The 1976 Oppenheimer lectures: Critical problems in plasma astrophysics. I. Turbulence and nonlinear waves*

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The J. Robert Oppenheimer Lectures are given annually at the Institute for Advanced Study, Princeton, New Jersey, by a distinguished visiting scientist who has made outstanding research contributions in a field of current interest and who is recognized for his ability to communicate progress in his area of expertise to a broad spectrum of physical scientists. The lecturer summarizes the progress and significance of work in his field in a series of two talks. The lectures are named in honor of J. Robert Oppenheimer, former Professor in the School of Natural Sciences, Institute for Advanced Study, and the third Director of the Institute. This paper presents the text of the first lecture on the topic of turbulence and nonlinear waves.

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I. INTRODUCTION

A proper understanding of many basic phenomena in both space and astrophysical applications depends critically on our ability to analyze the highly turbulent, nonlinear behavior of plasma. If one compares the plasma medium with the simple incompressible liquid—the old familiar subject of conventional hydrodynamic turbulence theory—it might appear at first glance that the plasma turbulence should be much more complicated. This is discouraging, since not even the hydrodynamic turbulence theory can be as yet entirely constructed from a first principles approach. For example, without various *ad hoc* hypotheses, it is not known how to systematically truncate the infinite hierarchy of n-point velocity correlation functions of the fluid without doing violence to one or more of the important experimentally measurable effects such as the inertial range energy spectrum.

The plasma medium seems more complicated than the simple fluid because the plasma has more collective degrees of freedom, or modes of oscillation (electron and ion plasma waves, magneto-hydrodynamic Alfvén waves, etc.). In the conventional incompressible fluid, the role of the "elementary excitations" is played by strongly interacting hydrodynamic vortices, the superposition of which represents the turbulent motion. In terms of a Fourier expansion, we would describe these elementary excitations by wave vectors **k**, with corresponding eigenfrequencies $\omega_k \simeq 0$. These vortex and convective cell motions are present in the plasma as well. However, there are many other plasma modes characterized by finite eigenfrequencies. It is, in fact, just this feature which makes plasma turbulence much more tractable from the point of view of a first principles development. The essential simplification derives from the fact that in many cases, where the turbulence is not too strongly excited, the plasma motion can be described as a superposition of real frequency eigenmodes, with amplitudes changing only slowly in time. This variation is due to the hierarchy of nonlinear interactions. It can be considered as slow if $\mathrm{Im}\,\omega_k \ll \omega_k$, where $\mathrm{Im}\,\omega_k$ represents the growth (or damping) rate of the mode, due to the nonlinear coupling between modes. Then, if sufficiently many modes are excited, the random phase approximation is often invoked. The resulting so-called "weak turbulence theory" (Sagdeev and Galeev, 1969; Kadomtsev, 1965) is widely accepted as the proper approach when the amplitude of the waves is small enough.

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II. WEAK TURBULENCE THEORY

Conceptually, it has proved very helpful to view these problems of weakly turbulent waves from a physical point of view which emphasizes their structural similarity to other familiar problems, e.g., problems in quantum mechanics, the solid state, and the kinetic theory of gases. In other words, we can regard the collection of randomly phased, turbulent waves as an ensemble of weakly interacting quasiparticles (waves) having quasienergies (frequencies) ω_k and quasimomenta (wave vectors) **k**. The evolution of such an ensemble is governed by kinetic equations for the distribution functions $\mathcal{N}_k(t)$ of these quasiparticles. (We can have different kinetic equations for different types of plasma waves.) For present purposes, we can define $\mathcal{N}_k$ as the energy density $\mathcal{E}_k$ of the collective mode of oscillation, divided by its natural frequency: $\mathcal{N}_k \equiv \mathcal{E}_k/\omega_k$. The kinetic equations are derived by an appropriate perturbation expansion of the Vlasov or fluid equations. In such a procedure, a parameter something like $\mathrm{Im}\,\omega_k/\omega_k$ appears, which represents the weak coupling between modes. The nonlinear mode–mode interaction then appears as an effective collision integral between the quasiparticles.

In some cases, it is also essential to retain the kinetic description for the (real) particles. Their interaction with the electric and magnetic fields of the waves $\mathcal{N}_k$ can also be described in terms of a collision integral, now between the particles and the quasiparticles. In the kinetic equation for the particle distribution, this effect usually takes the form of a Fokker—Planck-type diffusion of electrons or ions in velocity space. Discussion of the details of these effects—for example, the formulas for the collision integrals, or the appropriate way to treat resonant denominators—would take us too far afield. However, these details are not necessary to an understanding of the general structure and advantages of the weak turbulence theory. We can summarize the principle features of the derivation as follows:

I. Plasma Model

A. Consists of either

1. Vlasov equations for each species, or

2. a fluid model;

in both cases coupled self-consistently to

B. Maxwell's equations.

II. Assumptions

A. Small parameter expansion;

B. Random phase approximation (stochasticity of waves and particles).

III. Kinetic equations for

A. Particles (if Vlasov description is used) and

B. Quasiparticles (waves).

It is quite obvious that the structure of the wave–wave collision integrals must depend first of all on the simple kinematic properties of the elementary interactions. For example, if the dispersion law $\omega_k = \omega_k(\mathbf{k})$ allows three-wave collisions

$$\omega \rightarrow \omega_1 + \omega_2 ,$$

$$\mathbf{k} \rightarrow \mathbf{k}_1 + \mathbf{k}_2 ,$$

then the collision integral should be quadratic:

$$C\{\mathfrak{N}\} \rightarrow \int d\mathbf{k}_1 d\mathbf{k}_2 A(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, \mathfrak{N}_{k_1} \mathfrak{N}_{k_2} .$$

If at least four waves are involved in an elementary interaction, that is,

$$\omega \rightarrow \omega_1 + \omega_2 + \omega_3 ,$$

$$\mathbf{k} \rightarrow \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 ,$$

then we must deal with a $C\{\mathfrak{N}\}$ cubic in $\mathfrak{N}$.

To illustrate the concept of weak turbulence, I should like to discuss an example from the hydrodynamics of incompressible fluids. We shall put aside the vortex motions of strong turbulence and look instead for an area where we can apply the approach already developed for plasma waves. Specifically, let us consider surface waves on the ocean and ask: How do we find the energy spectrum for chaotic oscillation of the ocean surface? Now the dispersion relation for gravity waves in the long-wavelength limit is well known: $\omega^2 = gk$. We expect that this will lead to a simple scaling for the turbulent spectrum. In fact, under quite general conditions, one can imagine the existence of the so-called inertial subrange (Fig. 1), the regime in k space where nonlinear wave-wave interactions transport energy, without dissipation, from small k (long wavelengths) to much larger k (short wavelengths), where it is finally dissipated. Of course, solution of the wave kinetic equation provides a straightforward, systematic way of finding

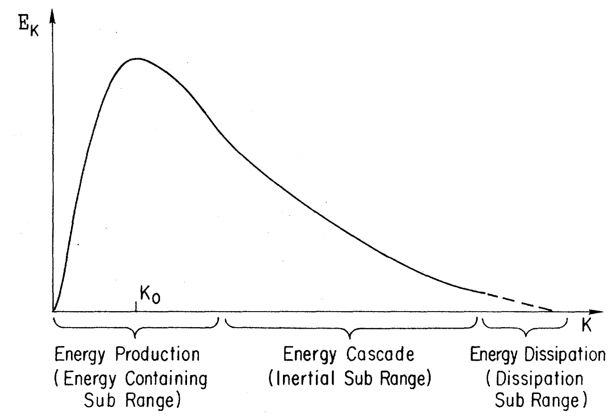

FIG. 1. The energy spectrum ($w = \int dk\ \mathcal{E}_k$) of a "typical" turbulent fluid.

the turbulence spectrum. To simplify things, however, we shall here develop only the scaling laws, which follow readily from dimensional arguments. We use the weak turbulence concept of wave-wave interaction; for gravity waves we need the four-wave process, or a cubic collision integral:

$$\frac{d\mathfrak{N}}{dt} \sim C\{\mathfrak{N}\mathfrak{N}\mathfrak{N}\} .$$

The inertial range scaling now follows in complete analogy to the conventional Kolmogoroff scaling (see Table I).

For the surface waves, we can also write the result in terms of the frequency spectrum $\mathcal{E}_\omega = \mathcal{E}_\omega(\omega)$. Using $\mathcal{E}_k\, dk = \mathcal{E}_\omega d\omega$, we get $\mathcal{E}_\omega \sim \omega^{-4}$. This is the form used more conventionally in oceanography.

Rigorous solution of the weak turbulence kinetic equation for the gravity waves produces exactly the same ω dependence as we found above (Zakharov and Filonenko, 1966). We conclude that, for weak turbulence, the inertial range spectrum can be derived in a self-consistent fashion. This contrasts with the strong vortex-type turbulence, for which the scaling was derived by Kolmogoroff only with the help of *ad hoc* arguments. Of course, even for surface waves, we can also imagine situations in which the modes would be so strongly excited that the dimensionless wave amplitudes could be of order one. This should certainly be regarded as the strong turbulence limit; it corresponds to the familiar picture of the ocean surface with breaking (overturning) waves. The wave breaking represents an additional mechanism of wave dissipation. To take crude account of it, one can use the following intuitive physical arguments. It is reasonable to assume that the breaking represents a self-regulatory process which gets rid of excess energy. We might thus expect that, in the strong turbulence regime, surface waves would come to a kind of threshold state $\tilde{v} \sim \omega/k$. With $\tilde{v} \sim (\varepsilon_k k^2)^{1/2}$ (where the k^2 gives the proper normalization of ε_k in this case, since ε_k is the energy per unit area rather than per unit volume), we find

$$\varepsilon_k \sim k^{-3}, \text{ or } \varepsilon_\omega \sim \omega^{-5} .$$

This is the so-called "Phillips spectrum." For an ex-

TABLE I. Comparative derivation of the inertial range energy spectra for gravity waves (left side of table) and Kolmogorov turbulence (right side of table).

Gravity wave turbulence	Kolmogoroff turbulence
Assume constancy of energy flux in k space (inertial range only) $$\epsilon_k \frac{dk}{dt} = \text{const.}$$ ($\int dk\epsilon_k = W$, the total energy per unit area for gravity waves, or per unit volume for Kolmogoroff turbulence.) Then assuming $$\frac{dk}{dt} \sim \frac{k}{\tau_k}$$ where τ_k is the characteristic time for energy transfer between adjacent wave numbers. We have	
$\tau_k \sim \frac{A(\omega, k)}{\epsilon_k^2}$,	$\tau_k \sim \frac{1}{k(\epsilon_k k)^{1/2}}$,
which follows from	
the mode-coupling equation $\frac{d\eta_k}{dt} \sim C(\eta^3)$ and the fact that, for gravity waves, ϵ_k is related to the surface density waves: $\epsilon_k \sim \eta_k$.	Euler's equation $\frac{\partial V}{\partial t} \sim -V \cdot \nabla V$ or $\tilde{V}/\tau_k \sim \tilde{V}^2 k$, $\tau_k \sim (k\tilde{V})^{-1}$ together with $\tilde{V} \sim (\epsilon_k k)^{1/2}$.
The mode-coupling coefficient must provide the proper dimensionality for τ_k, hence $\tau_k \sim \frac{1}{\omega_k}\left[\rho\left(\frac{\omega_k}{k}\right)^2 \frac{1}{k}\right]^2 \Big/ (\epsilon_k k)^2 \sim \epsilon_k^{-2} k^{-13/2}$. The factor of $1/k$ in the brackets arises since $\epsilon_k dk$ is a volume density; for the surface waves the fluctuations are confined to a layer depth $\sim 1/k$.	
Substituting τ_k in $\epsilon_k k/\tau_k = \text{const}$ yields immediately	
$\epsilon_k \sim k^{-5/2}$.	$\epsilon_k \sim k^{-5/3}$.

ample where three-wave collisions would be appropriate, we could consider the very-short-wavelength limit of surface waves where surface tension becomes important. This is the case of the so-called capillary waves, with dispersion law $\omega_k \sim k^{3/2}$.

There are also various space plasma applications of the weak turbulence theory. Particularly noteworthy are the turbulence properties of the solar wind plasma. The power-law spectra found for the gross solar wind turbulence suggest the existence of an inertial subrange. Both for acoustic waves and Alfvén waves, we can exploit the three-wave interaction estimate $\tau_k^{-1} \sim \omega_k(\varepsilon_k k)$ ($\varepsilon_k k$ could be normalized either to nT or to $B^2/8\pi$). Then, using the constancy of the energy flux $\varepsilon_k k \tau_k^{-1}$, we find $\varepsilon_k \sim k^{-3/2}$ (Zakharov and Sagdeev, 1970). This result agrees both with radioastronomical data and with "in situ" measurements.

Of course, determining the wave spectrum in the inertial range represents only part of the problem of plasma turbulence theory. The next part of the problem is concerned with the spectrum of the plasma particles, i.e., with their distribution function. This function evolves by interaction of the electrons and ions with the wave spectrum. The simplest kind of interaction, whose effect is a Fokker–Planck type of diffusion ("quasilinear diffusion"), is due to the Landau ($\omega = \mathbf{k} \cdot \mathbf{v}$) or cyclotron ($\omega \pm \omega_c = \mathbf{k} \cdot \mathbf{v}$) resonances of waves with particles. There are also interactions involving two waves and a particle. Essentially, these processes are equivalent to the induced scattering $\omega_2 \pm \omega_1 = (\mathbf{k}_2 \pm \mathbf{k}_1) \cdot \mathbf{v}$.

In the most general weak turbulence situation, all of the above interactions could be essential. Furthermore, the completely self-consistent problem should also include consideration of the spectrum not only in the inertial range, but in all k space, from the pump region to the damping region. Such a solution would provide the answer to important questions raised in experimental and observational plasmas. In many cases, for example, the transport properties of turbulent plasma are of paramount importance; these properties are directly related to the shape and overall magnitude of the turbulent spectrum.

Of course, there are many different types of waves in plasma, the turbulent state of which could form the subject of further study. However, in the remainder of this lecture I shall consider specifically aspects of only the two most fundamental modes: ion acoustic waves (phonons) and electron plasma waves (plasmons). Turbulence of ion acoustic waves is closely related to the important problem of anomalous electrical resistivity; plasmon turbulence introduces the intriguing problems of cavitons and collapse.

III. ANOMALOUS RESISTIVITY

Let me first review in general terms the present state of understanding of the electrical resistivity of turbulent plasma. Experimentally, anomalous resistivity generally appears when the electric current in the plasma exceeds some critical value: $j > j_c$. The onset of anomaly can be interpreted as the result of some current-driven instabilities, which lead to an additional loss of momentum by electrons due to coherent radiation of appropriate waves. Researchers have analyzed many different instabilities whose participation in anomalous resistivity was suspected. Although I will speak mostly of the ion acoustic instability, other familiar instabilities are related to this. For example, the Buneman instability is essential for problems involving very high relative drift of electrons through ions, $v_{\rm rel} \gtrsim v_{\rm te}$. However, this is believed to evolve eventually into ion acoustic instability as the plasma heats. A magnetic field $\omega_{ce} < \omega_{pe}$ does not usually change the zero field modes appreciably, but does bring in new modes. As an old example, the Drummond–Rosenbluth mode near the ion cyclotron frequency could be mentioned. It is comparatively slow-growing and is probably easily saturated by simple quasilinear plateau formation. This instability is inherent in the currents along $\mathbf{B}_0$. In contrast to this, we have the so-called modified Buneman instability, driven by perpendicular current. This mode also generalizes to a kinetic mode with resonant effects, as well as a dissipative mode. The latter is believed to play an important role in ionospheric phenomena. These instabilities have very small growth rates, but also low thresholds j_c.

While the instabilities can have different origins, most of them lead to quite general expressions for their net effect on anomalous resistivity. This effect can be found from simple considerations based on the exchange of momentum and energy between the electrons and waves. Quite generally, we may determine the electric conductivity σ of turbulent plasma in terms of an effective collision frequency ν^*:

$$\sigma \equiv \frac{ne^2}{m\nu^*}.$$

Specifically, ν^* is the electron collision frequency describing momentum loss due to wave generation. Let us consider the conservation of net momentum of a system of electrons and waves. The momentum loss per unit time is

$$mn\mathbf{u}\nu^* = -\mathbf{F}, \tag{1a}$$

where $\mathbf{F}$ is the collective drag force due to wave generation and $\mathbf{u}$ is the average drift velocity. If the waves have a spectral energy density ε_k, then $\varepsilon_k\,(\mathbf{k}/\omega_k)$ is their momentum density. Since the drag force is due to wave emission by electrons, we get

$$F = \int \frac{d\mathbf{k}}{(2\pi)^3}\gamma_k^{(e)}\mathcal{E}_k\,\frac{\mathbf{k}\cdot\hat{z}}{\omega_k}, \tag{1b}$$

where $\gamma_k{}^{(e)}$ is the electron contribution to the imaginary part of the frequency and $\hat{z}$ is the unit vector in the direction of the electric current. More rigorously, this formula follows from quasilinear theory (Sagdeev and Galeev, 1969). Thus,

$$\nu^* = (mnu)^{-1}\int \frac{d\mathbf{k}}{(2\pi)^3}\gamma_k^{(e)}\mathcal{E}_k\,\frac{\mathbf{k}\cdot\hat{z}}{\omega_k}. \tag{2}$$

If we understand $\gamma_k^{(e)}$ to be the linear growth rate, then the whole problem is reduced to determination of the wave spectrum.

If the resistivity is anomalous, the Joule heating is also anomalous:

$$Q = j^2/\sigma^*.$$

Such heating, which can be called "turbulent," generally heats the various species at different rates. As a rule, electrons are heated faster than ions. To see why, we can use the following arguments. The power associated with the friction force is the energy dissipated per second in the plasma:

$$Q = \mathbf{u}\cdot\mathbf{F} = mnu^2\nu^*,$$

$$= \int \frac{d\mathbf{k}}{(2\pi)^3}\gamma_k^{(e)}\mathcal{E}_k\,\frac{\mathbf{k}\cdot\mathbf{u}}{\omega_k}. \tag{3}$$

In the nonlinearly saturated steady state, the momentum of the waves (and therefore their energy as well) must be absorbed by the ions. The rate of ion heating is thus

$$\frac{dW_i}{dt} \sim \int \frac{d\mathbf{k}}{(2\pi)^3}\gamma_k^{(e)}\mathcal{E}_k. \tag{4}$$

From Eqs. (3) and (4) we then obtain

$$\frac{d(W_e+W_i)}{dW_i} \sim \frac{\int d\mathbf{k}\gamma_k^{(e)}\mathcal{E}_k\mathbf{k}\cdot\mathbf{u}/\omega_k}{\int d\mathbf{k}\gamma_k^{(e)}\mathcal{E}_k},$$

where we used $Q = d(W_e + W_i)/dt$. For an order of magnitude estimate, it is sufficient to put

$$\int d\mathbf{k}\gamma_k^{(e)}\mathcal{E}_k\,\frac{\mathbf{k}\cdot\mathbf{u}}{\omega_k} \simeq \frac{\langle\mathbf{k}\rangle\cdot\mathbf{u}}{\langle\omega_k\rangle}\int d\mathbf{k}\gamma_k^{(e)}\mathcal{E}_k,$$

where $\langle\mathbf{k}\rangle$ and $\langle\omega_k\rangle$ are a typical wave number and frequency, averaged over the spectrum. Thus we have

$$\frac{dW_e}{dW_i} \simeq \frac{\mathbf{k}\cdot\mathbf{u}}{\omega_k} - 1. \tag{5}$$

This formula has a universal character, as it does not depend on the details of the particular mode responsible for the turbulence. For a majority of modes, Eq. (5) indeed leads to a predominant heating of electrons, since usually $k\,u/\omega \gg 1$. For example, ion acoustic waves are unstable in this limit.

For ion acoustic turbulence, one can reduce Eq. (2) to a more conclusive form. The main contribution to (2) comes from wave numbers corresponding to the highest growth rates, so $k\lambda_D \lesssim 1$. Substituting $\gamma_k^{(e)} \simeq \omega_k(u/v_{\rm te})$, we obtain

$$\nu^* \simeq \omega_{\rm pe}\left(\frac{W}{nTe}\right),$$

where W is the total wave energy, summed over all wave numbers. Similar estimates are often used when a quick answer is needed.

From the point of view of a theoretical understanding of anomalous resistivity, ion acoustic turbulence is especially promising, since it qualifies as weak turbulence. We can exploit two different limiting cases. First, in the limit of sufficiently large electric fields the instability cannot be saturated by quasilinear effects, and one must take nonlinear mode-mode coupling and wave-particle interaction into account. We can regard the developed ion acoustic turbulence as a gas of nonlinearly interacting phonons. Due to the dispersion law $\omega_k^2 = k^2 c_s^2 [1+(k\lambda_D)^2]^{-1}$ for the phonons, resonant three-wave coupling is absent and the wave-wave collision integral should be cubic in $\mathfrak{N}_k$. However, the induced scattering of phonons on ions $[\omega_1 - \omega_2 = (\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{v}]$ provides quadratic terms in the wave kinetic equation and, in fact, dominates the nonlinear effects. Thus the phonon energy density first increases exponentially at low amplitudes at the linear ion acoustic instability growth rate. At higher amplitudes, nonlinear saturation should occur due to the quadratic induced scattering. In a simplified manner, we can represent this process of growth and saturation by the model nonlinear equation

$$\frac{d\varepsilon_k}{dt} = 2\gamma_k^{(e)} \varepsilon_k - A\,\varepsilon_k^2 .$$

Actually, the quadratic term is a rather complex integral expression, involving integration over all wave vectors. However, we can estimate its size by using dimensional arguments. From the linear stability theory, $\gamma_k^{(e)} \simeq \omega_k (u/v_{te})$. The product $A\varepsilon$ must have the dimensions of frequency. If we deal with a normalized energy density ε/nT_e, then $A \alpha \omega$. One may also expect that A is proportional to the small ratio T_i/T_e, since ion thermal motion is necessary for induced scattering. More precisely, we can argue as follows. First, we find the relative phase volume $\Delta\omega/\omega$ of interacting phonons. Since for induced scattering $\omega - \omega' \sim k v_{ti}$, we estimate $\Delta\omega/\omega \sim v_{ti}/c_s$. Next, at each scattering, the ions absorb only a fraction $\Delta\omega/\omega$ of the total phonon energy. Multiplying this by the phase volume of interacting phonons, we find that A should be proportional to $(\Delta\omega/\omega)^2 \sim T_i/T_e$. Thus $A \sim (T_i/T_e)\omega$. Balancing linear growth with nonlinear induced scattering, we find

$$W \sim (T_e/T_i)(u/v_{te})(nT_e) .$$

Indeed, more accurate weak turbulence calculations lead to exactly the same parametric dependence, with a numerical proportionality factor of $O(10^{-2})$. This leads to the formula for anomalous resistivity first derived by Sagdeev (1967):

$$\nu^* \simeq 10^{-2} \omega_{pi} (T_e/T_i)(u/c_s) .$$

More accurate considerations show that the spectral energy $W = \int d\mathbf{k}\varepsilon_k$ is logarithmically divergent at $k \to 0$. However, the momentum loss (expression for ν^*) is convergent, so we should not be too concerned about the divergence. Also, this divergence disappears in renormalized weak turbulence theory (Horton and Choi, 1974).

The above discussion of ion acoustic turbulence is concerned with electric fields so large that linear ion Landau damping is not sufficient to balance the electron growth term. Let us estimate the threshold field, where $|\gamma^{(e)}| \sim |\gamma^{(i)}|$. In order of magnitude, we have

$$\gamma^{(e)} \sim \omega (u/v_{te}) ,$$

$$\gamma^{(i)} \sim \omega (\omega/k v_{tl})^2 (\delta n/n) ,$$

where $\delta n/n$ is the fraction of ions in the tail of the ion distribution, the subscript "tl" stands for tail, and the formula for $\gamma^{(i)}$ comes from the usual ion Landau damping on the ion tail. To find the tail density, we argue as in Eq. (5) that the relative heating rate $(nT_e)^\circ / (\delta n T_{tl})^\circ \sim u/c_s$, or

$$\delta n/n \sim (T_e/T_{tl})(c_s/u) .$$

Balancing $\gamma^{(e)}$ with $\gamma^{(i)}$,

$$\left(\frac{u}{v_{te}}\right)^2 \sim \left(\frac{T_e}{T_{tl}}\right)\left(\frac{c_s}{v_{tl}}\right)^2 \left(\frac{c_s}{v_{te}}\right) .$$

We now maximize this rate by arguing that the ion tail will not be well formed unless $T_{tl} \gtrsim T_e$, so that $v_{tl} \gtrsim c_s$ for damping. Thus $\min(T_{tl}) \sim T_e$ from which

$$(u/v_{t_e})_{max} \simeq (m/M)^{1/4} ,$$

or

$$u/c_s \simeq (M/m)^{1/4} . \tag{6}$$

To find the threshold electric field, we use Ohm's law $j = \sigma E$, or

$$E = m\nu^* u/e . \tag{7}$$

Using

$$\nu^* \simeq 10^{-2} (u/c_s)^2 \omega_{pi} \tag{8}$$

$(T_e/T_i \sim u/c_s)$, we insert Eqs. (6) and (8) into Eq. (7) to find that the threshold field E_{thr} is of order

$$E_{thr} \simeq 10^{-2} [(mM^3)^{1/4}/e]\, \omega_{pi}\, c_s .$$

In the regime of weak fields $E \sim E_{thr}$, the most complicated problem is concerned with the great sensitivity of the threshold u_c to the form of the resonant ion distribution function—the far tail of the distribution. Even if the distribution were specified at $t=0$ (e.g., Maxwellian), it would soon be completely distorted, because only the small number of resonant ions would interact with the waves. The main point of difficulty then lies in learning the evolution of the ion distribution. In this regard, analysis is simplified by invoking self-similarity arguments (Vekshtein and Sagdeev, 1970). One argues that the long-time behavior of plasma in a regime of anomalous resistivity should become insensitive to the initial conditions. This should then lead to establishment of self-similarity in, for example, the shape of the distribution function. In this case, a fraction $(m/M)^{1/4}$ of ions are always resonant, obtaining thermal energy $\sim T_e$ and maintaining the threshold $u \sim (M/m)^{1/4} c_s$. The self-similar electron distribution has a simple universal form $f \sim \exp[-A(t)v^5]$. Numerical simulation supports the self-similar model for saturation of the ion acoustic instability (Dum *et al.*, 1974).

In the limit of extremely intense fields, the arguments based on induced scattering are no longer correct, since higher-order nonlinearities will become important. However, we can give a simple argument which gives the asymptotic limit of the anomalous resistivity

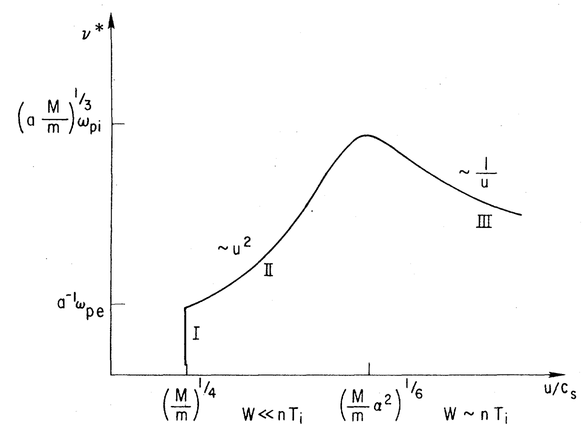

FIG. 2. Schematic drawing of the anomalous collision frequency ν^* for ion acoustic turbulence as a function of normalized drift velocity u/c_s.

as E becomes very large. Of course, we still have $\nu^* \sim \omega_{pe}(W/nT_e)$. However, the relation $T_e/T_i \sim u/c_s$ is not generally correct, but correct only for $nT_i \gg W$. When W becomes comparable to nT_i, we have more properly

$$nT_e/(nT_i + W) \sim u/c_s .$$

The maximum spectral energy then obeys $nT_e/W_{max} \sim u/c_s$, or

$$W_{max} \sim (c_s/u)nT_e .$$

This then gives the anomalous collision frequency in the very high field limit as (Galeev, 1976)

$$\nu^*_{max} \sim \omega_{pe}(c_s/u) ,$$

which corresponds to constant drag force or to a kind of runaway regime in the limit of a very high electric field. The complete predicted behavior of ν^* is sketched in Fig. 2.

One of the most crucial outstanding problems in anomalous resistivity concerns the electron interaction with the ion acoustic waves. The velocity dependence $\sim v^{-3}$ of electrons scattering on phonons reminds us of the situation with the Lorentz gas, where the electrons eventually run away. Does this lead to an eventual increase of u? Of course, this will not happen for currents $\perp B$. But for the currents parallel to a very strong magnetic field, it is certainly an important problem to be solved.

To end this part of the discussion, I would conclude that ion acoustic turbulence can indeed be considered self-consistently within the framework of weak turbulence theory, although the concrete realization of this program may represent a quite difficult task.

IV. PLASMON TURBULENCE

Let us now consider some of the very fascinating and important aspects of turbulence involving the second fundamental mode of plasma oscillation, electron plasma waves. This turbulence behaves quite differently from acoustic turbulence. In the present case, the turbulent state is a gas of nonlinearly interacting plasmons $\{\omega^2 = \omega_p^2[1+3(k\lambda_D)^2]\}$. These waves can be excited by many means, including electron beams, parametric pumping, etc. Since the plasmon dispersion law forbids three-wave interaction, the lowest-order nonlinear effect is the induced scattering of plasmons on the particles. This leads to a very interesting effect. The usual step-by-step hierarchy of nonlinear interactions would cause at each scattering the loss of both energy and momentum by the plasmon. Thus, as time increases, ω_k would approach ω_p and k would approach 0. But as k becomes smaller, so does the possibility of plasmon absorption by linear Landau damping. Thus one would predict an accumulation of plasmons at very small k in the long-time limit. Furthermore, the energy deposited in the particles by the induced scattering process represents only a small fraction of the energy which accumulates in the plasmons:

$$\Delta\mathcal{E} \lesssim [(\omega_k - \omega_p)/\omega_p]\,\mathcal{E} ,$$

where ω_k is the initial plasmon frequency. This follows immediately from the fact that the total plasmon "number" is conserved during induced scattering. Thus, if we have a continuous input of plasmons due to some instability mechanism, we would at first expect an indefinitely continued accumulation of plasmons with $k \to 0$. This reminds us of Bose–Einstein condensation. However, there is a weak link in the argument, namely that the weak turbulence criteria will be violated when the net plasmon energy becomes sufficiently large. This is a crucial problem, and we conclude that weak turbulence theory cannot provide a completely self-consistent description of plasma turbulence. Rather, we must resort to (or develop!) the methods of strong turbulence.

To understand further the breakdown of weak turbulence theory, let us consider the eventual fate of the plasmon condensate. We imagine a steady input of energy into the plasmons. When the plasma medium fills with sufficiently large-amplitude electron plasma oscillations, it is clear that the plasmon radiation pressure—a quantity hitherto neglected—becomes important. (The radiation pressure is that quantity whose gradient is the so-called ponderomotive force $-n_0 m\overline{\mathbf{V}\cdot\nabla\mathbf{V}} = -n_0 e^2\nabla\sum_k|E_k|^2/2m\omega_k^2$, where V is the electron velocity in plasma oscillations and superbar means an average over the fast electron plasma oscillation time scale.) Consider now a low-amplitude, long-wavelength density fluctuation. Plasmons will tend to concentrate in the regions of density minima in this perturbation, since these regions serve as dielectric cavities for the waves. The increased plasmon concentration produces additional radiation pressure, which in turn expels more plasma from the cavity. It is clear that we have here the beginning of an instability (Vedenov and Rudakov, 1964)—it has come to be called the modulational instability.

I will now give a very simple, heuristic derivation of modulational instability by using an analogy based on the Maxwell–Boltzmann distribution for charged particles in a potential ϕ (these arguments can be made precise by using the kinetic equation for plasmons). In such a potential, the electron distribution function would be

$$f(v) \sim \exp[-(1/2 m v^2 - e\phi)/T],$$

from which we get the electron density by velocity integration:

$$n = n_0 \exp(e\phi/T).$$

By analogy, let us consider a distribution function for plasmons:

$$\mathfrak{N}_k \sim N_0 \exp(-k^2/2\langle k^2\rangle).$$

This would be an equilibrium distribution, in the absence of any density perturbations. More generally, there will be a contribution from the density perturbations, which we can find by considering the plasmon quasienergy

$$\omega = \omega_p[1 + (3/2)(k\lambda_D)^2].$$

In the presence of a density perturbation δn,

$$\omega \simeq \omega_{p\,0}[1 + (3/2)(k\lambda_{D0})^2 + (1/2)\delta n/n_0],$$

so by analogy with the particle distribution with potential ψ the plasmon distribution should be

$$\mathfrak{N}_k \simeq N_0 \exp\{-[(1/2)k^2 + (1/3)k_D^2(\delta n/n_0)]/\langle k^2\rangle\}.$$

Integration over k then gives the total plasmon density

$$\int dk\mathfrak{N}_k = N_0 \exp\left[-(1/3)\frac{1}{\langle k^2\rangle\lambda_D^2}\left(\frac{\delta n}{n_0}\right)\right],$$

where N_0 is the equilibrium plasmon density. The fluctuation in plasmon density is then

$$\delta\mathfrak{N}_k \simeq -(1/3)N_0\frac{1}{\langle k^2\rangle\lambda_D^2}\left(\frac{\delta n}{n_0}\right),$$

in complete analogy to the electron density perturbation $\delta n_e \simeq n_0(e\phi/T)$. Thus the change in plasmon radiation pressure

$$\delta P_{\rm rad} \simeq -(1/3)P_0\frac{1}{\langle k^2\rangle\lambda_D^2}\left(\frac{\delta n}{n_0}\right)$$

is negative(!!). Instability will result when

$$|\delta P_{\rm rad}| > T\delta n,$$

so the instability criterion becomes

$$P_0 > 3\langle k^2\rangle\lambda_D^2(nT).$$

Thus, even for a very small radiation pressure $P_0/nT \ll 1$, we will get instability when $\langle k^2\rangle \to 0$. This picture of the turbulent state is quite different from the weak turbulence picture. Now, we have plasmons trapped in random cavities. The dynamics of the cavities (cavitons), in turn, correspond to a purely growing instability ($\mathrm{Re}\,\omega = 0$); the turbulence is no longer weak. We can estimate the growth rate by including the plasmon radiation pressure in the acoustic dispersion relation $\omega^2 \sim k^2(\delta P/\delta nM)$, which becomes

$$\omega^2 \sim k^2[T - P_{\rm rad}/n(k\lambda_D)^2]/M.$$

For $P_{\rm rad} \gg (k\lambda_D)^2 nT$, then

$$\mathrm{Im}\,\omega \sim (P_{\rm rad}/nM\lambda_D^2)^{1/2}.$$

Of vital concern, of course, is the nonlinear behavior of this instability (Zakharov, 1972). Very crudely, we can discuss this behavior as follows. First, when $\delta n/n \sim \langle k^2\rangle\lambda_D^2$, we know that most of the plasmons are trapped and coalesce in randomly placed cavities. Naturally, we expect some sort of saturation of the coalescence process. To examine this, we look first at the behavior of individual cavities. We will find that this behavior depends crucially on the number of spatial dimensions d. Let N be the number of plasmons in the cavity. By action conservation, this number will be constant. Since the change in ω_k is small, we have roughly

$$N \simeq \int d^d x E^2.$$

If $l(t)$ is the typical cavity dimension, then constancy of N gives the variation of the radiation pressure as

$$E^2 \sim 1/l(t)^d.$$

On the other hand, from the plasmon trapping condition $\delta n(t)/n \sim k(t)^2\lambda_D^2$ with $k(t) \sim 1/l(t)$, we have

$$\delta n(t) \sim 1/l(t)^2,$$

notably independent of dimensionality. Since the expelled pressure is proportional to $\delta n(t)$, it is clear that for $d = 1$, a balance will be reached between the plasma and radiation pressures, the collapse will cease, and we expect a soliton type of solution. For $d = 3$, the collapse will continue; and for $d = 2$, we have a marginal situation in which if the collapse starts for any reason, it cannot be stopped by the particle pressure. The collapse will not continue indefinitely, of course. We expect that when $l(t) \to \lambda_D$, strong linear Landau damping will set in and the plasmons will be suddenly damped (Fig. 3).

Before saying more about the final stage of collapse, let us give a simple derivation of the scaling laws for the collapse and the associated inertial range spectrum (Galeev *et al.*, 1975). The low-frequency density fluctuations will obey (Zakharov, 1972)

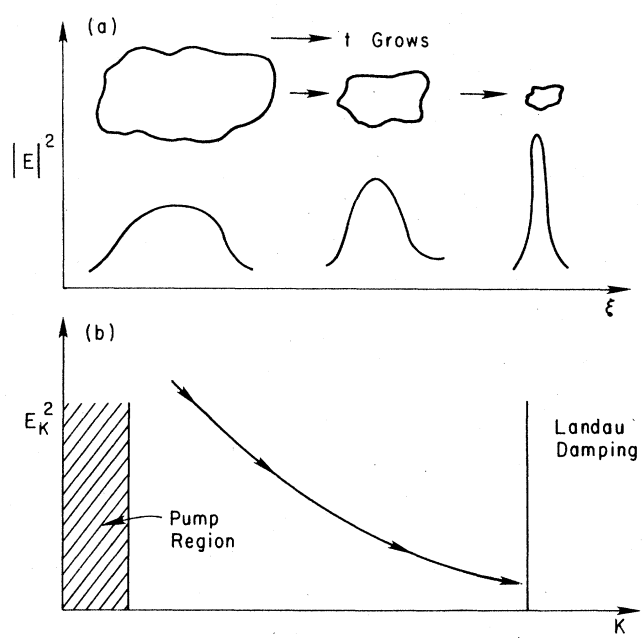

FIG. 3. Aspects of plasmon collapse. (a) Evolution of density and field fluctuations. (b) The spectrum.

$$\left[\frac{\partial^2}{\partial t^2} - \left(\frac{T_e}{M}\right)\nabla^2\right]\delta n = \frac{1}{16\pi M}\nabla^2 |E|^2, \tag{10}$$

which takes into account the influence of the radiation pressure (right-hand side) on the phonons (left-hand side). From

$$\delta\omega = \frac{\omega_p}{2}\left(\frac{\delta n}{n} + 3k^2\lambda_D^2\right) = \text{const}$$

or more rigorously from the nonlinear Schrödinger equation (Zakharov, 1972),

$$\text{div}\left[\frac{\partial E}{\partial t} + \tfrac{3}{2}\omega_p\lambda_D^2 \nabla\nabla\cdot\mathbf{E}\right] = \frac{\omega_p}{2}\,\text{div}\left[\left(\frac{\delta n}{n_0}\right)\mathbf{E}\right], \tag{11}$$

we find $\delta n \sim \nabla^2$. Thus, in the highly unstable limit where $|E|^2 \gg (k\lambda_D)^2 nT$, we can write

$$\frac{\partial^2}{\partial t^2}\,\delta n \sim \frac{1}{16\pi M}\nabla^2 |E|^2,$$

from which by "cancelling" δn and ∇^2, we estimate

$$|E|^2 \sim (t_c - t)^{-2} \sim l^{-d}. \tag{12}$$

From $\nabla^2 \sim 1/l^2 \sim \delta n$, we also get

$$\delta n \sim (t_c - t)^{-4/d}.$$

Finally, we can estimate the inertial range spectrum by again using the constancy of the energy flux:

$$\mathcal{E}_k \frac{dk}{dt(k)} = \text{const}.$$

Now, $dt(k)$ is the collapse time from $l(t) \sim 1/k$ to $l(t+dt) \sim 1/(k+dk)$, which we estimate using Eq. (12). The final result for the inertial range spectrum is

$$\mathcal{E}_k \sim 1/k^{1+d/2}.$$

We can also determine the magnitude of the spectral energy and the anomalous resistivity. Consider situations in which there is a well-defined inertial range, with an electric field source ("pump") E_0 at small k and an absorption region at large k. The anomalous collision frequency will quite generally obey

$$\nu^* = \alpha\omega_{pe}\left(\frac{W}{nT_e}\right), \tag{13}$$

where α is a numerical coefficient. The power absorbed will be $\nu^*(E_0^2/8\pi)$, while the power entering the inertial range will be W/τ_{mod}, where $\tau_{mod}^{-1} = \gamma_{mod}$ is the growth rate for the modulational instability:

$$\gamma_{mod} \simeq (W/nM\lambda_D^2)^{1/2}.$$

Equating the absorbed with the generated power, we find

$$\frac{W}{nT_e} \simeq \alpha^2\left(\frac{M}{m}\right)\left(\frac{E_0^2}{8\pi nT_e}\right)^2.$$

The anomalous collision frequency then follows by substitution back into Eq. (13). Of course, we have not found the numerical factors associated with these scaling laws.

The above argument breaks down when the turbulence is too strong. In particular, when W/nT_e becomes so large that $\langle k_{mod}^2\rangle\lambda_D^2 \sim W/nT_e \sim \alpha^2(M/m)(E_0^2/8\pi nT_e)^2$ becomes less than or of the order of $k'^2\lambda_d{}^2$ [k' is the characteristic value of k in the damping region $k' \sim (\frac{1}{3}$ or $\frac{1}{4})\lambda_D^{-1}$], then the source region overlaps with the Landau damping region and the inertial range disappears completely. Evidently, the critical pump field is

$$\frac{E_0}{(8\pi nT_e)^{1/2}} \sim \left(\frac{1}{\alpha^2}\;m/M\right)^{1/4} k'\lambda_D.$$

For pump fields exceeding this threshold value, we have as yet no analytic theory. On the other hand, essentially all the numerical simulations have been done in this regime of no inertial range (see, for example, Kruer *et al.*, 1970; Thompson *et al.*, 1973), since for weaker fields the run times become prohibitively long. There have been a few one-dimensional computations performed in the weak field limit (Galeev *et al.*, 1976). However, it is clear that a major outstanding problem is the satisfactory merging of analytic theory and numerical experiment for these important regimes of plasmon turbulence. A summary of what we know so far is sketched in Figs. 4.

Let me now make a few brief comments about the state of the plasma during the final stages of collapse. The sudden absorption of the plasmons as $l(t) \to \lambda_D$ will clearly leave density holes which would effectively generate short-wavelength phonons. If $T_e \gg T_i$ so that these phonons are long-lived, then the ultimate turbulent steady state would really consist of two interacting states, each with their own distinct properties. The short-wavelength phonons would influence the modulational instability and collapse process in at least two ways: (1) they would scatter plasmons, and (2) they would produce a new channel for damping, namely the conversion process l(plasmon) + s(phonon) $\to l'$ (short-wavelength plasmon), which would then be immediately absorbed (Galeev *et al.*, 1976).

As an interesting application, it seems necessary to revise our ideas about the electron-beam-plasma collective interaction (Sudan, 1973; Galeev *et al.*, 1977). Up until now, quasilinear theory has been used to predict the penetration length for the beam; it was found that $l \sim n_b^{-1}$, since the more dense the beam, the stronger the instability $\langle E^2\rangle$. Now, however, we can use our knowledge of modulational instability to predict satura-

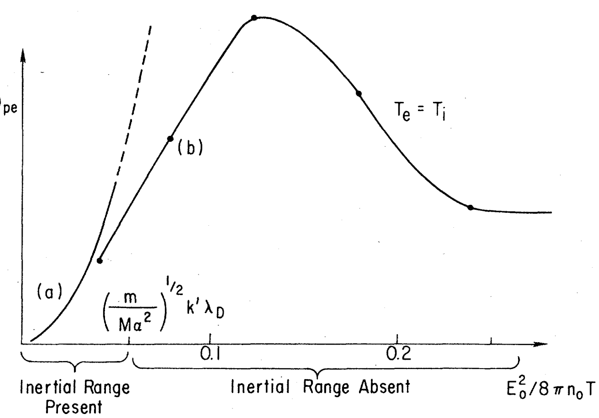

FIG. 4. Anomalous collision frequency ν^* as a function of pump field E_0 in the regime of modulational instability. (a) Theoretical scaling from inertial range arguments (see text). (b) Empirical dependence from numerical simulations (Kruer *et al.*, 1970).

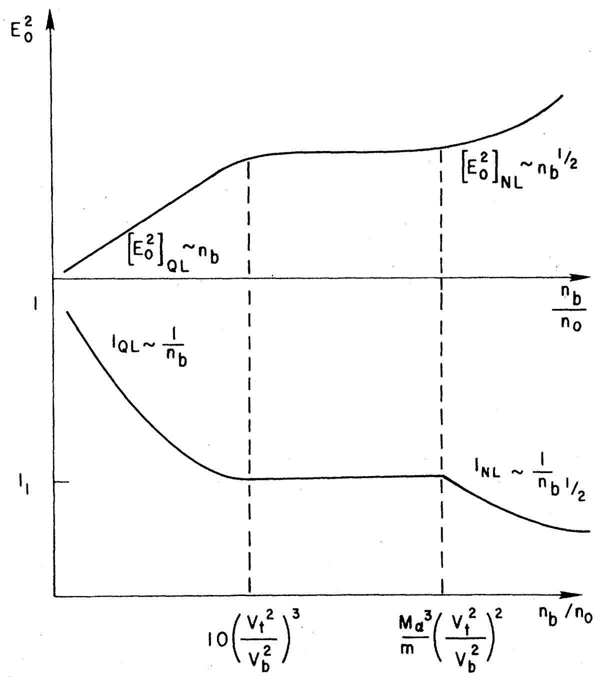

FIG. 5. Saturated fluctuation level $\langle E^2 \rangle$ for beam-plasma interaction as a function of beam density n_b.

FIG. 6. Beam penetration length l for beam-plasma interaction as a function of beam density n_b. l marks onset of modulation instability.

tion (Fig. 5) at $E^2/16\pi \sim (v_t/v_b)^2 nT$ [since $v_b^2 \sim (\omega/k_b)^2$ so $(v_t/v_b)^2 \simeq (k_b \lambda_D)^2$]. Thus the penetration length l of the beam as a function of beam density will look qualitatively like Fig. 6. (For details, see Galeev *et al.*, 1977). For sufficiently strong beams, we see that the penetration length of the beam is enhanced over the quasilinear result.

We should add a few cautions at this point. In spite of the intensive work which has been done on collapse problems in recent years, we have as yet no rigorous mathematical proof of collapse, although some computer simulations point in this direction. Even the proper equations to use are in some dispute. For example, very little is known about the interaction between cavitons. If this interaction is strong enough, it may prevent collapse completely—indeed, it would no longer make sense to speak of cavitons as distinct entities. Furthermore, it may be that other fundamental effects—for example, nonlinear frequency shifts—are important in a proper description of the cavitons. There are still very fascinating and important questions to be answered.

What conclusions can we draw from this very rapid survey of problems in turbulence theory? It appears that the weak turbulence theory can be developed entirely from first principles. Furthermore, there appear to be at least a few relevant problems, such as ion acoustic turbulence, which can be treated self-consistently within the weak turbulence framework. Our knowledge supports the concept of scaling and the inertial range for strong plasma turbulence. However, there is a great need for a more rigorous theory of strong turbulence. Even more difficult are problems falling into the gap between weak and strong turbulence, where the random phase approximation is not valid and we need precise knowledge of higher-order correlation functions, and so on. It is clear that much work remains. However, it seems that we are beginning to isolate the important unifying features of the turbulent state, and we can thus be hopeful of further progress. It is almost superfluous to mention that understanding of many astrophysical and laboratory phenomena, as well as such practical questions as predicting the confinement properties of magnetic bottles, are limited by our knowledge of turbulent phenomena. The qualitative understanding we now possess must be augmented by much analysis, computation, and experiment to become quantitative.

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