Pion fields in nuclear matter

A. B. Migdal

L. D. Landau Institute for Theoretical Physics, Academy of Sciences, Moscow, USSR

A comprehensive review is presented dealing with pion excitations and condensation in nucleon matter. A discussion of the behavior of bosons in scalar electric and nuclear fields is given along with a consideration of the possible existence of superdense and supercharged nuclei. The applications to nuclei and neutron stars are given.

CONTENTS

I. INTRODUCTION

The main purpose of this review is to investigate the physical consequences ensuing from the restructuring of the pion field in a sufficiently dense nucleon medium. Inasmuch as the energy gained thereby is proportional to the volume of the system, we are dealing with a phase transition ("pion condensation"). The most important physical consequence of this phase transition is the feasibility, in principle, of the existence of superdense nuclei, in which the energy gained in the phase transition offsets the energy loss due to contraction. The estimates of the condensation, as we shall show, are insufficiently accurate for any reliable conclusion concerning the existence of such anomalous nuclei. Further experiments will make it possible to redefine the parameters introduced into the theory, and to make more reliable statements.

The uncertainty in the estimate of the critical density does not exclude the possibility that a phase transition has already taken place in ordinary nuclei. In this case, the presence of the π condensate would be manifest in the presence, in the nucleus, of a periodic structure of the spin density of the nucleons with the wave vector $k_0 \simeq p_F$, which would exert an influence on the scattering of the nucleons and electrons by the nuclei. In addition, in second order in the amplitude of such a standing spin wave, a periodic structure can arise in the density of the neutrons and protons, with wave vector $2k_0$ and could explain the anomaly of the electric form factor of the nucleus, which manifests itself in the elastic scattering of the electrons.

Regardless of whether a phase transition has occurred or not, the closeness of the nuclei to condensation is revealed by a large number of experimental facts, namely, in all the phenomena in which an important role is played by exchange of one-pion excitation. The closeness to condensation makes the pion degree of freedom "soft," and this leads to an enhancement of the matrix elements which have pion symmetry. Among the phenomena greatly influenced by the decrease of the pion energy in nuclear matter are the following: the shift of the levels $0^-, 1^+, 2^-, \ldots$ in comparison with their position in the shell model, enhancement of $M1$ transitions with change of the orbital angular momentum by two units (l-forbidden transitions) and enhancement of the Gamow-Teller transitions. The softening of the pion degree of freedom should also be taken into account in calculations of the suppression of the spin part of the magnetic moments in the nucleus. Closeness to condensation exerts a particularly strong influence on the intensity of the l-forbidden transition—the intensity of these transitions exceeding in many cases the calculated values obtained without allowance for the closeness to condensation. The decrease of pion energy in the nucleus, predicted by the theory, manifests itself directly in the spectral data of the π-atom.

Pion condensation has interesting implications for the structure of neutron stars. At a neutron density noticeably lower than nuclear ($n_c \simeq 0.5n_0$), quasiparticles are produced which are bound states of a proton and a neutron hole, and which have the quantum number of the π^+ meson (we shall call them π_s^+ mesons). As a result of the condensation of such π_s^+ mesons, the equation of state of the neutron star becomes "softened," and the pressure becomes smaller at the same density. At a higher density, production of $\pi^-\pi_s^+$ meson pairs begins, as a result of which the compressibility can change the sign and a density jump takes place.

At the outer radius of the star there is a phase with a density $n \simeq n_c \simeq n_0$ and inside there is a superdense phase with $n \sim (3-5)n_0$.

To understand all these phenomena, it is very useful to trace the mechanism of π condensation, using first simple examples of condensation in an external scalar or electric field, and then the most interesting case of π condensation in a nucleon medium.

Pion condensation in an external field is of independent physical interest, apart from methodology, in connection with the possible existence of supercharged nuclei, in which the energy gain due to condensation in the electric field of the nucleus is partially offset by the energy loss due to the Coulomb field.

In the next section of this chapter an attempt will be made to present a simple and illustrative description of π condensation, both in external fields and in a nucleonic medium.

A. The physical nature of π condensation

1. Restructuring of the vacuum in strong fields

Problems connected with the restructuring of the vacuum in strong fields have recently been intensively investigated. This restructuring takes place in those cases in which the energy of an individual particle drops below $-mc^2$, so that the production of particles from the vacuum becomes possible. Such a restructuring of an electron–positron vacuum takes place in the field of a nucleus having a large charge, $Z > Z_c$ ($Z_c \simeq 170$) when the energy level of the K electron drops to a value $-mc^2$.

At $Z > Z_c$ the ground state of the vacuum corresponds to a charge $-2e$, distributed in the vacuum with a density close to the density in the K shell. With further increase of Z, the next restructuring of the vacuum takes place when the energy level of L electrons reaches $-mc^2$— the ground state of the vacuum will then correspond to a charge equal to the sum of the charges of the K and L shells. Thus the vacuum is so restructured that there is one vacuum electron charge for each nondegenerate level crossing the boundary $-mc^2$.

A much more appreciable restructuring of the vacuum takes place in the case of Bose particles, since there is no longer a limitation on the number of produced particles due to the Pauli principle. The stability of the system has to come from the interactions between the bosons, and particle production will stop when the density of bosons is such that the interaction energy compensates the energy given by the external field. Therefore this interaction between particles has to be repulsive. In fact, in the case of attraction between the Bose particles, the vacuum will be unstable even without an external field. Indeed, at a sufficiently large particle density, the energy loss to particle production (mc^2) is offset by the gain due to the attraction, and the system energy decreases with further production of particles.

The vacuum instability manifests itself in simplest form in the case of scalar field in the form of a broad square well. The influence of the external field in this case reduces to a change in the particle mass ($c=1$)

$$\tilde{m}^2 = m^2 - U_0 \tag{1.1}$$

where U_0 is the depth of the scalar well [the scalar field U is connected with the V of the Schrödinger equation $U=2mV$ (see below)]. When the effective mass $\tilde{m}$ vanishes, the vacuum becomes unstable. Particles are produced until the repulsion between them leads to no further energy gain. An analogous instability can arise in an electric field, but in this case two types of vacuum restructuring are possible, depending on whether the charge of the system is constant or can change via successive β decays. In the former case, only pairs of particles with opposite charges can be produced, requiring that the sum of the energies $\omega^+ + \omega^- = 0$. We shall henceforth consider, for the sake of argument, π^+ and π^- mesons and assume that the electric field is produced by the positive charges, that is, constitutes a well for the π^- mesons. Then instability, generally speaking, sets in when the level of the π^- meson reaches a value $-m_\pi c^2$ (as we shall see in Sec. II.A, for a narrow square well the instability sets in at an even smaller drop of the level). As the well becomes deeper, $\pi^+\pi^-$ pairs will accumulate to dangerous levels until the repulsion between them causes the process to stop.

On the other hand, if the charge of the system can vary, for example as the result of β decay of the protons that produce the electric field, then the instability sets in at a smaller depth of the well, namely, when the energy of the π^- meson vanishes—in this case the restructuring of the vacuum begins with accumulation of π^- mesons—the positive charge is carried away by the β positrons. This case could be realized in supercharged nuclei if the latter exist (see Sec. I.B).

Greatest interest attaches to restructuring of the pion field in a rather dense nucleon medium. We shall first explain the mechanism whereby the instability of the pion field is produced in this case.

2. Instability of the pion field in a nucleon medium

We regard the nucleon medium as the source of a field acting on the pions. The pion energy ω as a function of the momentum k can be obtained from the known relation ($\hbar = c = m_\pi = 1$)

$$\omega^2 = 1 + k^2 - 4\pi n F(k)\,, \tag{1.2}$$

where n is the nucleon density and $F(k)$ is the forward pion-nucleon scattering amplitude. The first two terms yield the energy of the free pion, and the third term constitutes the effective field acting on the pions in the nucleonic medium. For simplicity, we have dispensed with the isotopic indices. The scattering amplitude F, for both π^+ and π^- mesons, has the sign corresponding to attraction ($F>0$), and therefore at sufficient density the frequency can vanish, meaning instability of the pion field. However, $F(k)$ is small at small k and instability sets in at $k=k_0$, which corresponds to the minimal value of $k^2 - 4\pi n F(k)$. The instability condition is $\omega^2 = 0$ or

$$1 + k_0^2 = 4\pi n F(k_0)$$

When the condition $\omega^2=0$ is satisfied, pions of a given type will accumulate at the corresponding level ($k=k_0$) for any one of the three pion types. The relation (1.2) does not take into account the possible excitation of the nucleonic medium by the moving pion—the nucleons are regarded as an external field (the "gas" approximation).

This approach describes the phenomenon only in rough outline. For a more exact calculation it is necessary to take into account the particle-hole excitations of the nucleon medium. The pion energy as a function of the momentum is written in the form

$$\omega^2 = 1 + k^2 + \Pi(k,\omega) \tag{1.3}$$

where the quantity $\Pi(k,\omega)$, called the polarization operator, contains, in addition to terms of type (1.2) a term that takes into account the possibility of virtual production of particles and holes in nucleons with a Fermi distribution.

As shown by a theoretical analysis, the polarization operator is determined by the following processes: (1) S-wave scattering of pions by nucleons of the medium; (2) P-wave scattering of pions by nucleons with formation of the N^*_{33} resonance in the intermediate state; these two terms are contained in Eq. (1.2). However, the resonant scattering amplitude F_R is not the experimental one, but must be obtained theoretically "off the mass shell" (i.e., when $\omega^2 \neq 1+k^2$); (3) P scattering of a pion by a nucleon with one nucleon in the intermediate state. This "pole" part of the polarization operator, after taking the Pauli principle into account in the intermediate state, turns out to be a complicated function of ω and k.

After substitution of these three terms, Eq. (1.3) is transformed into a transcendental equation for the energies $\omega(k)$ of all the excitations with the quantum numbers of the pion. This equation has in general several solutions producing several branches for the excitation spectrum. Therefore beside the pion branch which becomes the free pion branch by turning off the $\pi - N$ interaction, there are other branches with the quantum numbers of the pion (0^-, $T=1$).

For π condensation there are two important branches: (i) the pion branch, (ii) the branch that corresponds to collective excitations of the nucleon medium with the quantum numbers of the pion and can be called the spin-isospin sound branch. To clarify matters, we recall that in a Fermi system, in the case of a repulsion interaction of the particles at the Fermi surface, there exist collective excitations called "zero sound," which can be interpreted as particle-hole bound states. These excitations can be of four types: (1) scalar—ordinary sound, (2) spin—representing a spin-density wave, (3) isotopic—corresponding to an isotopic-spin wave, and finally (4) spin-isospin waves having the pion quantum numbers.

The instability of the pion field in an isospin symmetric medium ($N=Z$) manifests itself in the fact that at a definite nucleon density the frequency $\omega_s^{+,-,0}$ of the spin-isospin branch vanishes at a definite $k=k_0$. (In order to distinguish the spin-isospin sound branch from the pion branch we shall label the corresponding quantities with the subscript s.) This means that at a high density in a medium, a periodic inhomogeneity of the spin density of the nucleons with wave vector k_0 is produced.

As will be shown later (Secs. II.A and VI.A), in a finite system the amplitude of this wave executes zero-point oscillations, so that in the ground state the expectation value of the field is zero, and only the expectation values of the even powers of the field differ from zero. Thus, the instability of the pion field in a medium with $N=Z$ corresponds to simultaneous vanishing of $\omega_s^2(k)$ for the three pion types.

The instability picture is much more complicated in a medium with $Z \ll N$ (neutron star). In this case Eq. (1.3) for neutral pions retains the same form as in the case of a medium with $Z=N$, and the instability of the neutral pion field ($\omega_s^2 \leq 0$) manifests itself in formation of a standing spin-density wave of the nucleons with wave vector k_0. The instability sets in at $n \sim n_0$.

For charged pions we have to consider the two charge states at the same time when solving Eq. (1.3), as should be the case for equations describing relativistic particles, solutions associated with antiparticles coming in with a minus sign. The criterion for the selection of solutions with the quantum numbers of the π^+ meson is:

$$2\omega^+ - (\partial\Pi/\partial\omega)_{\omega=\omega^+} > 0 . \tag{1.4}$$

A similar condition exists for the π^- meson. By analyzing the solutions of Eq. (1.3) in neutron matter we get the following results: At a certain density a spin-sound branch appears with a wave vector k of the order of p_F and a negative energy ($\omega_s^+ < 0$). When the density is larger than $n_c^+ \simeq 0.4 n_0$ the energy of this branch exceeds in magnitude the neutron Fermi energy ($\omega_s^+ < -\epsilon_F^{(n)}$) corresponding to an instability with respect to π_s^+ production according to the reaction $p \to n + \pi_s^+$. As will be shown in the next section, this gives rise to a "condensate" of π_s^+ mesons. In the language of nucleonic excitations, this condensate is made of particles that are bound states of a proton and a neutron hole, with binding energy $|\omega_s^+|$.

For π^- mesons, there is no spin-sound branch. With further increase of the density, at $n = n_c^{\pm} \simeq n_c$, the sum of the energies $\omega^- + \omega_s^+$ vanishes, indicating the onset of instability with respect to the production of π_s^+, π^- meson pairs.

3. Model of π condensation

Let us explain now how the pion field becomes restructured after instability sets in. For this analysis, it is irrelevant in which field the instability has set in. It is only important that the frequency of some degree of freedom passes through zero. Inasmuch as the "condensation" consists in the fact that the field φ_k corresponding to this degree of freedom is "strong," we can neglect the influence of the fields corresponding to all other degrees of freedom. Then the energy of the condensate can be written in the form

$$H = \int dr \left\{ \frac{\dot\varphi_k^2 + \omega^2 \varphi_k^2}{2} + \frac{\lambda \varphi_k^4}{4} \right\} . \tag{1.5}$$

At $\omega^2 = 1 + k^2$ and $\lambda = 0$, Eq. (1.5) goes over into the known expression for the energy of a free meson field. We have introduced phenomenologically the effective repulsion between the pions in the nucleon medium ($H' = \lambda\varphi^4/4, \lambda > 0$), which is necessary for the stability of the system.

The interaction between pions and the nucleon medium is the sum of their interaction in vacuum and the interaction due to exchange of excitations of the nucleonic medium. The determination of this interaction is a complicated problem, to which a separate chapter of the review is devoted. Near the transition point, when the field φ_k is not very strong, assuming that the condensate field is characterized by one wave vector k, the real $\pi\pi$ interaction takes the form assumed in (1.5) with a constant λ that depends on the parameters of the (NN) interaction ($\lambda \sim 1-10$).

Near the instability point, the frequency of the considered degree of freedom can be written in the form

$$\omega^2 = \eta(n_c - n), \quad \eta > 0 . \tag{1.6}$$

The quantity η is simply related to the polarization operator. At $n > n_c$, when $\omega^2 < 0$, a static condensate field is produced, and its value can be obtained by minimizing (1.5) with respect to $\varphi^2 k$.

Using (1.6), we obtain

$$\langle \varphi^2 k \rangle = -\frac{\omega^2}{\lambda} = \frac{\eta}{\lambda}(n - n_c) . \tag{1.7}$$

The energy density $\mathcal{E}_\pi$ of the condensate is obtained by substituting (1.7) in (1.5).

$$\mathcal{E}_\pi = -\frac{\omega^4}{4\lambda} \equiv -\frac{\beta(n-n_c)^2}{2} \tag{1.8}$$

We note that this is precisely the scheme used to construct the Landau theory of second-order phase transitions, in which the free energy was expanded in powers of an "order" parameter. The phase transition corresponded to vanishing of the coefficient of the linear term. In our case, the quantity $\varphi^2 k$ plays the role of the "order" parameter, and H plays the role of the free energy. Since the order parameter $\varphi^2 k$ increases from a zero value, we are dealing with a second-order phase transition.

The total energy density can be written in the form

$$\mathcal{E}(n) = \mathcal{E}_N(n) + \mathcal{E}_\pi(n) .$$

According to (1.8), a jump of the compressibility (a jump of $d^2\mathcal{E}/dn^2$) takes place at $n=n_c$. If this jump exceeds in absolute magnitude the compressibility of the nuclear matter prior to condensation, then after the condensation the compressibility turns out to be negative, and the density of the system will increase until a stable state is reached.

B. Physical consequences of π condensation

We begin our analysis of condensation phenomena with the simplest case, namely condensation in unbounded matter. A physical example of such a case is a neutron star.

1. Condensation mechanism in homogeneous nuclear matter

Let us first introduce some refinements into the simple condensation model considered above. It is known that allowance for the increasing role of fluctuations near the critical point alters significantly Landau's simple theory of phase transitions. Similarly, near the π condensation point, the constant of the (π,π) is strongly influenced by exchange of "soft" excitations, the frequency of which vanishes at the critical point.

It turns out that near the transition point, at $n<n_c$, the constant λ reverses sign. It is necessary to take into account the next higher terms of the expansion of the energy of the $(\pi\pi)$ interaction in powers of the field φ. It is easily seen, by minimizing the energy, that a finite field φ is produced at the transition point. Thus the transition is not of second but of first order. However, for numerical reasons, the discontinuity in the value of the field φ^2 turns out to be small and the formulas obtained assuming a second-order phase transition are in error only in the immediate vicinity of the critical point. It is therefore possible to use Eqs. (1.7) and (1.8) for estimates and forget this refinement.

In a medium with $N\cong Z$, all three types of pions are under identical conditions (isotopically symmetrical medium), and the condensation sets in simultaneously for the π_s^+, π_s^- and π_s^0 mesons. Expressions (1.7) and (1.8) remain in force if φ^2 is taken to mean the sum of the squares of all three fields.

The picture of condensation in a neutron star is much more complicated. In this case the instability sets in originally for the π_s^+ mesons. When the density n_c^+ is reached a spin-sound branch with energy $\omega_s^+\lesssim-\epsilon_F^{(n)}$ appears and the protons present in neutron matter (because of β equilibrium) are transformed according to the process:

$$p\to n+\pi_s^+ .$$

For this process to be possible the π_s^+ excitation has to give enough energy to promote the proton at the top of the neutron Fermi sea. The charge of the produced π_s^+ mesons is offset by the charge of the electrons present prior to the transition. With further increase of the neutron density as a result of the β process

$$n\to n+\pi_s^+ +e^- +\tilde{\nu}\,, \tag{1.9}$$

the density of the π_s^+ mesons and the electron density, which is equal to it, will increase together with increasing $|\omega_s^+|$, since at equilibrium, in accordance with (1.9), the Fermi energy of the electrons is equal to $|\omega_s^+|$.

For densities close to n_c^+, the energy density of the π_s^+ condensate is given by

$$\mathcal{E}_\pi=n_s^+\omega_s^+ +|\omega_s^+|^4/4\pi^2\,, \tag{1.10}$$

where the second term is the kinetic energy of the relativistic electrons ($\epsilon_F^{(e)}\gg m_e c^2$).

The density of the condensate is (using $p_F^{(e)}=\epsilon_F^{(e)}$)

$$n_s^+=n_e=|\omega_s^+|^3/3\pi^2\,. \tag{1.11}$$

Using (1.11), we obtain

$$\mathcal{E}_\pi=-|\omega_s^+|^4/12\pi^2\,. \tag{1.12}$$

Thus the energy of the condensate is discontinuous with respect to the density. However, this jump is compensated by the change of the nucleon energy, so that the total energy of the system remains continuous (see Sec. V.A).

We see that near n_c^+ the density of the condensate and the condensate energy are limited not by the repulsion between the pions, but by the Pauli principle for the electrons. With further increase of the density, the increase of $|\omega_s^+|$ with density is slowed down by the influence of the repulsion between the pions. Furthermore, as we have seen, an instability for the production of the $\pi^-\pi_s^+$ pairs sets in (at $n>n_c^\pm$), as a result of which a π^--meson field appears in the condensate in addition to the π_s^+.

Owing to the influence of the π_s^+ condensate, the energy of the condensate acquires a complicated dependence on the density. However, since the numerical factor in the denominator of (1.12) is large, the influence of the π_s^+ condensation is small, and Eq. (1.8) can be used at $n>n_c^\pm$.

If we use Eq. (1.8) for an estimate of the condensate energy, then it is easy to verify that at a density n ranging from $n_c^\pm$ to $n_c^\pm$ the compressibility first vanishes and then becomes negative. Compressibility is proportional to the second derivative of the energy density with respect to density. The condensate term (1.8) of the energy density makes a negative contribution to compressibility, which is larger in absolute value (at a density n_0) than the nucleon contribution to compressibility. With further increase of the density, the repulsion between nucleons at short distances assumes an ever increasing role, and in addition, when the pion field becomes strong enough, the growth of the condensate energy slows down, as a result of which the sign of the compressibility is restored. As we shall see in the section devoted to superdense nuclei, a minimum corresponding to a superdense state of nuclear matter can appear on the $\mathcal{E}(n)$ plot at a certain density $n=n_{\min}$. The possible existence of superdense neutron nuclei is discussed in Secs. VII.A and VII.B. If these nuclei are stable, neutron stars of a small size should exist. Their stability would be provided not by gravitation, as in usual stars, but by hadronic forces.

Since a state with negative compressibility is unstable, a sharp boundary separates matter with density $n\simeq n_0$ from matter with density $n=n_{\min}$.

The sharp change of the nucleon density along the radius of a neutron star is accompanied by a sharp change in the energy of the π_s^+ mesons, and consequently, of the Fermi energy of the electrons. But a change in the end-point energy of the electrons means that a change takes place in the depth of the electric potential well $V(r)$ that retains the electrons.

At equilibrium we have

$$\epsilon_F^{(e)}(r) + V(r) = \text{const} \tag{1.13}$$

$$\epsilon_F^{(e)}(r) + \omega_s^+(r) = 0 \ . \tag{1.14}$$

As a result, strong electric fields are produced, which can be obtained from the relation

$$\frac{dV}{dr} = \frac{d\omega_s^+}{dr} \ . \tag{1.15}$$

Thus π condensation exerts a strong influence on the structure of neutron stars.

2. Possibility of π condensation in ordinary nuclei

An estimate of the critical density for π condensation in symmetric nuclear matter $(\omega_s^{+,-,0}(k_0) = 0)$ gives a value $n_c \cong n_0$. The inaccuracy of this estimate is due to the inaccuracy of the constants of the (NN) and (πN) interactions in the medium, which were introduced in the theory. The uncertainty of the estimate n_c is such that it admits fully the possible existence of a π condensate in ordinary nuclei. It is therefore of great interest to analyze the experimental data in which a π condensate might appear, and the experiments that make it possible to establish the degree of proximity of the nuclei to condensation if the condensation has not yet taken place; such analysis will also improve our knowledge of the constants introduced into the theory, a particularly important step in the assessment of the possible existence of superdense nuclei.

To this end it is necessary first of all to consider π condensation in a finite system. Such an analysis shows that in medium and heavy nuclei one obtains a condensate-energy density that differs from the case of an infinite system only in a thin layer $\delta \ll R$ near the boundary of the nucleus. A periodic flat structure of the condensate field is realized

$$\varphi = a(r) \cos k_0 z \ . \tag{1.16}$$

The amplitude $a(r)$ is constant inside the volume and vanishes in the layer δ at the boundary of the nucleus. In the case of a deformed nucleus, the layers are oriented perpendicular to the major axis.

The additional surface energy connected with the π condensation is proportional not to the total surface of the nucleus, but to the smallest equatorial section. Consequently the condensation contributes to prolation of the nucleus and in principle could lead to the appearance of a second minimum on the plot of the nuclear energy against the deformation, i.e., to shape isomerism.

The structure of the condensate [Eq. (1.16)] induces a layer structure for the neutrons proportional to the square of the field amplitude and with a wave vector $2k_0$.

$$\begin{aligned} n_{n,p} &= n_{n,p}^0 (1 + \xi^2 \cos 2k_0 z) \\ \xi^2 &\sim a^2 \ . \end{aligned} \tag{1.17}$$

The layered structure (1.17) may cause a rotational spectrum to appear in nuclei that are spherical in the sense of the deformation parameter. In addition, a layered structure of the proton densities should influence the nuclear electric form factor that appears in electron scattering. In elastic scattering of electrons there are observed anomalies that offer evidence of a periodic structure of the proton density with wave vector $q \sim 3F^{-1}$. This value agrees with the value of $2k_0$ obtained from the $|\omega^2(k_0)|_{\min}$ condition and should correspond to a π condensate. However, an open question still remains: Is not the observed structure due to shell fluctuations of the density?

The strong decrease of pion energy in the nucleus, predicted by the theory, manifests itself in a number of experimental facts. Thus the spectral data of the π atom yield the "optical" potential of the pion in the nucleus (i.e., the effective potential well of the pion). It is clear that the optical potential is directly connected with the polarization operator $\Pi(k, \omega)$, introduced in (1.3). Reasonable agreement is obtained between the theoretical optical potential and the experimental one. The comparison makes it possible to refine the constants that enter into the theory.

To check on the expression employed for $\Pi(k, \omega)$ and to refine the constants it is important to compare with experiment the energies of the levels having pion symmetry. Such states include the levels $0^-, 1^+, 2^- \ldots$ The shift of the energies of these levels in comparison with values obtained in the shell model is determined to a great degree by the interaction of the nucleons via exchange of a "softened" pion. Satisfactory agreement with experiment is obtained.

Closeness to condensation exerts a particularly strong influence on the l-forbidden $M1$ transitions (transitions with change of orbital angular momentum by two units).

The intensity of such transitions contains a term due to one-pion exchange and having a pole at the critical point $(\omega_s(k_0) = 0)$. The intensities of these transitions exceed in some cases by many times the calculated values obtained without allowance for the exchange of the "soft" pion. This fact offers evidence of the proximity of the system to condensation, but leaves open the question of whether a phase transition took place.

Important information is provided by the still uncompleted analysis of the influence of one-pion exchange on the magnetic moments (in this case the influences are not very great) and on the probabilities of the Gamow–Teller β transitions. It is of great interest to search for anomalies in the scattering of nucleons by nuclei, and also to analyze the nuclear magnetic form factor obtained in experiments on large-angle electron scattering. In these experiments, the spin structure of the nucleon density can manifest itself (unlike the electric form factor, which is determined by the structure of the charge density).

Thus, an analysis of the available experimental data confirms the main conclusions of the theory and so far does not contradict the assumption that a condensate exists in the nuclei.

One can assume that a more careful analysis of the available facts, and also of the data obtained in experiments on scattering, will make it possible to resolve the question of the existence of a condensate in nuclei and at any rate will help to refine the constants in the theory so that more definite predictions can be made concerning the possible existence of superdense nuclei.

3. Possible existence of superdense and neutron nuclei

We consider first the possible existence of superdense neutron nuclei. The energy density of nuclear matter, as a function of the neutron density, is, according to (1.8)

$$\mathcal{E}(n)=\mathcal{E}_N(n)-\frac{\beta(n-n_c^{(+-)})^2}{2}\ , \tag{1.18}$$

where $\mathcal{E}_N(n)$ is the energy density of the nucleons in the absence of a condensate.

As already mentioned, Eq. (1.8) underestimates somewhat the condensate energy, with no account taken of the contribution of the π_s^+ condensation. Calculations show that $\beta\cong 1$. We shall show that near $n=n_c$ the energy $\mathcal{E}(n)$ has a maximum. According to calculations of the energy of neutron matter in the absence of a condensate, we have, in pionic units ($\hbar=c=m_\pi=1$) at $n=n_c$

$$d\mathcal{E}_N/dn\cong 0.2\ .$$

Therefore $\mathcal{E}(n)$ has a maximum near $n_c^{(+-)}\sim n_0$, the position of which is determined by the condition

$$d\mathcal{E}(n)/dn=0;\ n_{\max}-n_c^{(+-)}=0.2/\beta\ . \tag{1.19}$$

With further increase of the nucleon density, the rigidity of the nuclear matter increases sharply, the growth of the condensate energy slows down, and at $n=n_{\min}(n_{\min}\sim 5n_0)$ a minimum appears on the $\mathcal{E}(n)$ curve, meaning that a superdense state of nuclear matter exists. In order for a nucleus with such a nucleon density to be stable, it is necessary that its energy be lower than the sum of the energies of the free nucleons.

Calculations show that the stable state corresponds to reasonable values of NN and πN interaction constants, but at the same time no definite conclusion can be drawn concerning the existence of such a stable state, inasmuch as the energy of the nucleus constitutes a small difference between two large numbers—the positive nucleon energy and the negative condensate energy. Definite conclusions call for either direct experiments aimed at finding such anomalous nuclei or experiments that refine the constants introduced into the theory.

Calculations show that superdense nuclei with $N\cong Z$ should have a higher binding energy than nuclei with $Z\ll N$. Therefore neutron nuclei should undergo a cascade of β decays, until they reach the composition $N\cong Z$. The energy of the β electrons at the start of the cascade is 100–200 MeV, which corresponds to a lifetime $\sim 10^{-6}$–10^{-8} sec.

Under certain assumptions concerning the constants, one may obtain "neutron" nuclei that are stable with respect to β decay and fission, with $Z\ll N$ at $N>10^3$–10^5. Such nuclei could be observed in cosmic rays in the form of large fragments. With the same choice of constants, superdense nuclei with $Z\cong N$ and a relatively small number of particles would also be stable with respect to β decay and fission.

4. Supercharged nuclei

If the Coulomb potential well is deeper than $m_\pi c^2$, π condensation becomes possible. This effect may appear in supercharged nuclei with $Ze^3>1$ which corresponds to $Z>1600$. If the condensation energy gain exceeds the Coulomb energy, such nuclei may be stable. The question of stability of such nuclei with the screening effect of electrons and π^- mesons taken into account is considered in Sec. VII.A. It is shown that electric condensation does not provide the stable state but the effect of nucleon field apparently may lead to stable nuclei with normal density and with a charge $Z\gtrsim 1600$.

C. Guide to the review and remarks on the literature

The first sections of the Introduction present a qualitative picture of all the phenomena considered in the review. Readers not interested in details can confine their reading to this part of the review. Since the main concepts have already been defined, we proceed here to a more detailed exposition of the plan of the review and to a discussion of the references employed.

Section II of the review is an abbreviated exposition of an earlier paper (Migdal, 1971), which contains the first investigation of π condensation in external fields and in nucleonic matter, and the first discussion of the possible existence of superdense nuclei. Section II begins with an examination of the behavior of Bose particles in strong scalar and electric fields. We discuss limitations of the single-particle approach due to particle production from the vacuum. It becomes necessary here to make use of quantum field-theoretical methods. It is shown that a vacuum of Bose particles in strong external fields becomes restructured, leading to a lowering of the energy. The value and the energy of the field of Bose particles in the new ground state of the system is determined.

In the case of an external field in the form of a broad square well ($R\gg\hbar/m_\pi c$), the energy gained from restructuring of the boson vacuum is proportional to volume of the system, i.e., this restructuring can be regarded as a phase transition (π condensation). The question of π condensation in an electric field is considered here in greater detail than in the earlier paper (Migdal, 1971). This result is used in Sec. VII to estimate the possible existence of supercharged nuclei.

Section II deals next with conditions of the instability of pions in a nucleon medium and discusses the simplest π-condensation model introduced in the earlier paper (Migdal, 1971). The condensate field and the condensate energy near the critical point are obtained.

The influence of the nucleon medium on pions is considered in Sec. II in the gas approximation. Such a treatment gives only an approximate estimate of the critical density and of the condensation energy, since no account is taken of the possible excitation of the nucleonic medium by the moving pion. In Sec. III, methods associated with the many-body problem are applied to a quantitative study of pion motion in the nucleon, with account taken of all the essential processes that influence the pion motion. The method of calculating the polarization operator of the pion in the nucleon medium, developed by Migdal (1972), is described. This method is based on subdividing all quantities into two types: (1)

quantities that vary slowly with the momentum (characteristic range m_N), and (2) quantities with a rapid variation (characteristic range m_π). Quantities of the first type are replaced by constants that must be determined from experiment, and quantities of the second type are calculated exactly and are expressed in terms of introduced constants.

Migdal (1972) obtained in this manner a quantitative expression for the polarization operator at $N \cong Z$. Besides the process of the virtual transition of a pion into a nucleon and a nucleon hole, account is taken of the very appreciable contribution of transitions into an N_{33}^* isobar and a nucleon hole. In addition, the (NN) interaction is quantitatively taken into account, and the characteristic constant is determined from nuclear experimental data. An analogous calculation of the polarization operator for a medium with $Z \ll N$ (neutron star) was carried out by Migdal (1973). Account was taken of S scattering of a pion by a nucleon (which is inessential at $N=Z$), a method was established for selection of the physical solutions of the equation for $\omega(k)$, and instability conditions were obtained for the π^+, π^-, π^0 mesons (see below).

The first part of Sec. III is devoted to an elucidation of processes that determine the polarization operator, and to its calculation. Account is taken here of S and P scattering of a pion by a nucleon, and also of the influence of nucleon correlations. An expression is obtained for the polarization operator both for the case $N \cong Z$ (nucleus) and for the case $Z \ll N$ (neutron star).

In the second part of this Sec. (III.B) we discuss the influence of the "pion degree of freedom" in the nucleus on the nuclear phenomena, and present a possible scheme for a consistent allowance for this degree of freedom in the theory of nuclear matter.

Section III.C is devoted to a discussion of the various branches of the spectrum of excitations having the pion quantum numbers. These branches are obtained by solving the transcendental equation (1.3) with the polarization operator obtained in the first part of Sec. III. In addition to the physical branches, as already mentioned, "superfluous" solutions are obtained, corresponding to antiparticles. To select the physical solutions, a scheme is developed for quantizing the field in a medium with arbitrary polarization operator, and the branch selection criterion

$$2\omega^{(+,-)} - (\partial\Pi^{(+,-)}/\partial\omega)_{\omega^{(+,-)}} > 0 \ ,$$

obtained by Migdal (1973) is derived; references to later papers are given in Sec. III.C.

In analyzing the branches of the pion spectrum we find that instability of the spin-acoustic branch sets in at a definite density of the nuclear matter.

For the case $N=Z$, this instability was first observed by Dover and Lemmer (1968). They have shown that the (π, N) interaction can lead to an instability of the nuclear medium with respect to the formation of a nucleon spin-density wave, which in fact corresponds to condensation of the spin-acoustic branch.

For the case of a neutron star $(Z \ll N)$, the instability of the pion field was investigated by Sawyer and Scalapino (1972). They considered a simplified but exactly solvable model of π condensation in a neutron star. This paper has exerted a significant influence on the development of more realistic methods, which we shall discuss later, of treating a strongly developed condensate.

The connection between the instability obtained by Sawyer and Scalapino (1972) and the instability obtained by the more realistic model of Migdal (1973) is discussed at the end of Sec. III.

Sawyer and Scalapino (1972) worked from the assumption that, starting with a certain density, neutron matter becomes unstable with respect to the reaction

$$n \rightarrow p + \pi^- \ .$$

For the realization of this idea, they considered the Hamiltonian of nucleons that interact with a classical field of π^- mesons. Assuming that the π^--meson condensate is a running wave with a single wave vector $k = k_0$, the problem can be solved exactly. At a definite nucleon density, an instability sets in, and this instability was interpreted as a confirmation of the initial idea. Actually, the instability obtained by Sawyer and Scalapino (1972) does not correspond to the reaction $n \rightarrow p + \pi^-$. This instability would remain even if the interaction of the pions with the nucleons were to be turned off, whereas the instability observed by Sawyer and Scalapino vanishes in this case.

As shown (see Sec. III.C) by Migdal (1973) and by Migdal, Markin, and Mishustin (1974), the Sawyer–Scalapino instability consists in vanishing of the sum of the energies $\omega^- + \omega_s^+$, where ω_s^+ is the energy of the nucleonic spin-sound excitation with the quantum numbers of the π meson (see also Anderson *et al.*, 1975).

The calculation of the polarization operator, given in III.A is an abbreviated exposition of the paper of Migdal, Markin, and Mishustin (1974). That paper contains also a detailed analysis of the objections made to the method developed by Migdal (1972) for the calculation of the polarization operator (this analysis and the corresponding references are given in Sec. VI.B).

The critical nucleon density corresponding to the instability of the spectrum, for different values of the constant g^- characterizing the influence of the (NN) interaction on the polarization operator, is given in Table I, which lists also the values of the pion momentum k_c at which instability sets in.

To obtain the magnitude and energy of the condensate field it is necessary to find the effective pion interaction in nuclear matter. This is the subject of Sec. IV. Migdal (1971) introduced this interaction phenomenologically in the form

$$H' = \lambda\varphi^4/4; \ \lambda > 0 \ . \tag{1.22}$$

The true interaction, as shown in Sec. IV, takes this form only for a weakly developed condensate near the transition point.

In the Scalapino–Sawyer model, the condensate energy could be obtained for a condensate field in the form of a running wave with arbitrarily large amplitude. Therefore this model was developed and improved in a number of studies.

Sawyer and Scalapino (1972) considered only π^- mesons, an approach corresponding to describing the pions with the aid of the Schrödinger equation rather than the Klein–Gordon–Fock (KGF) equation. Sawyer and Yao (1973) introduced, to get rid of this shortcoming, π^- mesons whose density was determined by a variational method. Migdal (1973b) developed a method of finding the effective Lagrangian of pions, corresponding to a consistent relativistic description of pions in a nucleon medium, and obtained an expression for the energy of a strong pion field in a model that takes into account (πN) interaction only. This method is described in Sec. IV.A. The problem of finding the energy of a developed pion condensate was solved by Baym and Flowers (1974) with the aid of the Hamiltonian function for the pion field.

In the same paper, account was taken of the vacuum pion-pion and pion-nucleon interaction for strong pion fields in the form of a traveling wave, which follows from Weinberg's nonlinear Lagrangian.

A method for considering strong pion fields, with account taken of the influence of the N^* resonance, was proposed by Campbell, Dashen, and Manassah (1975). They obtained a nonlinear Lagrangian that includes the interaction of the N^* resonance, and makes it possible to obtain the condensate energy for a pion field in the form of a traveling wave for a large range of amplitude. The nucleon correlations were taken into account in this model by Baym, Campbell, Dashen, and Manassah (1975).

The results of these studies are described in Secs. IV.A and IV.B.

In the case of a weak condensate field, which corresponds to proximity to the critical point, a nonlinear Lagrangian for a periodic condensate field with wave vector $k=(\omega, \mathbf{k})$ leads to an expression for the energy in the form (1.22), with replacement of λ by the function $\Lambda(\omega, k)$. The calculation of the function $\Lambda(\omega, k)$ is the subject of Sec. IV.B.

In a realistic model, the calculation can be carried out only for a condensate field in the form of a traveling wave. A method of calculating the condensate field in the Thomas–Fermi approximation is described in Sec. IV.C. This method makes it possible, in principle, to calculate $\Lambda(\omega, k)$ for a condensate field of arbitrary configuration. So far, calculations have been performed in a model that takes into account only the (π, N) interaction. Secs. IV.B and IV.C are in fact expositions of the work of Migdal, Markin, and Mishustin (1976).

Section V is devoted to the calculation of the energy and the magnitude of the condensate field, for both the case of a weak field (near the condensation point) and a strongly developed condensate. Use is made in this section of the effective (π, π) interaction obtained in Sec. IV. The character of the modulations of the spin density and nucleon density, due to the condensation, is determined, and the (π, π) interaction singularities due to exchange of soft excitations (Dyugaev, 1975) near the condensate point are discussed.

In Sec. V.B, the method developed by Campbell, Dashen, and Manassah (1975) and by Baym *et al.* (1975) is used, and an expression is presented for the energy of a strongly developed condensate (the model of the limiting field), which is used in Sec. VII.B for estimates connected with the possible existence of superdense nuclei.

Section VI presents the arguments for and against the existence of a condensate in ordinary nuclei, and also discusses experiments which make it possible to establish the closeness of nuclei to condensation when condensation has not taken place.

The assumption that pion condensation is possible in ordinary nuclei was first advanced by Migdal (1972) and subsequently discussed in a number of papers (the appropriate references are given in Secs. VI.A and VI.B).

Section VI contains a brief exposition of work by Migdal, Kirichenko, and Sorokin (1974), in which the pion condensation problem is solved for a sufficiently large but finite system (medium and heavy nuclei), and the effect of the condensate (if it does exist in ordinary nuclei) on the deformation of nuclei and on rotational levels is discussed.

The conditions for pion instability in nuclei were investigated by Sapershtein, Tokonnikov, and Fayans (1975). They employed the methods of the theory of finite Fermi systems, which made it possible for them to consider light nuclei as well.

The critical value of the spin-isospin constant, $g'=g^-/2$, at which instability sets in, was determined. For medium and light nuclei, the values obtained for g'_0 were the same as in an infinite system.

Section VI.A deals also with Goldstone excitation modes that result from condensation. This question was first considered by Migdal (1974). Kirichenko and Sorokin (1975) have shown that the lowest mode of the Goldstone oscillations corresponds to rotation of the direction of the condensate layers relative to the major axis of the deformed nucleus. These oscillations might be observed in heavy nuclei.

It is shown in Sec. VI.A that, owing to the quantum character of the condensate field, the existence of a condensate does not violate the law of parity conservation in the nucleus.

The main task of Sec. VI.B is to consider experiments which can establish the degree of proximity of the nuclei to condensation, and at the same time refine the values of the constants introduced into the theory. The first to be analyzed are the nuclear experimental facts, and it is shown that our assumptions concerning condensation do not contradict the known data. The methods of the theory of finite Fermi systems are next used to investigate the influence of one-pion exchange on the spectra and probabilities of the transitions. This problem was investigated by Sapershtein and Troitskii (1975), who showed that proximity to condensation exerts a particularly strong influence on l-forbidden $M1$ transitions.

Next, in Sec. VI.B, the influence of the distortion of the pion energy in the nucleus on the optical potential of the pion in the π atom is explained.

A detailed analysis is made of the possible influence of the condensate on the elastic scattering of electrons by nuclei (Migdal, 1974), and experiments on the scattering of nucleons and electrons by nuclei, in which π condensates can appear, are discussed.

Section VII is devoted to perhaps the most interesting but the least secure task of the review, namely to questions connected with the possible existence of anomalous nuclei, such as superdense nuclei with $Z\cong N$ and superdense "neutron" stars (with $Z\ll N$), as well as supercharged nuclei ($Z \gtrsim (137)^{3/2}$).

In the first part of Sec. VII, the question of the possible existence of superdense nuclei is considered and a model proposed by Migdal (1974), representing a development of his first approach (1971). In these papers, an expression is used for the condensate energy near the critical point, and the change in compressibility of the nucleon medium, resulting from condensation, is determined. If the compressibility reverses sign, then the system should be compressed until it goes over into a new denser state. (The stability of this new state is examined in Sec. VII.B.) Furthermore, in VII.A, an assessment is made of the possible existence of supercharged nuclei in which the energy gained by condensation in the electric field offsets the Coulomb repulsion. This equation was discussed by Migdal (1972, 1974), who suggested the possibility of condensation with production of π^+, π^- pairs. As shown in Sec. II.4, however, in the discussion of condensation in an electric field, π^--meson condensation takes place at an even smaller nuclear charge, and as to condensation of $(\pi^+\pi^-)$ pairs, it must be considered with allowance for the screening of the nuclear field by the electrons resulting from the restructuring of the electron-positron field near a nucleus with charge $Z \gg 170$. The π^--condensation energy is estimated in VII.A and found to be of the order of the Coulomb energy. The question of the stability of such nuclei therefore still remains open, until more exact calculations are made.

The end of VII.A deals with the feasibility of a superdense state of nuclear matter, due to instability of the nucleon–antinucleon vacuum (Lee, 1974). It is shown that if such an instability is possible, it sets in at densities $n \sim 100 n_0$.

An expression for the condensate energy near the critical point was used in the estimates of the energies of superdense and neutron stars given in Sec. VII.A. To obtain the plot of energy against density and to assess the stability of the superdense state, we must have an expression for the condensate energy and for the nucleon energy at a density noticeably higher than n_c. These problems are discussed in VII.B.[1] To determine the π-condensate energy at high density, the limiting-field model (V.B) is used. The energy of the nucleon subsystem is estimated from the theory of neutron matter at high density (Pandharipande, 1971). Separate interpolation formulas are obtained for the nucleon and condensate energy (at an arbitrary ratio Z/N), and these go over into the known expressions for low and high densities.

An equation of state is obtained for the matter of neutron stars. The results are compared with calculations made by Hartle, Sawyer, and Scalapino (1972), Au and Baym (1974), and Weise and Brown (1975).

We analyze the existence and the stability of superdense ($Z \simeq N$) and neutron ($Z \ll N$) nuclei for different choices of the parameters of the theory.

[1]The exposition follows the paper of Migdal, Markin, Mishustin, and Sorokin (1976).

II. BOSE-PARTICLE CONDENSATION IN AN EXTERNAL FIELD

The problem of the motion of a single boson becomes physically meaningless in strong fields, when boson production energy approaches zero. Here we have the possibility of real or virtual particle production, so that a many-particle problem arises. Instead of the single-particle equation it is necessary to use the methods of quantum field theory, which yield the field of the produced or virtual particles. The energy of one particle in this approach is defined as the difference between the field energy in a state with the quantum numbers of one particle and the vacuum energy.

We ascertain below the conditions under which the single particle becomes unstable in the presence of scalar and electric external fields, and determine the critical field. We solve the quantum field-theoretical problem for bosons with interaction $H' \sim \lambda\varphi^4$, $\lambda > 0$, in an arbitrary potential well and show that in strong external fields the bosons produce a screening field such that the effective sum of the external and screening fields does not reach the critical value.

These results will be needed to assess the quantum nature of the condensate field in a finite system (as we shall show, only in a sufficiently large system can the condensate field be regarded as classical).

Of particular importance in what follows is the possible instability of the pion field in a nucleon medium treated as an external field. Using a very simple model as an example, we ascertain the mechanism of π condensation in a nucleon medium.

A. Bosons in scalar and electric fields

The condition for the instability of the single-particle problem for bosons moving in a scalar or in an electric field is obtained. The problem of finding the pion field in external fields exceeding the critical value is solved. Condensation of bosons in strong scalar and electric fields is considered.

1. Instability of the single-particle problem

Let us find first, for a number of very simple cases, the critical parameters of the external field at which the single-particle problem become unstable

Bose particles are described by the Klein–Gordon–Fock equation (KGF). In the absence of an external field, this equation takes the form ($\hbar = m = c = 1$)

$$\Delta\psi + [\omega^2 - 1]\psi = 0 \ . \tag{2.1}$$

For states with a definite momentum we have $\omega = \pm(1 + k^2)^{1/2}$. The positive root corresponds to the usual relativistic relation between the energy and momentum of the particle.

The negative sign corresponds to the antiparticle energy taken with a minus sign. Thus Eq. (2.1) describes simultaneously the behavior of particles and antiparticles.

In the simplest case of a static scalar field $U(r)$ we have

$$\Delta\psi + (\omega^2 - 1 - U)\psi = 0 \ . \tag{2.2}$$

The scalar field is added to a scalar quantity—the square of the mass (=1). By changing notation $\omega^2-1=2E$ and $U=2V$ we can write (2.2) in the form of the Schrödinger equation

$$\Delta\psi+2[E-V]\psi=0 \ .$$

It must be remembered, however, that Eq. (2.2), unlike the ordinary Schrödinger equation, describes simultaneously two types of particle—the particle and the antiparticle.

The scalar field U differs from the potential V of the Schrödinger equation by a factor $2mc^2$ (i.e., by a factor of 2 in our units.) Multiplying (2.2) by ψ and integrating, we obtain

$$\omega^2=1+\overline{p^2}+\overline{U} \ , \tag{2.3}$$

where $\overline{p}^2$ is the momentum squared averaged over the wave function of the considered state ($\overline{p}^2=\int\psi p^2\psi d\mathbf{r}=\int|\nabla\psi|^2 d\mathbf{r}$), and $\overline{U}$ is the mean value of $U(r)$.

If the field U is of the form of a broad square well with radius R and depth $U_0=-U$, then for the ground state we have $\nabla\psi\sim\psi/R$ and $p^2\sim R^{-2}$. For the energy we get the expression

$$\omega^2=1-U_0+O(1/R^2) \ .$$

At $U_0\cong 1$, the external field "eats up" the mass of the particle and the system becomes unstable—particles can be produced from the vacuum and accumulate in the considered state. In the case of a narrow well ($R\ll 1$), the criterion for the vanishing of ω will coincide in order of magnitude with the criterion for the appearance of a bound state in a square well,

$$U_0R^2\gtrsim 1 \ .$$

Recognizing that at $\omega=0$ the wave function outside the well takes the form $\psi=c_0(e^{-r}/r)$ (c_0 is a number on the order of unity), we obtain from Eq. (2.3) the estimate

$$\omega^2=1+(c_1/R)-c_2U_0R \ ,$$

where c_1 and c_2 are numbers of the order of unity. At $U_0R^2>c_1/c_2+R/c_2$ the sign of ω^2 is reversed. In the case of a square well it is easy to solve the problem exactly for an arbitrary well radius.

A plot of energy against the depth of the well takes the form shown in Fig. 1. The dashed line marks the branch emerging from -1, which yields the antiparticle energy with the minus sign. At $U_0=U_0^c$, the curve terminates. In the case of a deeper well, single-particle solutions exist only for those levels whose frequencies $\omega_\nu>0$.

In the case of a static electric field, the potential is not a scalar, but the fourth component of a potential 4-vector, and therefore the field must be added to the fourth component of the 4-vector (i.e., to the 4-component of the energy-momentum 4-vector).

The Klein–Gordon–Fock equation takes the form

$$\Delta\psi+[(\omega-V)^2-1]\psi=0 \ . \tag{2.4}$$

It is assumed that V has the form of a potential well (and consequently the form of a potential hump for antiparticles). Multiplying (2.4) by ψ and integrating, we obtain the quadratic equation

$$\omega^2-2\omega\overline{V}-1+\overline{V^2}-\overline{p^2}=0 \ ,$$

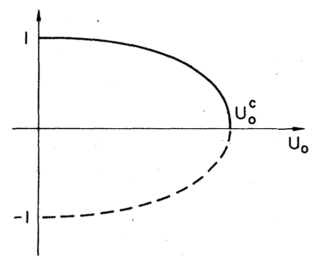


FIG. 1. Dependence of energy 1S level of a scalar particle in a scalar field on a parameter proportional to the depth of the well.

whence

$$\omega=\overline{V}\pm(1+\overline{p^2}+\overline{V}^2-\overline{V}^2)^{1/2} \ . \tag{2.5a}$$

The plus sign corresponds to a particle, for when the field is turned off, the energy ω must go over into the free-particle energy:

$$\omega^-=\overline{V}+(1+\overline{p^2}+\overline{V}^2-\overline{V^2})^{1/2} \ . \tag{2.5b}$$

The antiparticle energy is obtained from this expression by reversing the sign of the charge, i.e., by changing the sign of $\overline{V}$

$$\omega^+=-\overline{V}+(1+\overline{p}^2+\overline{V}^2-\overline{V}^2)^{1/2} \ . \tag{2.5c}$$

Thus Eq. (2.5a) yields the antiparticle energy taken with the minus sign. It can be shown that for a sufficiently deep square well the radicand vanishes. We confine ourselves to presenting plots of the particle and antiparticle energies as functions of a parameter ζ proportional to the depth of the well (Fig. 2). At a value $\zeta=\zeta_c$, $\omega^++\omega^-=0$, and the vacuum becomes unstable with respect to pair production.

It is amazing that for a sufficiently deep square well ($\zeta\gtrsim\zeta_1$), the bound state ($\omega<1$) is produced not only for particles (for which V is a potential well), but for antiparticles, for which the potential $-V$ corresponds to repulsion. The formal reason can be easily seen by writing (2.4) in the form of a Schrödinger equation, putting

$$\omega^2-1=2E; \ -V^2+2\omega V=2U \ . \tag{2.6}$$

Then Eq. (2.4) becomes

$$\Delta\psi+2(E-U)\psi=0 \ .$$

As seen from Eq. (2.6), the effective "potential" of the Schrödinger equation, for any sign of the charge, contains a term $-1/2V^2$, corresponding to attraction. (The

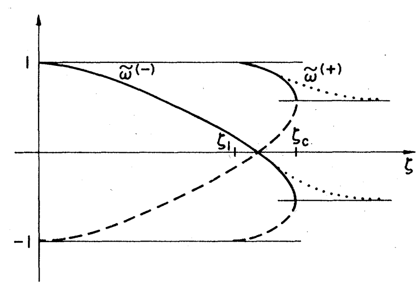


FIG. 2. Dependence of energy one 1S level of a scalar particle in an electric field on a parameter proportional to the depth of the well. The curves $\tilde{\omega}^-$ and $\tilde{\omega}^+$ correspond to particles for which $V<0$ and $V>0$.

quantity U can be termed a potential only arbitrarily, since it depends on E). At a sufficient well depth, the first term of U, corresponding to attraction, becomes more substantial than the second, and a bound state is produced also for $V>0$ (antiparticle).

2. Determination of the boson field

How must one solve the problem of the motion of bosons in a strong external field, after the onset of the instability?

We first explain how the instability of the single-particle problem manifests itself in field theory by considering the case of scalar bosons in a scalar external field. Let the bosons be described by the field φ.

We obtain the Lagrangian $L=\int \mathcal{L} d\mathbf{r}$ of the meson field φ in the presence of the external field U. The density $\mathcal{L}$ of the Lagrange function is defined such that the least-action requirement $\delta S=0$ ($S=\int L dt=\int \mathcal{L} d\mathbf{r} dt$) leads to Eq. (2.2) for φ. This corresponds to a Lagrange density $\mathcal{L}_0$ equal to (in units of $m=\hbar=c=1$)

$$\mathcal{L}_0=\tfrac{1}{2}(\dot{\varphi}^2-(\nabla\varphi)^2-(1+U)\varphi^2) . \quad (2.7)$$

Indeed, it is easily seen that the condition

$$\int \frac{\delta\mathcal{L}}{\delta\varphi}\, d\mathbf{r}dt=0$$

leads directly to Eq. (2.2).

The field energy is determined from $\mathcal{L}$ with the aid of the relation

$$H=\int\left\{\dot{\varphi}\,\frac{\delta\mathcal{L}}{\delta\dot{\varphi}}-\mathcal{L}\right\} d\mathbf{r} . \quad (2.8)$$

For the meson-field energy we obtain from (2.7) and (2.8)

$$H_0=\int \frac{\dot{\varphi}^2+(\nabla\varphi)^2+(1+U)\varphi^2}{2}\, d\mathbf{r}. \quad (2.9)$$

The problem of finding the field φ and determining the possible values of the system energy becomes much simpler if it is recognized that the main contribution to the field φ is made by particle creation and annihilation processes on a "dangerous" level, i.e., on that level whose energy goes through zero.

It can be verified that if ω^2 is negative for the "dangerous" state the influence of the remaining states introduces small corrections. If we retain only the dangerous state ψ, then the coordinate dependence of the field $\varphi(\mathbf{r})$ is known and the operator $\hat{\varphi}(\mathbf{r})$ can be written in the form

$$\hat{\varphi}(\mathbf{r})=\hat{q}\psi(\mathbf{r}); \int |\psi|^2 d\mathbf{r}=1 . \quad (2.10)$$

As we shall see presently, the problem of determining possible values of the system energy reduces to an oscillator problem in which q plays the role of the coordinate and the quantum number n (the oscillator excitation number) determines the number of bosons in the considered state.

Substituting (2.10) in (2.9) and using (2.2), we obtain

$$H_0=\frac{\dot{q}^2+\omega^2q^2}{2}$$

where ω^2 is defined by Eq. (2.3). We have thus reduced the problem to that of an oscillator with classical frequency ω equal to the energy of the "dangerous" state. When ω^2 goes through zero, the problem becomes meaningless—there exist no stationary solutions with finite values of q and the meson field is infinitely large. For the problem to become meaningful at $\omega^2<0$, it is necessary to introduce interaction between the bosons. We shall take this interaction into account below in a phenomenological form. We begin with consideration of the simplest model of the interaction, by adding to the field energy a term

$$\int \frac{\lambda\varphi^4}{4}\, d\mathbf{r},\ \lambda>0 .$$

The Lagrangian density takes the form

$$\mathcal{L}=\mathcal{L}_0-\lambda\varphi^4/4 . \quad (2.11)$$

For the field energy we obtain

$$H=\frac{\dot{q}^2+\omega^2q^2}{2}+\frac{\lambda_1q^4}{4} , \quad (2.12a)$$

where

$$\lambda_1=\lambda\int|\psi|^4 d\mathbf{r}>0 . \quad (2.12b)$$

Thus the solution of the field-theoretical problem reduces to the determination of the eigenfunctions and eigenvalues of an anharmonic oscillator. This problem can be solved either numerically or in the semiclassical approximation, which can be formally used for high excited states, but gives fairly accurate results even for the ground and first excited states (Migdal, 1971).

Of particular interest is the region of large negative values of ω^2.

In this case the oscillator potential energy $U(q)=(-|\omega^2|/2)q^2+\lambda_1q^4/4$ takes the form shown in Fig. 3. The first energy levels correspond to motion near q_0 and $-q_0$. The positions of the minima are determined by the condition $dU/dq=0$, whence

$$q_0^2=|\omega^2|/\lambda_1 . \quad (2.13)$$

We can approximate $U(q)$ in the vicinity of $q=q_0$ by:

$$U(q)\cong-\omega^4/4\lambda_1+|\omega^2|(q-q_0)^2 . \quad (2.14)$$

The eigenfunctions break up into two types, symmetrical and antisymmetrical with respect to the reversal of the sign of q. If the barrier separating the two minima has low penetrability, then the energies of the symmetrical states will be lower than the energies of the antisymmetrical states by a small amount proportional to the barrier penetrability. The solution within each of the wells corresponds to the problem of an oscillator with frequency $\omega_1=\sqrt{2}\,|\omega|$.

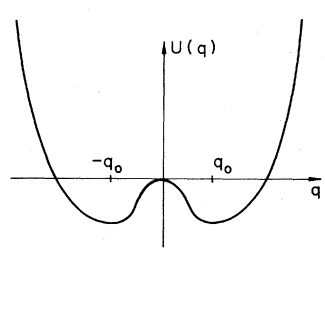


FIG. 3. Potential-energy curve of anharmonic oscillator.

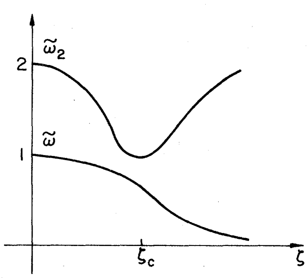


FIG. 4. Energy of one-boson ($\tilde\omega$) and two-boson ($\tilde\omega_2$) excitations.

If we neglect the exponentially small increments, then the energy of the ground state, as follows from the expansion (2.14), is equal to

$$E_0 \cong -\frac{\omega^4}{4\lambda_1} + \frac{1}{2}\sqrt{2}\,|\omega|\,, \qquad (2.15)$$

and the energy of the state with quantum number n' of the oscillator in one of the wells is

$$E_{n'} = -\frac{\omega^4}{4\lambda_1} + \left(n' + \frac{1}{2}\right)\sqrt{2}\,|\omega|\,. \qquad (2.16)$$

Each level of this oscillator has a "fine structure" corresponding to the symmetrical and antisymmetrical states.

The ground state corresponds to a symmetrical function that goes over into the ground state of this oscillator near the minimum of each of the wells.

The next state, corresponding to one-meson excitation, is given by the corresponding antisymmetrical function. The energy of the one-meson excitation is determined by the expression (Migdal, 1971)

$$\tilde\omega = E(1) - E(0) = \frac{|\omega|\sqrt{2}}{\pi}\exp\left[-\frac{\pi|\omega|^3}{4\lambda_1}\right], \qquad (2.17)$$

where $E(n)$ is the energy of the n meson excitation. It is easy to verify that the first antisymmetrical state goes over in the harmonic limit ($\omega^2 > 0$) at $\omega^3 \gg \lambda_1$ into the first excited state of the harmonic oscillator, i.e., it must be interpreted as one-meson excitation. Let us consider two-meson excitation. In the harmonic region $\omega^3 \gg \lambda_1$ it corresponds to the second level of the harmonic oscillator with energy $\omega_2 = 2\omega$. If the problem parameters are varied adiabatically, this state remains the second excited state, i.e., it corresponds in the transcritical region to a symmetrical state with $n' = 1$, where n' is the number of the state in an individual well. Its energy in the transcritical region is equal to

$$\tilde\omega_2 = E(2) - E(0) = \sqrt{2}\,|\omega|\,.$$

Plots of $\tilde\omega$ and $\tilde\omega_2$ against the parameter ζ proportional to the well depth are shown in Fig. 4.

As can be seen from Fig. 4, the energy $\tilde\omega$ tends to zero in the transcritical region, whereas the energy of $\tilde\omega_2$ decreases and then increases with increasing ζ.

3. Condensation in a scalar field

The most important result obtained so far is that the rearrangement of a boson vacuum in a sufficiently strong field (when $\omega^2 < 0$) lowers the system energy. Let us consider a large homogeneous system. We then have from (2.12b)

$$\lambda_1 = \lambda/\Omega\,,$$

where Ω is the volume of the system ($|\psi|^2 = 1/\Omega$). According to (2.15), the energy gained from the rearrangement of the vacuum is proportional to the volume of the system. It follows from (2.13) that a "condensate" field is produced

$$\langle\varphi^2\rangle = \langle q^2\rangle|\psi|^2 = -\omega^2/\lambda\,. \qquad (2.18)$$

The "condensate" energy density is equal to

$$\mathcal{E}_\pi = -\omega^4/4\lambda\,. \qquad (2.19)$$

The energy decreases with increasing $|\omega^2|$, i.e., with increasing depth of the well.

We noted that the expectation value of the field φ in the ground state is equal to zero:

$$\langle\varphi\rangle = \langle q\rangle\psi = 0\,.$$

Since the first excited state has an energy that tends exponentially to zero with increasing system volume [Eq. (2.17)], degeneracy arises in large systems. By mixing the ground and first excited states we can obtain states with a nonzero average field $\langle\varphi\rangle$,

$$\langle\varphi_1\rangle \cong q_0\psi;\ \langle\varphi_2\rangle \cong -q_0\psi\,.$$

The states $\langle\langle 1\rangle\rangle$ and $\langle\langle 2\rangle\rangle$ correspond to the system's being located in the right-hand and left-hand well of our oscillator, respectively.

4. Condensation in an electric field

Now let us consider the polarization of a vacuum of charged bosons in a static electric field. The Lagrangian of the system is

$$\mathcal{L} = \left(\frac{\partial}{\partial t} - iV\right)\hat\varphi^+\left(\frac{\partial}{\partial t} + iV\right)\hat\varphi - \nabla\hat\varphi^+\nabla\hat\varphi - \mu^2\hat\varphi^+\hat\varphi - \frac{\lambda}{2}(\hat\varphi^+\hat\varphi)^2\,,\quad \lambda > 0\,. \qquad (2.20)$$

We have assumed a very simple interaction between the bosons. It can be shown that the vacuum can also be stabilized by a purely electric interaction, but it is natural to propose that the hadronic interaction is more substantial, i.e., $\lambda \gg e^2$. It will become clear that modification of the form of the hadronic interaction does not influence the qualitative results.

The solution of the problem becomes exceedingly simple if it is recognized that the main contribution to the field φ is made by "dangerous" states with energies that tend to zero.

We write $\hat\varphi$ in the form

$$\hat\varphi = \hat q\psi/\sqrt{2}\,, \qquad (2.21)$$

and obtain the equation for ψ from the exact equation for the operators $\hat\varphi$

$$\Delta\hat\varphi + \left[-\left(\frac{\partial}{\partial t} + iV\right) - \mu^2\right]\hat\varphi - \lambda\hat\varphi^+\hat\varphi^2 = 0\,. \qquad (2.22)$$

The equation for ψ is obtained from the condition

$$q_{01}\psi/\sqrt{2} = \hat\varphi_{01},\ i\left(\frac{\partial\hat\varphi}{\partial t}\right)_{01} = ([H,\hat\varphi])_{01} = \tilde\omega_1\hat\varphi_{01}\,.$$

The matrix elements are taken over the exact states—

vacuum and vacuum plus one particle. Then ψ is an exact (normalized) function, and $\tilde{\omega}_1$ is the exact energy of one particle with allowance for the interaction. The equation for ψ follows from (2.22)

$$\Delta\psi + [(\tilde{\omega}_1 - V)^2 - \mu^2]\psi - \lambda\psi^3 \frac{(q^+q^2)_{01}}{2q_{01}} = 0 . \tag{2.23}$$

It remains to diagonalize the Hamiltonian, which depends on q and $\dot{q}$, and which we shall presently derive. It is then straightforward to determine the matrix elements $(q^+q^2)_{01}$ and q_{01}.

Substituting Eq. (2.21) into the Lagrangian density (2.20), and integrating over the volume, we obtain

$$L = \int \mathcal{L} d\mathbf{r} = -(\mu^2 + \overline{p^2} - \overline{V^2})\frac{q^+q}{2} + \frac{\dot{q}^+\dot{q}}{2} + \frac{i\overline{V}(\dot{q}^+q - q^+\dot{q})}{2} + \frac{\lambda_1}{4}(q^+q)^2 , \tag{2.24a}$$

where the bar denotes averaging over ψ

$$\lambda_1 = \frac{\lambda}{2}\int \psi^4 d\mathbf{r} \quad \overline{p^2} = \int (\nabla\psi)^2 d\mathbf{r} . \tag{2.24b}$$

We introduce in place of q and q^+ the Hermitian operators q_1 and q_2:

$$q = q_1 + iq_2; \quad q^+ = q_1 - iq_2 .$$

The generalized momenta corresponding to q_1 and q_2 are

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1 - \overline{V}q_2; \quad p_2 = \frac{\partial L}{\partial \dot{q}_2} = \dot{q}_2 + \overline{V}q_1 .$$

We obtain the Hamiltonian

$$H = \dot{q}_1\frac{\partial L}{\partial \dot{q}_1} + \dot{q}_2\frac{\partial L}{\partial \dot{q}_2} - L = \frac{p_1^2 + p_2^2 + \omega_0^2(q_1^2 + q_2^2)}{2} + \frac{\lambda_1}{4}(q_1^2 + q_2^2)^2 + \overline{V}(p_1q_2 - p_2q_1) . \tag{2.25}$$

Here

$$\omega_0^2 = \overline{V}^2 - \overline{V^2} + \mu^2 + \overline{p^2} . \tag{2.26}$$

Thus the problem has been reduced to that of a two-dimensional anharmonic oscillator with a potential energy that does not depend on the angle. The stability of the problem is ensured by the fact that $\lambda > 0$. The energy of such an oscillator depends on two quantum numbers, on the radial quantum number n and on the angular momentum m about an axis perpendicular to the plane of the oscillator. It is easy to verify from the expression for the particle density

$$\rho = \left\{\frac{1}{i}(q^+\dot{q} - \dot{q}^+q) - 2Vq^+q\right\}|\psi|^2$$

that

$$m = p_1q_2 - p_2q_1 = \int \rho d\mathbf{r} = Z_\pi$$

i.e., the angular momentum m has the meaning of the total boson charge. The radial quantum number n has the meaning of the number of pairs.

The energy $E(n, m)$ is determined in terms of the energy eigenvalue of the equation for the radial function of the oscillator

$$\chi'' + 2\left[E'(n, m) - U(\xi) - \frac{m^2 - \frac{1}{4}}{2\xi^2}\right]\chi = 0 ,$$

where

$$\xi = (q_1^2 + q_2^2)^{1/2}, \quad U(\xi) = \frac{\omega_0^2\xi^2}{2} + \lambda_1\left(\frac{\xi^4}{4}\right) ,$$

and $E'(n, m)$ is connected with $E(n, m)$ by the relation

$$E(n, m) = \overline{V}m + E'(n, m) . \tag{2.27}$$

In the semiclassical approximation, which can be seen from the example of the scalar well to be fairly accurate even for the ground state, we have

$$\int_0^{\xi_1}\left\{2\left[E'(n, m) - U(\xi) - \frac{m^2}{2\xi^2}\right]\right\}^{1/2} d\xi = (n + \tfrac{1}{2})\pi . \tag{2.28}$$

The Langer correction for a centrifugal potential is taken into account (see, e.g., Migdal, 1975).

We consider first the case of a weakly charged system ($m \sim 1$) and assume that the quantity ω_0^2 [Eq. (2.26)] vanishes at a certain value of the parameter ζ proportional to the well depth. This case is realized in a square well, and also for other potentials that decrease sufficiently rapidly as $r \to \infty$ (Mur and Popov, 1976).

We represent ω_0^2 near such a critical point in the form

$$\omega_0^2 = -\gamma(\zeta - \zeta_c)$$

and assume that $|\omega_0^2| \gg \lambda_1^{2/3}$. Then the potential energy $U(\xi)$ has a minimum at

$$\xi^2 = \xi_0^2 = -\omega_0^2/\lambda_1 \tag{2.29}$$

and the problem, as before, reduces to that of a harmonic oscillator with frequency $\omega' = (2|\omega_0^2|)^{1/2}$.

For $E(n, m)$ we readily obtain

$$E(n, m) = \overline{V}m + U(\xi_0) + \frac{m^2}{2\xi_0^2} + (n + \tfrac{1}{2})(2|\omega_0^2|)^{1/2} = -\frac{\omega_0^4}{4\lambda_1} + \frac{m^2}{2\omega_0^2}\lambda_1 + (n + \tfrac{1}{2})(2|\omega_0^2|)^{1/2} + \overline{V}m . \tag{2.30}$$

It follows therefore that the energies of the oppositely-charged particles are

$$\tilde{\omega}^- = E(0, -1) - E(0, 0) = \overline{V} + \frac{\lambda_1}{2|\omega_0^2|} ,$$

$$\tilde{\omega}^+ = E(0, 1) - E(0, 0) = -\overline{V} + \frac{\lambda_1}{2|\omega_0^2|} .$$

The course of the energies $\tilde{\omega}^+$ and $\tilde{\omega}^-$ as functions of ζ are shown by points in Fig. 2. The pair energy does not depend on $Z_\pi = m$, and increases with increasing distance from the critical point

$$E(1, m) - E(0, m) = (2|\omega_0^2|)^{1/2} .$$

Just as in the case of a scalar field, condensation has set in. The energy of the system decreases as a result of the rearrangement of the ground state. In a large system, according to Eq. (2.24b), we have $\lambda_1 = \lambda/2\Omega$, and the decrease of the energy density is equal to $\mathcal{E}_\pi = -\omega_0^4/2\lambda$.

We note that the terms (2.30) containing m influence significantly the system energy only if the charge Z_π is proportional to volume of the system. The condensate field in a homogeneous system ($|\psi|^2 = 1/\Omega$) is determined by the relation

$$\langle\varphi^2\rangle = \langle\xi^2\rangle \frac{|\psi|^2}{2} = \frac{|\omega_0^2|}{\lambda} \; . \tag{2.31}$$

These results were obtained with a fixed charge of the system. Physically this corresponds to exclusion of β decay. If the system exists a sufficiently long time, then the β decay will eventually produce in it a charge corresponding to the energy minimum.

We assume that the electric field is produced by a positive charge. Then a π^- meson condensate will decrease the energy of the system. There are two possibilities here: (1) the positrons and the neutrinos produced together with the π^- leave the system. This case is realized for condensation in the field of supercharged particles (see Sec. VII.A). (2) The system is large enough to satisfy the charge-neutrality constraint. Then the positrons remain and fill the Fermi sphere in such a way that their charge cancels the π^- meson charge. This case could be realized in supercharged stars if the latter were to exist.

In the first case, on the basis of (2.27) and (2.28), the system energy as a function of ξ and Z_π takes the form

$$E(\xi, Z_\pi) = \overline{V} Z_\pi + \frac{Z_\pi^2}{2\xi^2} + \frac{\omega_0^2 \xi^2}{2} + \frac{\lambda_1 \xi^4}{4} \; . \tag{2.32}$$

Here $\overline{V} < 0$. We have neglected in this expression the term $\dot{\xi}^2/2$, which makes a small contribution $(\sim(|\omega^2|)^{1/2})$ that does not contain the volume of the system.

Minimizing $E(\xi, Z_\pi)$ with respect to ξ^2 and Z_π, we obtain

$$\xi^2 = \frac{\overline{V}^2 - \omega_0^2}{\lambda_1} > 0 \, , \tag{2.33}$$

$$Z_\pi = -\overline{V} \, \frac{\overline{V}^2 - \omega_0^2}{\lambda_1} \; , \tag{2.34}$$

$$E_\pi = -\frac{(\overline{V}^2 - \omega_0^2)^2}{4\lambda_1} \; . \tag{2.35}$$

As seen from these expressions, in a large system ξ^2, Z_π, and E_π increase in proportion to the volume of the system. Such a "charged" condensation takes place regardless of the sign of ω_0^2. For a large class of potentials (when ω_0^2 does not go through zero at any well depth), ω_0^2 is estimated as

$$\omega_0^2 = 1 + \overline{V}^2 - \overline{V}^2 + \bar{p}^2 \cong 1 + O\left(\frac{1}{R^2}\right) \, .$$

For such a system, the critical field of a "charged" condensate is

$$\overline{V}_c \cong 1 \, .$$

In a large system, if ω_0^2 vanishes, it does so at $\overline{V} \geq 2$, but at $\overline{V} = 1$ we have $\omega_0^2 \cong 1$ (this result can be easily obtained in a broad potential well with a flat bottom and an arbitrarily shaped "diffuse" edge). The expression for the critical field of a "charged" condensate therefore corresponds in all cases to $\overline{V}_c \cong 1$. Let us now consider the second case, in which positrons remain in the system and cancel out the charge of the π^-. The solution of this problem will help to solve a similar problem in neutron stars where the π_s^+ charge is compensated by electrons. In this case it is necessary to add to the energy $E_\pi(\xi, Z_\pi)$ the energy of the positrons, the density of which is $n_e = n_\pi$. The positron energy density takes the form

$$\mathcal{E}_e = \frac{\mu_e^4}{4\pi^2} = \frac{3}{4} n_\pi \mu_e \, ,$$

where μ_e is the chemical potential of the positrons equal to

$$\mu_e = (3\pi^2 n_\pi)^{1/3} \, .$$

The total energy density is, according to Eqs. (2.24b) and (2.32),

$$\mathcal{E} = \mathcal{E}_\pi + \mathcal{E}_e = \overline{V} n_\pi + \frac{n_\pi^2}{2\eta^2} + \frac{\omega_0^2 \eta^2}{2} + \frac{\lambda}{8}\eta^4 + \frac{3}{4} n_\pi \mu_e \, , \tag{2.36}$$

where $\eta^2 = \xi^2/\Omega$. Minimizing (2.36) with respect to n_π and η, we obtain

$$\eta^2 = 2\, \frac{(\overline{V} + \mu_e)^2 - \omega_0^2}{\lambda} > 0 \, , \tag{2.37}$$

$$n_\pi = -2\, \frac{(\overline{V} + \mu_e)[(\overline{V} + \mu_e)^2 - \omega_0^2]}{\lambda} \; , \tag{2.38}$$

$$\mathcal{E} = -\frac{[(\overline{V} + \mu_e)^2 - \omega_0^2]^2}{2\lambda} - \frac{1}{4} n_\pi \mu_e \, . \tag{2.39}$$

Equation (2.38) is an equation for the determination of μ_e and, by the same token, n_π.

It is convenient to introduce the frequencies ω^+ and ω^-. According to (2.5b) and (2.5c)

$$\overline{V} = \frac{\omega^- - \omega^+}{2} \; ; \quad \omega_0 = \frac{\omega^+ + \omega^-}{2} \; .$$

The condition $\eta^2 > 0$ takes the form

$$\eta^2 = -2\, \frac{(\omega^+ - \mu_e)(\omega^- + \mu_e)}{\lambda} > 0 \, .$$

Thus the condensation corresponds to a negative value of $\omega^- + \mu_e$. Near the transition point $(|\omega^-|, \mu_e \ll \omega^+)$, Eq. (2.38) is written in the form

$$\mu_e = -\omega^-\left(1 - \frac{\lambda}{3\pi^2} \, \frac{\omega_-^2}{\omega_+^2}\right) \, .$$

The critical field V_c corresponds to the condition

$$\overline{V}_c = -\omega_0 \, ,$$

$$\omega^- = \overline{V} + \omega_0 = \overline{V} - \overline{V}_c \, .$$

Equation (2.39) then becomes

$$\mathcal{E} = -\tfrac{1}{4} n_\pi \mu_e + O((\xi - \xi_c)^6) \cong -\frac{1}{12\pi^2} (\overline{V} - \overline{V}_c)^4 \, . \tag{2.40}$$

In this case the condensate energy is limited not by the repulsion between the pions, but by the Pauli principle for the positrons.

B. Pion fields in a nucleon medium

The instability of a pion field in a nucleon medium is discussed in the "gas" approximation.

A very simple model of π condensation in a nucleon medium is presented and makes it possible to explain the physical meaning of the condensation. This treatment makes it easy to proceed to the realistic description of pion motion in a nucleon medium as presented in the following sections.

1. Instability of pion fields in a nucleon medium

Can a nucleon medium play the role of a certain external field for pions, and can the pion field become unstable thereby? To answer these question, Migdal (1971) considered the influence of nucleons on pions in the gas approximation. The meson frequency was obtained from the formula

$$\omega^2 = 1 + k^2 - 4\pi n F(k, \omega) = 1 + k^2 + nA(k, \omega), \qquad (2.41)$$

where n is the nucleon density, and F is the forward scattering amplitude of pions with momentum k and energy ω by a nucleon of the medium (we omit the isotopic indices for the time being). Here $A(k, \omega) = -4\pi F(k, \omega)$ is the scattering amplitude in the "energy" normalization.

It is easy to derive (2.41) by considering the Klein–Gordon–Fock equation, treating the nucleus as a dilute gas of noninteracting scattering centers and assuming that the π-nucleon scattering amplitude is known.

Since at $k = 0$, the interaction of the nucleons with the pions is small ($F \ll 1$), an instability can set in only if $F(k, \omega) > 0$ at some $k \gtrsim 1$. In particular the contribution of N^* resonance to the scattering amplitude has such a behavior—the sign of the amplitude F corresponds to attraction between the pions and the nucleons ($F > 0, k \gtrsim 1$).

Moreover, at a sufficiently high density n, when the principal role, as we shall show below, is played by a resonant scattering of pions by nucleons, Eq. (2.41), which was used by Migdal (1971), accounts sufficiently well for the real change of the pion energy.

According to (2.41), at sufficiently high density n, the value of $\omega^2(k)$ can vanish for some $k = k_0$. The pion field instability described above then sets in, and to solve the problem it is necessary to introduce the interaction between the pions. This interaction in a nucleon medium differs greatly from pion-pion interaction in vacuum, and will be considered in greater detail below. As a simple model we can use the interaction $H' = \lambda\varphi^4/4$, which was introduced above, and regard λ as a phenomenological constant which will be estimated below.

It is easy to verify that at $N = Z$ the nucleon medium acts in the same fashion on the π^+, π^-, and π^0 mesons, i.e., the field produced by the medium is an isotopic scalar. According to (2.41), the problem reduces to that solved above, concerning instability in a scalar field. In the case of a neutron star ($Z \ll N$), the instability condition must be investigated separately for each type of pionic excitation. We consider below in detail all possible types of pion field instability in a nucleon medium with arbitrary ratio Z/N.

2. Simplest model of π condensation

To explain the physical idea of π condensation, it suffices to consider the case of an isotopic scalar nucleon field. The energy in the transcritical region is given by (2.15). The term

$$E_\pi = -(\omega^4/4\lambda_1)$$

yields the energy gained as a result of the "condensation." "Condensation" means that at a large value of $|\omega^2|$ the mean square of the field in the ground state exceeds the square of the zero-point oscillation field. According to (2.10) and (2.13), we have in the transcritical region

$$\langle\varphi_0^2\rangle \cong |\psi(\mathbf{r})|^2 q_0^2 = -(\omega^2/\lambda_1)|\psi(\mathbf{r})|^2 .$$

The mean value of the field φ is then equal to 0.

In the case of a system of large dimension $R \gg 1$ (the case of interest to us from now on), the frequency squared ω_k^2 becomes a negative quantity for a large number of states with values of k close to the value k_0 for which $|\omega_k|^2$ is maximal. The minimum of the system energy, as follows from (2.15), corresponds to the state at which the "condensation" has occurred in the state $k_0(|\omega_{k_0}|^2$ is maximal). All the remaining degrees of freedom, when the field φ_0 is taken into account, will then have positive frequencies. Indeed, the Hamiltonian for an oscillator with frequency ω_k ($k \neq k_0$) becomes, after making the substitution $\varphi \to \varphi_k + \varphi_0$

$$H_k = \int \left\{ \frac{\dot{\varphi}_k^2 + \omega_k^2 \varphi_k^2}{2} + \frac{3\lambda\varphi_k^2\langle\varphi_0\rangle^2}{2} + \frac{\lambda\varphi_k^4}{4} \right\} d\mathbf{r} . \qquad (2.42)$$

We shall show that the third term is positive and larger than the second negative term. Thus condensation in the state k_0 stabilizes all the remaining states for which $\omega_k^2 < 0$. The value of the additional term in (2.42) depends on the form of $\psi(\mathbf{r})$. For a homogeneous unbounded system, the most general form of $\psi(\mathbf{r})$ for a real field corresponding to a wave vector $\mathbf{k}_0$, is

$$\psi(\mathbf{r}) = \sum_\lambda a_\lambda \sin \mathbf{k}_\lambda \mathbf{r},$$

$$k_\lambda^2 = k_0^2 .$$

In our simplest case it is easy to verify that the minimum energy E corresponds to the case when there is only one term in this sum. In the real formulation of the problem, the determination of the geometrical and isotopic structures of $\psi(\mathbf{r})$ is a complicated problem to which we shall return in Sec. IV.

Using the expression for φ_0 and the connection (2.12b) between λ_1 and λ at $\psi = (2/\Omega)^{1/2} \sin k_0 x$, we find that the resultant frequency of the oscillation with the wave vector $k \neq k_0$ is

$$\tilde{\omega}_k^2 = 2|\omega_{k_0}^2| - |\omega_k^2| > 0 .$$

Thus all the oscillations are stable in the transcritical region.

As seen from (2.15), the energy E_π gained through π condensation is proportional to the volume of the system, inasmuch as

$$\lambda_1 = \lambda \int \psi^4 d\mathbf{r} \sim \frac{\lambda}{\Omega} .$$

At $\psi(r) = (2/\Omega)^{1/2} \sin k_0 x$, we have $\lambda_1 = (\frac{3}{2})(\lambda/\Omega)$ and the condensation energy is equal to

$$E_\pi = -\frac{\omega_{k_0}^4}{6\lambda}\Omega .$$

Near the critical density, when $\omega_{k_0}^2$ passes through zero, we have

$$\omega_{k_0}^2=\eta(n_c-n), \quad \eta>0.$$

The value of η is easily determined from (2.41). Putting

$$\frac{\eta^2}{3\lambda}=\beta$$

we obtain for the energy density $\mathcal{E}_\pi$ the expression

$$\mathcal{E}_\pi=-\frac{\beta(n-n_c)^2}{2}\,\Theta(n-n_c). \tag{2.43}$$

The Θ function expresses the fact that $\mathcal{E}_\pi=0$ at $n<n_c$. Estimates, which will be detailed later on, yield $\beta\sim 1$.

III. PION EXCITATIONS IN NUCLEAR MATTER

In this section we investigate in detail processes that determine the change of the pion energy in a nucleon medium with an arbitrary ratio Z/N. A consistent realistic method is developed for the calculation of the spectrum of the π^+, π^-, and π^0 mesons in nuclear matter, using the methods of the many-body problem. Our calculation is based on separation of processes that depend essentially on the 4-momentum of the pion and on replacing the slowly-varying quantities by constants, in a manner similar to that used in the Fermi liquid theory. These constants must be obtained from experiment, or else estimated with the aid of the theory of nuclear matter. We discuss the influence of the pion degree of freedom on the interaction of the nucleons in nuclear matter and obtain Fermi liquid theory equations that take into account the role of one-pion exchange. A scheme is proposed for the inclusion of pion degrees of freedom in nuclear matter theory.

The energy spectrum of the π^+, π^-, and π^0 mesons is investigated in a medium with $N\approx Z$ (nucleus) and in a medium with $Z\ll N$ (neutron star). It is shown that besides the pion excitation branch, which corresponds to the free pion energy when the (πN) interaction is turned off, additional spectrum branches with the quantum numbers of the pion are produced in the system and must be interpreted as bound states of a nucleon with a nucleon hole (spin-sound branches). In the case $N\simeq Z$, starting with a certain density, there appears simultaneously for the three types of pions (π^+, π^-, and π^0) a region of values of the wave vector k such that $\omega^2(k)<0$, for the spin-sound branch, indicating instability of the nucleon-pion system with respect to the production of π_s^0 and π_s^+, π_s^- meson pairs. (The symbol s denotes spin-sound excitation with the quantum numbers of the pion.)

In the case of a system with $Z\ll N$, a π_s^+ branch with energy $\omega_s^+<-\epsilon_F^{(n)}$ is first produced, making possible the process $p\to n+\pi_s^+$, and all the protons of the medium change into neutrons and the bound state ($p\bar{n}$). At larger density, instability sets in with respect to the production of (π^-,π_s^+) pairs and π^0 mesons.

The nature of the instability observed in the model of Sawyer and Scalapino (1972) is explained. It is shown that this instability has the same nature as the instability with respect to production of (π^-,π_s^+) pairs that appears in the more realistic model (Migdal, 1973; Migdal, Markin, and Mishustin, 1974).

A. Determination of the pion polarization operator in nuclear matter

In this section we analyze the possibility of using methods associated with the many-body problem for the study of excitations having pion quantum numbers (pion degrees of freedom). All the processes that are essential for the calculation of the pion polarization operator are investigated in detail. An expression is obtained for the polarization operator with account taken of nucleon correlations, N^* resonance, and S scattering.

1. Use of the methods of the many-body problem

We have shown in the last section how to determine the pion spectrum in the gas approximation. Here this problem is considered in a more realistic formulation, with account taken of all the essential excitations of the medium. We recall from Eq. (1.3) that the pion energy ω in homogeneous nuclear matter as a function of the momentum k is written in the form ($\hbar=m_\pi=c=1$)

$$\omega^2=1+k^2+\Pi(k,\omega).$$

The quantity $\Pi(k,\omega)$ takes the polarization of the medium into account.

In the case of an electromagnetic field, the analogous quantity $\Pi^{(\gamma)}(k)$ is directly connected with the dielectric constant $\epsilon(k,\omega)$, inasmuch as in this case

$$\omega^2=\frac{k^2}{\epsilon(k,\omega)}=k^2\left(1+\frac{1}{k^2}\,\Pi^{(\gamma)}(k,\omega)\right).$$

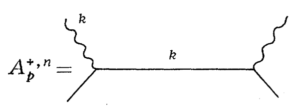
This analogy is frequently used to obtain, in the case of pions, a formula similar to the Lorenz–Lorentz formula (Ericson and Ericson, 1966). It must be assumed here that the amplitude of the virtual πN scattering (i.e., off–the-mass shell) is δ-like and does not differ from the real amplitude. These assumptions certainly are not satisfied in nuclear matter with nuclear density. Yet, as we shall see, there exists a consistent method of determining the polarization operator, free of these restrictions. Of course, the exact calculation of the polarization operator in a medium of strongly interacting particles is an insoluble problem. It is easy, however, to separate the slowly varying quantities, which can be regarded as constants and determined from experiment, and express in terms of them other quantities that vary significantly in the region of interest, in analogy with the procedure used in Fermi liquid theory (Migdal, 1967). This method is based on the fact that all processes that determine $\Pi(k,\omega)$ can be divided into two classes: those occurring at distances smaller than or of the order of $1/m_N$, and those occurring at distances on the order of unity in pion units. Processes of the former type, in a medium with density that is low in comparison with $m_N^3\sim 300$, proceed just as in a vacuum, whereas processes of the latter type are appreciably distorted by the medium. Thus, for example, the local pion-nucleon interaction vertex, as can be verified by considering the relevant diagrams, is determined by the small distances $r_0\sim 1/m_\rho$ or $1/m_N$, and consequently the πN-interaction constant in a medium of nuclear density differs little from the interaction in a vacuum.

Let us make a few remarks concerning the diagrammatic calculation method that will be frequently employed here.

Graphs or diagrams are primarily a convenient method of illustrating the occurring processes. They can be given the meaning of quantitative relations by assuming that each diagram describes a definite transition amplitude. Then, according to the superposition principle, the total transition amplitude is the sum of all the possible physically different amplitudes. In addition, if the diagram involves intermediate states, the corresponding amplitude is given by the product of "elementary" amplitudes corresponding to each subdiagram, integrated over the intermediate times. If we introduce time-independent amplitudes, then this statement corresponds to the known quantum-mechanical formula

$$A_{01}=\sum \frac{B_{0i}C_{i1}}{E_0-E_i}\ . \tag{3.1}$$

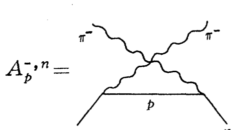
Any process, no matter how complicated, is made up by consecutive use of several simple amplitudes, which can be obtained once and for all by comparing the corresponding element of the diagram with perturbation theory. Thus the graphic method in the form in which we shall use it constitutes a simple utilization of the formulas of ordinary quantum mechanics and calls for no additional knowledge. For example, the pole part of the forward scattering amplitude of a π^+ meson by a neutron at rest can be written in the form



In the intermediate state there is a proton with momentum k. According to (3.1), this amplitude is equal to

$$A_p^{+,n}=\frac{|\Gamma|^2}{\omega+m_N-E(k)}\,,$$

where Γ is the amplitude for the absorption of the pion by a nucleon, ω is the pion energy, and $E(k)$ is the nucleon energy. In the case of pole scattering of a π^- meson by a neutron, the only possible diagram is



which corresponds to the fact that the final meson is emitted first, and the initial meson is absorbed next. The amplitude in this case is

$$A_p^{-,n}=\frac{|\Gamma|^2}{\omega+m_N-(2\omega+E(k))}=\frac{|\Gamma|^2}{-\omega+m_N-E(k)}.$$

More complicated diagrams will be explained as they appear.

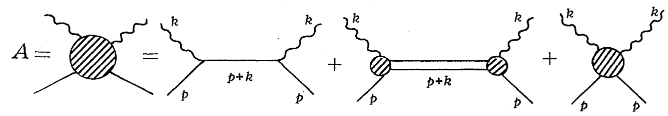
2. Diagrams that determine the polarization operator

The contribution of the medium to the square of the pion energy is expressed in the gas approximation in terms of the zero-angle scattering amplitude in the energy normalization [Eq. (2.41)]. Since the polarization operator is in fact this contribution, in the gas approximation we have

$$\Pi(k,\omega)=nA(k,\omega). \tag{3.2}$$

The normalization of the amplitude A is determined by the fact that in the Born approximation A becomes the volume integral of the energy of the perturbation due to one nucleon. To get rid of the gas approximation, it is necessary to introduce in place of the total density of the nucleons the Fermi distribution density $n(p)$ for the neutrons and protons and to take into account in the calculation of A the Pauli principle and the interaction between the nucleons in the intermediate states. As a result, the A amplitude itself turns out to depend on the distribution $n(p)$.

Before we proceed to the calculation of $\Pi(k,\omega)$, let us ascertain which of the processes determine the πN scattering amplitude in the vacuum. It is known that the πN scattering at low pion energies $\omega\sim 1$ is described with good accuracy by the following processes:



(3.3)

The first of the diagrams corresponds to one nucleon in the intermediate state (the "pole" term of the scattering). The second diagram corresponds to a transition to the N^*_{33} resonance (the resonant part of the scattering). We shall show that both terms describe P scattering. The last of the terms in (3.3) is S scattering. It is represented by a point, meaning that it can be regarded as a local interaction and consequently does not depend on the pion momentum. Indeed, it can be verified that S scattering is due to intermediate states that have high energies and momenta, and therefore depend little on the pion momentum at low pion energies. For the same reason, we use points to represent the vertices $N\pi N$ and $N\pi N^*$, in the first and second graphs. There is an additional part of P scattering due to the distant resonances, and one should take into account the off-mass-shell dependence of the S scattering. The corresponding contribution in the scattering amplitude is determined below from the experimental scattering data. To find the off-mass-shell amplitude the results of current algebra are used. The $N\pi N$ vertex is written in the form (see, for example, Gasiorowicz, 1966)

$$\Gamma(N\pi N)=f\bar{\psi}\gamma_\nu\gamma_5\tau_\alpha\psi\partial_\nu\varphi_\alpha\,. \tag{3.4}$$

where ψ is the wave function of the nucleon, γ_ν are Dirac matrices, τ_α are the nucleon isospin matrices, and φ_α are the components of the pion field. The fields of the π^+, π^-, and π^0 mesons are connected with φ_α by the relation

$$\varphi^{\pm}=\frac{\varphi_1\pm i\varphi_2}{\sqrt{2}};\ \ \varphi^0=\varphi_3. \tag{3.5}$$

The constant f in (3.4) equals $g/2m_N$, where g is a dimensionless interaction constant; $g^2/4\pi\cong 14$, and in pion units $m_N=6.7$ and $f\cong 1.0$.

For nonrelativistic nucleons, Eq. (3.5) simplifies to

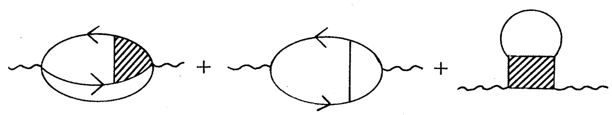
$$\Gamma(N,\pi,N) \cong f\psi^{+}\sigma_{\alpha}\tau_{\beta}\psi\nabla_{\alpha}\varphi_{\beta}, \tag{3.6}$$

where σ_{α} is the nucleon spin matrix.

As follows from (3.6), the vertex is proportional to the pion momentum, and the first term of (3.3) describes p scattering. Since the spin of the N^* isobar is 3/2 [the resonance $N_{33}^*(1232)$], the second term also corresponds to p scattering and its vertex is also proportional to the wave vector of the pion; the proportionality coefficient can be obtained sufficiently accurately from the cross section for the scattering of pions with energy close to resonance.

Accordingly, for the third process of (3.3), which determines the $(\pi N)S$ scattering amplitude, the polarization operator at pion 4-momenta $\omega \sim 1, k \sim 1$, as will be shown, is determined by the same πN scattering mechanisms in a medium. The pole (resonant) interaction of the pion with the nucleons of the medium can be represented in two ways: either as scattering of a pion with a transition of the nucleon into a state lying above the Fermi surface (an isobar), or as the production of a nucleon (an isobar) and the appearance of a hole in the nucleon Fermi sea. The second approach is for many reasons more convenient than the first and is in fact the one used in the many-body problem and in the Fermi liquid theory, the results of which we shall use.

Thus the polarization operator is represented by a sum of three diagrams:

$\Pi(k,\omega) =$  (3.7)

$= \Pi_k + \Pi_p + \Pi_S$

Lines with arrows directed to the left and to the right represent holes and particles, respectively. The shaded triangles represent vertices that take into account the NN and NN^* correlations in nuclear matter. Expressions connecting these vertices with the constants of the NN and NN^* interactions will be given later on. The first term, designated Π_R, corresponds to production of a nucleon hole in the Fermi sea and the isobar $N_{33}^*(1232)$ ("resonant term"). The "pole" term Π_p corresponds to excitation of a particle-hole type in the medium. The third term takes into account the S scattering. All the remaining diagrams that have no parts connected by a particle and hole or by a hole and isobar are determined by the large 4-momenta of the intermediate states ($\sim m_N$). They may make a small contribution, or differ little from the corresponding vacuum diagrams, which were already taken into account in the observed pion mass or they may be contained in the effective mass m^* of the nucleon, which will be used below ($m^* \approx 0.9 m_N$). In other words, these diagrams are characterized by spatial dimensions $\sim 1/m_N$ and are not greatly distorted in nuclear matter, where the distance between particles is of the order of m_π^{-1}. These diagrams depend little on the 4-momenta of the incoming particles, since we are interested in 4-momenta $\sim m_\pi$. They can therefore be replaced by constants, which should be obtained from experiment.

As is well known, the same idea is used in Fermi liquid theory to introduce the constants that determine the interaction near the Fermi surface, and also to introduce the effective mass and the effective local "charge" of the quasiparticles in an external field (Migdal, 1967).

As an illustration let us estimate the pion-mass error resulting from the fact that the incoming pions in Π are taken not on the mass shell, but at $k^2 - m_\pi^2 \cong m_\pi^2$. Since the vacuum part of the polarization operator changes significantly at momenta on the order of m_N^2 or on the order of the squared mass of the corresponding resonance, it follows that

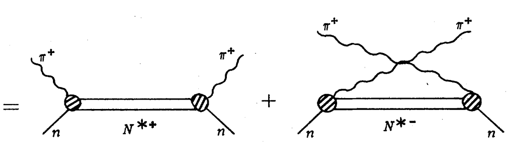
$$\delta m_\pi^2 \sim \frac{\delta \Pi_{\mathrm{vac}}}{\delta k^2}(k^2 - m_\pi^2) \sim \left(\frac{m_\pi}{m_N}\right)^2 m_\pi^2 .$$

We see that this error is small.

Thus, the use of the methods of the many-body problem makes it possible to separate and calculate diagrams that vary strongly in the range of variables of interest to us, and replace the remaining diagrams by constants obtained from experiment.

3. Resonant part of the polarization operator

The isotopic structure of the $N\pi N^*$ vertex is somewhat more complicated than for the $N\pi N$ vertex. We consider first the amplitude of the $\pi^+ n$ scattering. The second term of (3.4) must be written out in greater detail, since it consists of two parts

$A_R^{+,n} =$ 

We denote the matrix element of the vertex by $\sqrt{a}\,k$. The first term corresponds to the second-order energy correction and is equal to the square of the vertex divided by the energy difference between the initial and intermediate states:

$$(A_R^{+,n})_1 = \frac{ak^2}{\omega - \omega_R + i\gamma_0 k^3} .$$

The damping of the resonance is taken into account here, and the kinetic energy of the N^* particle in the intermediate state has been discarded:

$$\omega_R \cong m_{N^*} - m_N + \frac{k_R^2}{2m_{N^*}} \cong 2,4. \tag{3.8}$$

The denominator of the second term is equal to

$$\omega - [2\omega + \omega_R - i\gamma_0 k^3].$$

It can be easily verified from laws for the addition of the isotopic angular momentum that a factor equal to $\sqrt{3}$ appears in the second diagram. As a result

$$A_R^{+,n} = -ak^2\left(\frac{1}{\omega_R - \omega - i\gamma_0 k^3} + \frac{3}{\omega_R + \omega - i\gamma_0 k^3}\right). \tag{3.9}$$

We note that the second term is usually omitted, since at $\omega \cong \omega_R$ it amounts to $\sim 1\%$ of the first term. Far from resonance, however, when $\omega \ll \omega_R$, it must be taken into account.

The "optical theorem" (the unitarity condition) establishes a connection between the imaginary part of the zero-angle scattering amplitude and the corresponding cross section integrated over the angles. It follows from this relation that $a(k) = 4\pi\gamma_0(k)$. The function $a(k)$ is chosen such that Eq. (3.9) describes the observed resonant scattering. It is customary to use the empirical expression [all parameters were taken from the paper of Carter *et al.* (1971)]

$$a(k) = 4\pi\frac{0.08}{1+0.23k^2} = \frac{1.01}{1+0.23k^2}. \tag{3.10}$$

The strong k^2 dependence of $a(k)$ does not correspond to the theoretical expectations for the form factor $g^*(k^2)$ of the πNN^* vertex. The reasonable form of $g^*(k^2)$ is

$$g^{*2} = 1 - k^2/\Lambda^2, \quad k \ll \Lambda,$$

where $\Lambda \sim (1/2)m_N^2$. Therefore the empirical expression (3.10) takes into account the additional P scattering provided by distant resonances. To extract the N^*-resonance P scattering we shall use a value a which is found below: $a = 1.8a(k^R)$ (the form factor momentum dependence is omitted).

We are interested in values $\omega < \omega_R$ and $k \lesssim 1$, at which the damping can be omitted from (3.9). Therefore the resonant part of the polarization operator of a π^+ meson in a neutron medium takes the form

$$\Pi_R^{+,n} = -\frac{4nak^2\left(1 - \dfrac{\omega}{2\omega_R}\right)\omega_R}{\omega_R^2 - \omega^2}. \tag{3.11}$$

The polarization operator for the π^- meson can be obtained from Π^+ by using the crossing-symmetry requirement. Crossing-symmetry means that any transition amplitude (and, in particular, a polarization operator), must not change if we go from particle to antiparticle and simultaneously reverse the signs of the energy and momentum (the absorption of the particle with momentum **k** is equivalent to production of an antiparticle with momentum $-\mathbf{k}$). Using in addition isotopic invariance, we obtain the relations

$$\Pi_R^{+,n}(\omega,\mathbf{k}) = \Pi^{-,n}(-\omega,-\mathbf{k}) = \Pi^{-,n}(-\omega,\mathbf{k})$$
$$\Pi^{+,n}(\omega,\mathbf{k}) = \Pi^{-,p}(\omega,\mathbf{k}). \tag{3.12}$$

We have also made use of the fact that the polarization operator is a function of k^2. In a medium with an arbitrary ratio N/Z we obtain in accordance with (3.11) and (3.12)

$$\Pi_R^+ = -\frac{4ak^2\Gamma_R\omega_R}{\omega_R^2-\omega^2}\left[n_p\left(1+\frac{\omega}{2\omega_R}\right) + n_n\left(1-\frac{\omega}{2\omega_R}\right)\right] \tag{3.13}$$

We have introduced in (3.13) the factor Γ_R, which takes into account the change of the $N_\pi N^*$ vertex, owing to NN^* interaction in nuclear matter. The factor Γ_R can be approximately written in the form

$$\Gamma_R = \frac{1}{1+\nu n}, \tag{3.14}$$

where ν is a constant characterizing the NN^* interaction and $n = n_p + n_n$. We shall derive (3.14) below after calculating the influence of the NN interaction on the pole part of the polarization operator. Unfortunately, it is theoretically difficult not only to estimate the factor Γ_R, but even to determine whether it is larger or smaller than unity (i.e., to determine the sign of ν). If it is assumed that the NN^* interaction is of the same order as NN then at $n = n_0 = 0.5$ we have $\Gamma_R \cong 0.8-0.9$. On the other hand if NN^* does not contain repulsion at short distances and is determined by one-pion exchange, then $\Gamma_R \cong 1.1-1.2$. Favoring this assumption is the large cross section of the reaction $(pn; nN^+)$ at large momentum transfers, at which the NN^* repulsion at short distances should manifest itself (Mountz *et al.*, 1975). The data on the spectra of the π-atom seem to yield $\Gamma_R \cong 1$(VI.B).

We note that the expression we have used for the resonance scattering amplitude at low pion energies turns out to be smaller than the total amplitude of P scattering. This means that we need to add to the resonant part of the polarization operator the contribution from remote P resonances. This contribution can be taken into account by representing the P-scattering amplitude at low pion energies in the form

$$A_{RR'} = A_R + A_{R'}.$$

The first term corresponds to resonance scattering N_{33}^*, and the second takes into account the contribution of all the remaining P scatterings.

To determine A_R, we can use the expression given below for the zero-angle P-scattering amplitude obtained from a detailed analysis of experiments on πN scattering

$$A_{RR'}^+ \cong -\left(0.6 + \frac{1.6}{1-\omega^2/\omega_R^2}\right)\mathbf{k}\cdot\mathbf{k}', \quad A_{RR'}^- \cong 0.2\mathbf{k}\cdot\mathbf{k}'$$

where

$$A^s = \frac{1}{2}(A^{+n} + A^{-n})$$
$$A^a = \frac{1}{2}(A^{+n} - A^{-n}) \tag{3.15}$$

are the isotopic symmetric and antisymmetric amplitudes. At $\omega \ll \omega_R$ we have

$$\Pi_{RR'} \cong \Pi_R - 0.6nk^2.$$

Thus, the correction due to the distant resonances, which is negligibly small at $\omega \cong \omega_R$, turns out to be quite appreciable at small $\omega \lesssim 1$.

4. S-wave scattering

Since the S-scattering amplitude is δ-function-like, it should not change noticeably in the medium, and consequently we can use formula (2.41)

$$\Pi^{\pm}_{\text{loc}} = (n_n + n_p)\tilde{A}^s \pm (n_n - n_p)\tilde{A}^a \qquad \Pi^{0}_{\text{loc}} = (n_n + n_p)\tilde{A}^s \tag{3.16}$$

These equations are valid for all local contributions to Π. Here $\tilde{A}$ is the local scattering amplitude in vacuum in the energy normalization. On-the-mass shell ($\omega^2 = 1 + k^2$), $A^{s,a}$ should agree with the experimentally obtained scattering amplitude. It is necessary to obtain $A^{s,a}(\omega, k)$ off-the-mass shell. It follows from crossing symmetry that the difference between the $\pi^+ n$ and $\pi^- n$ scattering amplitudes is an odd function of ω:

$$\tilde{A}^a = A_1\omega + A_3\omega^3 + \ldots .$$

It follows from current-algebra considerations that the expansion is in powers of m_π/m_N.

The value of A_1 obtained from π^+ and π^- scattering by protons depends little on k at $k \ll m_N$, and is equal to

$$A_1 = -2\pi\,\frac{m_N + 1}{m_N}\left(F_S^{\pi^+,n}(m_\pi) - F_S^{\pi^-,n}(m_\pi)\right) = -2\pi(0.21) = -1.3$$

Here $F_S^{\pi^+,n}(m_\pi)$ is the amplitude in the usual normalization.

We note that the theoretical value of A_1 obtained from current-algebra considerations (Heisenberg *et al.*, 1969) agrees well with this:

$$A_1 = -\frac{2f_\pi^2}{g_A^2} = -1.4;\; g_A = 1.2 .$$

We thus have

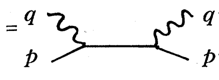
$$\Pi_S{}^a = (n_n - n_p)\tilde{A}^a = 1.4(n_n - n_p)\omega .$$

To determine the isotopically-symmetric part A_S^s of the S-scattering amplitude it is necessary to use the results obtained in current algebra for the πN-scattering amplitude off the mass shell. However it is more convenient to discuss the total amplitude without separating S and P-waves. We denote the incoming and outgoing 4-momenta of the pion and nucleon by q, q' and p, p', and introduce the usual notation $s = (p+q)^2 = (p'+q')^2$, $t = (q-q)^2$, and $u = (p-q')^2$. Let the nucleon be on the mass shell $p^2 = p'^2 = m^2$. The scattering amplitude can be regarded as a function of the variables t, $\nu = (s-u)/4m = \omega + t/4m$, $v = (q^2 + q'^2)/2$, where ω is the meson energy in the laboratory frame.

According to the "self-consistency condition" (Adler, 1965) the scattering amplitude, after subtracting the pole part, should vanish at $q^2 = 1$ and $q' \to 0$, i.e., at $t = 1$, $\nu = 0$, and $v = \frac{1}{2}$

$$\tilde{A} \equiv (A - A_P)\big|_{\substack{\nu=0\\ t=1\\ v=1/2}} = 0. \tag{3.17}$$

The pole term $A_P^{s,a}$ can be easily calculated

$A_P^{s,a} =$ 

$$= -\frac{g^2}{2m}\,\nu_B\left(\frac{1}{\nu_B - \nu} \pm \frac{1}{\nu_B + \nu}\right),$$

where

$$\nu_B = -\frac{qq'}{2m} = -\frac{\omega^2 - \mathbf{k}\cdot\mathbf{k}'}{2m},\quad A_P^s \underset{kk' \ll \omega^2}{=} \frac{f^2}{m}\omega^2 - \frac{2f^2}{m}\,\mathbf{k}\cdot\mathbf{k}' .$$

Since we have subtracted from A the only rapidly-varying (pole) term, we can confine ourselves in A to the terms linear in t and in v.

In the paper by Nagels *et al.* (1976), the amplitude $\tilde{A}$ at $v = 1$ is represented in the form (the quantity $\tilde{C}$ introduced there differs from A in sign)

$$\tilde{A} = a_{00} + (a_{01} + a_{11}\nu^2)t + (a_{10} + a_{20}\nu^2)\nu^2 + \alpha(v - 1) \tag{3.17a}$$

The coefficients $a_{mn}^{s,a}$ are approximately equal to

	a_{00}	a_{01}	a_{11}	a_{10}	a_{20}
$\tilde{A}^s$	1.5	−1.2	−0.15	−1.1	−0.2
$\nu^{-1}\tilde{A}^a$	−1.5	+0.1	0	0.2	0

The value of a_{00}^a was chosen such as to obtain the correct value of A^a at the threshold. The expression for A^s enables us to separate the contribution of the N^* resonance. If it is assumed that the dependence on $\nu^2 \approx \omega^2$ is determined by the N^* resonance, then $\tilde{A}$ can be represented in the form (at $v = 1$):

$$\tilde{A} = a_{00} + \left(a_{01}^{R'} + \frac{a_{01}^R}{1 - \omega^2/\omega_R^2}\right)t + \frac{a_{10}\omega^2}{1 - \omega^2/\omega_R^2} . \tag{3.17b}$$

Indeed, calculation of the diagram corresponding to N^* resonance leads to an expression in the form

$$\tilde{A}_R = a_R + \frac{b_R t}{1 - \omega^2/\omega_R^2} + \frac{c_R\omega^2}{1 - \omega^2/\omega_R^2} .$$

Accordingly, the ratio of the coefficients a_{20}/a_{10} is equal, with good accuracy, to $1/\omega_R^2$.

The contribution of the resonance to the term proportional to t is determined from the value of a_{11} and from the experimental value of the P-scattering length a_P^+ at the threshold

$$a_{01}^R/\omega_R^2 = a_{11};\; 2\left(a_{01}^{R'} + \frac{a_{01}^R}{1 - 1/\omega_R^2}\right) = -2.6 .$$

After adding the P-scattering pole term, which is equal to $-2f^2/m \approx 0.3$, this result agrees with the experimental value $a_P^+ = 0.21$.

We obtain:

$$a_{01}^{R'} = -0.3;\, a_{01}^R = -0.8 .$$

If the scattering were to be determined entirely by the N^* resonance, then the value of a_{01}^R would be 40–50% larger, in accord with the thorough analysis carried out by Höhler *et al.* (1972). Comparing the value of a_{01}^R with the quantity a introduced into the N^*-resonance amplitude [Eq. (3.13)], we get

$$-4a/\omega_R = 2a_{01}^R,\; a \cong 0.9$$

in place of the value $a(k = k_R) = 0.5$ given above. The reason for the discrepancy is that the simple resonant expression used by us ceases to be valid, owing to the large resonance width, long before the point $k = k_R$ is approached.

To find the constant α, we use the self-consistency condition (3.17)

$$\alpha = 2(a_{00} + a_{01}') \cong 0.8,$$

where

$$a'_{01} = a^{R'}_{01} + a^{R}_{01} \cong -1.1 .$$

The expression for the S scattering is obtained from (3.17b) by replacing t with $-2k^2$. For simplicity we present only A^s_S at $\omega = 0$

$$\tilde{A}^s_S|_{\omega=0} = 0.7 + 1.4k^2 .$$

We obtain an expression for the zero-angle scattering amplitude ($t=0$) which enters in the polarization operator without a breakup into the S and P amplitudes. From (3.17b) we get

$$\tilde{A}^s(t=0) = -(a_{00} + 2a'_{01}) - 2(a_{00} + a'_{01})k^2 + \left[2(a_{00} + a'_{01}) - a_{00}\frac{1 - 1/\omega_R^2}{1 - \omega^2/\omega_R^2}\right]\omega^2 .$$

We have used the fact that the amplitude at the threshold is small and becomes even smaller after subtracting the pole term ($= f^2/m = 0.15$), i.e.,

$$a_{00} + \frac{a_{10}}{1 - 1/\omega_R^2} = 0 .$$

Substituting the numerical values of the coefficients we have

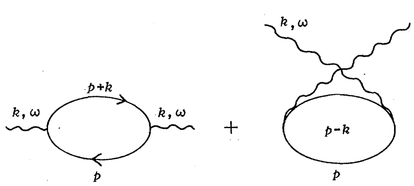
$$\tilde{A}^s(t=0) = 0.7 - 0.8k^2 + \left[0.8 - \frac{1.2}{1 - \omega^2/\omega_R^2}\right]\omega^2 , \qquad (3.17c)$$

$$\tilde{A}^a(t=0) = -1.5\omega(1 - 0.13\omega^2) .$$

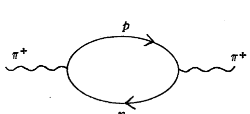
This expression, in the region of interest differs not very much from the value $\tilde{A}^+(t=0)$ obtained without allowance for the additional contribution to the P resonance and S-scattering (Migdal, Maikin, Mishustin, 1974).

5. Pole part of the polarization operator

The pole part of the polarization operator, without the correlations taken into account, is given by the diagrams

$\Pi_P(k,\omega) =$  (3.18)

We consider first a neutron medium. For $\Pi_p^{+,n}$ we obtain

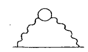
$\Pi_P^{+,n} =$ 

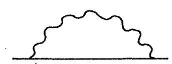
The second graph of (3.18) is allowed by the charge conservation law only if the medium contains protons (a proton hole must be produced).

Using (3.1) and (3.6), we easily obtain

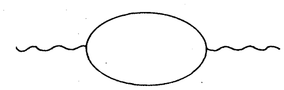
$$\Pi_p^{+,n} = 2f^2k^2 \int \frac{n^{(n)}(p)d^3p}{\omega - \epsilon^{(p)}(\mathbf{p}+\mathbf{k}) + \epsilon^{(n)}(p)} \frac{2}{(2\pi)^3} . \qquad (3.19)$$

We have confined ourselves to the case of nonrelativistic nucleons.

Equation (3.19) becomes rigorous if the intermediate state is taken to mean not the state of a free nucleon and a free hole, but states consisting of a quasiparticle and a quasihole. By the same token we have taken into account all the diagrams that correct the nucleon motion in the medium, e.g., of the type  . Diagrams of the type

 ,

in which the wavy line represents a free pion, have already been taken into account in the observed nucleon mass. Similarly, diagrams of the type

 ,

where the lines correspond to a nucleon and antinucleon (and not to a particle and a hole) are taken into account in the observable pion mass. The change over to quasiparticles complicates the function $E(p)$, but for momenta not too far from the Fermi surface it is possible to characterize the nucleonic excitations by two numbers, the Fermi energy and the nucleon effective mass. In a medium with $N \neq Z$, these quantities are different for the neutron and proton. In a medium with $N \cong Z$, these quantities are sufficiently well known from nuclear data. The effective mass of a nucleon quasiparticle is $m^* \cong 0.9m$ (Migdal, 1967), and the Fermi energy is determined by the density of the nuclear matter and is equal to

$$\epsilon_F = \frac{p_F^2}{2m^*} = \frac{(1.5\pi^2 n)^{2/3}}{2m^*} \cong 45\mathrm{MeV}.$$

As a rough estimate we can assume that in a neutron medium $m^* \cong m$ and determine the difference of the chemical potential of the neutrons and protons at $Z \lesssim N$ (disregarding for the time being the Coulomb field) from calculations for nuclear matter (Pandharipande, 1971). Equation (3.19) can be written in the form

$$\Pi_p^{+,n} = -4f^2 k^2 \frac{m^* p_F}{2\pi^2} \phi_1(k,\omega) \qquad (3.20)$$

where

$$\phi_1 = \frac{m^{*2}}{2k^3 p_F}\left\{\frac{a^2 - b^2}{2} \ln\frac{a+b}{a-b} - ab\right\},$$

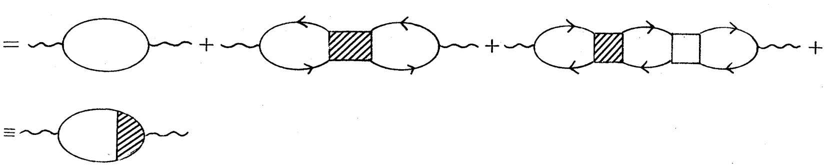
$$a = \omega - \frac{k^2}{2m^*} , \quad b = kv_F . \qquad (3.21)$$

At $|a| \gg b$ we have

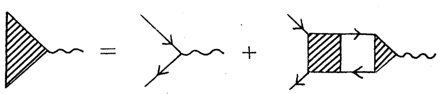
$$\phi_1 = -\frac{\pi^2 n}{m^* p_F} \frac{1}{\omega - \dfrac{k^2}{2m^*}} . \qquad (3.22)$$

6. Nucleon correlations

To take the interactions between nucleons into account it is necessary to replace one of the vertices of each of the diagrams in (3.18) by an exact vertex. Indeed, to take the NN interaction into account, it is necessary to sum the diagrams

$\Pi =$  (3.23)

The shaded triangle denotes here the exact vertex $\mathcal{T}_1$, containing no parts joined by one pion line. The equation for the vertex $\mathcal{T}_1$ in nuclear matter was investigated in detail by Migdal (1967), both for the case of a system of finite size (nucleus), and for the case of unbounded nuclear matter. We confine ourselves here to a brief explanation of the method of finding the vertex. For $\mathcal{T}_1$ we can write the symbolic equation

$\mathcal{T}_1 =$  $= \mathcal{T}_0 + \mathfrak{F}_1 A \mathcal{T}_1$ (3.24)

Here $\mathcal{T}_0$ is the "bare" vertex, $\mathfrak{F}_1$ is the local interaction of the quasiparticles in the nuclear matter, and A is the amplitude of the transition of the quasiparticle and quasihole (the product of the Green's functions of the quasiparticle and the quasihole). A coincides with Eq. (3.19) divided by the square of the bare vertex (i.e., by the quantity $2f^2k^2$. Thus, in our case, the basis of (3.20), the value of A reduces to

$$A = -\frac{m^* p_F}{\pi^2}\,\phi_1(k,\omega). \tag{3.25}$$

The local interaction $\mathfrak{F}_1$ is expressed in terms of dimensionless constants, which should be obtained from experiment or calculated from the theory of nuclear matter. These constants are introduced in the following manner (Migdal, 1965)

$$(dn/d\epsilon_F)_{p_F=p_0}\mathfrak{F}_1 = \{f + f'\boldsymbol{\tau}_1\boldsymbol{\tau}_2 + (g + g'\boldsymbol{\tau}_1\boldsymbol{\tau}_2)\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\}\delta(\mathbf{r}_1 - \mathbf{r}_2)\,. \tag{3.26}$$

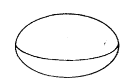
Here $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are the isospin matrices, $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the spin matrices of the two nucleons, while $p_0 = 1.92$ is the Fermi momentum in the nucleus. The local interaction was assumed to be δ-like. The only reason for the noticeable deviation of this interaction from a δ function is the contribution of the one-pion exchange. But from the definition of the polarization operator the diagrams corresponding to one-pion exchange in the particle-hole channel do not enter the interaction represented by the shaded triangles in (3.23).

In the case of a δ-function interaction, Eq. (3.24) reduces to an algebraic one. Since in our case the bare vertex, according to (3.6), is proportional to the spin and isospin matrices of the nucleon, Eq. (3.24) contains only the spin–isospin term (3.26). Using (3.24), (3.25), and (3.26), it is easy to obtain

$$\mathcal{T}_1 = \frac{\mathcal{T}_0}{1 + g^-(p_F/p_0)\phi_1(\omega,k)}\,. \tag{3.27}$$

Here $g^- = g^{nn} - g^{np} = 2g' = 1.6$. This numerical value was obtained from an analysis of the nuclear data on the suppression of the spin part of the magnetic moment in spherical and deformed nuclei and from a renormalization of the Gamow–Teller matrix elements of the β decay (Osadchev and Troitskii, 1968). In a medium with $Z \ll N$, the corresponding constant is unknown and can be estimated generally from the theory of nuclear matter. However, in the case of a neutron star, $\phi_1 \ll 1$ for the relevant values of k and ω, and the influence of the correlation is not very appreciable. As a reasonable estimate we can take the value of g^- in vacuum, which does not differ very strongly from the value in nuclear matter with $N = Z$.

An expression analogous to (3.27) can be obtained also for the vertex $\boldsymbol{\Gamma}_R$ introduced in (3.14). To this end it suffices to replace the quantity A in (3.24) by $A^{(R)}$

$A^{(R)} =$  $= \int \frac{n(p)2d^3p}{\omega + \epsilon^{(N)}(p) - \epsilon^{(N^*)}(p+k)}\,\frac{1}{(2\pi)^3}\,.$

Assuming that $\omega_R \gg kv_F$ and $\omega_R \gg \omega$, we obtain

$$A^{(R)} = -\frac{n}{\omega_R}\,.$$

Assuming the NN^* interaction to be δ-like and denoting the coefficient of $\delta(\mathbf{r}_1 - \mathbf{r}_2)$ by v, we obtain

$$\boldsymbol{\Gamma}_R \cong \frac{1}{1 + n(v/\omega_R)}$$

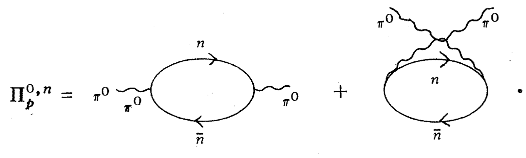
which corresponds to Eq. (3.14) with $\nu = v/\omega_R$. Repulsion ($v > 0$) corresponds to a positive sign of $\nu(\boldsymbol{\Gamma}_R < 1)$, whereas attraction ($v < 0$) yields $\nu < 0$ and $\boldsymbol{\Gamma}_R > 1$. As already mentioned (page 126), we can expect the interval of $\boldsymbol{\Gamma}_R$ to be $\boldsymbol{\Gamma}_R \sim 0.8-1.2$. Substituting (3.27) in (3.23), we obtain

$$\Pi_p^{+,n} = -2f^2k^2\,\frac{m^*p_F}{\pi^2}\,\frac{\phi_1(\omega,k)}{1 + g^-(p_F/p_0)\phi_1(\omega,k)}\,. \tag{3.28}$$

The expression for the polarization operator of the π^- meson can be obtained in similar fashion, but recognizing that in this case only the second of the diagrams of (3.18) is present. It suffices, however, to use crossing symmetry and obtain Π_p^- directly from (3.28)

$$\Pi_p^{+,n}(\omega,k) = \Pi_p^{+,n}(-\omega,-k) = \Pi_p^{+,n}(-\omega,k)\,. \tag{3.29}$$

We now find the polarization operator of the π^0 meson in a neutron medium. In this case both diagrams of (3.18) take part. Without taking the (NN) interaction into account, we have



Therefore, without allowance for the NN correlations we obtain in place of (3.20)

$$\Pi_p^0 = -f^2 k^2 \frac{m^* p_F}{\pi^2} \phi(\omega, k),$$

where

$$\phi(\omega, k) = \phi_1(\omega, k) + \phi_1(-\omega, k). \tag{3.30}$$

Allowance for the nucleon correlations, as can be easily obtained by a procedure analogous to that used in the case of the π^+ meson, yields

$$\Pi_p^0 = -f^2 k^2 \frac{m^* p_F}{\pi^2} \frac{\phi(k, \omega)}{1 + g^{nn}(p_F/p_0)\phi(\omega, k)} . \tag{3.31}$$

We consider the case of a medium with $N = Z$. In this case the polarization operators of the π^+ and π^- mesons, just as of the π^0 meson, will contain both diagrams of (3.18). Since the medium is isotopically invariant, it follows that

$$\Pi^{\pi^+}(\omega, k) = \Pi^{\pi^-}(\omega, k) = \Pi^{\pi^0}(\omega, k).$$

The expression for $\Pi^0(\omega, k)$ differs by a factor of 2 from Eq. (3.31), and in addition, the denominator will contain in place of g^{nn} the same combination as in the case of $\pi^{+,-}$ in a neutron medium, namely $g^- = g^{nn} - g^{np}$. Thus, if $N = Z$ we have

$$\Pi_p^{\pm,0} = -2f^2 k^2 \frac{m^* p_F}{\pi^2} \frac{\phi(k, \omega)}{1 + g^- \phi(k, \omega) p_F/p_0} \tag{3.32}$$

where $p_F^{(n)} = p_F^{(p)} = p_F$. The quantity $\Phi(\omega, k)$ is given by (3.30) and (3.21). The expression for the pole part of the polarization operator in the case of an arbitrary ratio N/Z is given in the paper of Migdal, Markin, and Mishustin (1974).

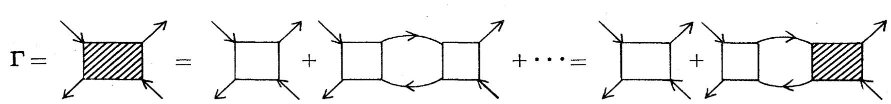
B. Pion degree of freedom in nuclear matter

The influence of one-pion exchange on the interaction of nucleons in nuclear matter is discussed. The equations of Fermi liquid theory in nuclear matter are obtained with allowance for one-pion exchange. It is shown that the distortion of the pion propagator in nuclear matter, which is not taken into account in the usual approaches to the theory of nucleon matter, alters significantly the nucleon interaction. A possible scheme of a consistent allowance for the pion degree of freedom in the theory of nucleon matter is presented.

1. Allowance for one-pion exchange in Fermi liquid theory

Equation (1.3) for the spectrum of the pion excitations can also be obtained from consideration of the poles of the NN scattering amplitude in a medium (correlation function).

Let us recall how the equation for the NN scattering amplitude is derived in the Fermi liquid theory (Migdal, 1965). In the case when the momentum $k = (\omega, \mathbf{k})$ in the particle-hole channel is small, this equation is determined by summation of the particle-hole diagrams



In symbolic form we have

$$\Gamma = \mathfrak{F} + \mathfrak{F} A \Gamma. \tag{3.33}$$

Here $\mathfrak{F}$ is the sum of all the diagrams that do not contain parts joined by a single pair, and A is the particle-hole propagator.

The reasoning behind this form of the expression is that the block $\mathfrak{F}$ incorporates all the diagrams that are not sensitive to the value of the momentum k. Therefore $\mathfrak{F}$ can be expressed in terms of constants that are determined from experiment and are the same for all nuclei and all processes, while the propagator A, which depends essentially on k and on the singularities of the shell structure, can be calculated exactly. According to the reasoning presented above (page 129), the quantity $\mathfrak{F}$, determined in terms of diagrams that contain more than one pair, is characterized by large intermediate momenta and changes significantly at $k \sim m_N$. The only exception is the one-pion exchange diagram, which, as we shall see, changes significantly at momenta $k \sim m_\pi$.

For the case of very small momenta, this diagram makes no contribution, since the πN interaction vertex is proportional to k. For momenta k of the order m_π, however, in which we are interested, it is necessary to separate out the diagram of the one-pion exchange and reduce to constants only the remaining part of $\mathfrak{F}$, which depends less strongly on k.

We note that an analogous extraction of the strongly k-dependent part of the interaction is made when the Coulomb screening in a plasma is evaluated (for a simple exposition see Migdal, 1975).

Since the πN interaction is $\sim \sigma_\alpha \tau_\beta$, one-pion exchange influences only those terms of the interaction $\mathfrak{F}$ which are of the form $(\sigma_1\sigma_2)(\tau_1\tau_2)$. Therefore the interaction $\mathfrak{F}$ can be written in the form (3.26), with the constant g' replaced by the function $g_t'(k, \omega)$, which takes into account one-pion exchange in the considered channel:

$$g_t' = g' + \left(\frac{dn}{d\epsilon_F}\right)_0 \frac{f^2 k^2}{\omega^2 - (1 + k^2 + \Pi'(k, \omega))} . \tag{3.34}$$

By definition, $\mathfrak{F}$ and consequently g_t' does not contain particle-hole graphs. Therefore $\Pi' = \Pi - \Pi_p$ (see the next section).

Inasmuch as the interaction $\mathcal{F}$ is δ-like in the coordinate representation with respect to all the coordinate differences that enter in these quantities, Γ in an infinite homogeneous system depends only on the difference between the incoming and outgoing coordinates. In the momentum representation Eq. (3.33) becomes algebraic. Using for A at $N = Z$ an expression analogous to (3.25)

$$A = -(dn/d\epsilon_F)_0 \phi(k, \omega)$$

and substituting in (3.33) the interaction (3.34), we obtain

$$\Gamma = \frac{g'}{1 + 2g_t'(k, \omega)\phi(k, \omega)} . \qquad (3.35)$$

The pole of this expression corresponds to the condition

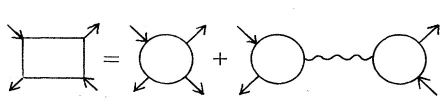
$$1 + 2g_t'(k, \omega)\phi(k, \omega) = 0. \qquad (3.36)$$

The quantity g^- introduced above is equal to $g^- = 2g'$.

Equation (3.36), with allowance for (3.34), coincides exactly with Eq. (1.3) for the frequency of the pionic excitations. It follows from (3.36) that for densities of the order of the nuclear density n_0, the function $g_t'(k, 0)$ reverses sign at $k \cong 1 (g_t = 0$ at $k = 0.8$ for $g' = 0.8$ and $n = n_0)$. A similar investigation was carried out by Backman and Weise (1975). This behavior of $g_t'(k, \omega)$, as we shall see (VI.B), is confirmed by the experimental data.

In (VI.B) below we shall consider the influence of one-pion exchange on the effective-field equation that leads to an enhancement of the probability of the transitions having pion symmetry. Since the effective field can be expressed in terms of Γ, the one-pion exchange can be taken into account in the same manner as in the problem considered here.

It is more convenient, however, especially when it comes to generalization to the case of a finite system, to express the amplitude Γ in a different form. We shall gather into blocks all the graphs that do not contain a pion line in the particle-hole channel (channel with zero baryon charge). Then the NN scattering amplitude is written in the form

$\Gamma =$  $= \Gamma_1 + \mathcal{T}_1 D \mathcal{T}_1,$ (3.37)

where Γ_1 is a scattering amplitude that contains no pion excitation in the considered channel, and $\mathcal{T}_1$ is the vertex that transforms the particle-hole into a pion excitation and also contains no pion pole. A similar separation of the term containing the pole that corresponds to a definite excitation is frequently used in the theory of finite Fermi systems (Migdal, 1967) to study collective levels.

The vertex $\mathcal{T}_1$ is determined by the relation (3.27). Analogously, the equation for Γ_1 differs from the equation for Γ in that the total interaction $\mathcal{F}$ is replaced by the interaction $\mathcal{F}_1$ which does not contain one-pion exchange in the considered channel.

Changing over in (3.37) to the momentum representation, we obtain

$$\Gamma(k, \omega) = \Gamma_1(k, \omega) + |\mathcal{T}_1(k, \omega)|^2 D(k, \omega) \qquad (3.38)$$

where D is the pion propagator in nuclear matter. For Γ_1 and $\mathcal{T}_1$ we have

$$\mathcal{T}_1^\alpha = \frac{f(\mathbf{k}\boldsymbol{\sigma})\tau_\alpha}{1 + 2g'\phi(k, \omega)} , \qquad (3.39)$$

$$\Gamma_1 = \frac{g'}{1 + 2g'\phi(k, \omega)} . \qquad (3.40)$$

The second term of (3.38) has a pole at $\omega \to 0$ and $k = k_0$. As $\omega \to 0$, the denominator of D takes the form

$$D^{-1} = \left(1 - \frac{\partial \Pi}{\partial \omega^2}\right)\omega^2 - \tilde{\omega}^2(k) - \mathrm{Im}\Pi$$

$$\tilde{\omega}^2(k) = 1 + k^2 + \Pi(k, 0)$$

Thus the pole of D corresponds to imaginary ω, i.e., it does not correspond to real oscillations. As we shall see, the situation is different in a finite system, where relations (3.24) and (3.33) are integral equations. These equations for fields with different symmetry were solved with a computer (Migdal, 1967). Inasmuch as the observed quantities are expressed in terms of matrix elements of vertices of the type $\mathcal{T}_1$ (when there is no one-pion exchange), a comparison with experiment makes it possible to determine the constants f, f', g, and g' of the theory.

In a homogeneous infinite system, the quantities Γ_1 and $\mathcal{T}_1$ depend only on the difference between the incoming and outgoing coordinates. In a finite system this is not the case, since an important role is played by the reflection of the particles from the system boundary. In a sufficiently large system, the reflected waves have large phase shifts and their averaged contribution to the observed quantities is small. As shown by the calculations, even heavy nuclei are not large enough for the reflection from the boundary to play a minor role. Therefore, the propagator A in the kernel depends on two variables, $A = A(\mathbf{r}, \mathbf{r}')$. The same pertains to the quantities Γ_1 and $\mathcal{T}_1$. Nonetheless, for the wave vectors $k \sim p_F$ of interest to us, Eq. (3.38) retains its form also in a finite system. Indeed, at distances $|\mathbf{r} - \mathbf{r}'| \ll r$, corresponding to $k \gg 1/R$, all the quantities are functions of $\mathbf{r} - \mathbf{r}'$ only. However, $\mathcal{T}_1$ and Γ_1 must be determined not from relations (3.39) and (3.40), but by the corresponding integral equations in which account is taken of the fact that the system is finite, as is done in the theory of finite Fermi systems. In addition, the fact that the system is finite introduces an appreciable change in the form of D near the pole. Namely, if ω is smaller than the energies of the first particle-hole excitations, then $\Pi(k, \omega)$ has no imaginary part. Moreover as it follows from the dispersion relation for $\Pi(\omega)$ in the regions of small damping $\partial\Pi/\partial\omega^2 < 0$.

The quantity $\partial\Pi/\partial\omega^2$ should be calculated with the account for pairing and finite radius of the system. Expressions (3.21) correspond to large damping due to the particle-hole creation and give the wrong sign for $(\partial\Pi/\partial\omega^2)_{\omega=0}$. The pole in D is determined by the relation

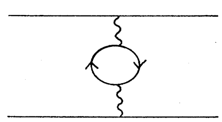
$$[1 - (\partial\Pi/\partial\omega^2)]\omega^2 - \tilde{\omega}^2(k) = 0,$$

i.e., it corresponds to the real oscillation branch. We shall see below (VI.1) that even in a nucleus close to condensation the lowest frequency of this branch is apparently much higher than the frequency of the single-particle excitations.

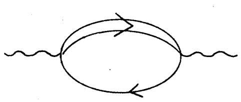
2. A scheme for a consistent theory of nuclear matter

The motion of the pions in nuclear matter, as we have seen, is greatly distorted by interaction with the nucleons. As shown by Migdal (1972), this distortion is not taken into account in the usual approach to the theory of nuclear matter. This approach is based on the assumption that the nucleus can be regarded as the gas of nucleons with a pair interaction obtained from an analysis of nucleon–nucleon scattering in a vacuum. Yet part of the vacuum nucleon–nucleon interaction, corresponding to the one-pion exchange diagrams, is distorted by the change of the pion propagator in the medium. In a symmetric medium ($N \simeq Z$) when S-wave πN interaction can be neglected, the distortion of the pion propagator due to resonant πN scattering is essential.

Indeed, diagrams of the type



which describe the decay of a pion in a particle and a hole are taken into account in the usual approach, since these diagrams correspond to interaction between two nucleons with a virtual excitation of a nucleon of the Fermi sea. However, all the diagrams that contain pion lines are affected by N^* resonance



are completely lost in the usual methods of calculating nuclear matter.

A correct theory of nuclear matter should be constructed in accordance with the following scheme. It is necessary to subtract from the vacuum pair interaction the one-pion exchange graph. The remaining part of the interaction is included in the Hamiltonian as an NN pair interaction. Added to this interaction in the Hamiltonian of the system is the πN interaction with the vacuum constant f (see the reasoning on page 123). Also added is the pion-field Hamiltonian which contains the vacuum $\pi\pi$ interaction. Of course, this problem of interacting nucleon and pion fields cannot be solved exactly. Taking into account the hypothesis concerning the equality of the local quantities in the medium to their vacuum values, it is possible to simplify the formulation of the problem greatly and to develop a consistent theory that is applicable up to fairly large densities.

All the NN interaction graphs, with the exception of the one-pion exchange graph, are assumed to be δ functions and reduce to constants that can be obtained from the vacuum interaction after subtracting from it the one-pion exchange graph. The δ-function interaction obtained in this manner is supplemented with a one-pion exchange, taking into account the distortion of the pion propagator in the medium. Our first problem is to express the spin-spin NN interaction in nuclear matter in terms of its vacuum value and in terms of an interaction corresponding to exchange of one distorted pion.

Qualitatively, the situation was formulated in the review by Rho (1975) "The nucleus is not a gas of nucleons, but a soup of pions." To verify that we are not dealing here with small corrections but with a significant modification of the theory of nuclear matter, we present for the pion energy the expression that follows from the formulas for the polarization operator at small k and ω in nuclear matter see (3.17c)

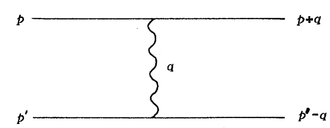
$$\omega^2 = a + bk^2 + c\omega^2, \quad a = 1 + 0.35\frac{n}{n_0},$$

$$b = 1 - 0.4\frac{n}{n_0}, \quad c = 0.2n/n_0$$

This corresponds to a propagator

$$D = \frac{1}{(1-c)\omega^2 - (a + bk^2)}$$

Exchange of such a "pion" over distance $r \gg 1$ leads to a nucleon–nucleon interaction that differs strongly from that in a vacuum. By considering elastic scattering of two nucleons



corresponding to exchange of one "pion" with $q = (0, k \ll 1)$, and changing over to the coordinate representation, we readily obtain

$$V(\mathbf{r}_1 - \mathbf{r}_2) = \frac{1}{b} f^2 (\nabla_1 \sigma_1)(\nabla_2 \sigma_2) \frac{\exp[-|\mathbf{r}_1 - \mathbf{r}_2|]}{|\mathbf{r}_1 - \mathbf{r}_2|} .$$

This expression differs from the vacuum value by a factor $\cong 1.3$ in the argument of the exponential, and by multiplication by the factor $1/b \cong 1.7$.

C. Pion spectrum and conditions for instability of a pion field

It is shown in this section that when the pion field in a medium is quantized the coefficients of the plane-wave expansion of the field contain the factor $[2\omega_k - \partial\Pi/\partial\omega]^{-1/2}_{\omega_k = \omega(k)}$ which goes over in vacuum into the usual factor $(2\omega_k)^{-1/2}$. The condition for the stability of the field is that the radicand be positive for each type of pion excitation. A general selection rule is given for the physical solutions of the dispersion equation for the pion energy. The solutions of this equation are analyzed both for $Z \cong N$ and $Z \ll N$. A physical interpretation is presented of the possible excitation modes for each type of pion. It is shown that at $Z \cong N$ the instability of the π^+, π^-, π^0 meson field manifests itself in the vanishing of the frequency $\omega_s^{+-0}(k_0)$. The symbol s denotes here the spin–isospin sound excitation with the quantum numbers of the corresponding pions. At $Z \ll N$, the instability of the π^0-meson field is of the same kind, and for the π^+ and π^- mesons the instability corresponds to vanishing of the sum of the energies $\omega_s^+ + \omega^- = 0$. In addition, instability to the reaction $p \to n + \pi_s^+$ appears and causes all the free protons of a neutron star, at densities $n \gtrsim n_0/2$, to go over to a bound state $\pi_s^+ = (p\bar{n})$ of the proton-neutron hole type. A connection between the results obtained here and those of Sawyer and Scalapino (1972) is established.

1. Quantization of the pion field in a medium

In the preceding sections we have investigated in detail the properties of the polarization operator $\pi(k,\omega)$ of pions in a nucleon medium. We now discuss the main properties of the solution of the equation for a pion field. We begin with the case of charged pions, for which[2]

$$[\omega^2 - 1 - k^2 - \Pi^{\pm}(k,\omega)]\varphi^{\pm}_{k,\omega} = 0\,.$$

We introduce the complex field that combines φ^+ and φ^-

$$\Psi = \sum_k \{c_k^+ a_k \exp[i(\omega_k^+ t - \mathbf{kr})] + c_k^- b_k^+ \exp[-i(\omega_k^- t - \mathbf{kr})]\}\,,$$

where a_k and b_k are the π^+ and π^- meson annihilation operators. If, at a given momentum k, there exist in the medium several excitation modes with the quantum numbers of the π^+ or π^- meson, then the summation implies throughout also summation over the excitation modes.

The coefficients c^+ and c^- are determined in the following manner.

The Lagrangian of the field Ψ is given by

$$\mathfrak{L} = \sum_k \Psi_k^+ [\omega_k^2 - 1 - k^2 - \Pi^{(+)}(k,\omega_k)]\Psi_k\,, \qquad (3.42)$$

where ω_k is an independent variable which must not be confused with the solution $\omega(k)$ of the dispersion equation. Using the usual method of treating Lagrangians with time derivatives of arbitrary order, we easily obtain the following formula for the component T_{44} of the energy-momentum tensor:

$$T_{44} = \mathcal{H} = \sum_k \left(\omega_k \frac{\partial \mathfrak{L}}{\partial \omega_k} - \mathfrak{L}\right)_{\omega_k = \omega(k)}\,. \qquad (3.43)$$

Let us illustrate this relation using as an example an electromagnetic field in a medium with a permittivity $\epsilon(\omega)$ and a magnetic permeability $\mu(\omega)$. The time-averaged Lagrange function, expressed in terms of the vector potential $\mathbf{A}(\mathbf{B} = \operatorname{curl}\mathbf{A}, \mathbf{E} = -\dot{\mathbf{A}}, A = A_0 \sin\omega t)$, takes the form

$$\bar{\mathfrak{L}} = \sum_k \left(\epsilon\omega^2 - \frac{k^2}{\mu}\right)\frac{A_0^2}{16\pi}\,.$$

From this we obtain with the aid of (3.43), for the average field energy,

$$\overline{T}_{44} = \frac{d(\epsilon\omega)}{d\omega}\frac{E_0^2}{16\pi} + \frac{d(\mu\omega)}{d\omega}\frac{H_0^2}{16\pi}$$

in agreement with the formula of Landau and Lifshitz (1959).

From (3.42) and (3.43) we easily obtain an expression for the Hamiltonian $\mathcal{H}$ of the pion field (we omit, for simplicity, the labels of the excitation modes):

$$\mathcal{H} = \sum_k \Psi_k^+\left[\omega_k\left(2\omega_k - \frac{\partial\Pi^+}{\partial\omega_k}\right)\right]_{\omega_k = \omega(k)}\Psi_k$$

$$= \sum_k \{(c_k^+)^2\omega^+(k)\Omega^+(k)a_k^+ a_k + (c_k^-)^2\omega^-(k)\Omega^-(k)b_k^+ b_k\}$$

$$\Omega^{\pm}(k) = \left(2\omega_k - \frac{\partial\Pi^{\pm}(k,\omega)}{\partial\omega_k}\right)_{\omega_k = \omega^{\pm}(k)} \qquad (3.44)$$

(we have used the equality $\Pi^+(k,\omega) = \Pi^-(-k,-\omega)$. We stipulate that the Hamiltonian be of the form

$$\mathcal{H} = \sum_k \{\omega^+(k)a_k^+ a_k + \omega^-(k)b_k^+ b_k\}\,,$$

where $\omega^{\pm}(k)$ is the $\pi^{\pm}$ meson energy. It follows from this condition that

$$(c_k^+)^{-2} = \Omega^+(k)\,,\ (c_k^-)^{-2} = \Omega^-(k)\,. \qquad (3.45)$$

Thus

$$\Psi(z,t) = \sum_k \left\{\frac{a_k \exp[i\omega^+(k)t - i\mathbf{kr}]}{[\Omega^+(k)]^{1/2}} + \frac{b_k^+ \exp[-i\omega^-(k)t + i\mathbf{kr}]}{[\Omega^-(k)]^{1/2}}\right\}. \qquad (3.46)$$

The same result can also be obtained in another way. The method used to obtain the expression for T_{44} yields as well the current 4-vector. From (3.42) we obtain for the charge density

$$j_0 = e\frac{\partial\mathfrak{L}}{\partial\omega_k} = e\sum_k \Psi_k^+\left(2\omega_k - \frac{\partial\Pi^+}{\partial\omega_k}\right)_{\omega_k = \omega(k)}\Psi_k. \qquad (3.47)$$

Stipulating that j_0 take the form

$$j_0 = e\sum_k (a_k^+ a_k - b_k^+ b_k) \qquad (3.48)$$

we again obtain (3.39). The factor $2\omega - \partial\Pi/\partial\omega$ in (3.47) is obtained also from Ward's theorem, according to which the fourth component of the electromagnetic vertex is

$$\Gamma_4^{el} = \frac{\partial D^{-1}}{\partial\omega} = 2\omega - \frac{\partial\Pi}{\partial\omega}\,,$$

where D is the pion propagator in the nucleon medium,

$$D^{-1} = \omega^2 - 1 - k^2 - \Pi(k,\omega)\,.$$

As follows from (3.46), the function $\omega^+(k)$ for π^+ mesons should be such as to satisfy the condition

$$\left[2\omega - \frac{\partial\Pi^+(k,\omega)}{\partial\omega}\right]_{\omega = \omega^+(k)} > 0\,.$$

For π^- mesons we have

$$\left[2\omega - \frac{\partial\Pi^-(k,\omega)}{\partial\omega}\right]_{\omega = \omega^-(k)} > 0$$

or

$$\left[2\omega - \frac{\partial\Pi^+(k,\omega)}{\partial\omega}\right]_{\omega = -\omega^-(k)} < 0\,.$$

We thus obtain the following selection rule for the solutions (Migdal, 1973; Migdal, Markin, and Mishustin, 1974). Assume that the solutions $\omega(k)$ have been obtained

[2]Here we will use the notations Π^+, Π^- instead of Π^{π^+}, Π^{π^-}.

for the equation

$$\omega^2 = 1 + k^2 + \Pi^+(k, \omega).$$

The solutions situated in the region

$$2\omega - \partial \Pi^+(k, \omega)/\partial \omega > 0$$

correspond to π^+ mesons. Those solutions which are situated in the region

$$2\omega - \partial \Pi^+(k, \omega)/\partial \omega < 0$$

become, following the substitution $\omega \to -\omega$, the dispersion law for the π^- mesons.[3]

It is easy, in analogy with the derivation of Eq. (3.46) for the charged-meson field ψ, to obtain the corresponding expression for the π^0-meson field. The π^0-meson density is

$$\rho_3 = \sum_k \left(2\omega_k - \frac{\partial \Pi^0}{\partial \omega_k}\right)_{\omega_k = \omega(k)} \varphi_3^2 .$$

Here Π^0 is an even function of ω.

From the spectral representation we see that $\partial \Pi^0/\partial \omega^2 < 0$ [this is evident also from (3.31) and (3.32), for example]. The physical solutions for the π^0 mesons therefore correspond to the condition $\omega > 0$.

2. Pion spectra in the cases *Z=0* and *Z=N*

We consider first the case $Z = 0$. Using Eqs. (3.16), (3.17b), and (3.28), we obtain the following equation for the determination of the π^+-meson energy:

$$\begin{aligned}\omega^2 &= 1 + k^2 + \Pi^{+n}_{\text{loc}} + \Pi^{+n}_p \\ &= 1 + k^2 + n\left\{0.7 - 0.8k^2 + \left[0.8 - \frac{1.2}{1 - \omega^2/\omega_R^2}\right]\omega^2 \right. \\ &\qquad \left. - 1.5\omega(1 - 0.13\omega^2)\right\} \\ &\quad - 2f^2k^2 \frac{m^* p_F}{\pi^2} \frac{\phi_1(\omega, k)}{1 + g^-(p_F/p_0)}\end{aligned} \tag{3.49}$$

Figure 5 shows the result of a numerical solution of Eq. (3.49) for $n < n_c^+$. Here and below in this section the numerical results are taken from the paper by Migdal, Markin, Mishustin (1974) where slightly different expression for Π have been used. The solid lines are the spectral branches with $2\omega - \partial \Pi^+/\partial \omega > 0$, i.e., corresponding to π^+ mesons, and the dashed lines are the sections for which $2\omega - \partial \Pi^+/\partial \omega < 0$, i.e., corresponding to π^- mesons (we recall that the sign of ω is reversed for the π^- mesons). Starting with a density $n_c^+ \cong 0.2$, a solution with $\omega < -\epsilon_F^{(n)}$ appears in the π^+-meson spectrum (see Fig. 6). The physical meaning of this π^+-excitation mode is that it represents a bound state of a proton and a neutron hole—spin-isospin sound excitation (see the next section). This branch vanishes when the πN interaction is turned off, whereas the second branch remains and coincides with the free-pion branch. It is natural to call the latter the pion branch, and the former the "spin-isospin-sound" branch or "spin-sound" for short (see Migdal, 1972, 1973; Migdal, Markin, and Mishustin, 1974). To avoid confusion, we shall label the spin-sound excitations (π^+ mesons) with a subscript "*s*." The second branch is missing from the π^--meson spectrum. A similar interpretation is arrived at also by Anderson *et al.* (1975), who call the π_s^+ excitations "spin-isospin waves." The presence of such solutions leads to instability of the protons in the neutron medium ($p \to n + \pi_s^+$). We note that at $n < n_c^{\pm}$ the pion energies are such that $\omega_s^+ + \omega^- > 0$.

The spectrum of the charged pions in a medium with density $n \geq n_c^{\pm} = 0.4$ (Fig. 7) has a distinguishing feature,

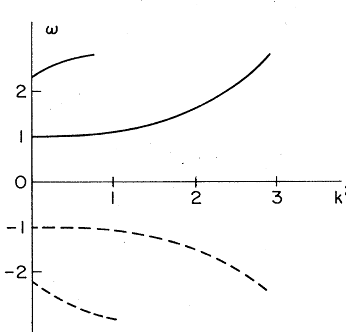


FIG. 5. Spectrum of charged pions in a neutron medium with density $n = 0.1 < n_c^+(g^- = 0.8)$. Solid line, energy $\omega^+(k)$ of π^+ mesons; dashed line, energy $-\omega^-(k)$ of π^- mesons with the sign reversed. For all k we have $\omega^+ + \omega^- > 0$ and $\omega^+ > 0$.

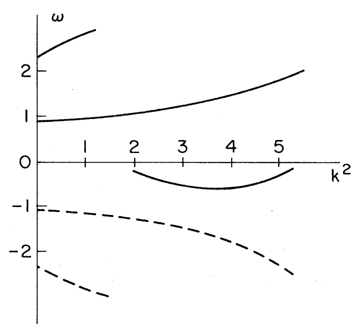


FIG. 6. Spectrum of charged pions in a neutron star with density $n = 0.3 (n_c^+ < n < n_c^{\pm},\ g^- = 0.8)$. For $2 < k < 5.5$, the energy $\omega_s^+ < \epsilon_F^n$, leading to instability of the protons in such a medium with respect to the process $p \to n + \pi_s^+$. The quantity $\omega_s^+ + \omega^-$ is positive throughout.

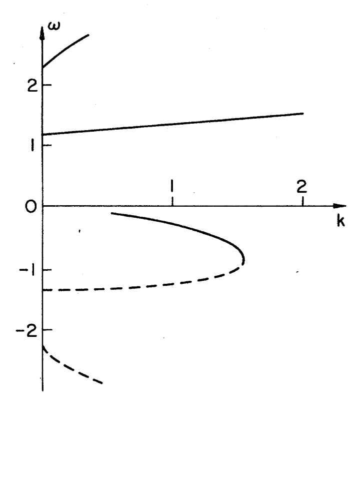


FIG. 7. Spectrum of charged mesons in a neutron medium with density $n = 0.5 > n_c^{\pm}(g^- = 0.8)$. At $k = k_c^+ = 1.6$ we have $\omega_s^+ + \omega^- = 0$. A system with such a density is unstable to production of $\pi_s^+\pi^-$-meson pairs.

[3]These conditions are derived anew from time to time (see, for example, Bertsch and Johnson, 1974, 1975).

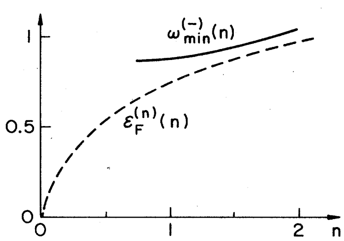


FIG. 8. Solid line, dependence of the minimal energy ω^- (n) of π^- mesons in a neutron medium on the density n; dashed line, energy $\epsilon_F^n=(3\pi^2 n)^{2/3}/2m$ of the neutron Fermi boundary; ω^- (n) $>\epsilon_F^n$, so that the process $n\to p+\pi^-$ is impossible.

namely a point at which $\omega_s^+ + \omega^- = 0$. At this point $2\omega - \partial\Pi^+/\partial\omega = 0$ and $\partial\omega/\partial k=\infty$, i.e., a system having this density is unstable to the production of $\pi_s^+\pi^-$ meson pairs, in analogy with the situation in a strong electric field (Sec. II.A).

Figure 8 shows a plot of the minimum energy $\omega^-_{\min}(n)$ of the π^- meson against the neutron density for $n>n_c^{\pm}$; it is seen from the figure that even if no account is taken of the stabilizing action of the condensate we have $\omega^-_{\min} - \epsilon_F^{(n)} > 0$. It follows therefore that a second-order phase transition with formation of a π^- condensate $(n\to p+\pi^-)$ is impossible.

The dispersion law for π^0 mesons takes the form

$$\begin{aligned}\omega^2 &= 1+k^2+\Pi^{0n}_{\rm loc}+\Pi^{0n}_p \\ &= 1+k^2+n\left\{0.7-0.8k^2+\left[0.8-\frac{1.2}{1-\omega^2/\omega_R^2}\right]\omega^2\right\} \\ &\quad -f^2k^2\frac{m^*p_F}{\pi^2}\,\frac{\phi(k,\omega)}{1+g^{nn}(p_F/p_0)\phi(k,\omega)} \qquad (3.50)\end{aligned}$$

A numerical solution of this equation yielded the $\omega^2(k^2)$ spectrum shown in Fig. 9 for $n<n_c^0=0.4$. At the density $n>n_c^0$ (Fig. 10), a region with $\omega^2<0$ appears, indicating instability of the system to the production of neutral pions.

We proceed now to the case $Z=N$. We have already stated that in such a medium, by virtue of isotopic invariance, the results are the same for all pions. The pion energy $\omega(k)$ is determined by the equation

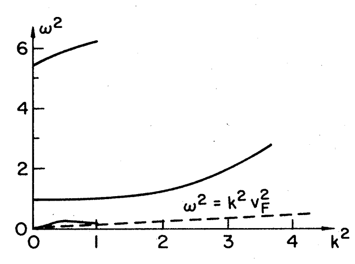


FIG. 9. Spectrum of π^0 mesons in a neutron medium with density $n=0.3<n_c^0(g^{nn}=1)$. The three branches of the spectrum correspond to the three possible types of excitations: of the isobar-hole-type ["resonant branch," $\omega(k=0)=1$], and particle-hole-type ("spin = sound branch"). For all k we have $[\omega(k)]^2>0$.

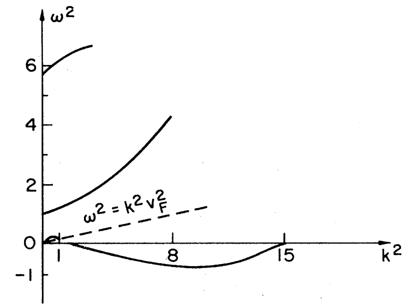


FIG. 10. Spectrum of π^0 mesons in a neutron medium with density $n=0.9>n_c^0(g^{nn}=1)$. At $1.5\leq k^2\leq 1.5$ there exists a branch with $[\omega(k)]^2<0$. A system with such a density is unstable with respect to production of π_s^0 mesons.

$$\begin{aligned}\omega^2 &= 1+0.7n+(1-0.8n)\,k^2+n\left[0.8-\frac{1.2}{1-\omega^2/\omega_R^2}\right]\omega^2 \\ &\quad -\frac{2mp_F}{\pi^2}f^2k^2\phi(k,\omega)\left[1+\frac{p_F}{p_0}g^-\phi(k,\omega)\right]^{-1}. \qquad (3.51)\end{aligned}$$

It takes the same form as the equation for the neutral pions at $Z=0$.

At $n<n_c$ the spectrum is similar to Fig. 9; at $n>n_c=0.3$, just as before, a region with $\omega^2<0$ appears (see Fig. 11), but at $N=Z$ this means already instability with respect to simultaneous production of π^+, π^-, and π^0 mesons.

The table lists the characteristic parameters for different values of g^- and g^{nn}. The columns k_c^0 and k_c contain those values of k for which $\omega^2=0$ and accordingly $n=n_c^0(Z=0)$ and $n=n_c(Z=N)$; $k_c^{\pm}$ is the value of k for which $\omega_s^+ + \omega^- = 0$ (at $Z=0$).

It is seen from Figs. 5–11 that there are three branches of pion spectra in accord with the three types of excitations: the pion branch $(\omega(k=0)\simeq 1)$, the resonant branch (excitations of the isobar—hole type) $(\omega(k=0)\approx\omega_k)$, and a particle-hole branch, which we shall call spin-sound. The resonant branch is of interest in problems connected with the scattering of pions by nuclei in the region of the N_{33}^* resonance. The two others are significant in the study of stability problems.

In an isotopically symmetrical medium $(N=Z)$, as already mentioned, the polarization operator is an even

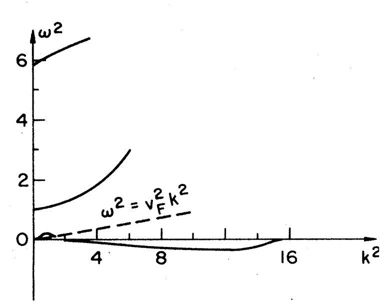


FIG. 11. Spectrum of mesons in a medium with $N=Z$ at a density $n=n_0\cong 0.5>n_c$ (n_0 is the nuclear density, $g^-=1.6$). At $2\leq k^2\leq 15$ we have $\omega^2<0$. The system is unstable with respect to production of π_s^+, π_s^-, and π_s^0 mesons.

TABLE I. Critical values of the nucleon density (n_c) and of the condensate-field momentum (k_c) for the various types of instability corresponding to different values of spin-spin nucleon interaction constants (g^{nn}, g^-).

	$Z=0$					$Z=0$			$N=Z$	
g^-	$n_c^\pm$	$k_c^\pm$	n_c^+	k_c^+	g^{nn}	n_c^0	k_c^0	g^-	n_c	k_c
0	0.37	1.4			0	0.2	2.4	0	0.10	1.60
0.3	0.39	1.6						0,4	0,13	1.78
0.6	0.41	1.6	0.2	P_F	1	0.4	2.5	0.8	0.18	1.93
								1.2	0.25	2.07
0.8	0.43	1.6			1.6	0.6	2.5	1.6	0.34	2.13
								2.0	0.43	2.10

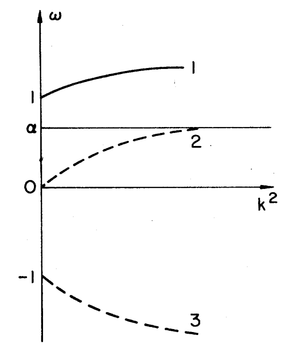


FIG. 12. Solution of dispersion equation for charged pions in neutron matter in the high-frequency approximation $\omega \gg kv_F^n$. The solid lines show the branches of the π^--meson spectrum, and the dashed lines show the branches of the π^+-meson spectrum with the energy sign reversed.

function of the frequency, and the physical solutions correspond to $\omega > 0$. Starting with $n = n_c \cong 0.3$, a solution with $\omega^2 < 0$ appears for all pions (the k^2 interval in which $\omega^2 < 0$ increases from zero with increasing n). If this estimate were correct, then it would follow that the nucleus should contain the π condensate, the presence of which would influence the calculation of the different characteristics of the nucleus (see Sec. VI.B).

For $Z \ll N$ (neutron star), at a certain density, there appears for the π^+ mesons a branch with energy $\omega_s^+ < 0$. At a density $n > n_c^+ \cong 0.2$, the energy of this branch reaches a value $\omega_s^+ = \epsilon_F^{(p)} - \epsilon_F^{(n)}$. This leads to an important consequence: A neutron star contains an admixture of protons whose charge is cancelled by electrons. Since $\omega_s^+ + \epsilon_F^{(n)} - \epsilon_F^{(p)} < 0$, the protons will be transformed into π_s^+ mesons and neutrons ($p \to n + \pi_s^+$). The equilibrium values of the π_s^+ mesons and electrons is determined by the equation

$$\omega_s^+ + \epsilon_F^{(e)} = 0 .$$

At a density $n = n_c^\pm \cong 0.4$, the energy $\omega_s^+ + \omega^-$ of the $\pi^+\pi^-$ meson pair vanishes at $k_c^\pm = 1.6$, and this leads to formation of the $\pi_s^+\pi^-$ condensate. The π^0-meson condensate is produced at approximately the same density. As shown by Migdal (1971, and later in 1972), the appearance of the condensate makes the system stable. The formation of the condensates corresponds to second-order phase transitions.

The minimal π^--meson energy prior to the appearance of the $\pi_s^+\pi^-$ and π^0 condensates is larger than $\epsilon_F^{(n)}$, so that a second-order transition with formation of a π^- condensate is impossible. Upon formation of the condensates, the minimal energy of the π^- mesons increases, as shown by Migdal (1973), so that $\omega - \epsilon_F^{(n)} > 0$, at least up to very high densities; thus the instability detected by Sawyer and Scalapino (1972) does not correspond to the reaction $n \rightleftharpoons p + \pi^-$. The nature of the instability observed by Sawyer and Scalapino is discussed below.

3. Pion spectrum in a simple model

To explain the results obtained in the case $N \gg Z$, we simplify the problem of determining the spectrum, retaining only the pole part of the polarization operator (Migdal, Markin, and Mishustin, 1976). In addition, we assume that the frequency $\omega \gg kv_F$. While this is a crude assumption at $n \sim n_c$, it does hold true, for in the case $Z \ll N$ the significant values are $\omega \sim 1$, whereas $kv_F \sim \epsilon_F \simeq 0.3$. The dispersion equation is then so simplified that it can be solved analytically. The dispersion equation for the determination of the energies of the pion quasiparticles in the normal phase of neutron matter is given by expressions (3.20) and (3.22)

$$D^{-1}(k, \omega) = \omega^2 - \omega_k^2 + \frac{2nf^2k^2}{\omega} = 0 \qquad (3.52)$$

From here and beyond $\omega_k = 1 + k^2$. The last term in this equation is the polarization operator of the π^- meson in the approximation $\omega \gg kv_F^{(n)}$, taken with a minus sign.

The solution of (3.52) can be written in the form

$$k^2(\omega) = \omega^2 - 1\Big/\left(1 - \frac{\alpha}{\omega}\right),$$

where $\alpha = 2nf^2$. At $\alpha < 1$ there are three branches of the spectrum (Fig. 12).

As shown above, the classification of the branches is determined by the sign of the residue $D(k, \omega)$, i.e., by the sign of the quantity

$$\frac{\partial D^{-1}}{\partial \omega} = 2\omega - \frac{\partial \Pi^-}{\partial \omega} = 2\omega - \alpha \frac{k^2}{\omega}.$$

The branches on which $(2\omega - \partial\Pi/\partial\omega) > 0$ correspond to π^- mesons, and branches on which $(2\omega - \partial\Pi^-/\partial\omega) < 0$ yield the dispersion law of the π^+ mesons after replacing ω by $-\omega$. Branch 1 of Fig. 12 corresponds to π^- mesons, inasmuch as $(2\omega - \partial\Pi^-/\partial\omega) > 0$ on it. On branches 2 and 3 $(2\omega - \partial\Pi^-/\partial\omega) < 0$, so that after reversing the sign of ω these branches yield the dispersion law of quasiparticles with the quantum numbers of π^+ mesons. Thus, there are two types of π^+-meson excitations in the medium, π^+ and π_s^+. The quasiparticle spectrum takes the form shown in Fig. 13. When the pion-nucleon interaction is turned off, branches 1 and 3 go over into the vacuum

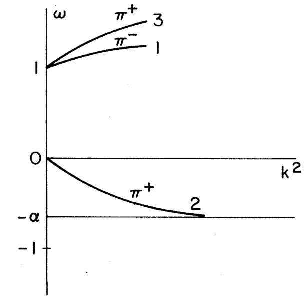


FIG. 13. Spectrum of charged mesons in a neutron medium in the high-frequency approximation, $\omega \gg kv_F^n$, at $n < n_c^\pm = 1/2f^2$. Curves 1 and 3—meson branches; curve 2—spin-isospin sound branch (π_s^+).

spectra of the π^- and π^+ mesons ($\omega \to \omega_k$), while branch 2 goes over into the spin-isospin-sound dispersion law. It follows from the exact calculations that the section of branch 2 with $|\omega| \lesssim kv_F^{(n)}$ vanishes, and at $\alpha < kv_F^{(n)}$ there is no π_s^+ branch at all. It is the appearance of this branch which signals the aforementioned instability with respect to the reaction $p \to n + \pi_s^+$.

With further increase of the density, the branches π^- and π_s^+ drop lower, the value of $(\omega_s^+ + \omega^-)_{\min}$ decreases, and for $\alpha = 1$ there is a value k_c at which $\omega_s^+ + \omega^- = 0$ appears on the spectrum for the first time. The value $\alpha = 1$ corresponds thus to the critical density $n_c = 1/2f^2$ of $\pi_s^+\pi^-$ condensation. It is obvious that at the critical point $(2\omega - \partial\Pi^-/\partial\omega) = 0$, which corresponds to coalescences of the two roots of Eq. (3.45), i.e., to a double pole of the pion propagator $D(k, \omega)$. At $\alpha = 1$ the spectrum is degenerate and splits into two curves, $\omega = 1$ and $k^2 = \omega(\omega + 1)$. The coordinates of the intersection point of these curves are the critical parameters of the $\pi_s^+\pi^-$ condensation in this model, namely $k_c^\pm = \sqrt{2}$ and $\omega_c = 1$.

At $\alpha > 1$, as can be easily verified, the spectrum takes the form shown in Fig. 14. It has one distinguishing singularity, namely, a break on the edges of which $d\omega/dk = \infty$. In the region of the break we have $(\omega_s^+ + \omega^-)^2 < 0$. This attests to the instability of the system to the formation of the $\pi_s^+\pi^-$ condensate. Thus this simplified model accounts correctly for the main results of the exact calculation.

We turn now to the models of Sawyer and Scalapino (1972, 1973). In this model they consider a system consisting of neutrons, protons, and π^- mesons populating a single state with wave vector **k**. The degrees of freedom are artificially restricted here—the π^+ mesons are not taken into account; this corresponds to describing the π^--meson field by the Schrödinger equation in lieu of the Klein–Gordon–Fock equation. As a result, the equation for the energy (chemical potential) of the π^- meson takes, in contrast to (3.52), the form

$$\omega = \omega_k - nf^2k^2/\omega_k\omega \,. \tag{3.53}$$

The second term in the right-hand side is the self-energy part $\Sigma^{(-)}$ of the π^- meson.

Equation (3.53) has two roots:

$$\omega = \frac{\omega_k}{2}\left[1 \pm \left(1 - \frac{4f^2k^2n}{\omega_k^3}\right)^{1/2}\right].$$

The upper sign corresponds to the π^- meson. For this solution, the residue of the π^--meson Green's function is positive $(1 - \partial\Sigma^{(-)}/\partial\omega > 0)$, as it should be. In addition, when the interaction is turned off we have $\omega \to \omega_k$. This is precisely the branch considered by Migdal (1973)

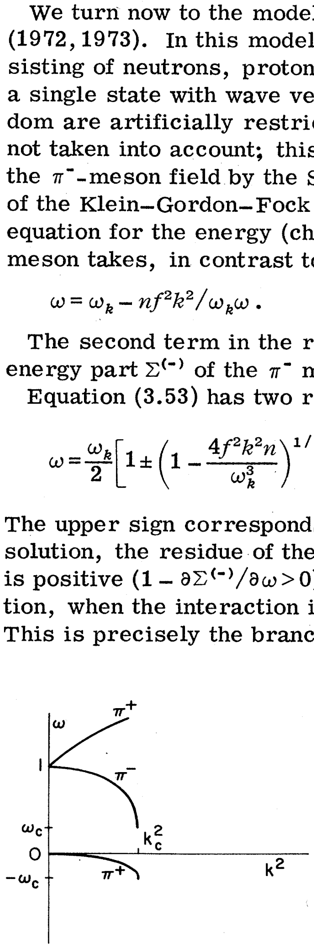


FIG. 14. Spectrum of charged pions at $n > n_c^\pm$.

to prove the stability of the system with respect to the reaction $n \to p + \pi^-$.

Within the framework of the model considered by Sawyer and Scalapino (1972, 1973), it is difficult to offer a reasonable physical interpretation of the second root of (3.53), for which $1 - \partial\Sigma^{(-)}/\partial\omega < 0$. It follows from the analysis of Eq. (3.52) above, the second root should be interpreted (after reversing the sign of ω) as the π_s^+-meson branch, i.e., as a bound state of a proton and a neutron hole $(p, \bar{n})$. If the condition $1 - 4f^2k^2n/\omega_k^3 = 0$, which coincides with the critical condition obtained by Sawyer and Scalapino (1972, 1973), is satisfied, the energy sum $\omega_s^+ + \omega^-$ vanishes (the two roots coalesce) and the system becomes unstable to the appearance of an electrically neutral $\pi_s^+\pi^-$ condensate. It is easy to obtain the values of the critical parameters of the $\pi_s^+\pi^-$ condensation in this model.

$$n_c^\pm = \frac{3\sqrt{3}}{8f^2} \simeq 1,3n_0, \; \omega_c = \frac{\sqrt{3}}{2}, \; k_c^\pm = \sqrt{2}\,.$$

We note that Sawyer and Scalapino take the term "pion" to mean a "bare" pion, whereas in our approach we have in mind pion quasiparticles, including also the spin-sound branch. The positive charge of a hadron system, in the language of bare particles, consists of the charges of the bare protons and π^+ and π^- mesons. In the language of pion quasiparticles, this charge is the difference between the number of the π_s^+ and π^- mesons (the baryon quasiparticles are not charged).

In terms of the bare pions, it is easy to make an error in the classification of the solutions of (3.49); thus, for example, the π_s^+ branch is ascribed by Sawyer (1973) to the π^- meson). For a clear physical interpretation, one should use the quasiparticle language, as is always done when studying phase transitions, rather than the bare particle terms. It is especially important in nucleon matter, since there are several branches of excitations carrying the pionic quantum numbers.

IV. EFFECTIVE INTERACTION OF PIONS IN NUCLEAR MATTER

The structure of the condensate, i.e., the amplitude, the coordinate dependence, and the isotopic structure of the condensate field are determined by the effective interaction of the pions in the nuclear matter. For the qualitative analysis we have used above (Sec. II) the $\frac{1}{4}\lambda\varphi^4$ phenomenological interaction model.

The task of the present section is to find the effective interaction and the structure of the π condensate in nuclear matter using a possibly more realistic model.

The effective pion interaction consists of the vacuum interaction, determined by the Weinberg Lagrangian, and the interaction due to exchange of nucleon excitations. It is convenient to describe the condensate field with the aid of an effective pion Lagrangian averaged over the motions of the nucleons. To obtain this Lagrangian it is necessary to find the energy of the nucleon subsystem as a functional of the condensate field. It is the nonlinear terms of such Lagrangians which constitute the effective interaction of the pions in the medium, replacing the $\lambda\varphi^4/4$ phenomenological interaction.

In the case of small amplitude of the condensate field, the expression for the effective Lagrangian becomes much simpler. The energy of the nucleon subsystem can then be obtained by perturbation theory in terms of the amplitude of the pion field, and the $\pi\pi$ interaction takes the form $\frac{1}{4}\Lambda(k,\omega)(\varphi_k\varphi_{-k})^2$. It is possible to obtain Λ by perturbation theory only by specifying a simple form of the condensate field.

We develop next a method for calculating Λ, using the Thomas-Fermi approximation, which makes it possible to obtain the energy of the condensate with less restricted assumptions concerning the condensate field than in the case of perturbation theory.

A. Pion effective Lagrangian

The method of finding the effective Lagrangian is illustrated with a simplified model in which only the πN interaction is taken into account.

The effective Lagrangian, even in the case of the simplified model, is a complicated function of the field and of its derivatives with respect to time. We investigate the character of the possible solutions for the pion field. We next present the Weinberg Lagrangian, which is used to determine the vacuum contribution to the $\pi\pi$ interaction.

In the case of weak fields, the effective interaction of the pions acquires the simple form $\frac{1}{4}(\varphi_k\varphi_{-k})^2$. This expression will be used to obtain the quantities characterizing the system near the phase-transition point. For a realistic model (with allowance for the N^* resonance and the nucleon correlations), the function $\Lambda(k,\omega)$ is determined in Sec. IV.B.

1. Procedure for determining the Lagrangian

The Lagrangian describing the system with interacting nucleons and pions can be represented in the form

$$\mathcal{L}=\mathcal{L}_\pi+\mathcal{L}_N+\mathcal{L}_{\pi N}+\mathcal{L}_{NN}+\mathcal{L}_{\pi\pi}\,, \tag{4.1}$$

where $\mathcal{L}_\pi$ and $\mathcal{L}_N$ are the free Lagrangians of the pion and nucleon fields, while $\mathcal{L}_{\pi N}$, $\mathcal{L}_{NN}$, and $\mathcal{L}_{\pi\pi}$ are the Lagrangians of the πN, NN, and $\pi\pi$ interactions. When describing the ground state of the system with the condensate, we can average in (4.1) over all the degrees of freedom of the nucleon and meson field, leaving only the classical part φ, which describes the condensate field. To this end it is necessary to introduce in place of $\mathcal{L}$ another quantity—the effective Lagrangian $\tilde{\mathcal{L}}$, which is obtained from (4.1) by averaging over the exact states of the nucleons in the condensate field. The role of the nucleon medium reduces here to a change of the spectrum of the pions and their interaction in comparison with the vacuum values. Since the interaction of the pion quasiparticles with one another and with the averaged field of the nucleons is connected with exchange of low-frequency particle–hole excitations of the medium, this interaction is essentially retarded, i.e., the effective Lagrangian contains high-order derivatives of φ with respect to time. If, however, φ describes a stationary state, i.e., it depends on the time like $e^{-i\omega t}$, this does not lead to any complications. Instead of a dependence on $\dot{\varphi}$, $\ddot{\varphi}$, etc. it is possible to introduce in the effective Lagrangian, as independent variables, the frequency ω and the amplitude $\varphi(\mathbf{r})$ of the condensate field.

Let us consider, by way of illustration, a Lagrangian containing only the πN interaction:

$$\mathcal{L}=\sum_p \Psi_p^+(w-H)\Psi_p+\frac{1}{2}\sum_k(\omega^2-\omega_k^2)\varphi_k\varphi_{-k}+if\sum_{pk}\Psi_p^+(\sigma\mathbf{k})\boldsymbol{\tau}\Phi_{\mathbf{p}-\mathbf{k}}\varphi_k\,. \tag{4.2}$$

Here w and ω are the frequencies of the nucleon and meson fields. We have omitted the isotopic symbols, and H is the Hamiltonian of one nucleon. Variation with respect to Ψ_p^+ and φ_{-k} yields the following system of equations for the field φ and the operator Ψ:

$$(w-H)\Psi_{\mathbf{p}}=if\sum_k(\sigma\mathbf{k})\boldsymbol{\tau}\Psi_{\mathbf{p}-\mathbf{k}}\varphi_{\mathbf{k}}$$

$$(\omega^2-\omega_k^2)\varphi_{\mathbf{k}}=if\sum_p\Psi_{\mathbf{p}}^+(\sigma\mathbf{k})\boldsymbol{\tau}\Psi_{\mathbf{p}+\mathbf{k}} \tag{4.3}$$

To find the effective Lagrangian of the pions we determine the energy of the system of nucleons in the field φ. The φ-dependent part of the nucleon energy plays the role of the "potential energy" for the mesons. Then the effective Lagrangian $\tilde{\mathcal{L}}$ in the momentum representation can be written in the form (Migdal, 1973; Migdal, Markin, and Mishustin, 1976):

$$\tilde{\mathcal{L}}=\frac{1}{2}\sum_k(\omega^2-\omega_k^2)\varphi_k\varphi_{-k}+\sum_p(w^{(n)}-\tilde{\epsilon}^{(n)}(p))\tilde{n}_p^+\tilde{n}_p+\sum(w^{(p)}-\tilde{\epsilon}^{(p)}(p))\tilde{p}_p^+\tilde{p}_p \tag{4.4}$$

Here $\tilde{\epsilon}^{(n)}$ and $\tilde{\epsilon}^{(p)}$ are the exact single-particle energies of the neutron and proton in the condensate field; $\tilde{n}_p^+$ and $\tilde{p}_p^+$ are the creation operators of the "new" neutron and the "new" proton defined such that

$$\tilde{n}_p^+\tilde{n}_p=\begin{cases}1, \tilde{\epsilon}^{(n)}(p)<\tilde{\epsilon}_F^{(n)}\\0, \tilde{\epsilon}^{(n)}(p)>\tilde{\epsilon}_F^{(n)}\end{cases};\quad \tilde{p}_p^+\tilde{p}_p=\begin{cases}1, \tilde{\epsilon}^{(p)}(p)<\tilde{\epsilon}_F^{(p)}\\0, \tilde{\epsilon}^{(p)}(p)>\tilde{\epsilon}_F^{(p)}\end{cases},$$

where $\tilde{\epsilon}_F^{(n)}$ and $\epsilon_F^{(p)}$ are the Fermi energies of the "new" particles, which are obtained from the condition that the total number of the nucleons be conserved in the condensate field. Thus, the problem is to determine the changes of the nucleon energy in the external field, the role of which is assumed by the field of the pion condensate.

Equation (4.4) gives the correct equations of motion: the Schrödinger equation for the nucleon field and the Klein–Gordon–Fock equation for the meson field altered by the πN interaction.

It is easily seen that for a weak field Eq. (4.3) yields a Klein–Gordon–Fock equation in which ω_k^2 is replaced by $\omega_k^2+\Pi_p(k,\omega)$, where $\Pi_p(k,\omega)$ coincides exactly, as it should, with the pole part obtained in Sec. III for the polarization operator [without allowance for the NN interaction].

The energy density of the system (the average Hamiltonian density) is connected with the effective Lagrangian (4.4) by the equation (Migdal, Markin, and Mishustin, 1974; Migdal, 1973)

$$\tilde{H} = w^{(n)}\frac{\partial \tilde{\mathfrak{L}}}{\partial w^{(n)}} + w^{(p)}\frac{\partial \tilde{\mathfrak{L}}}{\partial w^{(p)}} + \omega\frac{\partial \tilde{\mathfrak{L}}}{\partial \omega} - \tilde{\mathfrak{L}}. \quad (4.5)$$

We shall also need a formula relating the 4-vector of the charged-quasiparticle current with the effective Lagrangian (4.4) (Migdal, Markin, and Mishustin, 1974; 1976)

$$j_\mu = e\frac{\partial \tilde{\mathfrak{L}}}{\partial k_\mu}, \quad (4.6)$$

where $k=(\omega, \mathbf{k})$ is the 4-momentum of the considered particle.

It should be noted that when (4.6) is taken into account for the zeroth component of the current 4-vector, relation (4.5) assumes a form similar to the relation between the free energy of the system and the thermodynamic potential Ω. Thus, the effective Lagrangian $\tilde{\mathfrak{L}}$ is equivalent to the potential $-\Omega$, in which the roles of the chemical potentials are assumed by the frequencies of the meson and nucleon fields.

It follows from (4.4) and (4.5) that to find the energy of the condensate it suffices to calculate the energy of the nucleons in the field

$$E_N[\varphi] = \sum \tilde{\epsilon}_n \tilde{n}^+ \tilde{n} + \sum \tilde{\epsilon}_p \tilde{p}^+ \tilde{p}\,.$$

Later on we shall add to (4.4) the interaction terms $\mathfrak{L}_{NN}$ and $\mathfrak{L}_{\pi\pi}$, and also the πNN^* interaction, which makes it possible to take the N^* resonance into account.

We use the effective-Lagrangian method to take into account the vacuum contribution to the total $\pi\pi$ interaction, both in the case of weak fields (Sec. IV.B) and for strong ones (Sec. V.B).

2. Character of possible solutions

As already noted, production of $\pi_s^+\pi^-$ pairs and π_s^0 quasiparticle pairs becomes possible in sufficiently dense neutron matter. The produced particles "populate" macroscopically the state of lowest energy, forming the condensate field. We shall henceforth consider a classical condensate field $\varphi(\mathbf{r}, t)$, comprising the mean value of the pion-field operator over the new ground state with broken symmetry. The quantity $\varphi(\mathbf{r}, t)$ will be the complex order parameter characterizing the new phase.

From the condition of thermodynamic equilibrium with respect to the creation and annihilation of the $\pi_s^+\pi^-$ meson pairs, it follows that the frequencies (chemical potentials) of the π^- and π_s^+ mesons in the condensate are connected by the relation

$$\omega^- + \omega_s^+ = 0$$

not only at the critical point, but also at $n > n_c^{+,-}$.

Thus the wave function of the condensate field $\varphi(\mathbf{r}, t)$, which describes simultaneously the π^- and π_s^+ components of the meson field, should take the form:

$$\varphi(\mathbf{r}, t) = \tfrac{1}{2}(\varphi_{\pi^-} e^{-i\omega^- t} + \varphi_{\pi^+}^* e^{i\omega_s^+}) = \varphi(\mathbf{r}) e^{-i\omega t}, \quad (4.7)$$

where

$$\omega \equiv \omega^- = -\omega_s^+, \quad \varphi_\pi \pm = \frac{\varphi_1 \pm i\varphi_2}{\sqrt{2}}$$

A time dependence of similar type is obtained for the condensate field in the Hamiltonian formalism (Sawyer, 1973). As to the coordinate dependence, the exact form of $\varphi(\mathbf{r})$ should be determined from the equation of motion, which in this case is a nonlinear integro-differential equation. Leaving aside the attempts to find an exact solution, we use a variational method — we specify various trial functions $\varphi(r)$ and choose the one corresponding to the lowest energy. Since the instability sets in for a nonzero momentum $k \sim m_\pi$, the trial functions should be periodic functions of r. The simplest functions are

$$\varphi(r) = \frac{1}{\sqrt{2}} a e^{i\mathbf{kr}}\ (traveling\ wave) \quad (4.8)$$

$$\varphi(r) = a \sin \mathbf{kr}\ (standing\ wave) \quad (4.9)$$

The amplitude "a" is defined in such a way that the mean value is $\overline{\varphi_1^2} + \overline{\varphi_2^2} + \overline{\varphi_3^2} = a^2$. Since the field of the π^0 mesons is real and static, the simplest solution for the neutral component of the condensate field takes the form (4.9).

The effective Lagrangian $\tilde{\mathfrak{L}}$ is a function of three independent variables ω, k, and a. The equation of motion reduces to an algebraic equation for the determination of the optimal amplitude a of the condensate field:

$$\frac{\partial \tilde{\mathfrak{L}}}{\partial a} = 0 \quad (4.10)$$

In some cases the electroneutrality condition for pion quasiparticles $j_0 = e(\partial\tilde{\mathfrak{L}}/\partial\omega) = 0$ is satisfied. Then, the energy minimalization implies $\partial E/\partial k = 0$ and it follows that $\partial\tilde{\mathfrak{L}}/\partial k = 0$, i.e., there is no 4-current in the ground state. Thus

$$\partial\tilde{\mathfrak{L}}/\partial\omega = 0; \quad \partial\tilde{\mathfrak{L}}/\partial k = 0 \quad (4.11)$$

From (4.10) and (4.11) we determine a, ω, and k.

In the limit as $a \to 0$, when we confine ourselves in the Lagrangian $\tilde{\mathfrak{L}}$ to terms of order a^2, the three equations in (4.10) and (4.11) go over into a system that determines the critical parameters of the $\pi_s^+\pi^-$ condensation, namely $n_c^\pm$, ω_c, and $k_c^\pm$. At the critical point, the coefficient of a^2 in $\tilde{\mathfrak{L}}$ reverses sign, and to determine the condensate parameters it is necessary to include in $\tilde{\mathfrak{L}}$ terms nonlinear in a^2. The calculation of these terms is rather difficult in a realistic formulation. This problem can be solved for a condensate field in the form (4.8), to which Sec. IV.B is devoted. Of course, the assumption that only one field harmonic remains in the case of strong fields is completely unfounded; therefore the results obtained by such a method are only qualitative in character.

3. Expansion in the amplitude of the condensate field

By virtue of the pseudoscalar character of the pions, the effective Lagrangian of the pion field should be an even function of the field $\varphi = \{\varphi_1, \varphi_2, \varphi_3\}$ $(\varphi_{\pi^\pm} = (\varphi_1 \pm i\varphi_2)/\sqrt{2}$; $\varphi_{\pi^0} = \varphi_3)$. Near the critical point, it can be represented in the form

$$\tilde{\mathfrak{L}}_\pi = \frac{1}{2}\sum_k D^{-1}|\varphi_k|^2 - \frac{1}{4}\sum_{kk'} \Lambda(\mathbf{k}, \mathbf{k}', \omega)|\varphi_k|^2|\varphi_{k'}|^2, \quad (4.12)$$

where D is the propagator of the pion in the nucleon medium

$$\mathcal{D}^{-1} = \omega^2 - (1+k^2) - \Pi(k,\omega)\,. \tag{4.13}$$

The function Λ in (4.12) describes the effective interaction of the pion quasiparticles in a nucleon medium. The condition for a second-order phase transition to really take place is that the second term in the right-hand side of (4.12) be positive-definite in the vicinity of the critical point, i.e., when the coefficient of $|\varphi_k|^2$ vanishes $\sum\Lambda|\varphi_k|^2|\varphi_{k'}|^2>0$. In the opposite case we get a first-order phase transition and the expansion in φ_k is not valid.

We define the quantity $\Lambda(k,k',\omega)|_{k=k'}$ such that the effective Lagrangian takes the same form for both a traveling and a standing wave:

$$\tilde{\mathcal{L}}_\pi = \frac{1}{2}D^{-1}a^2 - \frac{1}{4}\Lambda(k,\omega)\,a^4 + O(a^6)\,. \tag{4.14}$$

The amplitude a, the frequency ω, and the wave number k of the condensate field are determined from the three equations (4.10)–(4.11). The first equation is the equation of motion for the condensate field, which in this case is algebraic. The two other equations ensure the absence of a 4-current in the ground state of the system.

Near the second-order transition point, the amplitude of the condensate field $a\to 0$, and equations (4.10) and (4.11) take the form

$$\omega^2 = 1 + k^2 + \Pi(k,\omega)$$

$$2\omega - \frac{\partial\Pi}{\partial\omega} = 0 \quad 2k + \frac{\partial\Pi}{\partial k} = 0\,.$$

The first of them is the dispersion equation of the pion excitations. The second expression is the condition for the instability of the pion field with respect to the production of $\pi^+\pi^-$ pairs and π^0 mesons.

Thus, in the case of weak fields, the task of finding the Lagrangian reduces to a calculation of the quantity $\Lambda(k,k',\omega)$. To determine the condensate field it suffices to know the quantity $\Lambda(k_0,k_0,\omega_c)$, where ω_c is the frequency of the condensate field near the critical point (in a system with $N=Z$ we have $\omega_c=0$).

4. Vacuum interaction of pions

The nonlinear Lagrangians are obtained in the PCAC theory from the requirement of chiral symmetry as $m_\pi\to 0$. As shown by Weinberg (1966), the simplest nonlinear Lagrangian that includes the nucleon and pion fields can be obtained from the linear one by replacing the general derivatives

$$\partial_\mu\boldsymbol{\varphi} \to \mathcal{D}_\mu\boldsymbol{\varphi} = \frac{\partial_\mu\boldsymbol{\varphi}}{1+\varphi^2/F^2} \tag{4.15}$$

$$\partial_\mu\Psi \to \mathcal{D}_\mu\Psi = \left[\partial_\mu + \frac{i\boldsymbol{\tau}(\boldsymbol{\varphi}\times\partial_\mu\boldsymbol{\varphi})}{F^2+\varphi^2}\right]\Psi \tag{4.16}$$

where $F=1.35m_\pi = 189$ MeV is the pion decay constant. We note that the second term in the square brackets of (4.16) describes the S-wave πN interaction in the lowest order in φ. Its contribution to the polarization operation of the pions was investigated by Migdal (1973a, 1973b) and by Migdal, Markin, and Mishustin (1974). As to the mass term of the Lagrangian, which is proportional to m_π^2, there are several variants of its renormalization. Inasmuch as the results depend little on the specific choice by fields φ of small amplitude, we shall use the simplest variant proposed by Weinberg (1966):

$$m_\pi^2\varphi^2 \to \frac{m_\pi\varphi^2}{1+\varphi^2/F^2} \tag{4.17}$$

The Lagrangian of the system of nucleons and pions, without allowance for the nucleon correlations and N^* resonance, takes the form

$$\mathcal{L} = w\Psi^+\Psi - (\mathcal{D}_\alpha\Psi)^+(\mathcal{D}_\alpha\Psi)/2m_N + f\Psi^+\sigma_\alpha\tau_\beta\Psi\mathcal{D}_\alpha\varphi_\beta + \frac{\omega^2\varphi^2-(\nabla\varphi)^2}{(1+\varphi^2/F^2)^2} - \frac{1}{2}\frac{\varphi^2}{1+\varphi^2/F^2} \tag{4.18}$$

where $\alpha=1,2,3$. In the limit $\varphi\ll F$, this Lagrangian is identical to Eq. (4.2).

As shown by Au and Baym (1974), in the case of a traveling wave the transformations of the fields (4.15) and (4.16), which leads to the Weinberg Lagrangian, reduces to a shift of the nucleon 4-momenta. Putting

$$\theta = 2\arctan(a/F)$$

we obtain from (4.15) and (4.16) the following rule for the transformation of the 4-momenta of the nucleons and of the pion field

$$\left.\begin{aligned} &\mathbf{p}\to\mathbf{p}+\tau_3\mathbf{k}\sin^2\frac{\theta}{2}\\ &\epsilon(p)\to\epsilon\left(\mathbf{p}+\tau_3\mathbf{k}\sin^2\frac{\theta}{2}\right)-\tau_3\omega\sin^2\frac{\theta}{2}\end{aligned}\right\} \tag{4.19}$$

using (4.17) one finds

$$a^2\to(F^2/4)\sin^2\theta \tag{4.20}$$

The free Lagrangian of the condensate field takes as a result of the transformations (4.17) and (4.20) the form

$$\mathcal{L}_\pi = -\frac{F^2}{8}\left[(k^2-\omega^2)\sin^2\theta + 4\sin^2\frac{\theta}{2}\right] \tag{4.21}$$

Once the transformations (4.19) and (4.20) are made, it is not difficult to calculate the energy of the π condensate for an arbitrary amplitude (see Sec. V.B).

B. Interaction of pions in the case of weak fields

This section is devoted to the calculation of the function $\Lambda(k,\omega)$, which determines the interaction of the pions in weak fields

$$\tilde{\mathcal{L}}_{\pi\pi} = \frac{1}{4}\Lambda a^4$$

At first we calculate Λ in the simplified model considered above [see Eq. (4.2)], in which only the πN interaction is taken into account. An expression is obtained for the nucleon energy in the field of the condensate, accurate to a^4, which makes it possible, after substitution in (4.4), to determine Λ for both a traveling and a standing wave.

The effect of the nucleon correlations of the value of Λ is then taken into account. The principal effect of the nucleon interaction is that the value of Λ obtained in the simplified model is multiplied by the factor

$$\left[1+g'\frac{p_F}{p_0}\Phi(k,0)\right]^{-4}$$

The contribution made to Λ by the vacuum $\pi\pi$ interaction, as well as the influence of the N_{33}^* resonance, are then taken into account.

1. Calculation of the interaction parameter Λ

As follows from the Lagrangian (4.4), the coefficients of expansion in the amplitude a can be obtained by calculating the sum of the single-particle energies of the nucleons $\tilde{\epsilon}(p)$ in the condensate field φ.

Indeed, we write down the nucleon energy density in the condensate field in the form

$$\mathcal{E}_N(\varphi)=2\int_{\tilde{\epsilon}(p)<\tilde{\epsilon}_F}\frac{d^3p}{(2\pi)^3}\tilde{\epsilon}(p)=\mathcal{E}_N+\frac{\Pi a^2}{2}+\frac{\Lambda a^4}{4}+O(a^6) \tag{4.22}$$

where $\tilde{\epsilon}_F$ is the Fermi energy in the condensate field. For Π and Λ we have

$$\Pi=2\left(\frac{\partial^2\mathcal{E}_N}{\partial a^2}\right)_{a^2=0}$$

$$\Lambda=2\left(\frac{\partial^2\mathcal{E}_N}{\partial (a^2)^2}\right)_{a^2=0}$$

Thus, to determine Π and Λ it suffices to obtain $\tilde{\epsilon}(p)$ accurate to terms of fourth order in the field amplitude. The new Fermi surface is determined by the condition

$$\tilde{\epsilon}(p)=\text{const}=\tilde{\epsilon}_F$$

and the energy $\tilde{\epsilon}_F$ is obtained from the requirement that the particle number be conserved:

$$n=2\int_{\tilde{\epsilon}(p)<\tilde{\epsilon}_F}\frac{d^3p}{(2\pi)^3}=\frac{p_F^3}{3\pi^2},$$

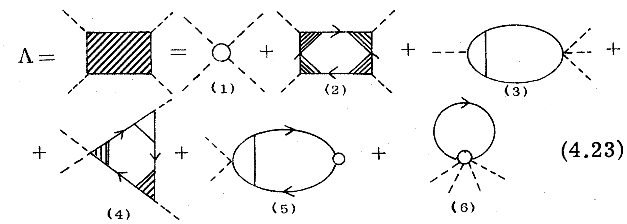
where p_F is the momentum of the old Fermi surface. We note that in all the cases, we have considered and in particular in the most realistic model given below, we have $\omega_{\pi^-}>\epsilon_F^{(n)}$. The reaction $n\to p+\pi^-$ is not allowed by energy conservation since, to transform a neutron quasiparticle into a proton quasiparticle and a π^- meson it is necessary to expend an energy $\Delta\epsilon=\omega_{\pi^-}-\epsilon_F^{(n)}$. On the other hand, in the case of $\pi_s^+\pi^-$ condensation, only the new neutron states are filled.

To verify this method of calculating Π and Λ, we can find an expression for the energy $\tilde{\epsilon}(p)$ in the model considered above, in which there is only πN interaction. In the case of a running wave it is easy to find an exact expression for $\tilde{\epsilon}(p)$, without expansion in powers of a, for in this case the problem reduces to that of two levels with momenta $\mathbf{p}$ and $\mathbf{p}-\mathbf{k}$ (Sec. V.B).

After finding the energy $\tilde{\epsilon}(p)$ up to terms a^4 inclusive, we obtain from (4.22) an expression for Π, which coincides exactly, as it should in this model, with the pole part Π_p of the polarization operator (without allowance for the nucleon correlations). This serves as a good check on the correctness of the calculations. Another method of finding Λ is to calculate directly the diagrams that determine this quantity.

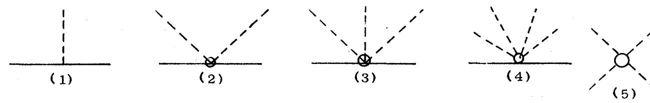
2. Processes that determine the $\pi\pi$ interaction

We begin by presenting the diagrams that determine the interaction Λ



(4.23)

The first of these diagrams corresponds to the vacuum $\pi\pi$ interaction. The nonlinear Weinberg Lagrangian expresses the vacuum vertices



(4.24)

in terms of the known constants f and F. The triangle in diagram 3 of (4.23) represents the already known vertex of the πN interaction in the medium

$$\mathcal{T}=\mathcal{T}_0\Gamma=\mathcal{T}_0\left[1+g'\frac{p_F}{p_0}\Phi(k,\omega)\right]^{-1}$$

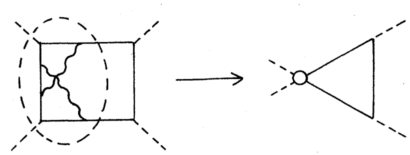
for $N=Z$, and

$$\mathcal{T}_1=\mathcal{T}_0\Gamma_1=\mathcal{T}_0\left[1+g'\frac{p_F}{p_0}\phi_1(k,\omega)\right]^{-1}$$

in the case $Z=0$.

The quantity Γ gives the quenching of the vertex in the nucleon medium. According to the arguments presented above (see page 125), all diagrams that are joined by more than one particle–hole pair are determined by the large 4-momenta of the intermediate states and differ little from their vacuum values. The vertices (4.24) in the medium can therefore be replaced by their vacuum values obtained from the Weinberg Lagrangian.

Thus, for example, a diagram of the type



should be regarded as equivalent to diagram (4) of Eq. (4.23).

In fact, the part of the diagram enclosed in the dashed circle was determined by large 4-momenta, and such diagrams, as we have already seen (page 125), are either small or are included in the corresponding vacuum vertex.

The first of the diagrams of (4.23) is obtained directly from the expansion of the Weinberg Lagrangian in powers of φ up to the φ^4 term.

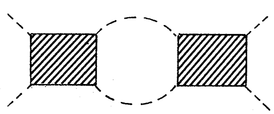
The second diagram is obtained either by direct calculation with the aid of a Green's function, or by finding the nucleon energy $\mathcal{E}_N(a^2)$ in the condensate field accurate to a^4 terms. The same result is obtained by both methods.

The third and fifth diagrams are expressed in terms of the pole part of the polarization operator. The fourth diagram of (4.23) is expressed in terms of the derivatives $\partial\Pi/\partial\omega$ and $\partial\Pi/\partial k$. These diagrams can also be calculated either with the aid of Green's functions or from the expansion (4.22), if we include in the Lagrangian the πN interaction determined by vertices 1, 2, and 3 of Eq. (4.24). Finally, diagram 5 is expressed in terms of vertex 4 of (4.24) and the nucleon density, inasmuch as a closed nucleon loop corresponds to a Green's function

$$G(\mathbf{r},\mathbf{r}) = \langle\Psi^+(\mathbf{r}),\Psi(\mathbf{r})\rangle = n(\mathbf{r})$$

Thus Eq. (4.23) for Λ can be obtained either directly with the aid of Green's functions, or by expanding, in terms of the field amplitude, the Lagrange function, in which we include not only the ordinary πN interaction but also the interactions corresponding to the vertices of (4.24).

As we shall see later (Sec. IV.C), an essential role near the critical point is played by diagrams of the type



A diagram of this type, with four pion lines, is determined by large 4-momenta and is taken into account in the first vacuum vertex of Eq. (4.23). With increasing distance from the critical point, the unaccounted-for part of this diagram becomes inessential. In addition to all the foregoing diagrams, an important role is also played by diagrams in which one or two nucleon lines are replaced by N^*-resonance lines.

3. Allowance for the N^* resonance

To obtain the diagrams containing N^* it is necessary to know the vacuum vertices of the type of figure (4.24), in which one or two nucleon lines are replaced by N^*-resonance lines. This calls for the introduction into the nonlinear Weinberg Lagrangian of not only the nucleons and pions, but also an additional particle with quantum numbers $S=T=\frac{3}{2}$. This problem was solved by Campbell, Dashen, and Manassah (1975) with the aid of the chiral-symmetry requirement. A connection was obtained between vertices of the type $N^*\pi N$ $N^*2\pi N$, $N^*\pi N^*$, $N^*2\pi N^*$ and the analogous nucleon vertices on the basis of the SU(4) quark model. The results can be written compactly in the form of a 6×6 matrix (in the basis of the $N^{*++}, N^{*+}, p, n, N^{*0}, N^{*-}$ states). The elements of this matrix give those vacuum-Lagrangian terms which determine the corresponding process. The problem was solved for a field of charged pions in the form of a running wave of amplitude $(F/2)\sin\theta$. It was assumed (without any justification whatever) that there was no condensate of neutral pions.

In that case, to determine the single-particle energies of the "new" particles it is necessary to diagonalize the matrix

$$H^* = \begin{vmatrix} (a+3b+9c+\Delta) & \left(\frac{id}{\sqrt{3}}\right) & \left(-\frac{4id}{\sqrt{6}}\right) & (0) & (0) & (0) \\ \left(-\frac{id}{\sqrt{3}}\right) & (a+b+c+\Delta) & (0) & \left(-\frac{4id}{\sqrt{18}}\right) & \left(\frac{2id}{3}\right) & (0) \\ \left(\frac{4id}{\sqrt{6}}\right) & (0) & (a+b+c) & \left(\frac{5id}{3}\right) & \left(\frac{4id}{\sqrt{18}}\right) & (0) \\ (0) & \left(\frac{4id}{\sqrt{18}}\right) & \left(-\frac{5id}{3}\right) & (a-b+c) & (0) & \left(\frac{4id}{\sqrt{6}}\right) \\ (0) & \left(-\frac{2id}{3}\right) & \left(-\frac{4id}{\sqrt{18}}\right) & (0) & (a-b+c+\Delta) & \left(\frac{id}{\sqrt{3}}\right) \\ (0) & (0) & (0) & \left(-\frac{4id}{\sqrt{6}}\right) & \left(-\frac{id}{\sqrt{3}}\right) & (a-3b+9c+\Delta) \end{vmatrix}, \qquad (4.25)$$

where

$$a=\frac{p^2}{2m}+\frac{1}{2}\omega;\quad b=\frac{1}{2}\left(\omega-\frac{\mathbf{pk}}{m}\right)\cos\theta,\quad c=\frac{k^2}{8m}\cos^2\theta$$

$$d=\frac{3}{10}g_A k\sin\theta,\quad \Delta=m_{N^*}-m_N\approx 2,2\ ,\quad \theta=\arctan\frac{a}{F}$$

Expanding the corresponding matrix elements in power of θ, we can obtain the vertices (4.24), both for the case of nucleon transitions and for transitions with participation of the N^* resonance.

It follows from (4.25) that the $p\pi n$ vertex (H^*_{pn}) is connected with the $N^{*+}\pi n$ vertex $H^*_{N^{*+}n}$ by the factor $(8/25)^{1/2}=0.56$. This is in fair agreement with the value 0.47 obtained for the ratio of these vertices at $k=k_R$ by using, as was done in Sec. III, the experimental data on the πN scattering in the region of the N^*_{33} resonance.

The method proposed by Baym *et al.* (1975) to take into account the nucleon correlations corresponds to the assumption that the NN, NN^*, and N^*N^* interactions are the same apart from the Clebsch–Gordan coefficients. Yet the experimental data on $(pp; N^*n)$ scattering with large momentum transfers, $q \sim 600$ MeV/c (Mountz *et al.*, 1975) seem to indicate a weak NN^* interaction at short distances. We shall therefore present estimates under two assumptions: (1) the same interaction between all the baryons, and (2) correlations exist only between the nucleons.

Equation (4.25) was used by Campbell *et al.* (1975) and by Baym *et al.* (1975) to describe π condensation in a pure neutron medium. They carried out an explicit diagonalization of H^* in the two interesting limiting cases, $\theta \to 0$ and $\theta \to \pi/2$. As $\theta \to 0$, the problem reduces to a determination of the critical parameters of the π condensation. The values obtained by Baym *et al.* (1975) for n_c, k_c, and ω_c turned out to be close to those calculated by Migdal, Markin, and Mishustin (1974) by another method. We shall return to the question of the contribution of the N^* resonance in the case of a developed condensate ($\theta \to \pi/2$) in Section V.B.

4. Results of calculation of Λ

The individual terms of expression (4.23) were calculated by several methods (with the exception of the first term, which is obtained directly from the Weinberg Lagrangian). The second term was obtained both with the aid of Green's functions and by expanding the effective Lagrangians up to terms a^4. Terms of type (3), with allowance for N^* resonance, reduce to the polarization operator, while terms of type (4) reduce to its derivatives. The last term is proportional to ω, and therefore vanishes at $N=Z$.

We present the results of the calculation of Λ.[4] As already noted, the calculations have been carried out for a condensate of charged pions in the running wave form (4.8). The corresponding expressions are also given for the polarization operator, since they differ somewhat from those obtained in Sec. III.

In a medium with $Z=0$, the formula for Λ turns out in the general case to be very cumbersome. We present here the result obtained in the approximation of high frequencies $\omega \gg kv_F$. Assuming that the local amplitudes of the NN, NN^*, and N^*N^* interactions are the same we have:

$$\Lambda(k,\omega) = -\frac{2}{F^2}[2(k^2-\omega^2)+1] - \frac{4}{F^2}\Pi^-(k,\omega) + 8(f\Gamma_1^* k)^2 n\omega \frac{\partial W_1}{\partial \omega} + 4(f\Gamma_1^* k)^4 n W_2 - \frac{4}{F^4}n\omega\,, \tag{4.26}$$

where

$$\Pi^-(k,\omega) = -2f^2k^2\Gamma_1^* nW_1,\quad \Gamma_1^* = \left[1 + g^- \frac{\pi^2}{mp_0} nW_1\right]^{-1},$$

$$W_1 = \frac{1}{\omega} + \frac{8}{25}\frac{1}{\Delta+\omega} + \frac{24}{25}\frac{1}{\Delta-\omega}\,,$$

$$W_2 = \frac{1}{\omega^3} + \frac{8}{25}\frac{1}{\omega^2}\left[\frac{1}{\Delta+\omega} + \frac{3}{\Delta-\omega} - \frac{1}{\Delta} - \frac{3}{\Delta+2\omega}\right] + \frac{8}{25}\frac{1}{\omega}\left[\frac{1}{(\Delta+\omega)^2} + \frac{3}{(\Delta-\omega)^2} - \frac{0.8}{\Delta(\Delta+\omega)} - \frac{1.2}{(\Delta+\omega)(\Delta+2\omega)} - \frac{2.4}{\Delta(\Delta-\omega)}\right]$$
$$+ \frac{64}{625}\left[\frac{1}{(\Delta+\omega)^3} + \frac{3}{(\Delta+\omega)^2(\Delta-\omega)} + \frac{3}{(\Delta-\omega)^2(\Delta+\omega)} + \frac{9}{(\Delta-\omega)^3} - \frac{0.5}{\Delta(\Delta+\omega)^2} - \frac{0.375}{(\Delta+2\omega)(\Delta+\omega)^2} - \frac{1.175}{\Delta(\Delta-\omega)^2} - \frac{1.5}{(\Delta-\omega)\Delta(\Delta+\omega)}\right].$$

If there are no NN^* and N^*N^* correlations, then the renormalization factor Γ_1^* is replaced by

$$\Gamma_1 = \left[1 + g^- \frac{\pi^2}{mp_0}\frac{n}{\omega}\right]^{-1},$$

and only the terms containing $1/\omega$, $1/\omega^2$, and $1/\omega^3$ in Π^- and Λ are renormalized by the factors Γ_1, Γ_1^2, and Γ_1^4.

In a medium with $N=Z$, for a condensate field in the form

$$\boldsymbol{\varphi}(\mathbf{r}) = a\{\cos\mathbf{kr}, -\sin\mathbf{kr}, 0\}$$

(running wave with $\omega=0$), with identical NN, NN^*, and N^*N^* correlations, we obtain the expression

$$\Lambda(k) = -\frac{2}{F^2}(2k^2+1) - \frac{4}{F^2}\Pi(k,0) + 8\frac{m^*p_F}{\pi^2}\left(\frac{f\Gamma^*k}{F}\right)^2\left[k\frac{\partial}{\partial k}\Phi\left(\frac{k}{2p_F}\right) - \frac{64}{75}\left(\frac{kv_F}{\Delta}\right)^2\right]$$
$$+ \frac{4(f\Gamma^*k)^4}{3\pi^2 v_F^3}\left[\Phi_2\left(\frac{k}{2p_F}\right) + \frac{384}{125}\left(\frac{\epsilon_F}{\Delta}\right)^2\Phi'\left(\frac{k}{2p_F}\right) + \frac{20}{3}\frac{\epsilon_F}{\Delta}\right)\right] \tag{4.27a}$$

Here

$$\Pi(k,0) = -\frac{2m^*p_F}{\pi^2}f^2k^2\Gamma^*\Phi^*\left(\frac{k}{2p_F}\right),$$

$$\Gamma^* = \left[1 + g^- \frac{p_F}{p_0}\Phi^*\left(\frac{k}{2p_F}\right)\right]^{-1}$$

$$\Phi(x) = \frac{1}{2} + \frac{1-x^2}{4x}\ln\left|\frac{1+x}{1-x}\right|,$$

$$\Phi^*(x) = \Phi(x) + \frac{128}{75}\frac{\epsilon_F}{\Delta}\,,$$

$$\Phi_2(x) = \frac{3}{16x^2}\left[\ln^2\left|\frac{1+x}{1-x}\right| - \frac{1}{x}\ln\left|\frac{1+x}{1-x}\right| + \frac{2}{1-x^2}\right].$$

[4]These results have been obtained by I. N. Mishustin.

If the correlations exist only between nucleons, and the local NN^* and N^*N^* amplitudes are equal to zero, then we have for $N=Z$

$$\Lambda(k) = -\frac{2}{F^2}(2k^2+1) - \frac{4}{F^2}\Pi(k,0) + 8\frac{m^*p_F}{\pi^2}\left(\frac{fk}{F}\right)^2\left[\Gamma^2 k\frac{\partial}{\partial k}\Phi\left(\frac{k}{2p_F}\right) - \frac{64}{75}\left(\frac{kv_F}{\Delta}\right)^2\right] - \frac{4(fk)^4}{3\pi^2 v_F^3}\left[\Gamma^4\Phi_2\left(\frac{k}{2p_F}\right) + \frac{384}{125}\left(\frac{\epsilon_F}{\Delta}\right)^2\left(\Gamma\Phi\left(\frac{k}{2p_F}\right) + \frac{20}{3}\frac{\epsilon_F}{\Delta}\right)\right]. \tag{4.27b}$$

Here

$$\Pi(k,0) = -\frac{2m^*p_F}{\pi^2}f^2k^2\left[\Gamma\Phi\left(\frac{k}{2p_F}\right) + \frac{128}{75}\frac{\epsilon_F}{\Delta}\right],$$

$$\Gamma = \left[1 + g^-\frac{p_F}{p_0}\Phi\left(\frac{k}{2p_F}\right)\right]^{-1}.$$

Equations (4.26) and (4.27) for the parameter Λ, which describes the interaction of the pion excitations in a nucleon medium, will be sufficient for our discussion. We shall use them subsequently in the numerical estimates. In the vicinity of the critical point, in all the cases considered, Λ is positive. Thus, at $n=n_0$, $k=p_F$, and $g^-=1.6$, Eqs. (4.27a) and (4.27b) give, respectively, the values 0.4 and 8.1.

We note that even the most complete model considered above cannot determine Λ quantitatively. No account was taken of the local scalar NN interaction, of many-particle correlations, and of many other factors indicated above. The NN^* and N^*N^* interactions are unknown, and in addition, we do not know the constants of the spin-spin interaction between the nucleons for a neutron medium. Finally, the possibility of a first-order phase transition was not investigated—such an investigation calls for the use of numerical methods.

In the next section we shall consider the Thomas-Fermi approximation, which is valid at $k^2/4p_F^2 \ll 1$. This method can be used to obtain an expression for Λ at an arbitrary form of the condensate field $\varphi(\mathbf{r})$. In addition, this approximation provides a good method of verifying the formulas obtained above in the limit as $k\to 0$.

C. Approximation of weakly varying fields

This section is devoted to the investigation of π condensation in isotopically-symmetrical nuclear matter $(Z=N)$, and also to π^0 condensation at $N \gg Z$. As already noted, owing to isotopic symmetry, instability in a system with $Z=N$ sets in simultaneously for all three types of pions, π^+, π^-, and π^0. This instability results in a static electrically neutral π condensate $\varphi=\{\varphi_1, \varphi_2, \varphi_3\}$ $(\varphi_{\pi\pm}=(\varphi_1 \pm i\varphi_2)/\sqrt{2},\ \varphi_{\pi^0}=\varphi_3)$, with the possibility of having all three components of the pion field different from zero. Therefore, on top of the difficulties in determination of the spatial structure of the condensate field, we encounter one more difficulty, connected with the choice of the optimal isotopic composition of the condensate. This raises a problem which is rather difficult to formulate realistically, that of finding the energy of the system in the presence of a condensate that has a complicated spatial and isotopic structure.

A convenient method for a qualitative solution of this problem is the Thomas–Fermi approximation. The condition for the applicability of the Thomas–Fermi method in the case of periodic potentials is the inequality

$$k^2/4p_F^2 \ll 1$$

(k is the wave vector of the field). Yet, according to calculations, condensation takes place at $k\approx p_F$, so that when we use the Thomas–Fermi approximation we cannot count on obtaining good quantitative accuracy. As we shall show, this approximation makes it possible to find the condensation parameters under less restrictive assumptions concerning the spatial and isotopic structure of the condensate than in the case of perturbation theory. Moreover, this method can be so developed that account is taken of all the vertices of (4.24) which result from the Weinberg Lagrangian.

The most favorable configuration turns out to be a three-dimensional lattice for a π^0 condensate in neutron matter. In symmetric nuclear matter $(N=Z)$ isospin asymmetric configurations can be lower in energy than the symmetric ones.

1. The Thomas-Fermi method

We consider a system of nucleons $(N=Z)$ interacting with a classical field of a static pion condensate. When we disregard the transitions of the nucleons into N_{33}^* states, such an interaction can be described by a potential

$$U = f\tau_\alpha\sigma_\beta\frac{\partial\varphi_\alpha}{\partial x_\beta} \tag{4.28}$$

which is an operator in the space of the spin and isospin variables.

The energy density of nucleons in the Thomas–Fermi approximation is given by the expression:

$$\mathcal{E}_N = \frac{2(2m)^{3/2}}{5\pi^2}\langle(\tilde{\epsilon}_F - U)^{5/2}\rangle + \frac{2(2m)^{3/2}}{3\pi^2}\langle U(\tilde{\epsilon}_F - U)^{3/2}\rangle \tag{4.29}$$

The brackets correspond to the averaging over space, spin, and isospin variables. The density and the spin density of particles are:

$$n(\mathbf{r}) = \frac{2(2m)^{3/2}}{3\pi^2}\frac{1}{4}Tr_\sigma Tr_\tau(\tilde{\epsilon}_F - U)^{3/2};$$

$$\mathbf{S}(\mathbf{r}) = \frac{2(2m)^{3/2}}{3\pi^2}\frac{1}{4}Tr_\sigma Tr_\tau[\boldsymbol{\sigma}(\tilde{\epsilon}_F - U)^{3/2}]. \tag{4.30}$$

In Eqs. (4.29) and (4.30) there is a factor 2 which takes into account the two types of particles.

The nucleon Fermi energy $\tilde{\epsilon}_F$ in the field U is determined by the conservation of the particles:

$$\bar{n}(\mathbf{r}) = \frac{2}{3\pi^2}p_F^3$$

The insertion of $n(\mathbf{r})$ in (4.30) gives

$$\tilde{\epsilon}_F = \epsilon_F - \frac{m}{2p_F^2}\langle U^2\rangle - \frac{3m^3}{8p_F^6}\left[\langle U^2\rangle^2 + \frac{1}{3}\langle U^4\rangle\right] + O(U^6) \tag{4.31}$$

The total energy density is the sum of the nucleon part (4.29) and the free energy density of the pion field.

$$\mathcal{E}=\mathcal{E}_N+\frac{1}{2}\langle\varphi^2\rangle+\frac{1}{2}\langle(\nabla\varphi)^2\rangle$$

For periodic fields $\varphi(\mathbf{r})$ with wave number k, minimizing with respect to the field amplitude, we obtain an expression accurate to φ^4 (Migdal, Markin, and Mishustin, 1976)

$$\mathcal{E}=\mathcal{E}(\varphi=0)+\mathcal{E}_\pi$$

$$\mathcal{E}_\pi=-\frac{\tilde{\omega}^4}{4\lambda}\,\frac{4}{3+\left\langle\left(\sigma_\beta\tau_\alpha\dfrac{\partial\varphi_\alpha}{\partial x_\beta}\right)^4\right\rangle\Big/\left\langle\left(\sigma_\beta\tau_\alpha\dfrac{\partial\varphi_\alpha}{\partial x_\beta}\right)^2\right\rangle^2} \tag{4.32}$$

where

$$\tilde{\omega}^2=1+k^2-f^2k^2\frac{dn}{d\epsilon_F}(1+g^{-}p_F/p_0)^{-1},\quad \lambda=\frac{f^4k^4n}{4\epsilon_F^3}\left(1+g^{-}\frac{p_F}{p_0}\right)^{-4}.$$

Here $\mathcal{E}_\pi$ describes the decrease, due to condensation, of the ground-state energy. The factor $(1+g^{-}p_F/p_0)^{-1}$ takes into account the main contribution of the nucleon correlations.

As a result of simple transformations we obtain

$$\left\langle\left(\sigma_\alpha\tau_\beta\frac{\partial\varphi_\beta}{\partial x_\alpha}\right)^4\right\rangle=\overline{(\nabla\varphi)^4}+\overline{(\nabla\varphi_1\times\nabla\varphi_2)^2}+\overline{(\nabla\varphi_1\times\nabla\varphi_3)^2}+\overline{(\nabla\varphi_2\times\nabla\varphi_3)^2} \tag{4.33}$$

$$\left\langle\left(\sigma_\beta\tau_\alpha\frac{\partial\varphi_\alpha}{\partial x_\beta}\right)^2\right\rangle=\overline{(\nabla\varphi)^2}$$

2. Spatial and isotopic structure of the condensate

Information on the isotopic and spatial structure of the condensate can be obtained by analyzing Eq. (4.33).

The last three terms vanish in the case of a one-dimensional (or spherically symmetrical) field, while the first term is in this case minimal if $(\nabla\varphi)^2$ does not depend on the coordinates:

$$(\nabla\varphi)^2=\text{const}. \tag{4.34}$$

Thus one-dimensional fields satisfying the condition (4.34) correspond to the minimum of the energy (4.32).

For such fields we have $\mathcal{E}_\pi=(\mathcal{E}_\pi)_{\min}=-\omega^4/4\lambda$. Examples of one-dimensional solutions of (4.34) are

$$\varphi(\mathbf{r})=a\{\cos kx,\,-\sin kx,\,0\}, \tag{4.35}$$

$$\varphi(\mathbf{r})=a\left\{\frac{\cos kx}{\sqrt{2}},\,\frac{\cos kx}{\sqrt{2}},\,\sin kx\right\},\ \ldots \tag{4.36}$$

We note that pion fields satisfying condition (4.34) modulate neither the neutron nor the proton density (the question of the modulations of the nucleon density is considered in V.1 in a more realistic model, without the use of the Thomas–Fermi approximation).

We proceed to the case of π^0 condensation in a neutron medium ($Z=0$). To this end we set $\varphi_1=\varphi_2=0$ in Eq. (4.33).

Table II gives the energies of the different configurations of the field $\varphi_3(\mathbf{r})$, comparing in particular one-dimensional, two-dimensional, and three-dimensional configurations. The lowest energy is possessed by a meson field in the form of a three-dimensional lattice:

TABLE II. Ratio of energies of different condensate configurations $[\varphi_3(r)]$ to the energy of the three-dimensional structure in the Thomas–Fermi approximation.

$\varphi_3(\mathbf{r})$	$\left\lvert\dfrac{E^{(\pi)}[\varphi]}{E^{(\pi)}[\varphi^{(1)}]}\right\rvert$
$\varphi^{(1)}(\mathbf{r})=\sin kx+\sin ky+\sin kz$	1
$\varphi^{(2)}(\mathbf{r})=\sin kx+\sin ky$	50/51
$\varphi^{(3)}(\mathbf{r})=\sin kx$	25/27
$\varphi^{(4)}(\mathbf{r})=\sin kx\,\sin ky\,\sin kz$	20/21

$$\varphi_3(\mathbf{r})=(2/3)^{1/2}a(\sin kx+\sin ky+\sin kz) \tag{4.37}$$

for which

$$a^2=-\frac{24}{25}\frac{\tilde{\omega}^2}{\lambda}\;;\quad \mathcal{E}_{\pi 0}=-\frac{24}{100}\frac{\tilde{\omega}^4}{\lambda}$$

In these expressions

$$\tilde{\omega}^2=1+k^2-f^2k^2\frac{dn}{d\epsilon_F}\left(1+g^{nn}\frac{p_F}{p_0}\right)^{-1};$$

$$\lambda=f^4k^4\frac{n}{4\epsilon_F^3}\left(1+g^{nn}\frac{p_F}{p_0}\right)^{-4} \tag{4.38}$$

Thus the most probable structure of the π^0 condensate in a neutron medium is the three-dimensional lattice (4.37). It should be stressed that this conclusion is based on an approximate calculation method (valid for $k^2/4p_F^2\ll 1$), and that for a final solution of the problem of the π^0 condensate structure in a neutron medium it is necessary to carry out the calculation for $k\simeq p_F$. In addition, we need to take into account the N_{33}^* resonance, the vacuum $\pi\pi$ interaction, and finally, the interaction of the π^0 condensate with the condensate of the charged mesons, which also influences significantly the structure of the condensate (Markin and Mishustin, 1974).

Using (4.30), we easily obtain the distribution of the particle density and of the neutron spin density in the field (4.37).

Expanding $n(\mathbf{r})$ and $S(\mathbf{r})$ up to the square of the potential, we obtain

$$n(\mathbf{r})=n[1+\xi^2(\cos 2kx+\cos 2ky+\cos 2kz)],$$

$$\xi^2=\frac{f^2k^2a^2}{8\epsilon_F^2}\left[1\Big/\left(1+g^{nn}\frac{p_F}{p_0}\right)^2\right] \tag{4.39}$$

with the spin density

$$S(\mathbf{r})=\frac{mp_F}{\pi^2}\frac{1}{(1+g^{nn}p_F/p_0)}\nabla\varphi_3(r). \tag{4.40}$$

V. PION CONDENSATION IN A NUCLEON MEDIUM

In the case of second-order phase transitions, the order parameter characterizing the new phase increases from a zero value, so that the problem has a small parameter near the transition point. Using the smallness of this parameter, we can find the equation of state of the new phase. In the theory of π condensation near the critical density, this small parameter is the amplitude $\varphi(\mathbf{r}, t)$ of the classical pion field. To study the properties of the condensate near the critical point we can therefore use the expansion obtained above for the effective Lagrangian function in powers of the field amplitude. With this expression we shall obtain the amplitude of the condensate field and the energy gained by condensation as a function of the nucleon density.

In the case of a neutron medium, the condensation mechanism is somewhat more complicated than for a medium with $N=Z$. First, at $n\simeq 0.4n_0$, π_s^+-meson condensation sets in, followed at $n\sim n_0$ by instability for production of $\pi^-\pi_s^+$ meson pairs. In the case of a neutron star, where the electroneutrality condition is satisfied and the charge of the π_s^+ mesons is cancelled by the electron charge, the density of the π_s^+ mesons and the energy of the π_s^+ condensate are small, and the π_s^+ condensation alters the expression for the $\pi^-\pi_s^+$ condensation energy only near $n=n_c^\pm$. Neutron systems of finite dimension, when the electroneutrality condition does not hold, are considered in Sec. VII.B in connection with an assessment of the existence of neutron nuclei. The amplitude of the spin density modulation and of the total density of the nucleons due to π condensation is determined.

Singularities in the effective pion interaction, due to the "softening" of the pion degree of freedom near the condensation point, are investigated. Because of these singularities, the second-order transition gives way to a first-order transition with a small discontinuity of the field φ. This phenomenon alters the results obtained by assuming a second-order transition only in the nearest vicinity of the phase transition point.

Section V.B is devoted to the determination of the condensate energy in the case of a strong condensate field. The expression given in this section for the condensate energy at a nucleon density $n\gg n_c$ (the limiting field model) is used in Sec. VII.B to assess the possibility of the existence of superdense nuclei.

A. Properties of the condensate near the critical point

The effective pion Lagrangian obtained in the case of weak fields (Sec. IV.B) is used to determine the amplitude and energy of the condensate near the critical point. Properties of π_s^+ condensation in a neutron medium are elucidated.

Expressions are obtained describing the modulation of the density and of the spin density in the condensate field. It is shown that, owing to quantum fluctuations, condensation corresponds to a first-order phase transition, but with a small discontinuity of the pion energy, such that the expressions obtained assuming a second-order phase transition cease to hold only in the immediate vicinity of the critical point.

1. Energy and amplitude of the condensate

To find the energy of the condensate near the critical point we can use the weak-field Lagrangian obtained in Sec. IV.B.

If we disregard the not-too-significant influence of π_s^+ condensation in the neutron medium, which will be considered in the next section, the expressions for the condensate energy in the cases $Z\cong N$ and $Z\ll N$ are of the same form and differ only in a numerical coefficient and in the value of the critical density.

We consider an electrically neutral condensate, when the pion electroneutrality condition $\partial\bar{\mathcal{L}}_\pi/\partial\omega=0$ is satisfied. The case of a charged condensate (π_s^+ condensation) will be considered in the next section.

From Eq. (4.5) between the energy and the Lagrange function it follows that if the pion field is electrically neutral then the energy density of the condensate differs only in sign from Eq. (4.14) for $\mathcal{L}_\pi$

$$\mathcal{E}_\pi(a^2)=-\tfrac{1}{2}D^{-1}(k,\omega)a^2+\tfrac{1}{4}\Lambda(k,\omega)a^4\,, \tag{5.1}$$

where $D(k,\omega)$ is the pion propagator

$$D^{-1}=\omega^2-\omega_k^2-\Pi(k,\omega)\,.$$

Minimizing $\mathcal{E}_\pi(a^2)$ with respect to a^2 and using the conditions (4.11), we obtain a system of equations for determining a^2, the wave vector $\mathbf{k}$, and the condensate frequency ω:

$$\begin{gathered}\omega^2-\omega_k^2-\Pi(k,\omega)-\Lambda a^2=0\\ \left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)a^2-\frac{1}{2}\frac{\partial\Lambda}{\partial\omega}a^4=0\\ \left(2k+\frac{\partial\Pi}{\partial k}\right)a^2+\frac{1}{2}\frac{\partial\Lambda}{\partial k}a^4=0.\end{gathered} \tag{5.2}$$

At the critical point, i.e., at $n=n_c$ and $a^2=0$, we get

$$\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)_c=\left(2k+\frac{\partial\Pi}{\partial k}\right)_c=0$$

$$D^{-1}=D_c^{-1}=\omega_c^2-\omega_{k_c}^2-\Pi(k_c,\omega_c)\big|_{n=n_c}=0$$

Near the critical point we have

$$\begin{aligned}D^{-1}&=\left\{\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)\frac{d\omega}{dn}-\left(2k+\frac{\partial\Pi}{\partial k}\right)\frac{dk}{dn}+\frac{\partial\Pi}{\partial n}\right\}_c(n-n_c)\\ &=\left(\frac{\partial\Pi}{\partial n}\right)_c(n-n_c)\,.\end{aligned} \tag{5.3}$$

Substituting this expression in the first one, we obtain from (5.2)

$$a^2=\alpha(n-n_c),\ \alpha=-\left[\frac{1}{\Lambda}\frac{\partial\Pi}{\partial n}\right]_c. \tag{5.4}$$

Substitution of (5.2) and (5.3) in (5.1) yields

$$\mathcal{E}_\pi=-\beta\frac{(n-n_c)^2}{2};\ \beta=\left[\frac{1}{2\Lambda}\left(\frac{\partial\Pi}{\partial n}\right)^2\right]_c \tag{5.5}$$

Using Eqs. (3.49)–(3.51) for Π, and also Eqs. (4.26) and (4.27) for Λ, obtained in the most realistic model for the the case of a running wave, we can calculate the parameters α, β, and η, which determine the properties of the condensate near the critical point. We present here values of these parameters for a medium with $N=Z$, obtained under the assumption that there are no NN^* and N^*N^* correlations, and that the NN correlation constant is $g^-=1.6$. Using the values $n_c=0.34$ and $k_c=2.13$ from Table I, we obtain

$$\alpha=0.4,\ \beta=1.3,\ \eta=6.2.$$

The parameters n_c and β, calculated under the assumption that the NN, NN^*, and N^*N^* interactions are the same for the cases $N=Z$ and $Z=0$, are listed in Table III as functions of the parameter γ that defines this interaction:

$$\gamma=g^-\frac{\pi^2}{2mp_0}\frac{1}{f'^2}.$$

Account was taken here of the possible difference between the constant f' of the πN interaction in a medium and the vacuum value f. It must be emphasized that in all cases β turns out to be close to unity.

2. π^+ condensation in a neutron medium

It has already been shown (Sec. III.C) that in a neutron medium, at a density $n>n_c^+\cong 0.4n_0$, the π_s^+-meson branch has an energy $\omega_s^+<-(\epsilon_F^n-\epsilon_F^p)$. This results in an instability with respect to the process $p\to n+\pi_s^+$. Thus a π_s^+-meson condensate is produced even before the onset of the instability connected with the production of $\pi^-\pi_s^+$ pairs.

Just as in the case of condensation in an electric field (Sec. II.A) the result depends on whether the electroneutrality condition is satisfied in the system. We consider first a neutron system of large dimensions (neutron star), in which this condition should be satisfied.

Prior to condensation, the proton density is equal to the electron density

$$n_p=n_e=\mu_e^3/3\pi^2.$$

The chemical potential (the Fermi energy) of the electrons in equilibrium prior to condensation is $\mu_e=\epsilon_F^n-\epsilon_F^p$, and after condensation we have $\mu_e=-\omega_s^+$. (This follows from the equilibrium with respect to the variations $n\rightleftarrows p+e+\bar{\nu}$, $n\rightleftarrows n+\pi^++e+\bar{\nu}$.)

The total energy density of the system can be written in the form

$$\mathcal{E}=\mathcal{E}_N+\mathcal{E}_\pi+(\mu_e^4/4\pi^2) \tag{5.6}$$

where the last term is the energy of the relativistic electrons.

Let us find the energy of the π_s^+ condensate. To this end, we write the Lagrangian of the pions in the form

$$\tilde{\mathcal{L}}_\pi=[\omega^2-\omega_k^2-\Pi(k,\omega)]^+\frac{a^2}{2}-\Lambda\frac{a^4}{4},$$

where a is the amplitude of the field of the π_s^+ mesons. In contrast to the case considered in the preceding section, here the pion electroneutrality condition is not satisfied ($\partial\tilde{\mathcal{L}}_\pi/2\omega\neq 0$).

According to (4.6), we have

$$n_\pi=\frac{\partial\tilde{\mathcal{L}}_\pi}{\partial\omega}=\left(2\omega-\frac{\partial\Pi}{\partial\omega}\right)^+\frac{a^2}{2}-\frac{\partial\Lambda}{\partial\omega}\frac{a^4}{4}. \tag{5.7}$$

From (4.5) and (4.14) we get for the pion energy density

$$\mathcal{E}_\pi=\omega_s^+\frac{\partial\tilde{\mathcal{L}}_\pi}{\partial\omega}-\tilde{\mathcal{L}}_\pi=\omega_s^+n_\pi+\frac{\omega_k^2+\Pi-\omega^2}{2}a^2+\Lambda\frac{a^4}{4}.$$

Using (5.2), we obtain

$$\mathcal{E}_\pi=\omega_s^+n_\pi-\Lambda\frac{a^4}{4}. \tag{5.8}$$

Near the critical point we can neglect the second term of (5.8). The energy density of the system after the condensation is then

$$\mathcal{E}=\omega_s^+n_\pi+\frac{(\omega_s^+)^4}{4\pi^2}+\mathcal{E}_N;\ n_\pi=n_e=\frac{|\omega_s^+|^3}{3\pi^2} \tag{5.9}$$

The energy density prior to condensation is

$$\mathcal{E}^0=\frac{(\epsilon_F^n-\epsilon_F^p)^4}{4\pi^2}+\mathcal{E}_N^0.$$

We shall neglect for simplicity the kinetic energy of the protons (at $n\sim n_0$ the proton density in a neutron star is $\sim 10^{-2}n_n$). Then the change of the nucleon energy upon condensation is equal to (n_p protons go over into the neutron state)

$$\mathcal{E}_N-\mathcal{E}_N^0=n_p\epsilon_F^n\cong(\epsilon_F^n)^4/3\pi^2.$$

Substituting this expression in (5.9), we obtain

$$\mathcal{E}=\mathcal{E}^0-\frac{(\omega_s^+)^4}{12\pi^2}+\frac{(\epsilon_F^n)^4}{12\pi^2}. \tag{5.10}$$

The energy of the system does not experience a jump at $n\approx n_c^+$, inasmuch as at $n=n_c^+$ we have $\omega_s^+=\epsilon_F^n-\epsilon_F^p\cong\epsilon_F^n$. With increasing density, $|\omega_s^+|$ increases more rapidly than ϵ_F^n, but with increasing a^2, according to (5.2), the growth of $|\omega_s^+|$ is slowed down by the repulsion between the pions ($\Lambda>0$). As a result no instability with respect to production of $\pi^-\pi_s^+$ pairs arises ($\omega^-+\omega_s^+=0$)—it arose only when no account was taken of the field of the π_s^+ condensate. This can be easily verified by using the expressions for n_π and for ω_s^+.

The result is a complicated dependence of the condensation energy on the density, but qualitatively the picture is perfectly analogous to that of condensation in an electric field, which was considered in Sec. II.

If we introduce the concept of "free" π^- and π_s^+ mesons (π^- and π_s^+ mesons not perturbed by the field of the π_s^+ condensate), then such pairs are produced at $n\gtrsim n_c^\pm$, when the condition $(\omega^-+\omega_s^+)_{a=0}=0$ is satisfied. This process, however, can be described also in terms of the exact π_s^+ mesons with allowance for the condensate field.

As we have seen with condensation in an electric field as an example [Eq. (2.35)], at the point of the instability for the production of "free" pairs $\omega_0^2=0$) no phase transition takes place—the process is described in terms of exact π_s^+ pions.

The expression for the condensate energy under the condition $-\omega_0^2\gg\bar{V}^2$ has changed over into the formula for the energy of the $\pi^+\pi^-$ pair condensate.

We can therefore expect in our case, starting with certain values of $(n-n_c^\pm)$, the formula for the π_s^+ condensate to go over into expression (5.5), which was obtained without taking the π_s^+ condensation into account. Moreover, since the expression for the energy of the

π_s^+ condensate contains, according to (5.10), a large number in the denominator, the transition to the simple expression (5.5) should occur even at a small excess of n above $n_c^{\pm}$.

Thus in spite of the fact that in a neutron star, according to our terminology, π_s^+ condensation does take place, we can use the expression obtained for $\pi_s^+\pi^-$ condensation to find the condensate energy.

The case of a finite system, when the electroneutrality condition is not satisfied, will be considered in Sec. VII.B in connection with the possible existence of "neutron" nuclei ($Z \ll N$). Numerical calculations of the π condensation energy in a neutron star under β-equilibrium condition, which is equivalent to allowance for the π_s^+ condensation, was carried out by Baym and Flowers (1974), Au and Baym (1974), and Au (1976).

3. Equation of state

The total energy density is

$$\mathcal{E}(n) = \mathcal{E}_N(n) - \tfrac{1}{2}\beta(n-n_c)^2\theta(n-n_c)\,, \tag{5.11}$$

where

$$\theta(x) = \begin{cases} 1, & x>0 \\ 0, & x<0\,. \end{cases}$$

At the critical point the quantity $d^2\mathcal{E}/dn^2$ has a discontinuity (second-order phase transition).

The pressure is connected with the energy density by the relation

$$P = n\frac{d\mathcal{E}}{dn} - \mathcal{E} = P_N - \tfrac{1}{2}\beta(n^2 - n_c^2)\theta(n-n_c).$$

The condition for the stability of the system is a positive compressibility

$$K = n(d^2\mathcal{E}/dn^2)\,.$$

Therefore, if the inequality

$$\beta > (d^2\mathcal{E}_N/dn^2)\,,$$

is satisfied at the phase transition point, then the nuclear matter will be compressed until the influence of the repulsion forces at short distances leads to satisfaction of the condition $d^2\mathcal{E}/dn^2 > 0$. Since the matter is unstable in the region where $d^2\mathcal{E}/dn^2 < 0$, a density discontinuity takes place. The jump in density is determined by the relation

$$P_N(n_c) = P_N(n_m) + P_\pi(n_m)\,,$$

where the $P_\pi = n(d\mathcal{E}_\pi/dn) - \mathcal{E}_\pi$ is the pion pressure ($P_\pi < 0$). A method of calculation of $\mathcal{E}(n)$ for large n is discussed in Sec. VII.B.

4. Modulation of density and spin density of nucleons

As shown in Sec. IV.C, the spin density $S(\mathbf{r})$ of nucleons is modulated in the condensate field with the wave vector k_0.

In Sec. IV.C this result was obtained in the Thomas–Fermi approximation, i.e., for the case $k_0 \ll 2p_F$. In this section we obtain the amplitude of the modulations of $n(\mathbf{r})$ and $S(\mathbf{r})$ without this assumption, and also take the influence of N_{33}^* resonance into account.

The ordinary quantum-mechanical approach to this calculation consists in finding the perturbation of the nucleon wave function in the condensate field, and then determining the changes of the average nucleon density.

It is much more convenient, however, to perform this calculation by a graphic method, i.e., with the aid of Green's functions. This approach makes it possible to take into account very simply the contribution of the nucleon correlations and the influence of transitions with production of the N_{33}^* resonance in the intermediate state. To this end, we introduce the nucleon density matrix ρ. The density matrix $\rho(\mathbf{r},\alpha;\mathbf{r}',\alpha')$, where α and α' are the spin indices, can be written in the form

$$\rho(\mathbf{r},\alpha;\mathbf{r}',\alpha') = \langle\psi_\alpha^+(\mathbf{r})\psi_{\alpha'}(\mathbf{r}')\rangle$$

The angle brackets denote the expectation value in the ground state of the system. From this expression and from the definition of the Green's function it follows that (see, for example, Migdal, 1967)

$$\rho(\mathbf{r},\mathbf{r}') = G(\mathbf{r},\mathbf{r}';\tau)\big|_{\tau\to -0} = \int\frac{d\epsilon}{2\pi i}\,G(\mathbf{r},\mathbf{r}';\epsilon),$$

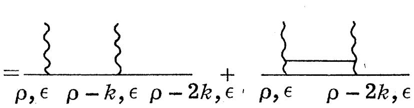
where $G(x,x';\epsilon)$ is the Fourier transform of $G(\mathbf{r},\mathbf{r}';\tau)$. It is convenient to change over in the calculations to the momentum representation and to introduce

$$\rho_{\lambda\lambda'} = (\Psi_\lambda^*(\mathbf{r})\rho(\mathbf{r},\alpha;\mathbf{r}',\alpha')\Psi_{\lambda'}(\mathbf{r}'))\,,$$

where $\psi_\lambda = \chi_\lambda e^{ipr}$, and χ is the spin function. The change of the density matrix in the condensate field is then

$$(\delta\rho)_{\lambda\lambda'} = \int\frac{d\epsilon}{2\pi i}\,[\tilde{G}_{\lambda\lambda'}(\epsilon) - G_\lambda(\epsilon)\delta_{\lambda\lambda'}]\,. \tag{5.12}$$

Here $\tilde{G}_{\lambda\lambda'}(\epsilon)$ is the exact Green function in an external field, in this case in the condensate field; $G_\lambda(\epsilon)$ is the Green's function at $\varphi=0$. Contributions to the scalar quantity $\delta n(\mathbf{r})$ can be made only by matrix elements corresponding to nucleon transitions with emission or absorption of an even number of condensate mesons. In an isotopically symmetrical field (4.28), in a medium with $N=Z$, such transitions are described by Green's functions of the type

$\tilde{G}(\mathbf{p},\epsilon;\mathbf{p}-2\mathbf{k},\epsilon) =$ 

$$= \frac{a^2f^2k^2}{2}\,G(\mathbf{p},\epsilon)\left\{G(\mathbf{p}-\mathbf{k},\epsilon)\;\frac{1}{[1+(g'p_F/p_0)\Phi(k/2p_F)]^2} - \frac{A_R^*(k,\omega)}{2}\right\}G(p-2k,\epsilon) \tag{5.13}$$

where $A_R^{(+)}$ is the resonant scattering amplitude given in Sec. III (page 41). The change of the nucleon density $\delta n(\mathbf{r})$ is expressed in terms of the change of the matrix $\delta\rho_{\lambda\lambda'}$ as follows

$$\delta n(\mathbf{r}) = \sum_{\lambda\lambda'} (\delta\rho)_{\lambda\lambda'} \Psi_\lambda^+(\mathbf{r})\Psi_{\lambda'}(\mathbf{r}) \ .$$

In our case only the terms with $\mathbf{p}' = \mathbf{p} \pm 2\mathbf{k}$ differ from zero, where $\mathbf{k}$ is the wave vector of the condensate.

Omitting the spin indices, we have

$$\delta n(\mathbf{r}) = \sum_{\mathbf{p}} \left[(\delta\mathbf{p})_{\mathbf{p},\mathbf{p}+2\mathbf{k}}\, e^{i2kz} + (\delta\rho)_{\mathbf{p},\mathbf{p}-2\mathbf{k}}\, e^{-i2kz}\right]$$

Confining ourselves to terms of order a^2, it is easy to obtain from (5.12) and (5.13), for the case of a standing wave

$$\delta n^{(n)}(\mathbf{r}) = \delta n^{(p)}(\mathbf{r}) = n\xi^2 \cos 2kz. \tag{5.14}$$

The quantity ξ^2 with allowance for the nucleon correlations is given by (Migdal, Markin, and Mishustin, 1976)

$$\xi^2 = \frac{3a^2}{v_F^2}\left[f^2\left[\Phi\left(\frac{k}{2p_F}\right) - \Phi\left(\frac{k}{p_F}\right)\right] \Big/ \left[1 + g^- \frac{p_F}{p_0}\, \Phi\left(\frac{k}{2p_F}\right)\right]^2\right.$$
$$\left. + \frac{k^2}{\omega_R}\, \frac{0.15}{1+0.23k^2}\, \Phi\left(\frac{k}{p_F}\right)\right], \tag{5.15}$$

where $\Phi(x)$ is given in Eq. (3.30).

At the critical values n_c and k_c and at the nuclear value $g^- \cong 1.6$ we have

$$\xi^2 = 0.5\,\frac{a^2}{v_F^2} \sim \frac{n-n_c}{n_c} \ .$$

We obtain in similar fashion the distribution of the spin density of the nucleons in the condensate field

$$\mathbf{S}(\mathbf{r}) = \sum_{\lambda\lambda'} (\delta\rho)_{\lambda\lambda'}\, \psi^+_{\lambda'}(\mathbf{r})\sigma\psi_\lambda(\mathbf{r}). \tag{5.16}$$

Contributions to $\mathbf{S}(\mathbf{r})$ are made only by the matrix elements $\rho_{\lambda\lambda'}$ corresponding to absorption or emission of an odd number of condensate pions. For the field (4.9) in the lowest order in a we obtain

$$S_z^{(n)}(\mathbf{r}) = fk\,\frac{2mp_F}{\pi^2}\left[\Phi\left(\frac{k}{2p_F}\right) \Big/ \left[1 + g^- \frac{p_F}{p_0}\, \Phi\left(\frac{k}{2p_F}\right)\right]\right] a\cos kz$$
$$= -S_z^{(p)}(\mathbf{r}).$$

In the case of $Z \ll N$, in a $\pi_S^+\pi^-$ condensate field of the standing-wave type (4.9), the nucleon density is also subject to modulation with a wave vector $2\mathbf{k}$ and an amplitude, near the critical point, equal to

$$\xi^2_{\pi\pm} \sim \frac{n-n_c}{n_c} \ .$$

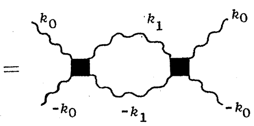
At the same time, in the field of a $\pi_S^+\pi^-$ condensate in the form of a running wave, the density of particles of each type remains uniform both at $Z=0$ and at $N=Z$, and the spin density is equal to zero. The reason is that no off-diagonal transitions $\mathbf{p} \to \mathbf{p} \pm 2\mathbf{k}$ exist in such a field, and the transitions with emission of one pion, for which $\mathbf{p} \to \mathbf{p} \pm \mathbf{k}$, are accompanied by a change in the isotopic index of the nucleon. It is easy to verify that in this case only the mean value of the operator $\sigma_z\tau_+$, which has the symmetry of the charged pion field, is different from zero (modulation of the spin-isospin density). A contribution to the modulation of the density and of the spin density is made by the field of the π^0 mesons (for which only a standing wave is possible).

5. Singularities in pion interaction near the critical point

As shown by A. M. Dyugaev (1975), a long range pion-pion interaction takes place near the phase transition point. As a result of this interaction, the constant of the effective 4-boson interaction Λ can reverse sign near the transition point at a density $n < n_c$. In this case, a first-order phase transition takes place. As we shall verify, this phase transition proceeds with a small jump of the condensate-field amplitude a and therefore differs very little from the second-order transition considered above.

We confine ourselves below to a consideration of symmetrical nuclear matter ($Z = N$).

Let us present the results of Dyugaev (1975). He considers first the $\pi\pi$ interaction diagram corresponding to exchange of two "dangerous" pions:

$\Lambda_0 =$ 

The quantity Λ_0 is defined by the expression

$$\Lambda_0 = \int \frac{i d\omega d^3k_1}{(2\pi)^4}\, \frac{\Lambda^2(k_0, -k_0; k_1, -k_1)}{[\omega^2 - \omega_{k_1}^2 - \Pi(k_1,\omega)]^2},\quad \omega_k^2 = 1 + k^2. \tag{5.17}$$

In this expression Λ should be taken to mean the sum of all the diagrams for which there are no two meson lines with small total momentum in any of the channels. This Λ will be local (δ-like) in the coordinate representation in terms of the differences of the coordinates of the input ends. At $k_1 = k_0$, the value of Λ coincides with $\Lambda(k_0)$ calculated in the preceding sections.

It is easily seen that near the transition point, the main contribution to the integral is made by the region $k_1 \approx k_0$, $\omega \to 0$. In order to verify this, we write down the expression for $\Pi(k,\omega)$ as $\omega \to 0$. In the real part of Π we can put $\omega = 0$:

$$Re\,\Pi(k, \omega \to 0) \simeq \Pi(k, 0)[1 + O(\omega^2)]$$

and the imaginary part is the result only of the pole term of Π

$$\mathrm{Im}\Pi(k, \omega \to 0) = \mathrm{Im}\Pi_s(k, \omega \to 0)$$
$$= -\frac{2m^* p_F}{\pi^2} f^2 k^2\, \frac{\pi|\omega|}{2kv_F(1+g^-)^2}$$

where m^* is the mass of the nucleon quasiparticle.

In the expression for ImΠ it was assumed for simplicity that $k \ll 2p_F$. Then the inverse pion propagator takes the form

$$D^{-1}(k, \omega \to 0) = -\tilde{\omega}^2(k) + i\chi|\omega|,$$

where

$$\tilde{\omega}^2(k) = \omega_k^2 + \Pi(k, 0) \underset{k \to k_0}{=} \omega_0^2 + \kappa \frac{(k^2 - k_0^2)^2}{4k_0^2} \tag{5.18}$$

and

$$\chi = \frac{2m^* p_F}{\pi^2} f^2 k^2 \frac{\pi}{2kv_F(1+g^-)^2} = f^2 \frac{m^{*2} k}{\pi(1+g^-)^2}\ .$$

Using the form $\tilde{\omega}^2(k)$ near k_0 (5.18), we obtain after integrating (5.17) with respect to ω

$$\Lambda_0 = -\frac{\Lambda^2}{(2\pi)^3} \frac{2}{\chi} \int \frac{4\pi k_0^2 dk}{\omega_0^2 + \kappa(k - k_0)^2}$$

$$= -\frac{\Lambda^2 (1+g^-)^2 k_0}{f^2 \pi m^{*2} \sqrt{\kappa}} \frac{1}{\omega_0} = -\Lambda \frac{\omega_1}{\omega_0} \tag{5.19}$$

At $m^* = 6.7$, $g^- = 1.6$, $k_0 = 2$, and $\kappa = 0.4$ we have $\omega_1 = 0.05\Lambda$.

Thus the quantity Λ_0 has a pole at the critical point, when ω_0 equals zero. Therefore diagrams of this type become significant even before the onset of the second-order phase transition, and must be taken into account in all orders in Λ.

An examination of an analogous diagram with momentum $\mathbf{q}$ in the horizontal channel would lead us to an expression of the type

$$\Lambda_0(q) \cong \Lambda \frac{\omega_1}{\sqrt{\omega_0^2 + \kappa q^2}}\ .$$

Changing over to the coordinate representation, we get

$$\Lambda_0(r) \sim -\frac{(\sin \kappa_0 r)^2}{r^3} \frac{\omega_1}{\omega_0} \exp\left(-\frac{2r\omega_0}{\sqrt{\kappa}}\right),\ r\omega_0 \gg 1.$$

This is indeed the long-range interaction referred to above.

The pion interaction energy in the coordinate representation takes the form

$$\tilde{H}_{\pi\pi} = \frac{1}{4} \int \Lambda(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \varphi(\mathbf{r}) \varphi(\mathbf{r}_1) \varphi(\mathbf{r}_2) \varphi(\mathbf{r}_3) d\mathbf{r} d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3\ .$$

The local interaction Λ corresponds to an energy term equal to

$$\frac{1}{4} \int \Lambda \varphi^4(\mathbf{r}) d\mathbf{r}\ .$$

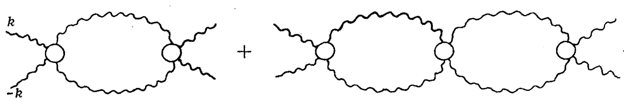
The long-range interaction diagrams correspond to a case in which two of the distances between the points $\mathbf{r}$, $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are small, and the two others large. There are three cases: any one of the three points $\mathbf{r}_1$, $\mathbf{r}_2$, or $\mathbf{r}_3$ lies close to the point $\mathbf{r}$, and the corresponding term in $\tilde{H}_{\pi\pi}$ takes the form

$$\frac{3}{4} \int \Lambda_1(\mathbf{r}_1 - \mathbf{r}_2) \varphi^2(\mathbf{r}_1) \varphi^2(\mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2 \approx \frac{3}{4} \Lambda_1 \varphi^2 \varphi^2 V,$$

where

$$\Lambda_1 = \int \Lambda_1(\mathbf{r}) d\mathbf{r} = \Lambda_1(q = 0).$$

The quantity Λ_1 is determined by the sum of the long-range diagrams

$\Lambda_1 =$  $+ \cdots = \Lambda_0 + \frac{\Lambda_0^2}{\Lambda} + \frac{\Lambda_0^3}{\Lambda^2} + \cdots$

$$= \frac{\Lambda_0}{1 - \Lambda_0/\Lambda}$$

Using (5.19), we obtain

$$\Lambda_1 = -\frac{\omega_1/\omega_0}{1 - \omega_1/\omega_0}\ .$$

Taking the field φ in the form $\varphi = a\sqrt{2}\sin kz$ and substituting in $\tilde{H}_{\pi\pi}$, we obtain the expression $\tilde{H}_{\pi\pi} = \tilde{\Lambda} a^4/4$, where $\tilde{\Lambda}$ is equal to

$$\tilde{\Lambda} = \frac{3}{2} \Lambda \frac{1 - \omega_1/\omega_0}{1 + \omega_1/\omega_0}\ . \tag{5.20}$$

It follows from this expression that $\tilde{\Lambda}$ reverses sign at $\omega_0 = \omega_1$, and with further increase of the density (decrease of ω_0) the system becomes unstable with respect to a first-order phase transition.

The transition point is determined by the condition $d\mathcal{E}(a^2)/da^2 = 0$. Writing down the energy in the form

$$\mathcal{E}(a^2) = \frac{\omega_0^2}{2} a^2 + \frac{\tilde{\Lambda}}{4} a^4 + \frac{\mu}{12} a^6,$$

where $0 < \mu \sim 1$, we get

$$a^4 + \frac{2\tilde{\Lambda}}{\mu} a^2 + \frac{2\omega_0^2}{\mu} = 0,$$

hence

$$a^2 = -\frac{\tilde{\Lambda}}{\mu} + \left(\frac{\tilde{\Lambda}^2}{\mu^2} - \frac{2\omega_0^2}{\mu}\right)^{1/2}.$$

At $\tilde{\Lambda} = -\omega_0\sqrt{2\mu}$, a real solution is obtained with $a^2 = \omega_0\sqrt{2/\mu}$, which corresponds to a minimum of $\mathcal{E}(a^2)$ (the minus sign in front of the square root corresponds to a maximum of $\mathcal{E}(a^2)$, i.e., to the unstable branch).

As shown by Dyugaev (1975), allowance for the quantum fluctuations of the meson field leads to singularities not only in Λ but also in other quantities. Thus, for example near the critical point, the quantum correction to the self energy of the nucleon becomes singular:

$$\left.\frac{\partial \Sigma}{\partial \epsilon}\right|_{\substack{|p| = p_F \\ \epsilon \to \epsilon_F}} \sim \frac{1}{\omega_0}\ .$$

It is clear that for a correct description of the vicinity of the transition point it is necessary to separate in succession the "dangerous" diagrams not only in Λ, but also in all the quantities having a singularity at $\omega_0=0$.

B. A developed condensate

In this section we investigate an exactly solvable model of π condensation, in which a condensate field in the form of a running wave (4.8) is assumed. The energy of this condensate can be obtained without assuming proximity to the critical point. We consider in detail the high-frequency approximation ($\omega \gg ku_F$) within the framework of which we can obtain analytically the critical parameters and the energy of the π condensate.

We discuss the model of a strongly developed condensate, proposed by Campbell, Dashen, and Manassah (1975) and by Baym *et al.* (1975).

1. Nucleon energy in a condensate field

We consider a $\pi_s^+\pi^-$ condensate field in the form of a running wave

$$\varphi(\mathbf{r},t)=\frac{a}{\sqrt{2}}\,e^{-i\omega t+i\mathbf{k}\mathbf{r}}=\frac{\varphi_1-i\varphi_2}{\sqrt{2}}\ ,$$

where a is real.

In such a field, all the π^- mesons will occupy a single state, with frequency ω and momentum $\mathbf{k}$, and all the π_s^+ mesons populate a state with frequency $-\omega$ and momentum $-\mathbf{k}$. The $N\pi N$ interaction vertex $\Gamma_0^\alpha=i\sqrt{2}f(\mathbf{k}\cdot\sigma)\tau^{+\alpha}$ transforms a neutron with momentum p into a proton with momentum $\mathbf{p}-\mathbf{k}$. Inasmuch as only these two nucleon states take part in the processes, the problem can be solved exactly. Writing down the system of two equations relating the proton and neutron states, we easily obtain an expression for the energies of the new neutrons and protons (Sawyer and Scalapino, 1972). We have

$$\tilde\epsilon^{(n)}(\mathrm{p})=\frac{\epsilon_p+\epsilon_{|\mathbf{p}+\mathbf{k}|}+\omega}{2}+\frac{\epsilon_p-\epsilon_{|\mathbf{p}-\mathbf{k}|}-\omega}{2}\times\left(1+\frac{4f^2k^2a^2}{(\epsilon_p-\epsilon_{|\mathbf{p}-\mathbf{k}|}-\omega)^2}\right)^{1/2},\qquad(5.21)$$

$$\tilde\epsilon^{(p)}(\mathrm{p})=\frac{\epsilon_p+\epsilon_{|\mathbf{p}-\mathbf{k}|}-\omega}{2}+\frac{\epsilon_p-\epsilon_{|\mathbf{p}+\mathbf{k}|}+\omega}{2}\times\left(1+\frac{4f^2k^2a^2}{(\epsilon_p-\epsilon_{|\mathbf{p}+\mathbf{k}|}+\omega)^2}\right)^{1/2}.\qquad(5.22)$$

The signs in front of the square roots are chosen such that the $\tilde\epsilon^{(n,p)}(\mathbf{p})$ coincides with the free particle energy $\epsilon^{(n,p)}=p^2/2m$ as $a\to 0$. Since the inequality $\omega_{\pi^-}>\tilde\epsilon_F^{(n)}$ holds for this model, the filling of the new proton states is not possible. Indeed, as shown by Migdal (1973), the transformation of a small number of neutrons into protons and exact excitations with quantum numbers of the π^- mesons ($n\to p+\pi^-$) leads to a change in the system energy

$$\delta\mathcal{E}=(\omega_{\pi^-}-\tilde\epsilon_F^{(n)})\tilde\nu_p,$$

where $\tilde\nu_p$ is the density of the new protons, and is equal to the density of the π^- mesons. The filling of the new proton states, which should be accompanied by formation of a π^- condensate, would be possible only under the condition $\omega_{\pi^-}<\tilde\epsilon_F^{(n)}$, which is shown by Migdal (1973) not to be satisfied up to very high neutron matter densities.

Thus, in $\pi_s^+\pi^-$ condensation, only the new neutron states are filled. The Fermi surface $\tilde S_F^{(n)}$ is then no longer a sphere as in the normal phase. Its equation is obtained from the condition

$$\tilde\epsilon^{(n)}(\mathrm{p})\big|_{\tilde S_F^{(n)}}=\tilde\epsilon_F^{(n)}\,.\qquad(5.23)$$

The value of the exact Fermi energy $\tilde\epsilon_F^{(n)}$ is determined by the requirement that the total number of nucleons remain unchanged after redistribution in the condensate field. The kinetic energy of the nucleons in the condensate field, which enters into the effective Lagrangian (4.4), is obtained from Eq. (4.22).

The problem of determining the system energy in the presence of a $\pi_s^+\pi^-$ condensate in the form of a traveling wave can be solved without assuming small amplitude of the condensate field. The calculations are carried out in the following manner. Equation (5.21) is used to obtain the exact neutron energy $\tilde\epsilon^{(n)}(p)$ as a function of the parameters n, ω, k, and a^2. This energy is then substituted in Eq. (4.22) for $\tilde{\mathcal{E}}N$. Equation (4.4) is used next to construct the effective condensate-field Lagrangian $\tilde{\mathfrak{L}}(a^2,k,\omega)$, from which the system of equations (4.10) and (4.11) is obtained for the determination of a^2, ω, and k. Substitution of these parameters in (4.5) yields the total system energy $\tilde{\mathcal{E}}(n)$ as a function of the nucleon density n.

Determining the condensate energy density entails computational difficulties and calls for the use of numerical methods even in the simplified theory variants considered above (Baym and Flowers, 1974; Au and Baym, 1974).

2. The high-frequency approximation: $w\gg kv_F$

To illustrate the method of calculating $\mathcal{E}_\pi(n)$ it is useful to consider a less approximate variant of the theory, but one that admits of a simple analytic solution—the high-frequency approximation (Migdal, Markin, and Mishustin, 1976; Baym *et al.*, 1975).

In this approximation, the frequency ω of the condensate field is assumed to be large in comparison with the difference of the kinetic energies of the nucleons $\epsilon_p-\epsilon_{|\mathbf{p}-\mathbf{k}|}$, i.e., $\omega\gg kv_F^{(n)}$. From (5.21) and (5.22) we obtain the single-particle energies of the "new" neutrons and protons at $\omega\gg\epsilon_p-\epsilon_{|\mathbf{p}-\mathbf{k}|}$:

$$\tilde\epsilon^{(n)}(p)=\epsilon_p-\frac{\omega}{2}(\gamma-1),\quad \tilde\epsilon^{(p)}(p)=\epsilon_p+\frac{\omega}{2}(\gamma+1)$$

$$\gamma=\left(1+\frac{4f^2k^2a^2}{\omega^2}\right)^{1/2}.$$

It is clear that in this approximation the neutron Fermi sea remains spherically symmetrical, and the new Fermi momentum $\tilde p_F^{(n)}$, by virtue of the conservation of the total number of nucleons, coincides with the Fermi momentum $p_F^{(n)}$ at $a=0$ and at the same total density:

$$\tilde p_F^{(n)}=(2m\tilde\epsilon_F^{(n)}+m\omega(\gamma-1))^{1/2}=p_F^{(n)}.$$

From (4.4), taking into account the fact that at $\omega>\tilde\epsilon_F^{(n)}$

there is no filling of new proton states, we obtain

$$\tilde{\mathfrak{L}}_\pi = \tilde{\mathfrak{L}}(a) - \tilde{\mathfrak{L}}(0) = \frac{n\omega}{2}(\gamma - 1) + (\omega^2 - \omega_k^2)\frac{a^2}{2}.$$

Varying $\tilde{\mathfrak{L}}_\pi$ with respect to a^2, we obtain an equation of motion that connects the frequency and amplitude of the condensate field

$$2\frac{\partial \tilde{\mathfrak{L}}_\pi}{\partial (a^2)} = \omega^2 - \omega_k^2 + \frac{2nf^2k^2}{\omega\gamma} = 0. \tag{5.24}$$

The electroneutrality conditions in the absence of an electric current (4.11) assume the form

$$\frac{\partial \tilde{\mathfrak{L}}_\pi}{\partial \omega} = 2\omega\frac{a^2}{2} - \frac{2nf^2k^2}{\omega^2\gamma} + \frac{n}{2}(\gamma - 1) = \omega a^2\left[1 + \frac{\omega^2 - \omega_k^2}{\omega^2(1+\gamma)}\right] = 0, \tag{5.25}$$

$$\frac{\partial \tilde{\mathfrak{L}}_\pi}{\partial k} = -2k\left[1 - \frac{2nf^2}{\omega\gamma}\right]\frac{a^2}{2} = -ka^2\frac{\omega^2 - 1}{k^2} = 0. \tag{5.26}$$

In the limit as $a \to 0(\gamma = 1)$, the system (5.25) and (5.26) coincides with the conditions that determine the critical parameters of the $\pi_s^+\pi^-$ condensation $n_c^\pm$, $k_c^\pm$, and ω_c

$$\omega^2 - \omega_k^2 - \Pi^{(-)}(k, \omega) = 0; \quad 2k + \frac{\partial \Pi^{(-)}(k, \omega)}{\partial k} = 0;$$

$$2\omega - \frac{\partial \Pi^{(-)}(k, \omega)}{\partial \omega} = 0. \tag{5.27}$$

Here $\Pi^{(-)}(k, \omega)$ is the polarization operator of the π^- meson in the nucleon medium. In the considered case $\Pi^{(-)}(k, \omega)$ coincides with the pole part of the polarization operator of the π^- meson in a pure neutron medium (without allowance for the nucleon correlations): $\Pi_p^{(-)}(k, \omega) \gg kv_F^{(n)} = -(2nf^2k^2/\omega)$. From the system (5.27) we obtain

$$n_c^\pm = \frac{1}{2f^2} \approx n_0, \omega_c = 1, k_c^\pm = \sqrt{2}. \tag{5.28}$$

At $n > n_c^\pm$, using Eqs. (5.24) (5.25), and (5.26) without assuming that we are close to the critical point, we obtain the parameters of the $\pi_s^+\pi^-$ condensate:

$$a^2 = \frac{1}{4f^2}\left(\frac{n}{n_c^\pm} - 1\right), \quad \omega = \omega_c = 1, \quad k_c^\pm = \left(1 + \frac{n}{n_c^\pm}\right)^{1/2}. \tag{5.29}$$

Substituting the values of (5.29) in Eq. (5.1) for the total energy of the system, we obtain the energy gain connected with the formation of the new phase:

$$\mathcal{E}^{(\pi)}(n) = -\frac{(n - n_c^\pm)^2}{4n_c^\pm}, \quad n \geq n_c^\pm. \tag{5.30}$$

It must be emphasized that Eq. (5.30) for $\mathcal{E}^{(\pi)}(n)$ is only a rough estimate. That this estimate is unsatisfactory is obvious even from the fact that at high densities $\mathcal{E}^{(\pi)} \propto (-n^2)$, whereas the kinetic energy of the nucleons increases like $n^{5/3}$, so that a system with such a condensate would be unstable with respect to a limited compression ("collapse"). This is connected primarily with the fact that the approximation used by us is quite crude. As seen from the presented values of $n_c^\pm$, ω_c, and $k_c^\pm$, even at the critical point the expansion parameter is not small: $(kv_F^{(n)}/\omega)^2 \approx 1/4$, and as the density is increased the approximation becomes even rougher. The main cause of the instability, however, is of course the fact that in the derivation of (5.30) we did not take into account the strong repulsion of the nucleons at short distances ("cores") and the vacuum $\pi\pi$ interaction, which limits the amplitude of the condensate field at high densities.

Even though it is crude, the high-frequency approximation turns out to be useful in tracing qualitatively the influence exerted on the parameters of the $\pi_s^+\pi^-$ condensate by such important factors as nucleon correlations, the N^* resonance, the S-wave πN interaction, and the vacuum $\pi\pi$ interaction (Baym *et al.*, 1975).

3. Model of a developed condensate

Calculation of the energy of a strongly developed π condensate with allowance for the N^* resonance and for the nucleon correlations was performed by Campbell, Dashen, and Manassah (1975) and by Baym *et al.* (1975), who used the chiral-symmetry approximation.

An expression was obtained for the π-meson field Lagrangian with allowance of the N^* resonance and the nucleon correlations. Without taking the N^* resonance into account, the same result can be obtained from the Weinberg Lagrangian (Au and Baym, 1974). We derive first the Lagrangian of the system without taking the correlations and the N^* resonance into account.

Substituting (4.19) in (4.18) and changing over to the nonrelativistic limit, we obtain

$$\mathfrak{L} = \sum_p \Psi_p^+\left[w - \frac{1}{2m}\left(\mathbf{p} - \tau_3\frac{\mathbf{k}}{2}\cos\theta\right)^2 - \tau_3\frac{\omega}{2}\cos\theta + \frac{1}{2}g_A(\sigma\mathbf{k})\tau_2\sin\theta\right]\Psi_p - \frac{F^2}{8}\left[(k^2 - \omega^2)\sin^2\theta + 4m_\pi^2\sin^2\frac{\theta}{2}\right], \tag{5.31}$$

where $g_A = fF$ is the axial weak-interaction constant.

Here Ψ_p is the Fourier transform of the nucleon-field operator $\Psi(\mathbf{r})$ with a momentum shifted by $-\mathbf{k}\tau_3/2$:

$$\Psi_p = \int d^3r e^{-i(\mathbf{p} - \frac{1}{2}\mathbf{k}\tau_3)\mathbf{r}}\Psi(\mathbf{r}).$$

In isospin space, $\Psi_\mathbf{p}$ is a spinor $(p_{\mathbf{p}+\mathbf{k}/2} n_{\mathbf{p}+\mathbf{k}/2})$, where $p_\mathbf{p}$ and $n_\mathbf{p}$ are the annihilation operators for the bare proton and the neutron with momentum $\mathbf{p}$. The terms proportional to π_3 in (5.31) describe the S-wave πN interaction, and the term proportional to τ_2 corresponds to P scattering.

Diagonalizing the nucleon terms in (5.31), we obtain the exact single-particle energies of the nucleons in the condensate field

$$\epsilon^\pm = \frac{p^2}{2m} + \frac{k^2}{8m}\cos^2\theta \pm \frac{1}{2}\left[\left(\omega - \frac{\mathbf{pk}}{m}\right)^2\cos^2\theta + g_A^2 k^2\sin^2\theta\right]^{1/2}. \tag{5.32}$$

The eigenstates corresponding to the two signs (±) are linear combinations of the neutron and proton states

$$|N^+\rangle = \cos\frac{\chi}{2}|p_{\mathbf{p}-\frac{\mathbf{k}}{2}}\rangle + i(\sigma\mathbf{k})\sin\frac{\chi}{2}|n_{\mathbf{p}+\frac{\mathbf{k}}{2}}\rangle, \tag{5.33}$$

$$|N^-\rangle = \cos\frac{\chi}{2}|n_{\mathbf{p}+\frac{\mathbf{k}}{2}}\rangle + i(\sigma\mathbf{k})\sin\frac{\chi}{2}|p_{\mathbf{p}-\frac{\mathbf{k}}{2}}\rangle, \quad (5.34)$$

where

$$\tan\chi = \frac{g_A k\tan\theta}{\omega - \mathbf{pk}/m}.$$

To find the effective Lagrangian and the energy of the system, it is necessary to find the sum of the exact energies of the nucleons over the new Fermi sea. This problem is particularly easy to solve in the case $\theta=\pi/2$ (limiting condensate field $a=F$), which describes condensation as $n\to\infty$. It is easy to verify that in this case only states of the type N^- are filled, which in this case ($\chi=\pi/2$) contain a neutron and a proton with equal weights.

It follows therefore that at $\theta=\pi/2$, regardless of the relation between N and Z in the normal phase, the number of bare neutrons and protons in the system is equal, and the charge density of the bare nucleons is equal to $n/2$. Since at $\theta=\pi/2$ there is no p_z dependence of $\epsilon_p^{\pm}$, the Fermi surface is spherically symmetrical and the calculations are trivial. As a result, the expression for the effective Lagrangian takes the form

$$\tilde{\mathcal{L}}_\pi = -\frac{1}{2}n(\omega - g_A k) - \frac{F^2}{8}[(k^2-\omega^2)+2m_\pi^2]. \quad (5.35)$$

The condition $\partial\tilde{\mathcal{L}}/\partial k_\mu = 0$ for the absence of the 4-current yields

$$\omega = 2n/F^2,\ k = 2g_A n/F^2. \quad (5.36)$$

The system energy change due to the π condensate then takes the form

$$\mathcal{E}_\pi\left(n,\theta=\frac{\pi}{2}\right) = -\frac{(g_A^2-1)}{2F^2}n^2 + \frac{F^2m_\pi^2}{4}. \quad (5.37)$$

This expression is valid at $n\gg n_c$.

One of the most important results of Campbell, Dashen, and Manassah (1975) is the extension of the σ model by inclusion of the isobar N_{33}^* (1232), which is regarded as an additional "elementary" particle. This has already enabled us above, in Sec. IV.B, to find the contribution of the N^* resonance to the function Λ and the energy of the condensate near the critical point. We shall now demonstrate how N^* influences the energy of the developed condensate.

The idea of including the N^* resonance is based on the following: The Lagrangian of the interaction of the nucleons with the pion field can be expressed in terms of vector and axial currents.

$$H' = k^\mu(V_\mu\cos\theta - A_\mu\sin\theta).$$

Added to these currents is the contribution of the N^* particles, calculated with the aid of the $SU(4)$ quark model. The matrix elements H' over the states N^{*++}, N^{*+}, p, n, N^{*0}, and N^{*-} are given in Section IV.B [Eq. (4.25)].

At $\theta=\pi/2$, the matrix (4.25) can be diagonalized analytically. Just as before, in this case one Fermi sea is filled, but now the baryon quasiparticles are superpositions of states N and N^*. The charge density of the baryon subsystem remains equal to $n/2$, where n is the total baryon density. For the condensate energy in this case we obtain the expression

$$\mathcal{E}_\pi\left(n,\theta=\frac{\pi}{2}\right) = \frac{\Delta}{3}n - \frac{n^2}{2F^2}\left(\frac{81}{25}g_A^2 - 1\right) - \frac{25}{54}\left(\frac{\Delta}{3}\right)^2\frac{F^2}{g_A^2} + \frac{F^2m_\pi^2}{4} \quad (5.38)$$

We recall $\Delta = m_{N^*} - m_N$. For simplicity we have confined ourselves in the derivation of (5.38) to the first two terms of the expansion of the parameter $\Delta/g_A k$, which is small at high density.

The next improvement of the developed-condensate model consists in taking the nucleon correlations into account. A method that can be used for this purpose was proposed by Baym *et al.* (1975) and by Weise and Brown (1975). The result of their calculation reduces to the fact that in the expression for the condensate energy it is necessary to renormalize the axial coupling constant

$$g_A \to g_A^* = g_A(1-\gamma)^{1/2}, \quad (5.39)$$

where γ is connected with the Fermi-liquid spin–spin interaction constant g^- by the relation $g^- = f^2(2mp_0/\pi^2)\gamma$ ($p_0=1.92$ is the Fermi momentum at normal nuclear density).

We note that the simple result (5.38) was obtained under the assumption that, up to Clebsch–Gordan coefficients, the local amplitudes of the NN, NN^*, N^*N^* interactions in a nucleon medium are the same. This assumption, however, is entirely arbitrary, inasmuch as at the present time there is no direct experimental information of the NN^* and N^*N^* interaction. As can be assessed from the experiments of Mountz *et al.* (1975) on the (pp, nN^*) reaction with large momentum transfer, the local NN^* interaction seems to be much weaker than the NN interaction. While an analysis of the spectral data of the π atom seems to offer evidence of a noticeable NN^* interaction. On the other hand, no account was taken above of the suppression of the vertices at large pion momenta, which leads to a decrease of the condensate energy. For the time being it is impossible to take consistent account of all these effects consistently, and we shall therefore use for our estimate Eq. (5.38) with g_A^* from (5.39).

In concluding this section, we present an expression for the condensate energy in the case when the system is not electrically neutral. This will be useful to us later on in the analysis of anomalous nuclei. We first connect the frequency (chemical potential) of the pions with the total charge of the system, Z. Considering bare particles the total charge is the sum of the bare baryon charge (equal to $A/2$) and the bare pion charge (which is negative). When discussing the excitations there are no charged baryon quasiparticles and the total charge equals the charge of the pion excitations (which is positive). Hence from (4.6) one finds

$$\frac{j_0^\pi}{e} \equiv n_{\pi^+} - n_{\pi^-} = \nu n = -\frac{\partial\mathcal{L}}{\partial\omega},$$

where $\nu = Z/A$.

Using (5.35) one obtains

$$\omega = (2n/F^2)(1-2\nu),\ \nu\equiv Z/A.$$

Now, knowing the effective Lagrangian and its connection with the energy of the system (5.38), we easily obtain

$$\mathcal{E}_\pi\left(n,\theta=\frac{\pi}{2}\right)=\frac{\Delta}{3}n-\frac{81}{50}\frac{g_A^2(1-\gamma)}{F^2}n^2-\frac{25}{54}\left(\frac{\Delta}{3}\right)^2\frac{F^2}{g_A^2(1-\gamma)}+\frac{F^2m_\pi^2}{4}+\frac{n^2}{2F^2}(1-2\nu)^2. \quad (5.40)$$

At $\nu=0$ this expression coincides with (5.38). From (5.40) we see that, at a given γ, the lowest energy is possessed by the system having a charge $A/2$, i.e., in the language of bare particles, by a system with a static electrically neutral condensate. In the language of pion excitations, this charge is equal to the difference between the charges of the π_s^+ and π^- quasiparticles.

VI. DOES A CONDENSATE EXIST IN ORDINARY NUCLEI?

So far, pion excitations and condensation have been considered in infinite homogeneous nuclear matter. In order to be able to apply these results to nuclei, it is necessary to discuss the pion condensation in a finite system. This task is facilitated by the fact that the instability of the pion excitation sets in at a wave vector $k\cong p_F\gg 1/R$. Therefore, even in middle nuclei, the influence of the nuclear surface on the condensation is not very large. We consider below condensation in a sufficiently large spherical system. It becomes necessary to solve a nonlinear equation for the pion condensate field in a nucleus in coordinate representation. Outside the nucleus, the field goes over into the solution of the free Klein–Gordon–Fock equation. An approximate analytic solution, valid near the critical point, is obtained. Inside the volume of the nucleus, a solution is obtained that agrees with the case of an infinite system and goes over, at a thickness $\delta\ll R$, into the value of the field on the surface of the nucleus. The volume part of the condensate energy coincides with the energy of the homogeneous system. The resultant surface condensate energy turns out to be proportional to the cross section area perpendicular to the direction of the wave vector of the condensate field, favoring the orientation of the wave vector along the largest axis of deformation of the nucleus. The condensation contributes to elongation of the nucleus and can lead to shape isomerism. The presence of a layered structure could lead to the appearance of rotational levels in spherical nuclei. Inasmuch as the condensate field violates translational and rotational symmetries, Goldstone oscillation modes set in. The lowest frequency corresponds to oscillations of the wave vector around the axis of elongation. The mean value of the condensate field in a nucleus with zero angular momentum is equal to zero. Only the mean-squared condensate field differs from zero. Therefore condensation would not lead to violation of parity conservation in nuclei.

To establish the existence of a condensate in nuclei, it is necessary first to ascertain whether this assumption would contradict the known nuclear facts, and then to indicate possible experiments capable of proving or refuting the existence of condensate in nuclei.

Even in the case when there is no condensate in the ground state of the nuclei, it is of great interest to determine the parameters characterizing the closeness of the nuclei to condensation.

It turns out that the presently known experiments do not contradict the existence of a condensate, and at least indicate that the nuclei are close to condensation.

Valuable information concerning the parameters of πN and NN interactions can be obtained from the positions of the levels 0^-, 1^+, and 2^-, which are strongly influenced by one-pion exchange. Information on the "resonant" part of the polarization operator (more accurately, the non-pole P-wave part) can be obtained from an analysis of data on the π atom.

Anomalously large values of the matrix elements of certain l-forbidden $M1$ transitions can be naturally explained as being due to the closeness of the nuclei to condensation. Indeed, the probability of an l-forbidden transition contains a term that has a pole at $|\omega_0^2|=0$. A rough estimate yields $|\omega_0^2|\sim 0.2$, but this still leaves open the question whether this value corresponds to a condensate or whether $\omega_0^2>0$ and the system is close to condensation.

In the scattering of electrons by nuclei, anomalies were observed and were explained by assuming a layered structure of the proton density. If a condensate exists, then it should lead to a layered structure having just the period which is observed in experiment (an anomalous behavior of the form factor is observed at a momentum transfer $q=3\ F^{-1}\cong 2k_0$).

Experiments on the scattering of nucleons and pions by nuclei, and also on the large-angle electron scattering, when the spin density of the nucleons acquires a structure, can yield valuable information on the parameters that determine the closeness of the nuclei to condensation. Some other experiments which can determine nuclear interaction parameters and can show the proximity of nuclei to the critical point are discussed.

A. Pion condensation in a finite system

An equation describing the condensate field in a finite system is solved. The influence of condensation on the deformation and on the moments of inertia of the nuclei is discussed. It is shown that condensation contributes to elongation of the nuclei and could lead to shape isomerism. Goldstone low-frequency modes produced as a result of condensation are investigated. The frequencies of the lowest oscillations are estimated. It is shown that condensation in a finite system does not violate parity conservation.

1. The condensate field in a finite system

The equation for the pion frequency, without allowance for condensation, at $\omega\ll 1$ and for $N\cong Z$, when $\Pi(k,\omega)$ is an even function of ω, can be written in the form (see Sec. III.B.1.

$$\left(1-\frac{\partial\Pi}{\partial\omega^2}\right)_{\omega=0}\omega^2=\tilde{\omega}_k^2,$$

where

$$\tilde{\omega}_k^2=1+k^2+\Pi(k,0),$$

$$\left(1-\frac{\partial\Pi}{\partial\omega^2}\right)_{\omega=0}>0.$$

These relations are determined by the harmonic part of the Lagrange function. The anharmonic term for weak fields is given by

$$\tilde{\mathfrak{L}}_\pi = \frac{\Lambda(k)}{4}\varphi^4.$$

In sufficiently weak fields and at low frequencies ($\omega \ll 1$), the effective pion Lagrangian can therefore be written in the form

$$\tilde{\mathfrak{L}}_\pi = \frac{(1-(\partial\Pi/\partial\omega^2))\dot{\varphi}^2 - \varphi\tilde{\omega}^2(k)\varphi}{2} - \frac{\varphi^2\Lambda(k)\varphi^2}{4}. \qquad (6.1)$$

The isotopic indices have been left out for simplicity.

In the coordinate representation, to which we have changed over, $\Lambda(k)$ and $\tilde{\omega}(k)$ should be taken to mean differential operators obtained after the substitution $k \rightarrow (1/i)\nabla$.

The Lagrange function (6.1) yields the following equation for φ.

$$\left(1-\frac{\partial\Pi}{\partial\omega^2}\right)\ddot{\varphi} + \tilde{\omega}^2(k)\varphi + \Lambda(k)\varphi^3 = 0 \qquad (6.2)$$

We consider a finite system with dimensions $R \gg 1/k_0$, where k_0 is the wave vector corresponding to the condensate field of an infinite system. Then the Fourier transform of the field $\varphi(r,t)$ in the finite system will contain the wave vectors $k = k_0 + \Delta k$, where $\Delta k \sim 1/R$. Therefore the functions $\tilde{\omega}^2(k)$ and $\Lambda(k)$ can be expanded about $k = k_0$. For $\Lambda(k)$, the zeroth term $\Lambda(k) = \Lambda(k_0) \equiv \lambda$ is sufficient, while $\tilde{\omega}^2(k)$ can be represented in the form

$$\tilde{\omega}^2 = -\omega_0^2 + \kappa\frac{(k^2-k_0^2)^2}{4k_0^2};\ \omega_0^2 > 0, \kappa > 0. \qquad (6.3)$$

For a static condensate field (we assume $N \cong Z$ and consequently the condensate is static) we obtain from (6.2)

$$\tilde{\omega}^2(k)\varphi_0 + \lambda\varphi_0^3 = 0. \qquad (6.4a)$$

Using (6.3) we obtain (Migdal, Kirichenko, and Sorokin, 1974)

$$(\Delta+1)^2 f(\mathbf{r}') = \epsilon\left(f(\mathbf{r}') - \frac{4}{3}f^3(\mathbf{r}')\right), \qquad (6.4b)$$

where

$$\epsilon = \frac{4\omega_0^2}{\kappa k_0^2};\ f = \frac{\varphi_0}{a_0};\ a_0^2 = \frac{4}{3}\frac{\omega_0^2}{\lambda};\ r' = k_0 r.$$

We have seen that a_0 is the amplitude of a periodic condensate field in an infinite medium. Near the critical point we have $\epsilon \ll 1$. We shall use below the smallness of this quantity for an approximate solution of Eq. (6.4b).

We are interested in those solutions of (6.4b) which have a constant amplitude a_0 far from the surface of the nucleus (only such solutions make it possible to minimize the energy). We consider two possibilities—spherical and plane layers $f = \sin\psi_0$, where $\psi_0 = r'$ and z', respectively. Then

$$f = a\sin(\psi_0 + \chi) \qquad (6.5)$$

where a and χ vary appreciably over distances r on the order of the $\delta \sim (1/\sqrt{\epsilon})(1/k_0)$. We see that the depth over which the amplitude of the condensate field varies is large in comparison with the distance $r_0 \sim 1/k_0$, between the nucleons, and consequently the character of the behavior of the density at the edge of the nucleus has little effect on the solution. It can be assumed for simplicity that the nucleus has a sharp boundary. Substituting (6.5) in (6.4b) and using the smallness of ϵ, we can easily obtain second-order equations for the quantities a and χ. The boundary conditions for the equation can be obtained by joining together our solution with the solution of the Klein–Gordon–Fock equation in a vacuum. It is easy to show that the slowly varying phase χ is inessential for the calculation of the energy, while the amplitude can be obtained with sufficient accuracy from the condition $a = 0$ on the surface of the system. For the system energy we obtain after integrating by part

$$E = -\frac{4\omega_0^4}{g\lambda}\int f^4 d\mathbf{r}.$$

In a spherical nucleus with radius $R \gg \delta$, Eq. (6.4) has a spherically symmetrical solution at $\epsilon \ll 1$ in the form

$$f(r') = \mathrm{th}\frac{(k_0 R - r')}{k_0\delta}\ \sin(r'+\chi);\ \delta = 2\sqrt{2}\,\frac{1}{\sqrt{\epsilon}\,k_0} = (2\kappa)^{1/2}\frac{1}{|\omega_0|}. \qquad (6.6)$$

It is easy to verify that this solution satisfies Eq. (6.4b) accurate to terms $\sim\sqrt{\epsilon}$. It corresponds to a spherical layered structure of the condensate. The solution (6.6) corresponds to a condensation energy

$$E = \mathcal{E}_0 V + \mathcal{E}_s S$$

where $\mathcal{E}_0$ is the volume density of the energy,

$$\mathcal{E}_0 = -\omega_0^4/6\lambda, \qquad (6.7)$$

V is the volume of the nucleus, S is the area of the surface, and $\mathcal{E}_s$ is the surface energy

$$\mathcal{E}_s = \tfrac{4}{3}\,\delta\,|\mathcal{E}_0|. \qquad (6.8)$$

The solution that corresponds to plane layers in the interior is of the form

$$f = \mathrm{th}\frac{(k_0^2R^2-\rho'^2)^{1/2} - |z'|}{k_0\delta}\ \sin(z'+\kappa) \qquad (6.9)$$

at

$$\frac{r}{R} \gg \sqrt{\epsilon},\ (R^2-\rho^2)^{1/2} \gg \delta,\ \rho' = k_0(x'^2+y'^2)^{1/2}.$$

The energy corresponding to (6.9) is

$$E = \mathcal{E}_0 V + 2\mathcal{E}_s S_e, \qquad (6.10)$$

where S_e is the area of the equatorial section of the nucleus. Expressions (6.7) and (6.10) are valid for spheroidal nuclei.

According to (6.10), in a deformed nucleus, the minimum surface energy corresponds to orientation of the wave vector of the layers along the major axis.

It is seen that in sufficiently large systems ($R \gg \delta$) the solution (6.9) with flat layers is realized, since it corresponds to the smaller surface energy.

The contribution of the condensate to the volume energy does not depend on the shape of the layers.

For light nuclei ($R \sim \delta$), this "macroscopic" approach is not suitable. As shown by Sapershtein *et al.* (1975), in this case the pion instability (as a function of the nucleon density) sets in first for states with zero angular momentum. In calculating the polarization operator of the pion, summation over the quantum numbers of the nucleon in a spherical nucleus was carried out instead of integration over the momenta of the nucleon states (as was done in an infinite system), and the constant $g'=g_c'$ at which the instability sets in was determined. The values of g_c' obtained for medium and heavy nuclei are practically the same as in an infinite system.

2. Deformation and moments of inertia of nuclei in the presence of a condensate

We consider a heavy nucleus in which a plane-layer structure (6.9) was produced. Since the condensate increment to the surface energy is proportional to the cross section of the nucleus, the presence of the condensate will contribute to elongation of the nuclei along the direction of the wave vector of the layers.

Inasmuch as the cross section contains a term that is linear in the nuclear deformation, the dependence of the nuclear energy on the quadrupole-deformation parameter β is given by (Migdal, Kirichenko, and Sorokin, 1974)

$$E(\beta)=\frac{\alpha(\beta)\beta^2}{2}-\frac{4\pi R^2\mathcal{E}_s}{3}\beta. \tag{6.11}$$

Account must be taken of the well known fact that rigidity to small deformations ($\beta<A^{-1/3}$) is determined by the disarrangement of the shell structure and is of the order of $\alpha(0)\sim\epsilon_F A$, whereas rigidity to large deformations ($\beta>A^{-1/3}$) is determined by the surface energy of the system and, as follows from the semiempirical formula for the binding energy of the nuclei, $\alpha(\beta)\sim(1/6)\epsilon_F A^{2/3}$ i.e., it is much less than $\alpha(0)$. It is easily seen that $E(\beta)$ has a minimum at low deformation $\beta_0=4\pi R^2\mathcal{E}_s/3\alpha_0$, and that at $8\pi R^2\mathcal{E}_s/3>(d(\alpha\beta^2)/d\beta)_{\min}$ a second minimum is produced at large deformation, and could lead to shape isomerism. We note that if the second minimum exists as a result of shell effects, then the condensate makes this minimum deeper. However, a minimum can appear even if it is not called for by the shell calculations. We note here also that the initial rigidity of the nucleus may turn out to be so large that, by virtue of the smallness of the equilibrium deformation β_0, the corresponding rotational band falls in the region of the single-particle energies, i.e., becomes unobservable. Choosing by way of estimate the values $\lambda\cong 10$ (see page 78 above) and using for the estimate of $a^2\sim|\omega_0^2|/\lambda$ the value $|\omega_0^2|=0.2$ (see p. 121), we obtain $\beta_0\sim 10^{-2}-10^{-3}$ for $A=100$.

However, even if $\beta_0\to 0$, the formation of flat layers leads to violation of spherical symmetry and consequently to the appearance of the moment of inertia in the spherical nuclei. For a noticeable moment of inertia to be produced it is necessary that splitting of the single-particle energies with respect to the projection of the angular momentum in the field of the layers be comparable with the pairing energy, or else, in the case of doubly magic nuclei, with the distance to the first levels (Migdal, 1974).

Using the results of Migdal (1959) it is easy to obtain an estimate of the ratio of the moment of inertia I to the rigid-body value I_0:

$$\frac{I}{I_0}\sim\frac{|\epsilon_{\nu,m}-\epsilon_{\nu,m\pm 1}|^2}{\Delta^2}, \tag{6.12}$$

where $\epsilon_{\nu,m}-\epsilon_{\nu,m\pm1}$ is the difference of the energies for neighboring projections of the angular momentum of the particle on the direction of the layers, and Δ is the pairing energy or the energy gap in the doubly magic nucleus. The difference between the single-particle energies is obtained from the expression for the nucleon energy in a periodic field φ_0

$$\epsilon(p^2,p_z)=\frac{p^2}{2m}+\frac{mf^2a^2k^2}{4}\frac{1}{p_z^2-(k^2/4)} \tag{6.13}$$

In the semiclassical approximation it is easy to connect p_z with the projection M of the angular momentum. Directing the x axis along the radius vector of the particle, we have

$$M_x=0, M_y=-rp_z, M^2=r^2p^2; M_y^2=r^2p_z^2, M_z^2=M^2\cos^2\theta,$$

whence

$$p_z^2=p^2\sin^2\theta\,\overline{\cos^2\varphi}=\frac{1}{2}p^2\left(1-\frac{m^2}{j^2}\right). \tag{6.14}$$

From (6.14) we get

$$\epsilon_{\nu,m}-\epsilon_{\nu,m-1}\sim\frac{f^2a^2k_0^2}{j\epsilon_F}\frac{1}{[1+g'\phi(k/2p_F)]^2}. \tag{6.15}$$

The last factor takes into account the interaction between the nucleons. Using the estimate of a^2 given above, we obtain

$$\left(\frac{I}{I_0}\right)^{1/2}\sim 0.1\frac{\epsilon_F}{j\Delta}. \tag{6.16}$$

In the case of ordinary deformation we have $(I/I_0)^{1/2}\sim\beta\epsilon_F/j\Delta$. Thus we see that the layered structure is equivalent, in the sense of the moment of inertia, to a deformation $\beta\sim 0.1$. Of course, these numbers are only by way of illustration.

3. Goldstone oscillation modes

The appearance of the field φ_0 signifies breaking of various symmetries. Owing to the coordinate dependence $\varphi_0(r)=a\sin k_0 r$, translational symmetry is broken. The existence of a preferred direction indicates violation of rotational symmetry. Finally, in the case of an isotopically asymmetric field $\varphi_0{}^\alpha$, isotopic symmetry is violated. It is seen that violation of continuous symmetry in an infinite system leads to the appearance of an oscillation mode with zero minimal frequency (the Goldstone theorem). In a finite system, this theorem leads to oscillations with frequencies that tend to zero like a certain power of the radius of the system. Thus, for example, violation of rotational symmetry in a deformed nucleus leads to the appearance of a rotational spectrum with a minimum frequency

$$\omega\sim\frac{1}{I}\sim\frac{1}{R^5}.$$

Let us trace the mechanism whereby the Goldstone modes are produced. To this end we derive an equation for the oscillating field. To obtain this equation in simple form it is necessary to represent φ in (6.2) in the form $\varphi=\varphi_0+\varphi'$ and retain the first term of the expansion in terms of φ'. Indeed, if an oscillation of a definite type, for example with fixed wave vector, is weakly excited, then its amplitude is of the order of the amplitude of the zero-point oscillations for one degree of freedom, i.e., it contains the square root of the volume of the system in the denominator. From (6.2) we obtain

$$\left(1-\frac{\partial\Pi}{\partial\omega^2}\right)\ddot{\varphi}'+\tilde{\omega}^2(k)\varphi'+3\lambda\varphi_0^2\varphi'=0. \qquad (6.17)$$

Here $k=(1/i\nabla)$.

If the wave vectors that play an important role in φ' are much smaller than k_0, then φ_0^2 in the third term of (6.17) must be replaced by its mean value. Using the relations obtained in (6.4b), we get

$$3\lambda\overline{\varphi}_0^2=\frac{3\lambda a^2}{2}=2\,|\omega_0^2|.$$

As a result we obtain an equation independent of r. The oscillation frequency, as a function of the wave vector q, is

$$\omega^2(q)=\frac{2\,|\omega_0^2|-|\tilde{\omega}^2(q)|}{1-\partial\Pi/\partial\omega^2)} \qquad (6.18)$$

The presence of the condensate consequently stabilizes the oscillation ($\omega^2>0$). This expression corresponds a finite minimal frequency ($\omega^2\sim\omega_0^2$) and is not related to the Goldstone oscillation mode. These oscillations exist as after as before the condensation and their frequency is strongly diminished by the factor $1/(1-\partial\Pi/\partial\omega^2)$ (see below). To see how the Goldstone modes arise, we differentiate Eq. (6.4a) for φ with respect to the coordinate $\mathbf{x}$ ($\mathbf{x}\parallel\mathbf{k}$) Migdal, 1973):

$$\tilde{\omega}^2(k)\frac{\partial\varphi_0}{\partial x}+3\lambda\varphi_0^2\frac{\partial\varphi_0}{\partial x}=0. \qquad (6.19)$$

Here again $k=(1/i\nabla)$.

Comparing this relation with (6.17), we find at $\varphi'\sim\partial\varphi_0/\partial x$ the oscillation frequency of φ' vanishes. In a finite system, the wave vector of the oscillation can be chosen to be equal to k_0 accurate to $\Delta k\sim 1/R$. Therefore the minimal oscillation frequency vanishes only as $R\to\infty$, and at finite R, as we shall show, it depends on R in power-law fashion.

To estimate the oscillation frequency, we represent φ' in the form

$$\varphi'=\frac{\partial\varphi_0}{\partial x}\chi$$

where χ is a slowly varying function. Multiplying (6.19) from the left by χ and substituting in (6.17), we obtain

$$\left(1-\frac{\partial\Pi}{\partial\omega^2}\right)_{\substack{k=k_0\\ \omega=0}}\frac{\partial\varphi_0}{\partial x}\ddot{\chi}+\tilde{\omega}^2(k)\frac{\partial\varphi_0}{\partial x}\chi-\chi\tilde{\omega}^2(k)\frac{\partial\varphi_0}{\partial x}=0. \qquad (6.20)$$

We consider first the case of an infinite system and let χ be characterized by a wave vector $\mathbf{q}$. Then, using (6.3), we can easily obtain from (6.20) the following expressions for the oscillation frequency:

$$\omega^2=\kappa\frac{q^4+4(qk_0)^2}{[1-(\partial\Pi/\partial\omega^2)]k_0^2}.$$

At small $\mathbf{q}$, the frequency depends essentially on the angle between $\mathbf{q}$ and $\mathbf{k}_0$. At $\mathbf{q}\parallel\mathbf{k}_0$ we have

$$\omega_\parallel^2\underset{q\to 0}{=}\frac{4\kappa q^2}{1-\dfrac{\partial\Pi}{\partial\omega^2}}, \qquad (6.21)$$

and at $\mathbf{q}\perp\mathbf{k}_0$

$$\omega_\perp^2=\frac{\kappa q^4}{[1-(\partial\Pi/\partial\omega^2)]k_0^2}. \qquad (6.22)$$

In a finite system, the lowest oscillation corresponds to $q_{\min}\sim R$, with $q_\parallel\sim q_\perp\sim 1/R$, so that the minimal oscillation frequency is determined by the terms containing $q_\parallel$

$$\omega_{\min}^2\sim\frac{\kappa}{1-(\partial\Pi/\partial\omega^2)}\frac{1}{R^2},$$

and for

$$\kappa=0.4;\,1-\frac{\partial\Pi}{\partial\omega^2}\sim|\Pi|\frac{1}{\Delta kv},$$

where Δ is the average distance between levels of pion symmetry (the exact value of $\partial\Pi/\partial\omega^2$ may be obtained only by the methods of the theory of finite Fermi systems; we have used $\Delta\sim 10-20$ MeV), we have

$$\omega_{\min}\sim\frac{0.2}{A^{1/3}}\sim\frac{30}{A^{1/3}}\,\text{MeV}.$$

Thus this oscillation mode is mixed with particle–hole excitations of the same symmetry and cannot be observed. Kirichenko and Sorokin (1976) have estimated the frequencies for various types of Goldstone oscillations, which should appear in the nucleus if a condensate were to exist.

They reached the conclusion that the lowest minimal oscillation frequency corresponds to oscillations of the directions of the layers relative to the elongation axis. The following expression is given for the frequency of such oscillations

$$\omega_{\text{rot}}^2=\frac{5\beta(2/\kappa)^{1/2}}{R(k_0R)^2(1-\dfrac{\partial\Pi}{\partial\omega^2})}\,|\omega_0| \qquad (6.23)$$

where β is the deformation parameter.

For $\beta=0.2$ and $\omega_0^2=-0.2$ (see page 121) we obtain $\omega_{\text{rot}}\sim 0.4$ MeV. It should be noted that Eq. (6.23) is valid only so long as ω_{rot} exceeds the frequency of the rotational levels ($\omega_{\text{rot}}>1/I$). Observation of such a level would be a serious argument in favor of the existence of a condensate.

4. Quantum character of a condensate field in a finite system. Parity conservation

In an infinite system, the condensate field can be regarded as classical, i.e., it has a definite value at each point of space.

In a finite system, as we have seen from the example of condensation in an external field (Sec. II), the ground state of the system is characterized by a wave function with zero mean value of the field. The condensation manifests itself in the fact that the mean square value of the field is large—positive and negative values of the field at a given point are equally probable. Indeed, let us represent the field operator $\hat{\varphi}$ in the form $\hat{\varphi}=\hat{q}\Psi$, where $\Psi(\mathbf{r})$ gives the coordinate dependence of the field in the ground state (the solution, corresponding to the lowest state, of the Klein–Gordon–Fock equation in a field). Then the ground and the excited states are determined, as we have seen, by the wave function $\chi(q)$, which describes the motion in two identical wells separated by a potential barrier. The ground state corresponds to a symmetrical wave function $\chi_0^{(s)}(q)$, and the first excited state to an antisymmetrical function $\chi_0^{(a)}(q)$, which constitute approximately a symmetrical and antisymmetrical superposition of the lowest states in each of the wells.

As we have seen, the energy of the first excited state is exponentially small

$$\tilde{\omega}^{(a)}=\frac{\omega\sqrt{2}}{\pi}\exp\left(-\frac{\pi\omega^3}{4\lambda_1}\right). \tag{2.17}$$

In a sufficiently homogeneous system we have

$$\lambda_1=\lambda\int\Psi^4 d\mathbf{r}\cong\lambda/V$$

and consequently the first-excitation energy decreases exponentially with the volume of the system. Thus, in a sufficiently large system, degeneracy sets in and instead of $\chi_0^{(s)}$ we can take the ground state to be $\chi_0^{(1,2)}=\chi_0^{(s)}+\chi_0^{(a)}/\sqrt{2}$. One of these states corresponds to predominantly positive values of q, and the other to predominantly negative values. The mean value of the field φ is nonzero and of opposite sign in these two states.

Equation (2.17) permits a reasonable estimate of the first-excitation energy (analogous to $\chi_0^{(a)}$) also for the case of a nucleus. For Ψ one should choose the solution

$$\Psi=\frac{2}{\sqrt{V}}\sin k_0 x$$

which is normalized to unity volume, and therefore

$$\lambda_1=\frac{\lambda}{V}\frac{3}{2}.$$

For the constant λ we can choose the value obtained from the estimates given above (Sec. IV, p. 78), $\lambda\approx 10$. The quantity ω^2 should be replaced by $-\omega_0^2=|\tilde{\omega}(k_0)|^2$.

The value of ω^2 is unknown. For the sake of argument, all estimates containing ω_0 here and below are made under the assumption that a condensate with $\omega_0^2=-0.2$ exists (see page 121). Approximately the same value corresponds ($\omega_0^2\approx-0.1$) as we shall show, to the assumption that the anomalies in the scattering of the electrons by nuclei are due to scattering by a layered condensate structure.

Substitution of these numbers in (2.17) yields

$$\omega^{(a)}=0.20e^{-0.96A/100}=28e^{-0.96A/100}\ \text{MeV}.$$

At $A=50$ we obtain $\omega^{(a)}=17$ MeV, i.e., a rather large value. At this energy, the decay into particle-hole excitations would lead to a smearing of this level, and it would be difficult to establish it. However, $\omega^{(a)}\cong 4.1$ MeV already at $A=200$, and a level can appear (a state with quantum numbers of the pion). Of course, this estimate is only illustrative, since the employed value of ω_0 is perfectly arbitrary, and the value of λ was estimated accurate to a factor ~1.

Both the ground and the excited states of the condensate, if it exists, have a definite parity and therefore the presence of the condensate with $\langle\varphi\rangle=0$ would not violate the classification of the nuclear levels by parity.

B. Experiments that make it possible to determine the closeness of nuclei to condensation

It is shown that the known nuclear facts do not exclude the possibility of condensation, and apparently offer evidence that the nuclei are close to the critical point.

By reproducing the positions of the levels $0^-, 1^+, 2^-, \ldots$ it is possible to refine the constants of the πN and NN interactions. The constants assumed by Migdal (1972, 1973) and by Migdal, Markin, and Mishustin (1974) do not contradict these data. An analysis of l-forbidden $M1$ transitions makes it possible to estimate the closeness of the nuclei to the critical point.

It is shown that the anomalies in electron scattering by nuclei might be attributed to the layered proton-density structure due to condensation. Possible experiments on nucleon and pion scattering by nuclei, capable of yielding information on the parameters that determine the condensation, are discussed.

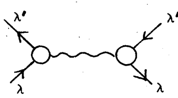
1. Does condensation contradict the known nuclear facts?

The assumption that a condensate may exist in nuclei (Migdal, 1972) has raised many objections (Barshay and Brown, 1973), which were analyzed by Migdal, Markin, and Mishustin (1974) and by Migdal (1973). Let us list the most important of these objections. A statement was made that allowance for the repulsion of the nucleons at short distances should lead to a strong decrease of the πN interaction in the nucleus and to a change of Migdal's estimate (1972) of n_c. As we have seen, the NN interaction is taken into account in the polarization operator by introducing the constant g^-. This means that the πNN vertex of the interaction is assumed to be weakened

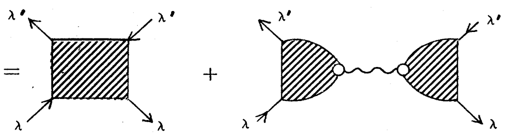
$$\mathcal{T}(\pi, NN)=\frac{\mathcal{T}_0(\pi NN)}{1+g^-\phi(k,0}\cong\frac{\mathcal{T}_0(\pi NN)}{2,6}. \tag{6.24}$$

It is this weakening which is caused by the repulsion of the nucleons at short distances. However, instead of calculating this repulsion with the aid of the NN interaction in vacuum, we determine the corresponding constant in the theory of finite Fermi systems empirically, from a comparison with other processes containing an interaction of the same symmetry as in the case of πNN. Indeed, the constant g^- enters in the renormalization, in the medium, of all the vertices having the symmetry of the pion (the vertex $\sim\sigma_\alpha\tau_\beta$). This method is much more reliable than calculations based on the theory of nuclear matter, since the latter does not take into account the distortions of the one-pion exchange due to many-particle interactions (see Sec. III A). It is surprising that the value of g^- obtained in such calculations (Barshay and Brown, 1973) turns out to be not greatly different from the empirical value ($g^-=1.2$—1.4 instead of $g^-=1.6$).

The example used to refute the πN approximation assumed by Migdal (1972) is the shift of the 0^- level in O^{16}. The level energy is $E = 12.78$ MeV, whereas the energy of the particle-hole excitation obtained from the mass differences of the neighboring nuclei of O^{16}, is 12.42 MeV. Thus, the interaction results in a positive energy shift $\Delta E = 0.36$ MeV. The shift was estimated with the aid of the one-pion exchange diagram

$\Delta E =$ 

The value $\Delta E = -4.8$ MeV (Barshay and Brown, 1973) was regarded as contradicting the interaction assumed by Migdal (1972). In this estimate no account was taken of repulsion at short distances. The level shift is determined by two types of diagrams

$\Delta E =$  (6.25)

The second of these diagrams gives the contribution of one-pion exchange with allowance for repulsion at short distances [Eq. (6.24)], which decreases the modulus of the foregoing value of $\Delta E(-4.8$ MeV), namely

$$\Delta E_\pi = -\frac{4.8}{(2.6)^2}\ \text{MeV}.$$

The first term in (6.25) is the sum of all the interactions that do not contain one-pion exchange in considered channel. That is, owing to the repulsion at short distances, the first term of (6.25) makes a positive contribution to the level shift, $\Delta E_{\text{h.c.}} \cong 1$ MeV. This results in a reasonable estimate of the observed shift.

We present below several examples of calculation of the energy shifts in different nuclei by the method of the theory of finite Fermi systems. It is shown that the interaction constants assumed above make it possible to explain the observed energy shifts.

Another objection was that the attraction due to one-pion exchange should have led to an enhancement of the spin part of the magnetic moment, whereas experiment has yielded a suppression. A detailed analysis of the influence of one-pion exchange on magnetic moments will be given later on. Here we present a simple qualitative reasoning, which shows that there is no contradiction to the suppression of the magnetic moment. First, one-pion exchange influences only that part of the magnetic moment which has $\sigma\tau$ symmetry. In spherical nuclei, as we shall show, this influence is noticeably weakened. Nonetheless, the role of one-pion exchange manifests itself in the fact that the weakening of the corresponding part of the magnetic moment is determined roughly by the relation

$$\mu^s = \frac{\mu_0^s}{1 + g_{\text{eff}}^-},$$

with a constant $g_{\text{eff}}^- \cong 1$ instead of $g^- = 1.6$. This weakening of the repulsion interaction is due to the influence of the one-pion exchange. Thus the phenomena considered here not only do not contradict the interaction constant values assumed above, but have provided a confirmation when a more thorough analysis is made.

Moreover, analysis of the probabilities of l-forbidden $M1$ transitions (see below) shows that the nuclei are either close to condensation or have a weak condensate. Nor is the possibility excluded that the observed anomalies in the scattering of the electrons by the nuclei are the result of the condensation-induced layered structure of the nuclear matter. We consider below possible experiments that can decide whether a condensate exists in nuclei, and if it does not, they can establish how close the nuclei are to condensation.

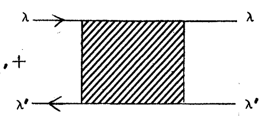
2. Influence of one-pion exchange on the spectra and probabilities of the transitions

The effective field acting on a nucleon inside a nucleus differs from the external field because of the polarization of the medium. The probabilities of single-particle transitions and the mean values of the additive quantities in the nucleus are expressed in terms of the matrix elements of the effective field V. In symbolic form, the connection between the effective field and the external field V_0 is given by

$$V = V_0 + \mathfrak{F}AV = V_0 + \Gamma A V_0 \tag{6.26}$$

Here $\mathfrak{F}$ is the effective interaction in the nuclear matter, A is the (quasiparticle—quasihole) pair propagator. The second term in (6.26) describes the additional field resulting from polarization of the medium. Γ is the NN scattering amplitude in the medium.

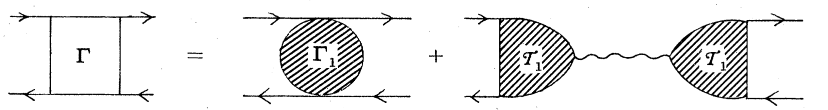
The energy levels are determined as the poles of Γ (or V). Particularly simple in form is the energy of the particle-hole level

$E_{\lambda\lambda'} = \epsilon_\lambda - \epsilon_{\lambda'} +$  , (6.27)

where ϵ_λ is the quasiparticle energy. The second term yields the level shift due to the interaction of the quasiparticle and the quasihole. In those cases when the effective field corresponding to the considered process has pion symmetry, it is necessary to take into account the contribution of the one-pion exchange to the NN interaction. The same pertains also to the energy shift of the particle-hole excitations.

If the state $\lambda\lambda'$ has pion quantum numbers, account must be taken of the contribution of one-pion exchange to the amplitude Γ. A comparison of the theory with experiment can serve as a check on the correctness of the chosen interaction constants and make it possible to assess the closeness of the nucleus to condensation or whether a condensate is present. Phenomena of these type include magnetic β transitions and the renormalization of the spin magnetic moment, Gamow-Teller transitions, and spectra of nuclei in states $T = 1, 0^-, 1^+, 2^-, \ldots$

We consider first the influence of one-pion exchange on the level position. The quantity Γ, as we have seen in Sec. III, is determined by the following diagrams



(6.28)

The quantities Γ_1 and $\mathcal{T}_1$, by definition, do not contain one-pion exchange in the considered channel, and the $\mathcal{T}_1$ blocks are connected by the exact pion propagator in the medium. According to (6.27), the single-particle level energy shift is

$$E_{\lambda\lambda'} - \epsilon_\lambda + \epsilon_{\lambda'} = \int \Psi_\lambda^*(\mathbf{r})\Psi_{\lambda'}(\mathbf{r})\Gamma(\mathbf{r},\mathbf{r}')\Psi_\lambda(\mathbf{r}')\Psi_{\lambda'}^*(\mathbf{r}')d\mathbf{r}d\mathbf{r}'$$

$$= \int (\Psi_\lambda^*\Psi_{\lambda'})_{-k}\Gamma(\mathbf{k},\mathbf{k}')(\Psi_\lambda\Psi_{\lambda'}^*)_{-k'}\frac{d\mathbf{k}d\mathbf{k}'}{(2\pi)^6}, \tag{6.29}$$

where $(\Psi_\lambda^*\Psi_{\lambda'})_k$ is the Fourier component of the product of the particle and hole wave functions.

For a rough estimate of the level shift we use for Γ_1 and $\mathcal{T}_1$ Eqs. (3.39) and (3.40), which were obtained for an infinite system. Thus the first term in (6.28) is positive, i.e., it corresponds to repulsion and leads to a raising of the level in comparison with the difference of the single-particle energies. The second term is equal to

$$\Gamma^{(\pi)}(k,\omega) = \mathrm{Sp}[\mathcal{T}_1^{(\alpha)}(k)]^2\frac{1}{\omega^2-\omega_k^2-\Pi(k,\omega)}.$$

By averaging over the spin variables and using Eq. (3.39), we obtain

$$\Gamma^{(\pi)}(k,\omega) = \frac{f^2k^2}{(1+g^-\phi)^2}\frac{1}{\omega^2-\omega_k^2-\Pi(k,\omega)}. \tag{6.30}$$

The level shift is determined by the value of Γ at $\omega=0$. Thus the term $\Gamma^{(\pi)}$ makes a negative contribution to the level energy. The amplitudes $\Gamma_1(k)$ and $\Gamma^{(\pi)}(k)$ become comparable at $k^2\sim 1$. $\Gamma^{(\pi)}(k,\omega=0)$ has a maximum at $k=k_0\cong p_F=2$, inasmuch as $\tilde{\omega}^2(k)=\omega_k^2+\Pi(k,0)$ has a minimum near k_0. Of course, Eq. (6.30) is valid only if condensation has not yet set in. In the presence of a condensate $\omega^2(k_0)<0$, but a term $2|\omega^2(k_0)|$ is added to $\Pi(k,0)$, so that the denominator has the same sign before and after the condensation.

The sign of the energy shift depends on the relative contribution of the large and small k to the Fourier expansion of the product of the wave functions

$$(\Psi_\lambda^*,\Psi_{\lambda'})_k = (\Psi_\lambda^*(\mathbf{r}), e^{i\mathbf{kr}}\Psi_{\lambda'}(\mathbf{r})).$$

In all the investigated cases it turned out that the contribution of the repulsion at small k is approximately equal to the contribution of the attraction at large k, so that the resultant shift turns out to be small, and can be of either sign. The shift of the single-particle 12.78 MeV 0^- level in O^{16} was estimated by Sapershtein and Troitskii (1976), whereas from the masses of the neighboring nuclei of O^{16} it follows that the energy difference $\epsilon_\lambda-\epsilon_{\lambda'}$ of the noninteracting quasiparticles is 12.42. The level shift is $\Delta E=0.36$. It turned out that the observed shift does not contradict the values of f and g^- chosen above.

Sapershtein and Troitskii (1976a) carried out a detailed calculation of the spectra of the considered type in ^{208}Pb and ^{208}Ti using the methods of the theory of finite Fermi systems. The values of the constant g' at which the calculated frequency coincides with the observed one were obtained. It was assumed that the renormalization of the constant f of the medium is equal to $f^*=0.9f$. The results for ^{208}Pb are summarized in Table III. The values obtained for ^{208}Ti at $E_{\exp}=4.21$ are $\Delta\epsilon=4.09$ and $g'=0.81$. A deviation is observed only for ^{208}Bi ($E_{\exp}=3.65$, $\Delta\epsilon=3.57$), where the shift is too small ($E-\Delta\epsilon=0.08$) to be explained. It was assumed in the calculation that Γ_R, which characterizes the change of the πN^*N vertex in the medium [Eq. (3.14)], is equal to unity.

The identical values of g' in the four cases offer evidence in favor of the chosen set of constants ($g^-=1.7$, $f=0.9$, $\Gamma_R=1$).

TABLE III. Comparison of the experimental shift of the single-particle levels with that calculated with allowance for one-pion exchange.

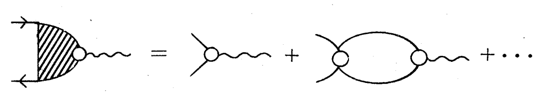
I^π	$E_{\exp}$	f'/f	g'	$\epsilon_\lambda-\epsilon_{\lambda'}=\Delta\epsilon$
0^-	5.28	0.9	0.85	$\Delta\epsilon=(P_{1/2}^{-n},S_{1/2}^{n})=5,46$
1^+	7.2	0.9	0.87	$\Delta\epsilon=(1i_{11/2}^{n},1i_{13/2}^{-n})=5,84$
2^-	4.2	0.9	0.82	$\Delta\epsilon=(2g_{9/2}^{n},2f_{5/2}^{-n})=4,00$

3. Magnetic moments

We proceed to an analysis of the magnetic moments and magnetic transitions. The operator of the spin part of the magnetic moment of the nucleon is of the form

$$\mu^{(s)} = \mu_0\left(\gamma_n\frac{1-\tau_3}{2}+\gamma_p\frac{1+\tau_3}{2}\right)\sigma.$$

Thus $\mu^{(s)}$ contains a term $\mu_0(\gamma_p-\gamma_n)\tau_3\sigma/2$, which has $\sigma\tau$ symmetry. To find the magnetic moment of the nucleus it is necessary to obtain the effective field corresponding to the bare fields σ and $\sigma\tau$. One-pion exchange and the local interaction $g^-=2g'$ enter only in an effective field of the form $\sigma\tau$. We consider first the effective field produced from an external field $\sim\sigma\tau$ when the one-pion exchange is turned off

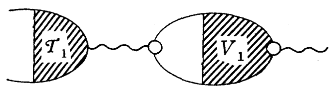
$V_1=$ 

The field V_1 does not contain pion diagrams in the horizontal channel. In the coordinate representation, the field V_1 consists of two terms

$$V_1^{(\alpha,\beta)}(r) = u_1(r)\sigma_\alpha\tau_\beta + v_1(r)\frac{(\mathbf{r}\sigma)r_\alpha}{r^2}\tau_\beta. \tag{6.31}$$

Calculations show that the functions $u_1(r)$ and $v_1(r)$ are of the same order. The NN interaction corresponding to one-pion exchange, Eq. (3.41), contains the angles between the spin vector and the wave vector. It is easy to verify that for this reason the one-pion exchange renormalizes only the second term of (6.31).

If the field $V_1(r)$ has been determined, then the effective field, with allowance for one-pion exchange, is equal to

$V = V_1 +$  $= V_1 + \mathcal{T}_1 D(\mathcal{T}_0 A V_1)$. (6.32)

At large K, the contribution of the one-pion exchange can be estimated by using, as was done earlier, the formulas of an infinite medium. We write V_1 and $\mathcal{T}_1$ in the form (we omit the isotopic symbols for simplicity)

$$V_1^\alpha(k) = u_1(k)\sigma_\alpha + v_1(k)\,\frac{(\bar{T}\mathbf{k})\,k_\alpha}{k^2}\ ,$$

$$\mathcal{T}_1 = \frac{\mathcal{T}_0}{1+2g'\phi} = \frac{(\bar{T}\mathbf{k})}{1+2g'\phi}\ ,$$

$$V^\alpha(k) = u(k)\sigma_\alpha + v(k)\,\frac{(\bar{T}\mathbf{k})\,k_\alpha}{k^2}\ .$$

Summing over the spin variables in the parentheses of (6.32), we readily obtain

$$u(k) = u_1(k)\ ,$$

$$v(k) = v_1(k) - \Pi_p(u_1(k) + v_1(k))\,\frac{1}{\tilde{\omega}^2(k)}, \qquad (6.33)$$

where Π_p^- is the pole part of the polarization operator. The magnetic moment in the state λ is determined by the matrix element $V_{\lambda\lambda}$ of the effective field. The part of the magnetic moment of interest to us ($\sim\boldsymbol{\sigma}\tau$) is

$$\mu_1^{(s)} = \mu_0\,\frac{\gamma_p - \gamma_n}{2}\,V_{\lambda\lambda}\ .$$

Integration over the angles yields

$$\overline{\frac{(\boldsymbol{\sigma}\mathbf{r})z}{r^2}} = \frac{1}{2j+2}$$

$$\bar{\sigma}_z = \frac{(-1)^l 2j}{2l+1}$$

at large j. Therefore the contribution of the term $\sim(\boldsymbol{\sigma}\mathbf{r})r_\alpha\tau_\beta$ is much weaker than that of the term $\sim\sigma_\alpha\tau_\beta$.

Thus one-pion exchange exerts an appreciable influence on the magnetic moments only in the case of small j. In states $S_{1/2}$ the experimental values of the magnetic moments deviate systematically from those calculated in the theory of finite Fermi systems without allowance for one-pion exchange. It is possible that allowance for one-pion exchange can explain these deviations and yield better values of the constants g' and f^*/f.

4. l-forbidden transitions

One-pion exchange plays a much more important role in the case of l-forbidden $M1$ transitions. The transition probability is determined by the matrix element $V_{\lambda\lambda'}$, and in the case of l-forbidden transitions we have $l' = l \pm 2$.

In this case, the only term of V that makes a nonzero contribution is of the type $(\boldsymbol{\sigma}\mathbf{r})r_\alpha$, which is greatly enhanced by one-pion exchange.

Qualitatively, this enhancement is determined by Eq. (6.33) with $u_1 = 0$. Indeed, the matrix element can be represented in the form

$$V_{\lambda\lambda'}^{\alpha\beta} = \int (\Psi_\lambda^*(\boldsymbol{\sigma}\mathbf{k})\tau_\beta\Psi_{\lambda'})_{-k}\,v(k,\omega)k_\alpha\,\frac{d^3k}{(2\pi)^3}\ , \qquad (6.34)$$

with $\omega = \epsilon_\lambda - \epsilon_{\lambda'}$. Neglecting the ratio $\omega^2/\tilde{\omega}^2$, we arrive at Eq. (6.33) under the sign of the integral with respect to k. If the nucleus is close to condensation, then $\tilde{\omega}^2(k)$ is close to zero at $k \cong k_0$, and under a favorable distribution of $(\Psi_\lambda^*(\boldsymbol{\sigma}\mathbf{k})\tau\Psi_{\lambda'})_{-k}$ with respect to k, an appreciable increase of the transition probability can occur. The influence of one-pion exchange on l-forbidden transitions was considered by Sapershtein and Troitskii (1975, 1976). A comparison of theory with experiment shows that the observed probabilities of the l-forbidden transitions exceed in some cases by dozens of times the results of calculation without allowance for one-pion exchange. Unfortunately, it is difficult to draw definite conclusions from this analysis concerning the values of the constants of the theory, but the general result is that the nuclei are very close to the phase transition, the possibility not being excluded that a small condensate is present.

Using information from the particle-hole level shifts and the l-forbidden transitions, it is possible to establish the correct values of the interaction constants and thus obtain the critical density. In the case when the l-forbidden transition is anomalously large, it must be attributed to the role of the second term in (6.33); moreover, a strong increase of the matrix element is possible only in the case of an anomalously small value of ω_0. Let us find the contribution made to the integral (6.34) with respect to k by the region $k \sim k_0$, where the pion propagator has a pole.

We write the matrix element of interest to us in the form

$$(\Psi_\lambda V(r,\omega)\Psi_{\lambda'}) = \int dk\left(A(k) + \frac{B(k)}{\tilde{\omega}^2(k)}\right)\ .$$

We separate the integral into regular and singular parts. Let the pole of the denominator correspond to $k = k_1$ ($k_1 \approx k_0$)

$$\int dk\{\ \ \} = \int dk\left\{A(k) + \frac{B(k) - B(k_1)}{\tilde{\omega}^2(k)}\right\} + \int dk\,\frac{B(k_1)}{\tilde{\omega}^2(k_1)}\ .$$

Inasmuch as the first term does not contain a pole, its contribution will correspond to the "background" of relatively small values of the transition probabilities. Large values can come only from the second term.

Using the expression $\tilde{\omega}^2(k) = \omega_0^2 + \kappa(k^2 - k_0^2)^2/4k_0^2$ and integrating with respect to k, we obtain

$$(\Psi_\lambda V\Psi_{\lambda'}) = (\Psi_\lambda V\Psi_{\lambda'})_{\text{Reg}} + B(k_1)/|\omega_0|\sqrt{\kappa}\ .$$

According to (6.33) we have $B(k_1) = A(k_1)\Pi_p(k_1)$ at $u = 0$. In addition, since the region of the integration with respect to k is bounded, on account of the properties of Ψ_λ, by the values $k = 2p_F$, we have $(\Psi_\lambda V\Psi_{\lambda'})_{\text{Reg}} \sim A(p_F)p_F$.

As a result we obtain the following estimate for the transition probability:

$$\frac{B(M1)}{\overline{B}(M1)} = \frac{|(\Psi_\lambda V \Psi_{\lambda'})|^2}{|(\Psi_\lambda V \Psi_{\lambda'})_{\rm Reg}|^2} \sim \Pi_p^2(k_0 \cong p_F)/|\omega_0^2| \sim \frac{p_F^2}{|\omega_0^2|}. \quad (6.35)$$

Thus the appreciable enhancement of the transition probability should be attributed to the smallness of the denominator in (6.35). At $B/\overline{B} \cong 20$ we obtain $|\omega_0^2| \sim 0.2$. If condensation has set in, then $\Pi(k, \omega)$ acquires a term $\lambda\overline{\varphi_0^2}$ that causes $|\omega_0^2|$ to replace ω_0^2 both before and after condensation, and Eq. (6.35) remains valid in the presence of a condensate. Therefore an analysis of the intensity of the l-forbidden transitions cannot answer the question of whether condensation took place, but can establish the proximity of the nucleus to the critical state. Is there another explanation of the anomalously large probabilities of the l-forbidden transitions?

It is possible to apply to Gamow-Teller β transitions also the arguments advanced concerning the magnetic moments and l-forbidden $M1$ transitions. For allowed transitions one-pion exchanges are important only for low angular momentum initial single particle state. One-pion exchange makes the largest contribution in the case of l-forbidden β transitions. In this case, resonant enhancement of the transition probability are also possible.

5. Pion optical potential

Information on the pion polarization operator in a nucleus can be obtained by analyzing data on the π-atom spectra.

From these data we can determine with high accuracy both the real and the imaginary part of the pion optical potential. Since small momenta play the significant role in the wave function of the π-mesic atom, it suffices to retain in this potential the constant term, which is connected with the S-wave πN scattering, and the term containing the square of the pion wave vector (i.e., $\Delta\Psi_\pi$). The pion optical potential is expressed in terms of the polarization operator $\Pi(k \ll 1, \omega \cong 1)$. The wave function of the pion satisfies the Klein-Gordon-Fock equation:

$$\Delta\Psi_\pi + [(\omega - V)^2 - \Pi(k, \omega - V) - 1]\Psi_\pi = 0, \quad (6.36)$$

where V is the potential of the electric field and $k = (1/i)\nabla$. At small k and at $\omega \cong 1$ the pole term makes a small contribution, since $\Phi(k \ll 1, \omega \cong 1) = k^2 v_F^2/3\omega^2$ [see Eqs. (3.3) and (3.22)]. We therefore have at $N = Z$

$$\Pi = \Pi_{\rm loc}(k \ll 1, \omega \cong 1) = n\{\tilde{A}_R^s(t=0)\Gamma_R + \tilde{A}_{R'}^s(t=0)\Gamma_{R'}\}$$

where Γ_R is the vertex, introduced above, for the $N\pi N^*$ interaction in a medium, $\Gamma_{R'}$ is the analogous vertex for the second term, which takes into account the contribution to P scattering by the distant resonances. If a condensate does exist in the nuclei, then a term $3\Lambda(k, k_0)\overline{\varphi_0^2}$ is added to the polarization operator. We have seen that $\Lambda \sim k^2$ at small k, so that the additional term takes the same form as the term due to P scattering, but differs from it in sign. If it were to turn out that Γ_R contradicts other experimental data, then this would be an argument in favor of the existence of the condensate (Troitskii *et al.*, 1975).

For simplicity we confine ourselves to the case $N = Z$. Then

$$\Pi = n\tilde{A}^s.$$

To determine the effective pion optical potential we use the expansion

$$\Pi(k, \omega) = \Pi(0, 1) + \frac{\partial\Pi}{\partial\omega^2}(\omega^2 - 1) + \frac{\partial\Pi}{\partial k^2}k^2.$$

Putting $\omega - V = \tilde{\omega}$ and substituting the expansion of $\Pi(k, \omega)$ in (6.36), we obtain

$$-k^2\frac{1 + (\partial\Pi/\partial k^2)}{1 - (\partial\Pi/\partial\omega^2)}\Psi_\pi + (\tilde{\omega}^2 - 1)\Psi_\pi = 0. \quad (6.37)$$

To separate the principle effects, we regard for the time being the operator k^2 as a number, i.e., we neglect the change of density at the edge of the nucleus [allowance for the $n(r)$ dependence introduces corrections of the order of $A^{1/3}$]. We have used in (6.37) the fact that $\Pi(0, \omega = 1)$ is negligibly small because of the smallness of the scattering amplitude at the threshold. Equation (6.37) can be rewritten in the form

$$\Delta\Psi_\pi + \left[\tilde{\omega}^2 - 1 - \frac{(\partial\Pi/\partial k^2) + (\partial\Pi/\partial\omega^2)}{1 - (\partial\Pi/\partial\omega^2)}k^2\right]\Psi_\pi = 0.$$

Changing over to the nonrelativistic limit, we obtain

$$\Delta\Psi_\pi + 2(E - U)\Psi_\pi = 0$$

i.e., the Schrödinger equation with energy

$$E = \frac{\omega^2 - 1}{2}$$

and with potential

$$U = -\frac{1}{2}V^2 + V + \frac{(\partial\Pi/\partial k^2) + (\partial\Pi/\partial\omega^2)}{2[1 - (\partial\Pi/\partial\omega^2)]}k^2,$$

where V is the Coulomb potential.

The role of the optical potential is played by the quantity (we neglect the small term $-V^2/2$)

$$U_{\rm opt} = n\frac{(\partial\tilde{A}^s/\partial k^2) + (\partial\tilde{A}^s/\partial\omega^2)}{2[1 - n(\partial\tilde{A}^s/\partial\omega^2)]}k^2.$$

We have neglected so far the variation of the scattering amplitude in the medium. Using (3.17c) we obtain

$$\frac{\partial\tilde{A}^s}{\partial k^2} = -0.8, \quad \frac{\partial\tilde{A}^s}{\partial\omega^2} = -1.1.$$

As a result, the coefficient of $k^2/2$ is equal to $-1.9n/(1 + 1.1n) = -0.6$ at $n = n_0$.

Experiment yields 0.7—0.9. It should be noted that the coefficients in (3.17c) had not been determined with sufficient accuracy.

6. Scattering of electrons by nuclei. Scattering of nucleons and pions

The nucleon-density modulation due to π condensation (Sec. V.A) exerts an influence on the electric form factor of the nucleus. It will be shown that these modulations could explain the experimentally observed anomalies in the elastic scattering of electrons. We shall consider the shell density fluctuations and show that they cannot account for the singularities of the scattering without a special choice of nucleon-nucleon interaction parameters. For the simplest configuration of the condensate field, in the form of a standing wave, the density of either the neutrons or the protons takes the form

$$n(\mathbf{r}) = n_0(1 + \xi^2 \cos 2k_0 z)$$

where ξ^2 is given by (5.15)

$$\xi^2 = 5.9 \quad \alpha^2 \cong \frac{n - n_c}{n_c}.$$

In Sec. VI.A we have shown that this result, obtained for an infinite system, should remain in force also in medium and heavy nuclei, and is distorted only in a layer $\delta \ll R$ near the surface of the nucleus.

As we have seen (Sec. VI.A), the direction of the layers is closely tied in with the direction of the nuclear deformation. In the rotational ground state, averaging takes place over the direction of the deformation and the direction of the layers. Indeed, the wave function of the deformed nuclei constitutes a product of the wave function in terms of the internal variables by the wave function in terms of the angles determining the nuclear orientation. Therefore in elastic-scattering experiments, when rotational levels are not excited, a density distribution averaged over the angles of the vector k_0 will be produced, namely

$$\bar{n}(r) = n_0(r)\left(1 + \xi^2 \frac{\sin 2k_0 r}{2k_0 r}\right). \tag{6.38}$$

Before we estimate the contribution of these density modulations to the amplitude of electron scattering by nuclei, we recall how these experiments were analyzed (see, for example, Bellicard *et al.*, 1967; Heisenberg *et al.*, 1969; Sinha *et al.*, 1973; Li *et al.*, 1974). The proton density distribution was chosen in the form

$$n_p(r) = n_p(0)\left| \frac{1 + (Wr^2/R^2)}{1 + \exp[(r - R)/\delta]}\right. \tag{6.39}$$

The constants W, R, and δ were chosen to obtain the best description of the experimental data in a wide interval of small $q(q < 2F^{-1})$, while the constant $n_p(0)$ is determined by the condition $Z = \int n_p d\mathbf{r}$. The scattering cross section for large q was calculated after finding the constants. The net result of all the experiments is that in a narrow interval of q, near $q = q_0 = 3\ F^{-1}$, a large deviation from the cross section calculated from the distribution (6.39) is observed, amounting sometimes to an order of magnitude. A similar phenomenon has been observed in proton scattering (Palevsky *et al.*, 1967; Alkhazov *et al.*, 1972). In this case, too, the cross section deviates at $q \cong 3\ F^{-1}$ from the calculated value obtained with the aid of the optical potential.

The variation of the cross section in a narrow interval of q seems to point to the existence of the periodic structure of the density, of the type (6.38), for all the investigated nuclei. Consider by way of illustration the Born approximation. The essential part of the Born form factor, corresponding to the distribution (6.39), is equal to (at $qR \gg 1$)

$$F(q) = -\frac{3}{q^2R^2}\left[\cos qR \cdot \Psi(\pi q\delta) - \frac{\xi^2}{2}\,\frac{\sin(q - 2k_0)R}{(q - 2k_0)R}\right], \tag{6.40}$$

where $\psi(x) = x/\sinh\lambda$. The experimentally observed change of F means that at $q = q_0$ we have

$$\xi^2 \cong 2\Psi(\pi q_0\delta)\cos q_0 R.$$

The experimental value of ξ^2 obtained from this condition is of the order of $(5-8)\times 10^{-2}$.

We shall show that shell fluctuations of the density can apparently not explain the observed variation of the cross section. Shell modulations of the density can be obtained analytically in the semiclassical approximation (see, for example, Kirzhnits and Shaptakovskaya, 1972). The corresponding form factor is a smooth function of q. This smooth function is taken into account automatically to a considerable degree in the analysis method used to reduce the scattering experiments. Addition of a smooth function of q leads only to a small change in the empirical distribution constants (6.39).

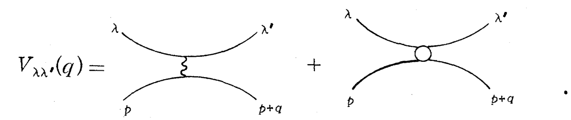
To verify this, the following computational experiment was performed. The proton distribution function for ^{208}Pb and ^{40}Ca were obtained with the aid of the Ψ functions of the individual nucleons in the Woods-Saxon model. This was followed by a Fourier analysis, i.e., the Born-approximation form factor corresponding to the obtained density distribution was determined. In addition, as is done in the analysis of the experiments, the distribution parameters (6.39) giving the best agreement with the form factor at $q < 2\ F^{-1}$ were found, after which the form factor was calculated for $q > 2\ F^{-1}$. The deviations of the form factor corresponding to a smooth distribution from the true form factor are small and are distributed in a broad interval of q.

It should be noted that there exist shell-fluctuation calculations in which, at the cost of introducing an interaction that contains arbitrary parameters, it becomes possible to reproduce the form factor at $q \simeq 3F^{-1}$, but at the same time, the agreement for smaller q becomes much worse (Friar, Negele, 1973; Bethe, 1974).

More complete information on the layered structure could be obtained in experiment on the scattering of electrons by oriented nuclei. Experiments of this kind were performed on ^{165}Ho (Urhane *et al.*, 1971), but the momentum transfers were too low. In these experiments, the orientation of the nuclei led to orientation of the quadrupole moment, inasmuch as the odd proton in Ho has an angular momentum projection on the elongation axis $m = j$ $(j = \frac{7}{2})$. Since the direction of the layers is connected with the elongation direction, the orientation of the nuclei means also orientation of the layers. This should lead to an increase of the diffraction by the layers in comparison with the case of layers averaged over the directions. It should be noted that according to Kirichenko and Sorokin (1976), in the case of a weakly developed condensate the direction of the layers executes zero-point oscillations relative to the elongation direction, and these can greatly weaken the effect.

We note that experiments on large-angle electron scattering would make it possible to observe the layered magnetic structure in the nucleus, corresponding to the periodicity of the spin density of nuclear matter. The corresponding maximum of the scattering cross section should correspond to $q = k_c = 1.4\ F^{-1}$, i.e., to a momentum transfer half as large as in the case of scattering by the charge distribution. The layered spin structure could also manifest itself in experiments on the scattering of π mesons and protons by oriented nuclei.

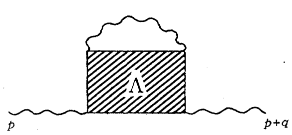
Significant information on the closeness of nuclei to condensation could be obtained from elastic scattering of protons by the pion field produced by the odd nucleon of the nucleus. The interaction of the scattered nucleon with the ood nucleon is determined by the diagrams



If the nucleus is close enough to condensation, then $V_{\lambda\lambda'}(q)$, as is readily seen, has a sharp maximum at $q(\omega=0, k=k_0)$. The separation of this maximum would make it possible to obtain the value of ω_0 and by the same token establish the parameters that determine closeness to condensation. It was shown by Troitskii (1977) that in the presence of the condensate with the amplitude $a^2=0.04$ the probability of one-nucleon π^- capture should be increased by 100 times.

Important information could be obtained from the angular distribution of the photoproduction on nuclei (M. Ericson, private communication). If the condensate exists, the photoproduction amplitude as a function of $\mathbf{q}=\mathbf{k}_\gamma-\mathbf{k}_\pi$ should have a maximum at $q=k_0$ which corresponds to a process involving the condensate.

The value of Λ characterizing the 4-pion interaction in a medium determines the amplitude of the condensate field, if condensation did take place. In addition, as we shall show in Sec. VII.A, knowledge of the value of Λ provides an answer to an important question—is there an instability point on the plot of the energy? Yet, as we have seen, it is only possible to estimate Λ. It would be therefore extremely important to determine this quantity directly from experiments of pion scattering by nucleons, the scattering amplitude containing the term



In addition, Λ could be determined from experiments of the type $(\pi, 3\pi)$ on nuclei. This would yield the interaction in the final state in nuclear matter rather than the vacuum, i.e., the value of Λ.

VII. π CONDENSATION AND POSSIBLE EXISTENCE OF ANOMALOUS NUCLEI

In the first half of Sec. VII we present arguments favoring the existence of superdense nuclei, by using the expression for the nucleon-system energies at densities close to n_c. This discussion follows that of Migdal (1971, 1974). For definite values of the constants of the theory, the compressibility reverses sign at the critical point and the system should be compressed until it goes over into a state with positive compressibility.

The energy, density, and stability conditions of this state cannot be obtained in such an approach—it is necessary to know the energy of the system at densities much higher than n_c (these questions are considered in Sec. VII.B). Section VII.A deals next with the possible stability of supercharged nuclei ($Z \gtrsim 137^{3/2}$). The results obtained by considering π condensation in an electric field are used. The energy gain due to π^- condensation offsets in part the Coulomb energy. The possibility of π^+, π^- condensation is limited by the screening of the nuclear field due to the restructuring of the electron-positron vacuum for a nucleus with a large charge.

The possibility of nucleon–antinucleon instability in dense nuclear matter (the Lee model, 1974), due to interaction of nucleons with scalar mesons, is analyzed at the end of Sec. VII.A. It is shown that the constant of this interaction, no matter how large it may be, is renormalized in the medium in such a way that instability arises, if it does at all, only at very high densities $n \gtrsim 100n_0$.

The second half of Sec. VII, which is based on work by Migdal, Markin, Mishustin, and Sorokin (1976), is devoted to clarification of the stability conditions of the superdense state relative to evaporation of particles, fission, and β decay. The results of Campbell, Dashen, and Manassah (1975) and of Baym *et al.* (1975) are used to determine the pion energy, and those of Pandharipande (1971) for the energy of the nucleon medium at large nucleon densities without allowance for the condensate. Interpolation formulas are set up separately for pion and nucleon energy, and combine the results for small and large densities. With the aid of these formulas, the energy and density of the superdense state of nuclear matter is determined, and the regions of stability with respect to fission and β decay are evaluated for different values of the constants of the theory.

Possible ways of observing superdense nuclei are discussed.

A. Anomalous states of nuclear matter

It is shown in this section that at reasonable values of the constants introduced into the theory, π condensation makes nucleon matter unstable–the compressibility becomes negative. This instability occurs both at $Z \cong N$ (superdense nuclei) and at $Z \ll N$ (neutron nuclei).

We discuss the question of the possible stability of nuclei with charge $Z \gtrsim 137^{3/2}$ (supercharged nuclei).

We analyze the possibility of instability of nucleon matter relative to production of nucleon–antinucleon pairs (the Lee model). An estimate $n_c \sim 100\, n_0$ is presented for the critical density of this process.

1. Superdense and supercharged nuclei

We shall show that allowance for π condensation can lead to the possible existence of two minima on the plot of the energy density $\mathcal{E}(n)$ against the density n. We write down the energy density of nuclear matter, with π condensation taken into account in the form

$$\mathcal{E}(n) = \mathcal{E}_N(n) + \mathcal{E}_\pi(n), \tag{7.1}$$

where $\mathcal{E}_N(n)$ is the nucleon energy density without allowance for the π condensate, and $\mathcal{E}_\pi(n)$ is the condensate energy density, which at densities $0 < n - n_c \ll n_c$ takes the form

$$\mathcal{E}_\pi(n) = -(\beta/2)(n-n_c)^2$$

As we have seen in Sec. VI.A, $n_c \cong n_0$ and $\beta \sim 1$.

The first term of $\mathcal{E}(n)$ has been calculated in numerous papers on the theory of nuclear matter (e.g., Bethe, 1974). It has turned out that at $N \cong Z$ the $\mathcal{E}_N(n)$ curve has a minimum at a density n_0^0 close to the nuclear density n_0.

We assume first that a condensate already exists at nuclear density, i.e., $n_c < n_0$. Then, expanding the energy density in powers of $n - n_0$, we obtain

$$\mathcal{E}(n) - \mathcal{E}(n_0) = \frac{1}{2}\left(\frac{d^2\mathcal{E}_N}{dn^2} - \beta(n_0)\right)(n-n_0)^2 + \frac{1}{6}\left(\frac{d^3\mathcal{E}_N}{dn^3} - 3\frac{d\beta}{dn}\right)(n-n_0)^3 . \quad (7.2)$$

The coefficients of $(n-n_0)^2$ is connected with the volume rigidity of the nuclear matter

$$\frac{K}{n_0} = \frac{d^2\mathcal{E}}{dn^2} \equiv \frac{K_0}{n_0} - \beta \equiv \beta_0 - \beta .$$

If $\beta(n)$ is an increasing function of n at $n \gg n_0$, then the coefficient of $(n-n_0)^3$ may turn out to be negative. In this case a maximum appears on the $\mathcal{E}(n)$ curve at

$$n_1 - n_0 = \frac{K}{n_0\kappa}$$

where κ is equal to

$$\kappa = \frac{3}{2}\frac{d\beta}{dn} - \frac{1}{2}\frac{d^3\mathcal{E}_N}{dn^3} .$$

With further increase of the density, the growth of the condensate energy weakens and the $\mathcal{E}(n)$ curve begins to grow. In addition, at a sufficient increase of the density the repulsion at short distances becomes appreciable and $\mathcal{E}_N(n)$ increases sharply. A second minimum should therefore appear on the $\mathcal{E}(n)$ curve (Fig. 15) at $n \gtrsim n_0$. This sign of κ is regarded as natural in the paper by Migdal (1974). Certain model calculations seem to argue against this possibility. For example, in a running-wave model without allowance for additional nuclear interactions, the condensation energy density takes the form $\mathcal{E}_\pi(n) = -(n-n_c)^2/4n_c$, i.e., $\beta = 1/2n_c$ and $d\beta/dn = 0$. Thus, in this model $\kappa < 0$ and consequently there is no second minimum at $n_c < n$.

It follows from (7.2) that at $n_0 > n_c$ the condition

$$K/n = \beta_0 - \beta > 0$$

should be satisfied. Owing to π condensation, the density n_0 differs from n_0^0. Writing $\mathcal{E}(n)$ at $n_0 > n_c$ in the form

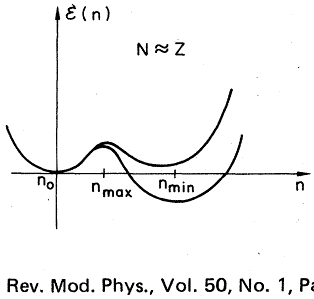


FIG. 15. Plots of energy density of a nucleus with $(N=Z)$ against the density n, with allowance for condensation.

$$\mathcal{E}(n) \cong \frac{\beta_0(n-n_0^0)^2}{2} - \frac{\beta(n-n_c)^2}{2} ,$$

we obtain from the condition $(d\mathcal{E}/dn)_{n=n_0} = 0$

$$n_0 - n_0^0 = (\beta/\beta_0)(n_0 - n_c) < n_0 - n_c . \quad (7.3a)$$

We now consider the case when $n_c > n_0^0$. Then $\mathcal{E}(n)$ has a minimum at $n = n_0^0$. If $\beta_0 > \beta$, then this minimum is a single one. On the other hand if $\beta > \beta_0$, then a second minimum must exist. Indeed, in this case, at $n = n_{max} < n_c$, a maximum appears, with n_{max} determined by an expression similar to (7.3a):

$$n_{max} - n_0^0 = (\beta/\beta_0)(n_{max} - n_c) . \quad (7.3b)$$

With further increase of the density, as already mentioned, the growth of $\mathcal{E}(n)$ resumes and $\mathcal{E}(n)$ is minimal at $n = n_{min} > n_{max}$.

If the first minimum corresponds to ordinary nuclei, then the nuclei should have a density n_0^0 and have no condensate. A discrepancy between the observed density n_0 and the calculated value n_0^0 is then attributable not to the condensate but to the inaccuracy of the calculations. On the other hand, if the second minimum is more stable and a density n_{min} corresponds to ordinary nuclei, rather than n_0, then the results should be noticeable inequality $n_0 = n_{min} > n_0^0$. The author does not plan to pass judgment as to whether the parameters introduced into the calculations of nuclear-matter energy can be chosen such that n_0^0 turns out to be, say, $n_0^0 \cong n_0/2$. A developed condensate should then exist in the nuclei. Metastable nuclei with anomalously low density could exist in such a case (with $n = n_0^0 < n_0$).

Thus, at $n_0 > n_c$ and at $\beta < \beta_0$ the second equilibrium state exists only at $\kappa > 0$ (which is doubtful). At $n_c > n_0$ and at $\beta < \beta_0$, there is one equilibrium state with $n = n_0^0$, and at $\beta > \beta_0$ there are two equilibrium states with densities n_0^0 and n_{min}. If the nuclei correspond to $n_0 = n_0^0$, then one should seek a second state with higher density. On the other hand, if the nuclei correspond to $n_0 = n_{min}$, then there should exist a state with an anomalously low density $n = n_0^0$.

2. Neutron nuclei

For simplicity we shall disregard the complicated density dependence introduced into the condensate energy of a neutron medium by π_s^+ condensation (Sec. V), and write the energy density $\mathcal{E}_\pi^{(n)}$ in the same form as for the case of nuclear matter with $Z \cong N$.

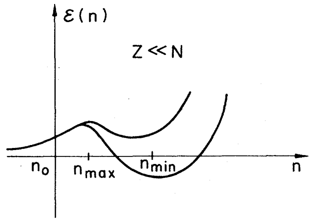
$$\mathcal{E}_\pi^{(n)}(n) = -\frac{\beta_n(n-\bar{n}_c)^2}{2} , \quad (7.4)$$

where $\beta_n \cong 1$, $n_c^+ < n_c^- < n_c^\pm$, n_c^+ and $n_c^\pm$ are the critical densities of the π_s^+ and $\pi_s^+\pi^-$ condensations, and $\bar{n}_c \sim n_0$. In contrast to the case $N = Z$, the function $\mathcal{E}_N^{(n)}$ has no minimum—no neutron nuclei exist without allowance for the condensate energy. From the calculations of the energy of nuclear matter we have

$$(d\mathcal{E}_N^{(n)}/dn)_{n=n_0} = 20 \text{ MeV} \cong 0.15 .$$

Then $\mathcal{E}^{(n)}(n) = \mathcal{E}_N^{(n)}(n) + \mathcal{E}_\pi^{(n)}(n)$ has a maximum at $n = n_{max}$, where n_{max} is determined by the condition

FIG. 16. Plot of the energy density of neutron nuclei against the density n, with allowance for condensation.



$$\frac{d\mathscr{E}^{(n)}(n)}{dn}=0,\ \frac{n_{\max}-\bar{n}_c}{n_0}=\frac{(d\mathscr{E}/dn)}{\beta_n n_0}\sim 0.3\,. \qquad (7.5)$$

With further increase of the density, the growth of the condensate energy becomes weaker, and in addition, repulsion forces come into play. As a result, a minimum is produced at $n=n_{\min}>n_{\max}$. Thus, at a density $n=n_{\min}$ there exists a nuclear-matter state with positive compressibility (the question of the stability of this state is discussed in Sec. VII. B). This means that with a sufficient number of neutrons, when the surface effects can be neglected, neutron nuclei with density $n\simeq n_{\min}$ can exist.

A pure neutron state will acquire a charge because of the process $n\to n+\pi_s^+ +e+\nu^-$ and, as we shall show in Sec. VII.B, can go over into a $\beta-$ stable state with $Z\ll N(N\gtrsim 10^5)$.

3. Supercharged nuclei

In a nucleus with $Ze^2/R>2m_\pi c^2$ which corresponds to $Ze^3\gtrsim 1$, $\pi^+\pi^-$ condensation should occur (Sec. II.A). The gain of the condensate energy may exceed the Coulomb repulsion energy and those nuclei would be stable. In fact the depth of the Coulomb potential well cannot reach the value $2m_\pi c^2$ due to the screening effect of the vacuum electrons created near the nuclei at $Ze^3\gtrsim 1$. The distribution of the vacuum electrons ("electron condensation") was considered in Müller, Rafelski (1975) and Migdal, Popov, and Voskresenskii (1976, 1977). At $Ze^3\gg 1$ the proton charge is screened by electrons inside the nucleus, and only the surface charge in a thin layer near the nuclear surface remains. Thus the Coulomb energy is considerably suppressed by electron screening. Nevertheless the kinetic energy of electrons makes a positive contribution to the nuclear energy and leads to the instability of these nuclei.

A greater release of energy is obtained in the case of one-pion condensation when the proton charge is screened by π^- mesons. But even in this case the energy of the nucleus is still positive [Chernoutsan, Voskresenskii (1977)]. The stability of supercharged nuclei apparently cannot be provided by the pure electric condensation. One should take into account the strong interaction and consider the condensation with the wave vector k_0 corresponding to the minimal value of $\omega^2(k)$.

If the critical density n_c only slightly exceeds the normal density n_0, then π^- condensation may appear in a normal nucleus. In fact, consider for simplicity the case $N=Z$. The π^- meson energy is determined by the equation

$$(1+k^2+\Pi(k,\hat{\omega})-\hat{\omega}^2)\psi=0\,, \qquad (7.6)$$

where

$$k=\frac{1}{i}\nabla,\ \hat{\omega}=\omega-V$$

Expanding $\Pi(k,\omega)$ in powers over ω and using only the first terms (which is correct if $|V|\ll kv_F$), one has

$$[(1-\partial\Pi/\partial\omega^2)\hat{\omega}^2-\tilde{\omega}^2(k^2)]\Psi=0\,. \qquad (7.7)$$

Here $\tilde{\omega}^2(k^2)=1+k^2+\Pi(k,0)$ has a minimum at $k=k_0$. Using this equation for $\tilde{\omega}^2$, multiplying it by Ψ, and integrating it is straightforward:

$$\omega^2-2\bar{V}\omega-\frac{\overline{\tilde{\omega}^2(k^2)}}{1-\partial\Pi/\partial\omega^2}+\overline{V^2}=0\,. \qquad (7.8)$$

The bar means averaging over $|\Psi^2|$ and $|\Psi|^2$ assumed to be normalized to 1. It follows from this equation that

$$\omega=\bar{V}+\left(\bar{V}^2-\overline{V^2}+\frac{\overline{\tilde{\omega}^2}}{1-\partial\Pi/\partial\omega^2}\right)^{1/2}\equiv\bar{V}+\omega_m\,. \qquad (7.9)$$

Thus the field V shifts the system towards condensation. The π^- condensation starts at $\omega=0$. Therefore the condensation may occur in the region of stable nuclei if

$$|\bar{V}|>\omega_m\,.$$

For highly charged nuclei, fission instability is most important. Fission stability is possible only if the Coulomb energy is considerably suppressed. This means that the π^- charge should be of the order of Z. As we have shown [Migdal, Voskresenskii, and Chernoutsan (unpublished)], at $\Lambda\sim 1$; $Z_\pi\simeq Z$ at $Ze^3\sim 1$. Thus the considerable suppression of the Coulomb energy at $Ze^3\sim 1$ can lead to the stability of supercharged nuclei. The relation between the energy of these nuclei and the energy of superdense nuclei with the same charge remains open. We hope to consider this problem in more detail elsewhere.

4. Instability of nucleon field (the Lee model)

Lee (1974) considered a system of nucleons interacting with a field of scalar mesons. The corresponding Hamiltonian is

$$H=H_N+g\bar{\psi}\psi\varphi+H_\varphi\,.$$

It was assumed that scalar mesons exist with mass $M_\varphi\cong M_N$ and with an interaction constant $g\simeq 15$. Of course, at such a tremendous interaction constant one cannot speak of determining the system energy in analytic form, and only guiding considerations are possible. Lee (1974) considered for this purpose an expression for the system energy in the self-consistent-field approximation, which is valid when $g\ll 1$. The main result of his approach was that at $n=n_c\simeq M_N M_\varphi^2/g^2$ the matter becomes unstable to the production of nuclear pairs, and this leads to the appearance of a new stable state of nuclear matter with density $n\cong n_c\cong 2n_0$.

We shall first repeat Lee's results in a form more convenient for our purposes, and then show that a more realistic analysis shifts the critical density into the region $n_c\sim 100n$, where repulsion at short distances plays a decisive role.

For simplicity we consider nonrelativistic nucleons. Then our results are exact at densities on the order of those of the nuclei, and will be of the correct order of magnitude in the region where the instability of the nucleon field appears. We introduce the energy density as a function of φ_0 and n (φ_0 is the static field)

$$\mathcal{E}(n, \varphi_0) = \mathcal{E}_N(n) + gn\varphi_0 + \frac{M_\varphi^2 \varphi_0^2}{2}$$

Minimizing with respect to φ_0, we obtain

$$\varphi_0 = -\frac{gn}{M_\varphi^2}, \ \mathcal{E}(n) = \mathcal{E}_N(n) - \frac{g^2 n^2}{2M_\varphi^2} \equiv \mathcal{E}_N + \mathcal{E}_\varphi \tag{7.10}$$

which corresponds to taking into account the interaction between the nucleons on account of φ-meson exchange (in the lowest order in g^2). The effective potential acting on the nucleon as a result of the field φ_0 is

$$V = \frac{\partial \mathcal{E}_\varphi(n)}{\partial n} = -\frac{g^2 n}{M_\varphi^2}$$

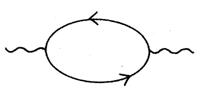
Instability sets in when this field "eats up" the mass of the nucleon and, consequently, the critical density n_c is equal to

$$n_c \cong M_N M_\varphi^2 / g^2 \ . \tag{7.11}$$

This formula coincides with the expression obtained by Lee (1974).

An analysis of this type is not convincing for a number of reasons. First, the interaction is very strongly renormalized in the medium. Let us find the parameters characterizing this renormalization.

We consider by way of example one of the diagrams determining the polarization operator of mesons in a medium at $\omega = 0$ and $k \to 0$. We obtain

$\Pi(0, k \to 0) =$  $= -g^2 \dfrac{dn}{d\epsilon_F}$.

The meson mass renormalization is given by

$$\tilde{M}_\varphi^2 = M_\varphi^2 - g^2 \frac{dn}{d\epsilon_F} = M_\varphi^2(1-\xi) \ . \tag{7.12}$$

Thus the role of the parameter determining the renormalization is played by the quantity $\xi = (g^2/M_\varphi^2)/(dn/d\epsilon_F)_{n=n_0} \cong 10$. Yet it follows from (7.12) that the equilibrium condition is $\xi < 1$. At $\xi = 1$, instability of the meson field sets in and the matter begins to become compressed. Indeed, the denominator of the second term of (7.10) will contain not M_φ^2, but the quantity $M_\varphi^2(1-\xi)$, which vanishes at $\xi = 1$, and $d^2\mathcal{E}/dn^2$ becomes less than zero (negative compressibility).

Of course, the calculation of all the essential diagrams at $\xi \geqslant 10$ is an insoluble problem, but the qualitative result is as follows: there are two possibilities: (1) allowance for all the diagrams leads to a change of sign of the interaction between the nucleons and to an effective interaction constant $g_{\rm eff}^2 \simeq 1$, (2) the system is unstable and will be compressed until repulsion at short distances cancels out the attraction. Thus, either the interaction constant $g_{\rm eff}$ is small ($g_{\rm eff}^2 \sim 1$), or else repulsion must be taken into account in addition to attraction.

The effective field V acting on the nucleon at a nuclear density n_0($n_0 = 0.5$) and $g = 15$ corresponds to a well depth equal to

$$V_0 = \frac{g^2 n_0}{M_\varphi^2} \cong 300 \text{ MeV} \ .$$

It is clear that this well has no connection whatever to reality, and consequently the second term in (7.10), if it does exist at all, should be almost completely canceled by the repulsion present in $\mathcal{E}_N(n)$.

At a constant $g = 15$ and $\xi = 10$, we must give up as hopeless any attempt to calculate the system energy. We can, however, carry out a very convincing phenomenological analysis that makes use of the properties of nuclear matter at densities $n = n_0$.

We first present a rough estimate of the lower limit n_c, which follows from the condition of equilibrium of nuclear matter at a nuclear density n_0.

Writing the total energy density near $n = n_0$ in the form

$$\mathcal{E} = \frac{3}{5}\epsilon_F n - \frac{\gamma n^2}{2}$$

we obtain from the stability condition

$$\frac{d^2\mathcal{E}}{dn^2} = \frac{2\epsilon_F}{3n_0} - \gamma > 0 \ ,$$

where γ includes all the nuclear interactions at $n \sim n_0$ (the interaction introduced above, $\gamma = g^2/M_\varphi^2$).

For the critical density we have

$$n_c = \frac{m_N}{\gamma} > \frac{3m_N n_0}{2\epsilon_F} \sim 30 n_0 \ .$$

Let us trace in greater detail the restrictions imposed by the condition of nuclear matter stability at $n = n_0$.

We rewrite $\mathcal{E}_N$ in the form

$$\mathcal{E}_N(n) = \frac{3}{5} n\epsilon_F + \phi(n) \ ,$$

where $\phi(n)$ contains the entire interaction, including the influence of repulsion at short distances.

Variation of $\mathcal{E}$ with respect to density yields

$$\delta\mathcal{E} = \left(\epsilon_F + \frac{d\phi}{dn} - \frac{g^2 n}{M_\varphi^2}\right)\delta n \ .$$

It is the increment to ϵ_F which is the effective field acting on the nucleon near the Fermi surface. Therefore the condition of instability with respect to pair production is determined by the relation

$$M \cong \frac{g^2 n_c^N}{M_\varphi^2} - \left(\frac{d\phi}{dn}\right)_{n = n_c^N} \ . \tag{7.13}$$

We estimate $d\phi/dn$ from the condition for the stability of nuclear matter at $n = n_0$. We have

$$\frac{d\mathcal{E}}{dn} = \epsilon_F + \left(\frac{d\phi}{dn}\right)_{n_0} - \frac{g^2 n_0}{M_\varphi^2} = 0$$

$$\frac{d^2\mathcal{E}}{dn^2} = \frac{2}{3}\frac{\epsilon_F}{n_0} + \left(\frac{d^2\phi}{dn^2}\right)_{n_0} - \frac{g^2}{M_\varphi^2} = \beta_0 = \frac{K_0}{n_0} > 0 \ .$$

It follows therefore that

$$n_0\left(\frac{d^2\phi}{dn^2}\right)_{n_0} - \left(\frac{d\phi}{dn}\right)_{n_0} = K_0 + \frac{1}{3}\epsilon_F \ .$$

Thus $d\phi(n)/dn$ increases in the region $n \sim n_0$, with increasing n, more rapidly than n. In the region of large n, the growth of $d\phi/dn$ increases because of repulsion at short distances. If no account is taken of π condensation, there are no grounds whatever for expecting a nonmonotonic increase of $d\phi/dn$. Therefore the quantity $g^2 n/M_\varphi^2 - d\phi/dn$ should decrease with increasing n and should reverse sign at $n \gtrsim n_0$. Thus the condition (7.13) is not satisfied, and the system remains stable with respect to nucleon-pair production. This phenomenological analysis, in contrast to that of Lee (1974), does not mean the use of perturbation theory, and the term $g^2 n/M_\varphi^2$ is separated from the interaction energy only to compare our conclusions with Lee's results. Lee attempted to take the repulsion into account by excluding the region of small distances between the nucleons. The influence of repulsion on the effective potential acting on the nucleon is then lost. It appears that this is the reason why the value obtained by Lee (1974) for the critical density is too low. Owing to π condensation, the growth of the effective field ceases to be monotonic, but even in this case the effective potential is of the order m_π and is insufficient to compensate for the nucleon mass. Thus there are no grounds for expecting nucleon instability, at least up to very high densities ($\sim 100\ n_0$), at which new phenomena can set in (for example, condensation of heavier resonances).

B. Stability of anomalous nuclei

Interpolation formulas are obtained for the energy of the baryon subsystem and for the energy of the condensate at an arbitrary density; at $n \sim n_0$ and $n \gg n_0$ these formulas coincide with the well known expressions. Conditions are formulated for the stability of the anomalous nuclei with respect to fission, particle evaporation and β decay. Two stability regions are possible: at small A with $Z/A \cong 1/2$ (superdense nuclei) and at large A with $Z/A \ll 1$ (neutron nuclei). Curves of the nuclear energy against density are plotted. At certain parameter values, a minimum corresponding to the existence of stable or β-active superdense and neutron nuclei can appear on these curves.

The accuracy of the theory is at present insufficient to conclude definitely that such nuclei exist, but this existence is feasible for a reasonable choice of nuclear constants. Possible ways of observing superdense and neutron stars are discussed.

1. Energy of a nucleus with allowance for condensation as a function of the density

The energy of a system of A nucleons with charge Z and with density n, reckoned from the sum of the nucleon masses, can be written in the form

$$E(n, A, \nu) = \epsilon_B(n, \nu)A + a_S(n, \nu)A^{2/3} + a_Q(n)\nu^2 A^{5/3} + \epsilon_\pi(n, \nu)A \ , \qquad (7.14)$$

where $\nu = Z/A$ and $Z = Z_B + Z_\pi$ is the sum of the baryon and pion charges. The terms proportional to $A^{2/3}$ and $A^{5/3}$ correspond to the surface and the Coulomb energy. The last term is the energy connected with the appearance of the pion condensate.

We shall neglect the corrections necessitated by pairing, deformation, and shell effects. The quantities in (7.14) were calculated for two limiting regions: $1 - 2\nu \ll 1$ and $\nu \ll 1$. As we shall see below, it is precisely these regions which are of greatest interest.

We consider the case $1 - 2\nu \ll 1$ and obtain first the baryon energy. In the density region $n - n_0 < n_0$, the volume part of the baryon energy can be expressed in terms of the compressibility of nuclear matter K. We have

$$\epsilon_B(n, \nu) = -\epsilon_0 + \alpha(n)(1 - 2\nu)^2 + \frac{K}{2}\left(1 - \frac{n}{n_0}\right)^2 , \qquad (7.15)$$

where $\epsilon_0 = 15.7$ MeV $= 0.11$; $\alpha(n_0) = 25$ MeV $= 0.18$. According to the theory of finite Fermi systems, K is expressed in terms of the constant f_0 of the NN interaction

$$K = \tfrac{2}{3}\epsilon_F(1 + 2f_0),$$

where $f_0 \approx 0.25$, whence $K = 40$ MeV $= 0.29$ (Osadchiev and Troitskii, 1968).

At large densities $n \gg n_0$, a strongly developed condensate is produced and the baryon system is substantially restructured: the two (neutron and proton) Fermi spheres are replaced by a single Fermi sphere of a baryon quasiparticle constituting a superposition of six baryons N^{*++}, N^{*+}, p, n, N^{*0}, and N^{*-} (see Sec. V.B). The potential energy of these quasiparticles is determined at high densities by the short-range repulsion of the baryons. If it is assumed that this repulsion is the same for all the participating baryons, then the baryon energy, at sufficiently high densities, should coincide with the energy of the neutron matter without allowance for condensation. This energy was calculated by Pandharipande (1971).

Under this assumption, the volume part of the baryon energy can be expressed, in the entire interval of the densities n, in the form of an interpolation formula:

$$\epsilon_B(x) = -0.11 + \frac{0.14x^2}{0.37x + 1} , \qquad (7.16)$$

where $x = (n - n_0)/n_0$. The function (7.16) at $x < 1$ and $\nu = 1/2$ coincides with expression (7.15), and at $x = 6.36$ ($n = 7.35 n_0 = 1.25 F^{-3}$) it is chosen such that it coincides, together with the first derivative $d\epsilon_B/dx$, with the results of the calculations of Pandharipande (1971). The surface term in (7.14) can be estimated under the assumption that the width of the surface layer depends little on the density. Then the surface energy is proportional to the energy per unit volume. We have

$$a_S = 0.13\left(\frac{n}{n_0}\right)^{1/3} \frac{\epsilon(n, \nu)}{\epsilon(n_0, \frac{1}{2})} \qquad (7.17)$$

at $n = n_0$ and $\nu = \frac{1}{2}$, this expression coincides with the corresponding term of the Weizsäcker formula. The factor a_Q, which determines the Coulomb energy under the assumption that the charge is uniformly distributed, has the following value in pion units:

$$a_Q = 5 \times 10^{-3}\left(\frac{n}{n_0}\right)^{1/3} . \qquad (7.18)$$

For the condensate energy at $\nu \cong \frac{1}{2}$ we have in the case of low densities $n-n_c \lesssim n_0$ (Sec. V.A)

$$\epsilon_\pi(n)=\frac{\mathcal{E}_\pi}{n}=-\frac{\beta}{2}\frac{(n-n_c)^2}{n} \quad . \tag{7.19}$$

In the case of high densities, we can use the expression obtained by Baym *et al.* (1975) for the case of a developed condensate (Sec. V.B).

We have

$$\epsilon_\pi(n,\nu)=\epsilon_\pi(n)+\alpha_\pi(1-2\nu)^2 \quad , \tag{7.20}$$

$$\epsilon_\pi(n)=-\left[\frac{81}{50}f^2(1-\gamma)n-\frac{\Delta}{3}\right], \tag{7.21}$$

$$\alpha^{(\pi)}=\frac{n}{2F^2}=0.14\frac{n}{n_0} \tag{7.22}$$

where $\Delta=m_{N*}-m=294$ MeV $=2.1$.

In these expressions, $F=1.35$ is the pion decay constant. The πN interaction constant is connected with the constant F and with the axial constant g_A by the relation $f=g_A/F$. As shown by comparison of theory with experiment, in ordinary nuclei, a weak renormalization $f\to f'=0.9f$ takes place (Migdal, 1965). This renormalization is taken into account in the calculation presented below.

The quantity γ takes into account the contribution of the nucleon correlations, and is connected with the constants g^- and g' introduced above by the relation

$$g^-=2g'=f'^2\frac{2mp_0}{\pi^2}\gamma ,$$

where p_0 is the Fermi momentum at normal density ($p_0=1.92$).

We have omitted from (7.21) the term $F^2/4n$, inasmuch as in the region of the densities of interest to us it is almost completely cancelled by the second term of the expansion of the condensate energy over the parameter $\Delta/g_A k$, where k is the wave number of the condensate field.

In the derivation of (7.21) it was assumed by Baym *et al.* (1975) that, up to Clebsch-Gordan coefficients, the local amplitudes of the NN, NN^*, and N^*N^* interaction in a nucleon medium are the same. This assumption, however, is theoretically unfounded. At present there is no direct experimental information on the NN^* and N^*N^* interactions. It appears that the local NN^* interaction is much weaker than the NN interaction, as follows from experiments on (pp, N^*n) scattering with large momentum transfers (Mountz *et al.*, 1975).

Allowance for this fact would lead to an increase of the condensate energy. On the other hand, Baym *et al.* (1975) did not take into account the suppression of the NN^* vertex, which leads to a lowering of the condensate energy.

It is at present impossible to take all these effects into account. We shall use (7.21) as a reasonable estimate of ϵ_π at high densities.

Equations (7.19) and (7.21) can be written in the form of an interpolation formula suitable in the entire range of values of n of interest to us (for $\nu\cong\frac{1}{2}$).

$$\epsilon_\pi=-\frac{\beta(n)}{2}\frac{(n-n_c)^2}{n} \quad , \tag{7.23}$$

TABLE IV. Parameters that determine the condensate energy [Eq. (7.23)].

$\nu=Z/A$	f'/f	γ	n_c	β	A	B	C
0.5	0.9	0.45	0.54	0.69	1.47	0.32	−1.10
0.5	0.9	0.5	0.65	0.81	1.34	0.49	−1.02
0.5	0.9	0.55	0.79	0.89	1.19	0.63	−0.93
0	0.9	0.45	0.69	0.80	0.91	−0.20	0.09
0	0.9	0.5	0.79	0.63	0.78	−0.21	0.06
0	1	0.4	0.48	1.19	1.42	−0.05	−0.18
0	1	0.45	0.54	1.11	1.26	−0.08	−0.07

where

$$\beta(n)=A+\frac{Bn_c}{n}+C\frac{n_c^2}{n^2} .$$

The coefficients A, B, and C are given in Table IV. They are such that the values of $\beta(n)$ coincide with the results given in Table IV at $n=n_c$, and that $\epsilon_\pi(n)$ together with $d\epsilon_\pi/dn$ coincide with (7.21) as $n/n_c\to\infty$.

We proceed now to the case $\nu\ll 1$, which we shall need in order to assess the stability of neutron nuclei.

For the baryon energy we can use, in the entire density interval, the results of calculations for neutron matter (Pandharipande, 1971).

For the pion energy we use the condensate energy near the critical point, calculated in Sec. V.A, and for large densities we use Eqs. (7.20), (7.21), and (7.22) with $\nu\ll 1$. Again, writing an interpolation formula for $\epsilon_\pi(n)$ in the form (7.23), but with different coefficients A, B, and C, we can find the condensate energy in the entire density interval.

2. Stability conditions

We formulate the equilibrium conditions that must be satisfied by a finite system of particles at zero pressure.

(1) *A positive mass defect*

$$-E>0. \tag{7.24}$$

This satisfies automatically the condition that the nucleons be bound, i.e., that the chemical potentials of the neutrons and the protons be negative

$$\mu_n=\left(\frac{\partial E}{\partial N}\right)_Z<0 \quad \mu_p=\left(\frac{\partial E}{\partial Z}\right)_N<0 .$$

It is easy to verify that at $E>0$ the chemical potentials of the nucleons are positive, and consequently the system is unstable with respect to particle evaporation.

(2) *β equilibrium (the electrons are free to leave)*

$$\left(\frac{\partial E}{\partial Z}\right)_A=\mu_p-\mu_n=0 . \tag{7.25}$$

(3) *Stability with respect to pressure*

$$\frac{Z^2}{A}<50f(n,\nu) . \tag{7.26}$$

The right-hand side of this inequality is determined by the ratio of the coefficients in the surface and Coulomb energies.

On the basis of (7.17) we have

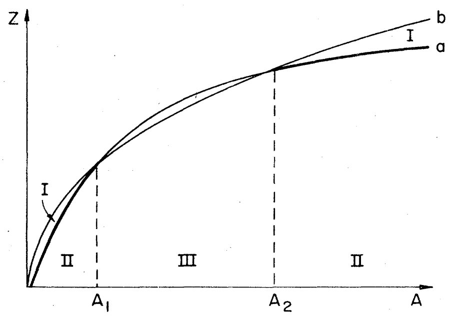


FIG. 17. Regions of existence of anomalous nuclei. (a) Curve corresponding to β stability; (b) limit of stability to fission (the region of stability is below curve b). The thicker parts of curve (a) correspond to stable anomalous nuclei. I, regions of β^+ activity; II, regions of β^- activity that lead to a stable state; III, region of β^- activity that leads to fission.

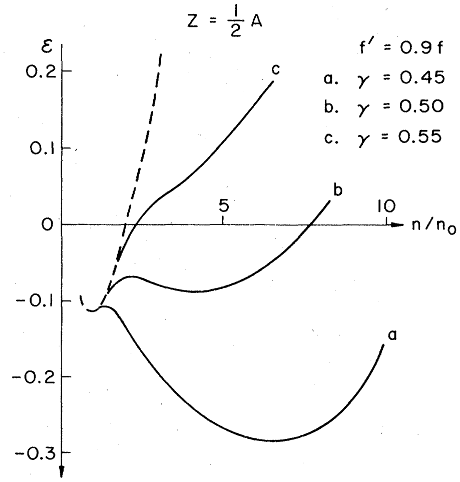


FIG. 18. Energy of nuclear matter per baryon in the case $Z/A = 1/2$. Dashed line, baryon energy [Eq. (7.16)]. Curves a, b, and c correspond to (NN) interaction constants $\gamma = 0.45$, 0.5, and 0.55, respectively.

$$f(n,\nu) = \frac{\epsilon(n,\nu)}{\epsilon(n_0,\frac{1}{2})} . \tag{7.27}$$

At $n = n_0$ and $\nu \cong \frac{1}{2}$ we obtain the known criterion of stability with respect to fission.

We assume now that condition (1) is satisfied; we then obtain from (7.14), (7.20), and (7.25) the equilibrium value of ν at high densities:

$$\nu = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{4}\frac{a_Q}{\alpha_\pi}A^{2/3}\right), & 1 - 2\nu \ll 1 \\ 2\frac{\alpha_\pi}{a_Q}A^{-2/3}, & \nu \ll 1 \end{cases} \tag{7.28}$$

From (7.26) and (7.28) we obtain two stability regions: (1) at $A < A_1 = 200 f(n, 1/2)$, and $\nu = 1/2$; (2) at $A > A_2 = 2\times 10^5 (n/n_0)^4 f^{-3}(n,0)$ and $\nu = 53(n/n_0)^{2/3}A^{-2/3}$.

The first region corresponds to superdense nuclei. The second region corresponds to neutron nuclei. These nuclei, in spite of the small Z/A ratio, have a charge high enough for the Coulomb energy to forbid β decay, but at the same time Z^2/A is small enough for fission to be impossible. Nuclei with a Z/A ratio different from the equilibrium value are β-active. The regions in which stable and β-active anomalous nuclei exist are shown in Fig. 17. We note that in the region of β^+ activity (I) and in the region of β^- activity (II) the evolution of the nucleus terminates at the stability line, whereas from region (III) the nucleus will go across the limit of fission stability.

It is seen from the foregoing expressions that A_1 and A_2 depend essentially on the parameters of the model.

3. Estimate of the density and of the binding energy of anomalous nuclei

We note first that at high densities the total energy of the system is a difference of two large numbers, the baryon-subsystem energy and the condensate energy, and these cancel each other to an appreciable degree. The accuracy of the calculations of each of the terms is at present low (at best we have order-of-magnitude estimates), and consequently the results of calculations of the total energy of the system should be regarded only as illustrations of the various possible cases.

Using the interpolation formulas obtained above for $\epsilon_B(n)$ and for $\epsilon_\pi(n)$, and substituting them in (7.14), we can obtain the $E(n,\nu)$ curve or the $\epsilon(n,\nu)$ curve at different values of the parameters entering in the problem, for two regions of the equilibrium values of ν ($\nu \cong \frac{1}{2}$—superdense nucleus, and $\nu \ll 1$—neutron nucleus). If the minimum corresponding to the anomalous nucleus lies below zero, then the system is bound.

We consider first superdense nuclei.

The results of the calculations of $\epsilon(n)$ for superdense nuclei ($\nu \cong \frac{1}{2}$) are shown in Fig. 18. Curve (a), calculated for $\gamma = 0.45$, demonstrates the case when the binding energy of the superdense nuclei exceeds the binding energy of the ordinary nuclei. If such a situation were actually to take place, then normal nuclei would be metastable with respect to transitions to the superdense state. The minimum corresponding to the superdense nuclei on curve (b), calculated at $\gamma = 0.5$, lies higher than the minimum corresponding to ordinary nuclei. In this case the superdense nuclei would be the metastable ones. We note also one important circumstance. In the calculations of curves (a) and (b) in Fig. 18 we used nuclear-constant values such that $n_c > n_0$. The condition $n_c < n_0$ is possible (Migdal, 1972; Migdal, Markin, and Mishustin, 1974); in that case the pion condensate exists in ordinary nuclei, and the constants that characterize the ordinary nuclei already contain the contribution of the condensate. It is most probable that in such a case no superdense nuclei exist.

The results of the calculation of the energy of neutron matter ($\nu \ll 1$), with the condensate taken into account, are shown in Fig. 19. The dashed curve is the energy of the neutron matter without condensation. Curves (a) and (b) were calculated with the same values of the nuclear constant as the corresponding curves of Fig. 18. It is seen that in this case the energy gain due to π condensation is insufficient for a bound state to be formed. However, one cannot exclude the possibility of a certain change of the constants with changing isotopic composi-

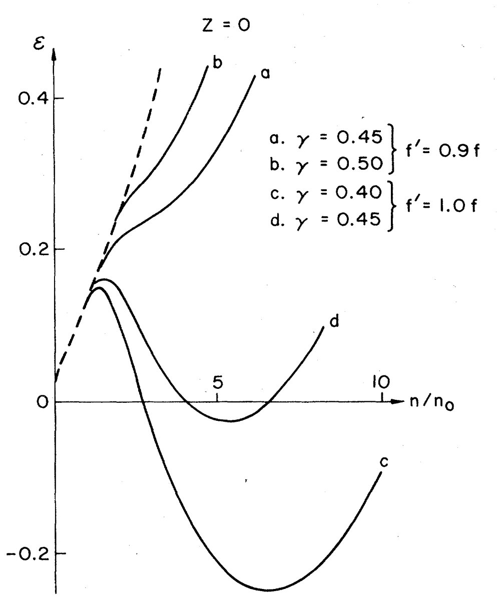


FIG. 19. Energy of neutron matter per baryon. Dashed line, energy of neutron matter without allowance for condensate. Curves a and b correspond to the same values of the parameters as in Fig. 18. Curves c and d, calculated for $f'=f=1.0$ and for $\gamma=0.4$ and 0.45, correspond to stable neutron nuclei.

tion of the medium. Curves (c) and (d) of Fig. 19, calculated at $g_A=1.36$ and at $\gamma=0.4$ and $\gamma=0.45$ illustrate the case in which a bound state appears for neutron nuclei.[5]

Thus our analysis depends essentially on the pion-nucleon and nucleon-nucleon interaction constants, which determine the π condensate energy.

Beside the uncertainty coming from the lack of knowledge on the interaction constants, the model used by us (Campbell, Dashen, and Manassah, 1975; Baym *et al.*, 1975) takes into account only the energy connected with a charge-pion condensate having a spatial running-wave structure. As shown by Migdal (1972, 1973) and by Migdal, Markin, and Mishustin (1974), in a nucleon medium at a density close to n_0, there should appear also a neutral-pion condensate, which leads to an additional energy gain. Moreover, the minimal energy of the system can correspond to a condensate-field spatial structure more complicated than a running wave (Migdal, Markin, and Mishustin, 1976; Markin and Mishustin, 1974; Sorokin, 1975). All these effects are additional factors favoring the existence of anomalous nuclei. On the other hand, the choice of a stiffer equation of state of the nucleon subsystem than that given by Pandharipande (1971), and also allowance for the suppression of the pion-nucleon vertices at large momentum transfers, would lead to an increase of the total system energy. At the present time it is impossible to take all these effects into account with the required accuracy, and the main conclusion that can be drawn on the basis of the foregoing analysis is that the existence of anomalous nuclei is theoretically not excluded, and the final solution of the problem can be provided only by experiment.

[5]As shown by Chernoutsan, Sorokin, Voskresenskii (1977), taking into account vacuum electrons, screening the Coulomb field of the nuclei with $Z\gtrsim 1/e^3$, significantly extends the region of stability for such a nuclei.

4. Possible ways of observing anomalous nuclei

We now make a few remarks concerning possible experiments aimed at observing anomalous nuclei.

If superdense nuclei do exist, it is not clear to which of the nuclei, normal or superdense, the larger binding energy corresponds. It is possible in principle that superdense nuclei have the larger binding energy. The experimental limitation on spontaneous transitions of normal nuclei into the superdense state are of particular interest in this connection. We note that so far, searches for nuclei with anomalously high binding energy have yielded no results. (Price and Stevenson, 1975; Holt *et al.*, 1975; Frankel *et al.*, 1976).

Searches for stable or short-lived β-active anomalous nuclei of small dimension ($A\cong 100$) in the fission products of ordinary nuclei are also of interest.

It is possible that superdense nuclei can be produced in collisions between heavy ions, at energy on the order of several hundred MeV per nucleon and affect the angular and energy distribution of the reaction products. This problem was considered by V. Ruck *et al.* (1976). A more detailed analysis of the effect of the phase transition on the shock wave dynamics was made by Galitskii *et al.* (1977). At sufficiently large β, the compressibility of the system becomes negative already at $n=n_c$. Therefore in order to initiate the formation of the superdense phase, it should be sufficient to compress the system to a density n_c. Regardless of whether stable superdense nuclei exist or not, pion condensation should greatly influence the dynamics of the collisions.

Finally, one can hope to observe anomalous nuclei in cosmic rays, as noted already by Migdal (1971).

It is interesting to note in this connection that the track ascribed to the magnetic monopole by Price *et al.* (1975b) can be interpreted as the track of an anomalous (neutron) nucleus. The possibility of observing in cosmic rays stable anomalous nuclei or their β-active fragments with anomalous Z/A, produced by interaction with nuclei of the atmosphere, should be taken into account when the experiments are performed and analyzed. It is also of interest to search for superdense nuclei of cosmic origin, accumulated over cosmologic time periods in the surface layers of lunar soil and in meteorites.

REFERENCES

Alkhazov, G., G. M. Amalsky, S. L. Belostatsky, A. A. Vorobyov, O. A. Domchenkov, Yu, V. Dotsenko, and V. E. Starodubsky, 1972, Phys. Lett. B **42**, 121.

Adler, S. L., 1965, Phys. Rev. B **137**, 1022.

Anderson, P. W., N. Itoh, A. M. Alpar, E. Tosatti, and R. G. Palmer, 1975, Nuovo Cimento Lett. **12**, 165.

Au, C. K., 1976, Phys. Lett. B **61**, 300.

Au, C. K., and G. Baym, 1974, Nucl. Phys. A **236**, 500.

Backman, S. O., and W. Weise, 1975, Phys. Lett. B **55**, 1.

Barshay, S., G. M. Vagradov, and G. E. Brown, 1973, Phys. Lett. B **43**, 359.

Barshay, S., and G. E. Brown, 1973, Phys. Lett. B **47**, 107.

Baym, G., 1973, Phys. Rev. Lett. **30**, 1340.

Baym, G., D. Campbell, R. Dashen, and J. Manassah, 1975, Phys. Lett. B **58**, 304.

Bellicard, J., P. Brounin, R. F. Frosch, R. Hofstadter, J. S. McCarthy, F. J. Uhrhane, M. R. Yearian, B. C. Clark, R. Herman, and D. G. Ravenhall, 1967, Phys. Rev. Lett. **19**, 527.
Bertsch, G., and M. Johnson, 1974, Phys. Lett. B **48**, 397.
Bertsch, G., and M. Johnson, 1975, Phys. Rev. D **12**, 1230.
Bethe, H., 1974, *Theory of Nuclear Matter*, Russian translation (Mir, Moscow).
Bethe, H., 1971, Annu. Rev. Nucl. Sci., p. 93.
Chernoutsan, A. I., D. N. Voskresenskii, 1977, "Pion condensation in an electric field of supercharged nuclei" (Preprint, Chernogolovka).
Chernoutsan, A. I., D. N. Voskresenskii, 1978, Yad. Fiz. **27**, 5.
Chernoutsan, A. I., G. A. Sorokin, D. N. Voskresenskii, 1977, Zh. Eksp. Teor. Fiz. Pis'ma Red. **25**, 495.
Campbell, D., R. Dashen, and J. Manassah, 1975, Phys. Rev. D **12**, 979.
Carter, A. A., J. R. Williams, D. V. Bugg, P. J. Bussey, and D. R. Dance, 1971, Nucl. Phys. B **26**, 445.
Dover, C., and R. Lemmer, 1968, Phys. Rev. **165**, 1105.
Dyugaev, A. M., 1975, Zh. Eksp. Teor. Fiz. Pis'ma Red. **22**, 181 [JETP Lett. **22**, 83 (1975)].
Ericson, M., and T. E. Ericson, 1966, Ann. Phys. (N.Y.) **36**, 323.
Frankel, S., W. Frati, V. Highland, O. Van Dyck, and R. Werbeck, 1976, Phys. Rev. C. **13**, 737.
Friar, J., and J. Negele, 1973, Nucl. Phys. **A212**, 93.
Galitskii, V. M., and I. N. Mishustin, 1977, "The influence of a phase transition on the dynamics of shock waves in nuclear matter" (Kurchakov Institute preprint).
Gasiorowicz, S., 1966, *Elementary Particle Physics* (Wiley, New York).
Hartle, J. B., R. F. Sawyer, and D.J. Scalapino, 1975, Astrophys. J. **199**, 471.
Heisenberg, J., R. Hofstadter, J. S. McCarthy, I. Sick, B. C. Clark, R. Herman, and D. G. Ravenhall, 1969, Phys. Rev. Lett. **23**, 1402.
Hohler, G., H. P. Jakob, and R. Strauss, 1972, Nucl. Phys. B **39**, 237.
Holt, R. J., J. P. Schiffer, J. Specht, L. M. Bollinger, G. E. Thomas, S. M. Fried, J. J. Hines, and A. M. Friedman, 1975, Phys. Rev. Lett. **36**, 183.
Kirichenko, N. A., and G. A. Sorokin, 1976, "Excited States of Pion Condensed Finite Systems," (Preprint, Chernogolovka, to appear in Phys. Lett.).
Kirzhnits, D. A., and G. Shpatakovskaya, 1972, Zh. Eksp. Teor. Fiz. **62**, 2082 [Sov. Phys.—JETP **35**, 1088 (1972)].
Landau, L. D., and E. M. Lifshitz, 1959, *Electrodynamics of Continuous Media* (Pergamon, New York).
Lee, T. D., 1975, Rev. Mod. Phys. **47**, 267.
Li, G., M. R. Yearian, and I. Sick, 1974, Phys. Rev. C **9**, 1861.
Markin, O. A., and I. N. Mishustin, 1974, Zh. Eksp. Teor. Fiz. Pis'ma **20**, 497 [JETP Lett. **20**, 226 (1974)].
Migdal, A. B., 1971, Zh. Eksp. Teor. Fiz. **61**, 2210 [Sov. Phys.—JETP **34**, 1184 (1972)].
Migdal, A. B., 1972, Zh. Eksp. Teor. Fiz. **63**, 1993 [Sov. Phys.—JETP **36**, 1052 (1973)].
Migdal, A. B., 1973, Nucl. Phys. A **210**, 421.
Migdal, A. B., 1973, Phys. Rev. Lett. **31**, 257.
Migdal, A. B., 1973, Phys. Lett. B **47**, 96.
Migdal, A. B., 1973, "π-condensation in nuclei and neutron stars" (Preprint, Chernogolovka).
Migdal, A. B., 1974, Zh. Eksp. Teor. Fiz. Pis'ma Red. **19**, 539 [JETP Lett. **19**, 284 (1974)].
Migdal, A. B., 1974, Phys. Lett. B **52**, 172.
Migdal, A. B., 1967, *Theory of Finite Fermi Systems* (Wiley, New York) (Russ. ed. 1965).
Migdal, A. B., 1975, *Qualitative Methods in Quantum Theory* (in Russian) (Nauka, Moscow).
Migdal, A. B., 1974, Phys. Lett. B **52**, 264.
Migdal, A. B., 1959, Zh. Eksp. Teor. Fiz. **37**, 249 [Sov. Phys.—JETP **10**, 176 (1960)].
Migdal, A. B., 1973, "Are there any reasonable objections to π condensation?" (Preprint, Chernogolovka).
Migdal, A. B., O. A. Markin, and I. N. Mishustin, 1973, Zh. Eksp. Teor. Fiz. **66**, 443 [Sov. Phys.—JETP **39**, 212 (1973)].
Migdal, A. B., N. A. Kirichenko, and G. A. Sorokin, 1974, Phys. Lett. B **50**, 411.
Migdal, A. B., O. A. Markin, and I. N. Mishustin, 1976 [Sov. Phys.—JETP **43**, (1976)].
Migdal, A. B., D. A. Markin, I. N. Mishustin, G. A. Sorokin, 1976, Phys. Lett. **65B**, 423; 1977, Zh. Eksp. Teor. Fiz. **72**, 1247.
Migdal, A. B., D. N. Voskresenskii, and V. S. Popov, 1976, Zh. Eksp. Teor. Fiz. Pis'ma Red. **24**, 186.
Migdal, A. B., V. S. Popov, and D. N. Voskresenskii, 1977, Zh. Eksp. Teor. Fiz., **72**, 834.
Mountz, J. D., G. A. Smith, A. J. Lennox, J. A. Poirer, J. P. Prukop, C. A. Rey, O. R. Sander, P. Kirk, R. D. Klem, and I. Spirn, 1975, Phys. Rev. D **12**, 1211.
Müller, B., J. Rafelski, 1975, Phys. Rev. Lett. **34**, 349.
Mur, V. D., and V. S. Popov, 1976, Teor. Mat. Fiz. **27**, 204.
Nagels, M., J. de Swart, H. Nielsen, G. Oades, J. Petersen, B. Tromborg, G. Gustafson, A. Irving, C. Jarlskog, W. Pfeil, H. Pilkuhn, F. Steiner, and L. Tauscher, 1976, Nucl. Phys. B **109**, 1.
Osadchiev, V. M., and M. A. Troitskii, 1968, Phys. Lett. B **26**, 421.
Pandharipande, V. R., 1971, Nucl. Phys. A **178**, 123.
Palevsky, H., J. L. Friedes, R. J. Sutter, G. W. Bennett, G. J. Igo, W. D. Simpson, G. C. Phillips, D. M. Corley, N. S. Wall, R. L. Stearns, and B. Gottschalk, 1967, Phys. Rev. Lett. **18**, 1200.
Price, P., and J. Stevenson, 1975, Phys. Rev. Lett. **35**, 487.
Price, P., E. K. Shirk, W. Z. Osborne, and L. S. Pinsky, 1975, Phys. Rev. Lett. **35**, 487.
Rho, M., 1975, in *High Energy Physics and Nuclear Structure*, AIP Conf. Proc., **25** (American Institute of Physics, N.Y.), p. 666.
Ruck, V., M. Gyulassy, and W. Greiner, 1976, Z. Phys. A **277**, 391.
Sapershtein, E. E., S. V. Tolokonnikov, and S. A. Fayans, 1975, Zh. Eksp. Teor. Fiz. Pis'ma Red., **21**, 554 [JETP Lett. **21**, 258 (1975)].
Sapershtein, E. E., and M. A. Troitskii, 1975, Zh. Eksp. Teor. Fiz. Pis'ma Red. **21**, 138 [JETP Lett. **21**, 62 (1975)].
Sapershtein, E. E., and M. A. Troitskii, 1975, Yad. Fiz. **22**, 257 [Sov. J. Nucl. Phys. **22**, 132 (1976)].
Sapershtein, E. E., and M. A. Troitskii, 1976, Izv. AN SSSR ser. fiz. **40**, 95, 103.
Sawyer, R. F., 1973, Phys. Rev. Lett. **31**, 1556.
Sawyer, R. F., 1972, Phys. Rev. Lett. **29**, 382.
Sawyer, R. F., and D. J. Scalapino, 1973, Phys. Rev. D **7**, 953.
Sawyer, R. F., and A. C. Yao, 1973, Phys. Rev. D **7**, 1579.
Scalapino, D. J., 1972, Phys. Rev. Lett. **29**, 386.
Sinha, B. B. P., G. A. Peterson, G. C. Li, and R. R. Whitney, 1973, Phys. Rev. C **6**, 1657.
Sinha, B. B. P., G. A. Peterson, R. R. Whitney, I. Sick, and J. S. McCarthy, 1973, Phys. Rev. C **7**, 1930.
Sorokin, G. A., 1975, Zh. Eksp. Teor. Fiz. Pis'ma Red. **21**, 96 [JETP Lett. **21**, 143 (1975)].
Troitskii, M. A., 1977, Zh. Eksp. Teor. Fiz. **73**, 1258.
Troitskii, M. A., E. E. Sapershtein, O. A. Markin, and I. N. Mishustin, 1975, Zh. Eksp. Teor. Fiz. Pis'ma Red. **21**, 96 [JETP Lett. **21**, 44 (1975)].
Urhane, F., J. S. McCarthy, and M. R. Yearian, 1971, Phys. Rev. Lett. **26**, 578.
Weinberg, S., 1966, Phys. Rev. Lett. **17**, 616.
Weinberg, S., 1966, Phys. Rev. **166**, 1568.
Weise, W., and G. Brown, 1975, Phys. Lett. B **58**, 300.