

# Feynman rules for lattice gauge theory\*

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We use functional techniques to give a simple derivation of Wilson's Feynman rules for strongly coupled gauge theory formulated on a lattice. Coupling sources to the various degrees of freedom, we obtain a compact formal expression for the Green's functions of the theory. The theory is rewritten in terms of creation and annihilation operators for quarks and "string bits" in a new space called "string space." This formulation emphasizes the close analogy with a lattice version of the string model. We also give a systematic diagrammatic procedure to evaluate all group integrations arising in the strong coupling expansion.

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## I. INTRODUCTION

Confinement is a conjectured property of a theory of quarks interacting with non-Abelian gauge fields (Politzer, 1974). If quarks are indeed confined, i.e., if free quarks do not exist, then we have a promising candidate for a theory of strong-interaction dynamics. This is all the more exciting in that it raises the possibility that all interactions realized in nature may be based on gauge theories.

Unfortunately little hard evidence exists for confinement in the strong-interaction gauge theory. Perturbation theory, historically the standard tool for studying interacting fields, has thus far failed to expose any clear indications of quark trapping (Appelquist *et al.*, 1976; Yao, 1976). Renormalization group arguments indicate that for low momenta, Green's functions in non-Abelian gauge theory will reflect a large effective coupling constant (Politzer, 1974). This in turn suggests that conventional perturbation theory may not be a reliable tool for investigating widely separated quarks. Furthermore, recently discovered nonperturbative classical solutions to gauge theories in Euclidian space suggests a much richer structure than that revealed by perturbation theory (Belavin *et al.*, 1975).

Thus there is a strong motivation for alternatives to the conventional perturbative approach. One problem immediately encountered upon leaving perturbation theo-

ry is how to handle the ultraviolet divergences occurring in a continuum field theory. Most renormalization techniques are based on the conventional Feynman diagram expansion. Any scheme avoiding perturbation theory must deal with these divergences so that the theory can be defined. This is the main reason for formulating a field theory on a lattice; it provides an ultraviolet cut-off that is independent of perturbation theory and yields a well defined theory. Indeed, as renormalizable theories are believed not to depend on the method of ultraviolet regularization, one can consider a lattice as just another cutoff procedure.

Believing that exact local gauge invariance is central to the question of confinement, Wilson formulated gauge theory on a lattice in a careful manner that keeps this symmetry (Wilson, 1974, 1975; Kadanoff, 1977). This provides a mathematically well defined theory in which one can search for approximation schemes alternative to conventional perturbation theory. When the coupling constant in this theory is made large while the lattice spacing is held fixed, an expansion in inverse coupling follows. This strong coupling expansion shows confinement because the gauge fields form themselves into strings with a finite energy per unit length. This results in an effective linear potential between widely separated quarks.

The strong coupling expansion is in terms of the bare gauge coupling defined with a cutoff in momentum space at the inverse lattice spacing. For an asymptotically free non-Abelian gauge theory, renormalization group arguments suggest that the bare coupling should go to zero as the cutoff is removed. Thus Wilson's expansion parameter is expected to become large in the continuum limit. One hopes that the confining nature of the theory will survive this limit, but conceivably one might encounter some nonanalyticity such as a phase transition to a theory with rather different properties, perhaps with unconfined quarks. To prove that such a transition does not occur for non-Abelian theories in four-dimensional space-time remains the major problem of lattice gauge theory.

Ordinary quantum electrodynamics in Wilson's formalism has the undesirable feature of confining electrons. The conjecture is that just such a phase transition to an unconfined phase does occur in the continuum limit of this theory. Mean field theory arguments (Balian *et al.*,

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1974, 1975) suggests that in enough space-time dimensions this type of transition will occur in both Abelian and non-Abelian theories. Migdal (Migdal, 1975; Kadanoff, 1976) has given approximate arguments that suggest four dimensions represent a critical case where the transition only occurs for the Abelian theory.

The strong coupling expansion of Wilson takes the form of a set of diagrammatic rules for calculating Green's functions of gauge-invariant operators. In this paper we rederive these rules in a functional notation. We couple all relevant degrees of freedom to external sources and obtain the diagrams of the theory by differentiation with respect to these sources. In addition, we show how all group integrations arising in the diagrams can be evaluated by further diagrammatic manipulations. Having derived the rules, we work out certain classes of diagrams that emphasize the equivalence of this model to a lattice version of the string model.

In Sec.II of this paper we review Wilson's locally gauge-invariant lattice theory. In Sec.III we introduce sources coupled to the degrees of freedom of the theory. We introduce the concept of "string space" in which these sources are annihilation operators for quarks and "string bits." Green's functions of Wilson's theory are expressed as matrix elements in this new space. In Sec.IV we develop a graphical algorithm for evaluating the group integrals occurring in the strong coupling expansion. Section V gives a listing of the complete set of Feynman rules. In Sec.VI we evaluate diagrams with topological structures emphasizing the analogy with the string model. We give a few concluding remarks in Sec.VII.

## II. THEORY

In this section we review the form of the lattice gauge theory to be used in later sections. The wave function of a particle interacting with a gauge field undergoes a path-dependent internal-symmetry rotation as the particle travels through space (Mandelstam, 1962; Yang, 1974). In Wilson's theory this concept of a nonintegrable group rotation provides the basic dynamical degrees of freedom for the gauge field. On a hypercubical space-time lattice a path is approximated by a series of straight line segments connecting nearest neighbor sites. Associated with each nearest neighbor pair of sites  $\{i, j\}$  is a group rotation matrix  $U_{ij}^{\alpha\beta}$ , where  $\alpha$  and  $\beta$  are the internal-symmetry matrix indices. Following a path in reverse direction gives the inverse rotation so we require

$$U_{ji}^{\alpha\beta} = (U_{ij}^{-1})^{\alpha\beta}. \tag{2.1}$$

The rotation associated with a particular path is the matrix product of the  $U_{ij}$  along the path.

The fermion degrees of freedom are described by a spinor field  $\psi_i^\alpha$  defined on each lattice site  $i$ . Here  $\alpha$  is the group index associated with the local gauge symmetry. We have suppressed both a four-valued spinor index and any additional indices describing ordinary unconfined quantum numbers such as isospin, strangeness, and charm.

We consider the gauge group SU(3) because this gives a simple construction of baryons out of three quarks.

This SU(3) represents a hidden symmetry under which all physical particles will be singlets, and must not be confused with the approximate SU(3) giving rise to the multiplet structure of observed hadrons. Under a global internal-symmetry rotation  $g^{\alpha\beta}$ , the Fermi field transforms as

$$\psi_i^\alpha \rightarrow g^{\alpha\beta} \psi_i^\beta, \tag{2.2}$$

where a sum over the index  $\beta$  is understood. The  $U_{ij}$  are all elements of SU(3); i.e., they are three-by-three unitary matrices of determinant one.

We treat fermions on the lattice with Wilson's projection operator technique. This is one of several methods of avoiding low-energy fermion states with momenta of the order of the inverse lattice spacing (Chodos and Healy, 1977; Drell *et al.*, 1976; Susskind, 1977). The theory is formulated in Euclidian space with the understanding that physical Green's functions are obtained by an analytical continuation to Minkowski space. The relation between this approach and a Hamiltonian version (Kogut and Susskind, 1975; Banks, *et al.*, 1977) has been discussed (Creutz, 1977; Luscher, 1977; Wilson, 1977b). We let  $a$  denote the nearest neighbor separation between sites on our four-dimensional hypercubical lattice. The action

$$\begin{aligned} S = & -a^4 \sum_{ij} \frac{A_{ij}}{2a} \bar{\psi}_i (1 - \gamma_\mu e_{ij}^\mu) U_{ij} \psi_j \\ & + a^4 \sum_i \bar{\psi}_i \left( \frac{4}{a} - m \right) \psi_i \\ & - \frac{1}{8g^2} \sum_{ijkl} P_{ijkl} \text{Tr}(U_{ij} U_{jk} U_{kl} U_{li}) \end{aligned} \tag{2.3}$$

defines the theory. Here we have suppressed internal symmetry indices,

$$A_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise,} \end{cases} \tag{2.4}$$

$e_{ij}^\mu$  is a unit vector pointing from site  $i$  to site  $j$ ,  $\gamma_\mu$  represents Euclidian Dirac matrices satisfying

$$\begin{aligned} [\gamma_\mu, \gamma_\nu]_+ &= 2\delta_{\mu\nu} \\ \gamma_\mu^* &= \gamma_\mu, \end{aligned} \tag{2.5}$$

a Euclidian sum over  $\mu$  is understood in  $\gamma_\mu e_{ij}^\mu$ ,  $g$  and  $m$  are the bare gauge coupling and quark mass matrix, and

$$P_{ijkl} = \begin{cases} 1 & \text{if } ijkl \text{ run around a "plaquette," an} \\ & \text{elementary square of side } a \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

The usual classical continuum gauge theory action follows from Eq. (2.3) by taking  $a$  to zero with

$$U_{ij} = e^{i a e_{ij}^\mu A_\mu}, \tag{2.7}$$

where  $A_\mu$  is a matrix representation of the vector potential for the gauge field. This limit is extensively discussed in Refs. 3 and 4.

The quantum theory is formulated in terms of Green's functions defined by the path integral

$$G(\psi_{i_1} \dots \psi_{i_n} \bar{\psi}_{j_1} \dots \bar{\psi}_{j_m} U_{k_1 i_1} \dots U_{k_m i_m}) = \frac{1}{Z} \int [d\psi dU] e^{-S} \psi_{i_1} \dots \psi_{i_n} \bar{\psi}_{j_1} \dots \bar{\psi}_{j_m} U_{k_1 i_1} \dots U_{k_m i_m}, \tag{2.8}$$

where

$$Z = \int [d\psi dU] e^{-S}, \tag{2.9}$$

We write the  $\psi$ 's and  $\bar{\psi}$ 's in a standard ordering that will be useful later in determining the overall sign of a strong coupling diagram. Because of baryon number conservation we need consider only Green's functions with an equal number of  $\psi$ 's and  $\bar{\psi}$ 's.

In this integral the Fermi fields are considered as anticommuting objects and are integrated over in a standard manner that will be made precise in the next section. The integration over the  $U_{ij}$  is done with the Haar measure over the group. For compact groups this measure has the properties

$$\int dg f(g) = \int dg f(g g_0) = \int dg f(g g_0^{-1}) = \int dg f(g^{-1}), \tag{2.10}$$

where  $g_0$  is an arbitrary group element. The measure is normalized such that

$$\int dg = 1. \tag{2.11}$$

The action in Eq. (2.3) is invariant under local gauge transformation. Given an arbitrary group element  $g_i$  for each site  $i$ , the substitution

$$\begin{aligned} \psi_i &\rightarrow g_i \psi \\ \psi_{ij} &\rightarrow g_i U_{ij} g_j^{-1} \end{aligned} \tag{2.12}$$

leaves the action unchanged. In terms of the Green's functions this means

$$G(\psi_{i_1} \dots \bar{\psi}_{j_1} \dots U_{k_1 i_1}) = G(g_{i_1} \psi_{i_1} \dots \bar{\psi}_{j_1} g_{j_1}^{-1} \dots g_{k_1} U_{k_1 i_1} g_{i_1}^{-1} \dots). \tag{2.13}$$

As this is true for arbitrary  $g_i$ , the Green's functions only depend on local gauge singlet combinations of the fields.

The theory has been defined without any reference to a gauge choice. In a previous publication (Creutz, 1977) we defined gauge fixing for the lattice theory and showed that Green's functions for gauge-invariant operators were unaffected by working in a particular gauge. The strong coupling rules are simpler if we do not select a gauge; therefore, in this paper we retain Eq. (2.8) which effectively integrates over all gauges.

As the lattice spacing  $a$  represents an ultraviolet cutoff, we ultimately desire to take it to zero. In the classical theory this limit can be taken with  $g$  and  $m$  held fixed; however, in the quantum theory it becomes necessary to allow these other parameters to vary with  $a$ . This point is familiar from renormalization theory for the conventional Feynman perturbation series where the bare coupling constant and masses are given a cutoff dependence such that physically measurable quantities remain finite as the cutoff is removed. Similarly, to take  $a$  to zero in the lattice theory, one should hold fixed

not  $g$  and  $m$  but rather a set of physical observables such as particle masses and physical couplings. This set should be large enough to determine  $g$  and  $m$  for any given value of  $a$ . In this way  $g$  and  $m$  will in general acquire dependence on the cutoff.

A hypothetical chirally symmetric world with massless pseudoscalar mesons and massive baryons may follow in the continuum limit if  $m$  is set identically to zero. This interesting theory depends on a single parameter  $g$  and thus only one observable need be fixed. This observable must be dimensional in order to set a mass scale; so we might as well choose it to be the nucleon mass. Thus follows the remarkable conclusion that when energies are measured in units of the nucleon mass, this theory has no remaining parameters. In particular the pseudoscalar meson coupling to the baryons must be uniquely determined in this chiral limit. The trading of a dimensionless coupling constant for a mass scale when the ultraviolet cutoff is removed is called "dimensional transmutation" (Coleman and Weinberg, 1973).

Conventional perturbative analysis of non-Abelian gauge theory shows that, if  $g$  for some finite cutoff is small enough that low-order perturbation theory in  $g$  applies, then  $g$  must go to zero as the cutoff is removed. This vanishing of the bare coupling in the continuum limit is called asymptotic freedom and has the physical consequence of a small effective quark-gluon coupling at short distances (Politzer, 1974). It is generally assumed that this is what occurs in the continuum limit of the strong-interaction gauge theory; however, it is conceivable that  $g$  never becomes small enough for conventional perturbation theory in  $g$  to apply but rather goes to some finite limit. In this case the theory would have qualitatively different phases corresponding to different directions from which  $g$  approaches its limiting value (Kadanoff, 1977). A major remaining problem is to show the absence of this multiple-phase alternative to the aesthetic picture of a single phase exhibiting both confinement and asymptotic freedom.

In what follows we derive an expansion in  $(1/g^2)$  at a fixed lattice spacing. If as  $a \rightarrow 0$ ,  $g$  also vanishes, then these rules are not in themselves phenomenologically useful in the continuum limit. Their value, rather, lies in the demonstration of confinement in one (hopefully the only) phase of the theory.

### III. SOURCES

The strong coupling diagrammatic expansion is most easily derived and compactly formulated through the introduction of external sources coupled to the degrees of freedom of the theory. In this approach, Green's functions are obtained by differentiating with respect to the various sources. Corresponding to the degrees of freedom  $U_{ij}^{\alpha\beta}$ ,  $\psi_i^\alpha$ , and  $\bar{\psi}_i^\alpha$  we introduce sources  $B_{ij}^{\alpha\beta}$ ,  $c_i^\alpha$ , and  $d_i^\alpha$  respectively. In the lattice theory the usual differential notation becomes rather cumbersome, so we shall use an equivalent operator formalism. We consider the sources as destruction operators in a new Hilbert space. We impose standard creation and destruction operator commutation relations

$$[B_{ij}^{\alpha\beta}, (B_{kl}^{\gamma\delta})^\dagger] = \delta_{\alpha\gamma} \delta_{\beta\delta} \delta_{ik} \delta_{jl}, \tag{3.1a}$$

$$[c_i^\alpha, (c_j^\beta)^\dagger]_* = [d_i^\alpha, (d_j^\beta)^\dagger]_* = \delta_{\alpha\beta} \delta_{ij}, \tag{3.1b}$$

$$[B_{ij}^{\alpha\beta}, B_{kl}^{\gamma\delta}] = 0, \tag{3.1c}$$

$$[c_i^\alpha, c_j^\beta]_* = [d_i^\alpha, d_j^\beta]_* = [c_i^\alpha, d_j^\beta]_* = 0, \tag{3.1d}$$

$$[B_{ij}^{\alpha\beta}, c_k^\gamma] = [B_{ij}^{\alpha\beta}, (c_k^\gamma)^\dagger] = [B_{ij}^{\alpha\beta}, d_k^\gamma] = [B_{ij}^{\alpha\beta}, (d_k^\gamma)^\dagger] = 0. \tag{3.1e}$$

Note that the fermion sources  $c$  and  $d$  are given anticommutation relations. In Eq. (3.1b) a unit matrix in spinor and other suppressed indices is understood.

Because of the forthcoming analogy with the string model, we call the new space in which these operators act "string space." The operator  $(B_{ij}^{\alpha\beta})^\dagger$  creates a "string bit" pointing from site  $i$  to site  $j$  with indices  $\alpha$  and  $\beta$  associated with its ends. Speaking somewhat loosely, we will say that  $(c_i^\alpha)^\dagger$  creates an antiquark with index  $\alpha$  at site  $i$ , while  $(d_i^\alpha)^\dagger$  creates a quark. Of course one should not confuse these "quark" states in string space with states in the physical Hilbert space of the Minkowski world. We use curving bras and kets to distinguish states  $|\psi\rangle$  in this new space from states  $|\psi\rangle$  in the physical Hilbert space. The general state in string space consists of quarks, antiquarks, and strings created by the operators  $c^\dagger$ ,  $d^\dagger$ , and  $B^\dagger$  acting on the "empty" state  $|0\rangle$  which satisfies

$$\begin{aligned} c_i^\alpha |0\rangle &= d_i^\alpha |0\rangle = B_{ij}^{\alpha\beta} |0\rangle = 0, \\ (0|0) &= 1. \end{aligned} \tag{3.2}$$

In a conventional continuum field theory, these operators could be regarded as creation and destruction operators for the ends of lines in a Feynman diagram (Creutz, 1975).

Coupling the sources to their respective fields, we define the generating state

$$\begin{aligned} (W| = (0| \int [d\psi dU] \exp \left\{ -S + \sum_i (c_i^\alpha \psi_i^\alpha + \bar{\psi}_i^\alpha d_i^\alpha) \right. \\ \left. + \sum_{i,j} A_{ij} B_{ij}^{\alpha\beta} U_{ij}^{\alpha\beta} \right\}. \end{aligned} \tag{3.3}$$

Here sums over repeated symmetry indices are understood and, as usual, spinor indices are suppressed. The utility of the generating state lies in the formula

$$\begin{aligned} G(\psi_{i_1} \dots \psi_{i_n} \bar{\psi}_{j_1} \dots \bar{\psi}_{j_m} U_{k_1 l_1} \dots U_{k_m l_m}) \\ = \frac{1}{Z} (W| d_{j_1}^\dagger \dots d_{j_n}^\dagger c_{i_n}^\dagger \dots c_{i_1}^\dagger B_{k_1 l_1}^\dagger \dots B_{k_m l_m}^\dagger |0), \\ Z = (W|0). \end{aligned} \tag{3.4}$$

Note that the order of Fermion operators is reversed in the right-hand side of this equation. Equation (3.4) means that functions of  $\psi$ ,  $\bar{\psi}$ , and  $U$  occurring in Green's functions can be replaced with functions of  $c^\dagger$ ,  $d^\dagger$ , and  $B^\dagger$  in matrix elements between the states  $(W|$  and  $|0\rangle$  in string space.

Perturbation theory formulated in string space begins by breaking  $S$  into two parts

$$S(\psi, \bar{\psi}, U) = S_0(\psi, \bar{\psi}, U) + S_I(\psi, \bar{\psi}, U). \tag{3.5}$$

We then write

$$(W| = (W_0| \exp \{-S'_I(c^\dagger, d^\dagger, B^\dagger)\}, \tag{3.6}$$

where

$$\begin{aligned} (W_0| = (0| \int [d\psi dU] \exp \{-S_0 + \sum_i (c_i \psi_i + \bar{\psi}_i d_i) \\ + \sum_{i,j} A_{ij} B_{ij}^{\alpha\beta} U_{ij}^{\alpha\beta}\}, \end{aligned} \tag{3.7}$$

and  $S'_I$  is just  $S_I$  with the order of all Fermi fields reversed. If the integrals defining  $(W_0|$  can be done, then a power series expansion of the exponential in Eq. (3.6) reduces the evaluation of a Green's function into an exercise in manipulation of creation and destruction operators.

Wilson's strong coupling expansion follows by considering

$$\begin{aligned} S_0 &= a^4 \sum_i \bar{\psi}_i \left( \frac{4}{a} - m \right) \psi_i, \\ S_I &= -a^4 \sum_{i,j} \frac{A_{ij}}{2a} \bar{\psi}_i (1 - \gamma_\mu e_{ij}^\mu) U_{ij} \psi_j, \\ &\quad - \frac{1}{8g^2} \sum_{ijkl} P_{ijkl} \text{Tr}(U_{ij} U_{jk} U_{kl} U_{li}). \end{aligned} \tag{3.8}$$

With this  $S_0$ , the  $\psi$  integral is easily done by completing the square

$$\begin{aligned} \int [d\psi] \exp \left\{ -a^4 \sum_i \bar{\psi}_i \left( \frac{4}{a} - m \right) \psi_i + \sum_i (c_i \psi_i + \bar{\psi}_i d_i) \right\} \\ = N \exp \left\{ \sum_i c_i (4a^3 - ma^4)^{-1} d_i \right\}. \end{aligned} \tag{3.9}$$

This equation may be regarded as a definition of the integral over the anticommuting Fermi fields, in which case the irrelevant normalization factor  $N$  can be defined to be one. We devote the next section to a discussion of the group integral  $dU$  in Eq. (3.7). For now we just define

$$\int [dU] \exp \sum_{i,j} A_{ij} B_{ij}^{\alpha\beta} U_{ij}^{\alpha\beta} = \prod_{\{i,j\}} D(B_{ij}, B_{ji}) \tag{3.10}$$

where  $\prod_{\{i,j\}}$  denotes a product over all nearest-neighbor pairs and

$$D(B_{ij}, B_{ji}) = \int dg \exp \{ B_{ij}^{\alpha\beta} g^{\alpha\beta} + B_{ji}^{\alpha\beta} (g^{-1})^{\alpha\beta} \}. \tag{3.11}$$

Putting all this together, we obtain the expression for the generating state

$$\begin{aligned} (W| = (0| \exp \left\{ \sum_i c_i (4a^3 - ma^4)^{-1} d_i \right\} \prod_{\{i,j\}} D(B_{ij}, B_{ji}) \\ \times \exp \left\{ -a^4 \sum_{i,j} \frac{A_{ij}}{2a} d_i^\dagger (1 - \gamma_\mu e_{ij}^\mu) B_{ij}^\dagger c_j^\dagger \right\} \\ \times \exp \left\{ \frac{1}{8g^2} \sum_{ijkl} P_{ijkl} \text{Tr}(B_{ij}^\dagger B_{ij}^\dagger B_{kl}^\dagger B_{li}^\dagger) \right\}. \end{aligned} \tag{3.12}$$

The four terms in this expression have a simple interpretation in string space. The first term destroys quark-antiquark pairs at a single site, the second term destroys sets of string bits associated with each nearest-neighbor pair of sites, the third term creates a quark-antiquark pair separated by one lattice spacing

and connected by a string bit pointing to the antiquark, and the last term creates elementary squares of string bits. This creation and destruction of quarks and string bits provides the basis of the diagrammatic rules to follow.

IV. GROUP INTEGRALS

In Eq. (3.11) we are left with the integral

$$D(B_{ij}, B_{ji}) = \int dg \exp \{ B_{ij}^{\alpha\beta} g^{\alpha\beta} + B_{ji}^{\alpha\beta} (g^{-1})^{\alpha\beta} \}. \quad (3.11)$$

This integral is a generating function for integrals of polynomials of group matrices through the relation

$$\begin{aligned} \int dgg^{\alpha_1\beta_1} \dots g^{\alpha_n\beta_n} (g^{-1})^{\gamma_1\delta_1} \dots (g^{-1})^{\gamma_m\delta_m} \\ = (0 | D(B_{ij}, B_{ji}) (B_{ij}^{\alpha_1\beta_1})^\dagger \dots (B_{ij}^{\alpha_n\beta_n})^\dagger (B_{ji}^{\gamma_1\delta_1})^\dagger \dots \\ \times (B_{ji}^{\gamma_m\delta_m})^\dagger | 0). \end{aligned} \quad (4.1)$$

Although we have not obtained a simple closed form for  $D(B_{ij}, B_{ji})$ , we will derive a straightforward diagrammatic algorithm for evaluating expressions of the form of Eq. (4.1) with the group SU(3).

We begin by noting that the inverse of an SU(3) matrix is given by its cofactors

$$(g^{-1})^{\alpha\beta} = \frac{1}{2} \epsilon^{\beta\gamma\delta} \epsilon^{\alpha\rho\sigma} g^{\gamma\rho} g^{\delta\sigma}, \quad (4.2)$$

where repeated indices are summed, and  $\epsilon^{\alpha\beta\gamma}$  is totally antisymmetric with  $\epsilon^{123} = 1$ . This allows us to eliminate  $B_{ji}$  from  $D(B_{ij}, B_{ji})$  with the formula

$$(0 | D(B_{ij}, B_{ji}) = (0 | D(B_{ij}) \exp \frac{1}{2} \epsilon^{\beta\gamma\delta} \epsilon^{\alpha\rho\sigma} B_{ji}^{\alpha\beta} (B_{ij}^{\gamma\rho})^\dagger (B_{ij}^{\delta\sigma})^\dagger | 0) \quad (4.3)$$

where

$$D(B_{ij}) = \int dg \exp(B_{ij}^{\alpha\beta} g^{\alpha\beta}). \quad (4.4)$$

To do this integral, observe that the invariance of the group integration measure implies

$$D(B_{ij}) = D(g_0 B_{ij} g_1^{-1}), \quad (4.5)$$

where  $g_0$  and  $g_1$  are arbitrary elements of SU(3). In Appendix A we prove that Eq. (4.5) implies that  $D(B_{ij})$  can only depend on the determinant of  $B_{ij}$ . We expand  $D(B_{ij})$  in powers of  $|B_{ij}| = 1/6 \epsilon^{\alpha\beta\gamma} \epsilon^{\delta\epsilon\rho} B_{ij}^{\alpha\delta} B_{ij}^{\beta\epsilon} B_{ij}^{\gamma\rho}$ ,

$$D(B_{ij}) = \sum a_n |B_{ij}|^n. \quad (4.6)$$

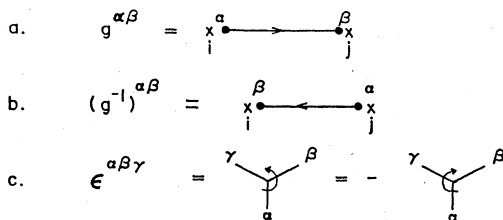


FIG. 1. Graphical representations of  $g_{ij}^{\alpha\beta}$ ,  $(g_{ij}^{-1})^{\alpha\beta}$ , and  $\epsilon^{\alpha\beta\gamma}$ .

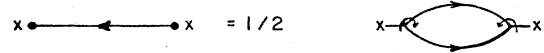


FIG. 2. Graphical representation of Eq. (4.2).

Since  $|g| = 1$  for  $g$  in SU(3), we have

$$(0 | D(B_{ij}) | B_{ij}^\dagger | = (0 | D(B_{ij}). \quad (4.7)$$

In Appendix B we prove the relation

$$(0 | |B_{ij}|^n | B_{ij}^\dagger | = n(n+1)(n+2)(0 | |B_{ij}|^{n-1}. \quad (4.8)$$

Combining this with Eqs. (4.6) and (4.7), we obtain

$$D(B_{ij}) = \sum_n \frac{2}{n!(n+1)!(n+2)!} |B_{ij}|^n. \quad (4.9)$$

With Eq. (4.3) this gives the result

$$\begin{aligned} (0 | D(B_{ij}, B_{ji}) = (0 | \sum_n \frac{2}{n!(n+1)!(n+2)!} |B_{ij}|^n \\ \times \exp \left\{ \frac{1}{2} \epsilon^{\beta\gamma\delta} \epsilon^{\alpha\rho\sigma} B_{ji}^{\alpha\beta} (B_{ij}^{\gamma\rho})^\dagger (B_{ij}^{\delta\sigma})^\dagger \right\}. \end{aligned} \quad (4.10)$$

The integral in Eq. (4.3) now becomes

$$\begin{aligned} \int dgg^{\alpha_1\beta_1} \dots g^{\alpha_n\beta_n} (g^{-1})^{\gamma_1\delta_1} \dots (g^{-1})^{\gamma_m\delta_m} \\ = \frac{2}{p!(p+1)!(p+2)!} (0 | |B_{ij}|^p \prod_{k=1}^n (B_{ij}^{\alpha_k\beta_k})^\dagger \\ \times \prod_{k=1}^m \frac{1}{2} \epsilon^{\delta_k\mu\nu\rho} \epsilon^{\gamma_k\sigma\tau} B_{ij}^{\mu\nu} (B_{ij}^{\rho\sigma})^\dagger (B_{ij}^{\tau\delta_k})^\dagger | 0), \end{aligned} \quad (4.11)$$

where  $p = 1/3(n + 2m)$  and must be an integer or the integral vanishes.

The manipulation of the  $B$ 's and  $B^\dagger$ 's necessary to evaluate Eq. (4.11) can be done graphically. For each  $g^{\alpha\beta}$  draw a directed line segment from site  $i$  to site  $j$  as sketched in Fig. 1(a). The index  $\alpha$  is associated with the  $i$  end of the line, and  $\beta$  with the  $j$  end of the line. For each  $(g^{-1})^{\alpha\beta}$  draw a directed line segment from  $j$  to  $i$  as in Fig. 1(b). Let the tensor  $\epsilon^{\alpha\beta\gamma}$  be denoted by a three-point vertex as in Fig. 1(c). With these conventions, Eq. (4.2) becomes Fig. 2. Using this result, any set of lines can be reduced to a set of lines oriented in one direction only. Such a set is then evaluated using Eq. (4.9) by grouping the lines into sets of three in all possible ways and combining the indices with  $\epsilon$  symbols as shown in Fig. 3. For  $3p$  lines there will be  $(3p)!/6^p p!$  terms in this expansion.

In evaluating these integrals, products of  $\epsilon$  symbols

$$\left( \begin{matrix} \uparrow \\ \uparrow \\ \uparrow \end{matrix} \right)^p = \frac{2}{(p+1)!(p+2)!} \left( \begin{matrix} \times \\ \times \\ \times \end{matrix} \right)^p + \text{PERMUTATIONS}$$

FIG. 3. Graphical evaluation of an integral of  $3p$  factors of  $g_{ij}$ .

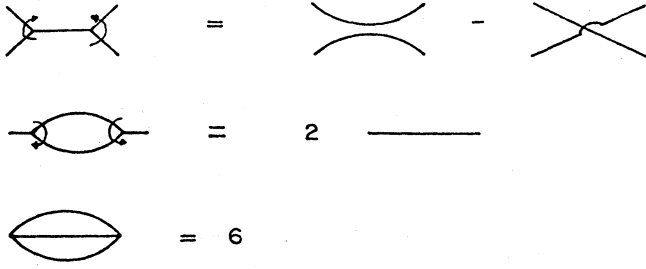


FIG. 4. Graphical representation of Eq. (4.14).

will occur. These can be simplified with the identities

$$\epsilon^{\alpha\beta\gamma} \epsilon^{\delta\rho\gamma} = \delta_{\alpha\delta} \delta_{\beta\rho} - \delta_{\alpha\rho} \delta_{\beta\delta}, \tag{4.12}$$

$$\epsilon^{\alpha\beta\gamma} \epsilon^{\delta\beta\gamma} = 2\delta_{\alpha\delta}, \tag{4.13}$$

$$\epsilon^{\alpha\beta\gamma} \epsilon^{\alpha\beta\gamma} = 6. \tag{4.14}$$

These identities are shown graphically in Fig. 4. The Kronecker delta symbols are represented by a nondirected line connecting two indices.

In the strong coupling diagrams for Wilson's theory, the indices of the  $U$ 's may be contracted in various ways. Several trivial identities which are useful in the graphical analysis are

$$UU^{-1} = 1 \tag{4.15}$$

illustrated in Fig. 5(a),

$$\epsilon^{\alpha\beta\gamma} U^{\alpha\delta} U^{\beta\rho} U^{\gamma\sigma} = \epsilon^{\delta\rho\sigma}, \tag{4.16}$$

illustrated in Fig. 5(b), and

$$\epsilon^{\alpha\beta\gamma} U^{\beta\delta} U^{\gamma\rho} = (U^{-1})^{\beta\alpha} \epsilon^{\delta\rho\alpha}, \tag{4.17}$$

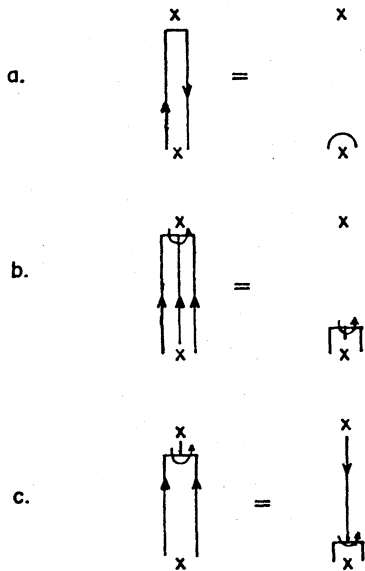


FIG. 5. Graphical representation of Eqs. (4.15)-(4.17).

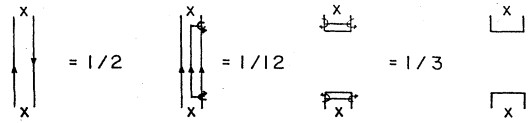


FIG. 6. Graphical representation of Eq. (4.18).

illustrated in Fig. 5(c). These identities are valid regardless of other lines present between the same sites.

To illustrate these rules, we work out the relation

$$\int dU U^{\alpha\beta} (U^{-1})^{\gamma\delta} = \frac{1}{3} \delta_{\alpha\delta} \delta_{\beta\gamma} \tag{4.18}$$

in Fig. 6. In Figs. 7 and 8 we give the graphical representation of the integrals

$$\int dU U^{\alpha\beta} U^{\gamma\delta} U^{\rho\sigma} = \frac{1}{6} \epsilon^{\alpha\gamma\rho} \epsilon^{\beta\delta\sigma} \tag{4.19}$$

and

$$\int dU U^{\alpha\beta} (U^{-1})^{\delta\gamma} U^{\rho\sigma} (U^{-1})^{\nu\mu} = \frac{1}{24} (3\delta_{\alpha\gamma} \delta_{\rho\mu} \delta_{\beta\delta} \delta_{\sigma\nu} + 3\delta_{\alpha\mu} \delta_{\gamma\rho} \delta_{\beta\nu} \delta_{\delta\sigma} - \delta_{\alpha\gamma} \delta_{\rho\mu} \delta_{\beta\nu} \delta_{\delta\sigma} - \delta_{\alpha\mu} \delta_{\gamma\rho} \delta_{\beta\delta} \delta_{\sigma\nu}). \tag{4.20}$$

We will use these integrals in particular diagrams in the following sections.

### V. STRONG COUPLING RULES

Equation (3.12) forms the basis of the graphical strong coupling expansion

$$\begin{aligned} \langle W \rangle = & \langle 0 | \exp \left\{ \sum_i c_i (4a^3 - ma^4)^{-1} d_i \right\} \times \prod_{\{i,j\}} D(B_{ij}, B_{ji}) \\ & \times \exp \left\{ -a^4 \sum_{i,j} \frac{A_{ij}}{2a} d_i^\dagger (1 - \gamma_n e_{ij}^\mu) B_{ij}^\dagger c_j^\dagger \right\} \\ & \times \exp \left\{ + \frac{1}{8a^2} \sum_{ijkl} P_{ijkl} \text{Tr}(B_{ij}^\dagger B_{jk}^\dagger B_{kl}^\dagger B_{li}^\dagger) \right\}. \end{aligned} \tag{3.12}$$

Expanding the last two exponentials in power series generates the strong coupling rules. Consider some particular Green's function

$$\begin{aligned} G(\psi_{i_1} \dots \psi_{i_n} \bar{\psi}_{j_1} \dots \bar{\psi}_{j_m} U_{k_1 l_1} \dots U_{k_m l_m}) \\ = \frac{1}{Z} \langle W | d_{i_1}^\dagger \dots d_{i_n}^\dagger c_{j_1}^\dagger \dots c_{j_m}^\dagger B_{k_1 l_1}^\dagger \dots B_{k_m l_m}^\dagger | 0 \rangle. \end{aligned} \tag{3.4}$$

The graphical rules for calculating this quantity can be read off from the above two equations:

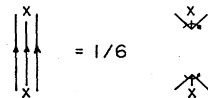


FIG. 7. Graphical representation of Eq. (4.19).

$$\begin{array}{c} \begin{array}{|c|} \hline x \\ \hline \uparrow \\ \hline \uparrow \\ \hline \uparrow \\ \hline x \\ \hline \end{array} = 1/24 \left( \begin{array}{c} \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \end{array} + 3 \begin{array}{c} \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \end{array} \right) \\ - \begin{array}{c} \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \\ \square \times \square \end{array} \end{array}$$

FIG. 8. Graphical representation of Eq. (4.20).

- (1) Draw a set of string bits, quarks, and antiquarks as created by the  $B^\dagger$ ,  $a^\dagger$ , and  $c^\dagger$  in Eq. (3.4).
- (2) Using the third factor in Eq. (3.12), create string bits connecting quark-antiquark pairs to produce a configuration where every site has an equal number of quarks and antiquarks. With several types of quarks, each species must balance separately. Every quark-string-antiquark combination generated by this rule gives a factor of  $(a^3/2)(1 - \gamma_\mu e_{ij}^\mu)$  to the amplitude. The spinor indices on these gamma matrices will be contracted in rule (4).
- (3) Use the last factor or "plaquette term" in Eq. (3.12) to create elementary squares of string bits, thus generating a configuration where every nearest neighbor pair of sites  $\{i, j\}$  has the number of string bits from  $i$  to  $j$  minus the number from  $j$  to  $i$  equal to a multiple of 3. Each plaquette created gives a factor  $1/2g^2$ . A set of  $n$  identical plaquettes gives a factor  $1/n!$ .
- (4) The first term in Eq. (3.12) now serves to connect the quarks and antiquarks at each site. In the process it connects string bit indices as well as spinor and any further indices specifying quark species. Each such "quark connection" gives a factor of  $(4a^3 - ma^4)^{-1}$  to the amplitude. The quark connections in a diagram will break up into sets corresponding to separate "quark lines" representing the path a particular quark travels through the diagram.
- (5) At this point we have a "strong coupling diagram." Now the group integrals are done by eliminating all string bits with the rules of the last section.
- (6) Some factors of minus one arise from the Dirac

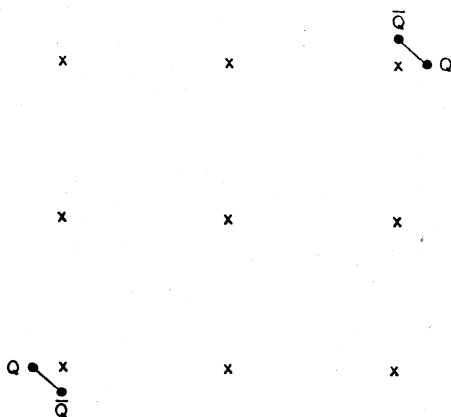


FIG. 9. The quark-antiquark pairs created by the Green's function in Eq. (5.1).

nature of the quarks. Each quark line forming an internal closed loop gives a factor  $(-1)$ . With  $G$  in the standard form of Eq. (2.8), if each  $\psi_i$  is connected by a quark line to  $\bar{\psi}_j$ , there are no more factors; otherwise multiply by minus one to the number of transpositions necessary to put the  $\bar{\psi}$ 's in the reverse order of the  $\psi$ 's that they are connected to. This corresponds to the rule for ordinary Feynman diagrams of giving an extra minus sign for each interchange of external fermion lines.

- (7) Sum over all distinct strong coupling diagrams, i.e., all different ways of carrying out steps 2 through 4.
- (8) Divide by  $Z$ , the sum of all vacuum fluctuation diagrams.

We now give an example to illustrate these rules. As the rules automatically pick out locally gauge-invariant combinations of the fields, we only consider Green's functions of such combinations. Taking a single quark species, we study

$$G(\bar{\psi}_i \gamma_5 \psi_i \bar{\psi}_j \gamma_5 \psi_j), \tag{5.1}$$

where  $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$ . This is the two-point function for the composite pseudoscalar field  $\bar{\psi}_i \gamma_5 \psi_i$ . Rule (1) instructs us to place quark-antiquark pairs at site  $i$  and site  $j$  as in Fig. 9. In this figure we let the vertical direction represent  $x_0$  and the horizontal direction represent  $x_1$ . In Fig. 10 we show one way of applying rule (2), thus adding quark-string-antiquark combinations so as to have all quarks paired with antiquarks. One dressing of the diagram with plaquettes by rule (3) is shown in Fig. 11. Making the quark connections with rule (4) gives Fig. 12. Finally rule (5) is carried out by repeatedly using the relations of Figs. 5(a) and 6 to give Fig 13. Combining the factors coming from the various rules, we obtain the contribution of this diagram  $D$  to the amplitude

$$G_D = -(a^3)^8 \left(\frac{1}{2g^2}\right)^3 (4a^3 - ma^4)^{-10} \left(\frac{1}{3}\right)^2 \text{Tr} \Gamma, \tag{5.2}$$

where  $\Gamma$  is the product of the Dirac projection operators around the diagram

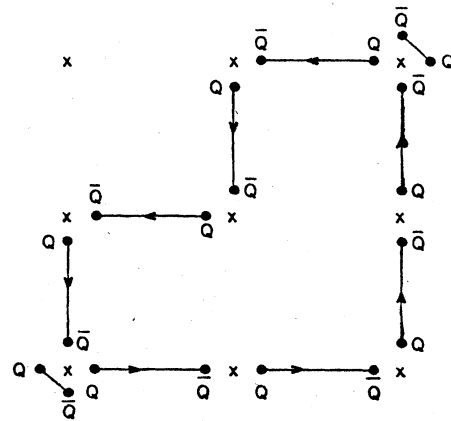


FIG. 10. A possible set of quark-string-antiquark combinations pairing all quarks with antiquarks.

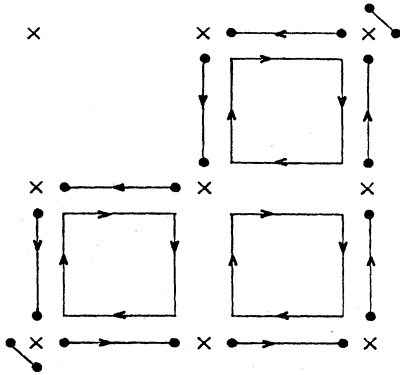


FIG. 11. Dressing the diagram with plaquettes.

$$\text{Tr}\Gamma = \frac{1}{2^8} \text{Tr}\{\gamma_5(1+\gamma_0)(1+\gamma_1)(1+\gamma_0)(1+\gamma_1)\gamma_5(1-\gamma_0)^2(1-\gamma_1)^2\} = \frac{1}{4}. \tag{5.3}$$

The overall minus sign in Eq. (5.2) is obtained by anti-commuting the Dirac fields into the standard ordering of Eq. (3.4) and then applying rule (6). Note that Eq. (5.2) can be put in the form

$$G_D = -3 \frac{1}{(4a^3 - ma^4)^2} \text{Tr}(\Gamma) \left(\frac{1}{4 - ma}\right)^P \left(\frac{1}{6g^2}\right)^A, \tag{5.4}$$

where  $P$  is the total quark line length in units of the lattice spacing,  $a$  and  $A$  is the area of the surface covered by plaquettes in units of  $a^2$ . This form provides the basis for the string analogy discussed in the next section. With several species of quark such that  $m$  is a matrix, the term  $(4 - ma)^{-P}$  becomes a matrix product around the loop.

VI. STRING ANALOGY

Equation (5.4) generalizes to all diagrams with the same topology as the diagram in Fig. 9, i.e., diagrams with a single surface of plaquettes bounded by a quark

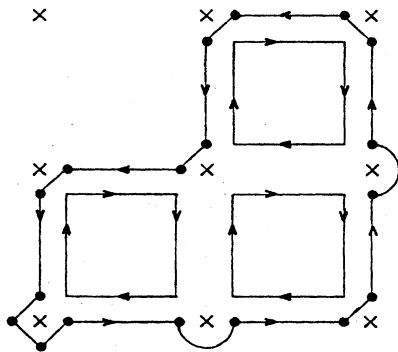


FIG. 12. Making quark connections.

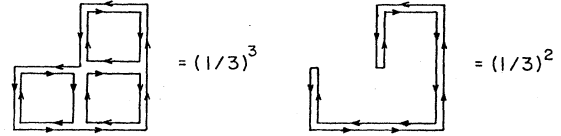


FIG. 13. Evaluating the group integrals.

line. This shows the striking connection between Wilson's theory and an oriented string model where the action associated with a particular world sheet swept out by a string contains a term proportional to its area. In two-dimensional space-time the connection of the string model with continuum non-Abelian gauge theory has been made precise (Bars, 1976). In the strong coupling limit the effective tension  $T$  in the string can be read off from Eq. (5.4),

$$T = \frac{1}{a^2} \log(6g^2). \tag{6.1}$$

The string analogy provides a useful topological classification of diagrams. For example, another topological class of diagrams contributing to the pseudoscalar two-point function of Eq. (5.1) is illustrated in Fig. 14. In this diagram the world sheet built up of plaquettes has a hole rimmed with a quark loop. The result for such a diagram is

$$G_D = +\frac{1}{3} \frac{3}{(4a^3 - ma^4)^2} \text{Tr}(\Gamma_E) \text{Tr}(\Gamma_I) \left(\frac{1}{4 - ma}\right)^P \left(\frac{1}{6g^2}\right)^A, \tag{6.2}$$

where  $\Gamma_E$  is the product of the Dirac matrices around the external loop, and  $\Gamma_I$  is the similar product around the internal loop. Here  $P$  is the total quark line length in units of  $a$ , including the internal quark loop. The factor of  $\frac{1}{3}$  in front of this expression forms the basis of the  $1/n$  topological expansion (t'Hooft, 1974). This factor, however, can be partially canceled when several species of quark contribute to the internal loop.

With  $SU(3)$  as the gauge group baryons can be constructed of three quarks. To study them consider the two-point function for a composite baryon field

$$G(\epsilon^{\alpha\beta\gamma} \psi_i^\alpha, a; \psi_j^\beta, b; \psi_k^\gamma, c; \epsilon^{\mu\nu\rho} \bar{\psi}_j^\mu, d; \bar{\psi}_k^\nu, e; \bar{\psi}_i^\rho, f). \tag{6.3}$$

Here the indices  $a \dots f$  denote previously suppressed spinor and quark indices. One topological class of graphs contributing to this Green's function is represented in Fig. 15. This diagram has three sheets of plaquettes intersecting in a line. With the help of Figs. 5(c)

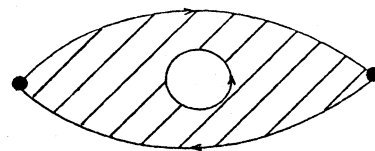


FIG. 14. A topological class of strong coupling diagrams.



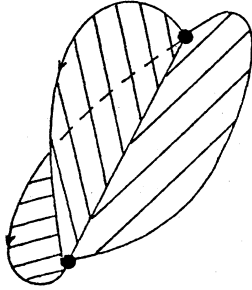


FIG. 15. A class of strong coupling diagrams contributing to baryon structure.

and 4 this diagram has the value

$$G_D = +6 \frac{1}{(4a^3 - ma^4)} \Gamma_{cd} \Gamma_{be} \Gamma_{af} \left( \frac{1}{4 - ma} \right)^P \left( \frac{1}{6g^2} \right)^A, \quad (6.4)$$

where the  $\Gamma$ 's are products of the Dirac projection matrices along their respective quark lines. As before,  $P$  is the length of the quark lines, and  $A$  is the total area of the surface covered by plaquettes, both in lattice units. Note the absence of any term directly dependent on the length of the line where the three sheets of plaquettes meet. Equation (6.4) also applies to "banana" diagrams as in Fig. 16. We see that in Wilson's model the "y" configuration of Fig. 15 and the "delta" configuration of Fig. 16 contribute to the structure of baryons; in general the proton is a resonant mixture of these configurations as well as more complicated ones.

Up to now we have only considered graphs with non-overlapping sheets of plaquettes. To complete the string analogy we need to discuss how overlapping strings interact. As a first step consider the topological configuration in Fig. 17. Here two sheets with quark edges intersect in a line. Along this line lie four string bits as in the left-hand side of Fig. 8. At first sight the right-hand side of Fig. 8 would seem to imply that the intersecting sheets would become irretrievably connected. Miraculously these interactions all cancel each other. Using the rule of Fig. 6 to reduce all string bits away from the intersection, one will find at the end of the intersection line that the string bits are connected in pairs as in Fig. 5(a). But the rule of Fig. 5(a) is valid

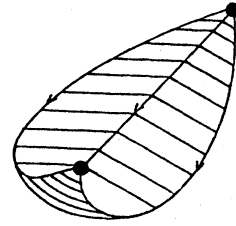


FIG. 16. Another diagram relevant for baryons.

independently of any additional string bits between the sites. Repeated use of this rule then completely reduces the intersection and thus we obtain the right-hand side of Fig. 17. Thus one sheet of the diagram can be evaluated without being affected by the other.

Now consider two surfaces intersecting as in Fig. 18 where the intersection is not terminated by a quark line. Doing the group integrals away from the intersection line again leaves an integral of a product of  $U$ 's along the intersection line as shown in Fig. 19. This remaining integral is evaluated using Fig. 8 on one link and then using Fig. 5(a) to reduce the others. This procedure gives a factor of two to the amplitude. This factor is just that needed to give the factorization indicated in Fig. 20.

When the world sheets of two strings intersect in an orthogonal manner, no net interaction occurs and the diagram factorizes. However, this noninteraction of strings breaks down when the sheets are not orthogonal, i.e., when their plaquettes share fundamental lattice squares. Straightforward application of the rule in Fig. 8 permits evaluation of diagrams which are superpositions of simple single-sheet diagrams. For parallel or antiparallel orientation of the superposed sheets, whenever several adjacent lattice squares are shared the resulting diagram is not simply the product of the superposed parts. In terms of the string analogy, this means there is a short-range string-string interaction. Thus strings can interact both through the quarks at their ends and by overlapping. A more intuitive interpretation of the latter interaction would be desirable. In particular how does one include such an interaction in a classical continuum string model?

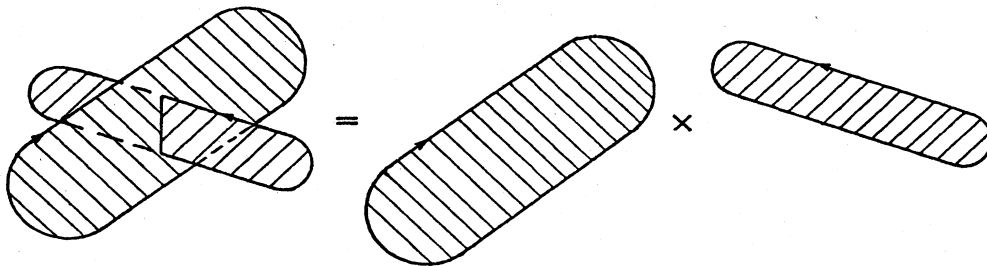


FIG. 17. A possible intersection of world sheets for two strings and its factorization.

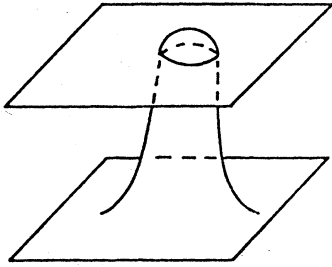


FIG. 18. An intersection without ends.

VII. CONCLUDING REMARKS

We have rederived Wilson's strong coupling rules for gauge theories formulated on a lattice. The main virtue of these rules is that they explicitly demonstrate the quark-confining nature of strongly coupled lattice gauge theory. Indeed the lattice theories remain the main motivation for believing that gauge theories can confine with a linear potential between quarks.

Confinement in the lattice theory follows because the theory is equivalent to a lattice version of the string model. This is a remarkable result in that the classical gauge and string theories appear to be totally unrelated. Working backwards, this equivalence may provide new insight into how to regulate divergences in the continuum string model.

Unfortunately, we feel that the rules are somewhat limited as a phenomenological tool, primarily because as the lattice spacing is made small the expansion parameter, the inverse bare coupling constant, becomes large. Nevertheless, some calculations of the mass spectrum have been made by going to the infinite  $g^2$  limit and treating the lattice spacing as a parameter representing the distance at which confinement forces become important (Wilson, 1977a). In this limit all the valence quarks of a hadron move together from site to site. The theory is not exactly solvable in this limit because hadrons can still interact by exchanging other hadrons. Wilson argues that neglect of these exchange processes is justified for large  $ma$ . In this simple limit the excitations of the theory consist of gauge singlet combinations of quarks moving from site to site together. Using this approximation and treating the lattice spacing and quark masses as adjustable parameters, Wilson was able to obtain approximate agreement with the observed hadronic spectrum.

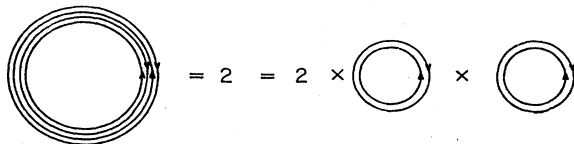


FIG. 19. A set of group integrals arising in evaluating the diagram in Fig. (18).

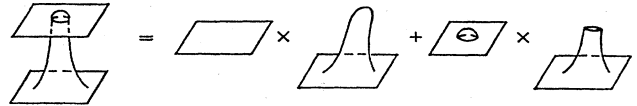


FIG. 20. Factorization of the diagram in Fig. (18).

APPENDIX A

*Theorem:* Let  $D(B)$  be an analytic function of an  $n$  by  $n$  matrix  $B$ . If  $D(B)$  satisfies

$$D(B) = D(g_0 B g_1) \tag{A1}$$

for arbitrary  $g_0$  and  $g_1$  in  $SU(n)$ , then  $D(B)$  is a function only of the determinant of  $B$ .

*Proof:* Because of the analyticity we can assume  $B$  to be real. Consider the element  $g$  of  $SU(n)$  defined by

$$g_{n1} = -g_{1n} = \frac{-B_{n1}}{(|B_{11}|^2 + |B_{1n}|^2)^{1/2}},$$

$$g_{11} = g_{nn} = \frac{B_{11}}{(|B_{11}|^2 + |B_{1n}|^2)^{1/2}},$$

$$g_{ij} = \delta_{ij} \text{ for } (i,j) \in \{(1,1), (n,n), (1,n), (n,1)\}. \tag{A2}$$

This satisfies

$$(Bg)_{n1} = 0. \tag{A3}$$

Repeating this process one can construct a  $g$  in  $SU(2)$  such that

$$(Bg)_{ij} = 0 \text{ for } i > j \tag{A4}$$

i.e.,  $B$  can be put in triangular form by multiplying by an element of  $SU(n)$ .

Assuming that  $B$  has been brought to triangular form, consider another element of  $SU(n)$

$$g_{ij} = \delta_{ij} e^{i\theta(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})}, \tag{A5}$$

where  $k$  and  $l$  will label two different rows of  $B$ . Taking this times  $B$  gives

$$(gB)_{ij} = B_{ij} e^{i\theta(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})} \tag{A6}$$

i.e., the  $i$ th and  $j$ th rows of  $B$  pick up a phase. Since  $D(B)$  is analytic we can expand it in a power series in the elements of  $B$ . Since replacing  $B$  by (A6) cannot change the value of  $D$ , each nonvanishing term in this expansion must contain an equal number of factors from rows  $k$  and  $l$ . The arbitrariness of  $k$  and  $l$  implies that all rows must have equal representation. A similar argument shows that all columns must also have equal representation. This coupled with the triangular nature of  $B$  implies that each term in the expansion of  $D(B)$  must be a power of the product of the diagonal elements of  $B$ , i.e., the determinant of  $B$ . This proves the theorem.

The analyticity of  $D(B)$  is essential; for a counterexample with nonanalytic  $D(B)$  consider

$$D(B) = \begin{cases} 1 & \text{if } B \in SU(n) \\ 0 & \text{otherwise} \end{cases}$$

Clearly the determinant of  $B$  does not determine if  $B$  is in  $SU(n)$  or not.

## APPENDIX B

Here we prove Eq. (4.8), which in more conventional differential notation reads

$$|\partial_{ij}| |B_{ij}|^n = n(n+1)(n+2) |B_{ij}|^{n-1}, \quad (\text{B1})$$

where

$$\partial_{ij} = \partial / \partial B_{ij}. \quad (\text{B2})$$

Note that

$$|B| = \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} B_{il} B_{jm} B_{kn}, \quad (\text{B3})$$

$$|\partial| = \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} \partial_{il} \partial_{jm} \partial_{kn} \quad (\text{B4})$$

$$\partial_{ij} |B| = \frac{1}{2} \epsilon_{ikt} \epsilon_{jmn} B_{km} B_{tn} = B_{ji}^{-1} |B|, \quad (\text{B5})$$

$$\partial_{ij} \partial_{kl} |B| = \epsilon_{ikm} \epsilon_{jln} B_{mn}, \quad (\text{B6})$$

$$|\partial| |B| = 6. \quad (\text{B7})$$

Counting the various ways the factors of  $\partial_{ij}$  can act on the factor of  $B_{ij}$  in  $|B|^n$ , we obtain

$$\begin{aligned} |\partial| |B|^n &= 6n |B|^{n-1} + n(n-1)(n-2) |B|^{n-3} |B^{-1}|B| \\ &\quad + 3n(n-1) |B|^{n-2} \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} B_{il}^{-1} |B| \epsilon_{jhp} \epsilon_{mnp} B_{pq}. \end{aligned} \quad (\text{B8})$$

Using

$$\epsilon_{ijk} \epsilon_{jhp} = 2\delta_{ip} \quad (\text{B9})$$

and some algebra we obtain Eq. (B1).

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