and Peebles, 1973), these effects may have been important in their evolution (White, 1978; Silk, 1978; White and Rees, 1978; Efstathiou and Jones, 1979). Indeed, Toomre (1977) has suggested that elliptical galaxies are made during the collision and merging of spirals. It is not difficult to include the merging of particles in the standard N-body codes in a way designed to test for its effect on the very small scale development of ξ. The results of such calculations will be presented soon (Aarseth and Fall, 1979). Finally, it seems appropriate to mention Fourier transform schemes. They may enable one to treat systems with up to 10^6 particles and thereby avoid the discreteness effects which have plagued recent N-body experiments. This seems a promising approach to future studies of the nonlinear development of clustering.

C. Fluctuation theory and kinetic theory

Up to this point, two extreme approaches to the problem of describing the nonlinear development of structure have been considered: the simple scaling arguments relating γ to n and the direct N-body experiments. Because of the various limitations from which each of these descriptions suffers, several recent research efforts have been devoted to developing a convincing description of clustering at an intermediate level. These studies are based on some form of fluctuation theory or kinetic theory or a hybrid of the two. In the first approach, the distribution of galaxies is treated as an expanding continuum with a fluctuating density field, and the dynamics of the system are assumed to be governed by the usual fluid-dynamical equations for conservation of mass, energy, and momentum (Euler's equations). Information about the velocity distribution is discarded or approximated in some way. In the second approach, the distribution of galaxies is treated as an expanding system of point masses and the statistical properties of the system are assumed to be governed by some set of kinetic equations for the position-velocity distribution functions of various orders. The kinetic theoretical approach is more fundamental than the fluid-dynamical approach but it is necessarily more complicated. In fact, both approaches are so technical and contain so few concrete results, in proportion to the mathematical effort required to derive them, that only a brief introduction to this part of the subject will be given here. The interested reader is urged to consult the literature cited below for further details.

Perhaps the most stimulating recent work on the nonlinear gravitational development of fluctuations is that of Press and Schechter (1974). In a somewhat different formalism, they gave what is essentially a more detailed version of the scaling arguments of Sec.III.C. In addition, they were able to derive the following multiplicity function for bound aggregates [cf. Eq. (34) above]:

$$\eta(m,z) \propto m^{-\beta} \exp\{-[m/m_c(z)]^{4-2\beta}\},$$
$$\beta = 3/2 - n/6, \quad m_c(z) \propto (1+z)^{1/(\beta-2)}, \tag{66}$$

where, as before, z denotes redshift and n is the index of the initial fluctuation spectrum.[33] In principle, this expression gives another indication of n, but unfortunately, the comparison with the distribution of galaxies on the sky involves several uncertain projection effects (cf., Gott and Turner, 1977b, and footnote 14). Although Press and Schechter's work is closely related to much of this subject, their aim was not to predict correlation functions. Recently, Lightman and Press (1978) and McClelland and Silk (1979) have developed interesting numerical schemes with this goal in mind. Their approaches are based on more elaborate versions of the scaling arguments, which are intended to include the effects of nonlinear condensations that have not reached equilibrium. Although these calculations illuminate some of the physical processes involved, it is not clear that their quantitative predictions are any better than the simple scaling arguments of Secs.III.C and IV.B.

As yet, not much work has been done on the fully nonlinear fluid-dynamical equations except to consider some of their special properties. For example, Peebles and Groth (1976) have studied a so-called "integral constraint" for the evolution of ξ which is a useful aid in the interpretation of N-body experiments. The focus of other studies has been mainly on determining the domain of validity of the scaling arguments. According to Eq. (66), the mass m_c of an inhomogeneity which is just beginning nonlinear condensation varies with proper time as $m_c \propto t^{4/(3+n)}$ at early times ($z > z_f$). In this connection, Peebles (1974e) and Doroshkevich and Zel'dovich (1975) have noted that any momentum-conserving process giving rise to the fluctuation spectrum prior to recombination will lead to the condition $n \leq 4$. Thus the minimum expected growth rate for the condensing mass scale is $m_c \propto t^{4/7}$. Since the mass within the horizon varies as $m_H \propto t$, fluctuations giving rise to galaxy clusters could have been set up by causal processes only if n is less than $+1$.

Apart from a sign difference in the two-body interaction potential, a system of self-gravitating particles is formally identical to an electromagnetically interacting plasma. This formal difference, however, gives rise to a wealth of different phenomena, mainly because of the lack of charge screening in the gravitational system. (For a review of kinetic theory as applied to non-expanding gravitational systems, see Haggerty and Severne, 1976.) Indeed, gravitational systems are often cited as the standard counterexample to many of the basic postulates and theorems of ordinary statistical mechanics and thermodynamics. Thus few if any of the results of plasma and liquid kinetic theory can be taken over directly without some modification. The statistical description of a gravitational system must therefore begin at the most fundamental level: Liouville's theorem, which merely states that the number of particles in the system is conserved.[34] Equivalent to this statement is the *Bogoliubov–Born–Green–Kirkwood–Yvon* (BBGKY) *hierarchy* of equations for the functions f_s (e.g., Ichimaru, 1973). The distribution function of order s is

[33]In Press and Schechter's notation, β is $1+\alpha$.

[34]In the cosmological context, this statement refers to a large comoving volume of space.

defined such that $f_s(\mathbf{x}_1, \mathbf{u}_1, \ldots, \mathbf{x}_s, \mathbf{u}_s) d^3\mathbf{x}_1 d^3\mathbf{u}_1 \ldots d^3\mathbf{x}_s d^3\mathbf{u}_s$ is proportional to the probability that particles 1 through s are in the position-velocity elements $d^3\mathbf{x}_1 d^3\mathbf{u}_1 \ldots d^3\mathbf{x}_s d^3\mathbf{u}_s$ at the phase point $(\mathbf{x}_1, \mathbf{u}_1 \ldots \mathbf{x}_s, \mathbf{u}_s)$, where the $\mathbf{x}_i$ are comoving position coordinates and the $\mathbf{u}_i$ are some kind of non-Hubble velocities (see below). These functions are assumed to be symmetrical with respect to the interchange of any pair of particle labels and are often denoted by $f_s(1, \ldots s)$. They are also functions of the proper time t and the masses of the particles if the system has a continuous mass spectrum.

The connection between the BBGKY hierarchy and the correlation approach to galaxy clustering is made explicit by introducing the usual position-velocity correlation functions of kinetic theory:

$$\begin{aligned} &f(1) \equiv f_1(1),\ g(1,2) \equiv f_2(1,2) - f(1)f(2)\,, \\ &h(1,2,3) \equiv f_3(1,2,3) - f(1)g(2,3) - f(2)g(3,1) \\ &-f(3)g(1,2) - f(1)f(2)f(3)\,, \end{aligned} \tag{67}$$

etc. The most natural normalizations for these functions are the following:

$$\int d^3\mathbf{u}_1 f(1) = a^3 n = \text{const}\,, \tag{68a}$$

$$\int d^3\mathbf{u}_1 d^3\mathbf{u}_2 g(1,2) = a^6 n^2 \xi(1,2)\,, \tag{68b}$$

$$\int d^3\mathbf{u}_1 d^3\mathbf{u}_2 d^3\mathbf{u}_3 h(1,2,3) = a^9 n^3 \zeta(1,2,3)\,, \tag{68c}$$

etc, where a is the expansion parameter, n is the time-dependent mean proper density and ξ and ζ are the time-dependent position correlation functions defined in Sec.II. The BBGKY equations turn out to have the simplest forms if $\mathbf{u}_i$ is taken to be $a\mathbf{v}_i$, where $\mathbf{v}_i$ is the proper peculiar velocity of the ith particle. In this case, the first two equations of the hierarchy for a system of identical particles of mass m are the following

$$\partial f(1)/\partial t = a^{-1} \int d^3\mathbf{x}_2 d^3\mathbf{u}_2 \Theta_{12} g(1,2)\,, \tag{69}$$

$$\begin{aligned} &[\partial/\partial t + a^{-2}(\mathbf{u}_1 - \mathbf{u}_2) \\ &\quad \times \partial/\partial(\mathbf{x}_1 - \mathbf{x}_2)] g(1,2) = a^{-1}\Theta_{12}[f(1)f(2) + g(1,2)] \\ &\quad + a^{-1}\int d^3\mathbf{x}_3 d^3\mathbf{u}_3 [\Theta_{13} f(1) g(2,3) + \Theta_{23} f(2) g(1,3) \\ &\qquad + (\Theta_{13} + \Theta_{23}) h(1,2,3)]\,, \end{aligned} \tag{70}$$

where t is proper time and the binary interaction operator Θ_{ij} is defined by the equation

$$\Theta_{ij} \equiv Gm\,|\mathbf{x}_i - \mathbf{x}_j|^{-3} (\mathbf{x}_i - \mathbf{x}_j) \cdot (\partial/\partial \mathbf{u}_i - \partial/\partial \mathbf{u}_j)\,. \tag{71}$$

These equations can be derived by first setting up a corresponding hierarchy in an inertial (non-expanding) frame of reference, then changing variables to the comoving coordinates $\mathbf{x}_i$ and the peculiar velocities $\mathbf{u}_i$ defined above and finally removing the mean field terms using Euler's equations (Fall and Severne, 1976). Alternatively, they can be derived from a Lagrangian principle (Davis and Peebles, 1977).

Several interesting results can be derived directly from the BBGKY equations in the form (69)–(71), including the basic energy balance equation (51) above (Gilbert, 1965a; Fall and Severne, 1976; Davis and Peebles, 1977). In order to actually solve the equations, some approximations are required. In the "weak coupling" approximation, where all terms but the $f(1)f(2)$ term on the right-hand side of Eq. (70) are neglected, the equations can be solved almost exactly for g if some form is assumed for the single-particle function f (Fall and Saslaw, 1976; Inagaki, 1976b; Yahil, 1976; Norman and Silk, 1978). The weak coupling term is responsible for classical two-body relaxation effects (Fall and Severne, 1976) and it vanishes in the continuum limit: $m \to 0$, $n \to \infty$, mn finite.[35] If the system is initially uncorrelated (i.e., if it has an initial Poisson distribution), the weak coupling approximation will be valid for at least some time until higher-order correlations become important; but it is not applicable to the recent evolution of clustering. Another approach is to drop all of the terms which vanish in the continuum limit and then to take velocity moments of the resulting equations (Saslaw, 1972; Inagaki, 1976a; Davis and Peebles, 1977). This drastically reduces the number of independent variables in the problem but only at the expense of neglecting two-body relaxation effects and of creating a hierarchy of moment equations in addition to the hierarchy of kinetic equations. If some closure scheme is adopted for the moment hierarchy, the result is essentially a set of fluid-dynamical equations for the position correlation functions. If this set can also be closed in some way, the resulting equations may be tractable, if only by numerical integration.[36]

The only serious attempt to date at a program like the one outlined above is that of Davis and Peebles (1977). Their goal was to calculate the evolution of ξ, particularly in the transition region ($\xi \simeq 1$). The basic assumptions of the Davis–Peebles theory are the following: (i) the background cosmology is Einstein–de Sitter ($\Omega = 1$); (ii) the initial power spectrum has power-law form with exponent n; (iii) discreteness effects are negligible (continuum limit); (iv) throughout the evolution, the relation between h and g is the same as the Peebles–Groth relation (20) between ζ and ξ with Q constant; (v) the distribution of relative velocities of pairs of particles has zero skewness about the mean; (vi) the large-scale form of ξ is given by the linear equations (42), (44), and the small-scale form of ξ is given by the results of the simple scaling arguments, Eqs. (47), (48). Assumptions (i), (ii), and (iii) mean that there

[35] If the system is considered to be completely collisionless, a Vlasov approach can be adopted (van Albada, 1960, 1961; Gilbert, 1965b; Bisnovatyi-Kogan and Zel'dovich, 1971). In this case, the single-particle distribution f is assumed to be a function of both position and velocity and all correlation functions are ignored. In a comoving system of variables, the Vlasov equation can then be solved for first-order perturbations about some basic state such as the adiabatically cooling Maxwellian distribution. The results show growth at the usual linear rate for fluctuations larger than the Jeans length and a kind of Landau damping on smaller scales.

[36] The Kirkwood closure scheme, often adopted for liquids and turbulent plasmas (Rice and Gray, 1965; Ichimaru, 1973), is not consistent with Eq. (20) (Davis and Peebles, 1977).

are no characteristic lengths, masses, or times in the problem and that the equations admit similarity solutions in terms of the single variable $s = x/t^{\alpha}$, with $\alpha = 2(1/\gamma - 1/3)$. Assumption (iv) closes the BBGKY hierarchy, and assumption (v), although difficult to justify, closes the moment hierarchy. In addition, several other approximations are made and the final equations are then solved numerically.[37] The results are various predictions for the shape of ξ and for some of the velocity moments. They are definite enough that they can be compared with observational data and with N-body experiments. One of the most attractive features of the theory is that it correctly predicts the value of Q.

Because of the large number of approximations required by the Davis–Peebles theory, checks against the N-body experiments would be especially useful. It would also be interesting to see the results of similar calculations made with some different assumptions (e.g., $\Omega \ll 1$). In this way, it may be possible to discard some of the less important terms and arrive at a simpler theory for the evolution of clustering. Another possibility might be to exploit the similarity between the fluctuations that develop in a self-gravitating system and the fluctuations that develop in a system near the critical point of a phase transition (Totsuji and Kihara, 1969; Saslaw, 1972; Liang, 1979). At the liquid-gas transition, for example, the homogeneity of the system is neutrally stable and density correlations have a long-range $1/r$ behavior (e.g., Landau and Liftshitz, 1959). Whether this is just an intriguing analogy or is somehow related to a deeper principle is mostly speculation at this stage.[38] It may, however, be an important key for further understanding of the development of structure in gravitational systems like the expanding one considered here.

V. CONCLUDING REMARKS

The nature of this subject does not readily lend itself to a list of specific technical conclusions. Instead, it seems best to emphasize the methods of the correlation approach to galaxy clustering. Much progress has been made on both the observational and the theoretical sides of the subject and many of the basic problems have been well-posed, if not actually solved. It should be clear, especially from what has been said in the previous section, that there are still some important gaps in our understanding of the distribution of galaxies and its evolution. The following is a list of specific research problems and programs which, when completed, would help to fill these gaps:

A. Observation

(a) Redshift samples. More redshifts are required. A direct estimate of the linear part of ξ may be possible. More velocity data will increase the accuracy to which Ω can be estimated.

(b) Deep samples. Larger samples and a better understanding of evolutionary corrections are required. An empirical test of the gravitational instability picture may be possible.

B. Theory

(a) Kinetic theory and fluctuation theory. Work in conjunction with computer experiments seems the most promising.

(b) Triplet correlation function. Why does ζ have the Peebles–Groth form (20)? Why is Q nearly equal to unity?

(c) Pre-recombination physics. Why does the fluctuation spectrum at recombination have the forms inferred from the scaling arguments ($0 \gtrsim n \gtrsim -2$)?

(d) Nongravitational effects. The possible influence of gas-dynamical and related processes on ξ should be studied in more detail.

C. Experiment

(a) N-body simulations. More care with initial conditions is required. Fourier transform techniques seem promising.

(b) Fluid-dynamical simulations. This may be the cheapest way to study the nonlinear development of fluctuations.

Finally, I cannot resist mentioning a nonscientific issue raised by the recent work on galaxy clustering. Studies of galaxy correlations have shown that despite the rich variety of clustering patterns, the distribution of galaxies has a remarkably simple underlying form. The philosopher Santayana discussed a similar issue from an aesthetic point of view in his famous book, *The Sense of Beauty* (1896). There, he considered the question: why is the night sky beautiful? Like the distribution of galaxies, the distribution of stars is not one of random placement and it is not one of repeating geometrical patterns. It is something more varied, and yet it seems to have a simple underlying form. Santayana thought this was the key to the question he had raised about the distribution of stars; it is beautiful because it has enough underlying regularity to satisfy our sense of the simple and it has enough diversity to satisfy our sense of the novel. Santayana also commented on the fact that music and speech have similar properties, but he wrote at a time when very little was known about the distribution of matter outside our own galaxy. His explanation, though, of why we find a complex pattern with a simple underlying form to be so pleasing seems especially intriguing in the light of recent studies of galaxy clustering.

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[37]The most worrisome of the supplementary approximations made by Davis and Peebles is that the velocity part of $g(1,2)$ can be decomposed into the form $G_1(\mathbf{u}_1 - \mathbf{u}_2)\, G_2(\mathbf{u}_1 + \mathbf{u}_2)$ [cf. their Eq. (59)]. In this case, the distribution of relative velocity $(\mathbf{u}_1 - \mathbf{u}_2)$ and the distribution of center-of-mass velocity $\frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2)$ are assumed to be independent. For weak correlations, this approximation is valid; but when triplet correlations are important, the relative velocities of pairs become correlated with their center-of-mass velocities through the tidal influence of other particles.

[38]It is interesting to note that the weak coupling solutions of the BBGKY equations (69)–(71) also have the form $\xi(r) \propto 1/r$, up to the distance particles could have traveled at the rms velocity in the elapsed time (e.g., Fall and Saslaw, 1976).

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My interest in this subject and my understanding of it have been influenced by too many individuals to mention here by name. I do, however, wish to record a debt of gratitude to Simon White and George Efstathiou for helpful suggestions during the writing of this article. Finally, I offer my apologies to anyone whose work I may have neglected or treated unfairly.

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