Current algebra formulation of radiative corrections in gauge theories and the universality of the weak interactions

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A current algebra formulation of the radiative corrections in gauge theories, with special applications to the analysis of the universality of the weak interactions, is developed in the framework of quantum chromodynamics. For definiteness, we work in the SU(2)×U(1) model with four quark flavors, but the methods are quite general and can be applied to other theories. The explicit cancellation of ultraviolet divergences for arbitrary semileptonic processes is achieved relying solely on the Ward identities and general considerations, both in the W and Higgs sectors. The finite parts of order $G_F\alpha$ are then evaluated in the case of the superallowed Fermi transitions, including small effects proportional to $\bar{g}_S^{2}(\kappa^2)$, which are induced by the strong interactions in the asymptotic domain. We consider here both the simplest version of the Weinberg–Salam model in which the Higgs scalars transform as a single isospinor, as well as the case of general symmetry breaking. Except for the small effects proportional to $\bar{g}_S^{2}(\kappa^2)$, the results are identical to the answers previously found on the basis of heuristic arguments. The phenomenological verification of Cabibbo universality on the basis of these corrections and the superallowed Fermi transitions has been discussed before and found to be in very good agreement with present experimental evidence. The analogous calculation for the transition rate of pion β decay is given. Theoretical alternatives to quantum chromodynamics as a framework for the evaluation of the radiative corrections are briefly discussed. The appendixes contain a generalization of an important result in the theory of radiative corrections due to L. S. Brown, G. Preparata, and W. I. Weisberger, an analysis of the hadronic contributions to the W and ϕ propagators, mathematical methods for evaluating the $\bar{g}_S^{2}(\kappa^2)$ corrections, and discussions of quark mass renormalization and the absence of operator "seagulls" in the hadronic correlation functions. Some of the methods discussed in this paper can also be applied to the study of radiative corrections of order $G_F\alpha$ to other processes affected by the strong interactions.

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I. INTRODUCTION

One of the most interesting theoretical advances in the last several years has been the development of unified models of weak, electromagnetic, and strong interactions on the basis of non-Abelian gauge theories.[1] Although these models are renormalizable, their application in the study of higher-order corrections to weak interactions has been hampered in many cases of physical interest by the complications stemming from the strong interactions.

In this paper a current algebra formulation is developed which, in principle, may be used as a framework to study the radiative corrections to arbitrary semileptonic processes. In fact, a current algebra formulation is probably our only hope of controlling the effects of the strong interactions in a clear and logical manner. Some of the main ideas and techniques of this approach have already been outlined by Sirlin (1974a) and applied to a discussion of the cancellation of divergences in semileptonic processes. In another paper (Sirlin, 1974b) the strategy for evaluation of many of the finite parts has been described, albeit rather briefly. Some further applications have been presented by Sirlin (1974c).[2]

[1]For reviews see, for example, Abers and Lee (1973), Bég and Sirlin (1974), Weinberg (1974), and Taylor (1976).

[2]A current algebra analysis with some similarities to the work of Sirlin (1974a) has been discussed in an interesting paper by T. Hagiwara (1974a) in the framework of the SO(3) model. There are, however, some important technical differences between this paper and the papers of Sirlin (1974a, 1974b, 1975), particularly in the treatment of the corrections to the external hadronic legs.

The current algebra formulation is applied in this paper to a detailed analysis of the problem of universality of the weak interactions. As is well known, in order to verify the principle of universality it is crucial to calculate the corrections of order α to the ratio of decay probabilities of superallowed Fermi β transitions and muon decay. Since the advent of renormalizable gauge theories, this problem has received renewed attention, and the corrections have been studied in the framework of simple hadronic extensions of the $SU(2)_L \times U(1)$ model. In fact, the corrections turn out to be quite large and of the sign and magnitude presently required to verify the universality of the weak interactions in the sense of Cabibbo (Sirlin, 1974b, 1974c, 1975; Angerson, 1974; Roos, 1974; Wilkinson, 1975a, b; Wilkinson and Alburger, 1976; Hardy and Towner, 1975; Raman *et al.*, 1975).[3] However, the original calculations of Sirlin (1974c) and Angerson (1974)[4] were heuristic in nature and did not take explicitly into account the effects of the strong interactions. It is one of the aims of the present work to analyze these effects in greater detail in the framework of the current algebra approach in order to develop a more complete and satisfactory picture of the radiative corrections.

In order to carry out the analysis of the finite parts of the corrections, we shall make certain assumptions, which we now proceed to enumerate, regarding the general properties of the underlying theory:

(i) We shall assume that the generators of the weak and strong gauge groups commute and that, accordingly, the conservation of the weak currents is broken only by quark mass terms.

(ii) We shall hypothesize that the strong interactions are described by a $SU(3)^c$ asymptotically free gauge theory (Politzer, 1974, and references cited therein) mediated by non-Abelian gluons [$SU(3)^c$ stands for SU(3) of color].

(iii) We shall accept the idea that the onset of the asymptotic behavior occurs early with respect to the mass scale of the intermediate bosons. That is, we shall assume that $\bar{g}_S^2(\kappa^2)/(4\pi^2)$ is already small at Euclidean momenta κ characterized by $\kappa^2 \ll m_W^2$, where $\bar{g}_S(\kappa^2)$ is the effective coupling constant of the strong interactions and m_W stands for the generic mass of the intermediate bosons.

(iv) We are interested in the corrections of order $G_F\alpha$ rather than $G_F\alpha(m^2/m_W^2)$, where m is a generic quark mass or the mass that sets the scale in the short-distance expansions. Thus terms of order $G_F\alpha(m^2/m_W^2)$ will be regarded as being of order G_F^2 and will be systematically neglected (Weinberg, 1973a). In particular this applies to the finite contributions arising from exchanges of virtual Higgs scalars, as their couplings to quarks and leptons will be considered to be of order gm/m_W.

(v) For definiteness we shall carry out our analysis in the familiar $SU(2)_L \times U(1) \times SU(3)^c$ theory, assuming the minimal scheme in which the quarks transform according to the fundamental representation of $SU(4) \times SU(3)^c$.

Assumptions (i) and (ii), together with the concepts of color confinement, still to be understood on a fundamental and dynamical basis, constitute the basic premises of quantum chromodynamics.[5] The set of assumptions (i)–(iv) is identical to that used by Weinberg (1973b) in his analysis of the radiative corrections of $O(\alpha)$ to strong interaction amplitudes. This set of assumptions constitutes a sufficient framework for our analysis. It is by no means clear to us that it is a necessary condition. As we shall see, the crucial factor regarding short-distance behavior that enters into our analysis is the assumption that the coefficients of the leading terms in the short-distance expansions of products of current operators (and such leading terms in our case involve also currents) are not affected significantly by the strong interactions. As high-energy physics has failed to uncover the existence of strong interactions at high momentum transfers (i.e., at short distances), it is not inconceivable that any theoretical development able to describe this situation may ultimately serve to justify the analysis of the radiative corrections. Finally, assumption (v) is motivated by the dual desiderata of simplicity and definiteness. We believe, however, that both the current algebra approach discussed in this paper and the main qualitative results of our analysis are applicable to other gauge models provided the $\Delta S=0$ and $\mu\nu W$ amplitudes involve only two parameters, g and the Cabibbo angle θ, and that otherwise the theories conform with properties (i) to (iv).

The results that emerge from the analysis of the present paper are indeed very simple and can be described as follows. The radiative corrections to pion β decay ($\pi^\pm \to \pi^0 + e^\pm + \nu$) and the superallowed Fermi transitions naturally divide into two parts: the photonic corrections, as computed in the local V-A theory with an additional convergence factor and an effective cutoff set equal to m_W, and nonphotonic corrections. Aside from universal corrections to the W propagator and from small effects of $O[\bar{g}_S^2(\kappa^2)]$ induced by the strong interactions in the asymptotic domain, the nonphotonic corrections of order $G_F\alpha$ [in contradistinction to those of order $G_F\alpha(m^2/m_W^2)$] turn out to be independent of the dynamics of the strong interactions. Moreover, many but not all of these contributions are universal and cancel when we consider the ratio of decay probabilities of β and muon decays. In discussing these nonphotonic contributions we distinguish two situations: (a) the case of arbitrary symmetry breaking in which the number of Higgs scalars and their representation content is left arbitrary and (b) the sim-

[3]A calculation of the radiative corrections to μ and β decays in the SO(3) gauge model has been given by T. Hagiwara (1974b, 1974c). Because of the existence of several mixing angles, e–μ and Cabibbo universalities are not natural in this model.

[4]In his significant paper, Angerson (1974) treated the corrections in the one-quartet quark model without color degrees of freedom and considered separately the cases of zero and finite Cabibbo angle. His calculations for $\theta=0$ coincide with the corresponding result in Sirlin (1974c), while in the case $\theta \neq 0$ his answers contain small additional terms proportional to $\sin^2\theta$. However, the additional terms in Angerson's 1974 paper are spurious (private communication from W. Angerson), so that the two calculations actually agree. Sirlin (1974c, 1975) further treated three-quartet quark models which incorporate the color degree of freedom.

[5]For a recent review see Marciano and Pagels (1978).

plest version of the theory in which the Higgs scalars are assumed to belong to a single isospinor representation. As is well known, in the latter case the masses m_Z and m_W of the intermediate bosons are determined as functions of the weak interaction angle θ_W, while in the general case m_Z and θ_W are independent parameters. Although the photonic corrections involve the contributions of both soft and hard photons, they can be analyzed with the aid of the powerful current algebra theorems discussed by many authors before the advent of the gauge theories.[6] We recall that to zeroth order in the lepton momenta, and with the exception of a small asymptotic contribution, the photonic corrections arising from the vector current are also independent of the dynamics of the strong interactions (Abers *et al.*, 1968; Dicus and Norton, 1970; Sirlin, 1967a, 1968a, 1969) and turn out to be very large in the SU(2)×U(1) (Sirlin, 1974b, 1974c, 1975). The photonic corrections involving the axial-vector current can be separated into an "asymptotic contribution" proportional to $\ln(m_W^2/m^2)$ (m here represents roughly the onset of the asymptotic behavior) and a nonasymptotic piece which is finite as $m_W \to \infty$. As we shall show, the coefficient of the asymptotic piece depends on the average charge $\overline{Q}$ of the u and d quarks but is otherwise independent of the dynamics of the strong interactions. The nonasymptotic piece is model dependent, but rough estimates indicate that it is small in comparison with the large logarithms from the other contributions.[7]

When the photonic and nonphotonic contributions are combined, the final answer that emerges in the case of general symmetry breaking is that, aside from the small asymptotic effects of $O(\overline{g}_S^2(\kappa^2))$, the radiative corrections are given by the same expression as in the local V-A theory, with the cutoff replaced by m_W plus an additional contribution which depends on $\ln(m_Z/m_W)$, θ_W, and $\overline{Q}$, and which is, moreover, positive definite provided that $\overline{Q} \geq -\frac{1}{2}$. As explained in Sirlin (1975), this result plus the fact that $m_W = 37.3$ GeV/$\sin\theta_W$ in the $SU(2)_L \times U(1)$ gauge theory, forces the radiative corrections to be quite large. In the simplest version of the theory the result simplifies further and, except for the small contributions of $O(\overline{g}_S^2(\kappa^2))$, the final answer becomes identical to that obtained before the advent of the gauge theories, with the cutoff replaced by m_Z.

In summary, the conclusion of our analysis is that the strong interactions have remarkably little effect on the radiative corrections to the Fermi β decay transitions. Phenomenologically this is a welcome situation, as the early estimates give rise to very simple answers which are in good agreement with experiment. Theoretically, however, the results are perhaps surprising. For example, it is well known that asymptotically free theories do not in general lead to the same estimates of the coefficients in the short-distance expansions as free field theories, but characteristically give rise to correction factors involving powers of logarithms (Politzer, 1974). Such logarithmic corrections arise because the relevant operators in the short-distance expansion have in general anomalous dimensions, and these approach zero as $\kappa \to \infty$ only too slowly. As we shall see, the absence of such correction factors in the β-decay case can be traced to the following: (a) the significant terms in the short-distance expansions involve only currents on both sides of the equation, and (b) in the underlying theory of strong interactions these are conserved or partially conserved currents and have, therefore, no anomalous dimensions.

The plan of the paper is the following: In Sec. II we review some basic properties of the underlying theory which are very important in the analysis of the radiative corrections. In Sec. III we discuss the corrections to the β-decay amplitude involving the Fourier transforms of products of three hadronic currents. We refer to these contributions as three-current correlation functions. Making use of the appropriate Ward identities we show how the three-current correlation functions can be reduced to expressions involving one- and two-current correlation functions plus some special contributions involving derivatives with respect to the momentum transfer, which we refer to as residual three-current correlation functions. Section IV is devoted to the evaluation of the relevant two-current correlation functions, including vertex and box diagrams and the asymptotic effects of $O(\overline{g}_S^2(\kappa^2))$. In Sec. V we consider the contribution of the residual three-current correlation functions in conjunction with the order α counterterms of the theory, and show that their combined effect does not affect the radiative corrections of order $G_F\alpha$. We also discuss briefly the corrections to the W propagator. In Sec. VI we demonstrate explicitly the cancellation of ultraviolet divergences associated with the Higgs sector to all orders in the strong interactions.[8] For simplicity we consider here the simplest version of the SU(2)×U(1) model. In Sec. VII we use the current algebra formulation to compare μ and β decays, which is necessary for the discussion of universality, and calculate the rate for pion β decay. Section VIII contains some observations regarding the applicability of the present approach to other theories and other semileptonic processes. Appendix A provides a missing argument in the derivation of an important result in the theory of radiative corrections due to Brown (1969) and Preparata and Weisberger (1968) and, using CP invariance, generalizes the discussion to parity-nonconserving perturbations. Appendix B discusses the absence of operator seagulls in the hadronic correlation functions. Appendix C develops mathematical methods for the evaluation of the small effects of order $\overline{g}_S^2(\kappa^2)$, while Appendix D discusses the quark mass renormalization. Finally, Appendix E studies the divergent parts of the hadronic corrections to the W and ϕ propagators and describes their cancellation.

Throughout the paper we concentrate our analysis on the most difficult parts of the calculation, namely the contributions of those diagrams which are affected by the strong interactions.

[6]See, for example, Bjorken, 1966; Abers *et al.*, 1968; Dicus and Norton, 1970; Sirlin, 1967a, 1968a, 1969; Preparata and Weisberger, 1968; Bég *et al.*, 1972; and papers cited therein.

[7]See Abers *et al.*, 1968; Dicus and Norton, 1970; Sirlin, 1968a, 1969; and Sec. VII.B of the present paper.

[8]The corresponding cancellations associated with the intermediate boson sector are discussed in Secs. IV and V.

II. GENERAL PROPERTIES

It will become apparent in the course of the discussion that, subject to our assumptions, the physically interesting corrections of order α stem from the virtual interchanges of the photon, the W and the Z vector mesons. We begin our analysis, therefore, by writing down the relevant part of the Lagrangian density in the SU(4) $\times$SU(3)^c model with fractionally charged quarks (Bég and Sirlin, 1974):

$$L_{\rm int} = -eA_\mu J^\mu_\gamma - \frac{g}{\sqrt{2}}\,(W^\dagger_\mu J^\mu_W + \text{h.c.}) - (g^2+g'^2)^{1/2} Z_\mu J^\mu_Z + \cdots, \tag{2.1}$$

where $\cdots$ refer to the leptonic and Higgs scalar terms, and $W^\dagger$ is the field which creates a W^+ vector meson.[9] The hadronic currents J^μ_γ, J^μ_W, and J^μ_Z are given by

$$J^\mu_\gamma = \bar\psi\gamma^\mu Q\psi\,, \tag{2.2}$$

$$J^\mu_W = \bar\psi\gamma^\mu a_- C_-\psi\,, \tag{2.3}$$

$$J^\mu_Z = \tfrac{1}{2}\bar\psi\gamma^\mu a_- C_3\psi - \sin^2\theta_W\bar\psi\gamma^\mu Q\psi\,, \tag{2.4}$$

$$Q = \left[\begin{array}{c|c} (\hat q+1)I & \\ \hline & \hat q I \end{array}\right], \tag{2.5}$$

$$C_- = \left[\begin{array}{cc|c} 0 & & 0 \\ \hline -s & c & \\ c & s & 0 \end{array}\right]\quad C_3 = \left[\begin{array}{c|c} I & \\ \hline & -I \end{array}\right], \tag{2.6}$$

where θ_W is the weak interaction angle, $c\equiv\cos\theta$, $s\equiv\sin\theta$ (θ is the Cabibbo angle), $\hat q$ is the charge of the d (or s) quark, I is the two by two unit matrix, and $a_-\equiv(1-\gamma_5)/2$. In Eqs. (2.2)–(2.4), $\psi\equiv\psi_{\alpha,i}$ [$\alpha=c, u, d, s$, and $i=1,2,3$ are the SU(4) and SU(3)^c degrees of freedom, respectively]; the matrices Q, C_-, and C_3 act on the SU(4) indices, and a summation over the color indices is implicit. Thus in the fractionally charged model under consideration J^μ_{el}, J^μ_W, and J^μ_Z are color singlets.[10]

The strong interactions are taken into account to all orders by working in a representation in which the quark and vector gluons satisfy the strong interaction field equations. In particular, the quark fields obey the equation

$$(i\gamma^\mu\partial_\mu - g_S\gamma^\mu T^A S^A_\mu - m)\psi = 0\,, \tag{2.7}$$

where S^A_μ stand for the strongly interacting vector fields, T^A are the SU(3)^c matrices corresponding to the ψ representation [i.e., the triplet representation], and m is the quark mass matrix which may be taken to be real and diagonal. As pointed out by Weinberg (1973b), m involves masses unrenormalized with respect to the strong interactions. They are proportional to a divergent constant which renders finite the hadronic matrix elements of mass terms bilinear in the quark fields.

The weak and electromagnetic currents satisfy the equal-time commutation relations of the associated current algebra. We quote the commutation relations relevant to our analysis

$$[J^0_W(x), J^\mu_Z(x')]_{x^0=x'^0} = \cos^2\theta_W J^\mu_W(x)\delta(\mathbf{x}-\mathbf{x}')\,, \tag{2.8}$$

$$[J^0_W(x), J^\mu_\gamma(x')]_{x^0=x'^0} = J^\mu_W(x)\delta(\mathbf{x}-\mathbf{x}')\,, \tag{2.9}$$

$$[J^0_W(x), J^{\dagger\mu}_W(x')]_{x^0=x'^0} = -J^\mu_3(x)\delta(\mathbf{x}-\mathbf{x}') + \text{S.T.}\,, \tag{2.10}$$

$$J^\mu_3(x) \equiv \bar\psi\gamma^\mu a_- C_3\psi = 2[\sin^2\theta_W J^\mu_\gamma + J^\mu_Z]\,, \tag{2.11}$$

where S.T. represents a c-number "Schwinger term." There are no operator Schwinger terms because the currents are invariant under local SU(3)^c, and in the underlying theory there are no hadronic gauge-invariant operators with dimensions $\leqslant 2$.[11]

It is also convenient to emphasize two important properties of the theory: (a) to zeroth order in g the conservation of the weak currents is broken only by the mass terms of the Lagrangian, so that their divergences are linear combinations of scalars and pseudoscalar densities and (b) these scalar and pseudoscalar densities are local operators in the sense that their equal-time commutators with the fourth components of the weak and electromagnetic currents are given by local operators with δ function coefficients. There are no operator anomalies in these commutators for the reasons given after Eq. (2.11).

Finally, we recall that in quantum chromodynamics the vacuum state is characterized by a phase parameter θ (Marciano and Pagels, 1978). As the strong interactions are P and T preserving, we assume implicitly that the physical vacuum corresponds to $\theta=0$.

III. THREE-CURRENT CORRELATION FUNCTIONS

In this section we discuss the contributions to the decay amplitude associated with the diagrams of Fig. 1. For simplicity we shall restrict ourselves to the case in which the initial and final hadrons are spinless, as this is the case in the most important applications, namely in pion β decay and the superallowed Fermi transitions. Some important aspects of the formulation for spin $\frac{1}{2}$ particles are discussed in Appendix A. For definiteness we discuss in this paper the case of a positron emitter.

It is particularly convenient to carry out the calculations in the t'Hooft–Feynman gauge in which the propagator of a vector meson of mass m_V and momentum p^μ_V takes the simple form $-ig_{\mu\nu}/(p_V^2-m_V^2)$. In this gauge the sum of the amplitudes depicted in Figs. 1(a) and 1(b) can be expressed as

$$\mathfrak{M}_{(1)} = \frac{ig^2}{2}\,\langle p'|J^\mu_W(0)|p\rangle'\,\frac{1}{q^2-m_W^2}\,(\bar u_\nu\gamma_\mu a_- v_e)\,, \tag{3.1}$$

[9]In the present paper the current J^μ_W carries $\Delta Q=-1$ and is the Hermitian adjoint of the current denoted by the same symbol in Sirlin, 1974a, 1974b, and 1975, e is the proton charge, and we otherwise follow the notational conventions of Bjorken and Drell, *Relativistic Quantum Fields* (1965).

[10]In the integer charge model of Pati and Salam (1974) J^μ_{el} contains a color octet part. In this model J^μ_Z obtains an additional contribution which cancels the octet color part of $-\sin^2\theta_W\bar\psi\gamma^\mu Q\psi$ in Eq. (2.4). Regarding this class of theories see remarks in Sec. VIII.

[11]Furthermore, as emphasized by Bég (1975), there are no coefficient anomalies in the once integrated algebra of current components because the currents are partially conserved and the strong interactions are asymptotically free.

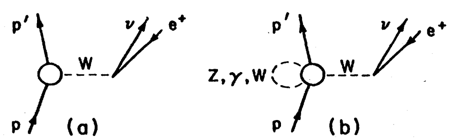

FIG. 1. Zeroth-order amplitude and diagrams involving correlation functions of three hadronic currents. The latter include a subtraction of mass insertions on the external hadronic lines (see Sec. III). Figure 1(b), for example, represents three different diagrams involving the exchange of Z, γ, or W, respectively.

where $\langle p'|J^{\mu}_{W}(0)|p\rangle'$ includes the order α radiative corrections associated with the emission and absorption of virtual γ, Z, and W along the hadronic line, p and p' are the momenta of the initial and final hadrons, and

$$q=p-p' \tag{3.2}$$

is the momentum transfer to the leptons. On general grounds of covariance we have

$$\langle p'|J^{\mu}_{W}(0)|p\rangle'=\tfrac{1}{2}[F_1(q^2)P^{\mu}+F_2(q^2)q^{\mu}], \tag{3.3}$$

where

$$P=p+p'. \tag{3.4}$$

If we now write

$$F_i(q^2)=F_i^{(0)}(q^2)+\sum_a \delta F_i^{(a)}(q^2) \quad (i=1,2;\ a=Z,\gamma,W), \tag{3.5}$$

where $F_i^{(0)}$ are the form factors to zeroth order in α, i.e., in the absence of the perturbation, and $\delta F_i^{(a)}(q^2)$ $(a=Z,\gamma,W)$ are the order α corrections to the form factors arising from the virtual emission and absorption of Z, γ, or W, we have

$$\tfrac{1}{2}[\delta F_{1(Z)}P^{\mu}+\delta F_{2(Z)}q^{\mu}]=\lim_{\bar{q}\to q} iT^{\mu}_{(Z)}(\bar{q},p,p'), \tag{3.6}$$

$$T^{\mu}_{(Z)}\equiv\frac{(g^2+g'^2)}{2(2\pi)^4}\times\int\frac{d^4k}{k^2-m_Z^2}\int d^4y\,e^{i\bar{q}\cdot y}\int d^4x\,e^{ik\cdot x}\times\langle p'|T[J^{\mu}_{W}(y)J^{\lambda}_{Z}(x)J_{Z\lambda}(0)]|p\rangle - B^{\mu}_{(Z)}, \tag{3.7}$$

where $-B^{\mu}_{(Z)}$ subtracts the pole terms at $(p'+\bar{q})^2=m_h^2$, and $(p-\bar{q})^2=m_{h'}^2$ of the first term in Eq. (3.7):

$$B^{\mu}_{(Z)}\equiv-\tfrac{1}{2}[F_1^{(0)}(\bar{q}^2)(2p'^{\mu}+\bar{q}^{\mu})+F_2^{(0)}(\bar{q}^2)\bar{q}^{\mu}]\frac{i}{(p'+\bar{q})^2-m_h^2}\times(\delta m_h^2)_{(Z)}-(\delta m_{h'}^2)_{(Z)}\frac{i}{(p-\bar{q})^2-m_{h'}^2}\times\tfrac{1}{2}[F_1^{(0)}(\bar{q}^2)(2p-\bar{q})^{\mu}+F_2^{(0)}(\bar{q}^2)\bar{q}^{\mu}]. \tag{3.8}$$

In Eqs. (3.7) and (3.8), m_h and $m_{h'}$ stand for the masses of the initial and final hadrons to zeroth order in α (but to all orders in the strong interactions), $p^2=m_h^2$, $p'^2=m_{h'}^2$, and

$$(\delta m_h^2)_{(Z)}\equiv-\frac{(g^2+g'^2)}{2(2\pi)^4}\int\frac{d^4k}{k^2-m_Z^2}\int d^4x\,e^{ik\cdot x}\times\langle p|T[J^{\lambda}_{Z}(x)J_{Z\lambda}(0)]|p\rangle \tag{3.9}$$

with an analogous expression for $(\delta m_{h'}^2)_{(Z)}$. Clearly, $(\delta m_h^2)_{(Z)}$ and $(\delta m_{h'}^2)_{(Z)}$ are the contributions to the mass shifts of the initial and final hadrons arising from virtual Z boson exchange. It is important to note that, because the pole terms have been subtracted, the limiting expression of Eq. (3.6) is well defined. Observe also that in Eqs. (3.7) and (3.8), which are explicitly of order α, p, and p' are constrained to the zeroth-order mass-shells, while in Eq. (3.3), which contains both zeroth- and first-order effects, they lie on the corrected mass shells.

In the case of particular interest in which the current is conserved to zeroth order in α, and p and p' refer to members of the same isospin multiplet, we have the simplifications: $F_2^{(0)}(q^2)=0$ and $m_h=m_{h'}$. Furthermore, for pion β decay and the superallowed Fermi transitions in which the initial and final hadrons belong to an $I=1$ multiplet, $F_1^{(0)}(0)=\sqrt{2}\cos\theta$. As is well known, in order to give a well defined meaning to expressions associated with individual Feynman diagrams, it is necessary to regularize the corresponding integrals with a method consistent with the Ward identities of the theory. An appropriate method is dimensional regularization. Therefore we shall implicitly assume in this paper that expressions associated with individual Feynman diagrams, such as Eq. (3.7), are regularized by dimensional continuation.[12]

Expressions analogous to Eqs. (3.6)–(3.8) hold for the corrections associated with the virtual emission and absorption of γ or W, with obvious modifications. For brevity we shall refer to all the contributions involving the Fourier transforms of three hadronic currents as three-current correlation functions.

On-mass-shell perturbation formulae such as Eqs. (3.6)–(3.8) have been used very often in the past in connection with the analysis of the photonic corrections to matrix elements of the weak interaction currents. Arguments justifying such expressions theoretically were outlined by G. Preparata and W. I. Weisberger (1968; see Appendix A of their paper), and a detailed derivation in the case of parity-conserving perturbations was given by L. S. Brown (1969) in his basic work on first-order corrections to strong interaction amplitudes. In Appendix A we study some aspects of the derivation of the radiative correction formula analogous to Eqs. (3.6)–(3.8) in the more complicated case in which the initial and final hadrons have spin $\frac{1}{2}$: we provide a missing argument in the analysis and, assuming $\mathcal{CP}$ invariance of the perturbing interaction, we generalize the result to the case of parity nonconservation. At the end of the Appendix we discuss briefly the simpler case in which the initial and final hadrons are spinless.

In order to reduce Eqs. (3.6)–(3.8) to a more tractable expression involving two-current correlation functions we contract $T^{\alpha}_{(Z)}$ with $\bar{q}_{\alpha}$ to derive the Ward identity

[12]See, for example, Taylor (1976) and Bardeen (1972) and papers cited therein.

$$\bar{q}_\alpha T^\alpha_{(Z)} = D_{(Z)} - \bar{q}_\alpha B^\alpha_{(Z)} + \int \frac{d^4k}{k^2 - m_Z^2} [v_{(Z)}(\bar{q}+k) + v_{(Z)}(p - p' - k)], \tag{3.10}$$

$$D_{(Z)} \equiv \frac{i(g^2+g'^2)}{2(2\pi)^4} \int \frac{d^4k}{k^2-m_Z^2} \int d^4y e^{i\bar{q}\cdot y} \int d^4x e^{ik\cdot x} \times \langle p'| T[\partial_\alpha J^\alpha_W(y) J^\lambda_Z(x) \times J_{Z\lambda}(0)]|p\rangle, \tag{3.11}$$

$$v_{(Z)}(k) \equiv \frac{ig^2}{2(2\pi)^4} \int d^4x e^{ik\cdot x} \langle p'| T[J^\lambda_W(x) J_{Z\lambda}(0)]|p\rangle, \tag{3.12}$$

where we have used Eq. (2.8) and the relation $(g^2+g'^2)\cos^2\theta_W = g^2$. Differentiating Eq. (3.10) with respect to $\bar{q}_\mu$ and then setting $\bar{q}_\mu = q_\mu$

$$T^\mu_{(Z)}|_{\bar{q}=q} = \left[-\bar{q}_\alpha \frac{\partial}{\partial \bar{q}_\mu} T^\alpha_{(Z)} + \frac{\partial}{\partial \bar{q}_\mu} D_{(Z)} - \frac{\partial}{\partial \bar{q}_\mu}(\bar{q}_\alpha B^\alpha_{(Z)}) + V^\mu_{(Z)}(\bar{q})\right]\Big|_{\bar{q}=q}, \tag{3.13}$$

where

$$V^\mu_{(Z)}(q) = \int \frac{d^4k}{k^2-m_Z^2} \frac{\partial}{\partial k_\mu} v_{(Z)}(q+k). \tag{3.14}$$

The last term in Eq. (3.13) involves, as desired, a two-current correlation function [see Eqs. (3.12) and (3.14)]. On the other hand, the first two terms on the right-hand side of Eq. (3.13) involve derivatives with respect to the momentum transfer of three-current correlation functions. Throughout this paper, contributions of this type will be referred to as residual three-current correlation functions. Finally the third term in Eq. (3.13) is a mass insertion contribution which is best discussed together with $\partial/\partial\bar{q}_\mu D_{(Z)}$, as their poles at $q=\bar{q}$ cancel.

For the contributions associated with the virtual exchanges of γ and W in Fig. 1(b) we find, repeating the steps leading from Eq. (3.6) to Eq. (3.13), the following expressions:

$$\tfrac{1}{2}[\delta F_{1(a)} P^\mu + \delta F_{2(a)} q^\mu] = \lim_{\bar{q}\to q} i T^\mu_{(a)}(\bar{q}, p, p') \quad (a=\gamma, W), \tag{3.15}$$

$$T^\mu_{(a)}|_{\bar{q}=q} = \left[-\bar{q}_\alpha \frac{\partial}{\partial \bar{q}_\mu} T^\alpha_{(a)} + \frac{\partial}{\partial \bar{q}_\mu} D_{(a)} - \frac{\partial}{\partial \bar{q}_\mu}(\bar{q}_\alpha B^\alpha_{(a)}) + V^\mu_{(a)}(\bar{q})\right]\Big|_{\bar{q}=q}, \tag{3.16}$$

where

$$T^\mu_{(\gamma)} = \frac{g^2\sin^2\theta_W}{2(2\pi)^4} \int \frac{d^4k}{k^2} \int d^4y e^{i\bar{q}\cdot y} \int d^4x e^{ik\cdot x} \langle p'| T[J^\mu_W(y) J^\lambda_\gamma(x) J_{\gamma\lambda}(0)]|p\rangle - B^\mu_{(\gamma)}, \tag{3.17}$$

$$D_{(\gamma)} = \frac{ig^2\sin^2\theta_W}{2(2\pi)^4} \int \frac{d^4k}{k^2} \int d^4y e^{i\bar{q}\cdot y} \int d^4x e^{ik\cdot x} \langle p'| T[\partial_\mu J^\mu_W(y) J^\lambda_\gamma(x) J_{\gamma\lambda}(0)]|p\rangle, \tag{3.18}$$

$$V^\mu_{(\gamma)}(q) = \int \frac{d^4k}{k^2} \frac{\partial}{\partial k_\mu} v_{(\gamma)}(q+k), \tag{3.19}$$

$$v_{(\gamma)}(k) = \frac{ig^2\sin^2\theta_W}{2(2\pi)^4} \int d^4x e^{ik\cdot x} \langle p'| T[J^\lambda_W(x) J_{\gamma\lambda}(0)]|p\rangle, \tag{3.20}$$

$$T^\mu_{(W)} = \frac{g^2}{4(2\pi)^4} \int \frac{d^4k}{k^2-m_W^2} \int d^4y e^{i\bar{q}\cdot y} \int d^4x e^{ik\cdot x} \langle p'| T[J^\mu_3(y)(J^{\dagger\lambda}_W(x) J_{W\lambda}(0) + \text{h.c.})]|p\rangle - B^\mu_{(W)}, \tag{3.21}$$

$$D_{(W)} = \frac{ig^2}{4(2\pi)^4} \int \frac{d^4k}{k^2-m_W^2} \int d^4y e^{i\bar{q}\cdot y} \int d^4x e^{ik\cdot x} \langle p'| T[\partial_\alpha J^\alpha_W(y)(J^{\dagger\lambda}_W(x) J_{W\lambda}(0) + \text{h.c.})]|p\rangle, \tag{3.22}$$

$$V^\mu_{(W)}(q) = \int \frac{d^4k}{k^2-m_W^2} \frac{\partial}{\partial k_\mu} v_{(W)}(q+k), \tag{3.23}$$

$$v_{(W)}(k) = -\frac{ig^2}{4(2\pi)^4} \int d^4x e^{ik\cdot x} \langle p'| T[J^\lambda_3(x) J_{W\lambda}(0)]|p\rangle. \tag{3.24}$$

In deriving Eqs. (3.15)–(3.24) we used Eqs. (2.9) and (2.10) and the relation $e^2 = g^2\sin^2\theta_W$. The quantities $B^\mu_{(a)}(a=\gamma, W)$ in Eqs. (3.17) and (3.21) subtract the pole terms in analogy with Eqs. (3.7) and (3.8).

It is to be understood that in Eq. (3.21) contributions of the form

$$\langle 0| T[J^\mu_W(y) J^{\dagger\lambda}_W(x)]|0\rangle \langle p'|J_{W\lambda}(0)|p\rangle,$$

$$\langle 0| T[J^\mu_W(y) J^{\dagger\lambda}_W(0)]|0\rangle \langle p'|J_{W\lambda}(x)|p\rangle,$$

and

$$\langle 0| T[J^{\dagger\lambda}_W(x) J_{W\lambda}(0) + \text{h.c.}]|0\rangle \langle p'|J^\mu_W(y)|p\rangle$$

have been subtracted from the T product. In fact, the first two terms represent corrections to the W propagator, discussed in Sec. V and Appendix E, rather than Fig. 1, while the third term is a nonconnected amplitude. Terms of the latter class are also to be subtracted from Eqs. (3.7), (3.9), (3.11), (3.17), (3.18), and (3.22). With these subtractions understood, the Ward identities are not affected by the c-number Schwinger terms, and the T products introduced in this section are covariant.[13]

The contributions of the two-current correlation functions $V^{\mu}_{(a)}(a=Z,\gamma,W)$ to the amplitude of Fig. 1 are particularly important for later discussion. Denoting these contributions by $\mathcal{V}_{(a)}$ and using Eqs. (3.1)–(3.5) we find

$$\mathcal{V}_{(Z)}=\frac{ig^4}{4(2\pi)^4}\,\frac{1}{(q^2-m_W^2)}\,L_\mu\int\frac{d^4k}{(k^2-m_Z^2)}\,\frac{\partial}{\partial k_\mu}\,T^{\lambda}_{(Z)\lambda}(k)\,, \tag{3.25}$$

$$\mathcal{V}_{(\gamma)}=\frac{ig^4}{4(2\pi)^4}\,\frac{\sin^2\theta_W}{(q^2-m_W^2)}\,L_\mu\int\frac{d^4k}{k^2}\,\frac{\partial}{\partial k_\mu}\,T^{\lambda}_{(\gamma)\lambda}(k)\,, \tag{3.26}$$

$$\mathcal{V}_{(W)}=\mathcal{V}^{(1)}_{(W)}+\mathcal{V}^{(2)}_{(W)}\,, \tag{3.27}$$

$$\mathcal{V}^{(1)}_{(W)}=\frac{ig^4}{4(2\pi)^4}\,\frac{1}{(q^2-m_W^2)}\,L_\mu\int\frac{d^4k}{k^2-m_W^2}\,\frac{\partial}{\partial k_\mu}\,T^{\lambda}_{(Z)\lambda}(k+q)\,, \tag{3.28}$$

$$\mathcal{V}^{(2)}_{(W)}=\frac{ig^4}{4(2\pi)^4}\,\frac{\sin^2\theta_W}{(q^2-m_W^2)}\,L_\mu\int\frac{d^4k}{k^2-m_W^2}\,\frac{\partial}{\partial k_\mu}\,T^{\lambda}_{(\gamma)\lambda}(k+q)\,, \tag{3.29}$$

where

$$L_\mu=\bar{u}_\nu\gamma_\mu a_- v_e\,, \tag{3.30}$$

$$T^{\lambda\rho}_{(Z)}(k)=\int d^4x e^{ik\cdot x}\langle p'|T[J^\lambda_Z(x)J^\rho_W(0)]|p\rangle\,, \tag{3.31}$$

$$T^{\lambda\rho}_{(\gamma)}(k)=\int d^4x e^{ik\cdot x}\langle p'|T[J^\lambda_\gamma(x)J^\rho_W(0)]|p\rangle\,. \tag{3.32}$$

In writing down Eqs. (3.25)–(3.32) we used translational invariance and Eq. (3.2) to simplify the final expressions. Observe that the two terms in the W contribution [Eqs. (3.27)–(3.29)] arise because we have expressed J^λ_3 in Eq. (3.24) in terms of J^λ_Z and J^λ_γ according to Eq. (2.11). As we shall see later, in order to separate out the traditional photonic corrections it is also very convenient to decompose the photon propagator in Eq. (3.17) as follows:

$$\frac{1}{k^2}=\frac{1}{k^2-m_W^2}+\frac{m_W^2}{m_W^2-k^2}\,\frac{1}{k^2}\,. \tag{3.33}$$

Accordingly we write

$$T^{\mu}_{(\gamma)}=T^{\mu}_{(\gamma>)}+T^{\mu}_{(\gamma<)} \tag{3.34}$$

$$\mathcal{V}_{(\gamma)}=\mathcal{V}_{(\gamma>)}+\mathcal{V}_{(\gamma<)}\,, \tag{3.35}$$

where, for instance, $\mathcal{V}_{(\gamma>)}$ and $\mathcal{V}_{(\gamma<)}$ are the contributions to $\mathcal{V}_{(\gamma)}$ arising from the first and second terms of Eq. (3.33), respectively. We may think of $\mathcal{V}_{(\gamma>)}$ as involving a "massive photon," while $\mathcal{V}_{(\gamma<)}$ corresponds to a "massless photon" with an additional convergence factor $m_W^2/(m_W^2-k^2)$.

[13]See the discussion at the end of this section and in Appendix B.

The contributions $\mathcal{V}_{(Z)}$, $\mathcal{V}_{(\gamma>)}$, $\mathcal{V}^{(1)}_{(W)}$, and $\mathcal{V}^{(2)}_{(W)}$ associated with the two-current correlation functions are discussed in Sec. IV in conjunction with the diagrams of Fig. 2. Finally the contributions of the residual three-current correlation functions are studied in Sec. V in combination with tadpole diagrams and the order α counterterms of the theory represented schematically in Fig. 3.

It is instructive at this stage to observe that in the cases of greatest interest, namely pion β decay and the superallowed Fermi transitions, the unperturbed current is the $\Delta S=\Delta C=0, |\Delta Q|=1$ vector current, which is conserved to zeroth order in α (we regard isospin breaking due to the quark mass matrix as being of order α), and the momentum transfer q^μ is very small. In fact, $q^\mu=O(\alpha)$. Inspection of Eq. (3.8) with the simplifications $F_2^{(0)}=0$ and $m_h=m_{h'}$ appropriate to these applications, shows that the third term in Eq. (3.13) is proportional to q. The same is obviously true of the contributions of $B^{\alpha}_{(\gamma)}$ and $B^{\alpha}_{(W)}$ in Eq. (3.16). As $T^{\alpha}_{a}(a=Z,\gamma,W)$ are free from singularities at $\bar{q}=q$ by construction, the first terms on the right-hand side of Eqs. (3.13) and (3.16) are also $O(q)$. Furthermore the effective photonic operator $J^\lambda_\gamma(x)J_{\gamma\lambda}(0)$ is diagonal in charm, strangeness, and parity; therefore in the expression for $D_{(\gamma)}$ in Eq. (3.18) $\partial_\alpha J^\alpha_W(y)$ can only contribute to the Fermi amplitude through the divergence of the $\Delta S=0$ vector current, which vanishes to zeroth order in α. Thus $D_{(\gamma)}=0$ to the order of our calculation, and we find $T^{\mu}_{(\gamma)\bar{q}=0}=V^{\mu}_{(\gamma)}(0)$, which is a well known result (Bjorken, 1966; Abers *et al.*, 1968; Dicus and Norton, 1970; Sirlin, 1967a, 1968a, 1969; Preparata and Weisberger, 1968; Bég *et al.*, 1972; Brown, 1969). In such a case, the photonic corrections associated with Fig. 1 reduce to $\mathcal{V}_{(\gamma)}$, given in Eq. (3.26). On the other hand, it is important to note that the effective operators $J^\lambda_Z(x)J_{Z\lambda}(0)$ and $J^{\dagger\lambda}_W(x)J_{W\lambda}(0)$ +h.c. are not diagonal and therefore $D_{(Z)}$ in Eq. (3.11) and $D_{(W)}$ in Eq. (3.22) can contribute to the Fermi amplitude through the nonconserved pieces of J^μ_W.

Aside from the three-current correlation functions discussed in this section, there are other three-point correlation functions affected by the strong interactions: namely, those involving Higgs scalars. This is illustrated in Figs. 4 and 5 in the case of the simplest version of the Salam–Weinberg model. Note that the diagrams of Figs. 1 through 4 represent radiative corrections to the vertex $Wh'h$ while Fig. 5 represents the corresponding corrections to the vertex $\phi h'h$ (h and h' denote here the initial and final hadrons, and ϕ is the unphysical Higgs scalar associated with W). The corrections involving virtual exchanges of Higgs scalars are studied in Sec. VI. Physically, they are less interesting than the contributions associated with current correlation functions because, subject to the assumptions of our paper, their finite contributions are of order G_F^2 rather than $G_F\alpha$ and can be safely neglected.

It is worth emphasizing that the absence of operator "seagull" terms in the time-ordered products of hadronic operators considered in this paper follows from the same considerations as the absence of operator Schwinger terms, namely the fact that there are no gauge-invariant hadronic operators with dimensions $\leqslant 2$. (See Sec. II.) (Note that in the asymptotically free theory these dimensional considerations are well defined, as the anomalous dimensions vanish in the "zero distance" limit.) There are, in general, c-number seagull terms but, for the amplitudes considered in this section, they are eliminated by the subtractions indicated on page 579. The absence of seagull terms in the present context is discussed in greater detail in Appendix B, where the situation is contrasted with the well known case of the radiative corrections of order α to the vacuum matrix element of the neutral hadronic axial-vector current.

Finally, it is interesting to observe that the analysis of this section is based on the Ward identities rather than the equal-time commutators of the current algebra.

IV. TWO-CURRENT CORRELATION FUNCTIONS

In this section we study the various contributions arising from the hadronic two-current correlation functions, including small effects of $O[\bar{g}_S^2(\kappa^2)]$ induced by the strong interactions in the asymptotic domain. We follow the general approach presented, rather schematically, in Sirlin (1974a and 1974b).

In Sec. IV.A we combine $\mathcal{V}_{(Z)}$, $\mathcal{V}_{(\gamma>)}$, and $\mathcal{V}_{(W)}$, discussed in Sec. III, with the diagrams of Fig. 2. We show that their divergent contributions reduce to a universal renormalization of the weak coupling constant g and, using the properties of the underlying theory of strong interactions, we evaluate their finite parts. In Sec. IV.B we discuss briefly the separation of the photonic corrections. We show that $\mathcal{V}_{(\gamma<)}$, introduced in Sec. III, combines with the photon exchange diagram in Fig. 6(a) and part of the electron field renormalization, to give rise to the photonic corrections to the Fermi amplitude as computed in the local V-A theory with an effective cutoff equal to m_W. We briefly recall some of the main features of the photonic corrections and point out that in the present context the coefficient of the asymptotic part of the model-dependent contribution induced by the axial-vector current can be rigorously determined. Finally in Sec. IV.C we study the Z exchange diagrams of Figs. 6(a), (b). Here we distinguish two different situations: (a) the case of the simplest version of the Salam–Weinberg model, and (b) the case of arbitrary symmetry breaking.

A. Two-current correlation functions associated with the vertex $Wh'h$

In this section we study the contributions to the vertex $Wh'h$ involving two-current correlation functions.[14]

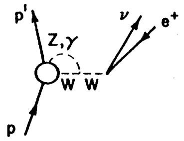

FIG. 2. Diagrams involving two-current correlation functions that contribute to the vertex $Wh'h$ (h,h' are the initial and final hadrons).

These are the diagrams of Fig. 2 and $\mathcal{V}_{(a)}(a=Z,\gamma,W)$ discussed in Sec. III. The amplitude for the Z exchange diagram in Fig. 2 is given in the t'Hooft–Feynman gauge by

$$\mathcal{U}_{(Z)}=-\frac{ig^4}{2(2\pi)^4}\,\frac{1}{q^2-m_W^2}\,L^\mu\int\frac{d^4k}{(k^2-m_Z^2)[(k-q)^2-m_W^2]}\times V_{\mu\lambda\rho}(k,q)T_{(Z)}^{\lambda\rho}(k), \quad (4.1)$$

where L^μ and the tensor $T_{(Z)}^{\lambda\rho}(k)$ are defined in Eqs. (3.30) and (3.31) and

$$V_{\mu\lambda\rho}(k,q)=(2k-q)_\mu g_{\lambda\rho}+(2q-k)_\lambda g_{\mu\rho}-(k+q)_\rho g_{\mu\lambda}\,. \quad (4.2)$$

To proceed further we need an estimate of the asymptotic behavior of the two-current correlation function $T_{(Z)}^{\lambda\rho}(k)$ for large k. The leading behavior for large k is determined by the operator of lowest dimensionality in the short-distance expansion of $T[J_Z^\lambda(x)J_W^\rho(0)]$ (Wilson, 1969). As this time-ordered product carries one unit of electric charge, it is easy to see that the operator of lowest dimensionality must be bilinear in the quark fields.[15] To any finite order of perturbation theory in a renormalizable theory of strong interactions the asymptotic behavior is determined, modulo powers of logarithms, by dimensional analysis in the scale-invariant limit. In an asymptotically free theory the discussion of the asymptotic behavior rests on a more secure basis, as the renormalization group approach allows one to draw conclusions independent of perturbation theory. The general qualitative result is, however, the same: modulo powers of logarithms the asymptotic behavior is determined by dimensional analysis in the scale-invariant limit. An elementary consideration shows then that, up to powers of logarithms, $T_{(Z)}^{\lambda\rho}(k)$ behaves as k^{-1} for large k. Inspection of Eqs. (4.1) and (4.2) tells us that the k integration diverges logarithmically. This implies that all contributions proportional to the momentum transfer q in Eq. (4.1) are finite and, in fact, of $O(G_F^2)$ rather than $O(G_F\alpha)$. Thus we can safely set $q=0$ in Eq. (4.1). Furthermore, we can use the Ward identities to study the contractions of the two-current correlation function in Eq. (4.1) with k_μ and k_σ.[16] In this way we obtain

$$\mathcal{U}_{(Z)}=-\frac{ig^4}{2(2\pi)^4}\,\frac{1}{q^2-m_W^2}\,L^\mu\int\frac{d^4k}{(k^2-m_Z^2)(k^2-m_W^2)}\Big\{2k_\mu T_{(Z)\lambda}^{\lambda}(k)+i\,2\cos^2\theta_W\langle p'|J_{W\mu}(0)|p\rangle$$
$$-i\int d^4x e^{ik\cdot x}\langle p'|T[\partial_\sigma J_Z^\sigma(x)J_{W\mu}(0)]|p\rangle$$
$$+i\int d^4x e^{-ik\cdot x}\langle p'|T[J_{Z\mu}(0)\partial_\sigma J_W^\sigma(x)]|p\rangle\Big\}\,, \quad (4.3)$$

[14]See discussion on page 578.

[15]This follows also from the requirement that the operator be color gauge invariant.

[16]The connection of some of the Ward identities associated with current algebra and the Slavnov–Taylor identities has been discussed by de Witt (1974).

where we have used Eq. (2.8) and the analogous expression for $[J_Z^0(x), J_{W\rho}(x')]$ at $x^0 = x'^0$. We now observe that in the underlying theory of strong interactions J_Z^μ and J_W^μ are partially conserved currents (i.e., their conservation is broken only by quark mass terms) and therefore their divergences are bilinear in the quark fields. Dimensional analysis tells us then that the correlation functions involving $\partial_\mu J_Z^\mu$ and $\partial_\sigma J_W^\sigma$ in Eq. (4.3) behave asymptotically as k^{-1} modulo powers of logarithms and that, therefore, their contributions are $O(G_F^2)$ rather than $O(G_F\alpha)$. Thus, to the order of our calculation, we may retain the first two terms of Eq. (4.3).

A similar analysis leads to the following expression for the γ-exchange diagram in Fig. 2

$$\mathfrak{U}_{(\gamma)} = -\frac{ig^4\sin^2\theta_W}{2(2\pi)^4}\frac{1}{q^2-m_W^2}L^\mu \times\int\frac{d^4k}{k^2(k^2-m_W^2)}\{2k_\mu T^\lambda_{(\gamma)\lambda}(k) + 2i\langle p'|J_{W\mu}(0)|p\rangle\},\tag{4.4}$$

where $T^{\lambda\rho}_{(\gamma)}(k)$ is defined in Eq. (3.32). Performing a partial integration in Eqs. (3.25)–(3.29) (the surface terms vanish on account of the regularization method) and combining judiciously the contributions of $\mathfrak{V}_{(Z)}$, $\mathfrak{V}_{(\gamma)}$, $\mathfrak{V}_{(W)}$, $\mathfrak{U}_{(Z)}$, and $\mathfrak{U}_{(\gamma)}$ we find

$$\mathfrak{V}_{(Z)} + \mathfrak{V}^{(1)}_{(W)} + \mathfrak{U}_{(Z)} = \frac{ig^4}{(2\pi)^4}\frac{L_\mu}{(q^2-m_W^2)} \times\left\{\frac{1}{2}\int d^4k k^\mu\left[\frac{1}{k^2-m_W^2} - \frac{1}{k^2-m_Z^2}\right]^2 \times T^\lambda_{(Z)\lambda}(k) + I^\mu_{(1)}\right\},\tag{4.5}$$

$$\mathfrak{V}_{(\gamma>)} + \mathfrak{V}^{(2)}_{(W)} + \mathfrak{U}_{(\gamma)} = \frac{ig^4\sin^2\theta_W L_\mu}{(2\pi)^4(q^2-m_W^2)} \times\left\{\int\frac{d^4k k^\mu}{(k^2-m_W^2)}\left[\frac{1}{k^2-m_W^2} - \frac{1}{k^2}\right] \times T^\lambda_{(\gamma)\lambda}(k) + I^\mu_{(2)}\right\},\tag{4.6}$$

where

$$I^\mu_{(1)} = -i\cos^2\theta_W\langle p'|J^\mu_W(0)|p\rangle\int\frac{d^4k}{(k^2-m_Z^2)(k^2-m_W^2)},\tag{4.7}$$

$$I^\mu_{(2)} = -i\langle p'|J^\mu_W(0)|p\rangle\int\frac{d^4k}{k^2(k^2-m_W^2)}.\tag{4.8}$$

In these expressions we have systematically neglected terms of $O(G_F^2)$. Inspection of Eqs. (4.5) and (4.6) in conjunction with the asymptotic estimates previously described reveals a rather striking result: the contributions of the two-current correlation functions have combined to give a finite answer! The terms involving $I^\mu_{(1)}$ and $I^\mu_{(2)}$ are divergent; however they are exactly proportional to the zeroth-order amplitude, and the coefficient of proportionality is universal, i.e., it is independent of the strong interactions and the nature of the initial and final particles h and h'. They can therefore be absorbed in a universal renormalization of the weak coupling constant (Sirlin, 1974a). In fact, it has been explicitly verified by Marciano and Sirlin (1975) that $I^\mu_{(1)}$ and $I^\mu_{(2)}$ coincide with the corresponding divergent contributions to the renormalization of g encountered in purely leptonic calculations.

We now turn our attention to the evaluation of the finite contributions involving the two-current correlation functions in Eqs. (4.5) and (4.6). We first note that the only contributions in these expressions which are of $O[G_F\alpha]$ rather than $O[G_F^2]$ are those arising from the terms in the two-current correlation functions which vanish as $k\to\infty$ no faster than $1/k$. These are precisely the leading terms in the short-distance expansion of the time-ordered products of the two currents and involve operators of dimension three. In fact, the next-to-leading terms are proportional to operators of dimension four; their contribution to the Fourier transforms behave for large k as $(1/k^2)$ modulo powers of logarithms, and it can be readily ascertained that they lead to terms of $O[G_F^2]$ rather than $O[G_F\alpha]$. On the other hand, the operators of lowest dimensionality must (i) be bilinear in the quark fields, (ii) be color singlets, (iii) carry one negative unit of charge, (iv) transform as Lorentz vectors and axial vectors because otherwise their contributions to Eqs. (4.5) and (4.6) would vanish after the x and k integrations are performed, and (v) have the appropriate quantum numbers to have nonvanishing matrix elements between the initial and final hadrons. Thus the leading operators in the short-distance expansion are currents and we can therefore write for large k:[17]

$$T^\lambda_{(Z)\lambda}(k) = \frac{k^\mu}{k^2}\sum_N C_N(-k^2, g_{SR}, m_R, \sigma)\langle p'|O^N_\mu(0)|p\rangle + \cdots,\tag{4.9}$$

where m_R and g_{SR} are the renormalized quark mass matrix and strong interaction coupling constant, σ is the renormalization point, the dots represent operators that either do not contribute to Eq. (4.5) or contribute to order G_F^2, and the N summation is over all currents $O^N_\mu(0)$ which satisfy properties (i) through (v) listed above.[18] As we shall perform a Wick rotation later on, it is sufficient to study the asymptotic behavior of the coefficient functions C_N for large Euclidean k^μ. In this regard it is very important to observe that the operators on both sides of Eq. (4.9) are conserved or partially conserved currents and, therefore, have no anomalous dimensions. This is a very welcome fact because it means that in the underlying asymptotically free theory, the two-current correlation functions approach the free field behavior as $\kappa^2 \equiv -k\to\infty$ without the logarithmic corrections associated with anomalous dimensions! In particular, in the

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[17] Setting $k^\mu = \lambda\hat{k}^\mu$ where $\hat{k}$ is a fixed four-vector, the limit of large k means the limit of large λ. In the expansion of Eq. (4.9) k^2 is assumed to be large, of $O(\lambda^2)$. [See Sec. VII. F of Wilson (1969).]

[18] In the particular case of the corrections to the Fermi amplitude, the 0^μ_N must be color singlet vector currents with $\Delta Q = -1$ and $\Delta S = \Delta C = 0$. Moreover, because in the assumed SU(4) $\times$SU(3)^c theory only one such current can be constructed (from the u and d quarks), the N summation reduces really to a single term, namely the $\Delta Q = -1$, $\Delta S = \Delta C = 0$ vector current of the weak interactions. It is convenient to use, however, the notation of Eqs. (4.9)–(4.12) as our arguments may then be applied to more general amplitudes.

deep Euclidean region the quark mass terms can be neglected and the dimensionless Wilson coefficients C_N tend to the limit

$$C_N(\kappa^2, g_{SR}, m_R, \sigma) \to C_N(\mu^2, \bar{g}_S(\kappa^2), \sigma) \tag{4.10}$$

where $\bar{g}_S(\kappa^2)$ is the effective strong interaction coupling constant and μ is a fixed momentum.[19] In an asymptotically free theory, as $\kappa \to \infty$, $\bar{g}_S \to 0$ and $C_N(\bar{g}_S) \to C_N(0) = C_{\rm free}$. An elementary free field theory calculation shows that in the scale-invariant limit as $\bar{g}_S \to 0$

$$\frac{k_\mu}{k^2} \sum_N C_N(0) O_N^\mu(0) = 2i \cos^2\theta_W \frac{k_\mu}{k^2} J_W^\mu(0). \tag{4.11}$$

Equation (4.11) can also be obtained using the Bjorken–Johnson–Low limit (Bjorken, 1966; Johnson and Low, 1966) with canonical evaluation of commutators, a procedure which is of course valid in the free field theory case. Thus all the $C_N(0)$ vanish except the one corresponding to $O_N^\mu = J_W^\mu$. Calling $A_{(1)}$ the first term between curly brackets in Eq. (4.5) and taking into account Eq. (4.9) we obtain

$$A_{(1)}^\mu = \frac{i}{8} \int d^4\kappa \left[\frac{1}{\kappa^2 + m_W^2} - \frac{1}{\kappa^2 + m_Z^2}\right]^2 \times \sum_N C_N(\kappa^2, g_{SR}, m_R, \sigma) \langle p' | O_N^\mu(0) | p\rangle + \cdots, \tag{4.12}$$

where we have performed a Wick rotation. At this stage we invoke our assumption that the onset of the asymptotic behavior occurs at $M^2 \ll m_W^2$. We note that the domain of integration $\kappa^2 \leq M^2$ gives contributions of higher order in G_F. In the region $\kappa^2 \geq M^2$ we set $\bar{g}_S = 0$ and, recalling Eqs. (4.10) and (4.11), find

$$A_{(1)}^\mu = \frac{-\pi^2}{4} \cos^2\theta_W \langle p' | J_W^\mu(0) | p\rangle \times \int_0^\infty \kappa^2 d\kappa^2 \left[\frac{1}{\kappa^2 + m_W^2} - \frac{1}{\kappa^2 + m_Z^2}\right]^2$$

$$= \frac{\pi^2}{4} \cos^2\theta_W \left[\frac{1+R}{1-R} \ln R + 2\right] \langle p' | J_W^\mu(0) | p\rangle, \tag{4.13}$$

where

$$R = m_W^2 / m_Z^2. \tag{4.14}$$

Calling $A_{(2)}^\mu$ the first term between curly brackets in Eq. (4.6), a similar analysis leads to

$$A_{(2)}^\mu = \frac{\pi^2}{2} m_W^2 \langle p' | J_W^\mu(0) | p\rangle \int_0^\infty d\kappa^2 \frac{1}{(\kappa^2 + m_W^2)^2}$$

$$= \frac{\pi^2}{2} \langle p' | J_W^\mu | p\rangle, \tag{4.15}$$

where we have again neglected terms of higher order in G_F.

It is important to observe that the contributions of Eqs. (4.13) and (4.15) are independent of the quark charges $\hat{q}$ introduced in Eq. (2.5). In fact, examination of the arguments leading to these results shows that all of these contributions are universal multiples of the zeroth-order amplitude, independent of the nature of the initial and final hadrons. As we shall see later, analogous contributions arise in muon decay. Thus in studying the ratio of decay probabilities of two different weak processes, we find that such contributions cancel and have no physical consequences, a state of affairs which was anticipated by more heuristic reasoning in Sirlin (1974b, 1974c, 1975). On the other hand, it is interesting to study the error made by setting $\bar{g}_S = 0$ in the calculations of the finite parts, as contributions depending on $\bar{g}_S^2$ are clearly nonuniversal. As explained in Appendix C, if we keep terms of this order and assume that the color group is $SU(3)^c$, in the region $\kappa^2 \geq M^2$ the integrands of Eqs. (4.13) and (4.15) must be multiplied by a factor $1 - \bar{g}_S^2(\kappa^2)/(4\pi^2)$. We recall that in the $SU(3)^c$ theory

$$\frac{\bar{g}_S^2(\kappa^2)}{4\pi^2} = \frac{g_{SR}^2}{4\pi^2}\left[1 + (11 - \tfrac{2}{3} f) \frac{g_{SR}^2}{16\pi^2} \ln\left(\frac{\kappa^2}{\mu^2}\right)\right]^{-1}, \tag{4.16}$$

where f is the number of quark flavors, and μ is the momentum at which $g_{SR}^2 = \bar{g}_S^2(\mu^2)$ is defined. Equation (4.16) is approximately valid provided that $\bar{g}_S^2(\kappa^2)/(4\pi^2)$ is sufficiently small. We take $\mu^2 = O(M^2) \ll m_W^2$. As $\bar{g}_S^2(\kappa^2)$ is a slowly varying function of κ, we may estimate its effect by replacing $\bar{g}_S^2(\kappa^2)/(4\pi^2)$ with $\bar{g}_S^2(K^2)/(4\pi^2)$, where K is a constant of $O(m_W)$. Setting $f = 4$ appropriate to SU(4), $K \approx m_W \approx 63$ GeV (which is the value of m_W in the simplest version of the Weinberg–Salam model for $\sin^2\theta_W = 0.35$), and using the values of g_{SR} and μ indicated by Altarelli *et al.* (1976), we obtain $\bar{g}_S^2(m_W^2)/(4\pi^2) = 0.0425$. Instead, if we apply the parametrization suggested by Barnett *et al.* (1976), $\bar{g}_S^2(m_W^2)/(4\pi^2) = 0.0496$. Thus we expect the $\bar{g}_S^2$ contributions to $A_{(1)}^\mu$ and $A_{(2)}^\mu$ to be approximately on the order of -5% relative to the results of Eqs. (4.13) and (4.15). In Appendix C we show how the effect of the terms of order $\bar{g}_S^2(\kappa^2)$ in the radiative corrections integrals can be calculated in a precise manner.[20] It is worthwhile to point out that the $\bar{g}_S^2$ corrections to $A_{(1)}^\mu$ and $A_{(2)}^\mu$ are very small in most applications. In fact, inserting Eqs. (4.13) and (4.15) into Eqs. (4.5) and (4.6), we see that to zeroth order in $\bar{g}_S$ the universal contributions of $A_{(1)}^\mu$ and $A_{(2)}^\mu$ to the amplitude are given by

$$\mathfrak{M}_{A_{(1)}} = \frac{\alpha}{8\pi} \mathfrak{M}^0 \cot^2\theta_W \{2 + [(1+R)/(1-R)] \ln R\} \tag{4.17}$$

and

$$\mathfrak{M}_{A_{(2)}} = \frac{\alpha}{4\pi} \mathfrak{M}^0, \tag{4.18}$$

respectively, where $\mathfrak{M}^0$ is the zeroth-order amplitude. Note that unless $\sin^2\theta_W \ll 1$ and or R is very different from 1, $\mathfrak{M}_{A_{(1)}}$ and $\mathfrak{M}_{A_{(2)}}$ are quite small. For instance,

[19] In Weinberg's approach to the renormalization group equations [see Politzer (1974), Weinberg (1973b), and papers cited therein] one may write $C_N(\kappa^2, g_{SR}, m_R, \sigma) = C_N(\mu^2, \bar{g}_S(\kappa^2), \bar{m}_S(\kappa), \sigma)$, where $\bar{m}_S(\kappa)$ is the effective quark mass matrix. In asymptotically free theories, for large κ, $\bar{m}_S(\kappa) \sim (1/\kappa)$ modulo powers of logarithms, so that asymptotically $\bar{m}_S$ can be neglected and we obtain Eq. (4.10). Note that $\bar{g}_S(\kappa^2)$ is really a function of κ^2/μ^2, as shown explicitly in Eq. (4.16).

[20] In particular, the $\bar{g}_S^2$ correction to $A_{(1)}^\mu$ is -0.0436 relative to the uncorrected integral of Eq. (4.13), rather than the naive estimate of -0.0425 given above.

in the simplest version of the model with $\sin^2\theta_W = 0.35$, they give rise to corrections of -3×10^{-5} and 1.2×10^{-3} to the transition probability, respectively. Clearly the $\bar{g}_S^2$ corrections, being -4.4% of the universal contributions, are extremely small in this case.

Thus far we have discussed all the two-current correlation functions associated with the vertex $Wh'h$ except the amplitude $\mathcal{V}_{(\gamma<)}$ introduced in Eq. (3.34). The latter will be discussed in Sec. IV.B in conjunction with the other photonic corrections.

B. Photonic corrections

The contribution of the photon box diagram in Fig. 6(a) is given by

$$\mathfrak{M}^{\text{box}}_{(\gamma)} = -\frac{ig^2e^2}{2m_W^2(2\pi)^4}\int d^4k\,\frac{1}{k^2}\left[\frac{m_W^2}{m_W^2-(k-q)^2}\right]T^{\lambda\rho}_{(\gamma)}(k) \times \bar{u}_\nu\gamma_\rho a_-\frac{1}{\not{k}-\not{l}-m_e}\gamma_\lambda v_e\,, \tag{4.19}$$

where l is the four-momentum of the outgoing positron, and the tensor $T^{\lambda\rho}_{(\gamma)}(k)$ is defined in Eq. (3.32). Using the asymptotic estimates we see that Eq. (4.19) is ultraviolet convergent. Neglecting the small q dependence in the W propagator, the factor in square brackets becomes $m_W^2/(m_W^2-k^2)$, the familiar Feynman cutoff function. It is then clear that Eq. (4.19) is the same as the photon exchange corrections calculated in the local theory with an effective Feynman cutoff equal to m_W. It is important to note that if we specialize the discussion to the corrections to the Fermi amplitude in the limit $q\to0$, an analogous statement is true for the contribution of $\mathcal{V}_{(\gamma<)}$. In fact, in Sec. III we pointed out that in this case the photon exchange corrections associated with Fig. 1 reduce to $\mathcal{V}_{(\gamma)}$ (with q set equal to zero). Our statement follows then from the observation that $\mathcal{V}_{(\gamma<)}$ is obtained from $\mathcal{V}_{(\gamma)}$ by means of the replacement

$$\frac{1}{k^2}\to\frac{1}{k^2}\,\frac{m_W^2}{m_W^2-k^2}\,.$$

We now consider the photonic corrections $\mathfrak{M}^{(e)}_{(\gamma)}$ associated with the field renormalization of the outgoing charged lepton. We again decompose the photon propagator according to Eq. (3.33) and write

$$\mathfrak{M}^{(e)}_{(\gamma)} = \mathfrak{M}^{(e)}_{(\gamma>)} + \mathfrak{M}^{(e)}_{(\gamma<)}\,, \tag{4.20}$$

where $\mathfrak{M}^{(e)}_{(\gamma>)}$ and $\mathfrak{M}^{(e)}_{(\gamma<)}$ are the contributions arising from the first and second terms of Eq. (3.33).[21] Equation (4.20) is, of course, the analog of Eq. (3.35). To complete the separation of the photonic corrections we must include the inner bremsstrahlung diagrams. As the only additional diagram involves the emission of a real photon by the intermediate boson (or intermediate ϕ), and the corresponding contributions are of higher order in G_F, the bremsstrahlung diagrams are the same as in the local theory.

If we restrict our attention to the corrections to the Fermi amplitude and neglect terms of $O(q)$, it is by now obvious that the contributions of $\mathcal{V}_{(\gamma<)}+\mathfrak{M}^{\text{box}}_{(\gamma)}+\mathfrak{M}^{e}_{(\gamma<)}$ to the transition probability, combined with the inner bremsstrahlung diagrams, lead to the same results as the calculation of the photonic corrections in the local theory, with the understanding that the cutoff that appears in such calculations is not a regulator parameter but is to be identified with m_W.[22]

The current algebra analysis of the photonic corrections to the Fermi amplitude in the local V-A theory has been carried out in considerable detail in the literature (Abers *et al.*, 1968; Dicus and Norton, 1970; Sirlin, 1967a, 1968a, 1969; Beg *et al.*, 1972). We recall that to zeroth order in the lepton momenta the photonic corrections arising from the vector current are independent of the dynamics of the strong interactions.[23] As the cutoff is replaced by m_W, which is $\geqslant 37.3$ GeV in the SU(2) $\times$U(1) gauge model, these corrections are indeed very large. On the other hand, the contributions induced by the axial-vector current are model dependent. We will show now that in an asymptotically free theory the coefficient of the asymptotic part of the latter corrections, i.e., the part proportional to $\ln m_W$, depends on the average charge $\bar{Q}$ of the u and d quarks but otherwise is independent of dynamical details. To this end we recall that these corrections arise from the $|\Delta Q|=1$, $\Delta S=\Delta C=0$ axial-vector current A^λ_W. Replacing J^λ_W by $-A^\lambda_W\cos\theta/2$ and setting $l=q=m_e=0$, Eq. (4.19) becomes

$$\mathfrak{M}^{\text{box}}_{(A;\gamma)} = \frac{-g^2e^2\cos\theta}{4(2\pi)^4}\,L^\mu\epsilon_{\sigma\lambda\rho\mu}\int\frac{d^4k}{k^4}\,\frac{k^\sigma}{(m_W^2-k^2)}A^{\lambda\rho}(k)\,, \tag{4.21}$$

where we have neglected all the terms which cannot contribute to the Fermi amplitude in the limit $q\to0$ and

$$A^{\lambda\rho}(k) = \int d^4x\,e^{ik\cdot x}\langle p'|T[J^\lambda_\gamma(x)A^\rho_W(0)]|p\rangle\,. \tag{4.22}$$

The subscript $(A;\gamma)$ reminds us that Eq. (4.21) represents the photonic corrections induced by the axial-vector current. Because the integral involves only one massive propagator, all the terms in the short-distance expansion of the time-ordered product contribute to order $G_F\alpha$. On the other hand, inspection of Eq. (4.21) shows that only the leading term behaves as k^{-1} for large k and can, therefore, contribute to order $G_F\alpha\ln m_W$. As it involves again partially conserved currents with no anomalous dimensions, we can calculate the corresponding corrections in a way similar to that explained in Sec. IV.A.[24] We write

$$A^{\lambda\rho}(k) = \epsilon^{\lambda\rho\alpha\beta}\frac{k_\alpha}{k^2-M^2}\sum_N d_N(-k^2, g_{SR}, m_R, \sigma) \times\langle p'|O^N_\beta(0)|p\rangle + \cdots\,, \tag{4.23}$$

[21] $\mathfrak{M}^{(e)}_{(\gamma>)}$ is a universal contribution analogous to $\mathcal{V}_{(\gamma>)}$ and it is best discussed in conjunction with the W and Z contributions to the νeW vertex.

[22] In the calculations in the local theory it is customary to neglect terms of order $G_F\alpha/\Lambda^2$ where Λ is the cutoff. In the present context such terms are finite contributions of $0(G_F\alpha/m_W^2) = 0(G_F^2)$ which are also negligible in the approach of the present paper.

[23] There is, however, a very small correction of $0[\bar{g}_S^2]$ which is discussed at the end of Appendix C.

[24] See also the discussion in Sec. V. B of Abers *et al.*, (1968).

where the O^N_β are appropriate vector currents and the dots indicate terms which do not contribute to the Fermi amplitude to order $G_F\alpha\ln m_W$. In Eq. (4.23) we have introduced the hadronic mass M to avoid the infrared divergence in Eq. (4.21). We now insert Eq. (4.23) into (4.21) and perform a Wick rotation. As $M^2 \ll m_W^2$ it is easy to see that the region $\kappa^2 \leq M^2$ does not contribute to order $G_F\alpha\ln m_W$,[25] while in region $\kappa^2 \geq M^2$ we can set $\bar{g}_S=0$. Thus, if we limit ourselves to calculating the terms of order $G_F\alpha\ln m_W$, we can work in the free field theory limit. An elementary calculation shows that in the scale-invariant limit as $\bar{g}_S \to 0$, the sum in Eq. (4.23) becomes

$$\sum_N d_N(\bar{g}_S=0)O^N_\beta(0) = -2\overline{Q}V_{W\beta}(0)\,, \tag{4.24}$$

where

$$\overline{Q}=(2\hat{q}+1)/2 \tag{4.25}$$

is the average charge of the u and d quarks and $V^\alpha_W(0)$ is the $\Delta S=\Delta C=0$ vector current. Inserting Eqs. (4.23) and (4.24) into Eq. (4.21) we finally obtain:

$$\mathfrak{M}^{\text{box}}_{(A;\gamma)} = -i\frac{\alpha}{16\pi}g^2\cos\theta L^\mu V_{W\mu}3\overline{Q}\int_{M^2}^\infty \frac{d\kappa^2}{(m_W^2+\kappa^2)\kappa^2}+\cdots$$
$$=\frac{\mathfrak{M}^0\alpha}{4\pi}6\overline{Q}\ln\left(\frac{m_W}{M}\right)+\cdots\,, \tag{4.26}$$

where the dots represent the nonasymptotic corrections of order $G_F\alpha$ induced by the axial-vector current. Equation (4.26) is, of course, a familiar result, first obtained in the framework of the local V-A theory (Abers *et al.*, 1968). As expected, the cutoff of the local theory calculation has been replaced by m_W. For a discussion of the nonasymptotic part of these corrections we refer the reader to Abers *et al.* (1968), Dicus and Norton (1970), Sirlin (1968a, 1969), and to Sec. VII.B of the present paper. If it is of order $(\alpha/2\pi)$ as indicated by the estimates of these papers and by the fact that no large logarithms are present, its contribution is indeed much smaller than the very large model-independent corrections arising from the vector current. In obtaining Eq. (4.26) we have set $\bar{g}_S=0$. As shown in Appendix C, if we keep terms of order $\bar{g}_S^2$ and assume that the color group is $\mathrm{SU}(3)^c$, the integrand of Eq. (4.26) must be multiplied by a factor $1-\bar{g}_S^2(\kappa^2)/(4\pi^2)$. Because Eq. (4.26) involves only one massive denominator and is sensitive to the lower as well as the higher domain of integration, in order to calculate the contribution of $\bar{g}_S^2(\kappa^2)$ it is necessary to carry out a detailed study. Choosing for simplicity $M=\mu$ [see Eq. (4.16)], this is done in Appendix C. It is worthwhile to note that the corrections to Eq. (4.26) induced by $\bar{g}_S^2$ contain a term of the form $\ln[\ln(m_W^2/\mu^2)]$. Using the parametrization of Altarelli *et al.* (1976) with $m_W\approx 63$ GeV and $M=\mu\approx 1$ GeV, we find that the $\bar{g}_S^2$ contributions are approximately of order -7.6% relative to the result of Eq. (4.26). For $\overline{Q}=\frac{1}{6}$ and the same values of m_W and M, Eq. (4.26) leads to a 4.8×10^{-3} correction to the transition probability. Therefore the $\bar{g}_S^2$ induced corrections to the lifetime are of order -3.6×10^{-4}, which is indeed very small.

[25]Note, however, that this region does contribute to order $G_F\alpha$ because Eq. (4.21) involves only one massive denominator.

C. Z-exchange box diagrams

The amplitude for the Z-exchange box diagram of Fig. 6(a) is given by

$$\mathfrak{M}^{\text{box}}_{(Z;a)} = -\frac{i(g^2+g'^2)g^2}{4(2\pi)^4}\int d^4k\frac{1}{[k^2-m_Z^2][(k-q)^2-m_W^2]}$$
$$\times T^{\lambda\rho}_{(Z)}(k)\,\bar{u}_\nu\gamma_\rho a_-\frac{1}{\not{k}-\not{l}-m_e}$$
$$\times\gamma_\lambda(2\sin^2\theta_W-a_-)v_e\,, \tag{4.27}$$

where $T^{\lambda\rho}_{(Z)}(k)$ is defined in Eq. (3.31). We have already seen in Sec. IV.A that $T^{\lambda\rho}_{(Z)}(k)\sim k^{-1}$ for large k. Thus Eq. (4.27) is convergent and, because it involves two massive denominators, the contributions proportional to the lepton momenta and mass are $O[G_F^2]$. Setting $l=q=m_e=0$ and using the familiar relation

$$\gamma_\rho\gamma_\sigma\gamma_\lambda = g_{\rho\sigma}\gamma_\lambda - g_{\rho\lambda}\gamma_\sigma + g_{\sigma\lambda}\gamma_\rho - i\epsilon_{\rho\sigma\lambda\alpha}\gamma^\alpha\gamma_5 \tag{4.28}$$

Eq. (4.27) reduces to

$$\mathfrak{M}^{\text{box}}_{(Z;a)} = -\frac{i(g^2+g'^2)}{4(2\pi)^4}g^2(2\sin^2\theta_W-1)$$
$$\times\int\frac{d^4k\,T^{\lambda\rho}_{(Z)}(k)}{k^2[k^2-m_Z^2][k^2-m_W^2]}$$
$$\times\bar{u}_\nu[k_\rho\gamma_\lambda - g_{\lambda\rho}\not{k}+k_\lambda\gamma_\rho+i\epsilon_{\lambda\rho\sigma\alpha}k^\sigma\gamma^\alpha]a_-v_e\,. \tag{4.29}$$

The contractions of the tensor $T^{\lambda\rho}_{(Z)}(k)$ with k_ρ and k_λ can be evaluated using the associated Ward identities: one obtains terms proportional to $\langle p'|J^\lambda_W|p\rangle$ and $\langle p'|J^\rho_W|p\rangle$ arising from the equal-time commutators and terms involving the Fourier transforms of $T[\partial_\lambda J^\lambda_Z(x)J_{W\rho}(0)]$ and $T[J^\lambda_Z(x)\partial_\rho J^\rho_W(0)]$. By an argument identical to that explained after Eq. (4.3) one shows that the latter contributions are $O[G_F^2]$, and we finally obtain

$$\mathfrak{M}^{\text{box}}_{(Z;a)} = \frac{i(g^2+g'^2)g^2\cos(2\theta_W)}{4(2\pi)^4}L^\mu$$
$$\times\int\frac{d^4k}{k^2[k^2-m_Z^2][k^2-m_W^2]}$$
$$\times[-2i\cos^2\theta_W\langle p'|J_{W\mu}(0)|p\rangle$$
$$-k_\mu T^\lambda_{(Z)\lambda}+i\epsilon_{\lambda\rho\sigma\mu}k^\sigma T^{\lambda\rho}]\,. \tag{4.30}$$

An identical analysis for the Z-exchange box diagram of Fig. 6(b) leads to

$$\mathfrak{M}^{\text{box}}_{(Z;b)} = \frac{i(g^2+g'^2)g^2}{4(2\pi)^4}L^\mu$$
$$\times\int\frac{d^4k}{k^2[k^2-m_Z^2][k^2-m_W^2]}$$
$$\times[-2i\cos^2\theta_W\langle p'|J_{W\mu}(0)|p\rangle$$
$$-k_\mu T^\lambda_{(Z)\lambda}-i\epsilon_{\lambda\rho\sigma\mu}k^\sigma T^{\lambda\rho}_{(Z)}]\,. \tag{4.31}$$

Thus, calling $\mathfrak{M}^{\text{box}}_{(Z)}$ the sum of Eqs. (4.30) and (4.31), we have

$$\mathfrak{M}^{\rm box}_{(Z)} = \frac{i(g^2+g'^2)g^2}{2(2\pi)^4} L^\mu \times \int \frac{d^4k}{k^2[k^2-m_Z^2][k^2-m_W^2]} \times \{\cos^2\theta_W[-2i\cos^2\theta_W\langle p'|J_{W\mu}(0)|p\rangle - k_\mu T^\lambda_{(Z)\lambda}] - i\sin^2\theta_W \epsilon_{\lambda\rho\sigma\mu}k^\sigma T^{\lambda\rho}_{(Z)}\}\,. \quad (4.32)$$

As implied by our discussion, terms of $O[G_F^2]$ have been neglected in Eqs. (4.30)–(4.32). To evaluate the contributions involving the tensor $T^{\lambda\rho}_{(Z)}$ in Eq. (4.32) we follow an analysis similar to that explained in Sec. IV.A. Noting that the integral in Eq. (4.32) involves two massive denominators and remembering $T^{\lambda\rho}_{(Z)} \sim k^{-1}$ for large k, we see that only the leading terms in the short-distance expansion of the time-ordered product of the two currents contribute to order $G_F\alpha$. These leading terms are again controlled by partially conserved currents with no anomalous dimensions. We insert the corresponding expressions into Eq. (4.32), note that the domain of integration $\kappa^2 \leqslant M^2$ contributes only to $O[G_F^2]$, and set $\bar{g}_S = 0$ in the region $\kappa^2 \geqslant M^2$. Thus, insofar as we restrict ourselves to corrections of $O[G_F\alpha]$ and neglect the small effects induced by $\bar{g}_S^2$ in the "asymptotic domain" $\kappa^2 \geqslant M^2$, we can evaluate $T^{\lambda\rho}_{(Z)}(k)$ in the scale-invariant limit of the free field theory. An elementary calculation shows that in this case

$$T^{\lambda\rho}_{(Z)}(k) = \frac{-ik_\tau}{k^2}\langle p'|J_{W\beta}(0)|p\rangle \times[\cos^2\theta_W(g^{\lambda\tau}g^{\rho\beta} - g^{\lambda\rho}g^{\tau\beta} + g^{\tau\rho}g^{\lambda\beta}) - 2i\sin^2\theta_W\bar{Q}\epsilon^{\lambda\rho\tau\beta}]\,, \quad (4.33)$$

where $\bar{Q}$ is defined in Eq. (4.25). Inserting this expression into Eq. (4.32) we obtain

$$\mathfrak{M}^{\rm box}_{(Z)} = \frac{-i\pi^2(g^2+g'^2)g^2}{2(2\pi)^4} L^\mu\langle p'|J_{W\mu}(0)|p\rangle \times \int \frac{d\kappa^2}{(\kappa^2+m_W^2)(\kappa^2+m_Z^2)} \times[(2+\tfrac{1}{2})\cos^4\theta_W + 3\bar{Q}\sin^4\theta_W]\,. \quad (4.34)$$

The contributions $2\cos^4\theta_W$ and $(\frac{1}{2})\cos^4\theta_W$ arise, respectively, from the first and second terms between curly brackets in Eq. (4.32), a distinction that will be useful later, while $3\bar{Q}\sin^4\theta_W$ has its origins in the third term. Performing the integration and recalling that $g^2+g'^2 = e^2/(\sin\theta_W\cos\theta_W)^2$, we see that Eq. (4.34) becomes

$$\mathfrak{M}^{\rm box}_{(Z)} = \frac{\alpha}{4\pi}\mathfrak{M}^0\left(\frac{R}{R-1}\right)\ln R[(2+\tfrac{1}{2})\cot^2\theta_W + 3\bar{Q}\tan^2\theta_W]\,, \quad (4.35)$$

where $\mathfrak{M}^0$ is the zeroth-order amplitude, and R is defined in Eq. (4.14). It is clear that the terms proportional to $\cot^2\theta_W$ are universal, i.e., independent of the nature of the initial and final particles represented by p and p' in Figs. 6(a), (b). In particular they are the same in β and muon decays and therefore cancel in the ratio of decay probabilities, a result which was anticipated in papers by Sirlin (1974b, 1974c, 1975). On the other hand, the last term in Eq. (4.35), previously derived in Sirlin (1975), clearly distinguishes between the leptonic decays (for which $\bar{Q}$ should be replaced by $-\frac{1}{2}$, as this is the average charge of ν_μ and μ^- which are the leptonic counterparts of the u and d quarks) and the semileptonic decays where $\bar{Q}$ is determined by the quark charges.[26] In the simplest version of the SU(2)×U(1) theory, in which the Higgs scalars transform according to a single isospinor representation of the weak group, there exists a relation between the vector meson masses and θ_W

$$m_W = m_Z\cos\theta_W\,. \quad (4.36)$$

In that case the nonuniversal part reduces to a simpler expression, namely, $3\bar{Q}(\alpha/2\pi)\mathfrak{M}^0\ln(m_Z/m_W)$, a result already obtained in Sirlin (1974c).

So far we have set $\bar{g}_S = 0$. In order to discuss the $\bar{g}_S^2$ corrections we recall that the contribution proportional to $2\cos^4\theta_W$ in Eq. (4.34) has its origin in the first term between curly brackets in Eq. (4.32), which was derived from the Ward identity associated with the time-time and time-space commutators. Thus it is not affected by strong interaction corrections and leads without approximations to the $2\cot^2\theta_W$ term in Eq. (4.35). On the other hand, the contributions proportional to $(\frac{1}{2})\cos^4\theta_W$ and $3\bar{Q}\sin^4\theta_W$ in Eq. (4.34) can be traced to the terms involving the tensor $T^{\lambda\rho}_{(Z)}$ in Eq. (4.32). As shown in Appendix C, if terms of order $\bar{g}_S^2$ are retained, the corresponding integrand is multiplied by $1-\bar{g}_S^2(\kappa^2)/(4\pi^2)$. The value of the corrections induced by $\bar{g}_S^2$ depends on m_W^2 and m_Z^2. A precise method by which to evaluate them is given in Appendix C. In the case of the simplest version of the Salam–Weinberg model with $\sin^2\theta_W = 0.35$, they amount to a -4.3% correction relative to the contributions of the second and third terms in the square bracket of Eq. (4.35). For $\bar{Q} = \frac{1}{6}$ the latter give a correction of 1.1×10^{-3} to the lifetime so that the $\bar{g}_S^2$ effect amounts only to -4.8×10^{-5}, which is again very small.

In summary, if we add all the corrections of order $\bar{g}_S^2$ induced by the strong interactions in the asymptotic domain, including the small photonic contribution discussed at the end of Appendix C, we find that in the simplest version of the theory with $\sin^2\theta_W = 0.35$ and $\bar{Q} = \frac{1}{6}$, they amount to a correction of -4.3×10^{-4} to the transition probability, which is negligible for most physical applications. For other values of the parameters and different choices of the number of flavors, the $\bar{g}_S^2$ corrections can be gleaned from the detailed expressions given in Appendix C. There are, of course, corrections of $O(\bar{g}_S^4)$ and higher, but the smallness of the $O(\bar{g}_S^2)$ terms is a strong indication that they can be safely neglected.

V. RESIDUAL THREE-CURRENT CORRELATIONS AND OTHER CORRECTIONS

In Sec. V.A we study the residual three-current correlation functions, introduced in Sec. III, in conjunc-

[26]As emphasized in Sirlin (1974c and 1975), in theories in which J_Y^μ is not a color singlet, $\bar{Q}$ includes an average over the three quartets. For instance, in the Pati–Salam model (Pati and Salam, 1974), the quarks have integer charges but $\bar{Q}$ has the same value as in the conventional model with $\hat{q} = 1/3$.

tion with the tadpole diagrams and order α counterterms of the theory. Our discussion here follows closely the approach of the Appendix of Sirlin (1975), in which the analogous problem was discussed in the framework of the free field theory. In Sec. V.B we discuss briefly the corrections associated with the W-meson propagator.

A. Residual three-current correlation functions

We note that the asymptotic behavior for large k of the residual three-current correlation functions of Eqs. (3.13) and (3.16) is controlled by two classes of singularities[27]: (a) those arising from $y\sim x\sim 0$, and (b) those originating from $x\sim 0$ for finite $y\neq 0$. In analyzing the first type of singularity, we recall that in the asymptotically free theory the leading asymptotic behavior is determined by dimensional analysis modulo powers of logarithms. As the time-ordered products in the relevant three-point functions carry one unit of charge, the operator of lowest dimensionality in the short-distance expansion of the three operators must be bilinear in the quark fields.[28] Observing that the $\partial/\partial\bar{q}_\mu$ derivatives in Eqs. (3.13) and (3.16) introduce a factor y^μ in the integrand and recalling that both the currents and $\partial_\alpha J_W^\alpha$ are bilinear in the quark fields, we learn from dimensional analysis that singularities of type (a) lead to coefficient functions that vanish at infinity no slower than k^{-3} modulo powers of logarithms.[29] This implies that the corresponding corrections are finite and, in fact, of $O[G_F^2]$ rather than $O[G_F\alpha]$.

The singularities of type (b) can be obtained from the short-distance expansion as $x\sim 0$ of bilinear products of operators such as $T[J_Z^\mu(x)J_{\mu Z}(0)]$. The only terms in this expansion which contribute to $O[G_F\alpha]$ are those whose Fourier transforms vanish as $k\to\infty$ no faster than k^{-2}. Therefore the relevant operators in the expansion must (i) have dimensions $\leqslant 4$ (ii) be color singlets and gluon gauge invariant (iii) be even under $\mathcal{CP}$ (we neglect the small $\mathcal{CP}$ violations in the weak interactions),[30] and (iv) transform as Lorentz scalars or pseudoscalars, as otherwise their contribution to $T^\mu_{(a)}$ and $D_{(a)}$ $(a=Z,\gamma,W)$ in Sec. III would vanish after the x and k integrations are carried out. Thus they are the same as occur in the analysis of the corrections of $O[\alpha]$ to strong interaction matrix elements. As shown by Weinberg (1973b), there are three classes of operators which at first hand satisfy the required properties and may appear in the short-distance expansions[31]: (1) $\bar{\psi}X^{(1)}\psi+\text{h.c.}$, (2) $\bar{\psi}X^{(2)}\gamma^\mu(i\partial_\mu - g_S T^A S_\mu^A)\psi+\text{h.c.}$, and (3) $G^A_{\mu\nu}G^{\mu\nu A}$, where S^A_μ is defined after Eq. (2.7)

$$G^A_{\mu\nu}=\partial_\mu S^A_\nu-\partial_\nu S^A_\mu-g_S C_{ABC}S^B_\mu S^C_\nu\,, \tag{5.1}$$

and the matrices $X^{(1)}$ and $X^{(2)}$ act on the SU(4) indices of the spinor field and may contain the Dirac matrices 1 and γ^5. When the short-distance expansions of the bilinear products of operators such as $T[J_Z^\mu(x)J_{\mu Z}(0)]$ are inserted into $T^\mu_{(a)}$ and $D_{(a)}$, the residual three-current correlation functions of Eqs. (3.13) and (3.16) are seen to be of the form

$$K_1^\mu=-\bar{q}_\alpha\frac{\partial}{\partial\bar{q}_\mu}\left[\sum_n c_n\int d^4y e^{i\bar{q}\cdot y}\langle p'|T[J_W^\alpha(y)O_n(0)]|p\rangle - b_n^\alpha\right]\Bigg|_{\bar{q}=q}+\cdots\,, \tag{5.2}$$

and

$$K_2^\mu=i\frac{\partial}{\partial\bar{q}_\mu}\sum_n c_n\left[\int d^4y e^{i\bar{q}\cdot y}\langle p'|T[\partial_\alpha J_W^\alpha(y)O_n(0)]|p\rangle - \bar{q}_\alpha b_n^\alpha\right]\Bigg|_{\bar{q}=q}+\cdots\,, \tag{5.3}$$

where $O_n(0)$ $(n=1,2,3)$ denotes collectively the three classes of operators listed above; c_n are divergent constants; the dots indicate terms that contribute to $O[G_F^2]$; and b_n^α, which has its origins in the vector $B_{(a)}$ of Sec. III,[32] subtracts the pole terms at $(p'+\bar{q})^2=m_h^2$ and $(p-\bar{q})^2=m_h^2$, of the first term of Eq. (5.2). As we shall see later, the divergent parts of these expressions are cancelled by the order α counterterms of the theory. We note that when we use equation of motion of the ψ field given in Eq. (2.7), the contributions of O_2 to Eqs. (5.2) and (5.3) can be reduced effectively to those of mass operators of type O_1.[33]

The operator $O_3=G^A_{\mu\nu}G^{\mu\nu A}$ conserves parity, strangeness, and charm. Because of this fact, for $n=3$ the first terms of Eq. (5.3) can only contribute to the Fermi amplitude when $\partial_\alpha J_W^\alpha(y)$ is the divergence of the $\Delta S=\Delta C=0$, $|\Delta Q|=1$ vector current, which vanishes to zeroth order in α. Recalling the discussion at the end of Sec. III, it is clear that the contributions of Eq. (5.2) and the second term of Eq. (5.3) are of $O(q)$. Thus, to first order in α and zeroth order in the momentum transfer, the operators of class (3) do not contribute to the Fermi amplitude.

In summary, the only operators that can, in principle, contribute effectively to the Fermi amplitude in Eqs. (5.2) and (5.3) are of type O_1, (i.e., mass operators).

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[27]The role played by these two classes of singularities can be illustrated by a generalization of the Bjorken–Johnson–Low limit to three-current correlation functions. See, for example, the expansions of Eqs. (8) and (16) in Sirlin (1968b).

[28]See also Footnote 15.

[29]A similar conclusion was reached by Preparata and Weisberger (1968) in their study of photonic corrections in the framework of the Abelian gluon theory.

[30]See the discussion in Appendix A.

[31]Operators of the form $i\partial_\mu(\bar{\psi}X^{(3)}\gamma^\mu\psi)+$ h.c. can be expressed as a combination of operators of class (2), while $\epsilon_{\mu\nu\alpha\beta}G^{\mu\nu A}_{(0)}G^{\alpha\beta A}_{(0)}$ is odd under $\mathcal{CP}$.

[32]The vectors b_n^α are given by expressions analogous to Eq. (3.8), except that the time-ordered product in Eq. (3.9) is replaced by the relevant terms in the short-distance expansion.

[33]It is interesting to observe that the perturbative correction induced on the weak vertex by 0_2 involves an amplitude $\int d^4y e^{i\bar{q}\cdot y}\langle p'|T^*[J_W^\alpha(y)0_2(0)]|p\rangle$ where T^* differs from the usual T product by a term proportional to $\delta^4(y)$, which is easy to verify in perturbation theory by considering single insertions of J_W^α and 0_2 and an arbitrary number of gluon insertions on a quark line. The additional term is related to the fact that 0_2 contains a derivative interaction, and leads to a contribution independent of $\bar{q}$. In the case of Eq. (5.2) [and similarly (5.3)] it is immaterial whether we use the T or T^* products as the $\delta^4(y)$ contributions cancel when the differentiations with respect to $\bar{q}_\mu$ are carried out.

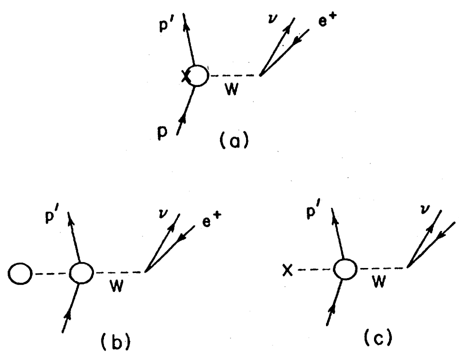

FIG. 3. Insertions of order α counterterms, tadpole terms, and tadpole counterterms in the hadronic line.

At this stage we consider the insertions on the hadronic line of the tadpole and order α counterterms of the theory, depicted in Fig. 3. As shown by Weinberg (1973b), their contributions also involve effectively operators of classes (1) and (3). Thus the corrections to the weak vertex arising from tadpole diagrams and order α counterterms are of the form

$$\delta T^{\alpha} = \sum_{n=1,3} \delta c_n \lim_{\bar{q}\to q}\left[\int d^4y e^{i\bar{q}\cdot y}\langle p'|T[J_W^{\alpha}(y)O_n(0)]|p\rangle - b_n^{\alpha}\right], \tag{5.4}$$

where the δc_n are divergent constants. Contracting Eq. (5.4) with $\bar{q}_\alpha$ we derive a Ward identity; differentiating with respect to $\bar{q}_\mu$ and then setting $\bar{q}=q$, we can write δT^{α} as the sum of two terms which, except for the constants δc_n, are identical to Eqs. (5.2) and (5.3).[34] The term $\delta c_1 O_1$ in Eq. (5.4) represents the contribution of the quark mass counterterm $\bar{\psi}\delta m\psi$ as well as the tadpoles and tadpole counterterms. In fact, the latter contributions are effectively given by "mass operators" of type (1). In the SU(2)×U(1) models discussed in this paper there are no zeroth-order natural relations involving the quark masses and/or mixing angles. As a consequence, δm is an arbitrary matrix acting on the flavor degrees of freedom and involving the Dirac operators 1 and γ_5, except for the restriction that it commutes with the electric charge operator Q. The simplest procedure is then to adjust $\delta c_n O_n$ to cancel the weak corrections of order α to strong interactions amplitudes (see Appendix D). As the corrections to weak amplitudes exhibited in Eqs. (5.2) and (5.3) involve exactly the same constants c_n and operators O_n, they will also be canceled automatically by the $\delta c_n O_n$ contributions. On the other hand, in theories with zeroth-order natural relations between the quark masses and/or mixing angles, δm is not arbitrary and one must follow the procedure outlined in Appendix D.

There is still another source of isospin violation that we must discuss in connection with the quark mass matrix. We note that in the SU(2)×U(1) models under consideration, the mass matrix m introduced in Eq. (2.7) involves four independent renormalized parameters, namely the quark masses. In particular, isospin symmetry which corresponds to the limit $m_u = m_d$, (or perhaps, $m_u = m_d = 0$) is not a natural symmetry in these models.[35] To study the corrections to isospin symmetry associated with the fact that $m_u \neq m_d$, it is convenient to write $m = m_{\rm sym} + \Delta m$, where $m_{\rm sym}$ is a diagonal matrix that conserves isospin exactly and Δm contains the symmetry-breaking part. One then regards $-\bar{\psi}m_{\rm sym}\psi$ as the "zeroth-order mass matrix" and the operator $-\bar{\psi}\Delta m\psi$ is included in the perturbing interaction. As is well known, the operator $-\bar{\psi}\Delta m\psi$ plays an important role in the understanding of isomultiplet mass differences and is formally regarded as being $O(\alpha)$. Note, however, that this operator is diagonal and therefore preserves strangeness, charm, and parity. But we have seen above, when we discussed O_3, that operators with such properties cannot induce first-order corrections to the conserved vector current vertex at zero momentum transfer. In fact, the result that mass terms diagonal in parity, strangeness, and charm do not renormalize the matrix elements of the vector current to first order in the limit of zero momentum transfer is the well known result of the nonrenormalization theorem (Behrends and Sirlin, 1960; Ademollo and Gatto, 1964).

The discussion of this section, based on the analysis of short-distance singularities, is clearly applicable to the residual three-current correlation functions involving massive vector-meson propagators. Thus, in the case of the photonic corrections, it is applicable to the residual amplitudes involving $T^{\mu}(\gamma>)$, defined in Eq. (3.34). For the contributions involving $T^{\mu}(\gamma<)$, which contain all the low-frequency components, the short-distance expansions are not particularly helpful. Two observations are relevant: (i) the factor $m_W^2/(m_W^2-k^2)$ in Eq. (3.33) insures the ultraviolet convergence of $T^{\mu}(\gamma<)$, and (ii) by the argument given in Sec. III, in the case of the corrections to the Fermi amplitude the residual corrections associated with $T^{\mu}(\gamma<)$ [as well as those involving $T^{\mu}(\gamma>)$] are of $O(G_F\alpha q)$.

We conclude that when the residual three-current correlation functions are combined with the order α counterterms and tadpole diagrams, they can only contribute to $O(G_F^2)$ provided that terms of $O(G_F\alpha q)$ are ignored.[36] Thus for our purposes they are completely negligible.

B. Corrections to *W* propagator

As the corrections to the W propagator in the t'Hooft–Feynman gauge have been studied in considerable detail in the literature (Appelquist *et al.*, 1972, 1973; Ross, 1973), they will be discussed only briefly here. On general grounds, the polarization tensor associated with the W propagator (see Fig. 7) is of the form

$$\Pi_{\mu\nu}(q) = A(q^2)g_{\mu\nu} + B(q^2)q_\mu q_\nu\,. \tag{5.5}$$

In the renormalizable gauges, the divergent parts of Eq. (5.5), as well as those of the propagator of the scalar ϕ associated with W and of the mixed $W-\phi$ propagator,

[34] See the analogous discussion in Preparata and Weisberger (1968).

[35] In these theories the fact that isospin breaking of "nonelectromagnetic origin" transforms as an isotriplet emerges naturally. What is not natural is the magnitude of isospin breaking which, in principle, may be arbitrary.

[36] In the superallowed Fermi transitions q is actually of $0(\alpha)$. Thus the neglected terms may be regarded as being of $0(G_F\alpha^2)$.

are cancelled by the appropriate renormalization counterterms of the unbroken theory. In particular, the divergences in Eq. (5.5) can be removed by performing the mass and field renormalization of the W meson, with the mass subtraction carried out at $q^2 = m_W^2$ so that the renormalized $A_R(q^2)$ satisfies $R_e A_R(m_W^2) = 0$. Once this is done it is clear that the contributions of $B_R(q^2) q_\mu q_\nu$ and of the propagators involving ϕ are $O[G_F^2]$. (Further they are proportional to extremely small factors such as $G_F \alpha (m_e m_\mu / m_W^2)$ in μ decay and even smaller in β decay). When $A_R(q^2) g_{\mu\nu}$ is inserted in the diagram of Fig. 7(a), it is "sandwiched" between two massive propagators of $O[m_W^{-4}]$. It can still give contributions of $O[G_F \alpha]$ because $A_R(q^2)$ will in general contain terms proportional to m_W^2 and m_Z^2 arising from diagrams involving the massive particles Z, W, ϕ^+, etc... or from subtraction terms. The important thing to realize (Ross, 1973) is that

$$A_R(q^2) = A_R(0)\left[1 + O\left(\frac{q^2}{m_W^2}\right)\right]. \tag{5.6}$$

The second term in Eq. (5.6) gives a contribution of $O[G_F^2]$. The first does contribute to $O[G_F \alpha]$, but if we again neglect the small q^2 dependence of the W propagators, its contribution is the same for all semileptonic and leptonic processes. Thus it is a universal factor without physical consequences. It should be clear that these considerations hold in the presence of the strong interactions. The hadronic contributions to the W propagator involve the polarization tensor of the J_W^μ current $\int d^4y e^{iq\cdot y} \langle 0 | T^*[J_W^\mu(y) J_W^{\dagger\nu}(0)] | 0 \rangle$, whose renormalization requires two subtractions. Our previous remarks are still valid, as the hadronic contributions $A_R(0)$ are universal and the terms of $O(q^2/m_W^2)$ are negligible. Although the hadronic contributions to the W propagator are not important for the problem of Cabibbo universality, they can also be computed under the assumptions of this paper. (A. Sirlin, unpublished). A detailed discussion of the cancellation of divergences in the hadronic contributions to the (WW), (W, ϕ), and $(\phi\phi)$ propagators, valid to all orders in the strong interactions, is given in Appendix E.

VI. CANCELLATION OF DIVERGENCES IN THE HIGGS SECTOR

In this section we discuss the cancellation of divergences in the Higgs sector. For definiteness, we work in the framework of the simplest version of the Weinberg-Salam model. However, the methods are general and can be applied to other theories. In Sec. VI.A we study the contributions of the Higgs scalars to the vertex $Wh'h$ and the box diagrams, while in Sec. VI.B we analyze the corrections to the vertex $\phi h'h$. Once the cancellation of divergences has been demonstrated, the finite parts of the corrections associated with Higgs exchanges can be neglected. In fact, subject to the assumptions of this paper, they only contribute to $O[G_F^2]$.

A. Corrections to the vertex *Wh'h* and box diagrams

In the simplest version in which the Higgs scalars transform as a single isospinor of the $SU(2)_L \times U(1)$ group, the hadronic interactions of the Higgs scalars are

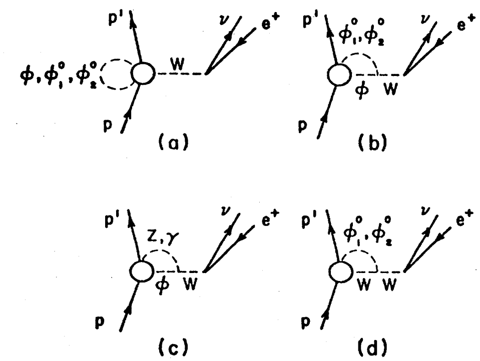

FIG. 4. Diagrams involving Higgs scalars which contribute to the vertex $Wh'h$ in the simplest version of the Salam–Weinberg model. The symbol ϕ_1^0 represents the physical Higgs scalar, while ϕ and ϕ_2^0 stand for the charged and neutral unphysical components.

described by[37]

$$\mathcal{L}_{\text{Higgs}} = -(g/2m_W)[\phi_1^0 S_1 + \phi_2^0 S_2 + \sqrt{2}(\phi^\dagger S + \text{h.c.})], \tag{6.1}$$

where

$$S_1 = \bar\psi m^0 \psi, \tag{6.2}$$

$$S_2 = 2\partial_\mu J_Z^\mu = -2i[Q_Z, \bar\psi m^0 \psi], \tag{6.3}$$

$$S = -i\,\partial_\mu J_W^\mu = -[Q_W, \bar\psi m^0 \psi], \tag{6.4}$$

the commutators being evaluated at equal times. In Eqs. (6.1)–(6.4), ϕ_1^0 is the physical Higgs scalar with ϕ_2^0 and ϕ are the unphysical counterparts associated with Z and W ($\phi^\dagger$ is the field which creates a ϕ^+ meson), m^0 is the bare mass matrix of the quarks, and Q_Z and Q_W are the charges associated with J_Z^μ and J_W^μ. We now write

$$m^0 = m - \delta m, \tag{6.5}$$

where δm are the order α mass counterterms discussed in Sec. V and Appendix D and m is the diagonal mass matrix of Eq. (2.7). To lowest order in α we may neglect δm and obtain

$$S_1 = \bar\psi m \psi, \tag{6.6}$$

$$S_2 = -i\bar\psi m C_3 \gamma_5 \psi, \tag{6.7}$$

$$S = \bar\psi \Gamma \psi, \tag{6.8}$$

$$\Gamma = \left[\begin{array}{c|c} 0 & 0 \\ \hline \rho & 0 \end{array}\right], \tag{6.9}$$

$$\rho = \begin{bmatrix} s(m_c a_+ - m_d a_-) & c(m_d a_- - m_u a_+) \\ c(m_s a_- - m_c a_+) & s(m_s a_- - m_u a_+) \end{bmatrix}, \tag{6.10}$$

where $a_+ = (1+\gamma_5)/2$ and otherwise we use the notation of Sec. II.

We now follow the methods developed in Secs. III and IV. Consider, for example, the ϕ-exchange diagram of Fig. 4(a). Its contribution to the decay amplitude is given by expressions analogous to Eqs. (3.1) to (3.6) with $T_{(Z)}^\mu$ replaced by

[37]For a discussion of the most general gauge-invariant Yukawa couplings in the standard model and their connection to the bare mass matrix m_0, see Marciano and Sirlin (1975).

$$T^{\mu}_{(\phi)} = -\frac{g^2}{4m_W^2(2\pi)^4}\int\frac{d^4k}{(k^2-m_W^2)}\int d^4y e^{i\bar{q}\cdot y}\int d^4x e^{ik\cdot x}\langle p'|T[J^{\mu}_W(y)(S^{\dagger}(x)S(0)+\text{h.c.})]|p\rangle - B^{\mu}_{(\phi)}, \quad (6.11)$$

where we used the fact that in the t'Hooft–Feynman gauge the propagator of ϕ is $i(k^2-m_W^2)^{-1}$. Contracting T^{α} with $\bar{q}_{\alpha}$ we derive a Ward identity, and differentiating with respect to $\bar{q}_{\mu}$ we obtain expressions analogous to Eqs. (3.13) and (3.16). In Sec. V we have seen that the divergences of the first three terms on the right-hand side of Eqs. (3.13) and (3.16) are cancelled by the order α counterterms and tadpole diagrams. Exactly the same will happen in the case of the corrections of Fig. 4(a).[38] Thus, in order to discuss the cancellation of the divergences, it is sufficient to retain the two-point correlation functions analogous to $V^{\mu}_{(a)}$ $(a=Z,\gamma,W)$ in Eqs. (3.13) and (3.16). The relevant equal-time commutators for deriving the Ward identities are

$$[J^0_W(y), S^{\dagger}(y')]_{y'^0=y^0} = \tfrac{1}{2}[S_1(y)-iS_2(y)]\delta^3(\mathbf{y}-\mathbf{y}'), \quad (6.12)$$

$$[J^0_W(y), S_1(y')]_{y'^0=y^0} = -S(y)\delta^3(\mathbf{y}-\mathbf{y}'), \quad (6.13)$$

$$[J^0_W(y), S_2(y')]_{y'^0=y^0} = iS(y)\delta^3(\mathbf{y}-\mathbf{y}'). \quad (6.14)$$

Calling $\mathcal{V}_{(b)}$ $(b=\phi, \phi_1^0, \phi_2^0)$ the contributions to the decay amplitude of the two-point correlation functions in the Ward identities, we find

$$\mathcal{V}_{(\phi)} = \frac{ig^4}{16m_W^2(2\pi)^4}\frac{1}{(q^2-m_W^2)}L_{\mu}\int\frac{d^4k}{k^2-m_W^2}\frac{\partial}{\partial k_{\mu}}\times(R_1-iR_2)(k+q), \quad (6.15)$$

$$\mathcal{V}_{(\phi_1^0)} = \frac{+ig^4}{16m_W^2(2\pi)^4}\frac{1}{(q^2-m_W^2)}L_{\mu}\int\frac{d^4k}{k^2-m_{\phi}^2}\frac{\partial}{\partial k_{\mu}}R_1(k), \quad (6.16)$$

$$\mathcal{V}_{(\phi_2^0)} = \frac{g^4}{16m_W^2(2\pi)^4}\frac{1}{(q^2-m_W^2)}L_{\mu}\int\frac{d^4k}{k^2-m_Z^2}\frac{\partial}{\partial k_{\mu}}R_2(k), \quad (6.17)$$

where m_{ϕ} is the mass of the physical Higgs scalar, and

$$R_i(k) = \int d^4x e^{ik\cdot x}\langle p'|T[S_i(x)S(0)]|p\rangle \quad (i=1,2). \quad (6.18)$$

Equations (6.15)–(6.18) are the counterpart of Eqs. (3.25)–(3.32). We have again simplified the final expressions by using Eq. (3.2) and translational invariance. Applying the asymptotic estimates we see that the $\mathcal{V}_{(b)}$ $(b=\phi, \phi_1^0, \phi_2^0)$ are logarithmically divergent. Next we consider the contributions of Fig. 4(b)

$$\mathcal{U}_{(\phi_1^0)} = \frac{-ig^4}{8m_W^2(2\pi)^4}\frac{1}{q^2-m_W^2}L_{\mu}\int\frac{d^4k(2k-q)^{\mu}}{(k^2-m_W^2)(k^2-m_{\phi}^2)}R_1(k), \quad (6.19)$$

$$\mathcal{U}_{(\phi_2^0)} = \frac{-g^4}{8m_W^2(2\pi)^4}\frac{1}{q^2-m_W^2}L_{\mu}\int\frac{d^4k(2k-q)^{\mu}}{(k^2-m_W^2)(k-m_Z^2)}R_2(k). \quad (6.20)$$

Performing a partial integration in Eqs. (6.15)–(6.17), neglecting finite terms of $O(q)$, and adding the contributions of Eqs. (6.19) and (6.20), we find

$$\mathcal{V}_{(\phi)}+\mathcal{V}_{(\phi_1^0)}+\mathcal{V}_{(\phi_2^0)}+\mathcal{U}_{(\phi_1^0)}+\mathcal{U}_{(\phi_2^0)} = \frac{ig^4}{16m_W^2(2\pi)^4(q^2-m_W^2)}L_{\mu}\times\int d^4k\,2k^{\mu}\left\{\left(\frac{1}{k^2-m_W^2}-\frac{1}{k^2-m_{\phi}^2}\right)^2R_1(k) - i\left(\frac{1}{k^2-m_W^2}-\frac{1}{k^2-m_Z^2}\right)^2R_2(k)\right\}. \quad (6.21)$$

Thus the two-point correlation functions contributing to the $Wh'h$ vertex have combined to give a finite answer! We note that the additional diagrams of Figs. 4(c)–(d) are individually convergent. The reason for this apparently miraculous cancellation can be understood by means of the following argument. A divergent contribution to the vertex $Wh'h$ arising from the Higgs scalars would be proportional to the quark masses. As we have already applied the order α mass counterterms [see discussion after Eq. (6.11)], the only way of canceling such divergences would be to absorb them in the definition of the weak coupling g. However, this is not possible because gauge invariance forces the renormalization of g to be the same as in the leptonic vertices of W, where the quark masses cannot be relevant.

As we mentioned before, subject to the assumptions of this paper, the finite corrections associated with the Higgs scalars are of $O[G_F^2]$ and therefore negligible. This includes the box diagrams involving Higgs scalars which are depicted in Fig. 8.

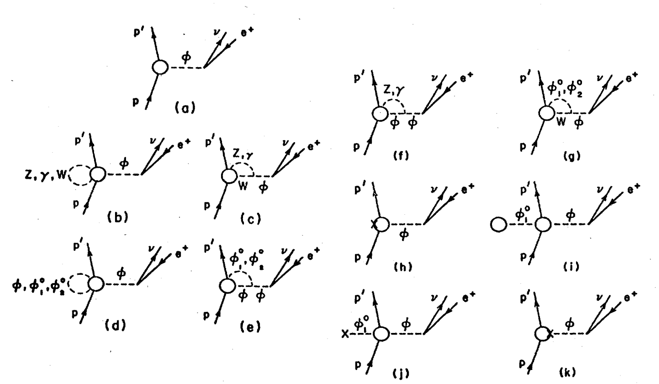

FIG. 5. Diagrams contributing to the $\phi h'h$ vertex in the simplest version of the Salam–Weinberg model.

[38]As shown in Marciano and Sirlin (1975), in order to cancel the divergences of Fig. 4(a) the mass counterterms must include off-diagonal terms proportional to $\alpha m^2/m_W^2$ where m is a generic quark mass.

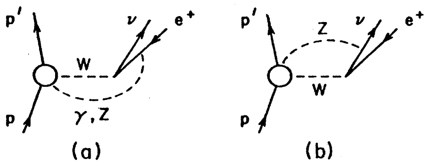

FIG. 6. Box diagrams involving the exchange of γ and Z between hadrons and leptons.

B. Corrections to the vertex $\phi h'h$

To simplify the analysis we assume in this section that ϕ_1^0 has been defined to have exactly zero vacuum expectation value, so that tadpole contributions such as Figs. 5(i), (j) can be disregarded (Taylor, 1976).

We first discuss the counterterms of order g^3 associated with the vertex $\phi h'h$. Referring back to Eq. (6.1) we note that in this equation g and m_W are bare quantities. Imagining that these constants are written as g^0 and m_W^0, performing the shifts $g^0 = g - \delta g$, $m_W^0 = m_W - \delta m_W$, remembering Eqs. (6.4) and (6.5), and including the field renormalization of ϕ, we see that up to terms of order g^3 the interaction of ϕ with the quarks can be expressed as

$$\mathcal{L}_{\phi\bar\psi\psi} = -\frac{g}{\sqrt{2}m_W} S\phi^\dagger + \delta\mathcal{L}_{\phi\bar\psi\psi} \tag{6.22}$$

$$\delta\mathcal{L}_{\phi\bar\psi\psi} = -\frac{g}{\sqrt{2}m_W}\left(\frac{\delta m_W}{m_W} - \frac{\delta g}{g} + \frac{Z_\phi - 1}{2}\right)S\phi^\dagger - \frac{g}{\sqrt{2}m_W}[Q_W, \bar\psi\delta m\psi]\phi^\dagger , \tag{6.23}$$

where g and m_W are now renormalized constants. The term $(Z_\phi - 1)/2$ arises because in order to cancel the divergent parts in the corrections to the W and ϕ propagators (see Fig. 7), we may choose to rescale the W and ϕ fields, and this rescaling induces a correction in the $\bar\psi\psi\phi$ vertex. Alternatively, we may choose not to rescale the fields, in which case the contribution $(Z_\phi - 1)/2$ arises through the combined effect of the unrenormalized diagrams of Fig. 7 (for details, see Appendix E). Note also that in our approach we do not absorb $\sqrt{Z_\phi}$ in a coupling constant renormalization because g and m_W are already renormalized constants.

We now turn our attention to the three-point correlation amplitudes depicted in Fig. 5(b). The Z-exchange

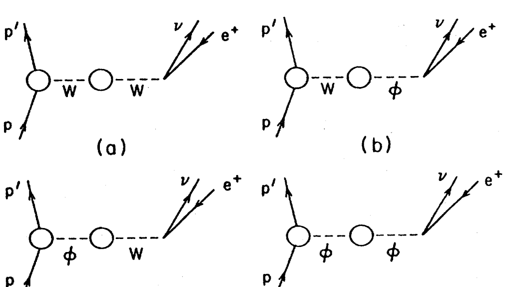

FIG. 7. Hadronic contributions to (WW), $(W\phi)$, and $(\phi\phi)$ propagators.

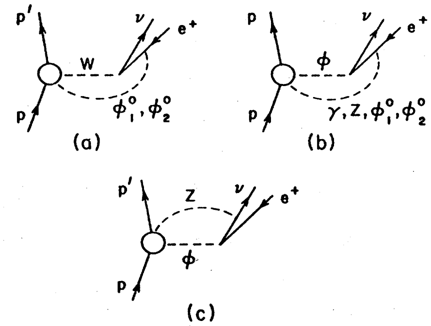

FIG. 8. Box diagrams involving Higgs scalars.

contribution to Fig. 5(b) is given by

$$\mathfrak{M}^{(Z)}_{(5b)} = \frac{-g^2}{2m_W^2}\lim_{\bar q\to q}[D_{(Z)} - p_{(Z)}]\frac{1}{(q^2 - m_W^2)} m_e \bar u_\nu a_+ v_e , \tag{6.24}$$

where $D_{(Z)}$ is defined in Eq. (3.11) and

$$p_{(Z)} = -d(\bar q^2)\frac{i}{(p' + \bar q)^2 - m_h^2}(\delta m_h^2)_{(Z)} - (\delta m_{h'}^2)_{(Z)}\frac{i}{(p - \bar q)^2 - m_{h'}^2} d(\bar q^2) \tag{6.25}$$

subtracts the pole terms of $D_{(Z)}$ at $\bar q = q$. To be precise, $d(\bar q^2)$ is the on-mass-shell matrix element of $i\partial_\alpha J_W^\alpha$ between the initial and final hadrons evaluated at $\bar q^2$

$$d(\bar q^2) = \tfrac{1}{2}[F_1^{(0)}(\bar q^2)(m_h^2 - m_{h'}^2) + F_2^{(0)}(\bar q^2)\bar q^2] \tag{6.26}$$

in the notation of Sec. III. Using the Ward identity of Eq. (3.10) and recalling Eqs. (3.8), (6.25), and (6.26), we find

$$\lim_{\bar q\to q}[D_{(Z)} - p_{(Z)}] = \frac{i}{2}F_1^{(0)}(q^2)[\delta m_{h'(Z)}^2 - \delta m_{h(Z)}^2] + q_\alpha T^\alpha_{(Z)} - 2\int\frac{d^4k}{k^2 - m_Z^2} v_{(Z)}(q+k) , \tag{6.27}$$

where $T^\alpha_{(Z)}$, $\delta m^2_{h(Z)}$, and $v_{(Z)}(k)$ are defined in Eqs. (3.7), (3.9), and (3.12). The first term on the right-hand side of Eq. (6.27) arises because the "pole terms" $\bar q_\alpha B^\alpha_{(Z)}$ and $p_{(Z)}$ don't cancel exactly but leave a finite remainder which is, as expected, free from the pole singularity at $\bar q = q$.

Next we consider the associated corrections to the vertex of $i\partial_\mu J_W^\mu$ induced by the order α counterterms [Fig. 5(h)]. These are given by

$$\delta\mathfrak{M}^{(Z)}_{(5h)} = \frac{-g^2}{2m_W^2}\lim_{\bar q\to q}[\delta D_{(Z)} - \delta p_{(Z)}]\frac{1}{q^2 - m_W^2} m_e\bar u_\nu a_+ v_e , \tag{6.28}$$

where

$$\delta D_{(Z)} = \sum_n \delta c_{n(Z)} i\int d^4y e^{i\bar q\cdot y}\langle p'|T[\partial_\mu J_W^\mu(y) O_n(0)]|p\rangle , \tag{6.29}$$

$$\delta p_{(Z)} = \sum_n \delta c_{n(Z)} d(\bar q^2)\frac{i}{(p' + \bar q)^2 - m_h^2}\langle p|O_n(0)|p\rangle + \sum_n \delta c_{n(Z)}\langle p'|O_n(0)|p'\rangle\frac{i}{(p - \bar q)^2 - m_{h'}^2} d(\bar q^2) . \tag{6.30}$$

In Eqs. (6.29) and (6.30) $\delta c_{n(Z)}O_n$ are the counterterms of class (1) and (3) (see Sec. V.A) associated with virtual Z exchange.[39] After applying the appropriate Ward identities we find

$$\lim_{\bar{q}\to q}(\delta D_{(Z)}-\delta p_{(Z)})=q_\alpha\delta T^\alpha_{(Z)}(q)-i\sum_n\delta c_{n(Z)}\langle p'|[Q_W(0),O_n(0)]|p\rangle+\frac{iF_1^0}{2}(q^2)\sum_n\delta c_{n(Z)}[\langle p|O_n(0)|p\rangle-\langle p'|O_n(0)|p'\rangle]\,, \tag{6.31}$$

where $\delta T^\alpha_{(Z)}(q)$ is the contribution from virtual Z exchange to δT^α defined in Eq. (5.4). As O_3 commutes at equal times with Q_W, only the term $\delta c_{1(Z)}O_1=\bar{\psi}\delta m_{(Z)}\psi$ survives in the second term on the right-hand side of Eq. (6.31) [$\delta m_{(Z)}$ is the quark mass counterterm associated with virtual Z exchange].

Combining Eqs. (6.27) and (6.31), we have

$$\begin{aligned}\lim_{\bar{q}\to q}\{D_{(Z)}-p_{(Z)}+\delta D_{(Z)}-\delta p_{(Z)}\}&=q_\alpha(T^\alpha_{(Z)}+\delta T^\alpha_{(Z)})\\&\quad-2\int\frac{d^4k}{k^2-m_Z^2}v_Z(q+k)-i\langle p'|[Q_W(0),\bar{\psi}(0)\delta m_{(Z)}\psi(0)]|p\rangle\\&\quad+\frac{iF_1^0}{2}(q^2)\Big\{\delta m^2_{h'(Z)}-\delta m^2_{h(Z)}+\sum_n\delta c_{n(Z)}[\langle p|O_n(0)|p\rangle-\langle p'|O_n(0)|p'\rangle]\Big\}.\end{aligned} \tag{6.32}$$

It is clear that for the γ- and W-exchange contributions to Fig. 5(b) and the associated corrections induced by the order α counterterms we obtain expressions analogous to (6.24), (6.28), and (6.32) with the subscripts Z replaced by γ and W, respectively.

At this stage, it is convenient to isolate the divergent parts of Eq. (6.32) and the analogous expressions for γ and W exchanges. As implied by Weinberg's analysis (1973a, 1973b), the order α counterterms cancel the divergences of order α in the mass shifts $\delta m^2_{h'(Z)}$ and $\delta m^2_{h(Z)}$, so that the expression between curly brackets on the right-hand side of Eq. (6.32) is finite. Next we prove that although the first and second terms on the right-hand side of Eq. (6.32) are divergent

$$\sum_{a=Z,\gamma,W}\left[q_\alpha(T^\alpha_{(a)}+\delta T^\alpha_{(a)})-2\int\frac{d^4k}{k^2-m_a^2}v_a(q+k)\right]=\text{finite}\,. \tag{6.33}$$

To prove this theorem we recall from the analysis of Sec. V that the order α counterterms cancel the ultraviolet divergences of the first three terms on the right-hand side of Eqs. (3.13) and (3.16). This implies that the divergent parts in $T^\alpha_{(a)}+\delta T^\alpha_{(a)}$ are given by $V^\alpha_{(a)}(q)$ defined in Eqs. (3.14), (3.19), and (3.23). Thus, in proving Eq. (6.33), it is sufficient to replace $T^\alpha_{(a)}+\delta T^\alpha_{(a)}$ by $V^\alpha_{(a)}(q)$. Remembering Eqs. (3.12), (3.20), and (3.24), it is convenient to cast the integrals in the form

$$2\int\frac{d^4k}{k^2-m_Z^2}v_Z(q+k)=\frac{ig^2}{(2\pi)^4}\int\frac{d^4k}{k^2-m_Z^2}\int d^4x e^{i(k+q/2)\cdot x}\times\langle p'|T\left[J_W^\lambda\left(\frac{x}{2}\right)J_{Z\lambda}\left(-\frac{x}{2}\right)\right]|p\rangle \tag{6.34}$$

$$2\int\frac{d^4k}{k^2}v_\gamma(q+k)=\frac{ig^2\sin^2\theta_W}{(2\pi)^4}\int\frac{d^4k}{k^2}\int d^4x e^{i(k+q/2)\cdot x}\times\langle p'|T\left[J_W^\lambda\left(\frac{x}{2}\right)J_{\gamma\lambda}\left(-\frac{x}{2}\right)\right]|p\rangle \tag{6.35}$$

$$2\int\frac{d^4k}{k^2-m_W^2}v_W(q+k)=\frac{-ig^2}{2(2\pi)^4}\int\frac{d^4k}{k^2-m_W^2}\int d^4x e^{i(k+q/2)\cdot x}\times\langle p'|T\left[J_3^\lambda\left(\frac{x}{2}\right)J_{W\lambda}\left(-\frac{x}{2}\right)\right]|p\rangle\,. \tag{6.36}$$

Expanding the exponentials $e^{iq\cdot x/2}$ in Eqs. (6.34)–(6.36) in powers of $q\cdot x/2$, it is easy to see that the terms linear in $q\cdot x/2$ cancel the divergent parts of the corresponding $q_\alpha V^\alpha_{(a)}$ in Eq. (6.33) while the terms of second and higher order are convergent. All that remains are the terms of zeroth order in $q\cdot x/2$. When we sum the Z,γ, and W contributions, these combine to give

$$\frac{ig^2}{(2\pi)^4}\int d^4k T^\lambda_{(Z)\lambda}\left(k+\frac{q}{2}\right)\left[\frac{1}{k^2-m_W^2}-\frac{1}{k^2-m_Z^2}\right]+\frac{ig^2}{(2\pi)^4}\sin^2\theta_W\int d^4k T^\lambda_{(\gamma)\lambda}\left(k+\frac{q}{2}\right)\left[\frac{1}{k^2-m_W^2}-\frac{1}{k^2}\right], \tag{6.37}$$

where $T^{\lambda\rho}_{(Z)}$ and $T^{\lambda\rho}_{(\gamma)}$ are defined in Eqs. (3.31) and (3.32). As these tensors behave asymptotically as $\sim 1/k$, we see that Eq. (6.37) is convergent. Thus the expressions on the left-hand side of Eq. (6.33) do indeed combine to give a finite answer! Referring back to Eq. (6.32) and the corresponding γ and W contributions, we conclude that the only divergent parts are the terms involving the equal-time commutators with Q_W

[39]To simplify the notation we suppress a subscript a ($a=Z$, W, $\gamma\ldots$) in the mass operators 0_1.

$$\sum_a \lim_{\bar{q}\to q} \{D_{(a)} - p_{(a)} + \delta D_{(a)} - \delta p_{(a)}\}^{\text{div}} = -i\sum_a \langle p'|[Q_W, \bar{\psi}\delta m_{(a)}\psi]|p\rangle\ , \quad (6.38)$$

where the superscript "div" means "divergent part" and $a = Z, \gamma, W$. Next we observe that when Eq. (6.38) is inserted in Eqs. (6.24), (6.28), and the analogous amplitudes associated with γ and W exchanges, it is exactly canceled by the contribution to the decay amplitude from the g^3 counterterm $-g/(\sqrt{2}m_W)\sum_{a=\gamma,Z,W}[Q_W, \bar{\psi}\,\delta m_{(a)}\psi]\,\phi^\dagger$ in Eq. (6.23)! In summary, we have shown that the divergent parts of Fig. 5(b) indeed cancel when combined with the contributions of the order α counterterms and the order g^3 counterterm given above.

It is worth noting that our strategy in treating the diagrams of Fig. 5(b) has been to relate them to those of Fig. 1(b) by means of the Ward identities and then use our knowledge of the latter and new detailed arguments to show explicitly the cancellation of divergences.

A completely analogous argument, which need not be repeated here, shows that the divergences associated with Fig. 5(d) cancel when combined with the corresponding order α and order g^3 counterterms.

Finally we consider the amplitudes of Figs. 5(c), (e), (f), (g). We observe that diagrams 5(c), (e) are convergent. Using the appropriate Ward identities we find for the divergent parts of Figs. 5(f), (g)

$$\mathfrak{M}^{\text{div}}_{(5,f+g)} = -\frac{g^4}{2m_W^2}\langle p'|S(0)|p\rangle \frac{m_e}{q^2 - m_W^2}\,\bar{u}_\nu a_+ v_e \times \frac{2i\pi^2}{4-n}\left[\sin^2\theta_W - \frac{(1-\tan^2\theta_W)}{4}(2\sin^2\theta_W - 1) + \tfrac{1}{2}\right], \quad (6.39)$$

where n is the parameter in the dimensional regularization. In the simplest version of the Salam–Weinberg model which we have adopted in this section, the expression between square brackets in Eq. (6.39) simplifies to $(\frac{1}{2})\times[1+1/(2R)]$ where R is defined in Eq. (4.14). Equation (6.39) is canceled by the contribution of the g^3 counterterm proportional to $S\phi^\dagger$ in Eq. (6.23), represented schematically in Fig. 5(k). Adjusting the parameters we obtain

$$\left(\frac{\delta m_W}{m_W} - \frac{\delta g}{g} + \frac{Z_\phi - 1}{2}\right)^{\text{div}} = \frac{g^2}{16\pi^2}\left(1 + \frac{1}{2R}\right)\frac{1}{n-4}\ . \quad (6.40)$$

It is interesting to note that although the divergent parts of δm_W, δg, and $(Z_\phi - 1)$ depend on dynamical details of the strong interactions on account of the hadronic contributions to the W and ϕ self-energies, the combination on the left-hand side of Eq. (6.40) does not. This can be understood from an analysis of these self-energies (see Appendix E). Equation (6.40) coincides exactly with the results of an unpublished calculation by Marciano (private communication), in which the renormalization of the vertex $\bar{u}d\phi^+$ (u and d stand for the u and d quarks) was studied in the quark model, neglecting the effects of the strong interactions.

Thus we have shown how all the ultraviolet divergences cancel in the renormalization of the $\phi h'h$ vertex. As the current algebra formulation can also be applied to processes involving spin $\frac{1}{2}$ hadrons, these results, in conjunction with our discussion of the renormalization of the $Wh'h$ vertex and the W and ϕ propagators, complete the proof of cancellation of divergences of order α in arbitrary semileptonic decays mediated by W and ϕ, to all orders in the strong interactions.

VII. COMPARISON WITH MUON DECAY AND CALCULATION OF RATE OF PION β DECAY

In Sec. VII.A we compare the decay rates for the superallowed Fermi transitions and μ decay, including the radiative corrections of order α, and briefly discuss the verification of the universality of the weak interactions in the sense of Cabibbo. In Sec. VII.B we use these results to calculate the rate of pion β decay.

A. Comparison with muon decay

In order to verify the universality of the weak interactions it is crucial to study the corrections of order α to the muon lifetime. Except for the universal hadronic contributions to the W propagator, these corrections are not affected by the strong interactions and can therefore be studied with the usual perturbative analysis based on individual Feynman graphs. However, as we are interested in comparing β and μ decays, it is very econonmical and convenient to discuss also μ decay in the framework of the current algebra approach. We must consider anew the diagrams of Figs. 1–8 with the understanding that μ^+ and ν_μ take the place of the initial and final hadrons of the semileptonic process.

Let us first discuss the diagrams of Fig. 1. The photonic corrections contribute only to the field renormalization of the muon. In discussing this term it is convenient to use the decomposition of Eq. (3.33). The contribution of the second term of Eq. (3.33) combines with the analogous contribution in the electron field renormalization and the γ-exchange graph of Fig. 6(a) to give the photonic corrections of the local V-A theory, except that there is an additional convergence factor $m_W^2/(m_W^2 - k^2)$. As is well known, the photonic corrections to μ decay are convergent in the local V-A theory (Kinoshita and Sirlin, 1959; Berman and Sirlin, 1962; and references cited therein). The additional convergence factor only generates additional terms of order $G_F\alpha(m_\mu^2/m_W^2) = O[G_F^2]$ which are, therefore, completely negligible. Thus these photonic corrections, when combined with the inner bremsstrahlung diagrams, reduce effectively to the traditional photonic corrections of the local V-A theory. The contributions of the "massive photon" propagator [first term of Eq. (3.33)] to Fig. 1 is best treated in conjunction with the virtual exchanges of the heavy bosons, W and Z. These corrections are described by quantities $\tilde{T}^\mu_{(a)}$ analogous to $T^\mu_{(a)}$ ($a = Z, \gamma>, W$) introduced in Eqs. (3.7), (3.17), (3.21), and (3.34).[40] In muon decay it is not necessary to subtract the pole terms $\tilde{B}^\mu(a)$ because this is automatically accomplished by the inser-

[40]In this section, amplitudes with a tilde are the analogs for μ decay of the corresponding expressions defined in Sec. III They can be obtained by replacing hadronic currents and states by leptonic currents and $|\mu_+\rangle$ and $|\nu_\mu\rangle$ in an obvious way.

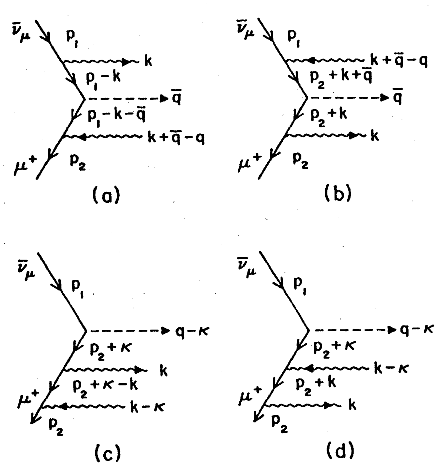

FIG. 9. Insertions associated with the three-current correlation functions in μ decay.

tions on the external legs of the mass counterterms, which in the absence of strong interaction complications are conventionally adjusted to cancel the $\bar{q}=q$ singularities. By using the Ward identities associated with the algebra of the leptonic currents we can derive for $\tilde{T}^{\mu}_{(a)}$ $(a=Z,\gamma>,W)$ expressions similar to Eqs. (3.13) and (3.16). We shall now show that the residual three-current correlation functions analogous to the two first terms in Eq. (3.13), when combined with the mass counterterms and tadpole diagrams, lead in the case of muon decay to corrections of $O[G_F^2]$. Consider, for example, the graphs involving virtual Z exchange. There are two types of diagrams contributing to $\tilde{T}^{\mu}_{(Z)}$ and $\tilde{D}_{(Z)}$: those in which the leptonic Z currents l_Z^{λ} enclose the vertex of the l_W^{μ} current or its divergence [see Figs. 9(a), (b)] and those in which the Z currents are attached to the μ^+ or ν_μ external legs, on either side of the W vertices. [Fig. 9(c), (d) illustrates the case in which the two currents are attached to the μ^+ external leg.] The contribution to Fig. 9(a) associated with virtual Z exchange is proportional to $\bar{v}_\mu \int d^4k/(k^2-m_Z^2) * 1/(\not{p}_1-\not{k}-\not{\bar{q}}-m_\mu) * 1/(\not{p}_1-\not{k}) * v_\nu$, where the $*$ represents the appropriate vertices of the Z and W currents (or the divergence of l_W^{μ}). As these vertices are momentum independent, the above integral is logarithmically divergent. However, when the differentiation with respect to $\bar{q}_\mu$ is carried out as indicated in the first two terms of Eqs. (3.13), the integral becomes convergent and, in fact, of $O(1/m_Z^2)$ so that its contribution to the decay amplitude is indeed of $O[G_F^2]$.

In discussing the contributions of Figs. 9(c), (d) it is convenient to set $\bar{q}=q-\kappa$ as indicated in the diagrams. Fig. 9(c) is proportional to

$$\tfrac{1}{2}\bar{v}_\mu\, \Sigma\,(p_2+\kappa)\, \frac{1}{\not{p}_2+\not{\kappa}-m_\mu} * v_\nu\,,$$

where $\Sigma\,(p_2+\kappa)=\int d^4k/(k^2-m_Z^2) * 1/(\not{p}_2+\not{\kappa}-\not{k}-m_\mu)*$, while Fig. 9(d) is given by the same expression with $\Sigma\,(p_2+\kappa)$ replaced by $\Sigma\,(p_2)$. Only the first two terms in the Taylor expansion of $\Sigma\,(P)$ in powers of P are divergent. An elementary consideration shows then that the terms of second and higher order in P are suppressed by a factor $(1/m_Z^2)$. Thus, to the order of our approximation, we may write: $\Sigma\,(P)=A+B\not{P}+C\not{P}\gamma_5$ where A, B, and C are independent of P. Summing the contributions of Figs. 9(c), (d) we obtain

$$\bar{v}_\mu\left[A+\frac{B}{2}\,(\not{p}_2+\not{\kappa}+\not{p}_2)+\frac{C}{2}\,(\not{p}_2+\not{\kappa}+\not{p}_2)\gamma_5\right]\frac{1}{\not{p}_2+\not{\kappa}-m_\mu} * v_\nu\,.$$

The insertion of tadpole diagrams and mass counterterms subtracts $\bar{v}_\mu\,\Sigma\,(\not{p}_2)v_\mu/(\bar{v}_\mu v_\mu)=A+Bm_\mu$ from the expression between square brackets (we use $\bar{v}_\mu\not{p}_2=\bar{v}_\mu m_\mu$) so that, after mass renormalization, the sum of the amplitudes of Figs. 9(c), (d) becomes $\bar{v}_\mu[(B/2)\not{\kappa}+(C/2)(\not{p}_2+\not{\kappa}+m_\mu)\gamma_5]\,1/(\not{p}_2+\not{\kappa}-m_\mu)*v_\nu=\bar{v}_\mu(B-C\gamma_5)/2*v_\nu$ which is the familiar field renormalization of the muon.[41] The important thing to note is that this expression is independent of κ so that it gives a vanishing contribution when the derivatives with respect to $\bar{q}_\alpha$ or, equivalently, κ_α are applied. Thus we conclude that the residual three-current correlation functions involving heavy boson propagators, when combined with tadpole diagrams and mass counterterms, give finite contributions of $O[G_F^2]$ to μ decay in complete analogy with the case of the superallowed Fermi transitions.

In summary, the only terms from $\tilde{T}^{\mu}_{(a)}$ $(a=Z,\gamma>,W)$ which contribute to order $G_F\alpha$ are the two-current correlation functions $\tilde{V}^{\mu}_{(a)}$. These lead to contributions $\tilde{\mathfrak{V}}_{(Z)}$, $\tilde{\mathfrak{V}}_{(\gamma>)}$, $\tilde{\mathfrak{V}}^{(1)}_{(W)}$, $\tilde{\mathfrak{V}}^{(2)}_{(W)}$ to the decay amplitude, analogous to the expressions we encountered in Sec. III. These terms can be studied following the discussion of Sec. IV.A. We first combine them with the contributions $\tilde{\mathfrak{U}}_{(Z)}$ and $\tilde{\mathfrak{U}}_{(\gamma)}$ arising from the diagrams in μ decay analogous to Fig. 2. As Eqs. (4.5) and (4.6) were derived using solely Ward identities, and as the time-time and time-space algebra of the leptonic currents is isomorphic to that of the hadronic currents, it is clear that Eqs. (4.5) and (4.6) are also valid for μ decay provided we replace all amplitudes by their "tilde counterparts." In particular, this demonstrates explicitly the universality of the divergent part in the renormalization of the weak coupling constant g. It is also easy to see that to order $G_F\alpha$ the finite parts of the corrections will be given by expressions $\tilde{A}^{\mu}_{(1)}$ and $\tilde{A}^{\mu}_{(2)}$ analogous to Eqs. (4.13) and (4.15). A quick way to understand this result is to remember that the expressions of Eqs. (4.13) and (4.15) were obtained effectively in the limit of the free field theory. As these contributions can be evaluated using the Bjorken–Johnson–Low limit (Bjorken, 1966; Johnson and Low, 1966) with canonical evaluation of commutators, the result follows again from the parallelism between the canonical hadronic and leptonic time-time and space-space commutators relevant to the derivation of these equations. As a consequence $A^{\mu}_{(1)}$ and $A^{\mu}_{(2)}$ and their tilde counterparts lead to corrections proportional to the zeroth-order amplitudes in β and μ decays, respectively, with identical coefficients. Thus these contributions are universal and cancel in the ratio of the decay probabilities, a result anticipated in a paper by Sirlin (1974c) on the basis of heuristic reasoning. On the other hand, the contributions proportional to $\bar{g}_S^2(\kappa^2)$ studied in Sec. IV are obviously induced by the strong interactions and do not have a counterpart in μ decay. However, as we pointed out, they are indeed very small.

[41]See, for example, Bailin (1973) and Bollini *et al.* (1973).

As explained in Sec. IV.C, the Z-exchange box diagrams corresponding to Figs. 6(a) and 6(b) are given by Eq. (4.35), with $\overline{Q}$ replaced by the average charge $-\frac{1}{2}$ of the ν_μ and μ^- fields. We recall that the first term on the right-hand side of Eq. (4.35) is a universal contribution, so only the term proportional to $\overline{Q}$ is significant in the discussion of universality. Finally, we note that the corrections to the W propagator (Sec. V.B) and to the $We\nu_e$ vertex can be disregarded as they obviously contribute equally to the two processes to order $G_F\alpha$.

In summary, the contributions which affect the ratio of decay probabilities of the superallowed Fermi transitions and muon decay fall into the following categories: (i) photonic corrections as calculated in the local V-A theory with a cutoff given by m_W; (ii) Z-exchange graphs which distinguish between the two processes because of the different values of $\overline{Q}$; (iii) small effects induced by the strong interactions in the asymptotic domain; and (iv) corrections of $O[G_F^2]$ which we neglect.

Taking into account these results, we are now in a position to write down explicitly the expression for the μ-decay probability as well as the spectrum for the superallowed Fermi transitions including the corrections of order α

$$\frac{1}{\tau_\mu}=\frac{\hat{G}_\mu^2 m_\mu^5}{192\pi^3}\left[1-\frac{8m_e^2}{m_\mu^2}\right]\times\left\{1+\frac{3}{5}\frac{m_\mu^2}{m_W^2}+\frac{\alpha}{2\pi}\left[\frac{25}{4}-\pi^2-\frac{3}{2}\tan^2\theta_W\frac{R}{R-1}\ln R+\cdots\right]\right\}, \tag{7.1}$$

$$Pd^3p_e=\hat{P}^0d^3p_e\left\{1+\frac{\alpha}{2\pi}\left[3\ln\left(\frac{m_W}{m_p}\right)+g(E,E_m)+6\overline{Q}\ln\left(\frac{m_W}{M}\right)+2C+3\overline{Q}\tan^2\theta_W\left(\frac{R}{R-1}\right)\ln R+\mathcal{A}_{\overline{g}}+\cdots\right]\right\}, \tag{7.2}$$

$$\hat{P}^0d^3p_e=\frac{\hat{G}_\mu^2\cos^2\theta}{8\pi^4}|M_v|^2(E_m-E)^2d^3pF(Z,E), \tag{7.3}$$

where the dots represent finite and universal contributions which do not affect the ratio of decay probabilities and terms of order α/m_W^2, which we neglect. In Eq. (7.1), the term $\frac{3}{5}(m_\mu^2/m_W^2)$ describes the effect induced by the vector-meson propagator to zeroth order in α. The term $(\alpha/2\pi)(25/4-\pi^2)$ is the photonic correction of the local V-A theory (Kinoshita and Sirlin, 1959; Berman and Sirlin, 1962). The last term arises from the Z-meson interchanges analogous to Fig. 6. In Eqs. (7.2) and (7.3) E_m is the end-point energy of the decay, E, p_e, and m_e are the energy, three-momentum, and mass of the electron, m_p is the proton mass, $g(E,E_m)$ is the function defined in Eq. (20b) of Sirlin (1967b), $F(Z,E)$ is the Fermi Coulomb function, R is defined in Eq. (4.14), and M_v is the vector matrix element ($M_v=\sqrt{2}$ for an isotriplet transition). The first two terms in the square bracket of Eq. (7.2): $3\ln(m_W/m_p)+g(E,E_m)$ are the universal photonic contributions arising from the vector current in the local V-A theory, with an effective cutoff $\Lambda=m_W$, while the third and fourth terms represent the asymptotic and nonasymptotic photonic corrections induced by the axial-vector current (for a discussion of the photonic corrections see Secs. IV.B and VII.B and references cited therein).[42] The fifth term in Eq. (7.2) arises from the Z-exchange graphs of Fig. 6, while $\mathcal{A}_{\overline{g}}$ are the very small effects proportional to $\overline{g}_S^2(\kappa^2)$ which are induced by the strong interactions in the asymptotic domain. They are discussed in Sec. IV and in Appendix C. The constant $\hat{G}_\mu$ in Eqs. (7.1) and (7.3) can be written as $\hat{G}_\mu/\sqrt{2}=g^2/8m_W^2$, where g and m_W are now renormalized coupling constants. For our purposes the exact definition of g is not important provided that the renormalization factors are universal, i.e., provided that g, and therefore $\hat{G}_\mu$, is defined in an identical manner in the two processes in terms of the bare constant g_0. It is worthwhile, however, to note that a straightforward and gauge-invariant procedure is to absorb in the definition of g the $(n-4)^{-1}$ pole terms arising from Eqs. (4.7) and (4.8) and from the field renormalization of W. The finite terms that remain in these contributions after subtraction of the pole terms are universal in character and can be included in the $\cdots$ contributions.

It is convenient at this stage to express Eqs. (7.1) and (7.2) in terms of a new coupling constant

$$G_\mu=\hat{G}_\mu\left[1-\frac{3\alpha}{8\pi}\tan^2\theta_W\frac{R}{R-1}\ln R+\cdots\right], \tag{7.4}$$

where the dots represent one-half of the universal contributions in Eqs. (7.1) and (7.2). Neglecting terms of order α^2 and $\alpha m_\mu^2/m_W^2$

$$\frac{1}{\tau_\mu}=\frac{G_\mu^2 m_\mu^5}{192\pi^3}\left[1-\frac{8m_e^2}{m_\mu^2}\right]\times\left\{1+\frac{3}{5}\frac{m_\mu^2}{m_W^2}+\frac{\alpha}{2\pi}\left[\frac{25}{4}-\pi^2\right]\right\}, \tag{7.5}$$

[42]The contribution of $g(E,E_m)$ to the total decay probability of ^{14}O and the choice of the constants C and M are given in Abers *et al.* (1968), Dicus and Norton (1970), and Sirlin (1974c). We recall that the function $g(E,E_m)$ contains a term $3\ln[m_p/(2E_m)]$ so that the sum of the first two terms in Eq. (7.2) is independent of m_p and its order of magnitude is, very roughly, $3\ln[m_W/(2E_m)]$.

$$Pd^3p_e = P^0 d^3p_e \left\{1 + \frac{\alpha}{2\pi}\left[3\ln\left(\frac{m_W}{m_p}\right) + g(E, E_m) + 6\bar{Q}\ln\left(\frac{m_W}{M}\right) + 2C + \frac{3}{2}\,\frac{R\ln R}{(R-1)}\tan^2\theta_W(1+2\bar{Q}) + \mathcal{A}_{\bar{g}}\right]\right\}, \qquad (7.6)$$

where P^0 is obtained from $\hat{P}^0$ by the replacement of $\hat{G}_\mu$ by G_μ. Equation (7.5) has now the traditional form of the radiative corrections in the local V-A theory (Kinoshita and Sirlin, 1959; Berman and Sirlin, 1962) and may be regarded as a definition of G_μ in terms of the experimental muon lifetime. In terms of the same constant, the spectrum for the superallowed Fermi transitions is given by Eq. (7.6) provided terms of order (α/m_W^2) are neglected. In the simplest version of the Weinberg–Salam model in which the Higgs scalars transform as a single isospinor, $[R/(R-1)]\tan^2\theta_W = -1$ and Eq. (7.6) reduces to

$$Pd^3p_e = P^0 d^3p_e \left\{1 + \frac{\alpha}{2\pi}\left[3\ln\left(\frac{m_Z}{m_p}\right) + g(E, E_m) + 6\bar{Q}\ln\left(\frac{m_Z}{M}\right) + 2C + \mathcal{A}_{\bar{g}}\right]\right\}. \qquad (7.7)$$

Except for the very small contribution $\mathcal{A}_{\bar{g}}$ induced by the strong interactions in the asymptotic domain, Eqs. (7.5), (7.6), and (7.7) coincide exactly with the results previously derived in Sirlin (1974c, 1975) on the basis of more heuristic reasoning. It is interesting to note that Eq. (4.36) and therefore Eq. (7.7) are still valid if the Higgs scalars belong to several isospinor representations.

The detailed verification of Cabibbo universality on the basis of Eqs. (7.6) and (7.7) (with $\mathcal{A}_{\bar{g}}$ set equal to zero) has been discussed in (Sirlin, 1974b, c; 1975; Roos, 1974; Wilkinson, 1975a, b; Wilkinson and Alburger, 1976; Hardy and Towner, 1975; Raman *et al.*, 1975). Because of its smallness (see end of Sec. IV.C) the inclusion of $\mathcal{A}_{\bar{g}}$ clearly has practically no effect on the main conclusions of these studies. As an example, in Sirlin (1974c) an analysis has been made on the basis of Eq. (7.7) with $\bar{Q} = \frac{1}{6}$,[43] which corresponds to the standard $SU(3)^c$ model with fractionally charged quarks ($\hat{q} = -\frac{1}{3}$). Using as input the ft value for ^{14}O decay, which is the lowest Z accurately measured transition, values for $\sin\theta$ in the range $0.222 \leqslant \sin\theta \leqslant 0.231$ for 74.6 GeV $\leqslant m_Z \leqslant$ 200 GeV were obtained. This range compares very well with the value of

$$f_+(0)\sin\theta = 0.220 \pm 0.002 \qquad (7.8)$$

derived from K^+_{e3} decays if, in fact, the second-order SU(3) breaking corrections to $f_+(0) - 1$ are reasonably small. It is also close to the value

$$\sin\theta = 0.232 \pm 0.003 \qquad (7.9)$$

obtained from the analysis of hyperon decays in the SU(3) limit (Roos, 1974). For a given G_μ, the radiative corrections in Eqs. (7.5) and (7.6) decrease the rate of muon decay by 0.42% and increase the rate of ^{14}O decay by 3.3 to 3.8% for the range of m_Z values mentioned above. The role of the corrections in the verification of universality can be cast in a more dramatic way by noting that, in their absence, the value of $\sin\theta$ derived from ^{14}O and μ decay would be $\sin\theta \approx 0.12$, in sharp disagreement with the $\Delta S = 1$ results.

Comparisons involving all the accurately measured superallowed Fermi transitions have been made by Wilkinson (1975a, b), Wilkinson and Alburger (1976), Hardy and Towner (1975), and Raman *et al.* (1975). In particular, in Wilkinson (1975a, b), the analysis of universality has been used to derive information about quark charges.

In the case of general symmetry breaking, the radiative corrections are given by Eq. (7.6), which depends now on two independent parameters, θ_W and R. In Sirlin (1975) this expression was applied to the case $\sin^2\theta_W = \frac{1}{6}$, which is the value suggested by a unified SU(5) theory of weak, electromagnetic, and strong interactions (Georgi *et al.*, 1974). It was noted that for the range $0.01 \leqslant R \leqslant 100$, the value of $\sin\theta$ derived from ^{14}O and μ decay still lies in the range $0.22 \lesssim \sin\theta \lesssim 0.23$ in satisfactory agreement with the $\Delta S = 1$ data.

B. Calculation of rate of $\pi\beta$ decay

As we mentioned in Sec. III, to zeroth order in α and q^2 the form factors for $\pi\beta$ decay have the values $F_1^{(0)}(0) = \sqrt{2}\cos\theta$, $F_2^{(0)}(0) = 0$. Thus the zeroth-order amplitude is given by

$$\mathfrak{M}^0 = -i\frac{G_\mu}{\sqrt{2}}\cos\theta\sqrt{2}\,(p+p')_\mu \bar{u}_\nu \gamma^\mu(1-\gamma_5)v_e. \qquad (7.10)$$

We note that, according to the observations after Eq. (3.9), p and p' are the physical π^+ and π^0 four-momenta and lie therefore on the corrected mass shells. Following the approach of Källén (1964) in calculating the uncorrected decay rate and including the effect of the radiative corrections we find

$$\frac{1}{\tau} = \frac{1}{\tau_0}(1+\delta), \qquad (7.11)$$

$$\frac{1}{\tau_0} = \frac{G_\mu^2\cos^2\theta}{30\pi^3}\left(1 - \frac{\Delta}{2m_+}\right)^3 \Delta^5 f(\epsilon, \Delta), \qquad (7.12)$$

$$f(\epsilon, \Delta) = \sqrt{1-\epsilon}\left(1 - \frac{9\epsilon}{2} - 4\epsilon^2\right) + \frac{15}{2}\epsilon^2\ln\left(\frac{1+\sqrt{1-\epsilon}}{\sqrt{\epsilon}}\right) - \frac{3}{7}\frac{\Delta^2}{(m_+ + m_0)^2}, \qquad (7.13)$$

where m_+ and m_0 are the masses of π^+ and π^0, $\Delta = m_+ - m_0$, $\epsilon = m_e^2/\Delta^2$, and δ represents the effect of the radiative corrections. In Eq. (7.13) we have included the leading correction in an expansion in powers of $\Delta^2/(m_+ + m_0)^2$. In the simplest version of the SU(2)×U(1) model, which we consider in this calculation, the radiative corrections are given by Eq. (7.7). The contribution $\bar{g}(E_m)$ of $g(E, E_m)$ to the integrated rate can be found from the tables of Wilkinson and Macefield (1977).[44] Using the

[43] The parameter ρ in Sirlin (1974c) equals $2\bar{Q}$ in the notation of the present paper.

[44] In Wilkinson and Macefield (1970) $(\alpha/2\pi)\bar{g}(E_m)$ is called $\delta^R(0, W_0)$. In many cases a good approximation for $\bar{g}(E_m)$ can be obtained from the expression

$$\bar{g}(E_m) \approx 3\ln\left(\frac{m_p}{2E_m}\right) + \frac{81}{10} - \frac{4\pi^2}{3}$$

which is an asymptotic formula for large E_m.

experimental value $\Delta = m_+ - m_0 = 4.6043 \pm 0.0037$ MeV, we find $E_m = 4.5293 \pm 0.0037$ MeV and $(\alpha/2\pi)\bar{g}(E_m) = 1.05 \times 10^{-2}$. In the case of $\pi\beta$ decay it is possible to obtain a rough estimate of the constant $2C$ on the basis of vector dominance and Weinberg sum rule arguments.[45] One considers the diagrams $\pi^+ \to \rho^+ + \omega \to \pi^0 + A_1^+ + \omega \to \pi^0 + W^+ + \gamma$ and $\pi^+ \to \rho^0 + A_1^+ \to \pi^0 + \omega + A_1^+ \to \pi^0 + \gamma + W^+$ with the W^+ and γ interacting with the leptons. The resulting corrections to the decay rate turn out to be a few times 10^{-4}, which is the same order of magnitude as $\mathcal{C}_{\bar{g}}$. As these effects are considerably smaller than the other terms and the uncertainties introduced by the experimental error in Δ, we neglect them. Setting $\bar{Q} = \frac{1}{6}$, $M = m_{A_1} = 1300$ MeV (Wilkinson, 1975a, b; Wilkinson and Alburger, 1976) and $m_Z = 78.2$ GeV, which corresponds to $\sin^2\theta_W = 0.35$, we obtain from Eq. (7.7) $\delta = 3.1 \times 10^{-2}$. Thus, in this model, the radiative corrections increase the rate of $\pi\beta$ decay by approximately 3%.

In order to calculate $1/\tau_0$, we shall take $\sin\theta = 0.226 \pm 0.009$. The center value is an average of the values of $\sin\theta$ extracted from K_{e3}^+ and hyperon decays in the SU(3) limit and the error range comprises both data [see Eqs. (7.8) and (7.9)]. Furthermore the center value is very close to the result obtained from Eq. (7.8) if one uses the Langacker–Pagels estimate of $f_+(0) - 1$ (Pagels, 1975). From here we obtain $\cos^2\theta = 0.949 \pm 0.004$. The constant G_μ is determined by comparing Eq. (7.5) with the experimental μ lifetime: $G_\mu = (1.4328 \pm 0.0001) \times 10^{-49}$ erg cm^3 $= (1.0268 \pm 0.0001) \times (10^{-5}/m_p^2)$ (Wilkinson, 1975a, b; Wilkinson and Alburger, 1976). Inserting the values of G_μ, $\cos^2\theta$, Δ, and δ in Eqs. (7.11) and (7.12) we find

$$1/\tau_0 = (0.391 \pm 0.003)\ \mathrm{sec}^{-1} \tag{7.14}$$

and

$$1/\tau = (0.403 \pm 0.003)\ \mathrm{sec}^{-1}, \tag{7.15}$$

where the error has been obtained by adding the uncertainties associated with $\cos^2\theta$ and Δ^5 in Eq. (7.12). The first source of error involves mainly the theoretical problem of determining corrections due to first- and second-order SU(3) breaking in hyperon and K_{e3}^+ decays, respectively. The considerable success of the Cabibbo approach suggests that SU(3) is a good approximate symmetry when applied to the relevant matrix elements of the $\Delta S = 0$ and $\Delta S = 1$ currents. In this context, it is worth noting that $(\delta \cos^2\theta)/(\cos^2\theta) = -\tan^2\theta(\delta \sin^2\theta)/\sin^2\theta$, so that the relative error in $\cos^2\theta$ is approximately one-twentieth of the relative error in $\sin^2\theta$. Thus, in the applications of universality, the uncertainties associated with our imprecise knowledge of SU(3) breaking are greatly diminished. The second class of errors mentioned above can be reduced by improving the experimental determination of $m_+ - m_0$.

Regarding the radiative corrections themselves, once the parameters of the gauge model have been chosen, the most important source of uncertainty appears to be our imprecise knowledge of the constant $2C$ in Eqs. (7.6) and (7.7), at least for $\pi\beta$ decay and the low-Z nuclei. In fact, as was the case before the advent of the gauge theories (Abers *et al.*, 1968; Dicus and Norton, 1970; Sirlin, 1968a, 1969), it is this nonasymptotic photonic contribution induced by the axial-vector current, the only contribution of order α which is not fully controlled in the analysis of universality. Fortunately, the discussions of Abers *et al.* (1968) and Sirlin (1974c), as well as the vector-dominance estimate mentioned in this section, suggest that this contribution is much smaller than the large corrections represented by the other terms of Eqs. (7.6) and (7.7).

[45]A. Sirlin (unpublished).

VIII. REMARKS

We have seen that in the underlying theory on which we based our considerations, the strong interactions have remarkably small effect on the radiative corrections to $\pi\beta$ decay and the superallowed Fermi transitions. In these studies the fact that the weak currents are partially conserved plays a very important role. In the evaluation of some of the finite parts we have used extensively the asymptotic freedom of the underlying theory and the assumption that the quasifree asymptotic behavior sets in early in comparison with the mass scale of the vector mesons. There are, however, some interesting gauge theories in which the strong interactions are not asymptotically free, such as, for example, that of Pati and Salam (1974). Our approach can, in principle, be applied to such theories but then one needs some special assumptions, related to approximate scaling, to justify our analysis of the finite parts in the two-current correlation functions. Thus, for example, one could assume that the renormalization group of the underlying theory has a fixed point at which $\bar{g}_S(\kappa^2)/(4\pi^2)$ is small and, again, that the quasifree asymptotic behavior sets in early. It should also be noted that in many of the non-asymptotically free models there are complicated mixings between the vector mesons of the weak and strong groups which, in general, depend on phenomenological angles. These particular complications must, of course, be handled in detail. However, it seems very likely that for wide ranges of these mixing angles and other parameters, the main features of the radiative corrections in these models are similar to those encountered in the present considerations.

Another interesting question is the possible application to other semileptonic processes. The theoretical framework of this paper is rather general and can, in principle, serve as basis for such discussions. Thus, for example, the explicit demonstration of the cancellation of divergences carried out in this work applies to arbitrary semileptonic decays. On the other hand, in the treatment of the finite parts associated with the traditional photonic corrections, we used extensively peculiar properties of the superallowed Fermi transitions and $\pi\beta$ decay, namely the fact that the momentum transfer is very small, being of $0(\alpha)$, and that $\Delta S = \Delta C = 0$ vector current is conserved to zeroth order in α. In a general process these simplifying characteristics are no longer extant and one must develop approximate methods to treat the photonic corrections. A similar problem arises when one considers the corrections induced in the matrix elements of the weak currents by the quark mass term Δm discussed in Sec. V. Because of the nonrenormalization theorem (Behrends and Sirlin, 1960; Ademollo and Gatto, 1964), which was established many years ago to answer precisely this question, the mass shifts do not alter the zero-momentum-transfer matrix element of the conserved vector current. In a general process, however, these renormalization effects will, in general, introduce nonvanishing corrections of order α. It seems clear, however, that the methods developed in this paper can be widely applied to study the radiative corrections of order $G_F\alpha$ to amplitudes not affected by photonic corrections or mass shifts, in arbitrary semileptonic processes. As an example, it is interesting to note that the current algebra formulation may be used to derive certain constraints imposed on a class of gauge theories by the absence of contributions of order $G_F\alpha$ in neutral $\Delta S=1$ processes such as $K_L\to\mu^++\mu^-$ (Hagiwara, 1977).[46]

In special circumstances, strong results on corrections to semileptonic processes may be obtained by other considerations. For instance, using gauge invariance and elementary considerations of analyticity, it has been possible to show that the coefficient of the logarithmic lepton mass singularity in the fractional corrections to the total decay rate $R(\pi\to l+\bar{\nu}_e)$ (where $l=e$ or μ) is not affected by the strong interactions (Marciano and Sirlin, 1976). This theorem has interesting implications for the prediction of $R(\pi\to e+\bar{\nu}_e)/R(\pi\to\mu+\bar{\nu}_\mu)$ and the verification of $e-\mu$ universality.

Finally, we should like to emphasize the importance of a detailed experimental analysis of pion β decay ($\pi^\pm\to\pi^0+e^\pm+\nu$) (Depommier *et al.*, 1968). Although measurements of the decay rate at the level of precision necessary to detect the effect of the radiative corrections seems difficult, the fact remains that this decay is one of the most elementary and fundamental processes of the weak interactions and should, in principle, be measured accurately.

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APPENDIX A: GENERALIZATION OF THE ANALYSIS OF BROWN, PREPARATA, AND WEISBERGER

As it was mentioned in Sec. III, the theoretical justification of the on-mass-shell perturbation formulae which served as the starting point of our discussion of the three-point correlation functions [see, for example, Eqs. (3.6)–(3.8)], is to be found in the work of G. Preparata and W. I. Weisberger (1968) and, in greater detail, in the basic paper of Lowell S. Brown (1969). In this appendix, we focus on the more complicated case in which the external particles are spin $\frac{1}{2}$ systems, provide a missing argument in Lowell Brown's discussion, and generalize the analysis to parity-nonconserving perturbations.

In our discussion we neglect possible $\mathcal{CP}$-violating interactions. In order to justify this approximation in the study of the three-current correlation functions, the following observations are relevant: (i) in the standard model with two left-handed quark doublets, four right-handed quark singlets, and one Higgs doublet, $\mathcal{CP}$ is automatically conserved (Kobayashi and Maskawa, 1973; Harari, 1976). (ii) If there is more than one Higgs doublet, $\mathcal{CP}$ may be violated in the Higgs sector (Weinberg, 1976). However, subject to our assumptions, the virtual exchange of Higgs mesons can only contribute to the radiative corrections to $O(G_F^2)$ which we regard as negligible. (iii) In models with right-handed doublets or more than four flavors, $\mathcal{CP}$ may be violated in the interactions of the vector mesons (Kobayashi and Maskawa, 1973; Harari, 1976). Although in such models the smallness of the $\mathcal{CP}$ odd amplitude is not explained naturally, the phenomenological analysis of $\mathcal{CP}$ violation implies that it is suppressed by a factor $\tilde{\epsilon}$ of $O(10^{-3})$, so that its contribution to the matrix elements of Fig. 1(b) is expected to be of $O(10^{-3}G_F\alpha)$. Furthermore, if the interaction of W with the usual $\Delta Q=1, \Delta S=\Delta C=\Delta C'=0$ current (C' stands for any additional additive quantum number beyond S and C) is invariant under $\mathcal{T}$, then the form factors induced by odd $\mathcal{T}$ perturbations contain a $\pi/2$ phase difference with the lowest-order form factors, so that there is no interference to order $G_F^2\alpha$ in the decay rate.[47] Thus we conclude that the neglect of the $\mathcal{CP}$ odd amplitudes in the study of the three-current correlation functions is an excellent approximation.

Returning to the discussion of the on-mass-shell perturbation formulae, we recall that Lowell Brown's method consists of the following steps. First one derives the perturbation formulae for the amplitudes of the weak currents, using canonical off-mass-shell perturbation theory, and considers the appropriate limits dictated by the reduction formulae. The second step is to verify that the on-mass-shell formulae epitomized by Eqs. (3.6)–(3.8), or their generalization to spin $\frac{1}{2}$ external systems, lead to the same answer. In order to do this it is necessary to consider in detail the insertions of the perturbation in the external legs and verify that they give rise to the field renormalizations of the external particles with the correct coefficients and to the alteration of the proper vertex of the weak current from the zeroth order to the perturbed mass shells as dictated by the reduction formulae.

Consider, for example, the perturbation associated

[46]These constraints were first discussed in Glashow and Weinberg (1977).

[47]This statement is rigorously true if one neglects the final-state Coulomb interaction or if one adopts the approximation that this interaction and the radiative corrections of order α factorize [see, for example, Dicus and Norton (1970)]. On general grounds, there may be interference effects of $O(\tilde{\epsilon}G_F^2Z\alpha^2)$, which are clearly negligible.

with virtual emission and absorption of W^μ along the hadronic line. The on-mass-shell perturbation formula analogous to Eq. (3.6) contains, before taking the limit $\bar{q}\to p-p'$, and performing the indicated subtractions, an amplitude proportional to

$$\frac{ig^2}{4(2\pi)^4}\int_k \frac{1}{k^2-m_W^2}\int_y e^{i\bar{q}\cdot y}\int_x e^{ik\cdot x} \times\langle p'|T[J_W^\mu(y)(J_W^\lambda(x)J_{\lambda W}^\dagger(0)+\text{h.c.})]|p\rangle$$

where the subscripts k, y, and x indicate the appropriate integrations, $p'^2=m_{h'}^2$, and $p^2=m_h^2$ (m_h and $m_{h'}$ are the masses of the initial and final hadrons to zeroth order in α). For our discussion, it is useful at this stage to separate out the contributions to the above amplitude which involve diagonal insertions of the perturbation in the external spinor line of momentum p'

$$C^\mu_{(W)}=-i\bar{u}_f(p')\Sigma(p',p-\bar{q})S_F'(p-\bar{q})\Gamma^\mu(p-\bar{q},p)u_i(p)\,, \tag{A1}$$

where u_f and u_i are the initial and final spinors, Γ^μ is the proper vertex of J_W^μ, S_F' is the propagator of the final spinor particle to all orders in the strong interactions and Σ, up to a factor, is the proper vertex of the perturbation:

$$S_F'(p_2)\Sigma(p_2,p_1)S_F'(p_1)=\frac{-g^2}{4(2\pi)^4}\int_k\frac{1}{k^2-m_W^2}\int_x e^{ik\cdot x}\int_u\int_v e^{ip_2\cdot u}e^{-ip_1\cdot v}\langle 0|T[\psi_{(u)}(J_W^\lambda(x)J_{\lambda W}^\dagger(0)+\text{h.c.})\bar{\psi}_{(v)}]|0\rangle\,. \tag{A2}$$

Here ψ is the field of the final spinor and p_1 and p_2 are arbitrary momenta not constrained to the mass-shell. In Eqs. (A1) and (A2) we have not considered off-diagonal insertions in which the perturbations mix different spinors in the external line, as these contributions are not singular in the limit $\bar{q}\to p-p'$. Because of this fact, such off-diagonal contributions will give obviously the same results as in the canonical perturbation theory and pose no problem. On the other hand, the diagonal insertions of Eq. (A1) are more delicate because the propagator S_F' is singular in the limit $\bar{q}\to p-p'$. For later use it is convenient to write

$$C^\mu_{(W)}=-i\bar{u}_f(p')\Sigma(p',p'+\kappa)S_F'(p'+\kappa)\Gamma^\mu(p'+\kappa,p)u_i(p)\,, \tag{A3}$$

where we have set

$$\bar{q}=p-p'-\kappa\,. \tag{A4}$$

Note that p and p' are on-shell momenta while $p'+\kappa$, p_2 and p_1 are in general off-shell. On general grounds of covariance

$$\Sigma(p_2,p_1)=a_1+a_2(\not p_1+\not p_2)+a_3(\not p_2-\not p_1)+a_4[\not p_2,\not p_1] +a_5\gamma_5+a_6(\not p_2+\not p_1)\gamma_5+a_7(\not p_2-\not p_1)\gamma_5+a_8[\not p_2,\not p_1]\gamma_5\,, \tag{A5}$$

where the $a_i (i=1,2\ldots8)$ are invariant function of p_2^2, $(p_2-p_1)^2$ and p_1^2. Using the transformation law under $\mathcal{CP}$

$$\mathcal{CP}J_W^\lambda(x)(\mathcal{CP})^{-1}=-J_{W\lambda}^\dagger(-\mathbf{x},x_0)\,, \tag{A6}$$

it is easy to derive the relation

$$\Sigma(p_2,p_1)=C\gamma_0\Sigma^T(-\tilde{p}_1,-\tilde{p}_2)\gamma_0C^{-1}\,, \tag{A7}$$

where C is the usual charge conjugation matrix, T means transpose, and $\tilde{p}_i(i=1,2)$ are the four vectors obtained from p_i by changing the sign of the space components. Inserting Eq. (A5) into Eq. (A7) we derive the relations

$$a_i(p_2,p_1)=\epsilon_i a_i(p_1,p_2)\,,\qquad \epsilon_i=\begin{cases}1 & (i=1,2,4,6)\\ -1 & (i=3,5,7,8)\end{cases}\,, \tag{A8}$$

where no summation over i is implied on the right-hand side of Eq. (A8). Equations (A8) are very important in comparing the results of the on-mass-shell perturbation formulae and the canonical off-mass-shell formulation. For example, in the latter approach what is relevant is the vertex $\Sigma(p_2,p_2)$ (that is, $p_1=p_2$ with p_2 generally off-shell), so that the form factors a_3, a_4, a_5, a_7, and a_8 obviously do not contribute. We must verify that the same is true for the on-shell perturbation formula in the limit $\bar{q}\to p-p'$ or, equivalently, $\kappa\to0$. To check this we need to prove that these form factors give vanishing contributions to Eq. (A3) as $\kappa\to0$. Writing the strong interaction propagator as $-iS_F'(p'+\kappa)=(\not p'+\not\kappa-m_{h'})^{-1}-\mathfrak{M}(\not p'+\not\kappa)$ where $\mathfrak{M}$ is not singular as $\kappa\to0$, we must show that

$$\lim_{\kappa\to0}\bar{u}_f(p')\{-a_3(p',p'+\kappa)\not\kappa+a_4(p',p'+\kappa)[\not p',\not\kappa]+a_5(p',p'+\kappa)\times\gamma_5-a_7(p',p'+\kappa)\not\kappa\gamma_5+a_8(p',p'+\kappa)[\not p',\not\kappa]\gamma_5\}\times\left[\frac{1}{\not p'+\not\kappa-m_{h'}}-\mathfrak{M}(\not p'+\not\kappa)\right]=0\,. \tag{A9}$$

Equations (A8) imply that $a_i(p_2,p_1)$ $(i=3,5,7,8)$ are proportional to odd powers of $p_1^2-p_2^2$ or, upon setting $p_2=p'$, $p_1=p'+\kappa$, that $a_i(p',p'+\kappa)$ $(i=3,5,7,8)$ are proportional to odd powers of $(p'+\kappa)^2-m_{h'}^2$. These factors cancel the singular part of the propagator, and the vanishing of the contributions of a_3, a_7, and a_8 then follows immediately because they involve additional cofactors proportional to κ. To show the vanishing of the a_5 contribution we note that $\mathfrak{M}(\not p'+\not\kappa)$ does not contribute because a_5 vanishes as $\kappa\to0$, we rationalize the singular part of the spinor propagator, and we anticommute with γ_5 so that the a_5 term becomes $-\lim_{\kappa\to0}\bar{u}_f(p')\not\kappa\gamma_5a_5(p',p'+\kappa)[(p'+\kappa)^2-m_{h'}^2]^{-1}$. This expression vanishes for the same reasons applied to a_3, a_7, and a_8. Because $a_4(p_2,p_1)$ is even under $p_2\leftrightarrow p_1$, we need a special argument to show the vanishing of its contribution. Again $\mathfrak{M}(\not p'+\not\kappa)$ is irrelevant because the cofactor of a_4 is proportional to κ. To treat the contribution of the singular part of the propagator we observe that after some elementary algebra it can be cast into the form $-\lim_{\kappa\to 0} a_4(p', p'+\kappa)\bar{u}_f(p')\{-\not{\kappa}+\kappa^2(2m_{h'}+\not{\kappa})/[\kappa^2+2p'\cdot\kappa]\}$ which obviously vanishes provided the limit is taken with some care.[48] We conclude that as $\kappa\to 0$, only the form factors a_1, a_2, and a_6 contribute, the same as in the canonical off-shell perturbation calculation. We note that the contribution of a_6 is not singular as $\kappa\to 0$: in fact,

$$\lim_{\kappa\to 0} a_6(p', p'+\kappa)\bar{u}_f(p')(\not{p}'+\not{p}'+\not{\kappa})\gamma_5\left[\frac{1}{\not{p}'+\not{\kappa}-m_{h'}}-\mathfrak{M}(\not{p}'+\not{\kappa})\right] = -a_6(p',p')\bar{u}_f(p')\gamma_5[1+2m_{h'}\mathfrak{M}(\not{p}')] \quad \text{(A10)}$$

which coincides with the answer provided by the canonical off-shell approach.

The contributions of a_1 and a_2 are more delicate because they are singular in the limit $\kappa\to 0$. They require the subtraction of a term

$$\delta C^{\mu}_{(W)} = \delta m_{h'}\bar{u}_f(p')\frac{1}{(\not{p}'+\not{\kappa}-m_{h'})}\mathcal{J}^{\mu}(\bar{q}), \quad \text{(A11)}$$

from Eq. (A3), where $\mathcal{J}^{\mu}(\bar{q})$ is the on-shell vertex of the current J^{μ}_W to zeroth order in α, evaluated at momentum transfer $\bar{q}^{\mu}$. By following the discussion of Lowell Brown one sees that, after subtracting this term and taking the limit $\kappa\to 0$, the contributions of a_1 and a_2 to Eq. (A3) contain the correct field renormalization of the external legs and the proper adjustment of the form factors from the zeroth order to the corrected mass shells. For physical applications, it is important to note that in this approach the zeroth-order contribution is given by

$$\Lambda^{\mu}_0 = \tfrac{1}{2}\bar{u}_f(p')\{f^0_1(q^2)\gamma^{\mu}+f^0_2(q^2)\sigma^{\mu\nu}q_{\nu}+f^0_3(q^2)q^{\mu} + [g^0_1(q^2)\gamma^{\mu}+g^0_2(q^2)\sigma^{\mu\nu}q_{\nu}+g^0_3 q^{\mu}]\gamma_5\}u_i(p), \quad \text{(A12)}$$

where p and p' are the four-momenta constrained to the corrected mass shells, $q^{\mu}=(p-p')^{\mu}$, and the form factors f^0_i and g^0_i $(i=1,2,3)$ are the zeroth-order form factors of the vector and axial-vector parts of J^{μ}_W evaluated at the zeroth-order mass shells. Equation (A12) is the counterpart of Eq. (3.3) when only the zeroth-order form factors $F^{(0)}_i$ are included, while Eq. (A11) (up to a $-i$ factor) is analogous to the second term in Eq. (3.8).

The discussion when the external hadrons are spin-0 objects is simpler because in that case the diagonal perturbation insertion analogous to $\Sigma(p_2,p_1)$ cannot have pseudoscalar components, as it is impossible to construct such quantities out of two momenta. Thus, for spin-0 external hadrons one can use directly the analysis of Brown (1969).

The discussion in this appendix complements Brown (1969) in two aspects when the external particles are spinors: for parity-conserving perturbations we provide an argument for the cancellation of the contribution of the a_4 form factor to Eq. (A3) which is important in establishing the validity of the method and, using $\mathcal{CP}$ invariance, we extend the on-mass-shell approach to parity-nonconserving interactions.

APPENDIX B: ABSENCE OF OPERATOR SEAGULL TERMS IN THE HADRONIC CORRELATION FUNCTIONS

We consider, for example, the amplitude $T^{\mu}_{(\gamma)}$ in Eq. (3.17), which represents the photonic corrections of order α to the vertex of the weak current J^{μ}_W. On general grounds such amplitude involves the matrix element $\langle p'|T^*[J^{\mu}_W(y)J^{\lambda}_{\gamma}(x)J^{\sigma}_{\gamma}(0)]|p\rangle$ where the T^* product may differ from the ordinary T product by terms proportional to $\delta^4(y-x)$, $\delta^4(y)$, $\delta^4(x)$, $\delta^4(y-x)\delta^4(x)$, and derivatives thereof. That is,

$$T^*[J^{\mu}_W(y)J^{\lambda}_{\gamma}(x)J^{\sigma}_{\gamma}(0)] = T[J^{\mu}_W(y)J^{\lambda}_{\gamma}(x)J^{\sigma}_{\gamma}(0)] + T^*[O^{\mu\lambda}_1(x)\delta^4(y-x)J^{\sigma}_{\gamma}(0)] + T^*[O^{\mu\sigma}_2(0)\delta^4(y)J^{\lambda}_{\gamma}(x)] + T^*[O^{\lambda\sigma}_3(0)\delta^4(x)J^{\mu}_W(y)] + O^{\mu\lambda\sigma}_4\delta^4(y-x)\delta^4(x)+\cdots, \quad \text{(B1)}$$

where the $\cdots$ represent possible terms involving derivatives of δ functions. Note that Eq. (3.17) is explicitly of order α. Thus to the order of our calculation all the operators in Eq. (B1) are to be considered to zeroth order in e or g. Dimensional analysis tell us that O_1, O_2, and O_3 must be of dimension 2, O_4 of dimension 1, while the operators represented by the dots can at most have dimension 1. However, in the underlying theory, the "color gauge-invariant" hadronic operators have dimensions ≥ 3. Thus operator seagull terms are not possible in Eq. (B1). As we mentioned in Sec. III, a c-number contribution to O_3 would correspond to a nonconnected amplitude and is subtracted. As there are no nontrivial seagulls and Schwinger terms, one expects on general grounds that the Ward identities we extensively used are free from anomalies. This can also be seen in a different way: anomalies in the Ward identities involving three operators usually arise because the equal-time commutator of two of the operators involves a limit $\epsilon\to 0$ and there is a contribution in which the third operator is "pinched" between the original two (Jackiw and Johnson, 1969). Whether or not an anomaly may possibly arise depends on whether this pinching contribution vanishes as $\epsilon\to 0$. This in turn depends on how the product of the three operators scales at short distances. In our case, the product of the three currents for $y\sim x\sim\epsilon$ cannot be more singular than ϵ^{-6} because it must be proportional to an operator of dimension 3. As the equal-time commutators in the Ward identities associated with $T^{\mu}_{(\gamma)}$ involve a seven-dimensional integration $[\int d^4y\int d^4x\,\delta(y_0)$, for example], the whole pinching contribution scales as ϵ and therefore vanishes as $\epsilon\to 0$.

It is interesting to contrast this situation with the well known case of the amplitude $e^2\langle 0|T^*[j^{\mu}_5(y)J^{\lambda}_{\gamma}(x)J^{\sigma}_{\gamma}(0)]|0\rangle$. Here the dimensionality of the product of the three currents is 9 and a c-number seagull proportional to a δ function times the derivative of another one is possible, because we consider a vacuum-to-vacuum amplitude. In fact, detailed study (Jackiw and Johnson, 1969) shows that such seagulls exist in perturbation theory and that, moreover, at least one of the Ward identities is anomalous. Note also that the analysis of the hadronic con-

[48]It is sufficient, for example, to write $\kappa=\xi\tilde{\kappa}$ where $\tilde{\kappa}$ is a fixed four-vector such that $p'\cdot\tilde{\kappa}\neq 0$ and then take the limit $\xi\to 0$.

tributions to the (WW), $(W\phi)$, and $(\phi\phi)$ propagators involves T^* products as c-number seagulls give nonvanishing contributions to vacuum expectation values.

APPENDIX C: EVALUATION OF THE CORRECTIONS OF ORDER $\bar{g}_S^2(\kappa^2)$

In this appendix we discuss the corrections of order $\bar{g}_S^2$ induced by the strong interactions in the evaluation of the finite parts. It is important to note that all the contributions derived by application of the Ward identities associated with the time-time and time-space algebra, without recourse to short-distance expansions, are completely unaffected by the strong interactions. These include most of the large photonic corrections associated with the vector current, which constitute by far the dominant contribution, and the term proportional to $2\cot^2\theta_W$ in Eq. (4.35). On the other hand, the finite contributions evaluated with the aid of the short-distance expansions are expected to be affected by small corrections induced by the strong interactions. These are the terms involving the tensors $T^{\lambda\rho}_{(Z)}$, $T^{\lambda\rho}_{(\gamma)}$, and $A^{\lambda\rho}_{(\gamma)}$ in Eqs. (4.5), (4.6), (4.21), and (4.32) and a small contribution to the photonic corrections discussed at the end of this appendix. In fact, although in the case of β decay these terms are not altered by the logarithmic factors associated with anomalous dimensions, the coefficient functions in the expansions do depend on $\bar{g}_S^2(\kappa^2)$. We recall that if the quarks transform according to the fundamental representation of the color group $SU(n)^c$, then in the asymptotically free theory the effective strong interaction coupling constant is given approximately for $\kappa^2 \geq \mu^2$ by

$$\frac{\bar{g}_S^2(\kappa^2)}{4\pi^2} = \frac{g_{SR}^2}{4\pi^2}\left\{1+\frac{g_{SR}^2}{48\pi^2}\,[11n-2f]\ln\left(\frac{\kappa^2}{\mu^2}\right)\right\}^{-1}, \qquad \text{(C1)}$$

where g_{SR}, μ, and f are defined after Eq. (4.16). We take $\mu^2 = O(M^2) \ll m_W^2$ where M is the mass scale discussed in Sec. IV. In the case $n=3$, we recover Eq. (4.16). As $\bar{g}_S^2(\kappa^2)$ approaches zero slowly, it is interesting to inquire about the magnitude of the $O[\bar{g}_S^2]$ corrections. As shown by Bég (1975) in a similar case, the terms of $O[\bar{g}_S^2]$ in the coefficient functions may be gleaned from the work of Adler and Wu-ki Tung on the perturbation theory contributions to the Bjorken–Johnson–Low limit, which was carried out in the Abelian model (Adler and Tung, 1969; Adler, 1970). In fact, to this order the only complication arising from the non-Abelian nature of the strong interactions is an additional factor $(\frac{1}{4})\lambda^a\lambda^a$ where the λ^a are the matrices corresponding to the fundamental representation of $SU(n)^c$ and the summation is over the n^2-1 generators of the group. Thus we obtain the coefficient of the $\bar{g}_S^2$ corrections by multiplying the Abelian result by the eigenvalue of the quadratic Casimir operator for the fundamental representation of $SU(n)^c$, namely $(n^2-1)/(2n)$. The corrections associated with Eq. (4.21) and the last term in Eq. (4.32) involve the components of $T^{\lambda\rho}$ which are antisymmetric in the Lorentz indexes. For these components one finds in the Abelian case a correction factor $1-3g_{SR}^2/16\pi^2$. The corrections of Eqs. (4.5), (4.6), and the second term in Eq. (4.32) involve the components of $T^{\lambda\rho}$ which are symmetric in the Lorentz indexes. The g_{SR}^2 contributions in this case are somewhat different (Adler and Tung, 1969; Adler, 1970). However, when the Lorentz indices are contracted to form $T^{\lambda}_{(Z)\lambda}$ and $T^{\lambda}_{(\gamma)\lambda}$ the same correction factor emerges. Therefore, if we wish to retain terms of $O[\bar{g}_S^2]$, the integrands in Eqs. (4.13), (4.15), (4.26), and the second and third terms between square brackets in Eq. (4.34) must be multiplied by

$$\left(1-\frac{3\bar{g}_S^2(\kappa^2)}{16\pi^2}\,\frac{(n^2-1)}{2n}\right).$$

For $SU(3)^c$ the correction factor reduces to

$$\left(1-\frac{\bar{g}_S^2(\kappa^2)}{4\pi^2}\right).$$

We now discuss a method to evaluate the relevant integrations when $\bar{g}_S^2(\kappa^2)/(4\pi^2)$ is inserted. We parametrize Eq. (4.16) or the more general expression of Eq. (C1) as

$$\frac{\bar{g}_S^2(\kappa^2)}{4\pi^2} = \frac{c_0}{1+c_1\ln(\kappa^2/\mu^2)} \qquad \text{(C2)}$$

and consider first

$$\Delta J_1(m_W^2) = -\int_{\mu^2}^{\infty} d\kappa^2\,\frac{1}{(\kappa^2+m_W^2)^2}\,\frac{c_0}{1+c_1\ln(\kappa^2/\mu^2)}, \qquad \text{(C3)}$$

which represents the correction to the integral in Eq. (4.15) induced by $\bar{g}_S^2$. As Eq. (C2) is only valid for $\kappa^2 \gtrsim \mu^2$ we have introduced a lower cutoff in the integral which, for simplicity, is assumed to be equal to μ^2. (Choice of a different cutoff $M^2 \ll m_W^2$ will simply give rise to differences of $O[G_F^2]$ which are negligible). Introducing a new variable $z=\ln(\kappa^2/\mu^2)$, ΔJ_1 can be cast in the form

$$\Delta J_1 = -\frac{1}{m_W^2}\int_0^{\infty}\frac{e^{z-\nu}}{(e^{z-\nu}+1)^2}\,\frac{c_0}{1+c_1 z}\,dz, \qquad \text{(C4)}$$

where

$$\nu = \ln(m_W^2/\mu^2). \qquad \text{(C5)}$$

Equation (C4) is very similar to integrals that appear frequently in statistical mechanics.[49] For large ν one can study these integrals by a variant of a method due to Sommerfeld (1928). As in that case the first factor in Eq. (C4) is sharply peaked at $z=\nu$, the idea is to expand the relatively slowly varying function $c_0/(1+c_1z)$ in a Taylor series about $z=\nu$. Using the identity

$$\frac{1}{1+c_1z} = \sum_{k=0}^{N-1}\frac{c_1^k(\nu-z)^k}{(1+c_1\nu)^{k+1}} + \left(\frac{c_1(\nu-z)}{1+c_1\nu}\right)^N\frac{1}{1+c_1z} \qquad \text{(C6)}$$

one readily obtains

$$\Delta J_1 = \frac{-1}{m_W^2}\left\{\sum_{k=0}^{N-1}\frac{c_0(-c_1)^k}{(1+c_1\nu)^{k+1}}\,I_k(\nu) + \mathcal{R}_N^{(1)}\right\}, \qquad \text{(C7)}$$

where $I_k(\nu)$ and the remainder $\mathcal{R}_N^{(1)}$ are given by

$$I_k(\nu) = \int_{-\nu}^{\infty}\frac{e^y}{(e^y+1)^2}\,y^k dy, \qquad \text{(C8)}$$

$$\mathcal{R}_N^{(1)} = \left(\frac{c_1}{1+c_1\nu}\right)^N\int_0^{\infty}\frac{e^{z-\nu}}{(e^{z-\nu}+1)^2}\,(\nu-z)^N\frac{c_0}{1+c_1z}\,dz. \qquad \text{(C9)}$$

Using the fact that $I_k(\nu)=I_k(\infty)+O(e^{-\nu})$ and $e^{-\nu}=\mu^2/m_W^2$

[49] See, for example, Sec. 11.1 in Huang (1963).

one finds

$$I_k(\nu) = O(\mu^2/m_W^2) \quad (k \text{ odd}), \tag{C10}$$

$$I_0(\nu) = 1 + O(\mu^2/m_W^2), \tag{C11}$$

$$I_k(\nu) = 2k!\,\zeta(k)(1 - \tfrac{1}{2}^{(k-1)}) + O(\mu^2/m_W^2) \quad (k \text{ even} \neq 0), \tag{C12}$$

where $\zeta(k) \equiv \sum_{l=1}^{\infty} 1/l^k$ is the Riemann's zeta function. An alternative representation for $\mathcal{R}_N^{(1)}$ is obtained by dividing the integration in Eq. (C9) in two domains: $z < \nu$ and $z > \nu$. Introducing $y = z - \nu$ and noting that $e^y(1+e^y)^{-2}$ is an even function we find

$$\mathcal{R}_N^{(1)} = \left(\frac{c_1}{1+c_1\nu}\right)^N \left\{ \int_0^{\nu} \frac{e^y}{(1+e^y)^2}\, y^N \frac{c_0\, dy}{1+c_1(\nu - y)} + \int_0^{\infty} \frac{e^y}{(1+e^y)^2} (-y)^N \frac{c_0\, dy}{1+c_1(\nu+y)} \right\}. \tag{C13}$$

Equations (C9) and (C13) show that $\mathcal{R}_N^{(1)}$ is manifestly positive for even N.[50] In this case a crude upper bound for $\mathcal{R}_N^{(1)}$ can be obtained by replacing $[1+c_1(\nu-y)]^{-1}$ with 1 and $[1+c_1(\nu+y)]^{-1}$ with $[1+c_1\nu]^{-1}$ in the first and second integrands, respectively, and extending the domain of integration in the first term to $y=\infty$. In this way we obtain the useful upper bound for $N \geq 2$[51]

$$\mathcal{R}_N^{(1)} < c_0\left(\frac{c_1}{1+c_1\nu}\right)^N \left(1 + \frac{1}{1+c_1\nu}\right) N!\,\zeta(N)\left(1 - \frac{1}{2^{N-1}}\right). \tag{C14}$$

The most practical way of evaluating ΔJ_1 for large ν is to retain just a few terms in the sum of Eq. (C7), neglect the terms in which k is odd [as these are of $O(\mu^2/m_W^2)$], and set an upper bound by applying Eq. (C14). For instance, using the values $\mu = 1$ GeV, $\bar{g}_S^2(\mu^2)/(4\pi) = \frac{1}{2}$ [given in Altarelli *et al.* (1976)], $m_W = 63$ GeV (which corresponds to the simplest version of the Salam–Weinberg model with $\sin^2\theta_W = 0.35$), setting $n=3$, $f=4$ in Eq. (C1), and $N=4$ in Eq. (C7) so that we retain only the terms $k=0$ and $k=2$ in the sum, we obtain $\Delta J = -0.0436 m_W^{-2}$. This represents a correction of -4.36% to the integral of Eq. (4.15) (which equals m_W^{-2}). It is close to the naive estimate of -4.25% given in Sec. IV.A, which corresponds to retaining only the $k=0$ term in Eq. (C7). Using Eq. (C14) and recalling that $\mathcal{R}_4^{(1)} > 0$ we learn that the relative error in the evaluation of ΔJ_1 is $< 0.64\%$ (in fact, we see that $-0.0439 m_W^{-2} < \Delta J_1 < -0.0436 m_W^{-2}$).

We now study

$$\Delta J_2(m_W^2) = -\int_{\mu^2}^{\infty} \frac{d\kappa^2}{(m_W^2+\kappa^2)\kappa^2}\, \frac{c_0}{1+c_1\ln(\kappa^2/\mu^2)} \tag{C15}$$

which represents the $\bar{g}_S^2$ correction to the integral in Eq. (4.26) (for simplicity we have set $M^2 = \mu^2$). Introducing again the variable z and performing a partial integration

$$\Delta J_2 = -\frac{1}{m_W^2}\,\frac{c_0}{c_1}\int_0^{\infty} \frac{e^{z-\nu}}{(1+e^{z-\nu})^2}\ln(1+c_1 z)\,dz\,. \tag{C16}$$

Integrating Eq. (C6) between ν and z, inserting the result in Eq. (C16), and carrying out the z integration we obtain

$$\Delta J_2 = -\frac{c_0}{m_W^2 c_1}\left[I_0(\nu)\ln(1+c_1\nu) - \sum_{k=1}^{N}\left(\frac{-c_1}{1+c_1\nu}\right)^k \frac{1}{k}\, I_k(\nu) - \mathcal{R}_N^{(2)}\right] \tag{C17}$$

where the remainder $\mathcal{R}_N^{(2)}$ is given by

$$\mathcal{R}_N^{(2)} = \left[\frac{c_1}{1+c_1\nu}\right]^N \int_0^{\infty} \frac{e^{z-\nu}}{[1+e^{z-\nu}]^2}\, dz \int_z^{\nu} (\nu - t)^N \frac{c_1}{1+c_1 t}\, dt\,. \tag{C18}$$

Equation (C18) shows that $\mathcal{R}_N^{(2)}$ is positive for odd N. In that case an upper limit can be derived by an argument similar to that used for $\mathcal{R}_N^{(1)}$. We divide the integration into two regions: $z<\nu$ where we set $(1+c_1 t)^{-1} \to 1$ and $z > \nu$ where we replace $(1+c_1 t)^{-1}$ by $(1+c_1\nu)^{-1}$; finally we extend the first region of integration to $z = -\infty$. This leads to the following upper bound for $N \geq 1$[52]

$$\mathcal{R}_N^{(2)} < c_1\left[\frac{c_1}{1+c_1\nu}\right]^N \left[1 + \frac{1}{1+c_1\nu}\right] N!\,\zeta(N+1)\left(1 - \frac{1}{2^N}\right). \tag{C19}$$

Using the same parameters as in the previous case and setting $N=3$ [so that we retain only the logarithmic and $k=2$ terms in Eq. (C17)] we obtain $\Delta J_2 = -0.628(1/m_W^2)$, while the uncorrected integral in Eq. (4.26) equals $8.286(1/m_W^2)$. Thus the $\bar{g}_S^2$ contribution in this case amounts to a -7.58% correction. This relatively large value is related to the fact that ΔJ_2 involves only one massive denominator and is therefore more sensitive than ΔJ_1 to the lower domain of integration. Using Eq. (C19) and recalling that $\mathcal{R}_3^{(2)} > 0$, we find that the relative error in our evaluation of ΔJ_2 is smaller than 0.13%.

Next we turn our attention to

$$\Delta J_3 = -\int_{\mu^2}^{\infty} d\kappa^2 \frac{1}{(\kappa^2+m_W^2)(\kappa^2+m_Z^2)}\, \frac{c_0}{1+c_1\ln(\kappa^2/\mu^2)} \tag{C20}$$

which represents the $\bar{g}_S^2$ corrections to the integral involving the second and third terms between square brackets in Eq. (4.34). Equation (C20) may be expressed in terms of ΔJ_2:

[50] Equation (C.7) is, of course, exact. If we let $N\to\infty$, it generates an asymptotic series. If terms of $O(e^{-\nu})$ are neglected we can stop the integration in the second term of Eq. (C13) at $y=\nu$. In this approximation, Eq. (C13) shows that $\mathcal{R}_N^{(1)} > 0$ also for odd N. At first hand this seems paradoxical because all the even k terms in the series between curly brackets in Eq. (C7) are also positive and grow in magnitude for sufficiently large k, so that the remainder cannot be positive for all N. What actually happens is that, for sufficiently large N, terms of $O(\mu^2/m_W^2)$ cannot be neglected and the sign of $\mathcal{R}_N^{(1)}$ for odd N is no longer fixed.

[51] For odd N, a slightly better upper bound is obtained by neglecting the second integral in Eq. (C13). For $N \geq 3$ this reduces to Eq. (C14) with $[1+(1+c_1\nu)^{-1}] \to 1$. For $N=1$, the last two factors in Eq. (C14) are replaced by ln2. See, however, the previous footnote regarding the odd N case.

[52] For even N we may obtain a slightly better upper bound by neglecting the $z>\nu$ domain in Eq. (C18). This leads to Eq. (C19) with $[1+(1+c_1\nu)^{-1}] \to 1$. For $\mathcal{R}_N^{(2)}$ with even N, observations analogous to $\mathcal{R}_N^{(1)}$ with odd N are applicable (see footnote 50).

$$\Delta J_3 = \frac{m_Z^2 \Delta J_2(m_Z^2) - m_W^2 \Delta J_2(m_W^2)}{m_Z^2 - m_W^2}\ . \tag{C21}$$

As an illustration, if we evaluate $\Delta J_2(m_Z^2)$ and $\Delta J_2(m_W^2)$ by setting $N=5$ in Eq. (C17) and use the values of m_Z and m_W corresponding to $\sin^2\theta_W = 0.35$, we find $\Delta J_3 = -0.0184(m_Z^2 - m_W^2)^{-1}$ with an error $\le 0.7\%$. The uncorrected integral in Eq. (4.34) equals $0.4308(m_Z^2 - m_W^2)^{-1}$. Thus the $\bar g_S^2$ contribution amounts to a -4.3% correction. This effect is easy to understand since in the above model $m_Z \approx 78$ GeV is not too different from m_W and we expect a result close to that found for ΔJ_1.

At this stage we note that the integral

$$\Delta J_4 = -(m_Z^2 - m_W^2)^2 \int_{\mu^2}^{\infty} \frac{\kappa^2 d\kappa^2}{(\kappa^2 + m_W^2)^2(\kappa^2 + m_Z^2)^2} \times \frac{c_0}{1 + c_1 \ln(\kappa^2/\mu^2)}\ , \tag{C22}$$

which represents the $\bar g_S^2$ correction to the integral in Eq. (4.13), can be expressed as

$$\Delta J_4 = (m_W^2 + m_Z^2)\Delta J_3 - m_Z^2 \Delta J_1(m_Z^2) - m_W^2 \Delta J_1(m_W^2)\ . \tag{C23}$$

If we evaluate the right-hand side of Eq. (C23) using the same parameters as before, we obtain an answer with a large relative error. [The reason is that Eq. (C23) involves in this application the subtraction of nearly equal quantities.] However, the precision is sufficient to establish that the $\bar g_S^2$ correction is smaller in absolute value than 7% of the uncorrected integral of Eq. (4.13). The latter contributes in this case a particularly small correction of -3×10^{-5} to the transition probability, so that the above upper bound is sufficient to show that ΔJ_4 is completely negligible.

Finally, we discuss a very small contribution of $O(\bar g_S^2)$ which arises in the study of the photonic amplitude $\mathcal{V}_{(\gamma<)}$. According to Eqs. (3.26), (3.33), and (3.35),

$$\mathcal{V}_{(\gamma<)} = \frac{ig^4}{4(2\pi)^4} \frac{\sin^2\theta_W}{(q^2 - m_W^2)} L_\mu \times \int d^4k\, \frac{m_W^2}{m_W^2 - k^2} \frac{1}{k^2} \frac{\partial}{\partial k_\mu} T^{\lambda}_{(\gamma)\lambda}(k)\ , \tag{C24}$$

where $T^{\lambda\rho}_{(\gamma)}$ is defined in Eq. (3.32).

Performing a partial integration, we find that two contributions $\mathcal{V}^{(i)}_{(\gamma<)}$ $(i=1,2)$ arise, proportional to $[m_W^2/(m_W^2 - k^2)][2k^\mu/k^4]$ and $-[m_W^2/k^2][2k^\mu/(m_W^2 - k^2)^2]$. As shown in Abers *et al.* (1968), in the limit of zero lepton momentum $\mathcal{V}^{(1)}_{(\gamma<)}$ cancels against part of the photon box exchange diagram and need not be evaluated explicitly. On the other hand, $\mathcal{V}^{(2)}_{(\gamma<)}$ is given by

$$\mathcal{V}^{(2)}_{(\gamma<)} = -\frac{ig^4}{2(2\pi)^4} \frac{\sin^2\theta_W}{(q^2 - m_W^2)} L_\mu \times \int d^4k\, \frac{m_W^2}{(m_W^2 - k^2)^2} \frac{k^\mu}{k^2} T^{\lambda}_{(\gamma)\lambda}(k)\ . \tag{C25}$$

We now observe that the integral in Eq. (C25) is the same that occurs in the first term of Eq. (4.6), that is, the integral $A^\mu_{(2)}$ evaluated in Eq. (4.15) and in the present appendix. In fact,

$$\mathcal{V}^{(2)}_{(\gamma<)} = -\tfrac{1}{2}\mathfrak{M}_{A(2)} = -\frac{\alpha}{8\pi}\mathfrak{M}^0(1 + m_W^2 \Delta J_1)\ , \tag{C26}$$

where $m_W^2 \Delta J_1$ represents the $O(\bar g_S^2)$ corrections discussed at the beginning of this appendix. The relative correction $-\alpha/(8\pi)$ was correctly included in Abers *et al.* (1968), where it was obtained heuristically by differentiation of an *ad hoc* cutoff function. On the other hand, the $O(\bar g_S^2)$ term $-(\alpha/8\pi)m_W^2 \Delta J_1$ represents a new, albeit very small, contribution.

APPENDIX D: QUARK MASS RENORMALIZATION

In this Appendix we discuss briefly the quark mass renormalization with respect to the weak and electromagnetic corrections.

In the "primed" representation in which the SU(2) doublets are expressed in terms of unmixed quark states, the bare mass term in the effective Lagrangian density is of the form $-\bar\psi' m_0' \psi' + \text{h.c.}$ where m_0' is, in general, a nondiagonal, non-Hermitian matrix acting on the flavor indices, which commutes with the electric charge operator Q and contains the Dirac matrices 1 and γ_5. The matrix m_0' is generated by spontaneous symmetry breaking from the most general gauge-invariant couplings of the quarks to the Higgs scalars. If the coupling constants of such Yukawa interactions are not restricted to being real or if more than one Higgs multiplet develops vacuum expectation values, m_0' will in general be also complex. Furthermore, in many cases of interest such as the minimal SU(2)×U(1) models with two doublets of left-handed quarks and four right-handed singlets, m_0' is an arbitrary matrix, safe for the requirement that it commutes with Q. As mentioned in Sec. V, this is related to the fact that in these models there are no natural zeroth-order relations among the particle masses and/or mixing angles. The arbitrariness of m_0' is easily verified. In the four-quark model a charge-preserving m_0' involves at most eight complex parameters. On the other hand, as the left- and right-handed quarks transform under the gauge group as doublets and singlets, respectively, the Higgs mesons which participate in the Yukawa couplings must necessarily be doublets. It is then sufficient to note that the Yukawa interactions of a single doublet involve eight independent coupling constants[53] (in general complex) which, after spontaneous symmetry breaking, contribute to m_0'.

To generate the mass counterterms we write $m_0' = m' - \delta m'$ where m' is regarded as the "renormalized" mass matrix[54] and $\delta m'$ represents the mass counterterms of order α. We may now diagonalize m' by applying independent unitary transformations to the left- and right-handed quark fields $\psi_{L,R} = U_{L,R}\psi'_{L,R}$ (Weinberg, 1973b). In the unprimed frame the mass terms in the Lagrangian become $-\bar\psi m \psi + \bar\psi \delta m \psi + \text{h.c.}$, where m is now real, diagonal, and free from γ_5 matrices. By choosing judiciously the phases of the quark fields, the charged and neutral currents of the four-quark model can be written, in the unprimed frame, in the form given in Sec. II (Kobayashi and Maskawa, 1973; Harari, 1976). Aside from the quark charges, they involved only two parameters, θ and θ_W.

[53]See, for example, Marciano and Sirlin (1975).

[54]Recall that m' is proportional to a divergent constant necessary for "strong" mass renormalization [see Weinberg (1973b) and Sec. II].

Since in our case the original matrix m_0' is arbitrary, so also is δm. In order to discuss the choice of δm, it is useful to make the following digression: We recall that the corrections of order α to strong interaction amplitudes can be classified as photonic and weak contributions. A convenient way to carry out the separation is to decompose the photon propagator according to Eq. (3.33). The contribution of the second term contains all the low-frequency components, and moreover the factor $m_W^2/(m_W^2-k^2)$ assures its ultraviolet convergence. It is then useful to think of this as the photonic contribution, while the corrections from the first term of Eq. (3.33) may be combined with the corrections arising from the virtual exchanges of the massive particles of the theory [W, Z, ϕ, etc...]. We shall refer to these latter contributions as "weak corrections." As implied by the analysis of Weinberg (1973b), the divergent parts as well as the finite parts of order α in the weak corrections involve the matrix elements of operators of class (1) and (3) discussed in Sec. V. This is not true of the finite parts of order α/m_W^2, but in the present context the latter are regarded as negligible. The simplest procedure is then to adjust the counterterms $\delta c_n O_n$ and therefore the arbitrary mass matrix δm, to cancel the weak corrections of order α to strong interaction amplitudes, both finite and divergent, as well as the divergent parts of $O(\alpha/m_W^2)$. That this subtraction can be done in a gauge-invariant manner follows from the following observations: (i) the sum of all corrections of order α to physical strong interaction amplitudes (including tadpole diagrams) must be gauge invariant and (ii) the only contributions of order α to such amplitudes that we have not subtracted are the photonic corrections which are separately gauge invariant.[55] As the corrections to weak amplitudes exhibited in Eqs. (5.2) and (5.3) involve exactly the same constants c_n and operators O_n as the weak corrections of order α to strong interaction amplitudes, they will be also canceled automatically by the $\delta c_n O_n$ contributions. Note that these cancellations are possible because $c_n O_n$ are independent of the nature of the hadronic states in the amplitude (although they do depend on properties of the underlying theory such as quark masses). With this choice of counterterms, the effective quark mass matrix to order α can be identified with the matrix m introduced in Eq. (2.7), which is real and diagonal, and the unprimed quark frame may be regarded as the physical frame. We recall that, in the models under consideration, the matrix m involve four arbitrary renormalized parameters m_α ($\alpha = u, d, s, c$). As pointed out in Sec. V, in order to study corrections to isospin symmetry, which is violated by the difference $m_u - m_d$, it is convenient to split m into symmetry-preserving and symmetry-breaking parts.

In theories with zeroth-order natural relations between the quark masses and/or mixing angles, δm is not arbitrary and $\delta c_1 O_1$ cannot be adjusted to cancel the weak corrections of order α. In this case there are finite weak contributions of order α to strong interaction amplitudes which can be described as arising from a correction δm_{wk} to the quark mass matrix. If the effective quark mass matrix $m+\delta m_{wk}$ is diagonal, the unprimed quark frame can be identified with the physical frame. Otherwise, in order to transform to the physical frame, it is necessary to diagonalize $m+\delta m_{wk}$ by means of suitable unitary transformations. Such transformations may, in principle, induce effects of order α on the weak interaction currents which must be included as part of the radiative corrections to the weak vertices.

[55]Alternative and physically equivalent procedures, in which the counterterms are adjusted to cancel only the divergent parts of the weak corrections to the quark mass matrix, have been discussed in the Appendix of Sirlin (1975) in connection with simple SU(2) ×U(1) models.

APPENDIX E: DIVERGENT PARTS OF THE HADRONIC CONTRIBUTIONS TO (WW), ($W\phi$), AND ($\phi\phi$) PROPAGATORS

In this appendix we discuss the hadronic contributions to the (WW), ($W\phi$), and ($\phi\phi$) propagators in the simplest version of the Weinberg–Salam model, show how the divergent parts are canceled or absorbed in the definition of the renormalized constants, and prove directly that the divergent part of $(\delta m_W/m_W - \delta g/g + [Z_\phi - 1]/2)$ does not depend on the dynamical details of the strong interactions, a result that was obtained indirectly in Sec. VI.B [see Eq. (6.40)]. Call $i(\Pi_h)_{\mu\nu}$, $-iC_h(q^2)q_\nu$, $-i\tilde{C}_h(q^2)q_\nu$, and $-i\Pi_h(q^2)$ the hadronic contributions to the unrenormalized (WW), ($W\phi$), (ϕW), and ($\phi\phi$) self-energies

$$(\Pi_h)^{\mu\nu} = -i\left(\frac{-ig}{\sqrt{2}}\right)^2 \int d^4x e^{iq\cdot x}\langle 0|T^*[J_W^\mu(x)J_W^{\dagger\nu}(0)]|0\rangle , \tag{E1}$$

$$-iC_h(q^2)q^\nu = \left(\frac{-ig}{\sqrt{2}}\right)^2 \frac{1}{m_W} \int d^4x e^{iq\cdot x}\langle 0|T^*[S(x)J_W^{\dagger\nu}(0)]|0\rangle , \tag{E2}$$

$$-i\tilde{C}_h(q^2)q^\nu = \left(\frac{-ig}{\sqrt{2}}\right)^2 \frac{1}{m_W} \int d^4x e^{iq\cdot x}\langle 0|T^*[J_W^\nu(x)S^\dagger(0)]|0\rangle , \tag{E3}$$

$$\Pi_h(q^2) = i\left(\frac{-ig}{\sqrt{2}}\right)^2 \frac{1}{m_W^2} \int d^4x e^{iq\cdot x}\langle 0|T^*[S(x)S^\dagger(0)]|0\rangle , \tag{E4}$$

where S is given in Eq. (6.8). Writing

$$(\Pi_h)_{\mu\nu} = A_h(q^2)g_{\mu\nu} + B_h(q^2)q_\mu q_\nu , \tag{E5}$$

contracting $(\Pi_h)_{\mu\nu}$ with q^μ, q^ν, and $q^\mu q^\nu$, and using Eqs. (6.4) and (6.12) we obtain the Ward identities

$$C_h(q^2)m_W = \tilde{C}_h(q^2)m_W = A_h(q^2) + B_h(q^2)q^2 , \tag{E6}$$

$$\Pi_h(q^2) = -[A_h(q^2) + B_h(q^2)q^2]\frac{q^2}{m_W^2} + \frac{g^2}{4m_W^2}\langle 0|S_1(0)|0\rangle , \tag{E7}$$

where $S_1(0)$ is defined in Eq. (6.2).

We recall that $B_h(q^2)$ and $A_h(q^2)$ require one and two subtractions, respectively. In fact, their divergent parts are of the form

$$A_h^{\text{div}}(q^2) = (\delta m_W^2)_h^{\text{div}} + (Z_W^h - 1)^{\text{div}}(q^2 - m_W^2) , \tag{E8}$$

$$B_h^{\text{div}}(q^2) = -(Z_W^h - 1)^{\text{div}} . \tag{E9}$$

The structure of Eqs. (E8) and (E9) follows from the observation that the divergent parts of (E5) can be removed, in principle, by W mass and field renormalization counterterms.

We shall now consider the effects of the various counterterms on the self-energies. For simplicity, we shall choose not to rescale the unrenormalized W and ϕ fields so that, except for the gauge-fixing term to be discussed later, we shall not be concerned with field renormalization counterterms. The W mass renormalization cancels the first term on the right-hand side of Eq. (E8) (we implicitly assume that tadpole counterterms have been adjusted to cancel the tadpole graphs).[56] Thus, after mass renormalization, the only divergent part of $A_h(q^2)$ is $(Z_W^h - 1)^{\text{div}}(q^2 - m_W^2)$. When this expression is inserted in the diagram of Fig. 7(a), its contribution is a multiple of the zeroth-order amplitude and $(Z_W^h - 1)^{\text{div}}$ is absorbed, in the usual manner, as part of the renormalization of the g_0's at the two ends of the W line. There are also counterterms associated with the mixed self-energies. These arise as follows: in the bare Lagrangian there is a term of the form $ig_0\lambda_0/2(W_\mu\partial^\mu\phi^\dagger - W_\mu^\dagger\partial^\mu\phi)$, where $g_0\lambda_0/2 = m_W^0$ is the unrenormalized W mass. In the t'Hooft–Feynman gauge, the gauge-fixing term is

$$-\left(\frac{\partial^\mu W_\mu}{\sqrt{Z_W}} + i\,\frac{m_W\phi}{\sqrt{Z_\phi}}\right)\left(\frac{\partial^\lambda W_\lambda^\dagger}{\sqrt{Z_W}} - i\,\frac{m_W\phi^\dagger}{\sqrt{Z_\phi}}\right),$$

where m_W is the renormalized W mass. The renormalization constants Z_W and Z_ϕ appear because W and ϕ represent here unrenormalized fields. (Recall that the gauge-fixing term does not involve counterterms when expressed in terms of the renormalized fields (Taylor, 1976). These $\sqrt{Z_W}$ and $\sqrt{Z_\phi}$ factors generate counterterms to the (WW), $(W\phi)$, (ϕW), and $(\phi\phi)$ self-energies. However, an elementary calculation shows that to the order of our calculation their contributions to the diagrams of Figs. 7(a)–(d) cancel among themselves. Thus, for our purposes, we can ignore these counterterms and imagine that $\sqrt{Z_W}$ and $\sqrt{Z_\phi}$ in the gauge-fixing term have been replaced by unity. After a partial integration the mixed contributions in the gauge-fixing term are given by $-im_W(W_\mu\partial^\mu\phi^\dagger - W_\mu^\dagger\partial^\mu\phi)$. When this is combined with the term $im_W^0(W_\mu\partial^\mu\phi^\dagger - W_\mu^\dagger\partial^\mu\phi)$ in the bare Lagrangian there remains a counterterm $-i\delta m_W[W_\mu\partial^\mu\phi^\dagger - W_\mu^\dagger\partial^\mu\phi]$. The effect of this counterterm is simply to replace $C(q^2) \to C(q^2) - \delta m_W$ in the mixed $(W\phi)$ and (ϕW) self-energies. There is also in the Lagrangian a counterterm of the form $\delta(|\phi|^2 + 2m_W^0\phi_1^0/g_0)$. The hadronic part δ_h of the renormalization constant can be adjusted so that the term linear in ϕ_1^0 cancels the hadronic contribution to the tadpole. Then δ_h is determined and $\delta_h|\phi|^2$ gives a contribution to the $(\phi\phi)$ self-energy which exactly cancels the zero-momentum-transfer correction represented by the second term on the right-hand side of Eq. (E7). In other words, we may ignore the tadpole graphs if at the same time we disregard the δ counterterms and the zero-momentum-transfer part of the $(\phi\phi)$ self-energy.[57]

We have already seen how the divergences in $A_h(q^2)$ are canceled or absorbed in the definition of the renormalized constant g. We now consider the contributions of the $B_h(q^2)$ term in Eq. (E5) in conjunction with the $(W\phi)$ and $(\phi\phi)$ self-energies and the relevant counterterms. When all these contributions are inserted in diagrams 7(a)–(d) they lead to the following amplitude

$$\mathfrak{M}_h^{(\phi)} = i\left(\frac{-ig}{\sqrt{2}}\right)^2 \langle p'|S|p\rangle\,(\bar{u}_\nu a_+ v_e)\left(\frac{-i}{q^2 - m_W^2}\right)^2 m_e \times\left\{B_h(q^2) - \frac{2}{m_W}\,[C_h(q^2) - (\delta m_W)_h] - \frac{1}{m_W^2}\left[\Pi_h(q^2) - \frac{g^2}{4m_W^2}\,\langle 0|S_1|0\rangle\right]\right\}, \tag{E10}$$

where the three terms between curly brackets represent contributions from the (WW), $(W\phi)$, and $(\phi\phi)$ diagrams, respectively. Using the Ward identities (E6) and (E7) to eliminate $C_h(q^2)$ and $\Pi_h(q^2)$, Eq. (E10) can be cast in the form

$$\mathfrak{M}_h^{(\phi)} = i\left(\frac{-ig}{\sqrt{2}}\right)^2 \langle p'|S|p\rangle\,(\bar{u}_\nu a_+ v_e)\left(\frac{-i}{q^2 - m_W^2}\right)^2 m_e \times\left\{[A_h(q^2) + (q^2 - m_W^2)B_h(q^2)]\,\frac{(q^2 - m_W^2)}{m_W^4} - \frac{[A_h(q^2) - (\delta m_W^2)_h]}{m_W^2}\right\} \tag{E11}$$

Recalling Eqs. (E8) and (E9) we see that the divergent part of $\mathfrak{M}_h^{(\phi)}$ is given by

$$(\mathfrak{M}_h^{(\phi)})^{\text{div}} = -\left(\frac{-ig}{\sqrt{2}}\right)^2 \langle p'|S|p\rangle\,(\bar{u}_\nu a_+ v_e)\frac{m_e}{m_W^2}\,\frac{i}{(q^2 - m_W^2)} \times\left[\frac{(\delta m_W^2)_h}{m_W^2} - (Z_W^h - 1)\right]^{\text{div}}. \tag{E12}$$

This can be written as

$$(\mathfrak{M}_h^{(\phi)})^{\text{div}} = \mathfrak{M}_0^{(\phi)}(Z_\phi^h - 1)^{\text{div}}, \tag{E13}$$

where

$$\mathfrak{M}_{(\phi)}^0 = \left(\frac{-ig}{\sqrt{2}}\right)^2 \langle p'|S|p\rangle\,\frac{i}{q^2 - m_W^2}\,\frac{m_e}{m_W^2}\,(\bar{u}_\nu a_+ v_e) \tag{E14}$$

is the zeroth-order amplitude of Fig. 5(a) and

$$(Z_\phi^h - 1)^{\text{div}} = \left[(Z_W^h - 1) - \frac{(\delta m_W^2)_h}{m_W^2}\right]^{\text{div}} = \frac{d\Pi_h^{\text{div}}(q^2)}{dq^2}. \tag{E15}$$

The second relation in Eq. (E15) follows from (E7), (E8), and (E9). Equation (E13) is equivalent to a counterterm of the form

$$\frac{-g}{\sqrt{2}m_W}\,\frac{(Z_\phi^h - 1)^{\text{div}}}{2}\,S\phi^\dagger$$

in the $\phi h'h$ vertex and an equivalent expression in the $\phi e\nu_e$ vertex. The counterterm

$$\frac{-g}{\sqrt{2}m_W}\,\frac{(Z_\phi^h - 1)^{\text{div}}}{2}\,S\phi^\dagger$$

may be included in $\delta\mathcal{L}\,\phi\bar{\psi}\psi$ of Eq. (6.23). Recalling that the only contribution to δg which is affected by the dynamical details of the strong interactions arises from the hadronic corrections to the W propagator and calling this part δg^h, we have

[56]See Sec. VI. B.

[57]See also the discussion in Taylor (1976) Sec. 14.6.

$$\left(\frac{\delta g^h}{g}\right)^{\rm div} = \tfrac{1}{2}(Z_W^h - 1)^{\rm div} .$$

Making use of Eq. (E15) we see that

$$\left(\frac{\delta m_W^h}{m_W} - \frac{\delta g^h}{g} + \frac{Z_\phi^h - 1}{2}\right)^{\rm div} = 0 . \tag{E16}$$

Thus, although the counterterms

$$\frac{\delta m_W^h}{m_W}, \frac{-\delta g^h}{g}, \text{ and } (Z_\phi^h - 1)/2$$

depend on the dynamical details of the strong interaction, their sum is finite.

In summary, the divergent part of $A_h(q^2)$ is canceled by the mass counterterm $(\delta m_W^2)_h$ and the renormalization of the weak coupling constant g, while the divergent parts of $B_h(q^2)$ and the hadronic parts of the $(W\phi)$ and $(\phi\phi)$ self-energies and associated counterterms reduce to the renormalization factor $(Z_\phi^h - 1)^{\rm div}/2$, which cancels against the counterterm

$$\left(\frac{\delta m_W^h}{m_W} - \frac{\delta g^h}{g}\right)^{\rm div}$$

in Eq. (6.23).

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