Quantum meaning of classical field theory*

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Recent researches have shown that it is possible to obtain information about the physical content of nontrivial quantum field theories by semiclassical methods. This article reviews some of these investigations. We discuss how solutions to field equations, treated as classical, c-number nonlinear differential equations, expose unexpected states in the quantal Hilbert space with novel quantum numbers which arise from topological properties of the classical field configuration or from the mixing of internal and space-time symmetries. Also imaginary-time, c-number solutions are reviewed. It is shown that they provide nonperturbative information about the vacuum sector of the quantum theory.

CONTENTS

I. INTRODUCTION

If quantum field theory is to be effective in describing physical processes, as an emerging consensus among theoretical physicists indicates, we must learn how to perform accurate but approximate calculations, since, due to their complexity, the relevant equations cannot be solved exactly. One approximation scheme, perturbation theory, has been extensively developed and is marvelously successful in some contexts, especially in lepton-quantum electrodynamics. Yet it is also clear that there are phenomena which can never be seen in the ordinary perturbative expansion. For example, there may be no small parameter in which to expand; or even if such a small parameter exists, it may be that the phenomenon we are studying is not described by formulas which can be expanded in that small parameter—such an expansion may be singular. In the last two years new approximation techniques have been developed for calculating in quantum field theory, which avoid some of the shortcomings of the perturbative expansion. These nonperturbative calculations have exposed an unexpectedly rich particle structure in the quantal Hilbert space; they have put into evidence novel effects, like emergence of fermions from bosons; and they have provided new mechanisms for spontaneous symmetry breaking without Goldstone bosons. Although the practical significance of all this for describing the present experimental data is obscure, we have clearly learned that a quantum field theory gives rise to phenomena of a much richer variety than had been believed heretofore.

This article reviews some of the relevant investigations. Since the topic is large and supports many different approaches, I shall mostly limit the discussion to research done in Cambridge (U. K. and U. S. A.). In all investigations, one begins with the first approximation in which quantal effects are ignored, and treats all equations as if they were describing classical field configurations, rather than quantum operator fields. Quantum mechanics is regained by quantizing the classical solution through semiclassical, Wentzel-Kramers-Brillouin (WKB) methods (Dashen, Hasslacher, and Neveu, 1974a,b; Korepin and Faddeev, 1975). Alternatively, the full quantum theory can be expanded in the Born-Oppenheimer fashion, for which the first term is computed classically, and quantum corrections are found in a series expansion (Goldstone and Jackiw, 1975). It is this second, systematic approximation that will be discussed here.[1]

In the first approximation we treat the Heisenberg operator field equations as c-number field equations and analyze them by methods of mathematical physics. Classical solutions may be categorized as follows: constant solutions (time—and space—independent); static solutions (time-independent but space dependent); time- and space-dependent solutions; and lastly solutions to modified field equations, where the modification consists of replacing the time variable by an imaginary-time variable ($t \to -ix_4$). The quantal significance of the

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[1]For previous alternate reviews see Jackiw, 1975; Rajaraman, 1975; Coleman, 1975c; Gervais and Neveu, 1976; Arnowitt and Nath, 1976.

constant solutions is known: they are first approximations to the vacuum expectation value of the quantum field and frequently signal spontaneous symmetry violation, i.e., the Goldstone phenomenon. The quantal meaning of static and time-varying solutions will be explained in Sec. II and III, first in the simple context of models in one spatial dimension, and then for realistic models in three spatial dimensions. It will be demonstrated that these classical solutions signal the presence of particle states which had not been previously seen in perturbative analyses. The imaginary time solutions, also analyzed in Sec. III, are evidence for quantum mechanical tunnelling—another nonperturbative effect.

II. MODELS IN ONE SPATIAL DIMENSION

In order to encounter first in a simple setting the ideas that I wish to review, let us consider a quantum field theory of a spinless field $\Phi(x,t)$ in one spatial dimension. The Lagrange density is assumed to be of the form

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi-U(\Phi). \tag{2.1}$$

For the energy to be positive definite, we take the field potential $U(\Phi)$ to be non-negative

$$U(\Phi)\geq 0. \tag{2.2a}$$

$U(\Phi)$ will in general depend on various numerical parameters (coupling constants). We wish to have a unique parameter g for systematic expansions, hence we assume that $U(\Phi)$ depends on g in a scaled fashion

$$U(\Phi)=U(\Phi;g)=\frac{1}{g^2}U(g\Phi;1). \tag{2.2b}$$

The operator equation satisfied by Φ is

$$\Box\Phi+U'(\Phi)=\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\Phi+U'(\Phi)=0. \tag{2.3}$$

(The prime denotes differentiation with respect to argument.) I shall discuss first static, c-number solutions to this field equation, and then their quantal meaning. Time-dependent solutions will be analyzed only briefly.

A. Static, *c*-number fields

Let us for the moment ignore the quantal nature of Eq. (2.3) and seek its static solutions. The c-number field $\varphi(x,t)$ satisfies

$$\Box\varphi+U'(\varphi)=0 \tag{2.4}$$

which for static configurations $\varphi(x)$ reduces to

$$\varphi''=U'(\varphi). \tag{2.5}$$

The solutions which are of interest to us are delimited by two requirements. Firstly we demand that if φ_c solves (2.5), then $E_c(\varphi_c)$ should be finite, where $E_c(\varphi)$ is the energy of a static field configuration φ

$$E_c(\varphi)=\int dx\,\{\tfrac{1}{2}(\varphi')^2+U(\varphi)\}. \tag{2.6}$$

The reason for this requirement is that $E_c(\varphi_c)$ will be identified with an approximation to the energy eigenvalue of a new state in the quantum theory, and obviously this should not be infinite. Secondly we demand classical stability. Classical stability may be discussed in two different ways. First we may return to the full, time-dependent equation (2.4) and write a time-dependent solution as

$$\varphi(x,t)=\varphi_c(x)+\psi_k(x)\exp(i\omega_k t) \tag{2.7}$$

where φ_c is our time-independent solution and the other term is a time-dependent perturbation, labeled by a parameter k. Substituting this *Ansatz* into the full time-dependent equation (2.4) and linearizing around the small perturbation ψ_k gives a Schrödinger-like equation for ψ_k

$$\left[-\frac{d^2}{dx^2}+U''(\varphi_c(x))\right]\psi_k(x)=\omega_k^2\psi_k(x). \tag{2.8}$$

Classical stability will ensue when the eigenvalues ω_k^2 are non-negative, so that small perturbations about φ_c do not grow exponentially in time. The other way of formulating classical stability is a variational one. Equation (2.5) can be obtained by demanding that the energy functional $E_c(\varphi)$, given in Eq. (2.6), be stationary with respect to variations of φ

$$0=\delta E_c(\varphi)/\delta\varphi(x)=-\varphi''+U'(\varphi). \tag{2.9}$$

For stability it is required that the second variation

$$\frac{\delta^2E_c(\varphi)}{\delta\varphi(x)\delta\varphi(y)}=\left[-\frac{d^2}{dx^2}+U''(\varphi)\right]\delta(x-y) \tag{2.10}$$

evaluated at the solution φ_c, be a non-negative differential operator. Clearly this means that all eigenfrequencies of (2.10) at $\varphi=\varphi_c$ must be non-negative, which again leads to a study of (2.8). So classical stability, in either formulation, demands that the eigenvalues of the Schrödinger equation (2.8) be non-negative. The demand for stability is motivated by the requirement that the corresponding quantum state be stable.

There are some things which can be said about our problem independently of the explicit form of $U(\varphi)$, and a very important statement is the following: The Schrödinger equation (2.8) always has a zero-frequency solution, called the "translation mode." For stability this must be the lowest mode. To prove the existence of the translation mode, we differentiate (2.5) and obtain [note that $U(\varphi)$ does not depend explicitly on the position x]

$$\left[-\frac{d^2}{dx^2}+U''(\varphi_c)\right]\varphi_c'=0, \tag{2.11}$$

which shows that φ_c' is a solution of Eq. (2.8) with $\omega_0=0$.

$$\psi_0\propto\varphi_c' \tag{2.12}$$

Moreover, this is a normalizable solution since $\int dx(\varphi_c')^2<\infty$, because the energy is finite. An equivalent way to understand the occurrence of a zero-frequency mode is the following: Translation invariance assures that if $\varphi_c(x)$ is a solution of (2.5), then $\varphi_c(x+x_0)$ will also be a solution. Expanding the latter $\varphi_c(x+x_0)=\varphi_c(x)+x_0\varphi_c'(x)$ and comparing with the expansion (2.7) $\varphi(x,t)=\varphi_c(x)+\psi_k(x)\exp(i\omega_k t)$ we see that φ_c' is a small fluctuation and the frequency of this fluctuation is zero.

It should be clear that the above generalizes to more than one dimension—if $\varphi_c(\mathbf{x})$ is a static solution, then the translation mode $\nabla\varphi_c(\mathbf{x})$ is a zero-frequency "small-oscillation" mode. Also symmetries other than the translation symmetry give rise to zero-frequency modes. If the energy functional is invariant under a transformation $\varphi \to \varphi + \delta\varphi$, then $\delta\varphi_c$ will be a zero-frequency mode.

We may now prove another general result: Only in one spatial dimension is it possible to find stable, static solutions of finite energy in a *spinless model* governed by the simple Lagrangian (2.1).[2] In more than one dimension one must necessarily deal with more complicated models, for example models with spin. This result becomes self-evident when it is realized that the zero-frequency, translation mode in d dimensions is d-fold degenerate (it is a p wave); on the other hand, for the ordinary Schrödinger equation the lowest eigenstate is nondegenerate. Hence for $d>1$ there must exist solutions to (2.8) with $\omega_k^2<0$. The same conclusion can be obtained by a scaling argument. Static field solutions $\varphi_c(\mathbf{x})$ in d dimensions stationarize the static energy $E_c(\varphi)=\int d\mathbf{x}\{\frac{1}{2}(\nabla\varphi)^2+U(\varphi)\}$. If $\varphi_c(\mathbf{x})$ is such a solution, and $\varphi_a(\mathbf{x})\equiv\varphi_c(\mathbf{x}/a)$, where a is a positive parameter, then $E_c(\varphi_a)$ must be stationary at $a=1$. A change of integration variable shows that $E_c(\varphi_a)=a^{d-2}E_T(\varphi_c)+a^dE_v(\varphi_c)$,

$$E_T(\varphi)=\int d\mathbf{x}\,\tfrac{1}{2}(\nabla\varphi)^2>0,\quad E_V(\varphi)=\int d\mathbf{x}U(\varphi)>0.$$

From

$$\left.\frac{\partial E_c(\varphi)_a}{\partial a}\right|_{a=1}=0,$$

a virial theorem may be deduced

$$E_V(\varphi_c)=\frac{2-d}{d}E_T(\varphi_c). \tag{2.13}$$

But now it follows that

$$\left.\frac{\partial^2E_c(\varphi_a)}{\partial a^2}\right|_{a=1}=2(2-d)E_T(\varphi_c),$$

so that $E_c(\varphi_a)$ will be minimized only for $2-d>0$, i.e., $d=1$.

It should be emphasized that this negative result applies only to static solutions of the simple model (2.1). In more than one dimension, it is easy to find static, stable, finite-energy solutions provided the model includes Yang–Mills fields—this will be discussed in Sec. III. The simple one-dimensional model remains an interesting laboratory for our theoretical ideas because all the problems of developing a quantum theory around a classical solution can be posed and answered. Moreover, the method of quantization carries over to higher dimensions, and will be employed in Sec. III for the Yang–Mills models.

We return therefore to the one-dimensional Eq. (2.5), whose first integral may be given for arbitrary U

$$\tfrac{1}{2}(\varphi')^2=U(\varphi). \tag{2.14}$$

An integration constant does not appear in (2.14) so that the energy will be finite. [Equation (2.14) is consistent with the virial theorem (2.13) at $d=1$: $E_T=E_V$.] To integrate (2.14), we need an expression for $U(\varphi)$. There are many formulas for $U(\varphi)$ which lead to static, stable solutions with finite energy. I shall discuss two examples explicitly; however, our theory is independent of the specific form of $U(\varphi)$

$$\varphi^4\text{ theory;}\quad U(\varphi)=\frac{m^4}{2g^2}\left[1-\frac{g^2\varphi^2}{m^2}\right]^2. \tag{2.15a}$$

Sine–Gordon (SG) theory;

$$U(\varphi)=\frac{m^4}{g^2}\left[1-\cos\left(\frac{g}{m}\varphi\right)\right]. \tag{2.15b}$$

Note that both theories possess discrete symmetries

$$\varphi^4\text{ theory;}\quad \varphi\to-\varphi \tag{2.16a}$$

$$\text{SG theory;}\quad \varphi\to\pm\varphi+2\pi n(m/g);\quad n=0,\pm1,\ldots \tag{2.16b}$$

but the minima of the potentials, $U'(\varphi_0)=0$ [these are constant solutions to (2.4)]

$$\varphi^4\text{ theory;}\quad \varphi_0=\pm m/g, \tag{2.17a}$$

$$\text{SG theory;}\quad \varphi_0=2\pi n(m/g);n=0,\pm1,\ldots, \tag{2.17b}$$

indicate that the symmetries are spontaneously broken by the vacuum state. Thus, according to the usual procedure, we assign to the quantum field Φ the vacuum expectation values

$$\varphi^4\text{ theory;}\quad \langle 0|\Phi|0\rangle=m/g, \tag{2.18a}$$

$$\text{SG theory;}\quad \langle 0|\Phi|0\rangle=0, \tag{2.18b}$$

and upon expanding $U(\varphi)$ about the vacuum value of φ we learn that the mass of the "mesons" in the φ^4 theory is $2m$, while in the SG theory it is m.

Position-dependent solutions to (2.14) are the following:

$$\varphi^4\text{ theory;}\quad \varphi_c(x)=\pm(m/g)\tanh m(x-x_0), \tag{2.19a}$$

$$\text{SG theory;}\quad \varphi_c(x)=\pm4(m/g)\tan^{-1}\exp(\pm m(x-x_0)). \tag{2.19b}$$

The occurrence of the parameter x_0 is a consequence of translation invariance; frequently we shall set it to zero. The classical energy of the solution is finite

$$\varphi^4\text{ theory;}\quad E_c(\varphi_c)=\tfrac{4}{3}(m^3/g^2) \tag{2.20a}$$

$$\text{SG theory;}\quad E_c(\varphi_c)=8(m^3/g^2). \tag{2.20b}$$

The stability equation (2.8) becomes

φ^4 theory;

$$\left[-\frac{d^2}{dx^2}+4m^2-\frac{6m^2}{\cosh^2mx}\right]\psi_k(x)=\omega_k^2\psi_k(x) \tag{2.21a}$$

SG theory;

$$\left[-\frac{d^2}{dx^2}+m^2-\frac{2m^2}{\cosh^2mx}\right]\psi_k(x)=\omega_k^2\psi_k(x) \tag{2.21b}$$

These Schrödinger equations can be completely solved. They are $L=2$ and $L=1$ cases of the equation

$$\left[-\frac{d^2}{dz^2}+L^2-\frac{L(L+1)}{\cosh^2z}\right]\psi_k(z)=\omega_k^2\psi_k(z) \tag{2.22}$$

with very simple properties. One finds a continuous spectrum for $\omega_k^2=k^2+L^2$, $k^2>0$ with $\psi_k(z)\sim\exp(ikz)$ multiplied by a Jacobi polynomial of degree L in $\tanh z$.

[2] This theorem is well known to investigators of nonlinear field equations. An early reference which emphasizes the relevance to particle physics is Hobart, 1963.

(There is no reflection, only transmission.) In addition ω_k^2 takes the discrete values L^2-n^2, $n=L$, $L-1,\ldots,1$. For (2.21) this means that in the φ^4 theory there is a zero-frequency solution φ_c', a second discrete eigenvalue, and a continuum beginning at $\omega_k^2=(2m)^2$; in the SG theory the zero-frequency state is again φ_c', and the continuum begins at $\omega_k^2=m^2$. The eigenvalues are non-negative; both solutions are stable. Note that in both cases the continuum begins as μ^2 where μ is the mass of the meson.

Let us observe that the static solutions (2.19) are $O(g^{-1})$, just as are the constant solutions (2.17). They interpolate between the constant solutions as x ranges from $-\infty$ to ∞. Also the energy density $\mathcal{E}=\frac{1}{2}(\varphi_c')^2+U(\varphi_c)=(\varphi_c')^2$ is localized at $x=x_0$. We shall call solutions that have a localized energy density for all time "solitons."[3]

The soliton, though arising in a classical *field* theory, looks very much like a classical *particle*. Its energy density is localized at a point, its total energy is finite, and it is stable. Moreover, because the field equations are Lorentz invariant, once we have the solution $\varphi_c(x)$, we also have the boosted solution $\varphi_c(x-vt/\sqrt{1-v^2})$ for arbitrary v, $|v|<1$. The soliton can move in space.

This then completes our discussion of static solutions in the simple examples. We now turn to the question of the quantal significance of such solutions.

B. Quantum meaning of static, c-number fields

In order to fit the static c-number solutions into a quantum theory, we shall posit postulates about the Hilbert space of states, and we shall verify self-consistently the validity of the postulates. Also a systematic expansion scheme will emerge, with which one can compute quantal amplitudes to arbitrary accuracy.

We postulate the existence of a vacuum state $|0\rangle$, which in our examples is degenerate. A particle space is built on one of the vacua—there is no tunnelling, hence we need not concern ourselves with the Hilbert space built upon the other vacua. There are of course "meson" states. The one-meson state $|k\rangle$ describes a stable, spinless boson with momentum k and mass μ, and there are also multi-meson states $|k_1,k_2,\ldots\rangle$. We call this the "vacuum" sector and calculations in the vacuum sector are performed in the standard way: The quantum field Φ is shifted by the constant solution φ_0 (m/g in the φ^4 theory, 0 in the SG theory)

$$\Phi(x,t)=\varphi_0+\hat{\Phi}(x,t) \tag{2.23}$$

and conventional perturbation theory may be used to calculate amplitudes of $\hat{\Phi}$. Note that φ_0 (when nonvanishing) is $O(g^{-1})$; it is the lowest-order (in g) approximation to $\langle 0|\Phi|0\rangle$. Multi-meson amplitudes involving $\hat{\Phi}$ are of higher order in g.

The above description of the quantum theory was, until recently, traditional, but it is incomplete since it does not take into account the static c-number solutions (Goldstone and Jackiw, 1975). In order that these solutions be properly included, we further postulate that, in addition to the meson states, there exist other particle states, the quantum soliton states. The one-soliton states $|P\rangle$ are momentum and energy eigenstates

$$\begin{aligned}&P|P\rangle=P|P\rangle,\\&H|P\rangle=E(P)|P\rangle,\\&E(P)=\sqrt{P^2+M^2},\end{aligned} \tag{2.24}$$

and we further postulate that, for small g, the quantum soliton is very heavy; specifically the soliton's mass M is taken to be

$$M=O(g^{-2}). \tag{2.25}$$

In addition there are of course one-soliton, multi-meson states $|P;k_1,k_2,\ldots\rangle$, where P is the total momentum, and the k_i are the asymptotic meson momenta. Also multi-soliton states exist, but we shall not be discussing them. We postulate that the soliton is absolutely stable against decay into mesons; this means that all matrix elements of the form

$$\langle\text{soliton, meson state}|\Phi\Phi\ldots|\text{no-soliton, meson state}\rangle$$

vanish identically. This sector of the Hilbert space is called the "soliton" sector.

Next it must be decided whether there is only one type of soliton, or whether there is a variety. To settle this we look at the variety of available static, c-number solutions with the same energy, which, as will be presently demonstrated, are relevant to the soliton sector (just as the constant, c-number solutions are relevant to the vacuum sector). Always there is a variety corresponding to the symmetries of the problem. In the examples considered, this is the variety labeled by x_0 arising from translational symmetry, as well as the variety of the sign of the solution corresponding to the field reflection symmetry; moreover, in the SG theory there is the variety of the inverse tangent's multiple branches, arising from the discrete field translation symmetry. Such symmetry-related varieties are of no consequence for distinguishing different types of solitons. If, however, there is a further variety to the classical solution, then we postulate that there are as many different types of solitons as there are varieties of static, c-number solutions with the same energy. Thus in the φ^4 theory, there is only one soliton; in the SG theory there are two, corresponding to the $\pm$ variety of the exponential. We may call them soliton and anti-soliton; in what follows we shall concentrate on the soliton sector, corresponding to +, with the understanding that there is also an anti-soliton sector which does not communicate with the soliton sector.

The last set of postulates sets the magnitudes of matrix elements of the quantum field Φ in the soliton sector. We shall show self-consistently the following to be true:

$$\langle P';k_1',\ldots,k_{n'}'|\Phi|P;k_1,\ldots,k_n\rangle_C=O(g^{n+n'-1}) \tag{2.26}$$

[Here and subsequently, it is understood that the field operator when written without argument is evaluated at

[3]The nomenclature, advocated by T. D. Lee, is a borrowing from the literature of applied mathematics and engineering. In those disciplines, however, "soliton" is used in a more restrictive sense. For a review of the older mathematical-engineering researches see Scott, Chu, and McLaughlin, 1973; Whitham, 1974.

the origin, *viz.* $\langle\Phi\rangle \equiv \langle\Phi(0)\rangle$.] The subscript C denotes the connected part, where only the mesons are disconnected. Thus according to the above

$$\begin{aligned}
&\langle P'\,|\,\Phi\,|\,P\rangle = O(g^{-1}),\\
&\langle P'\,|\,\Phi\,|\,P;k\rangle = O(g^{0}),\\
&\langle P';k'\,|\,\Phi\,|\,P;k\rangle = (2\pi)\delta(k'-k)\langle P'\,|\,\Phi\,|\,P\rangle + \langle P';k'\,|\,\Phi\,|\,P;k\rangle_C,\\
&\langle P';k'\,|\,\Phi\,|\,P;k\rangle_C = O(g^{1}).
\end{aligned} \tag{2.27}$$

To show that our postulates can be verified self-consistently, we begin by considering the one-soliton matrix element of Φ. We know that the quantum field satisfies the operator equation

$$\left(\frac{d^2}{dt^2}-\frac{d^2}{dx}\right)\Phi(x,t) = -U'(\Phi(x,t)) = 2m^2\Phi(x,t) - 2g^2\Phi^3(x,t). \tag{2.28}$$

(For definiteness we discuss the φ^4 theory, but the method is general) Let us take matrix elements of this equation between soliton states

$$\{[E(P')-E(P)]^2 - [P'-P]^2 + 2m^2\}f(P',P) = 2g^2\langle P'\,|\,\Phi^3\,|\,P\rangle = 2g^2\sum_{n,n'}\langle P'\,|\,\Phi\,|\,n\rangle\langle n\,|\,\Phi\,|\,n'\rangle\langle n'\,|\,\Phi\,|\,P\rangle, \tag{2.29}$$

where the field form factor f has been defined by

$$\langle P'\,|\,\Phi\,|\,P\rangle = f(P',P). \tag{2.30}$$

As a consequence of the soliton's stability, only states in the one-soliton sector contribute to the completeness sum in (2.29).

Equation (2.29) is exact; we now analyze to lowest order in g. The left-hand side is $O(g^{-1})$, since according to our postulate that is the magnitude of f, and the remaining factors are $O(g^0)$. On the right-hand side the factor g^2 requires that only terms of $O(g^{-3})$ be kept in the sum. But these can arise only from one-soliton intermediate states. Hence to $O(g^{-1})$ we may replace (2.29) by an integral equation for f

$$\{[E(P')-E(P)]^2 - [P'-P]^2 + 2m^2\}f(P',P) = 2g^2\int\frac{dP''dP'''}{(2\pi)^2}f(P',P'')f(P'',P''')f(P''',P). \tag{2.31}$$

Upon comparing the g-behavior of both sides of the equation one concludes that $f(P',P)$ is, indeed, self-consistently $O(g^{-1})$. In fact the equation may be simplified, and solved completely to $O(g^{-1})$.

The further simplifications are effected when it is recalled that $E(P')-E(P)=\sqrt{P'^2+M^2}-\sqrt{P^2+M^2}$. Since by hypothesis M is $O(g^{-2})$ the energy difference when expanded in powers of g is $O(g^2)$ and may be dropped in a lowest-order $O(g^{-1})$ calculation. The physical meaning is that the soliton is very heavy for weak coupling. Therefore to leading order it does not move, and its energy is just its rest mass, M. Next, observe that the function $f(P',P)$ depends only on the difference $P'-P$, to leading order in g. To see this, recall that Lorentz invariance insures that f can be only a function of the Lorentz scalars $(P'_\mu - P_\mu)^2$ and $\epsilon^{\mu\nu}P_\mu P'_\nu$. To leading order $(P'_\mu-P_\mu)^2 = [E(P')-E(P)]^2 - [P'-P]^2 \approx -[P'-P]^2$. Also $\epsilon^{\mu\nu}P'_\mu P_\nu = E(P)P' - E(P')P \approx M[P'-P]$. So $f(P',P)=f(P'-P)$, to leading order in g. As a consequence we may, to leading order, replace (2.31) by

$$\{-[P'-P]^2 + 2m^2\}f(P'-P) = 2g^2\int\frac{dP''dP'''}{(2\pi)^2}f(P'-P'')f(P''-P''')f(P'''-P). \tag{2.32}$$

In terms of a Fourier-transformed function $\phi(x)$, defined by

$$f(P'-P) = \int dx\,\exp[i(P'-P)x]\phi(x), \tag{2.33}$$

Equation (2.32) reads

$$\phi''(x) = -2m^2\phi(x) + 2g^2\phi^3(x) = U'(\phi(x)). \tag{2.34}$$

This is the static classical equation. As we know, its solution is

$$\phi(x) = \varphi_c(x) = \pm(m/g)\tanh m(x-x_0). \tag{2.35}$$

Therefore we have determined for the φ^4 theory the matrix element of the quantum field between one-soliton states:

$$\varphi^4 \text{ theory};\quad \langle P'\,|\,\Phi\,|\,P\rangle = \int dx\,\exp[i(P'-P)](m/g)\tanh mx + \text{higher powers in } g \tag{2.36a}$$

(The plus sign is taken in analogy with the vacuum sector, $\langle 0\,|\,\Phi\,|\,0\rangle = m/g +$ higher orders in g. The minus sign is relevant to a parallel Hilbert space of identical structure. Also the arbitrary origin, x_0, is physically uninteresting; it gives only an arbitrary phase to $\langle P'\,|\,\Phi\,|\,P\rangle$, which we henceforth set to zero.) A similar calculation for the SG theory also determines the field matrix element:

$$\text{SG theory};\quad \langle P'\,|\,\Phi\,|\,P\rangle = \int dx\,\exp[i(P'-P)x](4m/g)\tan^{-1}e^{mx} + \text{higher powers in } g \tag{2.36b}$$

This is the matrix element between soliton states; there is also a matrix element between anti-soliton states with a negative exponential; see (2.19b). There is no transition matrix element between the soliton and anti-soliton states.

Consequently a first result has been obtained: a classical, static solution can be fitted into the quantum theory provided we allow for new states—the soliton states. Then, in an expansion in powers of g, the Fourier transform of the classical solution is the first approximation to the field form factor $\langle P'\,|\,\Phi\,|\,P\rangle$ which is $O(g^{-1})$.

The next thing that has to be checked is whether the soliton's mass is, as postulated, of order g^{-2}; indeed we have to determine what the mass is. Let us compute the energy

$$\begin{aligned}
\langle P'\,|H|\,P\rangle &= E(P)(2\pi)\delta(P'-P),\\
&= \langle P'\,|\int dx\,\mathcal{H}(x,t)\,|\,P\rangle,\\
&= (2\pi)\delta(P'-P)\langle P\,|\,\mathcal{H}\,|\,P\rangle,
\end{aligned} \tag{2.37a}$$

where the Hamiltonian density $\mathcal{H}$ is given by

$$\mathcal{H}=\tfrac{1}{2}\dot{\Phi}^2+\tfrac{1}{2}(\Phi')^2+U(\Phi). \tag{2.37b}$$

Hence

$$E(P)=\langle P|\mathcal{H}|P\rangle=\sqrt{P^2+M^2}. \tag{2.37c}$$

We expand $E(P)$ in powers of g. The first two terms are $M+P^2/2M$. So $M\approx\langle P|\tfrac{1}{2}\dot{\Phi}^2+\tfrac{1}{2}(\Phi')^2+U(\Phi)|P\rangle$, where only the dominant, $O(g^{-2})$ terms are kept. Let us begin with the evaluation of $\langle P|\dot{\Phi}^2|P\rangle=\sum_n\langle P|\dot{\Phi}|n\rangle\langle n|\dot{\Phi}|P\rangle$. We need to compute this matrix element to order g^{-2}; therefore only the one-soliton states need be taken into account in the intermediate state sum. However, there is a time derivative which is equivalent to the energy difference. But to leading order, the energy difference between soliton states is zero, because to leading order the energy is just the mass M. So $\langle P|\dot{\Phi}^2|P\rangle$ may be dropped to $O(g^{-2})$. Next consider $\langle P|(\Phi')^2|P\rangle=\sum_n\langle P|\Phi'|n\rangle\langle n|\Phi'|P\rangle$. Again since we are computing this quantity to order g^{-2}, we need to keep only single-soliton states $\langle P|(\Phi')^2|P\rangle=\int dP'/(2\pi)\langle P|\Phi'|P'\rangle\langle P'|\Phi'|P\rangle=\int dP'/(2\pi)(P'-P)^2\langle P|\Phi|P'\rangle\langle P'|\Phi|P\rangle$. Substituting into this the expression for $\langle P|\Phi|P'\rangle$ in terms of $\phi(x)$, Eqs. (2.30) and (2.33), and carrying out all the integrations, one obtains $\langle P|(\Phi')^2|P\rangle=\int dx(\phi'(x))^2$. Finally taking matrix elements of $U(\Phi)$ and keeping only the single-soliton states in the intermediate states, which is all that is needed to $O(g^{-2})$, one arrives at a formula for M in terms of $\phi(x)$: $M=\int dx[\tfrac{1}{2}(\phi')^2+U(\phi)]$. Since ϕ has been shown to coincide with φ_c, we obtain:

$$M\approx E_c(\varphi_c)=M_0$$

$$\varphi^4 \text{ theory};\quad M_0=\frac{4}{3}\frac{m^3}{g^2} \tag{2.38a}$$

$$\text{SG theory};\quad M_0=8\,\frac{m^3}{g^2} \tag{2.38b}$$

We have thereby verified the self-consistency of the postulate (2.25) that the mass of the soliton is of order g^{-2}. Also we find that M coincides with the classical energy in lowest order. Of course, there are corrections of higher order in g.

The above calculations show that to lowest order in the coupling constant our postulates about the soliton sector are consistent. Various quantal objects can be computed; they are related to corresponding classical quantities. Moreover, we see that some quantum structures in the soliton sector (the field form factor, the soliton mass) are proportional to inverse powers of g—they are singular at $g=0$ and cannot be seen in ordinary perturbation theory. Nevertheless these irregular contributions can be isolated and completely calculated by the methods here developed. Moreover, corrections of higher order in g can be systematically computed.

We now give an example of such higher computations; we calculate everything to next order in g. As we shall see, an important quantum consistency condition emerges which establishes the Poincaré covariance of our method.

To evaluate first-order corrections we may still keep $E(P)$ independent of P, since the kinematical dependence enters in $O(g^2)$, two orders beyond the lowest $O(g^{-2})$. However, in the saturation by intermediate states we must keep the one-soliton one-meson state $|P;k\rangle$; an expression for $\langle P'|\Phi|P;k\rangle\equiv f_k(P',P)$ is needed. The exact equation for that quantity is (in the φ^4 theory)

$$\{[E(P')-E_k(P)]^2-[P'-P]^2+2m^2\}f_k(P',P)=2g^2\langle P'|\Phi^3|P\rangle \tag{2.39}$$

Here $E_k(P)$ is the energy of the one-soliton, one-meson state; P is the total momentum; k is the meson momentum. To lowest order we take $E_k(P)$ to be $M_0+\omega_k$. In saturating the right-hand side we keep the no-meson and the one-meson states, thus encountering the following matrix elements: $\langle P'|\Phi|P\rangle$, $\langle P'|\Phi|P;k\rangle$ and $\langle P';k'|\Phi|P;k\rangle$. The first is known to lowest order; the second is being calculated; the third we decompose into a connected and disconnected piece, as in (2.27). To the order we are computing only the disconnected piece is kept. Also we take $f_k(P',P)$ to be, in lowest order, a function of $P'-P$, the total momentum difference. With these simplifications (2.39) becomes

$$\omega_k^2 f_k(P'-P)=\{[P'-P]^2-2m^2\}f_k(P'-P)+6g^2\int\frac{dP''dP'''}{(2\pi)^2}f(P'-P'')\times f(P''-P''')f_k(P'''-P). \tag{2.40}$$

Upon introducing the Fourier transform

$$f_k(P'-P)=\int dx\exp[i(P'-P)x]\phi(k;x) \tag{2.41}$$

and using (2.33), we recognize that (2.40) is

$$\left[-\frac{d^2}{dx^2}-2m^2+6g^2\phi^2(x)\right]\phi(k;x)=\omega_k^2\phi(k;x)$$

$$\left[-\frac{d^2}{dx^2}+U''(\phi(x))\right]\phi(k;x)=\omega_k^2\phi(k;x). \tag{2.42a}$$

Since $\phi(x)=\varphi_c(x)$ in lowest order, (2.42a) becomes the Schrödinger equation (2.8):

$$\left[-\frac{d^2}{dx^2}+U''(\varphi_c(x))\right]\phi(k;x)=\omega_k^2\phi(k;x). \tag{2.42b}$$

Now we have a clear physical interpretation for the solutions of this equation. The continuum solutions, which begin at $\omega_k=\sqrt{k^2+\mu^2}$, where μ is the meson mass, are interpreted as meson–soliton scattering states. If there are discrete states, other than the zero-frequency state (as in φ^4 theory), they are excited states of the soliton. The zero-frequency solution of (2.42) is not associated with any state. For later convenience let us set $\phi(k;x)$ equal to $(1/\sqrt{2\omega_k})\psi_k(x)$ for all states with the exception of the zero-frequency state. The normalized zero-frequency state is $\varphi_c'(x)/\sqrt{M_0}$, since $M_0=\int dx(\varphi_c')^2$.

Is it consistent to exclude the zero-frequency mode; i.e., are the physical states complete even though we are excluding one of the functions which contribute to a set of mathematically complete functions? Note also that (2.42) does not determine the normalization of $\psi_k(x)$. To settle both these points, we consider matrix elements of the canonical commutator between one-soliton states

$$\langle P'|[\Phi(x,0),\dot{\Phi}(y,0)]|P\rangle=i\delta(x-y)(2\pi)\delta(P'-P). \tag{2.43}$$

Upon saturating with no-meson states and one-meson states, the contribution of the one-meson states can be shown to be

$$i\sum_k{}' \omega_k\left\{\int dz \exp[-i(P'-P)z] \times \frac{\psi_k^*(x-z)}{\sqrt{2\omega_k}}\frac{\psi_k(y-z)}{\sqrt{2\omega_k}} + x \leftrightarrow y\right\}. \quad (2.44a)$$

The prime on the sum indicates that the zero-frequency state is excluded. If we take the ψ_k's to be properly continuum normalized, then the sum can be evaluated by completeness. A delta function does not emerge, since the zero-frequency mode is excluded; rather we get

$$i\delta(x-y)(2\pi)\delta(P'-P) - i\int dz \exp[-i(P'-P)z] \times \frac{\varphi_c'(x-z)}{\sqrt{M_0}}\frac{\varphi_c'(y-z)}{\sqrt{M_0}}. \quad (2.44b)$$

Next the no-meson contribution is evaluated; here are encountered contributions of the form

$$\int\frac{dP''}{(2\pi)}\langle P'|\Phi(x,0)|P''\rangle\langle P''|\dot{\Phi}(y,0)|P\rangle = i\int\frac{dP''}{(2\pi)}[E(P'')-E(P)]\langle P'|\Phi(x,0)|P''\rangle\langle P''|\Phi(y,0)|P\rangle. \quad (2.44c)$$

The matrix elements in the right-hand side of (2.44c) are each $O(g^{-1})$; consequently we must retain $E(P'') - E(P)$ to order g^2, since we are computing the commutator which is $O(g^0)$. The energy difference is taken to be $(P''^2-P^2)/2M'$, where we have put a prime to distinguish the mass that occurs in the kinetic term from the rest mass; we shall prove that in fact $M'=M_0$. Evaluating the relevant integrals gives the no-meson contribution to $O(g^0)$:

$$\frac{i}{M'}\int dz \exp[-i(P'-P)z]\varphi_c'(x-z)\varphi_c'(y-z). \quad (2.44d)$$

Thus when $M'=M_0$, (2.44d) cancels the second term in (2.44b), and the canonical commutation relation, the hallmark of quantum mechanics, is regained.

By this exercise we have learned three things. First, the properly normalized matrix element is $\langle P'|\Phi|P;k\rangle = \int dx \exp[i(P'-P)x][\psi_k(x)/\sqrt{2\omega_k}]$ where $\psi_k(x)$ is a normalized solution of the Schrödinger equation, hence $O(g^0)$, consistent with the postulates (2.26) and (2.27). Second, the zero-frequency solution is not a state of the theory, rather it describes the first correction to the motion of the soliton. Third, to the order computed, the theory is Poincaré invariant since the rest mass coincides with the kinetic mass.

From the scattering solutions of (2.42) the meson-soliton S matrix can be found. For the φ^4 and SG theories there is no reflection, only transmission. The transmission amplitude T is a pure phase by unitarity:

$$T=\exp[2i\delta(k)]$$

$$\tan\delta(k) = -\sum_{n=1}^{L}\tan^{-1}\frac{k}{nm}+\frac{L\pi}{2}$$

$$\varphi^4 \text{ theory}; \quad L=2, \quad (2.45a)$$

$$\text{SG theory}; \quad L=1. \quad (2.45b)$$

Note that phase shift is independent of g.

With the one-meson matrix element determined, the first-order correction to the energy and soliton field form factor can be computed. Returning to (2.29) and retaining the one-meson states in the saturation of the right-hand side, we find that the equation satisfied by $\phi(x)$ is

$$\phi''(x) = U'(\phi)+\tfrac{1}{2}G(x,x)U'''(\phi)$$

$$G(x,y)=\sum_k{}'\psi_k^*(x)\frac{1}{2\omega_k}\psi_k(y). \quad (2.46)$$

Similarly, keeping the one-meson intermediate states in (2.37) gives

$$E(P)=E_c(\phi)+\frac{1}{2}\sum_k\omega_k. \quad (2.47a)$$

To order g^0, there is no kinetic energy—that arises in $O(g^2)$. Also according to (2.46) $\phi=\varphi_c+\delta\phi$ where $\delta\phi$ is $O(g^0)$. But $\delta\phi$ does not contribute to $E_c(\phi)$ since $E_c(\phi)$ is stationary at $\phi=\varphi_c$. Thus

$$M=M_0+\frac{1}{2}\sum_k\omega_k. \quad (2.47b)$$

The soliton's mass, through $O(g^0)$, is the classical energy plus half the sum of the small fluctuation frequencies—a completely reasonable, quantum mechanical formula. [Here again we see the need for non-negative eigenvalues in the Schrödinger equation (2.42): If $\omega_k^2 < 0$ then the soliton's mass becomes complex—it is an unstable particle.]

The equations (2.46) and (2.47) have to be renormalized. Firstly the (infinite) vacuum energy has to be removed from $\frac{1}{2}\sum_k\omega_k$. Moreover, the mass parameter in the theory m, has to be renormalized just as in the vacuum sector. (These are the only infinites of spinless theories in one spatial dimension without derivative couplings.) The mass formulas have been renormalized and evaluated (Dashen, Hasslacher, and Neveu, 1974b; 1975). The results are:

$$\varphi^4 \text{ theory}; \quad M=\frac{4}{3}\frac{m^3}{g^2}-m\left(\frac{3}{\pi}-\frac{1}{2\sqrt{3}}\right)+O(g^2), \quad (2.48a)$$

$$\text{SG theory}; \quad M=8\frac{m^3}{g^2}-\frac{m}{\pi}+O(g^2). \quad (2.48b)$$

Here m is the renormalized mass parameter.

It is of course possible, with increasing tedium of computation, to extend the above method to the next, indeed to arbitrary order in g^2.[4] Such computations have been performed; they provide an important verification of the consistency of our postulates about the soli-

[4]The study of a field theory through the equations satisfied by matrix elements of the quantum field between states of a very massive particle was pioneered in the context of nuclear physics by Kerman and Klein, 1963. The method was applied to the present problem by Goldstone and Jackiw, 1975. More recent developments are by Klein and Krejs, 1975, 1976; Klein, 1976; and Jacobs, 1976a.

ton sector (Jacobs, 1976a). I shall not review them here, since there is available an alternate approach, described in the following subsection, which gives a diagrammatic depiction of the series in g.

We turn next to the question of the soliton's stability: if it is heavy, why does it not decay into ordinary mesons? Stability is usually associated with an absolutely conserved quantum number. To see the existence of a conservation law in our models, observe that

$$J^{\mu}=\epsilon^{\mu\nu}\partial_{\nu}\Phi \tag{2.49}$$

is a conserved current, not because it arises by Noether's theorem from a symmetry of the theory, but rather because it is trivially conserved, since it is a divergence of an antisymmetric tensor. The charge associated with this current is

$$N=\int dx J^0=\int dx\,\Phi'=\Phi\big|_{x=+\infty}-\Phi\big|_{x=-\infty}. \tag{2.50}$$

In the vacuum sector the field tends to the same value as $x\to\pm\infty$, and N vanishes. In the soliton sector, the field tends to different values, N is nonzero, and its conservation renders the soliton stable. (In higher dimensions, an antisymmetric tensor, whose divergence is a conserved current, can be constructed with the help of the spin degrees of freedom.) Such currents and conservation laws are called "topological," since they arise from topological properties of field configurations and not directly from symmetries of the theory (Skyrme, 1961).

[It should be remarked that there exist localized, finite-energy solutions whose stability arises not from topological quantum numbers but from an ordinary, Noether conservation law. These solutions are necessarily time-dependent; I shall not discuss them here (Friedberg, Lee, and Sirlin, 1976).]

A final remark about the soliton: Observe that in the φ^4 example $\varphi_c(x)$ is an odd function of x, hence $f(P'-P)\approx\langle P'|\Phi|P\rangle$ will change sign when P' and P are interchanged. But by crossing symmetry an analytic continuation of $f(P'-P)$ should also describe the matrix element $\langle P'P|\Phi|0\rangle$. Antisymmetry of this matrix element indicates that the solitons in the φ^4 theory are fermions (Goldstone and Jackiw, 1975). This truly remarkable result—the emergence of fermions in a theory containing only Bose fields—will be encountered again when we study realistic three-dimensional models.

C. Quantization about static, *c*-number fields

Having established the existence of the soliton sector and demonstrated the feasibility of a systematic coupling constant expansion, it is appropriate to develop a diagrammatic perturbation theory, analogous to that in the vacuum sector. The approach there is to write $\Phi=\varphi_0+\hat{\Phi}$; that is, the quantum field Φ is shifted by φ_0, the constant, $O(g^{-1})$ solution to the field equations, and perturbation theory is developed in terms of the new field $\hat{\Phi}$. We would like to do something similar with the position-dependent solution. However, there are several problems. First, φ_c depends not only on x but also on x_0, the choice of the origin. Thus there are many classical solutions, parametrized by x_0, and the question is, by which classical solution should we shift? The second problem is that if we shift by a c-number which depends on x, then there will be great difficulty in implementing translation invariance because the momentum operator P commutes with the c-number solution. So, in order to maintain translation covariance, $\hat{\Phi}$ would have to transform in some very complicated way.

The problem that we are facing is that we have a theory with a symmetry. The symmetry is translational invariance. But our classical solution is not symmetric; a translation transforms it into another solution. To overcome this problem and to develop a quantum theory around the classical solution, the procedure is to take the whole class of classical solutions parametrized by x_0 and to promote x_0 in $\varphi_c(x-x_0)$, to a quantum variable $X(t)$. This new quantum variable is called a "collective coordinate." Therefore we write

$$\Phi(x,t)=\varphi_c(x-X(t))+\hat{\Phi}(x-X(t),t). \tag{2.51}$$

This can be viewed as a canonical transformation from the original set of variables Φ to a new set of variables $X,\hat{\Phi}$. Since Φ is a quantum field, it has an infinite number of quantum degrees of freedom; $\hat{\Phi}$ also has an infinite number of quantum degrees of freedom, and $\varphi_c(x-X(t))$ has one quantum degree of freedom in $X(t)$. In order not to increase the number of quantum degrees of freedom we should set a subsidiary condition, and we take it to be

$$\int dx\,\varphi_c'(x)\hat{\Phi}(x,t)=0. \tag{2.52}$$

This condition is very convenient because φ_c' is a small oscillation associated with the zero-frequency mode which does not correspond to a physical state. So the subsidiary condition insures that the quantum field $\hat{\Phi}(x,t)$ does not contain the unphysical zero-frequency mode.

To complete the specification of the canonical transformation we need to exhibit the transformation for the conjugate momenta. The canonical momentum conjugate to Φ is $\Pi=\delta\mathcal{L}/\delta\dot{\Phi}=\dot{\Phi}$. The transformation conjugate to (2.51) involves a momentum $P(t)$, conjugate to $X(t)$, and a field momentum $\hat{\Pi}(x,t)$, conjugate to $\hat{\Phi}(x,t)$. The transformation is complicated:

$$\Pi(x,t)=\hat{\Pi}(x-X(t),t)-\frac{1}{2}\left\{\frac{\varphi_c'(x-X(t))}{M_0+\xi(t)},\; P(t)+\int dx\,\hat{\Phi}'(x,t)\hat{\Pi}(x,t)\right\}_+, \tag{2.53}$$

where

$$\xi(t)=\int dx\,\varphi_c'(x)\hat{\Phi}'(x,t)$$

$$M_0=\int dx(\varphi_c'(x))^2.$$

As before we have to put a subsidiary condition on the canonical momenta so as not to increase the degrees of freedom

$$\int dx\,\varphi_c'(x)\hat{\Pi}(x,t)=0. \tag{2.54}$$

To verify that the above defines a canonical transformation, recall that the original variables satisfy $i[\Pi(x,t), \Phi(y,t)] = \delta(x-y)$. This formula is regained, provided the nonvanishing commutators of the new variables are taken to be[5]

$$i[P(t), X(t)] = 1$$
$$i[\hat{\Pi}(x,t), \hat{\Phi}(y,t)] = \delta(x-y) - \frac{\varphi_c'(x)\varphi_c'(y)}{M_0}. \tag{2.55}$$

The physical interpretation of $P(t)$ is that it is the total field momentum, which in terms of the old variables is $-\int dx \Pi(x,t)\Phi'(x,t)$. Upon expressing this in new variables and evaluating the integral with the help of the orthogonality relations (2.52) and (2.54), we find $P(t)$. (Since the total momentum is a constant of motion, we may drop the time dependence.)

Finally we exhibit the Hamiltonian in terms of the new variables, which in old variables reads

$$H = \int dx\{\tfrac{1}{2}\Pi^2(x,t) + \tfrac{1}{2}(\Phi'(x,t))^2 + U(\Phi(x,t))\}. \tag{2.56}$$

Substituting (2.51) for Φ, (2.53) for Π, and shifting the variable of integration from x to $x+X$, so that all fields become evaluated at x rather than $x-X$, yields

$$H = M_0 + \overline{P}^2/2M_0 + H(\hat{\Pi}, \hat{\Phi}) - \frac{1}{8M_0^2}\frac{1}{(1+\xi/M_0)^2}\int dx(\varphi_c'')^2 \tag{2.57a}$$

where

$$\overline{P}(t) = \frac{1}{2}\left\{P + \int dx\hat{\Phi}'(x,t)\hat{\Pi}(x,t), \frac{1}{1+\xi(t)/M_0}\right\}_+ \tag{2.57b}$$

$$H(\hat{\Pi}, \hat{\Phi}; t) = \int dx\{\tfrac{1}{2}\hat{\Pi}^2(x,t) + \tfrac{1}{2}(\hat{\Phi}'(x,t))^2 + U(\hat{\Phi}(x,t), \varphi_c(x))\} \tag{2.57c}$$

$$U(\Phi, \varphi_c) = U(\hat{\Phi} + \varphi_c) - \hat{\Phi}U'(\varphi_c) - U(\varphi_c). \tag{2.57d}$$

[The shift of integration of variable $x \to x+X$ involves a shift of a c-number by a q-number. Hence one must take into account the noncommutativity of X with P. The last term in (2.57a) arises from this quantum effect (Tomboulis, 1975).] Note that H is independent of X, hence commutes with P, which may be diagonalized and taken to be a c-number. The orders of magnitude in g are $M_0 = O(g^{-2}), P, \hat{\Pi}, \hat{\Phi} = O(g^0), \xi/M_0 = O(g)$. Therefore for a perturbative expansion, H may be separated in the following way:

$$H = M_0 + H_0 + H_I,$$
$$H_0 = \frac{1}{2}\int dx[\Pi^2 + (\hat{\Phi}')^2 + U''(\varphi_c)\hat{\Phi}^2] = O(g^0). \tag{2.58}$$

Here $H_I = H - M_0 - H_0$ is the interaction part.

When H_I is ignored, H_0 can be diagonalized in terms of the solutions of our Schrödinger equation (2.42), and in this approximation $\hat{\Phi}$ can be written in terms of (meson) creation and annihilation operators.

$$\hat{\Phi}(x,t) = \sum_k{}' \frac{1}{\sqrt{2\omega_k}}(a_k\psi_k(x)\exp(-i\omega_k t) + a_k^\dagger\psi_k^*(x)\exp(i\omega_k t)) \tag{2.59}$$

(The zero-frequency mode is not included in the sum since $\hat{\Phi}$ is orthogonal to it.) Substituting this expansion for $\hat{\Phi}$ in H_0 we regain (2.47b). The soliton states are labeled by P, the eigenvalue of momentum conjugate to $X(t)$, which is also the total momentum. One easily verifies that the field form factor is given to lowest order by the Fourier transform of φ_c, and other matrix elements of Φ follow the postulate (2.26).

It is clear that a systematic perturbation series can now be developed, with $M_0 + H_0$ the unperturbed part and H_I the perturbation. The perturbation theory is exactly the same as the one discussed in the previous subsection, but now it can be represented by familiar graphical methods. Various computations to high orders in g have been performed (Gervais, Jevicki, and Sakita, 1975; Jacobs, 1976a; Gervais and Jevicki, 1976a; de Vega, 1976).

Although the collective coordinate method for canonical quantization in the soliton sector has been presented for the simple example in one spatial dimension, it of course carries over to the three-dimensional theory as well. Moreover, collective coordinates have to be introduced for all degeneracies of the problem, and the corresponding zero-frequency modes have to be removed by subsidiary conditions like (2.52) and (2.54). The soliton states will then be labeled by eigenvalues of the momenta conjugate to the collective coordinate, which commute with the Hamiltonian, if they generate symmetry transformations. For example, in a model with charged fields Φ, Φ^*, or in a real basis Φ_1, Φ_2, if there is a classical solution $\boldsymbol{\varphi}_c = \binom{\varphi_1}{\varphi_2}$, then because of charge conservation one can obtain another solution by a charge rotation

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \to \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}.$$

In this case θ is the degeneracy parameter which describes the symmetry—charge conservation. There will be a zero-frequency mode $\delta\boldsymbol{\varphi}_c = \binom{\varphi_2}{-\varphi_1}$ and the collective coordinate is $\Theta(t)$ with conjugate momentum $Q(t)$, which in fact is time independent since it generates charge rotations. The soliton energy eigenstates are also eigenstates of Q; they carry charge. Since Q is conjugate to an angular variable, it is quantized. The energy will depend on Q in order g^2, and the dependence will be of the form $Q^2/2\int dx(\delta\boldsymbol{\varphi}_c)^2$ (Rajaraman and Weinberg, 1975).

The general expansion scheme here presented may be called a Born–Oppenheimer expansion for field theory,

[5] Collective coordinates are widely used in many-body physics; an early application to the polaron problem is by Bogoliubov and Tyablikov, 1949; and to meson physics by Pais, 1957. For the soliton they were first discussed by Gervais and Sakita, 1975; Callan and Gross, 1975; Korepin and Faddeev, 1975. These authors used a functional integral for the formulation; however, the functional integral obscures problems of quantum operator ordering. The correct formalism, developed through operator canonical transformations, was given by Christ and Lee, 1975; Tomboulis, 1975. For more recent developments see Creutz, 1975; Tomboulis and Woo, 1976a; Gervais and Jevicki, 1976a; Korepin and Faddeev, 1975; Jevicki, 1976; Abbot, 1977.

since it is very analogous to the approximate calculations of the quantal properties of molecules. In that context also, the first approximation describes a particle localized at an equilibrium point without kinetic energy, while the second approximation exposes the harmonic vibrational spectrum around the equilibrium. It is only in the next and higher approximations that rotational degrees of freedom are encountered.

D. Time-dependent c-number fields

The previous subsections were devoted to an exhaustive discussion of static solutions to classical field theory and of their quantum significance. An interpretation was given in terms of a new state in the theory—the soliton—and the one-soliton sector of the Hilbert space was analyzed. It is plausible to suppose that time-dependent solutions to the classical field equations of our examples have something to do with multi-soliton states. As yet a complete and systematic perturbation theory for the multi-soliton states has not been developed. However, using semiclassical methods for field theory some results have been obtained. We describe first the types of time-dependent c-number solutions that one may expect to find in a field theory.

There are of course the entirely trivial time-dependent solutions where a static solution has been boosted. This solution obviously continues to describe a soliton, but now in a moving reference frame. No new physical information can be obtained, and we need not concern ourselves with them.

In some models with complex (charged) fields, it is possible to find time-dependent, stable solutions with localized energy density—soliton solutions—even though no such static solutions exist. The time dependence is in a phase, corresponding to nonvanishing charge. It has been shown that these solutions can also be associated with a stable quantum particle state—the stability arises not from topological properties of the field configuration, but from ordinary charge conservation. These very interesting nontopological solitons may have something to do with the observed particles; but they are outside the scope of this review and will not be discussed (Friedberg, Lee, and Sirlin, 1976).

The types of time-dependent solutions which are relevant to the topological solitons under review here should have the property that as $t\to-\infty$ they describe several widely separated static solutions, moving towards each other. Clearly such solutions would be relevant to multi-soliton scattering. Also interesting are periodic solutions that could be interpreted as multi-soliton bound states. Unfortunately, even in one spatial dimension the nonlinear, partial differential equations are sufficiently complicated so that no general discussion of such solutions is at present available. However, for the SG theory it is possible to integrate the equations completely and all solutions are explicitly available. Of necessity, I confine the subsequent discussion to the SG solutions.

In the SG theory one can find the static soliton and anti-soliton solution, given in Eq. (2.19b). There are also time-dependent N soliton solutions with the following properties. The N soliton solution depends on $2N$ parameters. As $t\to-\infty$, the solution becomes a superposition of N one-soliton solutions and the $2N$ parameters correspond to asymptotic velocities $v^{(i)}$ and positions $x_0^{(i)}[i=1,\ldots,N]$ of the N solitons. As $t\to+\infty$, the solution again decomposes into a superposition of N one-soliton solutions. The asymptotic final velocities are the same as the initial ones, the asymptotic positions differ from the initial ones by an amount that can be ascribed to a time delay in the multi-soliton collision. By translation invariance, two constants of motion can be arbitrarily set to zero, and a third can also be made to vanish if the calculation is performed in the center-of-mass frame. For example, the two-soliton solution depends on one constant, u, the relative velocity of the two solitons. The explicit form of the two-soliton solutions is:

soliton, soliton;

$$\varphi_{ss}=\frac{4m}{g}\tan^{-1}\frac{u\sinh m\gamma x}{\cosh m\gamma ut}, \tag{2.60a}$$

soliton, anti-soliton;

$$\varphi_{s\bar{s}}=\frac{4m}{g}\tan^{-1}\frac{1}{u}\frac{\sinh m\gamma ut}{\cosh m\gamma x}. \tag{2.60b}$$

$$\gamma=(1-u^2)^{-1/2},\quad u^2<1 .$$

The total momentum of each solution is zero; the energy is $2M_0\gamma, M_0=8m^3/g^2$. Examination of the asymptotic forms of the two solutions shows that in both cases there is time delay

$$\Delta t(u)=(2/mu\gamma)\ln u. \tag{2.61}$$

There is another solution, the soliton, anti-soliton bound state or "breather" which is periodic in time. It is obtained from (2.60b) by taking $u=ia$, a real

$$\varphi_B=\frac{4m}{g}\tan^{-1}\frac{1}{a}\frac{\sin m\gamma at}{\cosh m\gamma x},$$
$$\gamma=(1+a^2)^{-1/2}. \tag{2.62}$$

The energy is $2M_0\gamma$. There is no soliton, soliton bound state.

E. Quantum meaning of time-dependent, c-number fields

Once we have in hand classical, time-dependent solutions for a field theory, which is a very rare circumstance indeed, we can do something different from the Born–Oppenheimer approximation scheme that we have described in connection with the static solutions. Rather, following Bohr and Wigner, we can perform WKB approximations as in quantum mechanics.

Let us recall the WKB approximation in quantum mechanics with one degree of freedom. When a particle is moving periodically with energy E in a smooth potential between two turning points q_1 and q_2, then the WKB quantization condition is

$$\int_{q_1}^{q_2}\sqrt{2E-2V(q)}\,dq=(n+\tfrac{1}{2})\pi=\int_{q_1}^{q_2}p\,(q)dq,$$

where $p(q)=\sqrt{2E-2V(q)}$ is the local momentum. This is valid for large n, and the WKB approximation gives the first *two* terms in a large n expansion of the energy.

The semiclassical, Bohr quantization condition, based on the correspondence principle, determines the *first* term, $n\pi$. The *second* term $\frac{1}{2}\pi$, follows from the details of motion in one degree of freedom. To generalize to many degrees of freedom we remain with the Bohr condition. [It is possible to generalize the full WKB approximation to many degrees of freedom (Maslow, 1970; Gutzwiller, 1971) and even to a field theory with infinite degrees of freedom (Dashen, Hasslacher, and Neveu, 1974a, b); I shall not be reviewing these.]

To develop the generalization of the semiclassical method, we first rewrite the Bohr condition as

$$\int_{q_1}^{q_2} p(q)dq = \int_{t_1}^{t_2} dt\, p(t)\dot{q}(t) = n\pi,$$

where the variable of integration has been changed from q to t. Note that $t_2 - t_1$ is the semiperiod of the motion. For N degrees of freedom, the generalization is

$$\int_{t_1}^{t_2} dt \sum_{i=1}^{N} p_i(t)\dot{q}_i(t) = n\pi$$

Clearly, for a field theory, with an infinite number of degrees of freedom, the Bohr, semiclassical quantization rule is

$$\int_{t_1}^{t_2} dt \int dx\, \Pi(x,t)\dot{\varphi}(x,t) = n\pi. \tag{2.63a}$$

An equivalent formula is

$$\int_{t_1}^{t_2} dt(L+H) = I_T(E) + ET = n\pi, \tag{2.63b}$$

where $I_T(E)$ is the classical action for a periodic solution with semiperiod T and energy E. We are instructed by (2.63) to find classical periodic solutions, φ_B, then to integrate the product of the canonical momentum with $\dot{\varphi}_B$ over a semiperiod and equate this to $n\pi$, thus achieving one quantization condition on the parameters of the solution.

In the SG problem there exists a periodic solution, Eq. (2.62), depending on the parameter a. Hence to quantize it, we integrate $\Pi\dot{\varphi}_B = \dot{\varphi}_B^2$ over all space, and then over the semiperiod $-\pi/2m\gamma a \leq t \leq \pi/2m\gamma a$. Equating the integral to $n\pi$ results in a quantization of the parameter a. Since the energy is also expressed in terms of a, $E = 2M_0(1+a^2)^{-1/2}$, this is equivalent to a quantization of the energy. The result is (Dashen, Hasslacher, and Neveu, 1975; Korepin and Faddeev, 1975)

$$E_n = 2M_0 \sin\frac{m}{2M_0}n, n = 1, 2, \ldots, \leq \frac{8\pi m^2}{g^2} = \frac{M_0\pi}{m}. \tag{2.64}$$

So in the quantum field theory there are soliton, antisoliton bound states.

Let us now perform a semiclassical analysis of scattering. Wigner has shown that twice the derivative of the phase shift in the semiclassical approximation is the time delay

$$d\delta(E)/dE = \tfrac{1}{2}\Delta t(E). \tag{2.65}$$

Hence we can compute the phase shift by integrating the time delay (2.61) with respect to energy

$$\delta(E) = \delta(E_{\text{th}}) + \tfrac{1}{2}\int_{E_{\text{th}}}^{E} dE'\, \Delta t(E'). \tag{2.66}$$

The constant of integration, the phase shift at threshold, may be evaluated as follows. Consider the classical action $I_T(E)$ for a solution to the equations of motion, with energy E, which passes from an initial configuration to a final configuration in time T. Since $dI_T(E)/dT = -E$, it follows that

$$\frac{d}{dE}(I_T(E) + ET) = \left(\frac{dI_T(E)}{dE} + E\right)\frac{dT}{dE} + T = T. \tag{2.67a}$$

That is, the time of flight can be expressed as an energy derivative. Total time delay is equal to the time of flight in the presence of forces, less the time of flight in the absence of forces, in the limit as T goes to infinity. But the time of flight in the absence of forces is given by (2.67a), where the term in parentheses on the left-hand side may be written as $p(E)[x_f(T) - x_i]$, with $p(E)$ being the relative momentum of the particles and x_i, x_f the initial, final position. Thus the total time delay is

$$\Delta t(E) = \lim \frac{d}{dE}(I_T(E) + ET - p(E)[x_f(T) - x_i]), \tag{2.67b}$$

and the phase shift can be taken to be

$$2\delta(E) = \lim(I_T(E) + ET - p(E)[x_f(T) - x_i]). \tag{2.68}$$

At threshold $p(E_{\text{th}}) = 0$ and therefore

$$2\delta(E_{\text{th}}) = \lim(I_T(E_{\text{th}}) + E_{\text{th}}T). \tag{2.69}$$

Next let us consider the quantization condition (2.63b). The total number of bound states is given by n_B, the maximum value of n, which occurs for E just below E_{th}, since at E_{th} the semiperiod becomes infinite. Hence it is true that

$$n_B\pi = \lim(I_T(E_{\text{th}}) + E_{\text{th}}T). \tag{2.70}$$

Comparing (2.69) and (2.70) we find

$$\delta(E_{\text{th}}) = (\pi/2)n_B, \tag{2.71}$$

which we call the semiclassical Levinson's theorem. [The exact Levinson's theorem is $\delta(E_{\text{th}}) - \delta(\infty) = (\pi/2)n_B$, where the factor of $\frac{1}{2}$ is peculiar to one-dimensional motion.]

Therefore the semiclassical phase shift is given by

$$\delta(E) = \tfrac{1}{2}n_B\pi + \tfrac{1}{2}\int_{E_{\text{th}}}^{E} dE'\, \Delta t(E'). \tag{2.72}$$

Using (2.61) as well as the fact that $n_B = 0$ for the soliton, soliton channel and $n_B = 8\pi m^2/g^2$ for the soliton, antisoliton channel, we find the following phase shifts (Jackiw and Woo, 1975):

$$\delta_{ss}(E) = \frac{16m^2}{g^2}\int_0^u dx\, \frac{\ln x}{1-x^2} = 2\frac{M_0}{m}\int_0^u dx\, \frac{\ln x}{1-x^2} \tag{2.73a}$$

$$\delta_{s\bar{s}}(E) = \frac{4\pi^2 m^2}{g^2} + \frac{16m^2}{g^2}\int_0^u dx\, \frac{\ln x}{1-x^2} = \frac{\pi^2}{2}\frac{M_0}{m} + 2\frac{M_0}{m}\int_0^u dx\, \frac{\ln x}{1-x^2}$$

$$E = (8m^3/g^2)(1-u^2)^{-1/2} = M_0(1-u^2)^{-1/2}. \tag{2.73b}$$

There are several interesting properties of the result. Note that for weak coupling (small g), the phase shifts are large. This means that the forces are large and that the solitons are strongly interacting even for weak coupling. In fact there are three scales of interaction for the theory when g is small. The conventional meson–meson interactions are weak, since they are proportional to g. The meson–soliton interactions are of intermediate strength—independent of g [compare (2.45)]. Finally the soliton–soliton forces are strong. Another interesting feature of (2.73) is that crossing symmetry is satisfied in the sense that δ_{ss} and $\delta_{s\bar{s}}$ are related by the crossing relation (Coleman, 1975b).

F. Quantization about time-dependent, c-number fields

In order to calculate contributions to the bound-state energies and phase shifts, beyond the lowest-order ones presented in the previous subsection, a systematic series must be constructed for expanding the quantum theory around a time-dependent, c-number solution. Such expansions have been found by functional WKB methods (Dashen, Hasslacher, and Neveu, 1975) or by time-dependent canonical transformations (Christ and Lee, 1975). The formalism requires explicit time-dependent solutions; consequently its applicability is confined to the SG theory where these are available. Moreover, it is sufficiently complicated so that only the first correction has been computed. I shall not review the theory, beyond quoting the results of the first-order computations which, in the end, turn out to be very simple (Dashen, Hasslacher, and Neveu, 1975; Gervais and Jevicki, 1976b; Lee and Gavrielides, 1976): Both the bound-state energies and the phase shifts are given by the same expressions as in the semiclassical limit, provided they are expressed in terms of the first-order soliton mass $M=(8m^3/g^2)-(m/\pi)$, rather than the lowest-order mass $M_0=(8m^3/g^2)$. It has also been possible to extend the WKB method to a calculation of the (exponentially small) reflection coefficient in multi-soliton scattering (Korepin, 1976).

The SG theory has continued to interest theorists, and several exact results have been obtained. It has been shown that the SG theory is equivalent to the massive Thirring model (Coleman, 1975a). The fermion fields of the Thirring model describe particles, which are identified with the SG solitons, and the fermion number current $\bar{\psi}\gamma^\mu\psi$ is proportional to the topological current $\epsilon^{\mu\nu}\partial_\nu\Phi$. Thus the SG solitons also are fermions, just as are the φ^4 solitons. Furthermore, it has been possible to obtain the bound-state spectrum of the solitons-fermions exactly, and it is found to agree with the WKB approximation (Luther, 1976).

These exact results validate the semiclassical approach. Also they expose a fascinating duality: the same physical reality has two equivalent descriptions, one bosonic with fermions appearing as coherent bound states; the other fermionic where the bosons are bound states. Unfortunately there is no indication at the present time that these marvelous features of the SG theory are also to be seen in realistic, three-dimensional models.

G. Effects of Fermi fields

The models discussed thus far involve only Bose fields. We now wish to summarize the new effects that arise when Fermi fields are included. We consider the Lagrange density

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi-\frac{m^4}{2g^2}\left(1-\frac{g^2}{m^2}\Phi^2\right)^2+i\bar{\Psi}\gamma^\mu\partial_\mu\Psi-Gg\Phi\bar{\Psi}\Psi \tag{2.74}$$

which leads to the operator field equations

$$\Box\Phi=2m^2\Phi-2g^2\Phi^3-Gg\bar{\Psi}\Psi, \tag{2.75a}$$

$$i\gamma^\mu\partial_\mu\Psi-Gg\,\Phi\Psi=0. \tag{2.75b}$$

In the absence of the Fermi fields, this is just the φ^4 model, with $O(g^{-1})$ soliton solutions. The Fermi field equation (2.75b) is linear in Ψ, whose interaction with the Bose field is of the form $g\Phi$, hence $O(g^0)$ when Φ is $O(g^{-1})$. [We take G to be $O(g^0)$.] Also the reaction of the Fermi fields on Φ is seen from (2.75a) to be $O(g)$. Hence for weak coupling (small g), we may ignore the Fermi fields in the first $O(g^{-1})$ approximation, solve the pure Bose equation, and then solve the Dirac equation in the external, static, c-number Bose potential, thus obtaining an $O(g^0)$ Dirac wave function. In other words, for the extended theory (2.74), a systematic coupling constant expansion may be given in which the Fermi fields enter only at the first correction to the lowest approximation.

Before describing the quantal significance of the c-number Dirac wave function, let us discuss the solutions of the Dirac equation (2.75b), where the Fermi field operator is viewed as a c-number wave function and $g\Phi$ is given by the static solution to the φ^4 theory. Because the potential is static, we may look for energy eigenfunctions

$$\psi(x,t)=\exp(-i\epsilon t)\psi_\epsilon(x), \tag{2.76a}$$

$$\left[\alpha\,\frac{1}{i}\,\frac{\partial}{\partial x}+\beta Gm\tanh mx\right]\psi_\epsilon(x)=\epsilon\psi_\epsilon(x), \tag{2.76b}$$

where $\alpha=\gamma^0\gamma^1,\beta=\gamma^0$. The equation possesses the usual positive and negative energy solutions. These are related by a fermion-number conjugation operation, which in a representation where $\alpha=\sigma^2$ and $\beta=\sigma^1$ is given by σ^3. However, in addition to these, there is a unique, normalizable, zero-energy solution

$$\psi_0(x)=\begin{pmatrix}[\cosh mx]^{-G}\\ 0\end{pmatrix}. \tag{2.77}$$

(We have assumed $G>0$; if $G<0$ there still is a solution, but with a nonvanishing lower component.) The zero-energy solution is self-conjugate

$$\sigma^3\psi_0=\psi_0. \tag{2.78}$$

We shall demonstrate that the occurrence of the zero-energy mode in the Fermion system has profound consequences for the physical interpretation of the quantum theory. Although the solution is here exhibited in a very specific model, in fact it is present in much more general situations. It appears, although this has not been proven, that any Dirac equation in a topologically interesting potential with a conjugation symmetry, possesses a normalizable, self-conjugate, zero-energy solution (Jackiw and Rebbi, 1976a, 1976c; 't Hooft, 1976a,b; Jacobs, 1976b; Grossman, 1977).

The quantum theory described by the Lagrangian (2.74) has the following structure. There is of course the conventional vacuum sector with mesons, fermions, and anti-fermions, there is also a soliton sector, and in the absence of Fermi fields we have demonstrated the existence of the one-soliton states $|P\rangle$. This sector is modified by the presence of Fermi fields which possess a zero-energy solution to the c-number Dirac equation. The zero-energy mode signals quantum mechanical degeneracy, and we postulate that as a consequence the soliton states are doublets $|P\pm\rangle$. The additional label, $\pm$, describes a twofold degeneracy which is required by the zero-energy fermion solution. We call the + state "soliton" and the − state "anti-soliton." (This bifurcation has no relation to any soliton multiplicity which is already present in the purely bosonic theory, as in the SG example. We are here describing a new degeneracy, which is a consequence of the fermions.) It must be stressed that we do not take the viewpoint that the solitons exist independently of the fermions, which then bind to them with zero energy. Rather we say that the soliton is doubly degenerate. The difference is that the first viewpoint would lead to four states: the original soliton, soliton plus fermion, soliton plus anti-fermion, soliton plus fermion and anti-fermion; our interpretation involves only two states.

The consistency of this picture is demonstrated by the same method as in the purely bosonic theories: we list relevant states in the soliton sector, postulate orders of magnitude for various field matrix elements, determine equations for the field form factors from the operator equations of motion (2.75), expand systematically in the coupling constant to regain the c-number equations, and finally normalize various solutions by quantum field commutation and anticommutation relations. Specifically we consider the states

$|P\pm\rangle$ soliton or anti-soliton, with energy $\sqrt{P^2+M^2}$,

$|P\pm;k\rangle$ soliton or anti-soliton plus one meson, with energy $\sqrt{P^2+M^2}+\omega_k$,

$|P\pm;p+\rangle$ soliton or anti-soliton plus one fermion with energy $\sqrt{P^2+M^2}+\epsilon_p$

$|P\pm;p-\rangle$ soliton or anti-soliton plus one anti-fermion with energy $\sqrt{P^2+M^2}+\epsilon_p$.

The field form factors are defined

$$\langle P'\pm|\Phi|P\pm\rangle=\int dx\,\exp[i(P'-P)x]\phi(x)=O(g^{-1}) \tag{2.79a}$$

$$\langle P'-|\Psi|P+\rangle=\int dx\,\exp[i(P'-P)x]u_0(x)=O(g^{0}) \tag{2.79b}$$

$$\langle P'\pm|\Psi|P\pm;p+\rangle=\int dx\,\exp[i(P'-P)x]u_p(x)=O(g^{0}) \tag{2.79c}$$

$$\langle P'\pm|\Psi^\dagger|P\pm;p-\rangle=\int dx\,\exp[i(P'-P)x]v_p(x)=O(g^{0}) \tag{2.79d}$$

$$\langle P'\pm|\Phi|P\pm;k\rangle=\int dx\,\exp[i(P'-P)x][\phi(k;x)/\sqrt{2\omega_k}]=O(g^{0}) \tag{2.79e}$$

One readily verifies that, to leading order in g, $\phi(x)$ continues to satisfy the classical, bosonic equation and $\phi(k,x)$ is the normalized solution of the Schrödinger equation with eigenvalue ω_k. Furthermore one finds that $u_0(x)$ is the normalized, zero-energy solution of the Dirac equation (2.76b); $u_p(x)$ is a normalized positive energy solution with eigenvalue ϵ_p; v_p is the conjugate of the normalized negative energy solution with eigenvalue $-\epsilon_p$. The normalization is determined by evaluating the equal-time Fermi field anti-commutator

$$\langle P'\pm|\{\Psi^\dagger(x,0),\Psi(y,0)\}_+|P\pm\rangle=\delta(x-y)(2\pi)\delta(P'-P). \tag{2.80}$$

A compact summary of the physical picture that is being presented can be given by the following expansion of the quantum fields. In order to account for the translational degrees of freedom, a collective position coordinate is introduced, as explained in Sec. IIC.

$$\begin{aligned}\Phi(x,t)&=\varphi_c(x-X(t))+\hat\Phi(x-X(t),t),\\ \Psi(x,t)&=\hat\Psi(x-X(t),t).\end{aligned} \tag{2.81}$$

The expansion for the new fields $\hat\Phi$ and $\hat\Psi$ is [compare (2.59)]

$$\begin{aligned}\hat\Phi(x,t)&={\sum_k}'\frac{1}{\sqrt{2\omega_k}}(a_k(t)\psi_k(x)+a_k^\dagger(t)\psi_k^*(x))\\ \hat\Psi(x,t)&=a(t)u_0+\sum_p(b_p(t)u_p(x)+d_p^\dagger(t)v_p^c(x)).\end{aligned} \tag{2.82}$$

The boson operators satisfy

$$[a_k(t),a_{k'}^\dagger(t)]=(2\pi)\delta(k-k'), \tag{2.83a}$$

and create or annihilate mesons in the soliton sector. The fermion operators satisfy

$$\begin{aligned}&\{a(t),a^\dagger(t)\}_+=1,\\ &\{b_p(t),b_{p'}^\dagger(t)\}_+=(2\pi)\delta(p-p'),\\ &\{d_p(t),d_{p'}^\dagger(t)\}_+=(2\pi)\delta(p-p'),\end{aligned} \tag{2.83b}$$

The b's create or annihilate fermions, while the d's perform the same task for the anti-fermions. The a operator however does not create or annihilate particles; it merely connects the soliton with the anti-soliton

$$\begin{aligned}&a|P+\rangle=|P-\rangle,\\ &a^\dagger|P-\rangle=|P+\rangle,\\ &a|P-\rangle=0\\ &a^\dagger|P+\rangle=0.\end{aligned} \tag{2.83c}$$

In the theory (2.74) fermion number is a good quantum number, arising from the conserved number current, $J^\mu=:\bar\psi\gamma^\mu\psi:$ which is odd under Fermi number conjugation. In the vacuum sector, the vacuum and the meson states have zero Fermi number; the one-fermion states carry Fermi number ±1, etc. In the soliton sector, we must conclude that soliton and anti-soliton differ by one unit of Fermi number, since the Fermi field matrix element connects the two; see (2.79) and (2.81). In order to preserve Fermi number symmetry, we must assign Fermi number $\pm\frac{1}{2}$ to the solitons, and one can easily verify that

$$\langle P'\pm|\int dx J^0|P\pm\rangle = \pm\tfrac{1}{2}(2\pi)\delta(P'-P). \tag{2.84}$$

The existence of states with fermion number $\pm\frac{1}{2}$ in a theory where all fundamental fields carry integer fermion number is a fascinating phenomenon which, as will be seen below, can also happen in realistic three-dimensional models (Jackiw and Rebbi, 1976a).

III. MODELS IN THREE SPATIAL DIMENSIONS

We have demonstrated that classical solutions to quantum field equations contain information about the corresponding quantum field theory. Moreover, a systematic coupling-constant expansion can be given for various physically interesting quantum objects. However, all our examples were confined to an unphysical world with one spatial dimension. Now we wish to discuss theories in three spatial dimensions.

In dimension greater than one, fields with spin degrees of freedom have to be used in order to support stable, static, classical solutions. It turns out that models that are known to possess soliton solutions in three dimensions are Yang–Mills–Higgs theories, precisely those theories which have recently become candidates for a fundamental description of natural processes. One of the several models which can exemplify the application of our methods is built on the SU(2) group. I shall review only this example.

The Lagrange density is

$$\begin{aligned}
\mathcal{L} &= -\tfrac{1}{4}F_a^{\mu\nu}F_{a\mu\nu} + \tfrac{1}{2}(D^\mu\Phi)_a(D_\mu\Phi)_a - U(\Phi_a\Phi_a),\\
F_a^{\mu\nu} &= \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + e\epsilon_{abc}A_b^\mu A_c^\nu,\\
(D^\mu\Phi)_a &= \partial^\mu\Phi_a + e\epsilon_{abc}A_b^\mu\Phi_c,\\
U(\Phi_a\Phi_a) &= \frac{m^4}{2g^2}\left(1-\frac{g^2}{m^2}\Phi_a\Phi_a\right)^2,
\end{aligned} \tag{3.1}$$

where

$$a = 1,2,3.$$

The gauge symmetry

$$\begin{aligned}
\delta\Phi_a &= \epsilon_{abc}\theta_b\Phi_c,\\
\delta A_a^\mu &= \epsilon_{abc}\theta_b A_c^\mu - \frac{1}{e}\partial^\mu\theta_a,
\end{aligned} \tag{3.2}$$

is spontaneously broken since the scalar field acquires a vacuum expectation value $[\varphi_0^2 = m^2/g^2]$. Consequently the vacuum sector contains a massive, scalar particle—the Higgs particle; a massive, charged doublet of vector mesons; and a massless vector meson—the photon. Static solutions and the corresponding soliton sector are discussed in three following subsections. Then two subsections are devoted to various extensions of the model. Finally, we delete all fields save the gauge fields and discuss imaginary-time solutions and their quantal significance in Yang–Mills theory.

A. Static, c-number fields

The theory (3.1) is known to possess a static solution ('t Hooft, 1974; Polyakov, 1974). Its form is

$$\begin{aligned}
A_a^0 &= 0\\
A_a^i &= \epsilon^{aij}\hat{r}_j a(r),\\
\varphi_a &= \hat{r}^a f(r).
\end{aligned} \tag{3.3}$$

Here $\hat{r}$ is the radial unit vector; $a(r)$ and $f(r)$ depend only on the radial magnitude. Notice that the index a on φ_a and A_a^i is an isospin index, but on $\hat{r}^a$ and ϵ^{aij} it is a spatial index. Hence the solution mixes spatial and internal degrees of freedom. The functions $a(r)$ and $f(r)$ satisfy nonlinear, coupled, differential equations which are obtained by substituting the *Ansatz* (3.3) into the complete Euler–Lagrange equations of the theory. The equations can only be solved numerically, but the asymptotic properties of the solutions are given explicitly

$$\begin{aligned}
a(0) &= 0,\\
f(0) &= 0,
\end{aligned} \tag{3.4a}$$

and

$$\begin{aligned}
a(r) &\xrightarrow[r\to\infty]{} -\frac{1}{er},\\
f(r) &\xrightarrow[r\to\infty]{} \frac{m}{g}.
\end{aligned} \tag{3.4b}$$

The complete functions, known only numerically, smoothly interpolate, without nodes, between their asymptotic values. With these asymptotic forms one can check that the total energy is finite, but due to the complexity of the equations no one has as yet verified stability, although it is generally believed that the solution is indeed stable. For a consistent expansion in terms of a unique coupling constant, we take g to be $O(e)$; the solutions are $O(e^{-1})$.

It is useful to exhibit the classical solution in a gauge where φ_a points in a fixed (rather than position-dependent) direction in isospace. Upon performing a gauge transformation so that φ_a lies in the third direction, the soliton solution becomes

$$\begin{aligned}
A_a^0 &= 0,\\
A_a^i &= e_a^i w(r), \quad a=1,2\\
A_3^i &= \frac{1}{2e}\left(\hat{n}\times\nabla\ln\frac{1+\hat{n}\cdot\hat{r}}{1-\hat{n}\cdot\hat{r}}\right)_i,\\
\varphi_a &= 0, \quad a=1,2\\
\varphi_3 &= f(r),\\
w(r) &= a(r) + \frac{1}{er}.
\end{aligned} \tag{3.5a}$$

Here $\hat{n}$ is a fixed unit vector in the z direction and the $\hat{e}_a$'s, $a=1,2$, are two unit vectors orthogonal to $\hat{r}$ so that

$$\hat{e}_2 = \frac{\hat{n}\times\hat{r}}{|\hat{n}\times\hat{r}|}, \quad \hat{e}_1 = \hat{e}_2\times\hat{r} \tag{3.5b}$$

Note that $\mathbf{A}_3$ is the vector potential for a magnetic monopole: $\mathbf{B} = \nabla\times\mathbf{A}_3 = \hat{r}/er^2$. Hence the soliton is also a magnetic monopole with pole strength $(1/e)$, and we recognize the $\hat{n}$ vector as the "Dirac string" which is always present in the vector potential for a magnetic monopole.

The gauge specification that φ_a should point in the third direction in isospace—the unitary gauge—does not fix the gauge completely, since it is still possible to make local gauge rotations about the third axis. Further gauge specification is achieved by requiring $\nabla\cdot\mathbf{A}_3=0$, a condition satisfied by (3.5). But this Coulomb condition still allows the gauge transformations $\mathbf{A}_3\rightarrow\mathbf{A}_3-1/e\nabla\theta$ with harmonic gauge functions $\nabla^2\theta=0$, and corresponding changes in $\mathbf{A}_a$, $a=1,2$. (Usually this additional gauge freedom is ignored since harmonic functions are always singular at the origin or at infinity, and singular field configurations usually are excluded. In the present context, however, we must allow for singular field configurations, since already $\nabla\times\mathbf{A}_3$ is singular at the origin.) One verifies that harmonic gauge transformations on (3.5) are equivalent to rotations of n. Hence to specify the gauge completely we may say that the "Dirac string" always points in the z direction, a choice already made in Eqs. (3.5).

A topologically conserved current may also be exhibited:

$$J^\mu=\frac{1}{4\pi}\partial_\nu{}^*F^{\mu\nu},$$
$${}^*F^{\mu\nu}=\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta},$$
$$F^{\mu\nu}=\partial^\mu A_3^\nu-\partial^\nu A_3^\mu. \tag{3.6}$$

The current is trivially conserved, since ${}^*F^{\mu\nu}$ is antisymmetric in $\mu\leftrightarrow\nu$. Nevertheless, the total charge

$$N=\int d\mathbf{r}J^0=\int\frac{d\mathbf{r}}{4\pi}\partial_i{}^*F^{0i}=\int\frac{d\mathbf{r}}{4\pi}\nabla\cdot\mathbf{B} \tag{3.7a}$$

is nonvanishing since $\nabla\cdot\mathbf{B}\alpha\delta^3(\mathbf{r})$ and

$$N=\frac{1}{e}. \tag{3.7b}$$

There are general topological analyses which make it possible to predict *a priori* when one may expect to find a topologically interesting solution to a gauge theory in three dimensions (Coleman, 1975c). Such an analysis, when applied to the problem at hand, predicts (3.7b), but is of no further aid in constructing the solution.

B. Quantum meaning of static, c-number fields

The classical solution (3.3) or (3.5) signals the presence of a soliton sector. To expose its properties, we first rewrite the Lagrangian density (3.1) in the unitary gauge, where the field operators are redefined as follows:

$$\Phi_a=\begin{pmatrix}0\\0\\1\end{pmatrix}\Phi \quad \text{(Higgs field)},$$
$$A_a^\mu=W_a^\mu,\quad a=1,2 \text{ (massive, charged vector meson field)},$$
$$A_3^\mu=A^\mu \quad \text{(photon field)},$$
$$F_a^{\mu\nu}=(\partial^\mu\delta_{ab}-eA^\mu\epsilon_{ab})W_b^\nu-\mu\rightarrow\nu\quad a=1,2,$$
$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu,$$
$$\epsilon_{ab}=-\epsilon_{ba},\epsilon_{12}=1, \tag{3.8a}$$

$$\mathcal{L}=-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}-\tfrac{1}{4}F_a^{\mu\nu}F_{a\mu\nu}+\tfrac{1}{2}(\partial_\mu\Phi)^2-\frac{e}{2}\epsilon_{ab}W_a^\mu W_b^\nu F_{\mu\nu}$$
$$-\frac{e^2}{4}(\epsilon_{ab}W_a^\mu W_b^\nu)^2+\frac{e^2}{2}\Phi^2W_a^\mu W_{\mu a}-U(\Phi^2). \tag{3.8b}$$

According to the general approach developed in the previous section, we expect that for the above model there exists, in addition to the usual, vacuum sector of the Hilbert space where N is zero, a soliton sector, with nonvanishing N, populated by heavy particles which are also monopoles. The quantum numbers describing the soliton-monopole states are the following. Translation invariance of the theory insures that we obtain new classical solutions by shifting the origin. Hence in the quantum theory, the energy eigenstate is described by a momentum quantum number—just as in our one-dimensional examples. Also because the theory (3.8) conserves charge (it is invariant under rotations in the two-dimensional charge space labeled by $a=1,2$), we obtain a new static solution by replacing $\hat{e}_a$ in (3.5) with

$$\begin{matrix}\hat{e}_1'\\ \hat{e}_2'\end{matrix}=\begin{pmatrix}\cos\theta\hat{e}_1+\sin\theta\hat{e}_2\\-\sin\theta\hat{e}_1+\cos\theta\hat{e}_2\end{pmatrix}.$$

Therefore, as explained in subsection II.C, we need a collective coordinate $\Theta(t)$, and the soliton–monopole states become labeled by the eigenvalues of the conjugate momentum, that is, by the total charge Q. In the quantum theory the monopole can move in space (the states carry momentum) and in charge space (the states carry charge). The classical monopole gives rise in the quantum theory to a tower of charged states—the monopole becomes a dyon. [In the classical theory, spatially moving, time-dependent solutions can be obtained by Lorentz-boosting static solutions. Similarly it has been possible to find time-dependent solutions with nonvanishing electric charge (Julia and Zee, 1975).[6]] Now we come to a subtle point: If we make a spatial rotation on the solution, (3.5), it appears that we arrive at a different solution, since the original equations are rotationally invariant. Does this mean that we need to introduce collective coordinates to describe rotations, and that the monopole states are further labeled by the conjugate variable which clearly would be the angular momentum? If this were the case, one would have to conclude that the monopoles have intrinsic spin. However, in fact the variety of solutions obtained by a rotation does not lead to new quantum degrees of freedom. The point is that a spatial rotation on (3.5) merely rotates the string $\hat{n}$, which in turn is equivalent to a gauge transformation. But our gauge specification requires that $\hat{n}$ always point in a fixed direction, so that to maintain this condition, the spatial rotation must be "undone" by a gauge transformation, and a new solution is not reached.

These considerations lead us to postulate that the monopole states are $|\mathbf{P},q\rangle$ (Goldstone and Jackiw, 1976)

[6]Julia and Zee exhibit their solution in a time-independent form with $A_a^0=\hat{r}_aW(r)$, and $W(r)$ constant at $r\rightarrow\infty$. This constant, however, may be removed by a gauge transformation which then renders the solution time-dependent, and convergent to zero at $r\rightarrow\infty$.

$$\mathbf{P}|\mathbf{P},q\rangle=\mathbf{P}|\mathbf{P},q\rangle,$$
$$H|\mathbf{P},q\rangle=E_q(P)|\mathbf{P},q\rangle, E_q(P)=\sqrt{M_q^2+P^2},$$
$$Q|\mathbf{P},q\rangle=q|\mathbf{P},q\rangle, \tag{3.9}$$

and the matrix elements of the quantum fields are, to order e^{-1},

$$\langle\mathbf{P}',q'|\Phi|\mathbf{P},q\rangle=\delta_{q';q}\int d\mathbf{r}\exp[i(\mathbf{P}'-\mathbf{P})\cdot\mathbf{r}]f(r),$$

$$\langle\mathbf{P}',q'|\mathbf{A}|\mathbf{P},q\rangle=\delta_{q';q}\int d\mathbf{r}\exp[i(\mathbf{P}'-\mathbf{P})\cdot\mathbf{r}]\times\frac{1}{2e}\left[\hat{n}\times\nabla\ln\frac{1+\hat{n}\cdot\hat{r}}{1-\hat{n}\cdot\hat{r}}\right],$$

$$\langle\mathbf{P}',q'|\frac{1}{\sqrt{2}}(\mathbf{W}_1+i\mathbf{W}_2)|P,q\rangle=\delta_{q';q-1}\int d\mathbf{r}\exp[i(\mathbf{P}'-\mathbf{P})\cdot\mathbf{r}]w(r). \tag{3.10}$$

Although the small-oscillation equation has not been solved, one may identify zero-frequency modes: there are zero-frequency modes associated with translations; they are the gradients of the classical solution and lead to the **P** dependence of the states; also there are zero-frequency modes associated with charge rotations $\delta\mathbf{W}_a=\epsilon_{ab}\mathbf{W}_b$ and these lead to the q dependence of the states. The energy acquires a dependence on **P** and q in order e^2.

C. Quantization about static, c-number fields

A systematic coupling constant expansion can be given by the collective coordinate method described in subsection II.C. The procedure is entirely similar to that in one dimension, with obvious modifications: there are three collective position coordinates, and now there is also a collective coordinate for the charge. The soliton's mass becomes, to $O(e^2)$,

$$M_q=M+\tfrac{1}{2}q^2\Delta M, \tag{3.11}$$

$$M=M_0+\frac{1}{2}\sum_k\omega_k+O(e^2), \tag{3.12}$$

where M_0 is the classical energy of the static field configurations. The charge-dependent correction ΔM is $O(e^2)$. It has contributions analogous to those arising from the momentum dependence of the energy: $\Delta M\sim 1/I, I=\int d\mathbf{r}\rho(r), \rho(r)=2w^2(r)=O(e^{-2})$. Also there is a further term describing the Coulomb interaction energy

$$\frac{e^2}{I^2}\frac{1}{4\pi}\int d\mathbf{r}d\mathbf{r}'\frac{\rho(r)\rho(r')}{|\mathbf{r}-\mathbf{r}'|}=O(e^2).$$

In spite of the rotationally nonsymmetric appearance of the matrix elements (3.10), one verifies that the theory is in fact rotationally covariant, and the monopoles are spinless. Therefore a completely consistent quantum mechanical description of the soliton–monopole sector is available (Tomboulis and Woo, 1976b; Hasenfratz and Ross, 1976; Christ, Guth, and Weinberg, 1976; Ansourian, 1976).

D. Effects of Fermi fields

The purely bosonic model (3.1) may be extended to include couplings to Fermi fields:

$$\mathcal{L}=-\tfrac{1}{4}F_a^{\mu\nu}F_{a\mu\nu}+\tfrac{1}{2}(D^\mu\Phi)_a(D_\mu\Phi)_a-U(\Phi_a\Phi_a) \tag{3.13a}$$

$$D^\mu\Psi=\partial^\mu\Psi-ieT^aA_a^\mu\Psi\quad+i\bar{\Psi}\gamma^\mu D_\mu\Psi-Ge\bar{\Psi}T^a\Psi\Phi_a. \tag{3.13b}$$

The multiplet of Fermi fields transforms under isospin rotations according to

$$\delta^a\Psi=iT^a\Psi,$$
$$[T^a,T^b]=i\epsilon^{abc}T^c. \tag{3.13c}$$

We assume that G is $O(e^0)$; therefore, the extended theory may be analyzed according to the program presented in subsection II.G. The Dirac equation in the external static field (3.3) is

$$\left[\alpha\cdot\frac{1}{i}\nabla+ea(r)T^a(\alpha\times\hat{r})_a+\beta Gef(r)T^a\hat{r}_a\right]\psi_\epsilon(\mathbf{r})=\epsilon\psi_\epsilon(\mathbf{r}),$$
$$\alpha=\gamma^0\gamma,\quad\beta=\gamma^0. \tag{3.14}$$

There exists a conjugation symmetry which takes positive energy solutions into negative energy solutions.

We recognize the situation here to be completely analogous to that described for the one-dimensional example, and we expect that (3.14) admits normalizable, zero-energy solutions. Indeed such a solution has been explicitly constructed for the case of isospinor fermions. In a representation where

$$\alpha=\begin{bmatrix}0&\sigma\\\sigma&0\end{bmatrix}\quad\beta=-i\begin{bmatrix}0&-iI\\iI&0\end{bmatrix} \tag{3.15}$$

the form of the zero-energy wave function is

$$\psi_0(\mathbf{r})=F(r)\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right\},$$

$$F(r)=\exp-e\int_0^r dr'[\tfrac{1}{2}Gf(r')-a(r')]. \tag{3.16}$$

Here the four-component spinors refer to the spin degrees of freedom, while the two-component ones refer to isospin. In the representation (3.15), the conjugation symmetry is effected by the matrix $\beta\alpha^2T^2$, and (3.16) is self-conjugate (Jackiw and Rebbi, 1976a).

The quantum description of the soliton–monopole sector is the extended theory follows the discussion of subsection II.G. The zero-energy fermion mode renders the solitons doubly degenerate; we append a ± label to the states and the Fermi quantum field effects a transition between them $\langle+|\Psi|-\rangle\neq 0$. The transition matrix element is given by the Fourier transform of (3.16), gauge-transformed to the unitary gauge. (Recall that the quantum theory for soliton-monopoles is developed in the unitary gauge where the particle content of the model is easily identified.)

Again the soliton states carry Fermi number $\pm\frac{1}{2}$. However, they remain spinless—enough c-number solutions to give a nontrivial representation of the rotation group are not available. It may appear contradictory that a Fermi, spin-$\frac{1}{2}$ field can have a nonvanishing matrix element between spinless states. Nevertheless this indeed is the case—Ψ is a gauge-dependent operator whose transformation properties in the soliton–monopole sector are consistent with the above results (Ansourian, 1976). (A more physical explanation is given in the next subsection.)

Zero-energy fermion solutions are found, not only for isodoublet fields, but also in other examples. For isovector fermions it is again possible to establish the presence of these zero-energy modes, but the complexity of the equations prevents explicit construction of the wave functions (Jackiw and Rebbi, 1976a). Also if the model is extended to SU(3), again zero-energy fermion solutions exist (Jacobs, 1976b).

E. Fermions from bosons, spin from isospin

In our study of one-dimensional solitons, we encountered the remarkable circumstance that the soliton particle is a fermion even though only Bose fields occur in the Lagrangian. This can also happen in realistic three-dimensional systems, where the phenomenon is even more startling since in three dimensions there is spin(intrinsic angular momentum), which for fermions must be half-integer valued—a requirement obviously absent in one dimension. I shall now describe how spin can arise from isospin, and correspondingly how fermions are created out of bosons (Jackiw and Rebbi, 1976b; Hasenfratz and 't Hooft, 1976; Goldhaber, 1976).

The basic mechanism for the emergence of half-integer spin in a theory where all fundamental particles carry integer spin, has been known since the late nineteenth century, when it was observed that the total angular momentum of a point particle with charge Q_e, mov-moving in a magnetic monopole field of strength Q_m, contains in addition to the usual orbital term a further contribution of magnitude Q_eQ_m, which can be a half-integer provided $Q_eQ_m=1/2$. Since we have shown that the SU(2) Yang–Mills–Higgs theory predicts the existence of monopoles of strength $1/e$, if we introduce particles with electric charge $e/2$ which bind to the monopole, the composite system will then have half-integer angular momentum—it will be a fermion. (This also explains how a Fermi field with spin 1/2 and charge $e/2$—as in the example of the previous subsection—can have a nonvanishing matrix element between spinless monopole states: the additional half-integer angular momentum resides in the monopole–fermion interaction.)

To exhibit in detail the emergence of half-integer spin in a model with only Bose fields, we enlarge the Yang–Mills–Higgs theory (3.1) by adding a spinless, isodoublet field B. Then

$$\begin{aligned}\mathcal{L}&=-\tfrac{1}{4}F^{\mu\nu}_{\alpha}F_{\alpha\mu\nu}+\tfrac{1}{2}(D^\mu\Phi)_a(D_\mu\Phi)_a-U(\Phi_a\Phi_a)\\&\quad+(D^\mu B)^\dagger(D_\mu B)-V(B^\dagger B,\Phi_a\Phi_a),\\D^\mu B&=\partial^\mu B-ie(\sigma^a/2)A^\mu_a B,\\V(B^\dagger B,\Phi^2)&=\mu^2B^\dagger B+\lambda^2(B^\dagger B)^2-h^2B^\dagger B\left(\frac{g^2}{m^2}-\Phi^2\right).\end{aligned}\tag{3.17}$$

For a range of values of μ^2, λ^2, and h^2, the solution in the vacuum section is $\varphi_0^2=m^2/g^2, B_0=0$; while in the soliton–monopole sector one may decrease the energy by a nonvanishing, position-dependent B. Thus we expect that the following static, degenerate solution exists

$$\begin{aligned}&A^0_a=0,\quad A^i_a=\epsilon^{aij}\hat{r}_j a(r),\\&\varphi_a=\hat{r}^a f(r),\quad B=u(r)\exp\left(-i\boldsymbol{\alpha}\cdot\frac{\boldsymbol{\sigma}}{2}\right)s.\end{aligned}\tag{3.18}$$

The arbitrary triplet $\boldsymbol{\alpha}$ parametrizes the degeneracy and s is a fixed, constant isospinor. The functions a and f differ only slightly from their forms in the absence of B; thus their asymptotic forms remain as in (3.4), while $u(r)$ is constant at the origin, and approaches exponentially its vacuum value, zero, as r tends to infinity. One verifies that the static solution (3.18) has finite energy, and one presumes that it is stable.

The static, stable finite-energy solution signals the presence of a soliton sector. The occurrence of $\boldsymbol{\alpha}$ shows that the degeneracies are now greater than in the absence of the scalar isodoublet. Consequently the soliton states will be labeled by momentum, charge, and a new quantum number related to $\boldsymbol{\alpha}$. The spinorial form of this degeneracy strongly suggests that the new quantum number is spin-angular momentum. (Recall that isospin symmetry is spontaneously broken in the model, hence the states cannot be isomultiplets.) We confirm that the soliton's spin is indeed $\frac{1}{2}$ by calculating the angular momentum operator $\mathbf{J}$ and by showing that it generates rotations of $\boldsymbol{\alpha}$ (Jackiw and Rebbi, 1976b).

[When the parameters of V in (3.17) are such that B vanishes also in the soliton sector—when no static B configuration is stable—the solitons remain of course spinless. However, one may then consider the states that arise from the binding of a B particle with the spinless soliton. These composite systems again possess half-integer spin (Hasenfratz and 't Hooft, 1976).]

Quantization for the soliton sector is performed in the unitary gauge where Φ_a points along the third isospin direction. Classical solutions in this gauge, obtained from (3.18) by an isorotation, are as in (3.5)

$$\mathbf{A}_a=\hat{e}_a w(r),\quad a=1,2\tag{3.19a}$$

$$\mathbf{A}_3=\mathbf{A}_D(\mathbf{r}),\tag{3.19b}$$

$$B=u(r)R\exp[-i\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}/2)],\tag{3.19c}$$

where R is an isorotation matrix effecting the desired transformation. $\mathbf{A}_D$ is a Dirac vector potential for a monopole of strength $1/e$: $\nabla\times\mathbf{A}_D=\hat{r}/er^2\,\nabla\cdot\mathbf{A}_D=0$. For the Lagrangian (3.17) $\mathbf{J}$ is given by

$$\mathbf{J}=-\int d\mathbf{r}\,\mathbf{r}\times\{\Pi^i_a\nabla A^i_a+\Pi_a\nabla\Phi_a+(\Pi^\dagger_B\nabla B+\text{h.c.})\}-\int d\mathbf{r}\,\boldsymbol{\Pi}_a\times\mathbf{A}_a,$$

$$a=1,2,3;\quad \Pi^i_a=F^{0i}_a;\quad \boldsymbol{\Pi}_a=(D^0\Phi)_a;\ \Pi_B=D^0B.\tag{3.20a}$$

In the unitary gauge, only $\Phi_3\equiv\Phi$ and $\Pi_3\equiv\Pi$ do not vanish. We redefine the charged, massive vector fields by $\mathbf{A}_a\equiv\mathbf{W}_a$, $a=1,2$; and the massless photon field by $\mathbf{A}_3\equiv\mathbf{A}$, $\boldsymbol{\Pi}_3\equiv\boldsymbol{\Pi}=\boldsymbol{\Pi}_T+\boldsymbol{\Pi}_L$. The longitudinal part $\boldsymbol{\Pi}_L$ satisfies a constraint equation

$$\nabla\cdot\boldsymbol{\Pi}_L=-e\epsilon_{ab}\boldsymbol{\Pi}_a\cdot\mathbf{W}_b-e[i\Pi_B{}^\dagger(\sigma^3/2)B+\text{h.c.}]=-ej^0,\quad a=1,2.$$

An integration by parts in (3.20a) eliminates $\boldsymbol{\Pi}_L$, and $\mathbf{J}$ is expressed by uncontrained variables in the unitary gauge

$$\begin{aligned}\mathbf{J}=-\int d\mathbf{r}\,\mathbf{r}\times\{&\Pi^i_a(\nabla\delta_{ab}+e\epsilon_{ab}\mathbf{A})W^i_b+\Pi\nabla\Phi\\&+[\Pi^\dagger_B[\nabla+(ie/2)\sigma^3\mathbf{A}]B+\text{h.c.}]\\&+[\boldsymbol{\Pi}_T-\nabla\nabla^{-2}ej^0]\times[\nabla\times\mathbf{A}]\}\\-\int d\mathbf{r}\,&\boldsymbol{\Pi}_a\times\mathbf{W}_a\\\nabla\cdot\mathbf{A}=0,\quad a&=1,2.\end{aligned}\tag{3.20b}$$

To calculate $\mathbf{J}$ in the soliton–monopole sector, the quantum fields are shifted by the classical solutions, with all degeneracy parameters promoted to collective coordinates, i.e., time-dependent quantum operators. We ignore the coordinates relevant to translations and charge rotations—their role has been amply discussed—and concentrate on the three-parameter degeneracy of (3.19c). To lowest order in the coupling constant, we need keep only the classical solutions. Thus we evaluate (3.20b) with

$$\Phi = f(r),\ \Pi = 0$$
$$\mathbf{W}_a = \hat{e}_a w(r),\ \mathbf{\Pi}_a = 0 \quad a = 1, 2$$
$$\mathbf{A} = \mathbf{A}_D(\mathbf{r}),\ \mathbf{\Pi}_T = 0$$
$$B = u(r)R\exp[-i\boldsymbol{\alpha}(t)\cdot(\boldsymbol{\sigma}/2)]s \equiv RB(r,t)$$
$$\Pi_B^\dagger = \dot{B}^\dagger(r,t)R^\dagger \equiv \Pi_B^\dagger(r,t)R^\dagger. \qquad (3.21)$$

Here Π_B is nonvanishing since B acquires time dependence from its collective coordinate $\boldsymbol{\alpha}(t)$. The other fields, not containing collective coordinates, are time independent and their conjugate momenta vanish. Now $\mathbf{J}$ becomes

$$\mathbf{J} = -\int d\mathbf{r}\,\mathbf{r}\times\Pi_B^\dagger(r,t)R^\dagger[\nabla + i\frac{e}{2}\sigma^3\mathbf{A}_D(\mathbf{r})]RB(r,t)$$
$$-\int d\mathbf{r}\,\mathbf{r}\,\Pi_B^\dagger(r,t)R^\dagger(i\sigma^3/2)RB(r,t) + \text{h.c.} \qquad (3.22a)$$

The rotation indicated by R is evaluated by comparison with (3.18) and (3.19); if we set $a(r) = -1/er$, $w(r) = 0$ in those equations we recognize that (3.22a) is equal to

$$\mathbf{J} = -\int d\mathbf{r}\,\mathbf{r}\times\Pi_B^\dagger(r,t)\left[\nabla + ie\frac{\boldsymbol{\sigma}}{2}\times\frac{\hat{r}}{er}\right]B(r,t)$$
$$-\int d\mathbf{r}\,\hat{r}\,\Pi_B^\dagger(r,t)i(\boldsymbol{\sigma}/2)\cdot\hat{r}B(r,t) + \text{h.c.}$$
$$= \int d\mathbf{r}\,\Pi_B^\dagger(r,t)[-\mathbf{r}\times\nabla - i(\boldsymbol{\sigma}/2)]B(r,t) + \text{h.c.} \qquad (3.22b)$$

The first term in the square brackets is the orbital angular momentum of B; for spherically symmetric fields, as in (3.21), it vanishes. The remainder is exactly the isospin generator $\mathbf{I}$. When terms that we have ignored are kept, $\mathbf{J}$ will of course also acquire conventional orbital and spin contributions. Thus we arrive at

$$\mathbf{J} = \mathbf{L} + \mathbf{I}. \qquad (3.23)$$

The total angular momentum is composed of the conventional orbital plus spin part, $\mathbf{L}$; also it acquires another contribution, $\mathbf{I}$.

For $\boldsymbol{\alpha}(t)$ pointing in a fixed direction i we have

$$B(r,t) = u(r)R\exp[-i\alpha(t)(\sigma^i/2)]s$$
$$\dot{B}(r,t) = -u(r)Ri\dot{\alpha}(t)(\sigma^i/2)\exp(-i\alpha(t)(\sigma^i/2))s$$
$$L = \int d\mathbf{r}\mathcal{L} = \frac{1}{4}\dot{\alpha}^2(t)\int d\mathbf{r}u^2(r) + \dots \qquad (3.24)$$

The momentum conjugate to α^i is $(\dot{\alpha}^i/2)\int d\mathbf{r}u^2(r)$, which also is $I^i = J^i$. What appears to be an isorotation is in fact a spin rotation—in a quantum theory spin has been created from isospin!

To go beyond the details of our example, it is clear that spin will always emerge from isospin whenever there is an isospin-degenerate, classical solution in the field of a monopole (Huang and Stump, 1977). Let us further stress that there is no doubt about the Poincaré covariance of this result. We do not add $\mathbf{I}$ to $\mathbf{L}$ to obtain a conserved quantity; rather we discover $\mathbf{I}$ as a contribution to $\epsilon^{ijk}\int d\mathbf{r}x^j\Theta^{0k}$. This is the correct Lorentz angular momentum expressed in terms of the symmetric energy-momentum tensor $\Theta^{\mu\nu}$. It satisfies, together with $\int d\mathbf{r}x^i\Theta^{00}$ and $\int d\mathbf{r}\Theta^{0\mu}$, the Poincaré algebra.

Since we are dealing with a local quantum field theory, we expect the spin-statistics connection to hold—not only is spin created from isospin, but we believe the half-integer spin solitons to be fermions. A nonrelativistic argument can be given to support this conclusion: When two dyons arise as Schrödinger equation bound states of identical magnetic and electrical poles, then the dyon wave function may violate the spin-statistics theorem, but it contains more Dirac strings than necessary. These suprefluous strings may be removed by a gauge transformation which introduces a new minus sign and restores the spin-statistics theorem (Goldhaber, 1976). We expect that this applies to the soliton bound states discussed here.

F. Imaginary-time, *c*-number fields

We have discussed constant, static and time-dependent c-number solutions to the Euler–Lagrange equations for field theories. There is one more type of solution that can be considered—that is a solution to the equations of motion, which have been continued from Minkowski space to Euclidean space. Such a continuation is effected by replacing $t = x^0$ by $-ix^4$ and replacing the time component of all vector fields A^0 by $-iA^4$. The equations retain the same form as in Minkowski space, but the metric becomes the identity. Consequently Lorentz invariance is replaced by $O(4)$ invariance and no distinction need be made between upper and lower indices.

Why might one be interested in classical solutions of this modified theory? A superficial answer recalls that practical computations of Feynman graphs are in fact most frequently performed by continuing the integrals from Minkowski space to Euclidean space. Moreover, an operator description of Euclidean field theory can be given, and it is known that Green's functions of that theory, upon analytic continuation to Minkowski space, reproduce the physically interesting Green's functions (Fubini, Hanson, and Jackiw, 1973). In fact, as shall be demonstrated below, there is a more profoundly physical place for Euclidean, classical solutions in the quantum theory.

In this subsection, I shall describe the Euclidean solutions. The theory is SU(2) Yang–Mills, governed by the Euclidean Lagrange density

$$\mathcal{L} = \tfrac{1}{4}F_a^{\mu\nu}F_{a\mu\nu}$$
$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + e\epsilon_{abc}A_b^\mu A_c^\nu. \qquad (3.25)$$

It is convenient to exhibit the equations of motion in matrix form. We define

$$\frac{1}{e}A^\mu = A_a^\mu \frac{\sigma^a}{2i}$$

$$\frac{1}{e}F^{\mu\nu} = F_a^{\mu\nu}\frac{\sigma^a}{2i}, \tag{3.26}$$

and it follows that

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + [A^\mu, A^\nu]. \tag{3.27}$$

The equation of motion is

$$\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0. \tag{3.28}$$

The theory is invariant under local gauge transformations; their finite form is

$$A^\mu \to g^{-1}A^\mu g + g^{-1}\partial^\mu g,$$

$$F^{\mu\nu} \to g^{-1}F^{\mu\nu}g,$$

where g is any position-dependent SU(2) matrix such that

$$g = \exp[i(\sigma^a/2)\theta_a(x)]. \tag{3.30}$$

A field configuration A^μ which is a pure gauge

$$A^\mu = g^{-1}\partial^\mu g \tag{3.31}$$

leads to vanishing $F^{\mu\nu}$.

The Euclidean action is

$$I = \frac{1}{e^2}\int d^4x S,$$

$$S = -\frac{1}{2}\mathrm{Tr}F^{\mu\nu}F_{\mu\nu}. \tag{3.32}$$

For this integral to exist, we must deman that $F^{\mu\nu}$ vanish faster than x^{-2} at infinity; however, it is not necessary that A^μ vanish faster than x^{-1}. Rather it is sufficient to demand

$$A^\mu(x) \underset{x\to\infty}{\longrightarrow} g^{-1}\partial^\mu g, \tag{3.33}$$

where g is nontrivial $[g \neq I]$. Therefore we seek solutions of (3.28) which are pure gauges at infinity.

Yang–Mills field configurations may be categorized as follows. Define the dual tensor

$$*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\kappa\beta}F_{\alpha\beta}. \tag{3.34}$$

Here $*F^{\mu\nu}$ always satisfies

$$\partial_\mu *F^{\mu\nu} + [A_\mu, *F^{\mu\nu}] = 0 \tag{3.35}$$

regardless of whether A^μ solves the field equations. Next define the "Pontryagin density" $*S$

$$*S = -\tfrac{1}{2}\mathrm{Tr}*F^{\mu\nu}F_{\mu\nu}. \tag{3.36}$$

One verifies from (3.27) and (3.34) that $*S$ is a total divergence

$$*S = \partial_\mu S^\mu$$

$$S^\mu = -2\,\mathrm{Tr}\epsilon^{\mu\alpha\beta\gamma}[\tfrac{1}{2}A_\alpha\partial_\beta A_\gamma + \tfrac{1}{3}A_\alpha A_\beta A_\gamma]. \tag{3.37}$$

Nevertheless the integral of $*S$ over all x need not vanish. Indeed when we define the "Pontryagin index" q by

$$q = \frac{1}{8\pi^2}\int d^4x *S \tag{3.38}$$

we see that as consequence of (3.37) q may be expressed as a surface integral at infinity. However, on that surface, one may use the asymptotic form (3.33) for A^μ. Hence an equivalent expression for q is

$$q = \frac{1}{24\pi^2}\int d\Omega_\mu \epsilon^{\mu\alpha\beta\gamma}\,\mathrm{Tr}(g^{-1}\partial_\alpha g)(g^{-1}\partial_\beta g)(g^{-1}\partial_\gamma g). \tag{3.39}$$

[Of course (3.39) is equivalent to (3.38) only when $*S$ is nonsingular; otherwise Gauss' law may not apply.] The integrand is exactly the invariant measure of the group, and q is an integer. (For fields A^μ which lead to infinite action, q need not be an integer.) Therefore, the integer-valued Pontryagin index is useful for categorizing Yang–Mills field configurations. We shall also see that it has great physical significance.

Let us now turn to the solutions of the field equation (3.28). Since $*F^{\mu\nu}$ always satisfies (3.35), we can find a solution to (3.28) by demanding that $F^{\mu\nu}$ be proportional to $*F^{\mu\nu}$. Since $*(*F^{\mu\nu}) = F^{\mu\nu}$, the proportionality constant must be ±1, and the second-order differential equations (3.28) can be solved by obtaining a solution to simpler, first-order equations which require $F^{\mu\nu}$ to be self-dual or anti-self-dual (Belavin, Polyakov, Schwartz, and Tyupkin, 1975)

$$F^{\mu\nu} = \pm *F^{\mu\nu} \tag{3.40}$$

[It is believed, though no one has proved this as yet, that all finite-action solutions of the second-order equations (3.28) are also solutions to the first-order equations (3.40).] It is clear that the Pontryagin index becomes proportional to the total action for solutions of (3.40); the self-dual solutions have positive Pontryagin index $g = N > 0$ and are called "pseudoparticles"; for the anti-self-dual ones, called "anti-pseudoparticles," $g = -N < 0$.

In order to exhibit explicitly pseudoparticle field configurations, it is useful to define the matrices $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$, which are anti-symmetric in $\mu \leftrightarrow \nu$

$$\sigma^{ij} = \tfrac{1}{2}\epsilon^{ijk}\sigma^k,$$

$$\sigma^{i4} = \tfrac{1}{2}\sigma^i;$$

$$\bar\sigma^{ij} = \sigma^{ij},$$

$$\bar\sigma^{i4} = -\sigma^{i4}. \tag{3.41}$$

The matrices are self-dual and anti-self-dual

$$*\sigma^{\mu\nu} = \sigma^{\mu\nu},$$

$$*\bar\sigma^{\mu\nu} = -\bar\sigma^{\mu\nu}, \tag{3.42}$$

and satisfy the following commutation relations

$$i[\sigma^{\mu\alpha}, \sigma^{\nu\beta}] = g^{\alpha\nu}\sigma^{\mu\beta} - g^{\mu\nu}\sigma^{\alpha\beta} + g^{\alpha\beta}\sigma^{\nu\mu} - g^{\mu\beta}\sigma^{\nu\alpha},$$

$$i[\bar\sigma^{\mu\alpha}, \bar\sigma^{\nu\beta}] = g^{\alpha\beta}\bar\sigma^{\mu\beta} - g^{\mu\nu}\bar\sigma^{\alpha\beta} + g^{\alpha\beta}\bar\sigma^{\nu\mu} - g^{\mu\beta}\bar\sigma^{\nu\alpha}. \tag{3.43}$$

The $N = 1$ pseudoparticle is (Belavin, Polyakov, Schwartz, and Tyupkin, 1975)

$$A^\mu = \frac{-i\sigma^{\mu\nu}2(x - x_0)_\nu}{(x - x_0)^2 + \lambda^2}. \tag{3.44}$$

Here x_0 is an arbitrary origin, and λ^2 is an arbitrary scale. We say that the pseudoparticle is located at x_0 and has the size λ. Note that five parameters are required to specify the solution. Without loss of generality we may take $x_0 = 0$; $\lambda^2 = 1$; thus the self-dual pseudoparticle is

$$A^\mu = -i\sigma^{\mu\nu}2x_\nu/(x^2 + 1), \tag{3.45a}$$

$$F^{\mu\nu}=i\sigma^{\mu\nu}[4/(x^2+1)^2]\,. \tag{3.45b}$$

(Anti-self-dual, anti-pseudoparticles are obtained by using $\bar{\sigma}^{\mu\nu}$ instead of $\sigma^{\mu\nu}$.) An alternate expression for (3.45a), which exhibits the asymptotic form (3.33), is

$$A^\mu=(x^2/1+x^2)g^{-1}\partial^\mu g\,, \tag{3.46a}$$

$$g=(x_4-i\mathbf{x}\cdot\sigma)/(x^2)^{1/2}$$

$$g^{-1}=g^\dagger\,. \tag{3.46b}$$

Yet another form obtained from (3.45a) or (3.46a) by gauge transforming with the gauge function g^{-1} is

$$A^\mu=-2i\bar{\sigma}^{\mu\nu}x_\nu/x^2(x^2+1)=i\bar{\sigma}^{\mu\nu}\partial_\nu\ln\left(1+\frac{1}{x^2}\right). \tag{3.47}$$

This expression, which is singular at $x^2=0$, will be useful in our discussion of N pseudoparticle configurations. Of course the action density S and the Pontryagin density $*S$ are gauge invariant, hence nonsingular. [The singular part of (3.47), $-2i\bar{\sigma}^{\mu\nu}(x_\nu/x^2)$, is a pure gauge, leading to vanishing $F^{\mu\nu}$.]

The pseudoparticle has an important set of invariances under coordinate transformations. Let us recall that in addition to the usual Poincaré symmetries—translations generated by $P^\mu[x^\mu\to x^\mu+a^\mu]$ and rotations generated by $M^{\mu\nu}[x^\mu\to\Lambda^{\mu\nu}x_\nu;\ \Lambda^{\mu\alpha}\Lambda_{\mu\beta}=g^{\alpha}_{\beta}]$—the Yang–Mills theory also possesses dilatation invariance, generated by $D[x^\mu\to e^a x^\mu]$, and special conformal invariance, generated by $K^\mu[x^\mu\to[x^\mu-c^\mu x^2/\sigma(c,x)],\ \sigma(c,x)=1-2cx+c^2x^2]$. The fifteen generators close to form the conformal group, which is an $O(5,1)$ group of invariances for the Yang–Mills theory.[7] Under an infinitesimal transformation, $\delta x^\mu=f^\mu(x)=a^\mu+\omega^{\mu\nu}x_\nu+ax^\mu+2cx-c^\mu x^2$, $\omega^{\mu\nu}=-\omega^{\nu\mu}$, the gauge potential transforms according to

$$\delta A^\mu=f^\alpha\partial_\alpha A^\mu+\partial^\mu f^\alpha A_\alpha\,. \tag{3.48}$$

The pseudoparticle solution (3.45) is obviously noninvariant under translations—they shift the position—and under dilatations—they rescale the size. Under rotations the expressions (3.45) appear also to be noninvariant, but the noninvariance may be compensated by a gauge transformation: a rotation parametrized by infinitesimal parameters $\omega_{\mu\nu}$ is compensated by the gauge transformation

$$g=\exp(-\tfrac{i}{2}\sigma^{\mu\nu}\omega_{\mu\nu})\,. \tag{3.49}$$

Thus the combined action of rotations and appropriate gauge transformations leaves (3.45) invariant. Since the algebra of the $\sigma^{\mu\nu}$ matrices follows that of the rotation group [see (3.43)], we may say that the pseudoparticle is $O(4)$-invariant. [It is easy to show that in fact (3.45) is the most general, $O(4)$-invariant solution.]

Under special conformal transformations, (3.45) is noninvariant; rather it transforms, with the help of a gauge transformation, into a solution with shifted origin and rescaled size. However, if one performs the infinitesimal transformation generated by

$$R^\mu=\tfrac{1}{2}[K^\mu+P^\mu] \tag{3.50}$$

with infinitesimal parameters a^μ, one finds that the noninvariant response of (3.45) can again be compensated by a local gauge transformation

$$g=\exp(-i\sigma^{\mu\nu}a_\mu x_\nu)\,. \tag{3.51}$$

Since R^μ together with $M^{\mu\nu}$ form a $O(5)$ subgroup of the $O(5,1)$ conformal group, we conclude that the pseudoparticle is $O(5)$ invariant and one can show that is is the most general, $O(5)$-invariant solution (Jackiw and Rebbi, 1976c).

The existence of this large invariance group allows for the explicit solution of many equations relevant to the pseudoparticle. One may analyze the small oscillation spectrum ('t Hooft, 1976b; Ore, 1977a). One can solve Klein–Gordon and Dirac equations in the external field of the pseudoparticle ('t Hooft, 1976b; Jackiw and Rebbi, 1976c; Ore, 1976a; Chadha, D'Adda, DiVecchia and Nicodemi, 1977). Even the propagator for a scalar, spinor, or vector particle can be exhibited in closed form (Jackiw and Rebbi, 1976c, Ore, 1977b). The computational technique makes use of a projection of the theory onto a five-dimensional hypersphere; then all the equations become free, harmonic equations on the sphere (Adler, 1972, 1973; Fubini, 1976; Jackiw and Rebbi, 1976c). For the fermions, one finds, just as in the soliton investigations, a normalizable zero-eigenvalue mode.

Pseudoparticles with $N>1$ have also been found, in $O(3)$ symmetric alignment—the pseudoparticles are positioned on a line (Witten, 1977)—as well as in more general configurations with no restriction on the positions ('t Hooft, 1977; Jackiw, Nohl, and Rebbi, 1977). The most general self-dual N pseudoparticle solution known at the present time makes use of the following *Ansatz* for the gauge field (Wilczek, 1977; Corrigan and Fairlie, 1977)

$$A^\mu=i\bar{\sigma}^{\mu\nu}\partial_\nu\ln\rho\,, \tag{3.52}$$

with ρ given by (Jackiw, Nohl, and Rebbi, 1977)

$$\rho=\sum_{i=1}^{N+1}\frac{\lambda_i^2}{(x-x_i)^2}\,. \tag{3.53}$$

(Anti-self-dual, anti-pseudoparticles make use of $\sigma^{\mu\nu}$ in place of $\bar{\sigma}^{\mu\nu}$.) In spite of the singularities present in ρ and A^μ, the action and Pontryagin densities are nonsingular [compare (3.47)]

$$S={}^*S=-\frac{1}{2}\Box\Box\ln\left(\sum_{i=1}^{N+1}\lambda_i^2\prod_{j\neq i}^{N+1}(x-x_i)^2\right) \tag{3.54}$$

and one easily finds that $q=N$.

The form of ρ, which specifies N pseudoparticles, is seen to depend on $5(N+1)$ parameters: the 4-vectors x_i^μ specifying the positions of the $N+1$ poles, and the residues λ_i^2. It is clear from (3.52) that A^μ is insensitive to an overall scale of ρ, hence the gauge potential depends only on $5N+4$ parameters. We must still determine if gauge-invariant quantities like S and $*S$ also exhibit a $5N+4$ parametric dependence, or whether some of this dependence is a gauge artifact. A lenghty analysis of the gauge freedom in the *Ansatz* (3.52) yields the following result. The *Ansatz* (3.52) is gauge invariant whenever the $N+1$ points x_i do not lie on a circle. However when they do lie on a circle, then there is a gauge freedom which moves the position of the poles around the circle. Thus for $N=1$, a threefold variety of circles may be drawn

[7] For a review of the conformal group see Treiman, Jackiw, and Gross, 1972.

through the two points x_1 and x_2, and the number of parameters is reduced by four, leaving the single pseudoparticle to depend on five parameters; this of course coincides with the result quoted previously [see (3.44)]. For $N=2$, one circle may be always drawn through the three points x_1, and x_2, and x_3, and our two-pseudoparticle solution depends in general on thirteen parameters. For $N>3$, a circle cannot in general be drawn through the $N+1>4$ x_i's, and the pseudoparticle configurations truly depend on $5N+4$ parameters (Jackiw, Nohl, and Rebbi, 1977). There is no obvious, simple relation between the positions and sizes of the N pseudoparticles and the $5N+4$ parameters that specify them. [However, it is possible to exhibit a less general, N pseudoparticle configuration which depends on only $5N$ parameters: take (3.53) and let λ_{N+1}, and x_{N+1} tend to infinity with $\lambda^2_{N+1}/x^2_{N+1}=1$. This leaves [compare (3.47)]

$$\rho = 1 + \sum_{i=1}^{N} \frac{\lambda_i^2}{(x-x_i)^2},$$

which also describes N pseudoparticles, and the x_i's and λ_i's can be interpreted as positions and sizes of the pseudoparticles ('t Hooft, 1977).

The solution (3.52) and (3.53) responds to special conformal transformations as follows. The correct transformation of the gauge potential, (3.48), is induced by transforming ρ as a scalar with scale dimension one

$$x^\mu \to \tilde{x}^\mu = (x^\mu - c^\mu x^2)/\sigma(c,x)$$
$$\rho(x) \to \tilde{\rho}(x) = [1/\sigma(c,x)]\rho(\tilde{x}) \tag{3.55}$$

provided the gauge readjustment (3.51) is also made. With little algebra one finds that (3.53) transforms into itself, with a redefinition of parameters

$$\tilde{\rho}(x) = \sum_{i=1}^{N+1} \frac{\tilde{\lambda}_i^2}{(x-\tilde{x}_i)^2},$$
$$\tilde{\lambda}_i^2 = \lambda_i^2/\sigma(-c, x_i),$$
$$\tilde{x}_i^\mu = (x_i^\mu + c^\mu x_i^2)/\sigma(-c, x_i). \tag{3.56}$$

It is trivial to verify that Poincaré transformations and dilatations also take (3.53) into itself; hence the solution is closed under the action of the full conformal groups. Consequently it is called the "conformal" solution.

The conformal solution is not the most general one. It has been proven that the most general N pseudoparticle configuration depends on $8N-3$ gauge invariant parameters (Jackiw and Rebbi, 1977a; Schwarz, 1977; Atiyah, Hitchin and Singer, 1977; Brown, Carlitz and Lee, 1977). The $8N$ parameters are understood to describe the position and size of each pseudoparticle ($5N$ parameters) and a global gauge specification for each pseudoparticle ($3N$ parameters). The total $8N$ is then decreased by 3 since an overall global gauge transformation may always be performed. Although the most general solution has not as yet been found explicitly, an infinitesimal deformation of the conformal solution, depending on $8N-3$ parameters has been given (Jackiw and Rebbi, 1977a, b).

The variety of further calculations that can be performed (small oscillation spectrum, coupling to other fields) have not as yet been completed. However the zero-eigenvalue modes of the Dirac equation in the N pseudoparticle conformal field have been found for isospinor and isovector fermions (Grossman, 1977; Jackiw and Rebbi, 1977b).

To conclude this discussion of imaginary-time Yang–Mills field configurations, we examine the energy-momentum tensor

$$\Theta^{\mu\nu} = -\frac{2}{e^2}\,\mathrm{Tr}\left[F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{g^{\mu\nu}}{4}F^{\alpha\beta}F_{\alpha\beta}\right]. \tag{3.57}$$

However, any self-dual or anti-self-dual matrix $m^{\mu\nu}$ has the property that

$$m^{\mu\alpha}m^{\nu}{}_{\alpha} = (g^{\mu\nu}/4)m^{\alpha\beta}m_{\alpha\beta}. \tag{3.58}$$

Consequently $\Theta^{\mu\nu}$ vanishes for all the field configurations that we have here considered. In particular the Euclidean "energy" $E=\int d^3x\Theta^{00}(x)$ is zero—our solutions are zero-energy solutions for imaginary time.

G. Quantum meaning of imaginary-time, c-number fields

Unlike the real-time classical solutions, the imaginary-time ones are not associated with new particle states in a new sector of the theory. Rather they are interpreted as evidence for quantum mechanical tunnelling in the conventional sector of the Hilbert space. We have noticed that our solutions carry zero (Euclidean) energy. Hence we expect that they are relevant to the ground state—the vacuum state—of the Yang–Mills theory.

In order to exemplify the role of imaginary-time classical solutions we begin by presenting an approximate analysis of the ground state in various one-particle quantum mechanical systems. Consider a particle moving in one dimension in a potential $V(q)$ shaped as in Fig. 1. In order to obtain an approximate description of the ground-state wave function $\Psi(q)$, we first determine the classical zero-energy configuration. That of course is just $q=q_0$. We know therefore that $\Psi(q)$ will be sharply peaked at $q=q_0$. Quantum fluctuations will give it some spread, so an accurate formula for $\Psi(q)$ is a Gaussian.

Next consider a potential with two minima and a symmetry $V(q)=V(-q)$, as in Fig. 2. [For definiteness we can take an explicit analytic expression for $V(q)$: $V(q) = (\lambda^2/2)(q^2-q_0^2)^2$]. There are now two classical zero-energy configurations: $q=\pm q_0$, and one may construct two Gaussian wave functions $\psi(q-q_0)$ peaked at $q=+q_0$, and $\psi(q+q_0)$ peaked at $q=-q_0$. However, neither is a parity eigenstate, so the true vacuum must be a superposition and

$$\Psi(q) = c_+\psi(q-q_0) + c_-\psi(q+q_0)$$

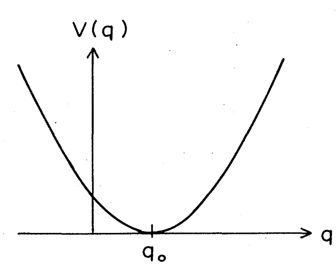

FIG. 1. An example of a simple potential for which semiclassical methods give an accurate picture of the ground state.

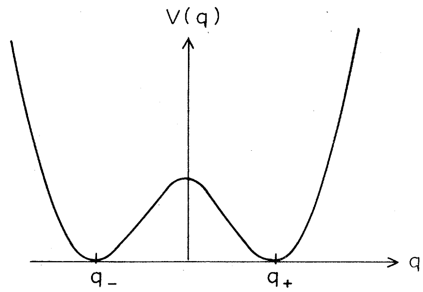

FIG. 2. An example of a potential for which semiclassical methods must be extended to give an accurate picture of the tunnelling properties of the ground state.

Demanding that $\Psi(q)$ be a parity eigenstate fixes the ratio $c_+/c_- = \pm 1$. At this stage it appears that the ground state is doubly degenerate, corresponding to the two parity eigenvalues. But we know that tunnelling will in fact remove the degeneracy, and one state (the antisymmetric one) will be higher in energy than the other, by an exponentially small amount.

Tunnelling corresponds to motion between $-q_0$ and $+q_0$, which is classically forbidden but quantum mechanically allowed. But it is possible to obtain a classical solution, even in the forbidden region, by the following trick. In the classical equation for zero-energy motion

$$\tfrac{1}{2}\dot{q}^2 = -V(q)$$

let us change time to imaginary time: $t \to -ix^4$. The equation now becomes

$$-\tfrac{1}{2}\dot{q}^2 = -V(q)\,,$$

which obviously has a solution $q(x^4)$, such that $q(-\infty) = -q_0$ and $q(\infty) = +q_0$. [In the explicit example, the imaginary-time, zero-energy solution is $q(x^4) = q_0 \tanh q_0\lambda x^4$.] Thus an imaginary-time, zero-energy solution, which interpolates between classically allowed zero-energy configurations, is evidence for the existence of quantum tunnelling (Freed, 1972; McLaughlin, 1972).

These two examples suggest the following strategy for obtaining an approximate, but accurate description of the ground-state wave function of a quantum system. First enumerate all classical zero-energy configurations. Next construct Gaussian wave packets peaked at each zero-energy configuration, and form linear superpositions if necessary to respect the symmetries of the problem. Finally look for imaginary-time, zero-energy solutions to the classical equations. If such solutions exist, and they interpolate in imaginary time between classically allowed zero-energy, real-time solutions, we conclude that there is quantum mechanical tunnelling, and the apparent degeneracy of the ground-state wave functions is lifted.

Before examining the Yang–Mills theory, let us consider a last example in particle quantum mechanics, where the potential is periodic, Fig. 3. We now have an infinite number of classical zero-energy configurations. $q = an$, $n = 0, \pm 1, \ldots$. Correspondingly there is an infinite number of Gaussians $\psi(q - an)$, each peaked at the zero-energy configuration $q = an$. The model possesses a symmetry: shifting q by a does not change $V(q)$. We demand that our states be eigenstates of the symmetry; therefore we must form a linear combination

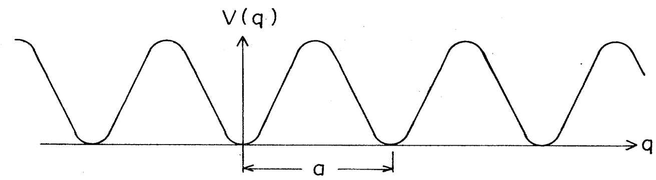

FIG. 3. An example of a potential which is periodic and of infinite extent. Tunnelling effects lead to a band structure in the energy spectrum.

$$\Psi(q) = \sum_n c_n \psi(q - an)\,.$$

The constants c_n are determined by the requirement that $\Psi(q+a)$ differ at most by a phase from $\Psi(q)$. We find $c_n = \exp(in\theta)$, and arrive at a family of states parametrized by θ

$$\Psi_\theta(q) = \sum_n \exp(in\theta)\psi(q - an)\,.$$

Of course there is tunnelling between the classical zero-energy configurations. This can be exposed classically by noting the existence of imaginary-time, zero-energy solutions. In a now familiar fashion this means that the energy eigenvalues associated with $\Psi_\theta(q)$ depend on θ; there is an energy band $E(\theta)$ parametrized by θ. Thus from our semiclassical considerations we arrive at a completely accurate description of a Bloch wave in a crystal.

The above method will now be used to analyze the vacuum structure of a Yang–Mills theory. We are discussing the model in Minkowski space, and use a Schrödinger representation for field theory. The Lagrange density is

$$\mathcal{L} = -\tfrac{1}{4}F_a^{\mu\nu}F_{a\mu\nu}\,, \tag{3.59}$$

and the Hamiltonian density is

$$\begin{aligned} \mathcal{H} &= \tfrac{1}{2}(\mathbf{E}_a^2 + \mathbf{B}_a^2)\,, \\ E_a^i &= F_a^{0i}\,, \\ B_a^i &= \tfrac{1}{2}\epsilon^{ijk}F_{ajk}\,. \end{aligned} \tag{3.60}$$

The analysis is most readily carried out in the gauge $A_a^0 = 0$ which we adopt. [This is not a necessary restriction; the theory may be developed in other gauges as well (Wadia and Yoneya, 1977).] The wave functional $\Psi(\mathbf{A})$ depends on the dynamical variables of the Yang–Mills theory—on the vector potentials $\mathbf{A}_a$. The Schrödinger equation is

$$\int d\mathbf{r}\left\{-\frac{1}{2}\,\frac{\delta^2}{\delta A_a^i(\mathbf{r})\,\delta A_a^i(\mathbf{r})} + \frac{1}{2}\mathbf{B}_a^2(\mathbf{r})\right\}\Psi(\mathbf{A}) = E\Psi(\mathbf{A}) \tag{3.61}$$

and scalar products of wave functionals as well as matrix elements of observables are realized by functional integration over $\mathbf{A}_a$.

To implement our strategy for studying the ground-state wave functional, we first enumerate the zero-energy, c-number field configurations. Zero energy requires $\mathcal{H} = 0$, hence $F_a^{\mu\nu} = 0$; the gauge potential must be a pure gauge; in matrix rotation this is

$$\mathbf{A}(\mathbf{r}) = g^{-1}(\mathbf{r})\nabla g(\mathbf{r})\,. \tag{3.62}$$

Here g is any unitary, position-dependent matrix. Of course $g=I$ gives $\mathbf{A}=0$ and $\mathcal{H}=0$; but there are infinitely many gauge copies of this configuration which also have zero energy, and we must decide whether the gauge copies are also physically interesting. For reasons that will be explained below, we require that

$$g(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} I. \tag{3.63}$$

At infinity, at least, the class of gauges which we are studying is trivial.

In a gauge theory, it is necessary to fix the gauge so that one avoids infinities, associated with the volume of the gauge group, in forming scalar products and matrix elements of observables. Without repeating details of the well-known gauge-fixing procedure, let us only recall that it removes from the functional integral over $\mathbf{A}_a$ configurations of the fields which can be joined by a *continuous* gauge transformation to configurations already counted. In particular, one does not integrate over potentials of the form (3.62) when g can be joined to the identity through a one-parameter continuous family of transformations $g(\mathbf{r},\alpha): g(\mathbf{r},1)=g(\mathbf{r});\ g(\mathbf{r},0)=I$. But it is important to realize that there are values of $\mathbf{A}_a$ that can be obtained from each other by gauge transformations which cannot be continuously joined to the identity transformation, with (3.63) always maintained. An example is

$$g_1(\mathbf{r})=\frac{r^2-\lambda^2}{r^2+\lambda^2}-\frac{2i\lambda\boldsymbol{\sigma}\cdot\mathbf{r}}{r^2+\lambda^2}, \tag{3.64a}$$

which gives origin to

$$\mathbf{A}_1(\mathbf{r})=g_1^{-1}(\mathbf{r})\nabla g_1(\mathbf{r}) = \frac{2i\lambda}{(r^2+\lambda^2)^2}\left[\boldsymbol{\sigma}(\lambda^2-r^2)+2\mathbf{r}(\boldsymbol{\sigma}\cdot\mathbf{r})+2\lambda\mathbf{r}\times\boldsymbol{\sigma}\right] \tag{3.64b}$$

and of course to vanishing energy. We postulate that values of the potentials like those of Eq. (3.64), although gauge equivalent to $\mathbf{A}=0$, should not be removed from the integrations over the field configurations by the gauge-fixing procedure, and indeed we shall show that physical effects are associated with them. (Further discussion of this postulate will be given below.)

Before proceeding, let us characterize the classes of gauge-equivalent, but not continuously gauge-equivalent, potentials. It is seen that a g satisfying (3.63) defines a mapping of three-dimensional space, with all the directions at ∞ identified, into the group space. From the topological point of view, the Euclidean space E^3 with points at ∞ identified is equivalent (homeomorphic) to a three-dimensional sphere S^3; but the manifold of SU(2) is also homeomorphic to S^3, so that g defines a mapping, $S^3 \underset{g}{\rightarrow} S^3$. It is known that these mappings fall into homotopy classes (mappings belonging to different classes cannot be continuously distorted into each other) classified by an integer n. The representative of the nth class is

$$g_n(\mathbf{r})=[g_1(\mathbf{r})]^n, \tag{3.65}$$

with g_1 given by (3.64a).

The above considerations bring us to the conclusion that the physically relevant zero-energy configurations comprise a denumerable set

$$\mathbf{A}_n(\mathbf{r})=g_n^{-1}(\mathbf{r})\nabla g_n(\mathbf{r}), \tag{3.66}$$

and we form functional Gaussians peaked around each $\mathbf{A}_n: \psi_n(\mathbf{A})=\varphi(\mathbf{A},\mathbf{A}_n)$. However, the theory is gauge invariant and we must form linear combinations to respect this symmetry:

$$\Psi(\mathbf{A})=\sum_n c_n\psi_n(\mathbf{A}). \tag{3.67a}$$

Gauge transformations can be of two kinds: firstly there are those that stay within a given homotopy class—these are "small" gauge transformations which can be ignored since by hypothesis they are treated by conventional methods; second, the "large" gauge transformations take $\mathbf{A}_n$ to $\mathbf{A}_{n+1}$. Such a gauge transformation changes ψ_n to ψ_{n+1} in (3.67a). Requiring $\Psi(\mathrm{A})$ to be stable against this change fixes the coefficients in (3.67a) to be pure phases, $c_n=\exp(in\theta)$. Thus we arrive at a family of vacuum wave functionals parametrized by an angle θ

$$\Psi_\theta(\mathbf{A})=\sum_n \exp(in\theta)\psi_n(\mathbf{A}). \tag{3.67b}$$

If we call the unitary operator which implements the large gauge transformation $\mathcal{G}$, we find

$$\mathcal{G}\Psi_\theta=\exp(-i\theta)\Psi_\theta. \tag{3.68}$$

The final step in the program is to ascertain whether all these states are degenerate in energy, or whether tunnelling lifts the degeneracy. Here the pseudoparticle solution (3.45) becomes relevant. We observe that by a gauge transformation which removes A^4 (remember we are in the gauge $A^0=0$, hence in Euclidean space $A^4=0$), we can cast the pseudoparticle into a form which vanishes as $x^4\to-\infty$, and tends to $\mathbf{A}_1(\mathbf{r})$ as $x^4\to+\infty$. Thus the pseudoparticle is an imaginary-time, zero-energy solution which interpolates, in imaginary time, between real-time, zero-energy configurations, and we conclude that tunnelling takes place; the vacua are not degenerate, rather there is an energy band $E(\theta)$. The Yang–Mills vacuum is a Bloch wave (Jackiw and Rebbi, 1976d; Callan, Dashen, and Gross, 1976)![8]

We must still comment upon two points which may not be entirely obvious. The first concerns the requirement (3.63) imposed on the gauge functions: $g(\mathbf{r})$ should tend to the identity at infinity. If the condition were relaxed, we would be considering zero-energy potentials which behave as $1/r$ at large distances. Such potentials are separated by an infinite-energy barrier from those that we are including, and presumably there is no tunnelling between two such configurations. To demonstrate that the energy barrier is indeed infinite, we should exhibit a real-time solution which interpolates, as t ranges from

[8]The connection between pseudoparticle solutions and tunnelling which removes the degeneracy of the ground state is also recognized by its absence in σ-type models of chiral symmetry breaking. For these theories one believes that the vacuum is degenerate and that tunnelling does not occur. Correspondingly, one may verify that no pseudoparticle solution exists. This is a consequence of the result in subsection II.A, where it is shown that spinless models in three spatial dimensions, like the σ model, do not support Euclidean solutions.

$-\infty$ to ∞, between $\mathbf{A}=0$ and the pure gauge potential which is $O(1/r)$ at large r. We should then compute the energy of this solution and find an infinite result. Unfortunately such a solution is not available at the present time, but we may present an approximate solution which puts the phenomenon into evidence. Consider the potential $\alpha(t)\mathbf{A}(\mathbf{r})$ where $\mathbf{A}(\mathbf{r})$ is a pure gauge, $O(1/r)$ at large r, and $\alpha(t)$ is a slowly varying function with $\alpha(-\infty)=0$, $\alpha(+\infty)=1$. This solves the equations approximately; the energy comes entirely from $\mathbf{E}_a=\dot{\alpha}\mathbf{A}_a$; but now we see that $\mathbf{E}_a^2$ is $O(1/r^2)$ for large r, hence the total volume integral diverges.

The second point for elucidation concerns our postulate that large gauges have observable consequences; or equivalently that the Schrödinger equation (3.61) be solved with the boundary condition (3.68). Is it possible to remove the angle from the theory, and to collapse the energy band by requiring that $\Psi_\theta(\mathbf{A})$ be invariant under $\mathcal{G}$? The answer is no: Even if we postulate that the Schrödinger wave functional satisfies

$$\mathcal{G}\Psi'(\mathbf{A})=\Psi'(\mathbf{A}) \tag{3.69}$$

there remains an angle in the theory. The point is that the Lagrange density (3.59) is not uniquely defined; one may always add a total divergence to it. Suppose we accept (3.69); we may, however, replace (3.59) by ('t Hooft, 1976b)

$$\mathcal{L}'=-\tfrac{1}{4}F_a^{\mu\nu}F_{a\mu\nu}+(1/32\pi^2)\,\theta\,{}^*F_a^{\mu\nu}F_{a\mu\nu}\,. \tag{3.70}$$

For constant θ, the addition is a total divergence [see (3.36), (3.37)]; thus the energy is, as before, (3.60). However the momentum conjugate to A_a^i is no longer E_a^i, rather it is $E_a^i+(1/8\pi^2)\theta B_a^i$. Hence the Schrödinger equation satisfied by $\Psi'(\mathbf{A})$ is

$$\int d\mathbf{r}\left\{\frac{1}{2}\left[\frac{\delta}{i\delta A_a^i(\mathbf{r})}-\frac{1}{8\pi^2}\,\theta B_a^i(\mathbf{r})\right]^2+\frac{1}{2}\mathbf{B}_a^2(\mathbf{r})\right\}\Psi'(A)=E\Psi'(A)\,. \tag{3.71}$$

Next define a new wave functional by

$$\Psi(\mathbf{A})=\exp[-i(\theta/2)W(\mathbf{A})]\Psi'(\mathbf{A})\,, \tag{3.72}$$

where [see (3.37)]

$$W(\mathbf{A})=-\frac{1}{4\pi^2}\epsilon^{ijk}\int d\mathbf{r}\,\mathrm{Tr}(\tfrac{1}{2}A_i\partial_jA_k+\tfrac{1}{3}A_iA_jA_k)=\frac{1}{8\pi^2}\int d\mathbf{r}S^0\,. \tag{3.73}$$

Since

$$\frac{\delta W(\mathbf{A})}{\delta A_a^i(\mathbf{r})}=\frac{1}{8\pi^2}B_a^i(\mathbf{r})\,,$$

we see that $\Psi(\mathbf{A})$ satisfies the original Schrödinger equation (3.61). What is the response of $\Psi(\mathbf{A})$ to a large gauge transformation when $\Psi'(\mathbf{A})$ is invariant? The transformation law will obviously be determined by the response of $W(\mathbf{A})$. It is easy to show that $W(\mathbf{A})$ transforms as

$$W(\mathbf{A})\underset{g}{\rightarrow}W(\mathbf{A})+\frac{1}{24\pi^2}\epsilon^{ijk}\int d\mathbf{r}\,\mathrm{Tr}(g^{-1}\partial_ig)\times(g^{-1}\partial_jg)(g^{-1}\partial_kg). \tag{3.74a}$$

For a large gauge transformation, we take $g=g_1$ and find

$$W(\mathbf{A})\underset{g_1}{\rightarrow}W(\mathbf{A})+1\,. \tag{3.74b}$$

Hence $\Psi(\mathbf{A})$ transforms according to (3.68). Therefore, even when states are required to be invariant under gauge transformations, the angle and the energy bands reappear if a total divergence is added to the Lagrangian. There is no apparent reason to exclude such an addition—note it is gauge invariant—and we must conclude that the phenomenon we have discussed is indeed present, and is gauge invariant. Also since ${}^*F_a^{\mu\nu}F_{a\mu\nu}$ is CP odd, we see that θ is a CP-violating angle.

When massless fermions are added to the Yang–Mills theory, the physical picture changes drastically—this is not a surprise in view of the startling effects that fermions have in soliton physics. The fermions give rise to the $U(1)$ axial-vector current

$$J_5^\mu=i\bar{\psi}\gamma^\mu\gamma_5\psi\,, \tag{3.75}$$

which would be conserved if anomalies did not intervene. However, quantum effects embodied in the triangle graph destroy the formal conservation law $\partial_\mu J_5^\mu=0$. The quantal result is (Adler, 1970; Treiman, Jackiw, and Gross, 1972).

$$\partial_\mu J_5^\mu=\frac{1}{8\pi^2}\,\mathrm{Tr}\,{}^*F^{\mu\nu}F_{\mu\nu}=-\frac{1}{4\pi^2}\partial_\mu S^\mu\,, \tag{3.76}$$

where the second equality follows from (3.36) and (3.37). Evidently a gauge-variant, but conserved current can be defined by

$$\tilde{J}_5^\mu=J_5^\mu+\frac{1}{4\pi^2}S^\mu\,. \tag{3.77a}$$

The conserved chiral charge is

$$\tilde{Q}_5=\int d\mathbf{r}\,\tilde{J}_5^0=Q_5+2W(\mathbf{A})\,, \tag{3.77b}$$

where Q_5 is the nonconserved, gauge-invariant fermionic axial charge

$$Q_5=\int d\mathbf{r}\,J_5^0\,, \tag{3.77c}$$

and the gauge-invariant addition, $W(\mathbf{A})$, is given in (3.73). Here $\tilde{Q}_5$ commutes with the Hamiltonian, but it does not commute with large gauge transformations [see (3.74)].

$$\mathcal{G}^{-1}\tilde{Q}_5\mathcal{G}=\tilde{Q}_5+2\,,\quad i[H,\mathcal{G}]=0\,,\quad i[H,\tilde{Q}_5]=0\,. \tag{3.78}$$

Our vacuum states diagonalize $\mathcal{G}$ [see (3.68)]; therefore they are not invariant under chiral transformations. Equation (3.78) demonstrates that $\tilde{Q}_5$ is a shift operator, hence

$$\exp[-i(\theta'/2)\tilde{Q}_5]\Psi_\theta=\Psi_{\theta+\theta'}\,. \tag{3.79}$$

Since $\tilde{Q}_5$ commutes with the Hamiltonian, the energy must be independent of θ when fermions are present. Thus coupling the Yang–Mills theory to Fermi fields suppresses the tunnelling, the energy band collapses to a single level, and the vacuum is degenerate ('t Hooft, 1976a; Jackiw and Rebbi, 1976d; Callan, Dashen, and Gross, 1976).

The degeneracy of the vacuum and its noninvariance under chiral transformations shows that the chiral $U(1)$ symmetry is spontaneously broken. There is no reason to suppose that a Goldstone boson is present (Goldstone's theorem cannot be proven in a gauge theory). Therefore it is plausible that the longstanding $U(1)$ problem of quark models has been solved (Weinberg. 1975; Pagels, 1976; 't Hooft, 1976a, b).[9]

H. Quantization about imaginary-time, *c*-number fields

Unlike solitons, pseudoparticles have not, as yet, been incorporated in a canonical quantization of the field theory. Since the classical configuration is given for imaginary time, it is not evident what role should be assigned to it in a real-time quantum theory. The suggestion has been made that theory by described by a functional integral, continued to Euclidean space, and then the functional integration variables be expanded around the stationary points of the classical, Euclidean action, *i.e.*, around the imaginary-time, *c*-number solution (Polyakov, 1975). The expansion around the trivial, vanishing solution gives ordinary perturbation theory (in Euclidean space), and it is hoped that expansion around the nontrivial pseudoparticle solution correctly gives other nonperturbative contributions. It is clear that these additional terms will have a common factor e^{-I_c}, where I_c is the action evaluated at the classical solution. For the pseudoparticles $I_c = 8\pi^2/e^2 |q|$; hence the $q = N = 1$ contribution is multiplied by $\exp[-(8\pi^2/e^2)]$, an obviously nonperturbative formula which is characteristic of tunnelling.

This calculational program can be successfully carried through in various simple, nonrealistic models (Polyakov, 1977). However, for the problem of interest, Yang-Mills theory in four space-time dimensions, an obstacle arises when the first quantum correction is computed. In order to eliminate the zero-frequency mode associated with the scale invariance of the theory and the scale noninvariance of the solution, a collective dilatational variable has to be introduced, and an integration over this variable is performed—the functional integral must be dominated by pseudoparticles of *all* sizes. Unfortunately the dilatational integral is infrared divergent, reflecting the infrared instability of the Yang-Mills theory. Thus numerical consequences of the pseudoparticle effect are not as yet available.

If this divergence is ignored (or cut off), results of calculations confirm the general considerations of the previous subsection: chiral $U(1)$ noninvariant amplitudes as well as CP-violating amplitudes are nonvanishing; tunnelling is suppressed in the presence of fermions ('t Hooft, 1976a,b). (The zero-eigenvalue fermion mode in the field of the pseudoparticle makes the functional integral vanish.) Nevertheless, quantitative assessment of these effects must wait for a resolution of the infrared problem in Yang-Mills theory.

[9]The relevance of the axial-vector current anomaly in pseudoparticle dynamics has been recently understood from algebraic geometry. It is now recognized that the anomalous divergence equation (3.76) is a local form of the Atiyah-Singer index theorem. This rare confluence of interest between pure mathematics and theoretical physics has led to further interesting calculations, which however are beyond the scope of this review. For an introduction see Jackiw and Rebbi, 1977b; Ansourian, 1977.

IV. CONCLUSION

It is very gratifying that it has been possible to put into evidence the rich nonperturbative phenomena of local quantum field theory. We see that, in the models studied, two general classes of results have been obtained. First, associated with soliton solutions, there are sectors of the Hilbert space populated by heavy, stable particles, with quantum numbers which are peculiar because they arise either from topological conservation laws or from mixing of internal and space-time symmetries. Second, as a consequence of pseudoparticle solutions, nonperturbative aspects of the vacuum sector of a quantum field theory have been understood.

The theory requires further development. For the solitons we must learn how to deal with multi-soliton processes; for the Yang-Mills pseudoparticles we must unravel the infrared structure of the model. In both cases it would be most useful to develop strong-coupling approximations. Also the peculiar role of zero-eigenvalue fermion modes should be better understood.

The utility of these ideas for physical theory is uncertain. About the solitons, the most conservative viewpoint is that the as yet undiscovered gauge theory of Nature possesses such solutions, which correspond to some heavy particles, whose properties we have begun to explore, but whose interest for phenomenological description of actual experiments is minimal. More venturesome is the notion that some of the observed "fundamental" particles are not bound states of a few elementary quarks, rather they are coherent bound states of the soliton variety. Finally the most speculative idea makes reference to the quarks themselves. Perhaps, as in the sine-Gordon example, one can formulate a theory in two equivalent ways—a "quark" model where the low-lying states are bound states of the quarks, or a "particle" model where the quarks emerge as super-heavy coherent bound states. There is no evidence that this fascinating duality, found in the sine-Gordon theory, actually occurs in realistic models; nevertheless a curious, formal similarity between the sine-Gordon solitons and colinear Yang-Mills pseudoparticles has been exposed (Dolan, 1977).

The pseudoparticles with the attendant quantum tunnelling appear to be more immediately relevant, since they describe phenomena in the vacuum sector and offer the possibility for a resolution of the longstanding $U(1)$ problem and a tantalizing hint for the origin of CP violation. More generally the occurrence of tunnelling in field theory forces us to reassess models for spontaneous symmetry violation, since tunnelling may restore an apparently broken symmetry. Also one may speculate that the pseudoparticle can somehow provide a mechanism for quark confinement (Polyakov, 1975, 1977; Callan, Dashen, and Gross, 1977). Such a dramatic physical effect must have a simple mathematical description and perhaps it can be found amongst the nonlinear phenomena that have now been uncovered. More generally, it is fascinating to observe that tunnelling does occur in quantum field theory, and gives rise to very weak effects which violate various symmetries.

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