Theory of dynamic critical phenomena

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An introductory review of the central ideas in the modern theory of dynamic critical phenomena is followed by a more detailed account of recent developments in the field. The concepts of the conventional theory, mode-coupling, scaling, universality, and the renormalization group are introduced and are illustrated in the context of a simple example—the phase separation of a symmetric binary fluid. The renormalization group is then developed in some detail, and applied to a variety of systems. The main dynamic universality classes are identified and characterized. It is found that the mode-coupling and renormalization group theories successfully explain available experimental data at the critical point of pure fluids, and binary mixtures, and at many magnetic phase transitions, but that a number of discrepancies exist with data at the superfluid transition of ^{4}He.

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I. INTRODUCTION

When a system is at or close to a critical point, anomalies occur in a wide variety of dynamic properties, as well as in the static properties most commonly discussed in the theory of critical phenomena. The dynamic properties of a system are quantities such as transport coefficients and relaxation rates, multi-time correlation functions, and the linear response to time-dependent perturbations, all of which depend on the equations of motion, and are not simply determined by the equilibrium distribution of the particles at a given instant of time. The static properties, by contrast, are quantities such as thermodynamic coefficients, single-time correlation functions, and the linear response to time-independent perturbations, which *are* determined by the single-time equilibrium distribution.

Dynamic properties may be measured by a variety of experiments. For example, time-dependent correlation functions are determined by inelastic scattering of neutrons or by frequency-resolved light scattering experiments. Relaxation rates may be measured by changing the temperature or some other parameter, and then monitoring the rate at which a system relaxes toward equilibrium. Relaxation rates may also be obtained indirectly from acoustic attenuation or from magnetic resonance experiments. Transport coefficients can be measured directly, or extracted from scattering experiments.

Over the years, a number of theoretical ideas have shaped our understanding of dynamic critical phenomena. We shall discuss in particular (i) the conventional theory of critical slowing down, (ii) the "mode-coupling" theory of transport anomalies, (iii) the hypotheses of dynamic scaling and universality, and (iv) the renormalization group approach to critical dynamics. [Brief reviews of these concepts may also be found in Halperin (1973a, 1976).]

The *conventional theory* of critical slowing down, which is due to Van Hove (1954) and Landau and Khalatnikov (1954), assumes that the *transport coefficient* or *kinetic coefficient* (defined below) for the order parameter remains finite at the critical point. Since relaxation rates are generally determined by the ratio of a transport or kinetic coefficient to a static susceptibility, and since the order parameter susceptibility diverges at the transition, the conventional theory implies that *there is at least one mode whose relaxation rate goes to zero at the critical point*. This slow relaxation of the order parameter can lead to observable anomalies in other transport coefficients and dynamic properties (Landau and Khalatnikov, 1954; Pitaevskii, 1958).

It is now known that the conventional theory is incorrect in most cases. Examples will be cited below in which transport coefficients and kinetic coefficients diverge, as well as examples where kinetic coefficients go to zero at the critical point. In no case, however, does a transport or kinetic coefficient diverge as strongly as the order parameter susceptibility, and the phenomenon of *critical slowing down* is still found in all cases, at least as applied to the relaxation rate of the order parameter.

The modern theories of dynamic critical phenomena which will be discussed below, provide a means of understanding the detailed behavior of relaxation rates in situations where the conventional theory fails. The starting point for all of these theories is the identification of the various *slow modes* of the system, i.e., the modes whose relaxation rates go to zero at long wavelengths near the critical point. These modes involve the *conserved densities* of the system, which enter the hydrodynamics, at sufficiently long wavelengths, for any temperature other than T_c. In addition, one must include the *order parameter* mode, if the order parameter is not itself one of the conserved densities.

Van Hove's original argument asserted that the transport and kinetic coefficients are essentially determined by interactions between particles on the length scale of interatomic separations, and that therefore these coefficients should have the same behavior as short-wavelength (equal-time) correlation functions, which are known to remain finite at T_c. The failures of the conventional theory occur because in many systems, there are important *nonlinear interactions* between the slow modes on a scale of wavelengths comparable to the *correlation length* ξ. This length diverges at the critical point, and is long compared to the interatomic spacing in the critical region.

The nonlinear interactions between the slow modes are of two types. The most important ones are *nondissipative* or reversible couplings, which have sometimes been called "convective terms" or "streaming terms" in the equations of motion. It is these terms which are responsible for the divergences occurring in various transport and kinetic coefficients. The magnitudes of the nondissipative couplings in the limit of long wavelengths are generally determined by Poisson-bracket relations (see below) among the various slow variables.

The second type of nonlinear interactions arise from the coupling terms in the free-energy functional, which are present in the theory of static critical phenomena,[1] and which lead to *dissipative* couplings of the dynamic modes. These couplings are responsible for the cases where a kinetic coefficient vanishes at the critical point.

Effects of the nondissipative interactions have been calculated (approximately) using a variety of formalisms, which we shall group together as coupled-mode or *mode-coupling* theories. The first such theory was proposed by Fixman (1962), and then reformulated in accordance with static-scaling concepts by Kadanoff and Swift (1968a), and especially Kawasaki (1967, 1970). Coupled-mode calculations, by these authors and others, have been successful in many areas of dynamic critical phenomena, but most notably for the critical points of simple fluids and binary fluid mixtures (see Kawasaki, 1976). A simplified illustration of the mode-coupling mechanism, which demonstrates the essential physics, will be given in Sec. II, below.

Parallel to the formulation of the coupled-mode theo-

[1] An extensive literature exists on the static renormalization group. Reviews may be found in Wilson and Kogut (1974), Wilson (1975), Fisher (1974), Ma (1973, 1976a), Barber (1977), Patashinskii and Pokrovskii (1977), and the various articles in Domb and Green (1976).

ries, a purely phenomenological approach, known as *dynamic scaling* was developed, in analogy with the scaling laws for static critical phenomena, by Ferrell *et al.* (1967, 1968) and the present authors (Halperin and Hohenberg, 1967, 1969a). The dynamic scaling hypotheses state, for example, that the wave vector- and frequency-dependent susceptibility of a ferromagnet near its Curie point may be expressed as a function independent of $|T-T_c|$, provided that the length and frequency scales, as well as the magnetization and magnetic field, are rescaled by appropriate powers of $(T-T_c)$. The implications and the applicability of the dynamic scaling hypotheses, will be discussed further in the following sections.

A recent development has been the application of *renormalization group* methods to dynamic critical phenomena (Halperin *et al.*, 1972). These techniques, which are generalizations of the methods developed by Wilson and others for static critical phenomena,[1] permit a proper treatment of both the dissipative and the nondissipative interactions. One may thus handle situations not correctly treated by the coupled-mode approach, and also justify dynamic scaling and mode coupling more systematically, where these are adequate. As in the static case, most of the practical calculations using the renormalization group have been performed as an expansion in the parameter $\epsilon = 4-d$, where d is the spatial dimensionality. For d greater than 4, the conventional theory holds in most systems, because the phase space for diverging long-wavelength fluctuations is too small to cause divergent renormalization of the transport and kinetic coefficients. In the case of the Heisenberg ferromagnet, however, the critical dimensionality below which divergences occur is $d=6$, and renormalization group expansions involve the parameter $6-d$ (see below). The renormalization group approach will be outlined in Sec. IV in the context of some simple relaxational models, where only disspative couplings are present. The results of renormalization group calculations for a variety of other systems will be discussed in the subsequent sections.

An important contribution of the renormalization group approach to static and dynamic critical behavior has been the elucidation of the concept of *universality*.[1,2] Since the study of critical phenomena concerns the behavior of a system whose correlation length ξ is very large compared to interatomic spacings, it is natural to suppose that many details of the microscopic Hamiltonian will be unimportant for the critical behavior. It follows that systems showing critical phenomena can be divided into broad groups known as *universality classes*, such that all members of a given class have "identical" critical properties. In this context, one may pose a number of fundamental questions, to which any theory of dynamic critical phenomena should address itself:

- What are the different universality classes for dynamic critical phenomena, and on what properties of the system do these depend?
- Within a given universality class, what are the numerical values of the exponents which characterize the singular behavior of various static and dynamic properties?
- Are there simple numerical relations (scaling laws) between the dynamic critical exponents and the various static exponents? Do scaling laws completely determine the dynamic critical exponents?
- In addition to exponents, one can construct various dimensionless functions or ratios of coefficients describing dynamic critical properties. Which of these are universal, within a given class? What are the numerical values of the universal functions (scaling functions) and ratios?

Beyond describing the asymptotic critical behavior, one may investigate the form of the leading *corrections*, as one moves away from T_c, or away from long wavelengths and low frequencies. These corrections may turn out to be very important in experimental situations.

Corrections to the asymptotic critical behavior are particularly interesting in the vicinity of a *multicritical point*, a special point at which a phase transition changes from one universality class to another, as a function of some parameter p in the Hamiltonian. When p is sufficiently close to its multicritical value p_m, there will be a regime of temperature in which properties are determined by the universality class of the multicritical point, followed by a regime of *crossover* to the asymptotic critical behavior, which is finally reached only very close to T_c. Modern theories of critical phenomena attempt to calculate properties in all of these regimes, when $|p-p_m|$ and $|T-T_c|$ are simultaneously small.

Let us mention briefly the properties which determine the dynamic universality classes. The remaining questions will be treated in later sections, as the individual classes are reviewed. It is known[1] that static critical phenomena depend on the spatial dimensionality d as well as on the symmetry of the order parameter (i.e., on the component number n, on the presence or absence of cubic anisotropies, etc.). Also, the statics will depend on whether or not long-range forces are present, whether impurities are present, and so forth. However, the dynamic properties will depend upon additional properties of the system which do not affect the statics. We know that conservation laws play an important role in determining the hydrodynamic behavior of a system away from T_c, as do "Poisson-bracket relations" (see below) among the conserved quantities. Naturally, we expect that these properties will also affect the long-wavelength low-frequency behavior at the critical point. We have already remarked that nondissipative couplings between the slow modes, which enter the coupled-mode calculations, depend on Poisson-bracket relations between the slow variables. It is hypothesized that the conservation laws and the Poisson-bracket relations among the order parameter and the conserved densities, together with the dimensionality, order parameter symmetry, and any other properties that affect the static

[2]Early discussions of universality may be found in Fisher (1966), Jasnow and Wortis (1968), and Watson (1969). Phenomenological formulations were given by Griffiths (1970), Kadanoff (1971), Betts *et al.* (1971), Stanley *et al.* (1971), and Stauffer *et al.* (1972). The mathematical mechanism for universality in static critical phenomena is explained in the references in footnote 1. See also Hohenberg *et al.* (1976a). Dynamic universality classes were discussed phenomenologically by Halperin and Hohenberg (1969a).

TABLE I. Some dynamical models treated by renormalization-group methods.

Model	Designation	System	Dimension order of parameter	Non-conserved fields	Conserved fields	Non-vanishing Poisson bracket
Relaxational	A	Kinetic Ising anisotropic magnets	n	ψ	None	None
	B	Kinetic Ising uniaxial ferromagnet	n	None	ψ	None
	C	Anisotropic magnets structural transition	n	ψ	m	None
Fluid	H	Gas–liquid binary fluid	1	None	ψ $\mathbf{j}$	$\{\psi,\mathbf{j}\}$
Symmetric planar magnet	E	Easy-plane magnet, $h_z=0$	2	ψ	m	$\{\psi,m\}$
Asymmetric planar magnet	F	Easy-plane magnet, $h_z\neq 0$ superfluid helium	2	ψ	m	$\{\psi,m\}$
Isotropic antiferromagnet	G	Heisenberg antiferromagnet	3	$\boldsymbol{\psi}$	$\mathbf{m}$	$\{\boldsymbol{\psi},\mathbf{m}\}$
Isotropic ferromagnet	J	Heisenberg ferromagnet	3	None	$\boldsymbol{\psi}$	$\{\boldsymbol{\psi},\boldsymbol{\psi}\}$

critical behavior, *suffice* to determine the universality class for critical dynamics. The extent to which this *universality hypothesis* has been tested by theory and experiment will be discussed below.

If one accepts the universality hypothesis then it suffices to study the simplest dynamical *model* consistent with the known static behavior and with a given set of conservation laws, Poisson-bracket relations, symmetry, etc. If we can solve that model for its critical behavior, then we know the critical behavior of all members of the class. In order to *test the universality assumption,* one may add all possible perturbations to the model, consistent with the conservation laws, etc., and try to convince oneself that these perturbations are "irrelevant" at the critical point. In practice, it has only been possible to study small perturbations from the simple systems, for both static and dynamic behavior.[3]

The field-theoretic models generally studied with coupled-mode or renormalization group methods are *semimicroscopic,* in that they describe the behavior of the variables remaining after one has first integrated over all variations with wavelengths short compared to a cutoff Λ^{-1}. The scale of length Λ^{-1} is intermediate between the microscopic scale $a\approx 1$ Å, and the macroscopic scale defined by the correlation length ξ near T_c. The transition from a microscopic Hamiltonian to the semimicroscopic models has not been carried out explicitly for either the statics or the dynamics in most cases. Indeed, the difficulties involved in such a *microscopic derivation* are comparable to the problem of calculating the transport coefficients and short-wavelength spectrum of a dense liquid away from the critical point, starting from interparticle potentials. Fortunately, however, this information does not seem to be needed to obtain the critical behavior, at least insofar as one is concerned with "universal" properties. The steps leading from a microscopic Hamiltonian to the corresponding semimicroscopic representation will be discussed further in Sec. IV.D, for a model of a displacive transition in an interacting phonon system.

Some semimicroscopic models belonging to different universality classes are listed in Table I, along with physical systems to which they correspond. These models will be described, and their dynamic critical behavior will be examined, in Secs. IV–VII below. Some key results of the renormalization group treatment of the models are summarized in Table II.

II. THE SYMMETRIC BINARY FLUID—A SIMPLE EXAMPLE

A. Hydrodynamics

Many of the physical ideas mentioned in the Introduction, can be illustrated with a simple example—the critical point for phase separation (consolute point) in a hypothetical binary fluid whose properties are symmetric with regard to interchange of the two constituents A and B. We shall consider the properties of this system for $T\geq T_c$, at the critical concentration which is $c=\frac{1}{2}$ by symmetry.

The binary fluid has six conserved densities,[4] viz., the energy density e, the mass densities ρ_A and ρ_B for the two constituents A and B, respectively, and the three components of the momentum density $\mathbf{j}$. Small, long-wavelength deviations of these densities from their con-

[3] It has never been rigorously demonstrated, for example, that the *static* critical behavior of the four-dimensional Ising model is necessarily the same as that of the weakly interacting four-dimensional continuum field theory, for which the renormalization group analysis is carried out.

[4] A conserved density is a quantity whose spatial integral over the whole system is a constant of the motion. If Q is a conserved density we can write its time derivative as the divergence of a current, $\partial Q/\partial t=-\nabla\cdot\mathbf{j}^Q$.

TABLE II. Some representative results of renormalization group calculations in $d=4-\epsilon$ dimensions.[a]

System	Quantity	Formula	ϵ expansion
Model A kinetic Ising anisotropic magnets	Relaxation rate	$\omega_\psi(\mathbf{k}=0)\propto\xi^{-z}$ $z=2+c\eta$	$c=0.7261(1-1.69\epsilon+\cdots)$
Model B kinetic Ising uniaxial ferromagnet	Transport coefficient	$\omega_\psi(\mathbf{k})=(\lambda/\chi_\psi)k^2$ $\lambda=$ const	Agrees
Model C ($n=1$) anisotropic magnets structural transitions	Relaxation rate	$\omega_\psi(\mathbf{k}=0)\propto\xi^{-z}$	
	Scaling law	$z=2+\alpha/\nu$	Agrees
	Amplitude ratio ($T>T_c$)	$\mu=\frac{\omega_m(\mathbf{k})}{\omega_\psi(\mathbf{k})}(k\xi)^{-2}$	$\mu=1+0.5004\epsilon+\cdots$
Model H gas–liquid (binary fluid)	Thermal conductivity (concentration conductivity)	$\lambda\propto\xi^{x_\lambda}$	$x_\lambda=\frac{18}{19}\epsilon(1-0.033\epsilon+\cdots)$
	Viscosity	$\bar\eta\propto\xi^{x}\bar\eta$	$x_{\bar\eta}=\frac{1}{19}\epsilon(1+0.238\epsilon+\cdots)$
	Scaling law	$x_\lambda+x_{\bar\eta}=\epsilon+\eta$	Agrees
	Amplitude ratio	$R=\frac{\lambda\bar\eta\xi^{d-2}}{k_BTC_p}$	$R=K_d\frac{19}{24\epsilon}(1+0.06\epsilon+\cdots)$
Model F superfluid ^{4}He	Scaling law	$z=\frac{1}{2}(d+\tilde\alpha/\nu)$ $\tilde\alpha\equiv\max(\alpha,0)$	Agrees
	Thermal conductivity, $T>T_c$	$\lambda_m\propto\xi^{x}\lambda$	
	Scaling law	$x_\lambda=\frac{1}{2}(\epsilon+\tilde\alpha/\nu)$	Agrees
	Amplitude ratio	$R_\lambda=\lambda_m/(\xi_+^{1/2}g_0C_p^{1/2})$	$R_\lambda=(K_d/\epsilon)^{1/2}(1+0.6\epsilon+\cdots)$
	Second-sound damping $T<T_c$	$D_s\propto\xi_T^{x_2}$	
	Scaling law	$x_2=\frac{1}{2}(\epsilon-\tilde\alpha/\nu)$	Agrees
	Amplitude ratio	$R_2=D_s/(2c_s\xi_T)$	$R_2^{2-\epsilon}=(K_d/\epsilon)[1+0.31\epsilon+\cdots]$
Model G isotropic antiferromagnet	Scaling law	$z=\frac{1}{2}d$	Agrees
	Amplitude ratio	$R_\Gamma=g_0^{-1}\omega_\psi\xi_+^{1/2}\chi_m^{1/2}$	$R_\Gamma=(3K_d/\epsilon)^{1/2}(1-0.6\epsilon+\cdots)$
	Amplitude ratio	$R_\lambda=g_0^{-1}\lambda_m\xi_+^{-1/2}\chi_m^{-1/2}$	$R_\lambda=(K_d/3\epsilon)^{1/2}(1+0.27\epsilon+\cdots)$
Model J isotropic ferromagnet	Scaling law	$z=\frac{1}{2}(d+2-\eta)$	Agrees for $\epsilon=6-d$
	Amplitude ratio	$R_\lambda=g_0^{-1}\lambda\xi_+^{(d-4)/2}\chi_\psi^{-1/2}$	

[a]The quantities $\omega_\psi(\mathbf{k})$ and $\omega_m(\mathbf{k})$ are characteristic frequencies for the order parameter and the auxiliary conserved density, respectively, χ_ψ and χ_m are the corresponding static susceptibilities, and λ_m is the transport coefficient for m. The transport coefficient for ψ when it is conserved is denoted as λ. The correlation length is written as ξ both above and below T_λ, except for models with continuous broken symmetry (F, G, and J), where it is written as ξ_+ above, and ξ_T below T_c. In the case of helium (model F) c_s denotes the second-sound velocity and $C_p\equiv\chi_m$ is the specific heat. All results listed are for $k\xi\to 0$, $T\to T_c$.

stant equilibrium values may be described by the equations of *linearized hydrodynamics* (Landau and Lifshitz, 1959). Solution of these equations leads to the five hydrodynamic modes—four diffusive modes with relaxation rates proportional to k^2, and a propagating sound wave, whose complex frequency is

$$\omega=\pm ck-i(D/2)k^2+O(k^3). \tag{2.1}$$

(Here c and D are the sound velocity and damping constant, respectively.) The diffusive modes consist of a thermal diffusion mode, a concentration diffusion mode, and the viscous relaxation modes for the two transverse components of the momentum density. The variables e, ρ_A, and ρ_B appear in various proportions in the thermal diffusion and concentration diffusion modes, as well as in the sound wave.

In the special case of the symmetric binary fluid at 50–50 concentration, there is a simplification in that the energy density e and the total mass density $\rho=\rho_A+\rho_B$ do not appear in the concentration diffusion mode, while the density difference

$$\psi\equiv\rho_B-\rho_A, \tag{2.2}$$

does not appear in the thermal diffusion or sound modes.

In order to study the dynamics of concentration fluctuations it is useful to introduce a fictitious potential Φ, coupled to the difference in densities of the two components. We may then write the Hamiltonian as

$$H = H_0 + \int \Phi(\mathbf{x}, t)\psi(\mathbf{x}, t)\, d^d x\,, \tag{2.3}$$

where H_0 is the Hamiltonian in the absence of Φ, i.e., the sum of the kinetic energy and the potential energy of interaction of the atoms. Let us first imagine an infinite system in thermal equilibrium with $\Phi = 0$ at time $t = 0$, and let us apply a potential $\Phi(\mathbf{x}) = -\mathbf{F}\cdot\mathbf{x}$ for $t > 0$. There is an applied force $\mathbf{F}$ per unit mass on each B atom in the fluid, and an equal and opposite force on each A atom. In the symmetric fluid there will be no net force per unit volume, and there will be no acceleration of the fluid as a whole. After a short time, a steady state will be reached in which there are equal and opposite currents of A and B atoms, with a net concentration current parallel to $\mathbf{F}$, of magnitude

$$\mathbf{j}^{\psi} = \mathbf{j}^B - \mathbf{j}^A = 2\mathbf{j}^B\,. \tag{2.4}$$

For small values of the applied force $\mathbf{F}$, we must have $\mathbf{j}^{\psi}$ proportional to $\mathbf{F}$, so we may write

$$\mathbf{j}^{\psi} = \lambda\mathbf{F}\,, \tag{2.5}$$

where λ is the *transport coefficient* for the density difference ψ.

Next consider a situation where there is no external force $\mathbf{F}$, but there are small gradients in the concentration. In this case the driving force for diffusion is the gradient of the chemical potential difference $\mu = \mu_B - \mu_A$, and we may write

$$\mathbf{j}^{\psi} = -\lambda\nabla\mu = -(\lambda/\chi_{\psi})\nabla\psi\,, \tag{2.6}$$

where the susceptibility χ_{ψ} is defined by a derivative at constant temperature and pressure

$$\chi_{\psi} \equiv \left.\frac{\partial\psi}{\partial\mu}\right|_{T,P}\,. \tag{2.7}$$

The coefficient λ must be the same in Eqs. (2.5) and (2.6), because if one has a long-wavelength spatial variation of both $\psi(\mathbf{x})$ and $\Phi(\mathbf{x})$, one must reach a thermal equilibrium state with $\nabla\cdot\mathbf{j}^{\psi} = 0$ when $\mu(\mathbf{x}) = \Phi(\mathbf{x}) + \text{constant}$. Equation (2.6) is just Fick's law for diffusion. In the symmetric binary fluid we need not worry about the coupling of $\mathbf{j}^{\psi}$ to any temperature gradients in the system, which would otherwise introduce additional terms in (2.6) (Landau and Lifshitz, 1959).

If we now combine (2.6) with the conservation law $d\psi/dt = -\nabla\cdot\mathbf{j}^{\psi}$, we see that the concentration relaxes at a rate

$$\omega_{\psi} = D_c k^2\,, \tag{2.8}$$

where the concentration diffusion constant D_c is given by

$$D_c = \lambda/\chi_{\psi}\,. \tag{2.9}$$

In a similar manner we have for the thermal and viscous diffusion constants, respectively,

$$D_T = \lambda_T/C_p\,, \tag{2.10}$$

$$D_v = \overline{\eta}/\rho\,, \tag{2.11}$$

where λ_T is the thermal conductivity, C_p is the constant-pressure specific heat per unit volume, and $\overline{\eta}$ is the shear viscosity. We may remark that $\overline{\eta}$ is the transport coefficient for momentum, and, because of Galilean invariance for a system with velocity-independent forces, ρ is the susceptibility for the momentum density, i.e., ρ^{-1} is the second derivative of the free energy per unit volume with respect to the momentum density.

B. Static critical behavior

The density difference ψ is the order parameter for the consolute point in the symmetric binary fluid. According to the scaling theory of static critical phenomena (see Stanley, 1971; Fisher, 1967) we must have

$$\chi_{\psi} \propto \xi^{2-\eta}\,, \tag{2.12}$$

where ξ is the correlation length which diverges according to the power law

$$\xi \propto (T - T_c)^{-\nu}\,. \tag{2.13}$$

The exponents ν and η have been calculated by static renormalization group methods[1] and other means (Fisher, 1967), and are believed to be roughly $\nu = 0.62$, $\eta = 0.04$ for the three-dimensional binary fluid ($d = 3$, $n = 1$).

C. Critical dynamics

If the conventional theory (Van Hove, 1954) held for the binary fluid, so that the transport coefficient λ was finite at the critical point, the diffusion constant D_c would be predicted to vanish proportional to $\xi^{-2+\eta}$. At sufficiently long wavelengths, therefore, the order parameter relaxation rate ω_{ψ} would go to zero as

$$\omega_{\psi} \propto \xi^{-z}(k\xi)^2\,, \tag{2.14}$$

with

$$z = 4 - \eta\,. \tag{2.15}$$

For wavelengths shorter than ξ, however, we know that the wave vector-dependent susceptibility $\chi_{\psi}(\mathbf{k})$ varies as $k^{-2+\eta}$ rather than $\xi^{2-\eta}$. In the spirit of the conventional theory one would expect the transport coefficient λ to be independent of wave vector, as long as the wavelength was large compared to the interatomic spacing. For $k > \xi^{-1}$, therefore, we would expect the order parameter relaxation rate to vary as

$$\omega_{\psi} \propto k^z\,, \tag{2.16}$$

with z given by (2.15).

We shall argue below that the conventional theory is incorrect for this system, and that λ *diverges* as $T \to T_c$. The order parameter relaxation rate in the regions $k < \xi^{-1}$ and $k > \xi^{-1}$ still has a homogeneous scaling form

$$\omega_{\psi}(k) = k^z\Omega(k\xi)\,, \tag{2.17}$$

which generalizes (2.14) and (2.16), but the exponent z is now *smaller* than $4 - \eta$. Equation (2.17) is the *dynamic scaling* assumption for the order parameter relaxation rate.

D. The coupled-mode theory

The reason for the divergence of λ can be understood by a simple coupled-mode argument, adapted from Arcovito *et al.* (1969). Let us consider the contribution to the transport coefficient λ, from a region of the fluid whose diameter is equal to the correlation length ξ. Let us define $\tilde{\psi}$ as the value of ψ averaged over the region in question, at some given instant of time. According to the equipartition theorem, the expectation value of $\tilde{\psi}^2$ is given by

$$\langle \tilde{\psi}^2 \rangle \approx k_B T \chi_\psi / \xi^d , \tag{2.18}$$

where ξ^d is the approximate volume of the region. In the presence of the field $\mathbf{F}$, introduced in Sec. II.A, there will be a net applied force on the region, of strength

$$\mathbf{f}_{\rm app} = \xi^d \tilde{\psi} \mathbf{F} . \tag{2.19}$$

If we assume that the region in question moves approximately as a rigid body, it will accelerate until $\mathbf{f}_{\rm app}$ is balanced by the viscous drag of the surrounding fluid,

$$\mathbf{f}_{\rm vis} \approx -\mathbf{v} \bar{\eta} \xi^{d-2} , \tag{2.20}$$

where $\mathbf{v}$ is the velocity of motion. The net current density in the region of diameter ξ is therefore

$$\mathbf{j}^\psi = \tilde{\psi} \mathbf{v} , \tag{2.21a}$$

$$\approx \tilde{\psi}^2 \xi^2 (1/\bar{\eta}) \mathbf{F} . \tag{2.21b}$$

Note that the contribution to $\mathbf{j}^\psi$ is independent of the sign of $\tilde{\psi}$. Taking account of (2.18), we see that the contribution to λ coming from fluctuations on the scale of ξ is given by

$$\lambda \approx (R/\bar{\eta}) k_B T \chi_\psi \xi^{2-d} \propto (1/\bar{\eta}) \xi^{4-d-\eta} , \tag{2.22}$$

where R is a dimensionless constant whose magnitude cannot be determined by our crude arguments. If we assume that the viscosity $\bar{\eta}$ is finite at T_c, then we predict that λ diverges roughly as ξ^{4-d}, for any $d<4$. [Recall that the exponent η is small, and vanishes[1] as $(4-d)^2$ for $d \to 4$.] For $d>4$, fluctuations on the scale of ξ make a vanishing contribution to λ. The value of λ is then determined by fluctuations on the interatomic distance scale, and λ will be finite at T_c as in the Van Hove theory.

We must now examine more closely the behavior of $\bar{\eta}$ in the vicinity of T_c. A coupled-mode argument similar to the above gives a divergence for $\bar{\eta}$ proportional to $\xi^{4-d-\eta}/\lambda$. [Indeed, the first mode-coupling calculation of Fixman (1962) was a prediction of this effect.] The two mode-coupling arguments therefore show that the product of the transport coefficients $\bar{\eta}\lambda$ diverges as $\xi^{4-d-\eta}$ (Kadanoff and Swift, 1968a), but these arguments do not determine the precise exponents for the separate divergences of the two coefficients. As will be discussed below, more careful mode-coupling analyses, as well as renormalization group expansions in $(4-d)$, show that most of the divergence is in λ, while $\bar{\eta}$ diverges relatively weakly at T_c. Thus λ diverges roughly as ξ in three dimensions, and $z \approx 3$ for this system [see Eqs. (2.14)–(2.17)].

Several aspects of the above derivation are worthy of comment. First we may note the important role of dimensionality in the divergence of λ. At sufficiently large d, the phase space for fluctuations with wave vectors on the order of ξ^{-1} is so small as to eliminate the contributions of these fluctuations as $\xi \to \infty$. Second, we may note the importance of conservation laws for the divergence of λ. Expression (2.20) for the viscous drag depends on the fact that the viscous force on an element of fluid is proportional to $\nabla^2 \mathbf{v}(\mathbf{x})$, a consequence of the conservation of momentum. If the momentum were not conserved (e.g., if the fluid were embedded in a porous medium with channels of microscopic size), then viscous forces would be proportional to $\mathbf{v}$ rather than $\nabla^2 \mathbf{v}$, and $\mathbf{f}_{\rm vis}$ would be proportional to ξ^d rather than ξ^{d-2}. This would remove a factor of ξ^2 from the right-hand side of (2.20), and there would be no divergence in λ for $d>2$.

We may also note the relevance of Poisson-bracket relations in the above calculation. It was essential to Eq. (2.21a) that a long-wavelength viscous motion of the fluid carries along with it any fluctuations in ψ that may be present. In a fundamental sense, this is a reflection of a Poisson-bracket relation between the total momentum $\mathbf{P}$ and any other density $A(\mathbf{x})$ in the system

$$\{A(\mathbf{x}), \mathbf{P}\} = \nabla A(\mathbf{x}) ; \tag{2.23}$$

i.e., the momentum operator is the infinitesimal generator of translations of the system.[5] It was also at this step that the *coupling* of modes came into play.

In the above analysis, we divided the system into regions of a definite size ξ^d, and considered only rigid motions of these regions. A more careful coupled-mode analysis would calculate the divergence of the transport coefficient λ as an integral over intermediate wave vectors $\mathbf{p}$, of contributions arising from the product of a concentration fluctuation at wave vector $\mathbf{p}$ with a transverse momentum fluctuation at wave vector $-\mathbf{p}$. Similar analyses for finite wave vector transport coefficients $\lambda(\mathbf{k})$ and $\bar{\eta}(\mathbf{k})$ involve the product of fluctuations at wave vectors $\mathbf{p}$ and $\mathbf{k}-\mathbf{p}$. One finds that the transport coefficients are independent of temperature, for $k > \xi^{-1}$, and that $\lambda(k)\bar{\eta}(k) \propto k^{d+\eta-4}$. This leads to a homogeneous scaling form for the order parameter relaxation rate, as in (2.17) (see Sec. V.B, below).

It is also interesting to note that a coupled-mode analysis similar to the above leads to a divergence in λ and other transport coefficients for a two-dimensional fluid, even away from any critical point.[6] To see this, one may replace ξ in Eqs. (2.18)–(2.22) by an arbitrary large length L, and note that far from T_c, χ_ψ is a constant independent of L.

The relationship of the symmetric binary fluid to critical points in real fluids will be discussed in Sec. V. It turns out that the complications resulting from the asymmetry in a real binary fluid do not change the divergences of λ and $\bar{\eta}$. Furthermore, one finds that the gas–liquid critical point of a pure fluid also belongs to the same universality class, where λ is now the thermal conductivity, and χ_ψ is the specific heat C_p.

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[5] Equation (2.19) is also a reflection of the Poisson brackets. See discussion in Sec. V.A below.

[6] The divergence of transport coefficients in two dimensions was noted by Yamada and Kawasaki (1967), and by Alder and Wainwright (1970). For more recent treatments see Pomeau and Résibois (1975), Forster *et al.* (1976), and Kawasaki and Gunton (1976a).

III. BASIC DEFINITIONS AND FORMALISM

A. Stochastic models

Consider a physical system whose state at any instant of time is described by a set of (real) classical variables $\tilde{Q}_\mu$. If the $\tilde{Q}_\mu$ are a *complete* set of variables for the system (including momenta), then they evolve in time according to a set of *deterministic* first-order differential equations[7]

$$d\tilde{Q}_\mu(t)/dt = \Phi_\mu[\tilde{\mathbf{Q}};\tilde{\mathbf{h}}], \tag{3.1}$$

where Φ_μ is a function of the variable set $\tilde{\mathbf{Q}}$ at time t, which may depend parametrically on a set of "applied fields" $\{\tilde{h}_\mu\} \equiv \tilde{\mathbf{h}}$. (We shall be interested in cases in which the applied fields are in general themselves functions of time.) Furthermore, according to classical mechanics[8] it is possible to define a Hamiltonian

$$H[\tilde{\mathbf{Q}};\tilde{\mathbf{h}}] = H_0[\tilde{\mathbf{Q}}] - \sum_\mu \tilde{h}_\mu \tilde{Q}_\mu, \tag{3.2}$$

and a set of "Poisson-bracket relations" among the variables

$$\{\tilde{Q}_\mu, \tilde{Q}_{\mu'}\} = -\{\tilde{Q}_{\mu'}, \tilde{Q}_\mu\} = W_{\mu\mu'}, \tag{3.3}$$

such that

$$\Phi_\mu = \sum_{\mu'} W_{\mu\mu'} \partial H/\partial \tilde{Q}_{\mu'}. \tag{3.4}$$

These equations are conservative in the sense that $dH/dt = 0$, if the $\tilde{h}_\mu$ do not vary in time. In the familiar case where the $\tilde{Q}_\mu$ are linear momentum and position variables, the $W_{\mu\mu'}$ are a set of constants, but in the more general case the quantities $W_{\mu\mu'}$ are themselves functions of the $\{\tilde{Q}_\mu\}$.

In discussing condensed systems it is often convenient to focus on a small subset[7] $\{Q_\nu\} \equiv \mathbf{Q}$ of the complete set $\tilde{\mathbf{Q}}$, and to take averages over the remaining variables. The ensuing equations of motion for the variables $\mathbf{Q}$, obtained by averaging (3.1), are then necessarily *stochastic*,[9] i.e., the value of $Q_\nu(t)$ for $t > 0$ is not completely determined by the values of the $\{Q_\nu\}$ at $t = 0$. In general, it is necessary to define a weight functional W which depends on the values of $\mathbf{Q}(t)$ for all times t, and which gives the relative probability of that particular time evolution of the system. The weight functional W also depends parametrically on any time-dependent fields $\mathbf{h}(t)$. One may then define expectation values $\langle Q_\nu(t)\rangle|_{\mathbf{h}}$, two-time correlation functions $\langle Q_\nu(t) Q_{\nu'}(t')\rangle|_{\mathbf{h}}$, etc., by taking the average over all possible time evolutions, weighted by W. The single-time expectation values $\langle Q_\nu(t)\rangle|_{\mathbf{h}}$, $\langle Q_\nu(t) Q_{\nu'}(t)\rangle|_{\mathbf{h}}$, etc., are entirely determined by the *probability distribution* $P[\mathbf{Q};t]$ for the values of $\mathbf{Q}$ at a single instant of time t.

A variety of formalisms exist for describing stochastic equations of motion, and for evaluating the necessary expectation values and correlation functions (at least formally)[9]; usually it is not necessary to consider the weight function W explicitly.

In discussing dynamic critical behavior, the variables Q_ν of interest are those with slow variations near thermal equilibrium. As mentioned in the Introduction, these are the long-wavelength components of the densities of the *conserved variables*[4] in the system, as well as the long-wavelength fluctuations of the *order parameter* for the transition, if it is not itself a conserved variable.

If the correlation functions $\langle Q_\nu Q_{\nu'}\rangle|_{\mathbf{h}}$ and the probability distribution $P[\mathbf{Q};t]$ are derived from a Hamiltonian, then there are a number of important properties which may be proven quite generally.[9] Alternatively, if we consider a stochastic model involving the restricted set of variables $\mathbf{Q}$, we should require that the correlation functions and probability distribution derived from that model also satisfy these conditions. The most important conditions are the following:

(i) Thermal Equilibrium. If the fields $\{h_\nu\}$ are independent of time, then the probability distribution is also independent of time, and it has the form

$$P[\mathbf{Q}] = P_{\rm eq}[\mathbf{Q}] = Z^{-1} \exp\left(-\left\{F[\mathbf{Q}] - \sum_i \mu_i \bar{Q}_i\right\}\Big/ k_B T\right), \tag{3.5}$$

where

$$\exp(-F[\mathbf{Q}]/k_B T) \equiv \text{tr}\, \exp(-H[\tilde{\mathbf{Q}}]/k_B T)|_{\mathbf{Q}}, \tag{3.6}$$

$$Z \equiv \text{tr} \exp\left(-\left\{F[\mathbf{Q}] - \sum_i \mu_i \bar{Q}_i\right\}\Big/ k_B T\right), \tag{3.7}$$

$$Z = \text{tr} \exp\left(-\left\{H[\tilde{\mathbf{Q}}] - \sum_i \mu_i \bar{Q}_i\right\}\Big/ k_B T\right). \tag{3.8}$$

The trace in (3.6) is taken over the "microscopic" variables $\tilde{\mathbf{Q}}$, with fixed values of the "macroscopic" variables $\mathbf{Q}$, which are a subset of the $\tilde{\mathbf{Q}}$. The quantities $\bar{Q}_i$ in (3.7) are the constants of the motion for the system, and they form a (small) subset of the $\{Q_\nu\}$. The constants μ_i are Lagrange multipliers conjugate to the $\bar{Q}_i$'s. On the coexistence curve of a system undergoing a phase transition, it is necessary to include the order parameter in the set of constants of the motion $\{\bar{Q}_i\}$, even if this order parameter is not conserved by the dynamics. In that case, the associated Lagrange multiplier μ_i will be infinitesimal.

The functional dependence of the equilibrium distribution function $P_{\rm eq}[\mathbf{Q}]$ and the partition function Z on the parameters $\{h_\nu\}$, $\{\mu_i\}$, and T must be consistent with statistical mechanics and must properly describe the static properties of the system. In particular the partition function and equilibrium expectation values must show the critical-point singularities appropriate to the spatial dimensionality and the symmetry of the order parameter.

(ii) Relaxation to Equilibrium. If the fields $\mathbf{h}$ are independent of time after some specified time t_0, the proba-

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[7] We shall use a vector notation $\tilde{\mathbf{Q}} \equiv \{\tilde{Q}_\mu\}$ to denote the set of all variables $\tilde{Q}_\mu$.

[8] Since the important fluctuations near T_c are at low frequencies, the condition $\hbar\omega/k_B T_c \ll 1$ is satisfied close to T_c. Consequently classical mechanics will be adequate to describe the asymptotic critical behavior of any physical system provided T_c is nonzero. The quantum case $T_c \to 0$ is mentioned briefly in Sec. VIII.D.2.

[9] See for example Van Kampen (1965, 1976), Lax (1960), Mori (1965), Martin (1968), Zwanzig (1972), Kawasaki (1976).

bility distribution $P[\mathbf{Q}, t]$ should *relax* at large times to an equilibrium distribution $P_{eq}[\mathbf{Q}]$, determined by the values of the constants of the motion $\{\overline{Q}_i\}$ (the energy E, the particle number N, etc.) at time t_0.

(iii) Causality. In the presence of a time-varying field $\mathbf{h}(t)$, the expectation values $\langle Q_\nu(t')\rangle|_h$ only depend on $\mathbf{h}(t)$ for $t<t'$ (Landau and Lifshitz, 1969).

(iv) Fluctuation-Dissipation Theorem. The fluctuation–dissipation theorem is a relation between equilibrium correlation functions and linear response functions, which may be proved for any system defined by a Hamiltonian (see Martin, 1968). We shall therefore only consider stochastic models in which this relation is obeyed in equilibrium. Specifically, let us assume that the $Q_\nu(t)$ are densities in space [a typical one being denoted as $Q(\mathbf{x}, t)$], and the conjugate field is $h_Q(\mathbf{x}, t)$. Then the linear response function $\chi_Q(\mathbf{k}, \omega)$ is defined by the relation

$$\langle Q(\mathbf{k}, \omega)\rangle_{h_Q} = \chi_Q(\mathbf{k}, \omega) h_Q(\mathbf{k}, \omega), \tag{3.9}$$

where Fourier transforms in space and time are given by

$$h_Q(\mathbf{x}, t) = \int \frac{d^d k}{(2\pi)^d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} h_Q(\mathbf{k}, \omega), \tag{3.10}$$

and the system is assumed to start from equilibrium at $t=-\infty$. The field h_Q is taken to be infinitesimal in (3.9), and the expectation value on the left-hand side is determined from the probability distribution $P[\mathbf{Q}; t]$ in the presence of $h_Q(\mathbf{x}, t)$. We may also define the correlation function

$$C_Q(\mathbf{x}, t) = \langle Q(\mathbf{x}, t) Q(0, 0)\rangle_{h_Q=0} - \langle Q(\mathbf{x}, t)\rangle_{h_Q=0} \langle Q(0, 0)\rangle_{h_Q=0}, \tag{3.11}$$

which is a two-time expectation value and depends on a joint probability $P_2[\mathbf{Q}, \mathbf{Q}'; t, t']$. We define the correlation function $C_Q(\mathbf{k}, \omega)$ as the Fourier transform of (3.11), and the *equal-time correlation function* $C_Q(\mathbf{k})$ as the spatial Fourier transform of (3.11) for $t=0$, or

$$C_Q(\mathbf{k}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, C_Q(\mathbf{k}, \omega). \tag{3.12}$$

According to the causality condition (iii), the response function $\chi_Q(\mathbf{k}, \omega)$ is an analytic function for complex frequencies in the upper-half plane, and its real and imaginary parts satisfy the Kramers-Kronig relations (Landau and Lifshitz, 1969). The fluctuation-dissipation theorem for classical systems[8] states that

$$C_Q(\mathbf{k}, \omega) = 2(k_B T/\omega)\, \mathrm{Im}\chi_Q(\mathbf{k}, \omega). \tag{3.13}$$

Similarly, the equal-time correlation function is related to the *static susceptibility* $\chi_Q(\mathbf{k}) \equiv \chi_Q(\mathbf{k}, \omega=0)$ by the equipartition theorem,

$$C_Q(\mathbf{k}) = k_B T \chi_Q(\mathbf{k}). \tag{3.14}$$

This theorem follows directly from (3.13) and the Kramers–Kronig relations.

Given the frequency spectrum of $\chi_Q(\mathbf{k}, \omega)$ we may define the *kinetic coefficient* $\Gamma_Q(\mathbf{k})$ by

$$\frac{1}{\Gamma_Q(\mathbf{k})} \equiv \frac{i\partial\chi_Q^{-1}(\mathbf{k}, \omega)}{\partial\omega}\bigg|_{\omega=0}. \tag{3.15}$$

and a (dissipative) *characteristic frequency* as

$$\omega_Q(\mathbf{k}) \equiv \Gamma_Q(\mathbf{k})/\chi_Q(\mathbf{k}). \tag{3.16}$$

From Eqs. (3.16) and (3.13) it is easy to show that[10]

$$\frac{1}{\omega_Q(\mathbf{k})} = \frac{-i}{\chi_Q(\mathbf{k})} \frac{\partial\chi_Q(\mathbf{k}, \omega)}{\partial\omega}\bigg|_{\omega=0} = \frac{C_Q(\mathbf{k}, \omega=0)}{2C_Q(\mathbf{k})}. \tag{3.17}$$

The quantity $1/\omega_Q$ is also the lifetime which enters the hysteresis loss at low frequencies. The power dissipated in response to a field $h_Q(\mathbf{k}, \omega)$ is

$$W = \tfrac{1}{2}|h_Q|^2 \chi_Q \omega_Q^{-1} \omega^2. \tag{3.18}$$

A different definition of the characteristic frequency makes use of the sum rule (3.12). We define the "median frequency" $\bar{\omega}_Q(\mathbf{k})$ by the relation (Halperin and Hohenberg, 1969a)

$$\int_{-\bar{\omega}_Q(\mathbf{k})}^{\bar{\omega}_Q(\mathbf{k})} \frac{d\omega}{2\pi}\, C_Q(\mathbf{k}, \omega) \equiv \tfrac{1}{2} C_Q(\mathbf{k}). \tag{3.19}$$

If the correlation function $C_Q(\mathbf{k}, \omega)$ has a Lorentzian spectrum centered about $\omega=0$, the two definitions (3.16) and (3.19) are identical. For a spectrum having a sharp peak at a finite frequency (propagating mode) the definition (3.16) is of course unsuitable. Note, also, that definitions in terms of positive frequency moments of $C_Q(\mathbf{k}, \omega)$ are generally unsuitable, since they are sensitive to high frequencies rather than to the low-frequency, long-time behavior which is of interest for hydrodynamic or critical properties.

If the variable Q is conserved by the dynamics[4] (i.e., if Q is the density of a constant of the motion), then the kinetic coefficient is proportional to k^2 at small k (for $T \neq T_c$), and we may define a transport coefficient λ_Q by

$$\lambda_Q = \lim_{k\to 0} k^{-2} \Gamma_Q(\mathbf{k}). \tag{3.20}$$

B. Linear and nonlinear hydrodynamics

In the limit of long wavelengths, for $T \neq T_c$, it is generally possible to write down a set of coupled differential equations, known as *hydrodynamic equations*, which govern the time evolution of the conserved densities of the system (Landau and Lifshitz, 1959). [For a system with a broken continuous symmetry, such as a Heisenberg ferromagnet or antiferromagnet below T_c, it is necessary to include, among the hydrodynamic densities, variables which describe gradients of the direction of orientation of the order parameter (see Halperin and Hohenberg, 1969b)]. Although the hydrodynamic equations are based on assumptions which cannot be rigorously justified, such as the possibility of making expansions in powers of the gradient operator at long wavelengths, those equations are nonetheless believed to be exact in the long-wavelength limit. In general, the equations contain a number of temperature-dependent coefficients whose values are not determined from macroscopic considerations. By taking advantage of the known symmetries of the system, however, and of the Poisson-

[10] Note that Eq. (3.17) differs from Eq. (3.9b) of Halperin *et al.* (1974a), which was in error.

bracket relations among the conserved variables, it is possible to derive relations among various hydrodynamic coefficients, and in some cases to express hydrodynamic coefficients directly in terms of static quantities.

When the deviations of the conserved densities from their equilibrium values are sufficiently small, the hydrodynamic equations may be *linearized*,[11] and solved to give a finite number of hydrodynamic modes for the time dependence of the densities at a given wave vector k. The frequencies of the hydrodynamic modes must go to zero (by definition) in the limit $k \to 0$.

If $Q(\mathbf{x}, t)$ is a conserved density, then the linear response function $\chi_Q(\mathbf{k}, \omega)$, and by (3.13) the correlation function $C_Q(\mathbf{k}, \omega)$, are completely determined by the linearized hydrodynamic equations, for $k \to 0$. In particular, $\chi_Q(\mathbf{k}, \omega)$ will have simple poles at the (complex) frequencies of one or more of the hydrodynamic modes, with residues determined by the hydrodynamic equations (see Landau and Lifshitz, 1959; Kadanoff and Martin, 1963; Forster, 1975). The response function $\chi_Q(\mathbf{k}, \omega)$ for a *nonconserved* variable will generally also have poles at the hydrodynamic frequencies, but it will have additional structure elsewhere in the lower half of the complex frequency plane.

In a similar way, the equations of *nonlinear* hydrodynamics may impose conditions on various nonlinear response functions and higher-order correlation functions such as $\langle Q_\nu(\mathbf{x}, t) Q_{\nu'}(\mathbf{x}', t') Q_{\nu''}(\mathbf{x}'', t'')\rangle$. In constructing stochastic models for the dynamic critical behavior of various physical systems, we shall take care that nonlinear as well as linear response functions conform to the hydrodynamic behavior of the Hamiltonian we wish to represent.[11] Specific theorems that will enter our considerations are the Larmor precession theorem for a Heisenberg or planar magnetic system, the Josephson relation for the rate of change of the phase of the order parameter of a superfluid, and the convective terms in the transport equations in a normal fluid. These theorems generally follow from the *Poisson-bracket* relations and the symmetries of the system.

C. Critical behavior

1. Static properties

Near the critical point, one of the variables Q under consideration has the largest fluctuations, i.e., has a susceptibility with a strong divergence as $T \to T_c$. We call this variable the order parameter ψ, and its conjugate field h_ψ (see Stanley, 1971; Fisher, 1967). The *scaling* hypothesis for thermodynamics is the assumption that the leading singularity in the free energy $F(h_\psi, T)$ can be written in the form

$$F_{\text{sing}}/Vk_BT_c = A_0|\Delta T|^{2-\alpha} f(K_0 \Delta T/|h_\psi|^{1/\beta\delta}), \tag{3.21}$$

where $\Delta T \equiv T - T_c$, A_0 and K_0 are constants, V is the volume of the system, and α, β, and δ are the usual critical exponents. Similarly, one can make a scaling assumption for the static response function

$$\chi_\psi(\mathbf{k}) = Vk_BT_cA_0K_0^{-2\beta\delta}|\Delta T|^{-\gamma} X\left(\frac{k}{\kappa}, \frac{K_0\Delta T}{|h_\psi|^{1/\beta\delta}}\right), \tag{3.22}$$

where κ is the inverse of the correlation length[12]

$$\xi \equiv \kappa^{-1} = |\Delta T|^{-\nu} \Xi\left(\frac{K_0\Delta T}{|h_\psi|^{1/\beta\delta}}\right), \tag{3.23a}$$

with $\gamma = \beta(\delta - 1)$ and $\nu = (2 - \alpha)/d$. For $h_\psi = 0$, Eq. (3.23a) reduces to

$$\xi = \xi_0^{\pm}|\Delta T/T_c|^{-\nu} = (\kappa_0^{\pm})^{-1}|\Delta T/T_c|^{-\nu}, \tag{3.23b}$$

The + and − signs corresponding to T greater or less than T_c, respectively. In the limit $k = 0$ the response function (3.22) must reduce to the thermodynamic susceptibility $\chi_\psi = \partial^2 F/\partial h_\psi^2|_T$. At the critical point ($\Delta T = 0$, $h_\psi = 0$, $\kappa = 0$) and for finite k, it behaves as

$$\chi_\psi(\mathbf{k}) \propto k^{-2+\eta}, \tag{3.24}$$

with $2 - \eta = \gamma/\nu$.

According to the hypothesis of universality,[2] the exponents are the same for different members of an equivalence class. Moreover, the scaling functions $f(x)$, $X(y,x)$, and $\Xi(x)$ in (3.20)–(3.22) are also the same, once the scales of F, and h_ψ (i.e., the constants A_0 and K_0) have been properly adjusted. In particular, the hypothesis of "two-scale-factor universality" states that the scale of lengths $\xi_0^{\pm}$ in (3.23) is not independent of the thermodynamic scales A_0 and K_0 (Stauffer *et al.*, 1972; Hohenberg *et al.*, 1976a).

2. Dynamic properties

The *dynamic scaling* assumption (Ferrell *et al.*, 1967, 1968; Halperin and Hohenberg, 1967, 1969a) states that the response function $\chi_\psi(\mathbf{k}, \omega)$ has the form

$$\chi_\psi(\mathbf{k}, \omega) = \chi_\psi(\mathbf{k})\, Y\left(\frac{\omega}{\Omega_0\kappa^z}, \frac{k}{\kappa}, \frac{K_0\Delta T}{|h_\psi|^{1/\beta\delta}}\right), \tag{3.25}$$

where Ω_0 is a constant which sets the scale of time, and z is the dynamic critical exponent. [By definition the function Y is normalized so that $Y(0, y, x) = 1$.] From the expression for the characteristic frequency $\omega_\psi(\mathbf{k})$ in (3.16), Eq. (3.25) implies the form

$$\omega_\psi(\mathbf{k}) = \Omega_0 k^z \tilde{\Omega}\left(\frac{k}{\kappa}, \frac{K_0\Delta T}{|h_\psi|^{1/\beta\delta}}\right), \tag{3.26}$$

with a similar form for $\bar{\omega}_\psi$ defined in (3.19). Ordinarily one considers the case $h_\psi = 0$, where (3.26) becomes

$$\omega_\psi(\mathbf{k}) = \Omega_0 k^z \tilde{\Omega}^{\pm}(k/\kappa), \tag{3.27}$$

with different functions above and below T_c. It is assumed that *at* T_c the characteristic frequency is unique, i.e.,

$$\tilde{\Omega}^+(\infty) = \tilde{\Omega}^-(\infty), \tag{3.28}$$

[11] It should be emphasized that the term "linear hydrodynamics" does not imply that the underlying equations of motion are linear, or that fluctuations are small, either in the microscopic equations or in the stochastic models. It is only necessary that the induced variations on the macroscopic length scale be small. Thus linear hydrodynamics may be used in the calculation of linear response functions, provided that the wavelength is sufficiently long.

[12] We shall use the notations ξ and $\kappa^{-1} \equiv \xi$ interchangeably throughout this paper.

or more generally, that the function $\tilde{\Omega}(\infty, x)$ in (3.26) is independent of the ratio $x = K_0 \Delta T / |h_\psi|^{1/\beta\delta}$, for $\Delta T \to 0$, $h_\psi \to 0$. According to universality, the exponent z and the dimensionless functions $Y(w, y, x)$ and $\tilde{\Omega}(y, x)$ are the same for all members of an equivalence class. The constant Ω_0 in (3.26) and (3.27), on the other hand, is nonuniversal, and it may or may not bear a simple relation to thermodynamic parameters.

The dynamic scaling hypothesis can also be made for variables other than the order parameter.[13] In systems with many hydrodynamic variables there are typically some which do *not* obey dynamic scaling. For example, in the case of the symmetric binary fluid treated in Sec. II, one finds $z \approx 3$ for the order parameter, while the relaxation rate for the momentum density $\mathbf{j}$ at long wavelengths goes as $\bar{\eta} k^2 \approx \kappa^{2-x_{\bar{\eta}}} (k/\kappa)^2$ with $x_{\bar{\eta}} \approx 0$. Furthermore, it can be shown that this relaxation rate does not have the scaling form (3.27) for $k/\kappa \approx 1$ (Kadanoff and Swift, 1968a).

We shall see below that in systems such as superfluid helium and the Heisenberg ferro- and antiferromagnet, where ψ has a *propagating* mode in the ordered state, the frequency $\omega_\psi(\mathbf{k})$ can be expressed purely in terms of static quantities at long wavelengths. In that case dynamic scaling is sufficient to relate the exponent z to static exponents. In other cases, such as anisotropic magnets, or the gas-liquid and consolute critical points, $\chi_\psi(\mathbf{k}, \omega)$ is dissipative and the dynamic scaling hypothesis does not determine z. In either case, dynamic scaling leads to *relations* between exponents characterizing the behavior in various regimes (e.g., $k \ll \kappa$ and $k \gg \kappa$). Let us note, finally, that the conventional theory is compatible with dynamic scaling, with $z = 2 - \eta$ for a nonconserved order parameter, and $z = 4 - \eta$ in the conserved case [see Eqs. (3.16) and (3.20)].

D. The dynamic universality classes

In the following sections we shall describe the main dynamic universality classes which have been studied to date, and some of the physical systems belonging to these classes. We shall explain the renormalization group in the context of the simplest systems, those with purely relaxational dynamics, since the fixed-point mechanism for dynamic scaling and universality can be adequately illustrated in that case. We have listed the universality classes discussed below, and some of their defining properties, in Table I.

IV. RENORMALIZATION GROUP FOR RELAXATIONAL MODELS

A. System with no conservation laws: Model A

We shall first study a set of models with dissipative equations of motion, which are often referred to as time-dependent Ginzburg–Landau models, although an equation of motion of this type seems to have been first employed by Landau and Khalatnikov (1954) in order to explain the anomalous attenuation of sound in helium near the λ-point. These models are purely relaxational, and they may be described by a Hermitian master equation.[14]

1. The model

The most elementary of these models is a system with no conservation laws, which we shall call *model A*. It is defined by the Markoffian equations of motion

$$\frac{\partial \psi_\alpha(\mathbf{x}, t)}{\partial t} = -\Gamma_0 \frac{\delta F_0}{\delta \psi_\alpha(\mathbf{x}, t)} + \theta_\alpha(\mathbf{x}, t) + \Gamma_0 h_\alpha(\mathbf{x}, t), \tag{4.1a}$$

$$F_0 = \int d^d x \left\{ \tfrac{1}{2} r_0 \psi^2 + \tfrac{1}{2} |\nabla \psi|^2 + u_0 \psi^4 \right\}, \tag{4.1b}$$

$$\psi^2 \equiv \sum_{\alpha=1}^{n} \psi_\alpha^2; \quad \psi^4 \equiv (\psi^2)^2 \tag{4.1c}$$

$$|\nabla \psi|^2 \equiv \sum_{\alpha=1}^{n} (\nabla \psi_\alpha)^2, \tag{4.1d}$$

where $\psi_\alpha(\mathbf{x}, t)$ is an n-component real order parameter depending on space and time, and it is assumed that $\psi_\alpha(\mathbf{x}, t)$ contains only variations with wave vector smaller than a specified cutoff Λ. The function $\theta_\alpha(\mathbf{x}, t)$ is a Gaussian white noise source with correlations

$$\langle \theta_\alpha \rangle = 0, \tag{4.1e}$$

$$\langle \theta_\alpha(\mathbf{x}, t) \theta_{\alpha'}(\mathbf{x}', t') \rangle = 2\Gamma_0 \delta(\mathbf{x} - \mathbf{x}') \delta(t - t') \delta_{\alpha\alpha'}, \tag{4.1f}$$

and the functions $h_\alpha(\mathbf{x}, t)$ are arbitrary external fields (we shall choose units such that $k_B T_c = 1$). Correlation and response functions for ψ_α may be obtained formally by solving Eqs. (4.1a, b) for ψ_α as a functional of the $\{\theta_\alpha\}$ and then averaging over all values of $\{\theta_\alpha\}$ according to Eqs. (4.1e–f). The first term in (4.1a) causes ψ to relax towards a configuration which minimizes the functional F_0, while the noise θ_α ensures that the proper equilibrium distribution is maintained, and that the fluctuation-dissipation theorem is satisfied (see Glauber, 1963). As in any Markoffian system, one can alternatively describe the model by a master equation for the time evolution of the probability distribution $P[\psi_\alpha; t]$ (Van Kampen, 1965; Lax, 1960). The master equation takes the form of a Fokker–Planck equation (or more properly a Smolukowski equation):

$$\frac{\partial P}{\partial t} = \Gamma_0 \sum_{\alpha=1}^{n} \int d^d x \frac{\delta}{\delta \psi_\alpha} \left[\frac{\delta P}{\delta \psi_\alpha} + P \frac{\delta F_0}{\delta \psi_\alpha} \right]. \tag{4.2}$$

For time-independent fields $\{h_\alpha\}$

$$P_{\text{eq}}[\psi_\alpha] = Z^{-1} \exp(-F_0[\psi_\alpha]), \tag{4.3}$$

$$Z = \int D\{\psi_\alpha\} \exp(-F_0[\psi_\alpha]), \tag{4.4}$$

where $\int D\{\psi_\alpha\}$ denotes a functional integral over the variations in ψ. The existence of the equilibrium solu-

[13] In the early formulation of dynamic scaling by the present authors (Halperin and Hohenberg, 1969a) the assumption (3.25) for the order parameter was referred to as *restricted* scaling, while the generalization to other variables was called *extended* scaling.

[14] The transport and kinetic coefficients will never diverge in such models, as was shown by Kawasaki (1966b). See also Kadanoff and Swift (1968b), Kawasaki (1972), and Halperin (1973b).

tion (4.3), whose free energy has the usual Ginzburg–Landau–Wilson form,[1] depends crucially on the fact that the same constant Γ_0 appears in (4.1a) and (4.1f). The model has a critical point for $h_\psi = 0$, and r_0 equal to a critical value $r_{0c}(u_0)$. For $r_0 > r_{0c}$, the model is in the "disordered phase" ($T > T_c$), while for $r_0 < r_{0c}$ the model is in the "ordered phase" ($T < T_c$). This model has no conserved variables, and thus does not have any hydrodynamic modes, in the sense of Sec. III.B.

If the coupling constant u_0 in (4.1b) is equal to zero, then the free-energy functional is quadratic in ψ, and in this Gaussian case it is easy to show that

$$\chi_\psi(\mathbf{k}, \omega) = \chi_0(\mathbf{k}, \omega) = (-i\omega/\Gamma_0 + r_0 + k^2)^{-1} = \chi_\psi(k)[1 - i\omega/\omega_\psi(k)]^{-1}, \tag{4.5}$$

$$\omega_\psi(k) = \Gamma_0(r_0 + k^2)^{-1} = \Gamma_0 \chi_\psi^{-1}(k); \tag{4.6}$$

i.e., the kinetic coefficient $\Gamma_\psi(\mathbf{k})$ is equal to a constant, Γ_0. In the interacting case ($u_0 \neq 0$), it is no longer possible to calculate $\chi_\psi(\mathbf{k}, \omega)$ exactly, since even the equilibrium problem is insoluble. If we define the kinetic coefficient $\Gamma_\psi(\mathbf{k})$ according to (3.15), we may still expect, at least for $r_0 > r_{0c}$, that $\Gamma_\psi(\mathbf{k})$ will be finite in the limit $k \to 0$, but the value of

$$\Gamma \equiv \Gamma_\psi(\mathbf{k} = 0) \tag{4.7}$$

will depend on u_0, Λ, and r_0, as well as on Γ_0. The relaxation rate $\omega_\psi(\mathbf{k} = 0) = \Gamma/\chi_\psi$ is also finite for $r_0 > r_{0c}$, but the spectrum of $\chi_\psi(\mathbf{k}, \omega)$ will no longer consist of a single pole at $\omega = -i\omega_\psi(\mathbf{k})$.

2. Perturbation theory

As we saw above, the dynamics of model A is trivial for $u_0 = 0$, since the conventional theory holds (with $\eta = 0$). Thus the interesting dynamic effects arise from the interaction u_0, as is also true for the static behavior (Wilson, 1972). We must therefore have a systematic formal procedure for expanding the response functions of model A in powers of u_0. There exist a number of equivalent formalisms for carrying out such expansions in terms of diagrams, and we refer the reader to the literature for details.[15] A typical formalism is given by Ma (1976a), where diagrams are constructed with four-point vertices u_0, and propagators represented by lines. The value assigned to a line is either $\chi_0(\mathbf{k}, \omega)$, Eq. (4.5), or $\omega^{-1}\mathrm{Im}\chi_0(\mathbf{k}, \omega)$, according to specified rules. The "internal" lines are integrated over frequency, and over the d-dimensional wave vector, up to the cutoff Λ. The effect of u_0 on the response function may be expressed in terms of a self-energy Σ given by

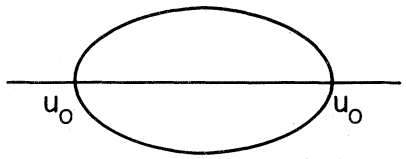

FIG. 1. Self-energy correction of order u_0^2. The lines represent propagation of an order parameter mode.

$$\chi_\psi^{-1}(\mathbf{k}, \omega) = \chi_0^{-1}(\mathbf{k}, \omega) + \Sigma(\mathbf{k}, \omega). \tag{4.8}$$

The kinetic coefficient Γ is obtained from (4.8) via the definition (3.15).

The leading contribution to the renormalization of Γ comes from the self-energy diagram indicated in Fig. 1. This contribution may be described in mode-coupling terms as arising from the nonlinear interaction of three order parameter fluctuations. The sign of the effect may be understood from a simple physical argument: The hysteresis loss in response to a slow time-varying field at wave vector $\mathbf{k}$ is proportional to ω^2 divided by the characteristic relaxation rate $\omega(\mathbf{k})$. Since our model is purely dissipative, coupling between the modes will tend to increase losses at low frequencies. This in turn can be interpreted as a decrease in $\omega(\mathbf{k})$, and therefore a decrease in the kinetic coefficient Γ.

After integration over internal frequencies, the contribution of this diagram is given by the expression

$$\Sigma(\mathbf{k}, \omega) = -\frac{3}{2}(n+2)u_0^2 \int_0^\Lambda \frac{d^d q}{(2\pi)^d} \int_0^\Lambda \frac{d^d p}{(2\pi)^d} \frac{1}{(r+p^2)(r+q^2)[r+(\mathbf{p}+\mathbf{q}-\mathbf{k})^2]} \left\{ \frac{\Gamma_0[3r + p^2 + q^2 + (\mathbf{p}+\mathbf{q}-\mathbf{k})^2]}{-i\omega + \Gamma_0[3r + p^2 + q^2 + (\mathbf{p}+\mathbf{q}-\mathbf{k})^2]} \right\}. \tag{4.9}$$

[Note that we have rearranged the perturbation theory by carrying out a "mass renormalization" (Wilson 1972), which replaces the parameter r_0 by its dressed value r, which vanishes at $T = T_c$, or $r_0 = r_{0c}$]. The kinetic coefficient may then be calculated for $k = 0$, $\omega = 0$, and one finds a contribution to Γ^{-1} proportional to $u_0^2 r^{d-4}$, which diverges at T_c for $d < 4$. Since higher-order diagrams will involve even more strongly divergent powers of r, it is not possible to take (4.9) at face value, and some other means must be found to extract sensible results for the model near T_c.

It was precisely in order to solve an analogous problem in the static case that the renormalization group methods[1] were developed, and it turns out that similar techniques apply here. The essential idea is to carry out integrals such as (4.9) step by step, over a range which excludes the origin $\mathbf{q} = 0$ at each step, and thereby toavoid the occurrence of divergent integrals at intermediate steps in the calculations, when $T = T_c$.

3. Recursion relations near $d = 4$

The renormalization group transformation we employ[16] is a generalization of the one used by Wilson and Kogut

[15] See, for instance, Tucker and Halperin (1971), Halperin *et al.* (1974a, 1976a), Ma (1976a), De Dominicis (1975), Patashinskii and Pokrovskii (1977), Suzuki and Tanaka (1974a), Suzuki (1973a), Kawasaki (1974), and Martin *et al.* (1973).

[16] Besides the exact renormalization group transformation defined here, there exist a number of alternative formulations (either exact or approximate), which have been applied to dynamics. See for instance De Dominicis *et al.* (1975), Suzuki and Tanaka (1974b), Kawasaki (1974, 1975), Kuramoto (1974), and Abrahams and Tsuneto (1975).

(1974) for static phenomena. It is defined by

$$R_b = R_b^s R_b^i , \tag{4.10}$$

where R_b^s is a simple change of scale

$$x \to x' = x/b , \tag{4.11a}$$

$$\Lambda \to \Lambda' = \Lambda b , \tag{4.11b}$$

$$\psi \to \psi' = b^a \psi , \tag{4.11c}$$

$$\omega \to \omega' = b^z \omega . \tag{4.11d}$$

Here b is an arbitrary constant greater than unity, and a and z are exponents to be determined. The operation R_b^i, which may be applied to the diagrammatic expansion of any quantity, consists of an integration over "internal" wave vectors in the domain $b^{-1}\Lambda < p < \Lambda$, and internal frequencies from $-\infty$ to $+\infty$.[17]

The effect of R_b^s on various terms in the equations of motion (4.1a) and the free-energy functional (4.1b) may be found by a simple "power-counting" procedure (Wilson and Kogut 1974; Ma 1976a; Halperin *et al.* 1974a, 1976a). For example, it follows from (4.1) and (4.11) that

$$R_b^s(r_0) = b^{d-2a} r_0 , \tag{4.12}$$

$$R_b^s(u_0) = b^{d-4a} u_0 , \tag{4.13}$$

$$R_b^s(\Gamma_0^{-1}) = b^{d-2a-z} \Gamma_0^{-1} . \tag{4.14}$$

The action of R_b^i is to add to the right-hand sides of Eqs. (4.12)–(4.14) contributions from all the appropriate diagrams with intermediate wave vectors restricted to the shell $b^{-1}\Lambda < p < \Lambda$. It is clear from the definitions given above that the operations R_b^s and R_b^i commute.

If the renormalization group transformation is iterated l times, the diagrammatic representation of the equations of motion will be given in terms of partially renormalized propagators and vertices, which are in principle much more complicated than the ones occurring in the starting equations (4.1). Nonetheless, we may identify *renormalized* constants r_l, u_l, and Γ_l by taking limits of the corresponding vertices as the external frequencies and wave vectors go to zero. These limits are well behaved, since the intermediate integrations were limited to a finite shell, excluding $\mathbf{p} = 0$. The renormalized constants r_l and Γ_l are defined by

$$r_l = \lim_{\substack{k \to 0 \\ \omega \to 0}} (R_b^s)^l [\chi_0^{-1}(k, \omega) + \Sigma_l(\mathbf{k}, \omega)] , \tag{4.15}$$

$$\Gamma_l^{-1} = \lim_{\substack{k \to 0 \\ \omega \to 0}} \frac{\partial}{\partial(-i\omega)} \{ (R_b^s)^l [\chi_0^{-1}(k, \omega) + \Sigma_l(\mathbf{k}, \omega)] \} , \tag{4.16}$$

where $\Sigma_l(\mathbf{k}, \omega)$ represents the contributions to $\Sigma(\mathbf{k}, \omega)$ coming from diagrams in which internal wave vectors are in the shell $\Lambda/b^l < p < \Lambda$. A definition similar to (4.16) may be given for u_l, in terms of the renormalized four-point vertex with external frequencies and wave vectors equal to zero (Halperin *et al.*, 1976a).

The above procedure leads to *recursion relations* for the quantities r_l, u_l, and Γ_l, which may be written down explicitly in *perturbation theory* in u_l. The expansion turns out to be meaningful for d near four, i.e., for $0 < \epsilon = 4 - d \ll 1$. The lowest-order contribution to Γ_l^{-1}, which comes from the self-energy diagram in Fig. 1 and Eq. (4.9), is proportional to $u_l^2 \Gamma_l^{-1} \ln b$, in four dimensions and for $r_l \ll 1$. [As mentioned earlier, this contribution is *finite*, because of the finite range of integration $\Lambda/b < p < \Lambda$.] The recursion relation for Γ_l^{-1} is thus

$$\Gamma_{l+1}^{-1} = b^{d-2a-z} \Gamma_l^{-1} [1 + c_1 u_l^2 \ln b] , \tag{4.17}$$

where

$$c_1 = 48(n+2) \ln(4/3) K_4^2 , \tag{4.18}$$

and

$$K_d \equiv 2^{1-d} \pi^{-d/2} \Gamma(d/2) \tag{4.19}$$

is $(2\pi)^{-d}$ times the surface of the unit sphere in d-dimensions $[K_4 = (8\pi^2)^{-1}]$. A similar procedure yields the well-known Wilson–Fisher (1972) recursion relations for r_l and u_l

$$r_{l+1} = b^{d-2a} \{ r_l + 2(n+2) K_4 u_l [\Lambda^2 (1 - b^{-2}) - 2 r_l \ln b] \} , \tag{4.20}$$

$$u_{l+1} = b^{d-4a} [u_l - 4(n+8) K_4 u_l^2 \ln b] , \tag{4.21}$$

where we have assumed $r_l \ll 1$.

In order to calculate the critical exponents to lowest significant order in ϵ, one determines the *fixed point* of the recursion relations (4.17)–(4.21). The exponent a is determined by requiring that the coefficient of the term proportional to $|\nabla\psi|^2$ in (4.1b) remain unchanged, which yields (Wilson and Kogut, 1974)

$$a = \tfrac{1}{2}(d - 2 + \eta) , \tag{4.22}$$

where η is the usual critical exponent for the correlation function at T_c. Equations (4.20)–(4.21) then go to a finite fixed point

$$u^* = \epsilon [4 K_4 (n+8)]^{-1} + O(\epsilon^2) , \tag{4.23a}$$

$$r^* = -\tfrac{1}{2} \epsilon (n+2) \Lambda^2 / (n+8) + O(\epsilon^2) , \tag{4.23b}$$

since η is of order ϵ^2 (Wilson, 1972). The fact that r^* is of order ϵ justifies the approximation $r_l \ll 1$ made in deriving Eqs. (4.20) and (4.21) (see Wilson and Kogut, 1974).

The recursion relation (4.17) for Γ_l will reach a finite nonzero fixed point Γ^* if the exponent z is given by

$$z = 2 - \eta + c_1 (u^*)^2 . \tag{4.24a}$$

Inserting u^* from (4.23a) and using the ϵ expansion of η (Wilson, 1972), we find (Halperin *et al.*, 1972)

$$z \equiv 2 + c\eta = 2 + [6 \ln(\tfrac{4}{3}) - 1] \eta + O(\epsilon^3) . \tag{4.24b}$$

By expanding the recursion relations (4.20) and (4.21) about the fixed point (4.23) it is readily seen that the fixed point is unstable to a perturbation of the form $r_l = r^* + \delta r_l$. On the other hand, the fixed point is stable with respect to small perturbations of the form $u_l = u^* + \delta u_l$, provided that r_l is adjusted slightly, in the form $r_l = r^*(1 + \delta u_l / u^*)$. The linearized recursion relations are said to have one *relevant* (unstable) eigenvalue, and one *irrelevant* (stable) eigenvalue, which, loosely speak-

[17] We have not found it necessary to introduce a cutoff in frequencies, since the propagators decay sufficiently rapidly for large $|\omega|$.

ing, we may identify with the variables $r_l - r^*$, and $u_l - u^*$, respectively. More generally, the fixed point (4.23) is reached from any positive starting value of u_0, provided r_0 lies on a critical curve

$$r_0 = r_{0c}(u_0). \tag{4.25}$$

When $r_0 > r_{0c}(u_0)$, the model corresponds to $T > T_c$, while for $r_0 < r_{0c}(u_0)$, the model corresponds to $T < T_c$.

It may be seen from Eqs. (4.21) and (4.22) that the rate of approach of u_l to its fixed point is slow for small ϵ; i.e., for large l we may linearize (4.21) to find,

$$(d/dl)(u_l - u^*) = -\epsilon(u_l - u^*)\ln b. \tag{4.26}$$

Thus the irrelevant variable u_0 leads to a *slow transient* u_l, near four dimensions, whose existence has important consequences for the expansion methods discussed in Sec. IV.A.5 below. At precisely $d = 4$ ($\epsilon = 0$), the linearized equation (4.26) yields no information, and one must go to quadratic terms in $u_l - u^*$. In that case u_l is referred to as a *marginal* variable.[1]

4. Scaling and universality: Stability of the fixed point to all orders in ϵ

The recursion relations (4.17)–(4.21) may be used to derive the scaling forms of the response functions discussed in Sec. III, for both statics and dynamics. Let us first discuss the temperature dependence of the kinetic coefficient as $T \to T_c^+$. According to the definition of R_b^s, we may write the physical coefficient Γ as

$$\frac{1}{\Gamma} = (R_b^s)^{-l}\left[\frac{1}{\Gamma_l} + \frac{1}{\Gamma_l'}\right] = b^{-(z-2+\eta)l}\left[\frac{1}{\Gamma_l} + \frac{1}{\Gamma_l'}\right], \tag{4.27}$$

where $1/\Gamma_l'$ represents the contribution to $[\partial\Sigma(0,\omega)/\partial(-i\omega)]|_{\omega=0}$ coming from intermediate wave vectors in the range $0 < p < \Lambda/b^l$. When the operation R_b is iterated many times, i.e., when l grows large, the quantity Γ_l approaches very close to its fixed-point value Γ^*, provided that $|r_0 - r_{0c}|$ is sufficiently small. Moreover, if we choose l such that $r_l \approx 1$, which occurs when

$$b^l \approx \Lambda/\kappa, \tag{4.28}$$

then both $1/\Gamma_l'$ and $\Gamma_l - \Gamma^*$ are finite and of order ϵ^2. Taking into account Eqs. (4.27) and (4.28), we see then that for large values of Λ/κ

$$\Gamma = (\Lambda/\kappa)^{z-2+\eta}\Gamma^*[1 + O(\epsilon^2)], \tag{4.29}$$

where the term of order ϵ^2 is independent of (Λ/κ) and is actually a universal constant for systems approaching the model A fixed point. This result may be combined with Eq. (3.16) (for $\mathbf{k} = 0$) to see that z is indeed the dynamic exponent defined in (3.27).

In order to obtain the scaling form (3.25) for finite k and ω, we repeat the above argument for the response function, which we write as

$$\chi_\psi^{-1}(\mathbf{k},\omega) = (R_b^s)^{-l}[(-i\omega/\Gamma_l) + r_l + k^2 + \Sigma_l'(\mathbf{k},\omega)], \tag{4.30}$$

$$\chi_\psi^{-1}(\mathbf{k},\omega) = b^{(-2+\eta)l}[(-i\omega b^{zl}/\Gamma_l) + r_l + k^2 b^{2l} + \Sigma_l'(\mathbf{k},\omega)], \tag{4.31}$$

where Σ_l' represents the contributions to the self-energy coming from intermediate wave vectors in the range $0 < p < \Lambda/b^l$, as well as contributions which are of higher order in ϵ than those retained in the recursion relations. The function $\Sigma_l'(\mathbf{k},\omega)$ may have singularities when either k or ω go to zero, but provided the renormalization group is well behaved, Σ_l' is *finite and of order* ϵ *or smaller*, when

$$\max\left[\frac{r_l}{\Lambda^2}, \left(\frac{k}{\Lambda}\right)b^l, \frac{\omega b^{zl}}{\Gamma_l\Lambda^2}\right] \approx 1. \tag{4.32}$$

If, for $r_0 \geq r_{0c}$ and finite k and ω, we choose l such that

$$b^{-l} \approx \max[\kappa/\Lambda, k/\Lambda, (\omega/\Gamma_0\Lambda^2)^{1/z}], \tag{4.33}$$

then Eq. (4.32) is satisfied, and Eq. (4.31) takes on the scaling form (3.25), (for $h_\psi = 0$),

$$\chi_\psi(\mathbf{k},\omega) = \chi_\psi(\mathbf{k})\, Y\left(\frac{\omega}{\Omega_0\kappa^z}, \frac{k}{\kappa}\right), \tag{4.34}$$

$$\chi_\psi(\mathbf{k}) = \chi_\psi X\left(\frac{k}{\kappa}\right). \tag{4.35}$$

The scaling functions (4.34) and (4.35) and the critical exponents are properties of the fixed point reached, and do not depend on the initial parameters r_0, u_0, Γ_0, and Λ. Thus these critical properties are *universal* for all systems which go to the same fixed point.

In order to ascertain that the renormalization group is well behaved, it is necessary to verify that the fixed point found in the lowest-order analysis is *stable* with respect to two types of perturbations. The first type involves the higher-order terms generated by the perturbation expansion in u_0, e.g., terms of order $u_0^2\omega^2$ in Σ_l of (4.15). The second class of perturbations are variations of the starting model, to approximate more closely a realistic physical system. Of course the two kinds of perturbations are closely related, and we shall represent both types by terms in the equations of motion involving regular powers of k, ω, and ψ, of arbitrarily high order. The essential feature which makes these terms *irrelevant* perturbations is that, under the operation R_b^s, the coefficients are reduced at each step by an amount b^{-x}, where x is at least $2 - O(\epsilon)$. Thus for d sufficiently close to 4, these are irrelevant perturbations, which decay rapidly, to values entirely determined by the slow transients and the relevant variables (Wilson and Kogut, 1974; Wilson, 1975; Halperin *et al.*, 1976a).

For concreteness, let us study an example in which we add to the inverse bare propagator $\chi_0^{-1}(k,\omega)$ a term $-M_0\omega^2$, representing a second derivative with respect to time in the equations of motion. We may define a coefficient M_l in terms of the second derivative with respect to ω of $\chi_0^{-1} + \Sigma_l$, in analogy to (4.16), and derive a recursion relation describing the action of R_b,

$$M_{l+1} = b^{d-2a-2z}M_l + m_l(1 - b^{-2}), \tag{4.36}$$

where the second term represents the contribution from $\Sigma_{l+1} - \Sigma_l$. Note that m_l is proportional to u_l^2, and so must be of order ϵ^2 or smaller, for large l. According to Eqs. (4.22) and (4.24) the exponent in (4.36) is equal to $-2 - (2c+1)\eta$ which is close to -2 for small ϵ. Moreover, for large l, m_l will be essentially independent of l. When R_b is iterated many times M_l can be written approximately in the form

$$M_l = (1 - b^{-2})(m_{l-1} + b^{-2}m_{l-2} + b^{-4}m_{l-3} + \cdots) \approx m_l , \quad (4.37)$$

which is of order ϵ^2, and *independent of the initial value* M_0. In fact, m_l, and therefore M_l, will be entirely determined by the fixed-point parameters r^*, u^*, and Γ^*, considered earlier. Furthermore, the nonzero value of M_l will only affect the recursion relation for Γ_l to order ϵ^4.

Arguments similar to the above for the stability of the fixed point under various perturbations have been elaborated in much more detail in Sec. 5 of Wilson and Kogut (1974), and Secs. II and V of Wilson (1975). These discussions are at the heart of the justification of the renormalization group, and of the derivation of scaling and universality near $d = 4$, *to all orders in* ϵ.

In addition to more complicated terms in the propagator $\chi_\psi(\mathbf{k}, \omega)$, the intermediate stages of the renormalization group will also contain k- and ω-dependent vertices of fourth and higher order in ψ. The fixed point will be stable with respect to these perturbations, for the same reason as discussed above. In particular, it was argued by Halperin *et al.* (1976a) that the frequency dependence of the four-point vertex $[u_4(\mathbf{k}_i, \omega_i)]_l$ was irrelevant, and that the static variable u_l was therefore also sufficient for a calculation of the dynamics. The argument depends crucially on the *regularity* of $[u_4(\mathbf{k}_i, \omega_i)]_l$ for small ω_i.

We may note, finally, that the frequency-dependent vertices and complicated propagators are reflected in non-Gaussian and frequency-dependent noise correlations, determined by fluctuation–dissipation theorems. It is clear that these features, which arise in intermediate stages of the renormalization group, could also have been introduced in the starting equations, and that the simple form of the model chosen in (4.1) is not a restriction on the applicability of the results.

5. Epsilon and 1/*n* expansions in higher orders

The recursion relations discussed in the previous sections are useful for illustrating the essentials of the fixed-point mechanism, and for analyzing the important properties of a given universality class. If, however, one wishes to carry out explicit calculations beyond the lowest significant orders, the recursion relations become quite cumbersome, and alternate methods have been developed which are more convenient. Most of these are procedures for extracting the critical exponents from ordinary perturbation theory, generally in the form of expansions in a small parameter. The earliest such method was that of Larkin and Khmel'nitzkii (1969), who studied static critical behavior in the "marginal" cases of short-range forces at $d = 4$, or dipolar-Ising systems at $d = 3$. Subsequently, Wilson (1972) developed his *Feynman-graph expansion* method for calculating the exponents in powers of ϵ. When truncated at the second order and extrapolated to $d = 3$, these expressions were in surprisingly good agreement with the best available series values. A similar expansion, in powers of $1/n$, valid when the number of components n of the order parameter is large, can be applied for $2 < d < 4$, and complements the ϵ expansion (see Ma, 1973, and references therein). More recently, a variety of field-theoretic methods based on the Callan–Symanzik equations have been devised, which permit calculations to rather high orders in the various expansions (see Brézin *et al.*, 1976). All of the above methods had been developed for the static case, but they may readily be generalized to dynamics (De Dominicis *et al.*, 1975; Murata, 1976a; Bausch *et al.*, 1976).

The Feynman-graph expansion for exponents determines the coefficients of various powers of ϵ by *matching* logarithmic terms in the perturbation theory expression for $\Sigma(\mathbf{k}, \omega)$ to the expected scaling behavior. It is important to note (Wilson, 1972) that this matching will only be correct if the slow transients in the approach to the fixed point have been eliminated. In the case of model A near four dimensions, the effect of the slow transient u_l can be eliminated from the perturbation theory by choosing the bare coupling constant u_0 to have a special value, $u_0 = u_0(\epsilon)$. As mentioned above, it turns out that u_l is the only slow transient for *both* the statics and and the dynamics, so the ϵ expansion for dynamics can also be carried out with u_0 fixed at its static value $u_0(\epsilon)$ (Halperin *et al.*, 1972).

The results for the dynamic exponent z in model A can be written in the form

$$z = 2 + c\eta , \quad (4.38)$$

where in general c is a function of d and n. Near $d = 4$, calculations yield (Halperin *et al.*, 1972; DeDominicis *et al.*, 1975)

$$c\eta = 0.7261(1 - 1.687\epsilon)\eta + O(\epsilon^4) , \quad (4.39)$$

which shows that c is independent of n, to the order which is presently available. For large n and $2 < d < 4$, the result to order $1/n$ is (Halperin *et al.*, 1972)

$$c = \left(\frac{4d}{4-d}\right)\left\{\frac{B(\frac{1}{2}d - 1, \frac{1}{2}d - 1)}{8\int_0^{1/2} dx[x(2-x)]^{d/2-2}} - 1\right\} . \quad (4.40)$$

This expression agrees with (4.39), in the limit $d \to 4$, and gives $c = \frac{1}{2}$ at $d = 3, n \to \infty$.

The properties of model A below T_c have been studied near $d = 4$ by Mazenko (1976).

It is also possible to make expansions about two dimensions for $n > 2$, since T_c vanishes for $d = 2$. De Dominicis *et al.* (1977) have calculated z to lowest order in $\epsilon' \equiv d - 2$, and they find a result of the form (4.38), with

$$c = (1 - \ln\tfrac{4}{3})\epsilon' , \quad (4.41)$$

$$\eta = \frac{\epsilon'}{n-2}\left[1 + O\left(\frac{\epsilon'}{n-2}\right)\right] . \quad (4.42)$$

It is interesting to note that in this domain c is also independent of n, and it agrees with the large-n expression extracted from (4.40) for $d \to 2^+$.

Values of z were also obtained for the two-dimensional kinetic Ising model (Glauber, 1963) from high-temperature series (Yahata and Suzuki, 1969; Yahata, 1971) and from Monte-Carlo calculations (Schneider *et al.*, 1972; Stoll *et al.*, 1973), and the results were consistent with the value $c = 0$, which would be obtained from (4.40) and (4.41), by assuming c to be independent of n down to $n = 1$. Recently, however, Rácz and Collins (1976) reanalyzed the series expansions using Padé approximants, and have concluded that $c \approx 0.5$ for the two-dimensional kinetic Ising model, suggesting a possible n-dependence for c.

We may note, finally, that in the case of model A, the dynamic exponent bears no simple relation to static exponents. This is in contrast to certain models with conservation laws which will be considered below.

6. Related models

It is known from the static theory[1] that various modifications of the free-energy functional F_0 will change the fixed point and the corresponding critical behavior. Examples are the introduction of cubic anisotropy for $n>3$, of dipolar forces, or of "long-range" spin interactions, which decay as $x^{-(d+\sigma)}$, with $0<\sigma<2$ (see Fisher, 1974). If one assumes that the dynamics remains purely relaxational, then the form of $\chi_\psi(\mathbf{k},\omega)$ will be similar to the short-range case considered above, but the values of the exponents and scaling functions will in general be modified. The critical dynamics of model A in the presence of long-range forces was discussed by Suzuki and Igarashi (1973), while the effects of cubic anisotropy were discussed by Yamazaki (1976a), and dipolar forces for $n=d=4-\epsilon$ were considered by Teitelbaum (1975).

It is well known that the addition of dipolar forces to an Ising-like ($n=1$) system has strong effects on the critical properties (Larkin and Khmel'nitzkii, 1969). The static critical behavior of the $d=3$ dipolar-Ising model is similar to that of the $d=4$ short-ranged model, and is mean-field-like, except for the occurrence of various fractional powers of $\ln(T-T_c)$ or $\ln k$ (see Fisher, 1974, and references therein). The critical dynamics of model A for the dipolar-Ising case at $d=3$, or for the short-ranged case at $d=4$, are similarly mean-field-like, as has been discussed by Siggia (1975) (see also, Müller and Merz, 1976).

The critical behavior of the s-state Potts model with relaxational dynamics has been studied by Trimper (1976).

B. Conserved order parameter: Model B

There exists a simple modification of model A, in which the order parameter is conserved (Kawasaki, 1966a). This model, which we call *model B*, is also defined by Eqs. (4.1), but with Γ_0 replaced by $-\lambda_0\nabla^2$. For $k\to 0$ and $r_0\neq r_{0c}$, the order parameter relaxation rate has the long-wavelength form

$$\omega_\psi(\mathbf{k})=(\lambda/\chi_\psi)k^2\,, \tag{4.43}$$

where the transport coefficient λ is in principle a function of the parameters λ_0, u_0, Λ, and r_0. The propagator for model B may be written in the form (4.8), with the bare propagator

$$\chi_0^{-1}(k,\omega)=-i\omega/\lambda_0 k^2+r_0+k^2\,. \tag{4.44}$$

By analyzing the perturbation expansion of $\chi_\psi(\mathbf{k},\omega)$ for $r_0\neq r_{0c}$, it may be shown that although $\Sigma(\mathbf{k},\omega)$ is frequency dependent, the quantity $\partial\Sigma/\partial(i\omega)$ remains finite for $k\to 0$, $\omega\to 0$. It follows that Σ does not contribute a term of order ω/k^2 to $\chi_\psi^{-1}(\mathbf{k},\omega)$, so that the exact transport coefficient λ is equal to its bare value λ_0 and the conventional theory holds in this case (Halperin *et al.*, 1972, 1974a, 1976a).

Alternatively, we may write down a recursion relation for λ_l of the form

$$\lambda_{l+1}^{-1}=b^{d-2a-z+2}\,\lambda_l^{-1}\,, \tag{4.45}$$

with no contribution from R_b^i in any order in ϵ. Using (4.22) we see that Eq. (4.45) will reach a finite nonzero fixed point λ^* if and only if

$$z=4-\eta\,, \tag{4.46}$$

as required by the conventional theory. The stability of the model-B fixed point to perturbations which *preserve* the conservation law may be studied in the same way as was discussed in Sec. IV.A.4.

It is interesting to note the instability of the fixed point to a perturbation which *violates* the conservation law, e.g., if the first term in the inverse bare propagator (4.44) has the form $-i\omega/(\lambda_0 k^2+\Gamma_0)$, with Γ_0 small.

$$\Gamma_{l+1}=b^{2a-d+z}\Gamma_l[1+O(u_l^2)]\,. \tag{4.47}$$

The exponent of b will be close to 2, if z is equal to $4-\eta$. Thus the perturbation Γ_l grows rapidly, and one is driven far away from the model-B fixed point ($\Gamma_l\equiv 0$), no matter how small Γ_0 was to start with. When Γ_l becomes much larger than $\lambda_l\Lambda^2$ one crosses over to the model-A fixed point, and it is more convenient to choose $z\approx 2$ so that Γ^* becomes finite. In that case Eq. (4.45) leads to a vanishing fixed-point value for λ_l, and λ_0 becomes irrelevant.

The fact that the dressed transport coefficient λ is precisely equal to its temperature-independent bare value λ_0, is actually an artifact of the class of models we have considered. A more general model, consistent with the requirements of a conserved order parameter, is obtained by assuming that the bare coefficient λ_0 is itself a function of ψ, such as

$$\lambda_0(\mathbf{x})=\lambda_{00}+\lambda_{02}\psi^2(\mathbf{x})\,. \tag{4.48}$$

It may be argued that this modification will lead to a transport coefficient of the form

$$\lambda=\lambda_{00}+\lambda_{02}\langle\psi(\mathbf{x})^2\rangle\,, \tag{4.49}$$

which implies a singularity in the *temperature derivative* of λ, proportional to the specific heat singularity $(T-T_c)^{-\alpha}$. Thus we see that the perturbation $\lambda_{02}\psi^2$ is irrelevant for the asymptotic fixed-point behavior of λ (which remains finite as in model B), but that the perturbation is important for predicting the first correction to this asymptotic behavior.

C. Coupling to an auxiliary conserved density: Model C

1. The model

In testing the stability of the fixed point in model A, we saw that any coupling terms in the free-energy functional involving high powers of $\psi(\mathbf{x})$ or high powers of gradients were irrelevant. Another possibility is to couple $\psi(\mathbf{x})$ to an auxiliary density, which itself varies slowly (Kadanoff and Swift, 1968b). We may introduce a conserved density $m(\mathbf{x},t)$ which couples to $|\psi|^2$ in the free-energy functional (4.1b), just as the "temperature" r_0. This density can represent the energy, or alternatively the concentration of a set of mobile impurities. The simplest such system (model C) is defined by the equations (Halperin *et al.*,

1974a)

$$\frac{\partial\psi_\alpha}{\partial t} = -\Gamma_0\left(\frac{\delta F_0}{\delta\psi_\alpha} - h_\alpha\right) + \theta_\alpha , \tag{4.50a}$$

$$\frac{\partial m}{\partial t} = \lambda_0^m\nabla^2\left(\frac{\delta F_0}{\delta m} + \delta\beta\right) + \zeta , \tag{4.50b}$$

$$\langle\zeta\rangle = 0 , \tag{4.50c}$$

$$\langle\zeta(\mathbf{x},t)\zeta(\mathbf{x}',t')\rangle = -2\lambda_0^m\nabla^2\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') , \tag{4.50d}$$

$$F_0 = \int d^dx\{\tfrac{1}{2}\tilde{r}_0\psi^2(\mathbf{x}) + \tilde{u}_0\psi^4(\mathbf{x}) + \tfrac{1}{2}|\nabla\psi(\mathbf{x})|^2 + \gamma_0\psi^2(\mathbf{x})m(\mathbf{x}) + \tfrac{1}{2}C_0^{-1}m^2(\mathbf{x})\} + \text{const.} + \text{const.}M , \tag{4.50e}$$

where M is the space integral of m, $\delta\beta(\mathbf{x},t)$ is an external field coupled to m, and the correlations of the Langevin noise source $\theta_\alpha(\mathbf{x},t)$ are still given by (4.1e) and (4.1f).

2. Static properties

The equilibrium distribution for the above model is given by a joint probability density ($k_BT_c = 1$)

$$P_{\text{eq}}[\psi, m] = Z^{-1}\exp(-F_0) , \tag{4.51}$$

where

$$Z = \int D\{\psi_\alpha\}\, D\{m\}\exp(-F_0) \tag{4.52}$$

is a functional integral over all $\psi_\alpha(\mathbf{x})$ and $m(\mathbf{x})$, whose variations have wave vectors less than Λ (see Halperin *et al.*, 1974a for details). Since the functional F_0 in (4.50e) is quadratic in m, we may integrate (4.52) over m to obtain a probability density for ψ alone. This density has the same form as in model A, Eq. (4.1b), with parameters

$$r_0 = \tilde{r}_0 - \text{const.} , \tag{4.53}$$

$$u_0 = \tilde{u}_0 - \tfrac{1}{2}\gamma_0^2C_0 . \tag{4.54}$$

In order to study the static renormalization group for model C, we may define a perturbation expansion in terms of the dimensionless four-point vertex u_0 in Eq. (4.54), and a dimensionless three-point vertex

$$v_0 \equiv K_d\gamma_0^2C_0 . \tag{4.55}$$

The recursion relations for u_l and r_l have the same form as Eqs. (4.20) and (4.21), and additional relations for C_l and v_l may be written in the form

$$C_{l+1}^{-1} = b^{d-2a_m}C_l^{-1}[1 - 2nv_l\ln b] , \tag{4.56}$$

$$v_{l+1} = b^{d-4a}v_l[1 - 8(n+2)u_lK_4\ln b - 2nv_l\ln b] , \tag{4.57}$$

where the exponent a_m is defined, in analogy with (4.11c), by

$$m' = b^{a_m}m . \tag{4.58}$$

In order for the functional F_l to reach a finite fixed point we must have

$$a_m = \tfrac{1}{2}(d - \tilde{\alpha}/\nu) , \tag{4.59}$$

where

$$\tilde{\alpha} \equiv \max(\alpha, 0) , \tag{4.60}$$

and α is the exponent for the *singular part* of the susceptibility χ_m, which is the specific heat if m is the energy density. From (4.59) and the recursion relations (4.56) and (4.57) it may be shown that

$$v^* = \tilde{\alpha}/2n\nu + O(\epsilon^2) , \tag{4.61}$$

with

$$\alpha/\nu = (4-n)\epsilon/(n+8) + O(\epsilon^2) . \tag{4.62}$$

Equation (4.61) implies that the effective coupling of the densities ψ and m vanishes asymptotically for $\alpha < 0$.

3. Dynamic properties

To study the dynamics we find the recursion relations for the variables λ_l^m and Γ_l,

$$(\lambda_{l+1}^m)^{-1} = b^{2-z+\tilde{\alpha}/\nu}(\lambda_l^m)^{-1} , \tag{4.63}$$

$$\Gamma_{l+1}^{-1} = b^{2-z}\Gamma_l^{-1}[1 + v_lA_l] , \tag{4.64}$$

$$A_l \equiv 4K_d^{-1}\int_{\Lambda/b}^{\Lambda}\frac{d^dp}{(2\pi)^d}\{(p^2+r_l)[p^2 + r_l + (\lambda_l^m/C_l\Gamma_l)p^2]\}^{-1} . \tag{4.65}$$

Since according to Eq. (4.61) v_l is of order ϵ, the fixed-point condition for Eq. (4.64) is different from (4.17) in that order. Evaluating A_l from Eq. (4.65) we find to lowest order in ϵ

$$A_l = 4(1+\mu_l)^{-1}\ln b \tag{4.66}$$

where

$$\mu_l = \lambda_l^m/\Gamma_lC_l \tag{4.67}$$

satisfies the recursion relation

$$\mu_{l+1} = \mu_l\left[1 + \left(\frac{\tilde{\alpha}}{\nu}\ln b\right)\left(\frac{2}{n(1+\mu_l)} - 1\right)\right] , \tag{4.68}$$

to first order in ϵ. Equation (4.68) has three possible fixed points,

$$\mu^{*(1)} = \infty , \tag{4.69}$$

$$\mu^{*(2)} = \frac{2}{n} - 1 , \tag{4.70}$$

and

$$\mu^{*(3)} = 0 . \tag{4.71}$$

a. The case n = 1

Let us first consider the case $n = 1$. Analysis of Eq. (4.68) shows that the fixed point $\mu^{*(2)} = 1$ is the *stable* one, from which it follows by Eqs. (4.64) and (4.61) that Γ_l will reach a finite nonzero fixed point if and only if we choose

$$z = 2 + \alpha/\nu . \tag{4.72}$$

(note that $\alpha > 0$ near $d = 4$ for $n = 1$). Comparing Eqs. (4.72) and (4.24), we see that the coupling of ψ to the conserved density m has changed the value of z in linear order in ϵ.

In fact, it may be shown that Eq. (4.72) is an *exact scaling relation*, which holds to all orders in ϵ for $n = 1$. It expresses the dynamic exponent z purely in terms of static exponents. The essential point in proving this scaling law is to verify that μ^* remains finite and non-

zero when higher-order terms are considered. For sufficiently small ϵ, this has been done (Halperin *et al.*, 1976a) by verifying that the recursion relations are regular near $\mu^* = 1$.

Associated with the scaling relation (4.72), there is a universal amplitude ratio, defined as a ratio of characteristic frequencies for ψ and m,

$$\mu \equiv \frac{\lambda_m \kappa^2 \chi_\psi}{\Gamma \chi_m} = \lim_{k \to 0} \frac{\omega_m(\mathbf{k})}{\omega_\psi(\mathbf{k})} \frac{\kappa^2}{k^2} . \tag{4.73}$$

To lowest order in ϵ, μ is just equal to $\mu^{*(2)} = 1$, and in the next order μ has been obtained from the Feynman-graph expansion (Halperin *et al.*, 1976a) as

$$\mu = 1 + \epsilon(9 \ln 2 - \tfrac{14}{3} \ln 3 - \tfrac{11}{18}) . \tag{4.74}$$

The exact scaling relation (4.72), which follows from the finite fixed-point value for the dimensionless constant μ_l and leads to the physical amplitude ratio (4.74), is the prototype for all the dynamic scaling relations to be discussed below.

Let us briefly consider the other fixed point values of μ_l, Eqs. (4.70) and (4.71) for the case $n = 1$. The fixed point $\mu^{*(1)} = \infty$ corresponds to model A, or to a situation in which the characteristic frequency for relaxation of m is large compared to that of ψ. In that case m effectively follows the fluctuations of ψ instantaneously. This fixed point is of course reached for $n = 1$ if one starts out with $\mu_0 \equiv \infty$ (or $\lambda_0^m \equiv \infty$), but it is unstable if one begins with any finite μ_0, no matter how large. In the latter case there will be a crossover from model A behavior to the model C fixed point.

The fixed point $\mu^{*(3)} = 0$ is also unstable for $n = 1$, as long as $\lambda_0^m > 0$ (or $\mu_0 > 0$), but it has significance if μ_0 is sufficiently small. Unfortunately, when $\mu_l \approx 0$, the validity of the recursion relation (4.68) is subject to considerable doubt. As discussed by Halperin *et al.* (1976a), there is a contribution to the renormalized four-point vertex which is a singular function of the frequency transfer ω, in the limit $\omega \to 0$, $\mu_l = 0$. A generalization of the recursion relations designed to keep track of this singular contribution does not lead to a well-behaved fixed point, so that the true critical behavior in the limit $\mu_0 \to 0$ is not yet known.

The limit $\mu_0 \ll 1$ will not occur when m is the energy density, since the thermal diffusion rate over a distance ξ is generally not small compared to the relaxation rate of the order parameter, in physical situations. A very small value of μ_0 can occur, however, when $m(\mathbf{x}, t)$ is a density of mobile impurities, whose diffusion rate may be arbitrarily slow. In order to ensure that we begin from a situation of thermal equilibrium, as is assumed in model C, we must require that the specimen be annealed at the measuring temperature T for a time long compared to the impurity diffusion time $\xi^2 \chi_m / \lambda_m$, before the order parameter relaxation is measured.

The dynamics of relaxational models with randomly distributed immobile impurities, representing a crystal in which the impurity positions are frozen in upon quenching from a temperature high compared to T, has been studied by Grinstein *et al.* (1976), Krey (1976, 1977a, 1977b), and Yamazaki (1976b, 1976c). In contrast to the annealed impurity case considered above, the dynamic renormalization group for the quenched models is found to reach a well-behaved fixed point whenever the static fixed point is well behaved. In particular, for impurities which couple to $|\psi|^2$, the critical exponents are different from those of the pure system, for values of n and d for which $\alpha > 0$ in the pure system.

b. The case $n > 1$

For any point in the $(n - d)$ plane where the specific heat exponent α is negative, the coupling constant v_l tends to zero for large l [Eq. (4.61)], and the conserved energy field has no effect on the asymptotic critical behavior. [The region $\alpha < 0$ occurs for $n > n_c(d)$, where $n_c = 4 - 4\epsilon + O(\epsilon^2)$ for $\epsilon \to 0$, $n_c(3) \approx 1.8$, and $n_c(2) = 2$; see Fisher (1974).] The critical behavior of model C is the same as that of model A in this case, and $\mu_l \to \infty$ for large l. In the domain $2 < n < 4, \epsilon \to 0$, the recursion relation (4.68) implies that the stable fixed point is $\mu^{*(3)} = 0$. Because of the difficulties with these recursion relations in the limit $\mu_l = 0$, however, it is not clear whether this result is in fact correct. Another possibility which cannot be ruled out is that the stable fixed point has a finite but very small value of μ^* in this region. A more complete discussion of the possible fixed points of model C, and their domains of stability, may be found in Halperin *et al.* (1974a, 1976b), Brézin and De Dominicis (1975), and Murata (1976a). The theoretical problems in the renormalization group when $\mu_l \to 0$ will arise again in the dynamics near the tricritical point of ^{3}He–^{4}He mixtures treated in Sec. VI.F, below.

D. Applications to physical systems

1. Structural phase transitions

a. Simple Hamiltonian model

Consider a system described by the classical *Hamiltonian*

$$H = \sum_i \left(\frac{p_i^2}{2M_0} + \tfrac{1}{2} r_0 \psi_i^2 + U_0 \psi_i^4 \right) + \frac{1}{4} \sum_{ij} J_{ij} (\psi_i - \psi_j)^2 , \tag{4.75}$$

where the subscripts i and j refer to points on a d-dimensional, simple cubic lattice with lattice constant unity, ψ_i is a scalar quantity describing the displacement of an atom in the i-th unit cell, and $p_i = M_0 d\psi_i/dt$ is the momentum conjugate to ψ_i.[18]

If r_0 is negative, while J_{ij} and U_0 are positive, then the ground state of H has a uniform distortion with $\langle \psi_i \rangle \neq 0$. Let us define

[18] When (4.75) is applied to a real structural transition, ψ may represent the displacement of an optical phonon mode at the center of the Brillouin zone, or it may be interpreted as a staggered displacement, corresponding to a soft phonon at the edge of the zone. Note, however, that if the zone center mode carries an electric dipole moment (ferroelectric case), the interaction J_{ij} will contain a long-range dipolar contribution, which will modify the static and dynamic critical behavior. For an Ising-like system with dipolar forces in three dimensions the dynamic critical behavior of model C will be mean-field-like, with logarithmic corrections (Siggia, 1975; see also Stauffer, 1977, and Müller and Merz, 1976).

$$J_q = \sum_j J_{ij} e^{i\mathbf{q}\cdot(\mathbf{x}_i - \mathbf{x}_j)}, \tag{4.76}$$

where $\mathbf{x}_i$ and $\mathbf{x}_j$ are the positions of the lattice sites i and j. If $|r_0|/J_0 \gg 1$, then the value of $|\psi_i|$ will be approximately $|r_0/4U_0|^{1/2}$ on each lattice site, both above and below the transition temperature. The partition function is essentially that of an Ising model, and the transition may be described as an *orientational* or an *order–disorder transition*. On the other hand, in the case $|r_0|/J_0 \ll 1$, one has a *displacive transition*, in which the phonon system is only weakly anharmonic. In general, the transition temperature T_c may be written as

$$k_B T_c = \text{const.}|r_0|\, J_0/U_0, \tag{4.77a}$$

where the constant of proportionality is a function of $|r_0|/J_0$, but remains finite in both the displacive and orientational limits (see Thomas, 1971; Halperin and Varma, 1976, and references therein). The quartic coupling constant u_0 of the previous sections may be identified as

$$u_0 = \text{const.}U_0 k_B T_c/J_0^2 = \text{const.}|r_0|/J_0, \tag{4.77b}$$

which is small in the displacive limit.

The only conserved density for the Hamiltonian (4.75) is the energy. According to the universality hypothesis, we would therefore expect the system to exhibit the critical dynamics of model C.[19] It is instructive to see how this may come about, i.e., to sketch a "derivation" of model C from a microscopic starting point.

If we assume that the interaction u_0 is small, a formal diagrammatic expansion of the response function $\chi_\psi(\mathbf{k}, \omega)$ in powers of u_0 can be developed for the Hamiltonian model, in much the same manner as for the simple relaxational case (model A) discussed above. The important difference is that the bare propagator $\chi_0(\mathbf{k}, \omega)$ is now taken to be

$$\chi_0^{-1}(\mathbf{k}, \omega) = -M_0\omega^2 + r_0 + J_0 - J_\mathbf{k}, \tag{4.78a}$$

$$= -M_0\omega^2 + r_0 + J_0'' k^2 + O(k^4), \tag{4.78b}$$

which has poles at the bare phonon frequencies $\pm\Omega_\mathbf{k} = \pm M_0^{-1/2}(r_0 + J_0 - J_\mathbf{k})^{1/2}$. The first step in the renormalization group approach is to eliminate fluctuations in the range $\Lambda_1 < p < \Lambda_0$, where $\Lambda_1 = \frac{1}{10}\Lambda_0$ say, and $\Lambda_0 = \pi$ is the maximum wave vector in the Brillouin zone. This leads to a partially dressed propagator (before rescaling by R_b^s) of the approximate form

$$\chi_1^{-1}(\mathbf{k}, \omega) = -M_1\omega^2 - i\omega/\Gamma_1 + r_1 + J_1'' k^2 + O(k^4) + O(\omega^3), \tag{4.79}$$

where r_1 is shifted from r_0 by a small amount of order $k_B T U_0/J_0^2 \approx u_0$; the coefficients M_1 and J_1'' are shifted from M_0 and J_0'' by a smaller amount (of order u_0^2), and $(1/\Gamma_1)$ is the phonon damping constant resulting from interactions with phonons of wave vector greater than Λ_1. If the term of order k^4 in Eq. (4.79) is sufficiently *positive*, so that the phonon spectrum curves upwards for $\Lambda_1 < p < 2\Lambda_1$, then $1/\Gamma_1$ may be calculated from the lowest-order diagram in Fig. 1, with the result[20]

$$1/\Gamma_1 = \text{const.}U_0^2 (k_B T)^2 M_0^{1/2} J_0^{-7/2}. \tag{4.80}$$

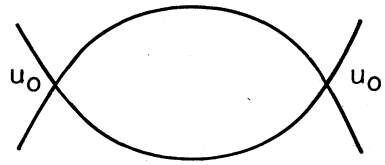

FIG. 2. The lowest correction to the four-point vertex. The lines represent propagation of an order parameter mode.

If one were to neglect the effect of this first renormalization operation on the four-point vertex U_1, then one would have at this stage a model with interaction vertices of the same form as model A, and with a propagator of the form (4.79), containing a small but finite phonon damping. Energy no longer is conserved for this model, and the universality hypothesis now predicts that the model should show the critical behavior of model A itself. Indeed, we may note that the rescaling operation R_b^s increases the importance of $-i\omega/\Gamma_1$ relative to $-M_1\omega^2$, and we expect that after many iterations of the renormalization procedure the phonon propagator will become overdamped, and the term $-M_1\omega^2$ may be neglected (see Sec. IV.A.4, and Murata, 1976a).

The change in propagator from (4.78) to (4.79) is, however not the only effect of the elimination of fluctuations with $p > \Lambda_1$. An important new feature arises from the contribution of Fig. 2 to the renormalized four-point vertex U_1. As a consequence of the absence of damping in the bare propagator (4.78), this diagram gives a singular function of the wave vector transfer $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$ and frequency transfer $\omega = \omega_1 + \omega_2$, in the limit k and $\omega \to 0$. If one is able to sum an appropriate series of diagrams, equivalent to a solution of the Boltzman equation for phonons with $p > \Lambda_1$, then the singularity in U_1 should take the form of a simple pole, at $\omega = -iD_1^m k^2$ (see Kwok and Martin, 1966). The quantity D_1^m is the thermal diffusion constant for the short wavelength phonon system, and is proportional to the phonon lifetime, and the square of the phonon velocity

$$D_1^m = \text{const.}J_0'' \Gamma_1, \tag{4.81}$$

$$U_1(\mathbf{k}, \omega) \underset{\omega\to 0}{\overset{k\to 0}{\simeq}} U_0 + \text{const.}\frac{U_0^2 k_B T_c}{J_0^2}\left[\frac{D_1^m k^2}{-i\omega + D_1^m k^2}\right]. \tag{4.82}$$

It may readily be seen that the rescaling operation R_b^s does not reduce the singular part of U_1, for d near 4; therefore one must keep track of the residue and position of the pole in U_1. These two parameters obviously correspond to the parameters v_1 and μ_1 in model C, and

[19] Note that there can be no reversible couplings between ψ and the energy density m because both densities are even under time reversal. The Poisson bracket of m and ψ need not be considered in this case.

[20] In a simple cubic lattice with nearest-neighbor forces only, the phonon spectrum curves downwards, and the contribution of Fig. 1 is found to vanish, because it is impossible to satisfy wave vector and frequency conservation in this process. In most physical systems, however, where the phonon spectrum has several branches, there will be a contribution to $1/\Gamma$, of order $U_0^2 T^2$ as in (4.80). There may also be a larger contribution (order T) in systems where three-phonon scattering processes are important.

indeed the introduction of a separate conserved energy field was essentially a device to keep track of the singular part of the retarded four-point interactions among the long-wavelength components of ψ. Although the calculations have not been carried out in complete detail, the preceding argument thus makes it plausible that application of the renormalization group to the microscopic phonon model (4.75) should carry it toward the fixed point of model C.

We may remark incidentally that the asymptotic critical behavior for the model in three dimensions does not seem to be very different, qualitatively, from the predictions of a time-dependent mean-field theory

$$\chi_{\psi}^{-1}(\mathbf{k},\omega) = -M_0\omega^2 - i\omega/\Gamma + r + J_0 - J_k\,, \tag{4.83}$$

where

$$r = r_0 + 12U_0\langle\psi_i^2\rangle = r_0 + 12U_0 k_B T/J_0 + \text{const.}(T - T_c)\,, \tag{4.84}$$

and Γ^{-1} is a phenomenological damping constant, which is generally taken to be independent of k and T. [See, for example, Halperin and Varma (1976).]

If the displacements ψ_i and momenta p_i in (4.75) are defined to be n-component *vectors*, the hydrodynamics and dynamic critical behavior of the model will be qualitatively different from the scalar case. The Hamiltonian in this case has a new set of conserved quantities, the angular momenta $L_{\alpha\beta} = \sum_i \psi_i^\alpha p_i^\beta - p_i^\alpha \psi_i^\beta$. The Poisson-bracket relations between $L_{\alpha\beta}$ and ψ_i are important here, and one expects that the system will behave very much like the planar ferromagnet or Heisenberg antiferromagnet systems, discussed in Secs. VI and VII below. (See Sasvári *et al.*, 1975; Janssen, 1976; Szépfalusy, 1976; Sasvári and Szépfalusy, 1977a, and 1977b.) On the other hand, if we add to the Hamiltonian anisotropic terms such as $\sum_\alpha \sum_i (\psi_i^\alpha)^4$, which will always be present for a real structural transition, angular momentum is no longer conserved, and we expect to find relaxational critical dynamics close to T_c, similar to that of model C, for $n > 1$.

Note added in proof. Bausch and Halperin (1977) have applied renormalization group methods to the critical dynamics of a Hamiltonian model for an antiferroelectric transition, in which the dominant contribution to the damping of the scalar order-parameter mode comes from cubic interactions with additional phonon branches. As expected, the model shows the same critical dynamics as model C, and results are in line with the discussion of the present section.

b. Experiments on structural phase transitions

Inelastic neutron scattering measurements of the phonon spectrum near displacive phase transitions in a variety of crystals have revealed a feature which is in striking contrast to the predictions of the previous paragraphs. The correlation function $C_\psi(\mathbf{k},\omega)$ is dominated by a very narrow *central peak* whose relative intensity diverges for $k \to 0$ and $T \to T_c^+$. This feature appears in addition to the damped or overdamped soft-phonon mode, whose frequency and intensity are found to be finite at T_c (see Riste, 1974; Müller and Merz, 1976). Indirect measurements of the central peak width in $SrTiO_3$ above T_c via electron-spin resonance (Müller *et al.*, 1974) and sound attenuation techniques (Rehwald, 1970), as well as direct measurements by inelastic neutron and γ-ray scattering (Töpler *et al.*, 1975; Darlington and O'Connor, 1975) suggest that the central peak is at least several orders of magnitude smaller than the typical phonon widths.

It has been pointed out that the qualitative behavior of the central peak can be described by a model in which a small concentration of *impurities* is responsible for pinning of local "domains" of the order parameter, on a time scale long compared to the fluctuation lifetimes in a pure material for $T > T_c$ (Shirane and Axe, 1973; Folk and Schwabl, 1974; Halperin and Varma, 1976; Schmidt and Schwabl, 1977.) Recent neutron measurements in $SrTiO_3$ have shown the strength of the central peak to be sample-dependent for temperatures $\gtrsim T_c + 10$ K, which clearly supports an extrinsic origin for the central peak, at least in that temperature regime (Currat *et al.*, 1977). Closer to T_c, however, the central peak strength was found to be sample independent and it is not clear how this relates to the impurity explanation.

Temperature-dependent central peaks below T_c have been observed recently in a number of materials, using light-scattering techniques. It appears that the central peak may have several temperature-dependent components in various situations, and that part of the central peak may be elastic while other parts are inelastic. See, for example, Fleury and Lyons, 1976, Lockwood *et al.*, 1977, Mermelstein and Cummins, 1977, Durrasula and Gammon, 1977, Lyons and Fleury, 1977, Courtens, 1976.

There have been a number of theoretical attempts to obtain a narrow central peak for a Hamiltonian similar to (4.75), without the introduction of impurities, in contradiction to the expectations of the previous subsection. (See the references cited in Halperin and Varma, 1976). In our opinion, these attempts have not been successful in predicting a central peak narrow enough to be identified with the one observed experimentally. Therefore, if one hopes to avoid an explanation in terms of impurities, one must probably consider some other model with two very different time scales built into the starting Hamiltonian.

2. Magnetic phase transitions

a. Hamiltonian examples

As a Hamiltonian example of a magnetic system, let us consider a three-dimensional anisotropic Heisenberg ferromagnet or antiferromagnet, in which the coupling constants for the three components of the spin are all different. In this system there is no conservation law other than conservation of energy, and the only low-frequency mode at long wavelengths, for $T \neq T_c$, is the thermal diffusion mode. Since the spin has a single easy direction, the order parameter is characterized by $n = 1$, and the static exponents are expected to be the same as those of the Ising model. Similarly we expect that the dynamic critical properties should be the same as those of the model with energy conservation, i.e. model C with $n = 1$.

In the uniaxial Heisenberg *ferro*magnet, the couplings of the x and y components of the spin are equal, but

smaller than the coupling of the z component. The order parameter is the z component of the spin, which is now conserved by the Hamiltonian. We expect that the dynamic critical behavior should be the same as that of case B,[21] provided dipolar interactions can be neglected.

In the uniaxial *anti*ferromagnet, the order parameter is the z component of the staggered magnetization which is not conserved. In addition to the energy, there is now a second conserved quantity, the z component of the total magnetization. However, under repeated application of the renormalization group, the (nonlinear) coupling of the order parameter and the energy to long-wavelength fluctuations of the z component of the magnetization becomes vanishingly small, and this variable will not affect the critical dynamics. Thus the uniaxial antiferromagnet should have the same critical behavior as model C.

In a real magnetic system coupling to the phonons or other degrees of freedom will also contribute to the time dependence of the spins. Since the thermal conductivity of the phonon system is often very high compared to that of the spins in a magnetic insulator, model A without energy conservation may be a better description of the magnetic system than model C. On the other hand, if the rate of energy transfer between the spin and phonon systems is very slow compared to the spin exchange frequencies, it may be a good approximation to consider the spins as a thermally isolated system with conservation of energy. Of course, in many cases the physical system will be intermediate between the cases described, and the dynamic critical behavior may reflect a crossover between the different regimes.

b. Experimental consequences in magnetic systems

Generally speaking, the relaxational models are rather well approximated by the conventional theory, since the deviations of the critical exponents from their conventional values are numerically small. Thus, in order to observe these deviations in magnetic systems, extremely accurate measurements would be required. There is some evidence in the NMR linewidth measurements of Gottlieb and Heller (1971) on the uniaxial antiferromagnet FeF_2, for a critical exponent z which is slightly larger than $2-\eta$, but the deviation from the conventional theory is by no means conclusive. Further discussion of these results is given by Halperin *et al.* (1974a).

Siggia and Nelson (1977) have noted that at a magnetic *tricritical* point, the kinetic coefficient should vanish much more strongly than at the ordinary critical point, provided the system can be described by relaxational dynamics with a conserved energy field, as in model C (see Sec. VIII.D.3, below). There may thus be a better chance of seeing the effects of energy conservation experimentally in the tricritical case.

[21] More precisely this case corresponds to *model D* of Halperin *et al.* (1974a, 1976a), in which both ψ and m are conserved. The order parameter relaxation obeys the conventional theory in that case, just as in model B.

3. Order-disorder transitions in alloys

A number of alloys such as β-brass (CuZn), Fe_3Al, and Ni_3Mn undergo an order-disorder transition, involving rearrangement of the atomic constituents. A given atomic species, which has an equal probability of lying on either of two equivalent sublattices in the disordered phase, tends to order on one of the two sublattices, for $T<T_c$. It is known that the equilibrium properties of these systems are well described by an antiferromagnetic Ising model (see Fisher, 1967), and the critical dynamics might be expected to be that of model A. (The energy diffusion rate is very large compared to the relaxation rate for the order parameter here.)

The order parameter relaxation in Ni_3Mn was studied by Collins and Teh (1973) by monitoring the relaxation of the appropriate Bragg peak intensity in neutron scattering after changing the temperature from one value to another, in the range $T<T_c$. Although the results seemed to differ significantly from the predictions of model A, the discrepancies may be due to nonlinear effects of the relatively large temperature changes ΔT (Rácz, 1976; see also Sec. VIII.B below). Another complication in the interpretation of these experiments arises because the mechanism for order parameter relaxation involves "catalysis" by a small density of mobile vacancies (Rácz and Collins, 1975).

V. GAS-LIQUID AND BINARY-FLUID CRITICAL POINTS

A. Model H

We shall first study the gas-liquid critical point of a pure fluid, and generalize to a binary mixture in Sec. V.E below. A pure fluid has four hydrodynamic modes, a soundwave, thermal diffusion, and two viscous diffusion modes (Landau and Lifshitz, 1959). For k of order ξ^{-1} the sound modes are at a significantly higher frequency than the diffusive modes near T_c. Therefore, if one considers fluctuations at frequencies ω small compared to ck and $c\xi^{-1}$, it is reasonable to employ a model in which sound waves are ignored, and fluctuations are considered to occur at constant pressure. Such a model (*model H*) is defined by the equations (Kawasaki, 1970; Halperin *et al.*, 1974b; Siggia *et al.*, 1976)

$$\frac{\partial\psi}{\partial t}=\lambda_0\nabla^2\frac{\delta F}{\delta\psi}-g_0\nabla\psi\cdot\frac{\delta F}{\delta\mathbf{j}}+\theta\,, \tag{5.1a}$$

$$\frac{\partial\mathbf{j}}{\partial t}=\mathcal{T}\cdot\left[\bar{\eta}_0\nabla^2\frac{\delta F}{\delta\mathbf{j}}+g_0(\nabla\psi)\frac{\delta F}{\delta\psi}+\zeta\right], \tag{5.1b}$$

$$F=F_0-\int d^dx\{h(\mathbf{x},t)\psi+\mathbf{A}(\mathbf{x},t)\cdot\mathbf{j}\}\,, \tag{5.1c}$$

$$F_0=\int d^dx\left\{\tfrac{1}{2}r_0\psi^2+\tfrac{1}{2}(\nabla\psi)^2+u_0\psi^4+\tfrac{1}{2}\,\mathbf{j}^2\right\}, \tag{5.1d}$$

where h and $\mathbf{A}$ are infinitesimal applied fields, $\mathcal{T}$ is a projection operator which selects the transverse part of the vector in brackets ($\mathcal{T}_{\mathbf{k}}^{\alpha\beta}=\delta_{\alpha\beta}-k_\alpha k_\beta/k^2$), and θ and ζ are appropriate Langevin noise sources. The order parameter ψ represents the linear combination

$$q(\mathbf{x},t)\equiv e(\mathbf{x},t)-(\bar{\mu}+T\bar{s})\,\rho(\mathbf{x},t)\,, \tag{5.2}$$

of the energy density $e(\mathbf{x},t)$ and the mass density $\rho(\mathbf{x},t)$ [$\bar{\mu}$ and $\bar{s}$ are the equilibrium chemical potential and entropy (per unit mass), respectively]. The quantity $\mathbf{j}(\mathbf{x},t)$ is the transverse part of the momentum density, and the constant g_0 is equal to unity in the usual units.[22]

The coupling terms in the equations of motion (5.1a) and (5.1b) are suggested by the Poisson-bracket relation (2.23), which follows from the fact that the momentum is the generator of translations in the system. Specifically, the term proportional to g_0 in (5.1a) corresponds to the convective term in the heat transport equations, and is necessary to obtain the correct nonlinear hydrodynamics. The g_0 term in the Navier-Stokes equation (5.1b) is then required so that the system will relax to the proper equilibrium distribution in the absence of external forces. This term corresponds to a contribution proportional to $\psi\nabla^2\psi$ in the stress tensor.[23]

At long wavelengths, for $T \neq T_c$, the linear response functions for ψ and $\mathbf{j}$ contain simple poles at frequencies

$$\omega_\psi(\mathbf{k}) = -i\lambda k^2/\chi_\psi \,, \tag{5.3}$$

and

$$\omega_j(\mathbf{k}) = -i\bar{\eta}k^2 \,, \tag{5.4}$$

respectively, where the transport coefficients λ and $\bar{\eta}$ are the "dressed" values of the thermal conductivity and shear viscosity in a pure fluid.

B. Mode-coupling treatment

The heuristic arguments presented in Sec. II may be applied to this model, and they suggest that the product $\lambda\bar{\eta}$ will diverge at T_c. A quantitative calculation which embodies these ideas was first performed by Kadanoff and Swift (1968a), and subsequently refined and generalized by Kawasaki (1969, 1970). The Kawasaki theory involves a self-consistent evaluation of the response functions $\chi_\psi(\mathbf{k},\omega)$ and $\chi_j(\mathbf{k},\omega)$ based on a first-order expansion of (5.1) in g_0. Although this was an uncontrolled approximation (the effective vertex is not small in three dimensions), the self-consistency condition ensures that dynamic scaling is preserved, and the results agree remarkably well with experiment (see below). The approximation, which is illustrated schematically in Fig. 3, may be written in the form

$$\chi_\psi^{-1}(\mathbf{k},\omega) = [-i\omega/\lambda(\mathbf{k})k^2] + \chi_\psi^{-1}(\mathbf{k}) \,, \tag{5.5}$$

$$\chi_j^{-1}(\mathbf{k},\omega) = [-i\omega/\bar{\eta}(\mathbf{k})k^2] + 1 \,, \tag{5.6}$$

with[24]

$$\lambda(\mathbf{k}) = \lambda_0 + g_0^2 k^{-2} \int \frac{d^d p}{(2\pi)^d}\, \chi_\psi(\mathbf{p}_+)\, \frac{(\mathbf{k}\cdot\mathcal{T}_\mathbf{p}\cdot\mathbf{k})}{\bar{\eta}(\mathbf{p}_-)p_-^2 + \lambda(\mathbf{p}_+)\chi_\psi^{-1}(\mathbf{p}_+)p_+^2} \tag{5.7}$$

$$\bar{\eta}(\mathbf{k}) = \bar{\eta}_0 + \frac{g_0^2 k^{-2}}{d-1} \int \frac{d^d p}{(2\pi)^d}\, \frac{\chi_\psi(\mathbf{p}_-)[\chi_\psi^{-1}(\mathbf{p}_+) - \chi_\psi^{-1}(\mathbf{p}_-)](\mathbf{p}\cdot\mathcal{T}_\mathbf{k}\cdot\mathbf{p})}{\lambda(\mathbf{p}_+)\chi_\psi^{-1}(\mathbf{p}_+)p_+^2 + \lambda(\mathbf{p}_-)\chi_\psi^{-1}(\mathbf{p}_-)p_-^2} \,, \tag{5.8}$$

$$\mathbf{p}_\pm = \mathbf{p} \pm \tfrac{1}{2}\mathbf{k} \,. \tag{5.9}$$

Note that in writing Eqs. (5.5) and (5.6) we have made a Lorentzian approximation for the frequency dependence of the response functions, i.e., we have evaluated the contributions of the diagrams in Fig. 3, in the limit of low frequency (see Kawasaki 1970, 1976; Siggia *et al.*, 1976). Since $\chi_\psi(\mathbf{p})$ is strongly divergent we may neglect the $\lambda(\mathbf{p}_+)$ term in the denominator of (5.7). Inserting a scaling form (3.21) for $\chi_\psi(\mathbf{p})$, and neglecting λ_0 and $\bar{\eta}_0$, it is apparent that Eqs. (5.7) and (5.8) have solutions in which $\lambda(\mathbf{k})$ and $\bar{\eta}(\mathbf{k})$ also have the scaling form

$$\lambda(\mathbf{k}) = k^{-x_\lambda} L(k\xi) \,, \tag{5.10}$$

$$\bar{\eta}(\mathbf{k}) = k^{-x_{\bar{\eta}}} E(k\xi) \,, \tag{5.11}$$

with

$$x_\lambda + x_{\bar{\eta}} = 4 - d + \eta \,. \tag{5.12}$$

Moreover, in the limit $k \to 0$ we have, using Eqs. (5.3) and (5.4), $\lambda \propto \xi^{x_\lambda}$, $\bar{\eta} \propto \xi^{x_{\bar{\eta}}}$, with

$$\lambda\bar{\eta} = g_0^2\, \chi_\psi(\mathbf{k}=0)\, \xi^{2-d} R \,, \tag{5.13}$$

where R is a (universal) numerical constant. Equation (5.13) may be rewritten in the "Kawasaki-Stokes" form, in terms of the thermal diffusivity $D = \lambda/\chi_\psi$ in physical units,[22] as (Kawasaki, 1970)

$$D = R k_B T/\bar{\eta}\xi^{d-2} \,. \tag{5.14}$$

This is precisely the result obtained in (2.22) by a simplified argument.

In order to calculate the exponents and scaling functions in (5.10) and (5.11) it is necessary to solve (5.7) and (5.8) numerically, or to make further approximations. The simplest of these follows from the experimental observation that $\bar{\eta}$ is at most weakly divergent. The approximation consists in setting $\bar{\eta}$ = constant, and

[22] In (5.1) we have chosen units in which the free energy F_0 is dimensionless ($k_B T_c = 1$), and the susceptibility $\chi_j = \rho^{-1}$ is set equal to unity. More generally, the field $\mathbf{j}$ is the transverse momentum density divided by $(\rho k_B T_c)^{1/2}$, and $g_0 = (\rho k_B T_c)^{-1/2}$.

[23] Note that a convective term of the form

$$g_0 \mathcal{T}\cdot\left[\left(\frac{\delta F_0}{\delta \mathbf{j}}\cdot\nabla\right)\mathbf{j}\right] \propto (\mathbf{j}\cdot\nabla)\mathbf{j}$$

also occurs in the nonlinear Navier-Stokes equation for $\partial\mathbf{j}/\partial t$. This term has been omitted from (5.1b) since it turns out to be irrelevant near T_c.

[24] In Eq. (A6) of Siggia *et al.* (1976), the factor $\{\chi_\psi^{-1}(\mathbf{p}_+) - \chi_\psi^{-1}(\mathbf{p}_-)\}$ in (5.8) was approximated as $\mathbf{p}\cdot\mathbf{k}$, which implies an Ornstein-Zernike assumption for $\chi_\psi(\mathbf{p})$ (i.e., $\eta = 0$).

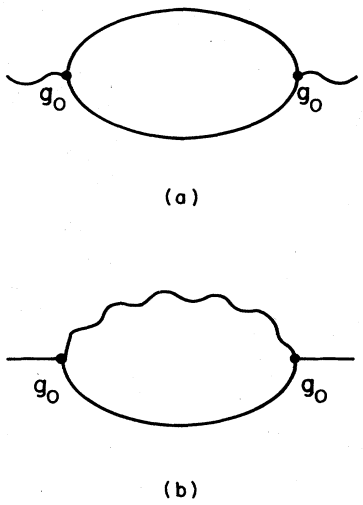

FIG. 3. Lowest-order self-energy diagrams for the critical dynamics of the fluid, in an expansion in the mode-coupling vertex g_0. The solid lines represent the order parameter mode, and the wavy lines are viscous relaxation modes. Near four dimensions these diagrams yield the correct asymptotic behavior at the critical point. In three dimensions they lead to the self-consistent Kawasaki approximation, which preserves scaling and agrees rather well with experiment.

using an Ornstein–Zernike form for $\chi_\psi(\mathbf{k})$ in (5.7), to calculate the scaling function $L(k\xi)$. From (5.12) with $d=3$ and $\eta=0$ we then obtain $x_\lambda=1$, and the characteristic frequency takes the form

$$\omega_\psi(\mathbf{k})=\lambda(\mathbf{k})k^2/\chi_\psi(\mathbf{k})\equiv(k_BT/6\pi\bar{\eta}\xi)\bar{\Omega}_K(k\xi)\,, \tag{5.15}$$

$$\bar{\Omega}_K(y)\equiv y^{-2}K_0(y)=\tfrac{3}{4}y^{-2}[1+y^2+(y^3-y^{-1})\arctan y]\,. \tag{5.16}$$

In the next approximation, the correction to $\bar{\eta}$ can be found from (5.8) by using (5.16). The result is[25]

$$\bar{\eta}=\bar{\eta}_0[1+(8/15\pi^2)\ln\xi]\,. \tag{5.17}$$

A more accurate solution of (5.7) and (5.8) may be obtained numerically, and the exponents for $d=3$ turn out to be (with $\eta=0$) (Ohta and Kawasaki, 1976)

$$x_{\bar{\eta}}=0.054,\quad x_\lambda=0.946\,, \tag{5.18}$$

which differs little from the lowest approximation $x_\lambda=1$, $x_{\bar{\eta}}=0$ in (5.16) and (5.17). A slightly different calculation was performed recently by Garisto and Kapral (1975, 1976), who did not make the Lorentzian approximation (5.5) and (5.6), and found $x_{\bar{\eta}}=0.07$. Furthermore, the scaling functions may be evaluated numerically from (5.7) and (5.8) (Kawasaki and Lo, 1972), and the results also show only small deviations from the lowest-order expression (5.16).

A result in the form (5.14) for the thermal diffusivity was first obtained by Kadanoff and Swift (1969a), except that they had a "high-frequency" viscosity η^* in place of the macroscopic transport coefficient $\bar{\eta}$, in the denominator. The correct form was presented by Kawasaki (1969, 1970, 1971), but he mistakenly argued that Eqs. (5.7) and (5.8) imply a finite $\bar{\eta}$ at the critical point. A mode-coupling argument including a logarithmically diverging viscosity in the form (5.17) was given by Ferrell (1970). A power-law divergence for the viscosity was first predicted by Halperin *et al.* (1974b), using renormalization group arguments near four dimensions (see below). The exponent $x_{\bar{\eta}}$ was subsequently calculated to second order in ϵ by Siggia *et al.* (1976), and self-consistently by Ohta and Kawasaki (1976), and Garisto and Kapral (1976), as in (5.18). More extensive references to the theoretical literature may be found in Swinney and Henry (1973), Kawasaki (1976), Siggia *et al.* (1976), and Kawasaki and Gunton (1976a).

C. Comparison with experiment

The results of the approximate mode-coupling calculations can be subjected to a number of tests, by comparing to inelastic light scattering experiments and direct macroscopic measurements of the transport coefficients. These have been reviewed extensively elsewhere (Swinney and Henry, 1973; Kawasaki, 1976; Sengers, 1971, 1973), so we shall only illustrate the main features. In comparing experiment and theory it is important to consider the regular background contributions to the transport coefficients, represented by λ_0 and $\bar{\eta}_0$ in Eqs. (5.7–8), since these do not become negligible until one reaches the asymptotic limit in which ξ is truly infinite (see Sengers, 1971, 1973; Swinney and Henry, 1973). For the viscosity in particular, the condition $\bar{\eta}\gg\bar{\eta}_0$ would only be satisfied in an unphysical temperature range, since the exponent $x_{\bar{\eta}}$ is so small. If $\bar{\eta}_0$ and λ_0 are retained in Eqs. (5.7) and (5.8) the solutions no longer have the scaling form (5.10)-(5.11) for finite values of k or ξ^{-1} (Oxtoby and Gelbart, 1976). In the Kawasaki approximation leading to (5.15) we may take λ_0 and $\bar{\eta}_0$ into account by writing

$$\omega_\psi(\mathbf{k})=\frac{\lambda_0k^2}{\chi_\psi(\mathbf{k})}+\frac{k_BT}{6\pi\bar{\eta}\xi}\,\bar{\Omega}_K(k\xi)\,, \tag{5.19}$$

where $\bar{\eta}$, the full viscosity, includes $\bar{\eta}_0$ as a multiplicative factor (Ohta, 1977).

The characteristic frequency $\omega_\psi(\mathbf{k})$ may be determined from the spectrum of inelastic light scattering in a fluid. In cases where the thermal conductivity has been measured independently, the constant λ_0 can be obtained from data far from the critical point. The susceptibility $\chi_\psi(\mathbf{k})$ may be replaced by the Ornstein–Zernike form $\chi_\psi(k)=C_p(1+k^2\xi^2)^{-1}$, and the quantities $\bar{\eta}$ and ξ can be measured directly. Thus, a comparison of Eq. (5.19) with experiments is possible, with *no adjustable parameters* (Swinney and Henry, 1973). The results for a number of fluids (including binary mixtures) are shown in Fig. 4, and represent in our opinion a truly remarkable success for the mode-coupling theory. Moreover, macroscopic measurements of λ and $\bar{\eta}$ are also consistent with the results (5.14) and (5.17) (Sengers, 1971, 1973).

The agreement between experiments and the simple Kawasaki theory is well within the 10%-20% accuracy with which the parameters C_p, $\bar{\eta}$, ξ, and λ_0 can be determined experimentally, and calculations or measurements of systematic deviations from this theory require great care. An important quantitative effect comes from

[25]Note that a factor of $\bar{\eta}_0$ was inadvertently left out of Eq. (4.13) of Siggia *et al.* (1976).

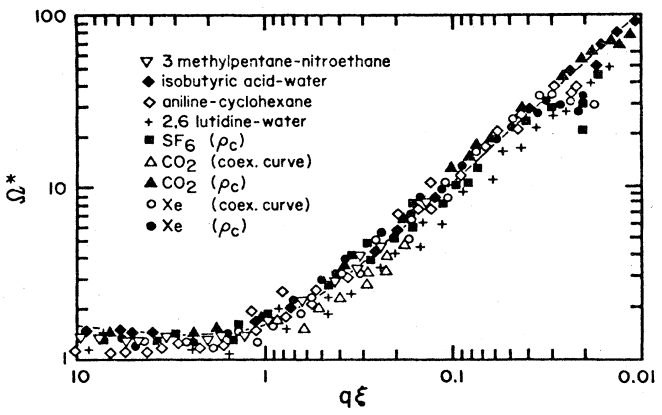

FIG. 4. Plot of the reduced order parameter diffusion rate, $\Omega^* \equiv (6\pi\bar{\eta}/q^3 k_B T)\omega_\psi(q)$ as a function of the reduced wave vector $q\xi$, for various fluids over a range of temperatures near their critical points. The data agree remarkably well with the prediction of the simple Kawasaki theory, Eq. (5.19). From Swinney and Henry (1973).

the corrections to scaling in λ and $\bar{\eta}$, as well as in the static functions (Wegner, 1972), and these may enter in rather subtle ways at the 10% level.

From a theoretical point of view, there are two ways to understand the success of the Kawasaki approximation (5.15). First of all, one may calculate the corrections by inserting the lowest-order expressions in a systematic expansion of the correlation functions. This has been done by a number of authors (Lo and Kawasaki, 1972; Perl and Ferrell, 1972a, 1972b; Garisto and Kapral, 1975), and quite generally the corrections are found to be less than 5% for the order parameter, and somewhat more for $\bar{\eta}$. [An example is the self-consistent calculation of x_λ and $x_{\bar{\eta}}$ in (5.18), which hardly changes their values, compared to (5.15)–(5.17).] Another way to investigate the validity of the mode-coupling theory is via the renormalization group, which provides a systematic expansion near $d = 4$.

D. Renormalization group

The treatment of model H follows rather closely the analysis given in Sec. IV for the relaxational models. The addition of nondissipative vertices proportional to g_0 makes the diagrammatic formalism considerably more complicated, but the general structure of the theory remains the same [see Siggia *et al.* (1976)]. One writes down recursion relations for the static parameters r_l and u_l, and for the dynamic quantities λ_l, g_l and $\bar{\eta}_l$.[26] It may be seen from Eqs. (5.1) that there is a dimensionless coupling constant

$$f_0 = K_d g_0^2 \Lambda^{d-4}/\lambda_0\bar{\eta}_0 . \tag{5.20}$$

It is thus useful to combine the recursion relations for g_l, λ_l, and $\bar{\eta}_l$ into an equation for f_l. The result is (Halperin *et al.*, 1974b)

$$\lambda_{l+1} = b^{z+\eta-4}\lambda_l(1 + \tfrac{3}{4} f_l \ln b) , \tag{5.21}$$

$$\bar{\eta}_{l+1} = b^{z-2}\bar{\eta}_l(1 + \tfrac{1}{24} f_l \ln b) , \tag{5.22}$$

$$f_{l+1} = b^{\epsilon-\eta} f_l(1 - \tfrac{19}{24} f_l \ln b) . \tag{5.23}$$

From Eq. (5.23) it follows that f_l reaches a stable fixed point

$$f^* = (24/19)\epsilon + O(\epsilon^2) , \tag{5.24}$$

for $d < 4$. An examination of the corrections to (5.24) shows that these are finite in every order when $l \to \infty$, so that f^* remains finite to all orders in ϵ, for small ϵ [Siggia *et al.* (1976)]. It then follows, by an analysis similar to the one leading from (4.70) to (4.72), that the transport coefficients are related by an *exact scaling law*, which is the one given in (5.12). This relation is also valid to all orders in ϵ, since it depends only on the finiteness of f^*. The dynamic exponent z may then be identified, using (5.21), as

$$z = 4 - \eta - x_\lambda . \tag{5.25}$$

The exponents x_λ and $x_{\bar{\eta}}$ and the universal constant R in (5.14) may be found to order ϵ from the recursion relations (5.21)-(5.23). For the higher-order corrections it is more convenient to use the Feynman-graph expansion, as was done by Siggia *et al.* (1976), who found

$$x_\lambda = \tfrac{18}{19}\epsilon[1 - 0.033\epsilon + O(\epsilon^2)] \approx 0.916 , \tag{5.26}$$

$$x_{\bar{\eta}} = \tfrac{1}{19}\epsilon[1 + 0.238\epsilon + O(\epsilon^2)] \approx 0.065 , \tag{5.27}$$

$$R = K_d \tfrac{19}{24}\epsilon^{-1}[1 + 0.06\epsilon + O(\epsilon^2)] . \tag{5.28}$$

A direct extrapolation of (5.28) to $d = 3$ yields $R = 0.042$, which is rather close to the result of the Kawasaki approximation $R_K = (6\pi)^{-1} \approx 0.05$, but probably not as accurate at $d = 3$. An extrapolation for R which combines mode coupling and the ϵ expansion, and should therefore be more accurate than either method separately, was devised by Siggia *et al.* (1976), and yields $R \approx 1.2R_K \approx 0.064$.

The ϵ expansion is also useful in understanding the success and the limitations of the mode-coupling approach. First of all, to the extent that one only examines *relations* between transport coefficients the scaling law (5.12) is correct to all orders in g_0 and is not limited to the lowest order. The further approximation made by Kawasaki (1970) to evaluate x_λ, namely the assumption that $x_{\bar{\eta}}$ is negligible, has its justification in experiments. However, it is also reflected in the smallness of the coefficient $\frac{1}{24} f_l$ in Eq. (5.22) compared to $\frac{3}{4} f_l$ in (5.21). The origin of this effect lies in the angular factors appearing in the integrand of (5.8), and it also leads to the small coefficient $8/15\pi^2$ in (5.17) for three dimensions. The weak effect of an order parameter fluctuation on transverse current fluctuations represents in some sense a small parameter in the theory, of order $x_{\bar{\eta}}/x_\lambda \approx 0.05$ (Siggia *et al.*, 1976). Although this parameter does not seem to lead to a systematic expansion, it appears in most of the higher-order diagrams and explains the success of the simple Kawasaki approximation (5.15) with $x_{\bar{\eta}} = 0, x_\lambda = 1$.

Another approximation made in all mode-coupling calculations (Gunton and Kawasaki, 1975; Kawasaki and Gunton, 1976b) is the neglect of the nondissipative cou-

[26] There is no static coupling between $\mathbf{j}$ and ψ in (5.1d), analogous to γ_0 in (4.50e), since the susceptibility $\chi_j \equiv \rho^{-1}$ does not have an α divergence at T_c.

pling u_0 (5.1d) which does not appear in linear order in g_0. The main effect of u_0 is in the statics,[1] where it changes the exponents from their mean-field values to their scaling values, but this is taken into account in the mode-coupling approach by using the experimental ξ and C_p. The remaining effects of u_0, on shape functions and dynamic exponents, are of order ϵ^2, and are small in three dimensions (roughly speaking one can say that these residual effects are of order η).

E. Real binary fluids

For a real binary fluid, where the symmetry assumed in Sec. II does not occur, it is convenient to use as a variable the quantity

$$c(\mathbf{x}) \equiv [\bar{\rho}_A \rho_B(\mathbf{x}) - \bar{\rho}_B \rho_A(\mathbf{x})]/\bar{\rho}^2 , \tag{5.29}$$

in addition to the transverse part of the momentum density $\mathbf{j}(\mathbf{x})$, and a "thermal fluctuation density" $q(\mathbf{x})$ [the symbols $\bar{\rho}_A$, $\bar{\rho}_B$, and $\bar{\rho}$ in (5.29) denote equilibrium values of ρ_A, ρ_B, and ρ, respectively]. The density $q(\mathbf{x})$ is a linear combination of ρ_A, ρ_B, and the energy density $e(\mathbf{x})$, chosen such that $q(\mathbf{x})$ and $c(\mathbf{x})$ are orthogonal to each other and to the pressure, in the sense that their equal-time cross-correlation functions vanish at long wavelengths, in equilibrium. The pressure fluctuations and the longitudinal part of the momentum density, which enter the sound mode, will be disregarded as before. Note that $c(\mathbf{x}) = [\rho_B(\mathbf{x})/\rho(\mathbf{x})] - \rho_B/\rho$, for small deviations from equilibrium, and $c(\mathbf{x})$ is loosely referred to as the *concentration fluctuation*. Because the binary fluid has two long-wavelength diffusive modes with the same symmetry properties, it is convenient to define a transport *matrix*, according to

$$\lambda_{ij} = \lim_{\substack{k,\omega\to 0 \\ \omega/k\to\infty \\ \omega/k^2\to 0}} \frac{\omega}{ik^2}\, \chi_{ij}(\mathbf{k},\omega) , \tag{5.30}$$

where i and j may be either c or q. The normal modes of these variables have relaxation rates D_1k^2 and D_2k^2, which are the eigenvalues of the matrix $\boldsymbol{\lambda}\boldsymbol{\chi}^{-1}k^2$, where $\boldsymbol{\chi}$ is the matrix of static susceptibilities. We remark that the diagonal matrix elements of $\boldsymbol{\lambda}$ reduce to the coefficient λ defined by (3.20), *only in the case where* $\boldsymbol{\lambda}$ and $\boldsymbol{\chi}$ commute. In general, this does not occur.

In the vicinity of the consolute point, the susceptibility $\chi_c(\equiv\chi_{cc})$ diverges strongly, as $\xi^{2-\eta}$, while χ_q has only a "specific-heat divergence" proportional to $\xi^{\alpha/\nu}$. On the basis of the mode-coupling theories, it has been found that one eigenvalue of $\boldsymbol{\lambda}$ diverges at T_c, while the other remains finite (Swift, 1968; Giterman and Gorodetskii, 1969; Mistura, 1975). Furthermore, the diverging part of $\boldsymbol{\lambda}$ is simply the matrix element λ_{cc}, so that in the asymptotic critical region, $\boldsymbol{\lambda}$ and $\boldsymbol{\chi}$ may be simultaneously diagonalized. The concentration c then enters the statics and dynamics in the same manner as the order parameter ψ for model H, or for the ordinary gas-liquid critical point. Although nonlinear couplings between c and q are present in the appropriate free-energy functional, the dissipative couplings have no effect on the critical dynamics of c, because the diffusion rate for q is fast compared to the diffusion rate for c. Thus it is found that the universality class for the asymptotic critical dynamics of the binary fluid is the *same* as for the ordinary fluid critical point.

A renormalization group analysis by Siggia *et al.* (1976) of a simple model containing a field analogous to q, in addition to $\mathbf{j}$ and ψ, confirms this picture, in agreement with the mode-coupling work cited above, and in disagreement with Papoular (1974).

VI. PLANAR MAGNET AND SUPERFLUID HELIUM

A. Models E and F

The usefulness of a pseudospin model to describe superfluid helium was first pointed out by Matsubara and Matsuda (1956). These authors showed that bose condensation in a quantum lattice gas was equivalent to the development of magnetic order in a spin-$\frac{1}{2}$ *"easy-plane" magnet*. The quantum field ψ corresponds to the operator[27] $(S_x - iS_y)$ in the magnet, and the density of the boson system corresponds to $(\frac{1}{2} - S_z)$. The easy-plane magnet has cylindrical symmetry about the z axis and has interactions which lead to ferromagnetic order in the $(x\text{-}y)$ plane, i.e., to a two-component order parameter $(n=2)$. It is to be noted that in both the planar magnet and the superfluid the order parameter is *not* a constant of the motion, i.e., it does not commute with the Hamiltonian. A constant of the motion in the planar magnet is the z component of magnetization, which we denote by M. This is proportional to the spin-angular momentum S_z, which is the generator of rotations of the order parameter in the $x\text{-}y$ plane. There is thus an important relationship between the symmetry of the order parameter and M, which we express by the Poisson-bracket relation

$$\{\psi, M\} = ig_0\psi . \tag{6.1}$$

This equation is crucial for the hydrodynamics and the critical dynamics of the system (Halperin and Hohenberg, 1969b; Anderson, 1966). In particular it is responsible for the Larmor precession theorem in the planar magnet, which states that the sole effect of a time-dependent uniform applied field $h_z(t)$ is to add to the motion of all the spins a precession at the angular frequency g_0h_z. In the superfluid, the corresponding theorem is the "Josephson equation" (Pitaevskii, 1958; Anderson, 1966)

$$d\phi/dt = \hbar^{-1}\mu , \tag{6.2}$$

where ϕ is the phase of the order parameter and μ is the chemical potential.

If the dynamics of the planar magnet arises from a Hamiltonian for the spins alone, then the energy of the spins is another constant of the motion. We shall, however, consider a model in which this energy is *not* conserved, as in model A of Sec. IV,[28] since such a model is applicable to superfluid helium. This is

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[27] For the systems treated in this section it is useful to introduce a *complex* order parameter $\psi(\mathbf{x},t)$. This requires certain trivial modifications in the formulas of Sec. III, which were derived for real variables.

[28] As mentioned in Sec. IV, such a model arises when one considers a spin system in contact with a heat reservoir having infinite heat capacity, or infinite thermal conductivity.

model F, defined by the equations (Halperin *et al.*, 1974b, 1976b; see also Stauffer and Wong, 1970; Okwamoto, 1976; Ginzburg and Sobyanin, 1976, and references therein)

$$\frac{\partial\psi}{\partial t} = -2\Gamma_0 \frac{\delta F}{\delta \psi^*} - ig_0\psi \frac{\delta F}{\delta m} + \theta\,, \tag{6.3a}$$

$$\frac{\partial m}{\partial t} = \lambda_0^m \nabla^2 \frac{\partial F}{\partial m} + 2g_0 \operatorname{Im}\left(\psi^* \frac{\delta F}{\delta \psi^*}\right) + \zeta\,, \tag{6.3b}$$

$$F[\psi, m] = F_0 - \int d^dx \{h_m(\mathbf{x}, t)m + \operatorname{Re}[h(\mathbf{x}, t)\psi^*]\}\,, \tag{6.3c}$$

$$F_0 = \int d^dx \{\tfrac{1}{2}\tilde{r}_0|\psi|^2 + \tfrac{1}{2}|\nabla\psi|^2 + \tilde{u}_0|\psi|^4 + \tfrac{1}{2}C_0^{-1}m^2 + \gamma_0 m|\psi|^2\}\,, \tag{6.3d}$$

where h_m and h are infinitesimal applied fields, and θ and ζ are the appropriate Langevin noise sources. The static properties of this model are precisely those of model C considered in Sec. IV.C, but the dynamic behavior is different, due to the presence of the nondissipative coupling g_0, and to the possibility of a *complex* value of Γ_0. It may be shown (Halperin and Hohenberg, 1969b; Halperin *et al.*, 1976b) that there is a propagating "spin-wave" mode below T_c, involving variations in m and the direction of ψ (in the complex plane), with frequency

$$\omega_\psi(\mathbf{k}) = \omega_m(\mathbf{k}) = c_s k\,, \tag{6.4}$$

$$c_s^2 = g_0^2\rho_s/\chi_m\,, \tag{6.5}$$

where the stiffness constant ρ_s is defined in Eq. (6.8) below. For $T > T_c$, the density m has a diffusive mode

$$\omega_m(\mathbf{k}) = (\lambda_m/\chi_m)k^2\,, \tag{6.6}$$

and the nonconserved field ψ has a characteristic frequency

$$\omega_\psi(\mathbf{k}) = \Gamma/\chi_\psi\,, \tag{6.7}$$

which does not vanish as $k \to 0$.

An important simplification arises in the planar magnet if one assumes that there is no applied field in the z direction so that the average value of $M = \int d^dx\, m(\mathbf{x})$ vanishes. This is the *symmetric* planar model, the symmetry here being rotation of 180° about an axis in the x-y plane. We can represent this by *model E*, which is obtained from F by settting $\gamma_0 = 0$. Then the equations are invariant under the transformation $m \to -m$ and $\psi \to \psi^*$, and it follows that $\langle M\rangle = 0$.

As mentioned above, for liquid helium the density ψ represents the expectation value of the quantum field. The conserved density $m(\mathbf{x}, t)$ is that linear combination of energy and mass which appears in the thermal diffusion mode in the normal fluid ($T > T_\lambda$), and in second sound in the superfluid ($T < T_\lambda$) (see, for example, Hohenberg and Martin, 1965). It follows that λ_m is the thermal conductivity, χ_m is the specific heat C_p, and c_s is the second-sound velocity. The quantity m is the same as q in Eq. (5.2), and it is orthogonal to the pressure variations at long wavelengths, both above and below T_λ.[29] Since a variation in $m(\mathbf{x})$ implies a change in the local value of $T - T_\lambda$ we expect to find couplings of the form $\gamma_0 m|\psi|^2$ in the free-energy functional, and we must represent helium by the asymmetric model (F) rather than the symmetric one (E).

An important distinction in the static properties, between systems with discrete ($n = 1$) and continuous ($n \geq 2$) symmetry, is that in the *ordered* state ($T < T_c$), the correlation function for ψ does not decay exponentially at large distances for systems with a continuous broken symmetry. Indeed, due to the invariance of the energy with respect to uniform rotations of $\langle\psi\rangle$ in this case, the free-energy functional contains a term proportional to $\rho_s|\nabla\phi|^2$, where ϕ is the phase of $\langle\psi\rangle$, and ρ_s is the stiffness constant. It follows that the static response has the long-wavelength form (Hohenberg and Martin, 1965)

$$\chi_\psi(\mathbf{k}) = |\langle\psi\rangle|^2/\rho_s k^2\,, \tag{6.8}$$

leading to a correlation function

$$C_\psi(\mathbf{x}) \sim \text{const}/x^{d-2}\,, \tag{6.9}$$

with power-law decay at large distances. It is then convenient to define the (transverse) correlation length by the relation (Josephson, 1966)

$$\lim_{x\to\infty} C_\psi(\mathbf{x}) \equiv A_d|\langle\psi\rangle|^2(\xi_T/x)^{d-2}\,, \tag{6.10}$$

where the numerical constant A_d is defined by

$$\int \frac{d^dk}{(2\pi)^d}\, k^{-2} e^{i\mathbf{k}\cdot\mathbf{x}} \equiv A_d x^{2-d}\,. \tag{6.11}$$

Comparing (6.10) with (6.8) we find ($k_B T_c = 1$)

$$\xi_T = \rho_s^{(2-d)^{-1}}\,. \tag{6.12}$$

The stiffness constant ρ_s is the superfluid density of liquid helium, and it may be obtained experimentally from the second-sound velocity (6.5), or form various measurements of the momentum carried by the superfluid in equilibrium.[30] Above T_λ, the correlation function decays exponentially, but the correlation length ξ_+ is not directly accessible to experiment. Nevertheless, one may estimate its amplitude from the measured specific heat and a calculation of a static universal ratio (see Hohenberg *et al.*, 1976a; Bervillier, 1976).

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[29] The velocity of ordinary (first) sound remains finite in helium at T_λ, so that this mode can be omitted from the model for the same reasons as at the gas-liquid critical point, discussed in Sec. V. Note also that we could have introduced mode coupling to the transverse momentum $\mathbf{j}$ in the equations of motion for ψ and m. These turn out to be irrelevant, however, in contrast to the situation in model H, because ψ is not conserved, and because the static susceptibility χ_m has at most a weak divergence in the present case.

[30] The equations of this section are written in units in which $k_B T_c = 1$ and the frequency scale is set by g_0. The usual superfluid density $\tilde{\rho}_s$, in units of mass per unit volume, is given by $\tilde{\rho}_s = (m_{\mathrm{He}}^2 k_B T/\hbar^2)\rho_s$, where m_{He} is the helium mass. The frequency g_0 is given by $g_0 = k_B T\sigma/\hbar$, where $\sigma = S/R$ is the dimensionless entropy per particle (R is the gas constant). Equations (6.12) and (6.5) become $\xi = m_{\mathrm{He}}^2 k_B T/\hbar^2\tilde{\rho}_s$ and $\tilde{c}_s^2 = \tilde{\rho}_s k_B^2 T\sigma^2/C_p m_{\mathrm{He}}^2$, where $C_p = k_B\chi_m$ is the specific heat per unit volume (see Halperin *et al.*, 1976b).

B. Dynamic scaling

The existence of a propagating critical mode below T_c, whose frequency is related to static quantities [see Eqs. (6.4), (6.5), and (6.8)], may be used to evaluate the exponent z of the dynamic scaling expression (3.27). Indeed, if the characteristic frequency for ψ is to have the general form (3.27) and the limiting behavior (6.4) for $k\xi_T \ll 1$, it then follows that $c_s \propto \xi^{1-z}$. But by (6.5) and (6.12) we have $c_s \propto \xi^{1-(d/2)+\tilde{\alpha}/2\nu}$, whence (Ferrell *et al.*, 1967, 1968; Halperin and Hohenberg, 1967, 1969a),

$$z = \frac{d}{2} + \frac{\tilde{\alpha}}{2\nu} , \tag{6.13}$$

where $\alpha \equiv \max(\alpha, 0)$ is the exponent of C_p. In the present case we see that z is expressed entirely in terms of static exponents, in contrast to the case of the gas-liquid critical point, Eq. (5.29).

The most interesting physical applications of the result (6.13) in liquid helium are to the damping of second sound below T_λ, and the thermal conductivity above T_λ (Ferrell *et al.*, 1967). If one includes the next term in k in the dispersion relation for second sound (6.4), one obtains (see Hohenberg and Martin, 1965)

$$\omega = \pm c_s k - (i/2) D_s k^2 , \tag{6.14}$$

where $D_s k^2$ represents the width of the second-sound peak in the correlation function. This width is also a characteristic frequency associated with the order parameter, and according to dynamic scaling it will scale with the same power z as $c_s k$. It follows that

$$D_s \propto \xi_T^{\epsilon/2-\tilde{\alpha}/2\nu} . \tag{6.15}$$

Similarly, consideration of thermal diffusion above T_λ (6.6) yields the prediction

$$\lambda_m \propto \xi_+^{\epsilon/2+\tilde{\alpha}/2\nu} . \tag{6.16}$$

Note that thermal diffusion is the characteristic frequency for m rather than ψ, so that the prediction (6.16) requires a dynamic scaling assumption for m, as well as for the order parameter (Halperin and Hohenberg, 1969a).

C. Renormalization group

The renormalization group may be formulated for model F in much the same way as for models C and H above (Halperin *et al.*, 1974b, 1976b). The static recursion relations for r_l, u_l, C_l, and $v_l \equiv K_d \gamma_l^2 C_l \Lambda^{d-4}$ are precisely the same as in Sec. IV.C, and it follows that the static susceptibility χ_m behaves as $k^{-\tilde{\alpha}/\nu}$ for $T = T_c$ in model F ($\gamma_0 \neq 0$). For the symmetric model ($\gamma_0 = 0$), χ_m remains constant (equal to C_0) as $k \to 0$ or $T \to T_c$. Recursion relations may also be developed for the dynamic quantities g_l, λ_l, and Γ_l, or for the combinations

$$f_l \equiv K_d g_l^2 \Lambda^{d-4}/(\lambda_l \mathrm{Re}\Gamma_l), \tag{6.17}$$

$$w_l \equiv \Gamma_l C_l/\lambda_l . \tag{6.18}$$

(Note that Γ_0 and Γ_l are in general complex.)

In the symmetric case (model E: $\gamma_0 = 0, \Gamma_0$ real), there are three dimensionless slow transients in the recursion relations, namely u_l, f_l, and w_l. To the lowest significant order, these reach the fixed-point values

$$f^* = \epsilon + O(\epsilon^2) , \tag{6.19}$$

$$w^* = 1 + O(\epsilon) , \tag{6.20}$$

while u^* is given by its static value (4.23a). It is then possible to show, by an argument quite analogous to the one leading to (4.72), that the frequencies $\omega_\psi(\mathbf{k})$ and $\omega_m(\mathbf{k})$ are both characterized by the same exponent

$$z = 2 - \tfrac{1}{2}\epsilon = \tfrac{1}{2} d . \tag{6.21}$$

This result is a scaling law, which holds to all orders in ϵ for d near 4, or more generally for any dimensionality at which the quantities f^*, w^*, and u^* are finite and nonzero.

In the asymmetric case (model F: $\gamma_0 \neq 0, \Gamma_0$ complex) there are two additional slow transients, v_l and $\mathrm{Im}w_l$. The fixed-point parameters are

$$f^* = \tfrac{6}{5}\epsilon + O(\epsilon^2) , \tag{6.22}$$

$$(\mathrm{Re}w)^* = 0.732 + O(\epsilon) , \tag{6.23a}$$

$$(\mathrm{Im}w)^* = 0.480\ \mathrm{sgn}\gamma^* + O(\epsilon) , \tag{6.23b}$$

$$v^* = \epsilon/20 + O(\epsilon^2) = \tilde{\alpha}/4\nu , \tag{6.24}$$

with u^* given in Eq. (4.23a). The dynamic exponent z then takes on precisely the value (6.13) predicted by the phenomenological scaling analysis.[31] The symmetric fixed point (6.19)–(6.20) is *unstable* with respect to the perturbation $\gamma_0 m|\psi|^2$ near $d = 4$. On the other hand, there is evidence that at $d = 3$, the specific heat exponent α is negative (see Ahlers, 1976), so that $v^* = 0$, in which case the stable fixed point for model F is the *symmetric* one.

The existence of five slow transients in model F will lead to a rather complicated structure in the leading *corrections* to scaling, for both $\alpha > 0$ and $\alpha < 0$. These corrections can be written in the form of power series in the variables $d_i \xi^{-x_i/\nu}$ ($i = 1, \ldots, 5$), where the x_i are correction exponents (Wegner, 1972) and the d_i are nonuniversal coefficients. Near the symmetric fixed point ($\alpha < 0$), the dynamic correction exponents have been evaluated to lowest order in ϵ by Halperin *et al.* (1976b), and to second order by De Dominicis and Peliti (1977). The results are

$$x_1 = \epsilon\nu(\tfrac{1}{4} - 0.1498\epsilon) + O(\epsilon^3) , \tag{6.25a}$$

$$x_2 = \epsilon\nu(\tfrac{3}{4} - 0.3618\epsilon) + O(\epsilon^3) , \tag{6.25b}$$

$$x_3 = \epsilon\nu(1 - 0.3135\epsilon) + O(\epsilon^3) . \tag{6.25c}$$

The static correction exponents

$$x_4 = \epsilon\nu + O(\epsilon^2) \tag{6.25d}$$

and

$$x_5 = -\alpha > 0 \tag{6.25e}$$

are the ones associated with the transients u_l and v_l, respectively (Wegner, 1972). Because the exponent x_5 is so close to zero in liquid helium ($\alpha \approx -0.02$, Ahlers,

[31] Recently, Kawasaki and Gunton (1976b) considered a model without dissipative couplings, but with a *singular* specific heat $C_0 \propto \xi^{\alpha/\nu}$ in the starting free-energy functional. This model leads to the scaling law (6.13) of the asymmetric model, but its universal amplitude ratios do not correspond either to those of model E or of model F, even in lowest order in ϵ.

1976), terms nonlinear in $d_5\xi^{-x_5/\nu}$ are expected to be important; it was therefore proposed by Halperin *et al.* (1976b) to eliminate the variable $d_5\xi^{-x_5/\nu}$ in favor of the measured quantity

$$\alpha_e \equiv \frac{d\ln C_p(T)}{d\ln\Delta T} \quad . \tag{6.26}$$

It is not clear, however, how useful such a procedure will be, especially in view of the existence of other small correction exponents, for instance $x_1 \approx 0.07$, as obtained from Eq. (6.25a). De Dominicis and Peliti (1977) have suggested the possibility that x_1 might even be negative in three dimensions, implying an instability of the fixed point and a breakdown of dynamic scaling.

The fixed-point mechanism leading to the scaling law (6.13) also implies that the response functions $\chi_\psi(\mathbf{k},\omega)$ and $\chi_m(\mathbf{k},\omega)$ will have a scaling form analogous to (3.25), with $h_\psi = 0$. These functions lead to a large number of universal amplitude ratios, which may in principle be calculated using the ϵ expansion and compared to experimental values. For example, in liquid helium the thermal conductivity λ_m may be written for $T > T_\lambda$ as

$$\lambda_m = R_\lambda g_0 C_p^{1/2} \xi_+^{1/2} , \tag{6.27}$$

where R_λ is a universal number. Since g_0 is determined by the entropy per particle σ and the value of T_λ,[30] Eq. (6.27) represents a universal relation between a dynamic quantity λ_m and purely static parameters.[32] This relation is predicted to be exact asymptotically close to T_λ, but it will contain correction terms of the form discussed above at finite temperature differences ΔT.

A universal amplitude ratio may also be defined from the damping constant of second sound below T_λ, namely

$$R_2 = D_s/2c_s\xi_T , \tag{6.28}$$

and another one from the characteristic frequency $\omega_m(\mathbf{k})$ at T_λ, given by

$$\omega_m(\mathbf{k}) = \Omega_m^\infty k^z \equiv R_m^{\rm crit}\, c_s \xi_T^{z-1} k^z . \tag{6.29}$$

Note that according to Eqs. (6.5), (6.12), and (6.13), $c_s\xi_T^{z-1}$ goes to a constant at T_λ, so $R_m^{\rm crit}$ also reaches a finite limit.[33]

The amplitude ratios defined above may be estimated in different ways. From the ϵ expansion one obtains, to second order (Halperin *et al.*, 1976b, Siggia, 1976),

$$R_\lambda = (K_d/\epsilon)^{1/2}[1 + 0.597\epsilon + O(\epsilon^2)] \approx 0.36 , \tag{6.30}$$

$$R_2^{2-\epsilon} = (K_d/\epsilon)[1 + 0.311\epsilon + O(\epsilon^2)] \approx 0.07 , \tag{6.31}$$

$$R_m^{\rm crit} = (K_d/\epsilon)^{1/2}[1 + 1.4\epsilon + O(\epsilon^2)] \approx 0.54 . \tag{6.32}$$

Alternatively, it is possible to calculate the universal scaling functions and amplitude ratios using a self-consistent, first-order perturbation theory, similar to the one depicted in Fig. 3 and described in Sec. V.B for the fluid. Such mode-coupling calculations were carried out in an approximate form both above and below T_λ, as a function of $k\xi$, by Hohenberg *et al.* (1976b) [see also Krueger and Huber (1970)]. The results for the dimensionless ratios discussed above are

$$R_\lambda = 0.19 , \tag{6.33}$$

$$R_2 = 0.09 , \tag{6.34}$$

$$R_m^{\rm crit} = 0.42 . \tag{6.35}$$

These values are consistent with the extrapolations of the ϵ expansion (6.30)–(6.32), to within the rather large (50–100%) inaccuracies inherent in either method of calculation. The mode-coupling calculations also yield the spectral shape functions defined in (3.25). An example for $T < T_c$ is shown in Fig. 5, for the function $C_m(\mathbf{k},\omega)$ which can be compared with inelastic light scattering measurements, as discussed below.

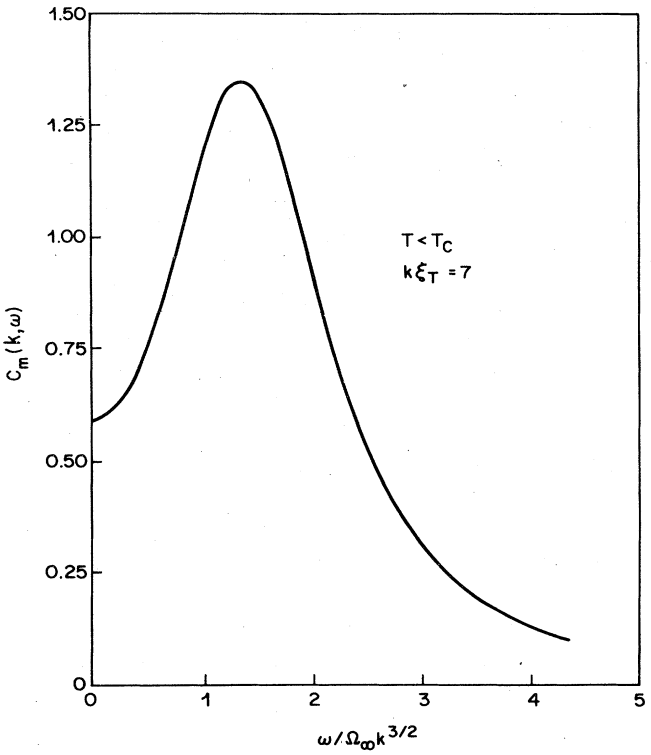

FIG. 5. The correlation function $C_m(\mathbf{k},\omega)$ in the symmetric planar magnet, obtained from a self-consistent approximation analogous to the one shown in Fig. 3, plotted as a function of the scaled frequency, for $T < T_c$. In superfluid ^{4}He, close to T_λ, this function has the same spectrum as the low-frequency part of the density-correlation function $C_\rho(\mathbf{k},\omega)$, which can be measured by inelastic light scattering experiments. From Hohenberg *et al.* (1976b).

[32]The possibility of expressing λ_m purely in terms of static parameters and a universal ratio depends on the existence of a propagating critical mode below T_c, and it also occurs in isotropic magnetic systems treated in Sec. VII. For the fluid or model C, which have relaxational critical modes, it is only combinations such as $\lambda\bar{\eta}$ or λ_m/Γ which can be expressed in this way, not the transport coefficients themselves. In the nomenclature of Kawasaki (1976), superfluid helium and isotropic magnets are "systems of the first kind," whereas the fluid and model C are "systems of the second kind."

[33]Since the spectrum of $\chi_m(\mathbf{k},\omega)$ is not Lorentzian at T_λ, the definition of the characteristic frequency $\omega_m(\mathbf{k})$ is not unique. The quantity in Eq. (6.29) is defined using Eq. (3.15); when the median frequency (3.19) is used instead, the corresponding universal ratio is designated R_λ^∞ [see Eq. (9a) of Siggia (1976), or Hohenberg *et al.* (1976b)].

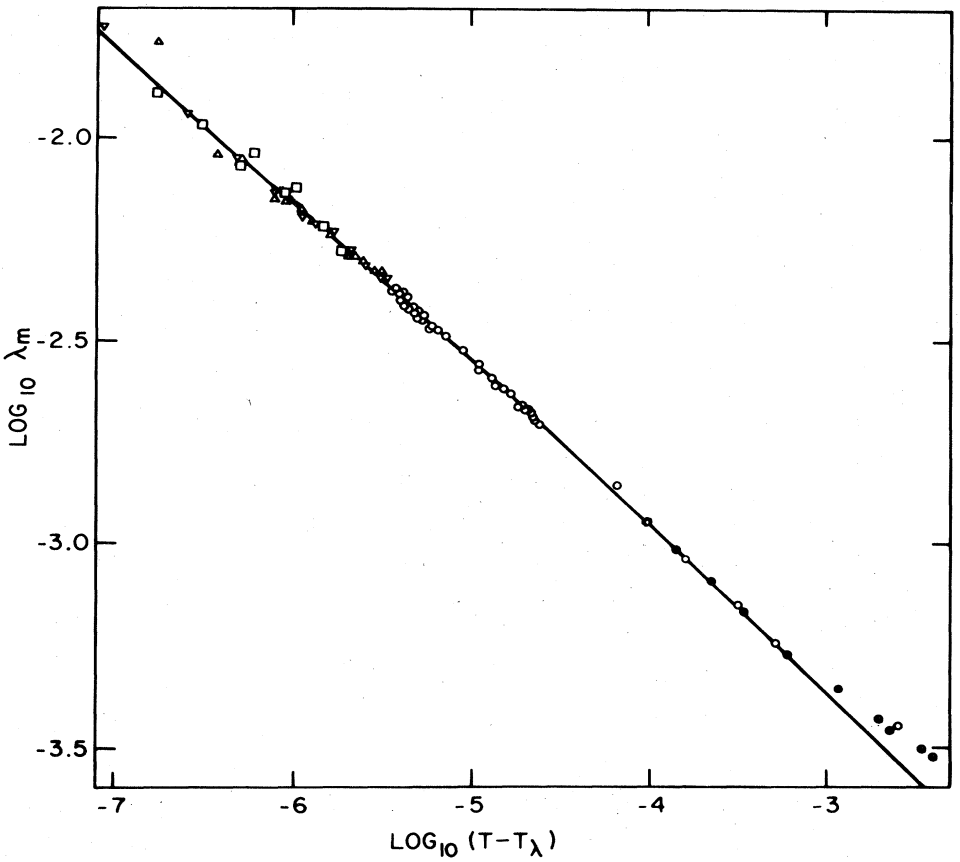

FIG. 6. The divergence of the thermal conductivity of superfluid ^{4}He for $T > T_\lambda$. The solid line represents Eq. (6.27), with R_λ adjusted to fit the data at one point. From Ahlers (1968).

D. Comparison with experiment

The earliest success of the dynamic scaling hypothesis was the verification of the predicted divergence of the thermal conductivity in liquid helium (Ferrell *et al.*, 1967; Ahlers, 1968). The observed temperature dependence, shown in Fig. 6, was consistent with Eq. (6.27) within the accuracy of the measurements, and the measured amplitude of the divergence yields a coefficient $R_\lambda^{\text{exp}} = 0.3$, in rather good agreement with the renormalization group calculations in Eqs. (6.30) or (6.33). However, subsequent, more accurate measurements indicated deviations from the exponent in Eq. (6.27) by about $\xi^{0.1}$ (Ahlers, 1971, and private communications). Moreover, some early experiments carried out in cells with heights less than 1 mm showed much larger deviations, which are as yet unexplained (Archibald *et al.*, 1968).

Macroscopic measurements of the attenuation of second sound have also been made, and they follow the temperature dependence predicted in Eq. (6.15) (Tyson, 1968). The coefficient R_2, however, has the value $R_2^{\text{expt}} \approx 0.5$, which is quite far from the theoretical estimates in (6.31) and (6.34).

Another way to test the theory for liquid helium is through inelastic light scattering measurements, which yield the spectrum of the density-correlation function. Near T_λ and at low frequencies, it may be shown that this function is proportional to $C_m(\mathbf{k}, \omega)$, calculated in the planar-spin model. Specifically, the correspondence holds for T close to T_λ and for $\omega \ll c_1 k$, where c_1 is the velocity of *first* sound, which remains finite at T_λ in helium. Given the spectrum of $C_m(\mathbf{k}, \omega)$, the characteristic frequencies may be evaluated for different values of $k\xi$, both above and below T_λ, and comparison can be made with the expressions of Sec. VI.C. The presently available experimental data, which span a sizeable range of $k\xi$ values, do not show any temperature dependence for the characteristic frequencies, in striking contrast to the dynamic scaling predictions (Winterling *et al.*, 1973, 1974; Vinen *et al.*, 1975; O'Connor *et al.*, 1975; Tarvin *et al.*, 1977). A comparison of experiment and theory is shown in Fig. 7.

Thus, although there is some evidence in favor of the theory outlined above, the overall situation remains quite unclear. It is hoped that further experimental and theoretical work will clarify the picture, since the λ transition in helium presents a unique case where accurate experiments are possible over a wide range of temperatures, frequencies, and pressures (see Ahlers, 1976). On the theoretical side, it is necessary to make quantitative estimates of the correction terms discussed above, in order to obtain realistic predictions in the experimentally accessible range.

E. Microscopic models

Various authors[34] have attempted to study the critical dynamics of helium by applying the renormalization

[34]See, for instance, Suzuki and Igarashi (1974), Yamashita and Tsuneto (1974), Suzuki (1975), Abrahams and Tsuneto (1975); see also the earlier Polyakov, 1969.

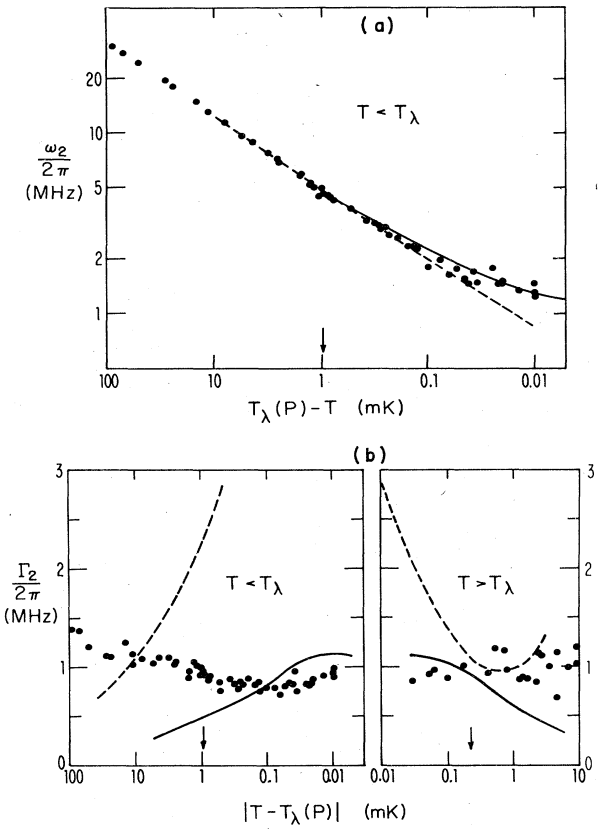

FIG. 7. Light scattering measurements of second sound and thermal relaxation in helium near T_λ, for $k = 1.8 \times 10^5$ Å^{-1} and $P \approx 23$ atm. The density correlation function $C_\rho(\mathbf{k}, \omega)$, at the given k, has been fit to the sum of two Lorentzians of width Γ_2 (half-width at half-maximum) centered at $\omega = \pm\omega_2$. The value of $\omega_2/2\pi$ thus determined is plotted vs temperature in Fig. 7(a), for $T < T_\lambda$, while $\Gamma_2/2\pi$ is plotted in Fig. 7(b) for $T < T_\lambda$ and $T > T_\lambda$. Dots represent the experimental values of Tarvin *et al.* (1977); solid curves represent a two-Lorentzian fit to the theory of Hohenberg *et al.* (1976b) for the symmetric planar-spin model. The broken curves are extrapolations from macroscopic measurements of the second-sound velocity (Greywall and Ahlers, 1973) and second-sound damping (Tyson, 1968) below T_λ, and of the thermal conductivity and specific heat (Ahlers, 1968) above T_λ. Arrows indicate temperature where $k\xi_T = 1$, for $T < T_\lambda$ and $k\xi_+ = 1$ for $T > T_\lambda$. Note that ω_2 is comparable to Γ_2 at T_λ, and the half-width at half-maximum of the density spectrum is approximately twice the value of Γ_2 at this point. The fitted value of ω_2 continues to drop for $T > T_\lambda$ (not shown in figure) and is indistinguishable from zero for $T - T_\lambda \geq 0.2$ mK. Figures taken from Tarvin *et al.* (1977).

group directly to a Hamiltonian model of helium involving only a single complex field ψ. The simplest example of such a model can be obtained from Eq. (6.3) by treating Γ_0 as pure imaginary (no dissipation) and dropping all couplings to the field m

$$\frac{\partial\psi}{\partial t} = 2i\Gamma_0'' \frac{\delta H}{\delta\psi^*} , \tag{6.36}$$

$$H[\psi] = \int d^dx \left(\tfrac{1}{2}r_0|\psi|^2 + \tfrac{1}{2}|\nabla\psi|^2 + u_0|\psi|^4\right), \tag{6.37}$$

with an appropriate cutoff at short wavelengths. If the renormalization group is properly applied to this model, however, one encounters a difficulty on the first iteration, similar to that found in the phonon Hamiltonian of Sec. IV.D. The renormalized four-point vertex $U(\mathbf{k}_i, \omega_i)$ is a singular function of the wave vectors and frequencies, when the net wave vector and frequency transfers go to zero. These singularities are due to the absence of damping in the bare propagators, and are reflections of the conservation laws in the system. Equations (6.36)–(6.37) conserve the total "particle number,"

$$N = \int |\psi|^2 d^dx , \tag{6.38}$$

as well as the total energy $E = H[\psi]$. Furthermore, as long as the cutoff is introduced in such a way that the Hamiltonian H is translationally invariant, the total momentum will also be conserved.

A microscopic derivation of stochastic model F, beginning with a model such as (6.36)–(6.37), or with the more realistic quantum Hamiltonian, would proceed along the lines suggested in Sec. IV.D for the interacting phonon system. One begins by integrating out all wave vectors larger than a cutoff Λ_1, which we choose to be less than the inverse mean free path for atomic collisions. The partially renormalized four-point vertex U_1 will then contain a number of simple poles near zero frequency transfer (when the momentum transfer is small), while the renormalized propagator for ψ will be damped and well behaved. A careful analysis of the singularities in U_1 would then involve keeping track of the positions and residues of these poles. We believe that this procedure is equivalent to introducing propagators and coupling constants for appropriate conserved densities, as has been done in the present approach.

If one attempts to remove the singularities in the vertex $U_1(\mathbf{k}_i, \omega_i)$ by simply introducing an imaginary part in the frequencies of the *initial* propagator for ψ, while keeping a structureless bare coupling constant u_0, one violates the conservation laws, and one is led by further iterations to the fixed point for a time-dependent Ginzburg–Landau model with no conserved quantity (model A), rather than to a fixed point appropriate to helium (De Dominicis *et al.*, 1975; Brézin and De Dominicis, 1975; Abrahams and Tsuneto, 1975; Tanaka, 1975).

It has been emphasized by Kawasaki and Gunton (1976b) that the fixed point of model C with $n = 2$ is *unstable* with respect to the introduction of reversible mode coupling ($g_0 \neq 0$) in the equations of motion. Thus, as long as there is a conserved field coupled via g_0 to the order parameter, the system will exhibit the critical behavior of model F, or E. On the other hand, the addition of terms in the Hamiltonian which violate the $n = 2$ symmetry of the order parameter, as well as the conservation of m, will lead to a crossover to relaxational dynamics, as in model A (Kawasaki and Hikami, 1976).

F. ^{3}He–^{4}He mixtures and tricritical dynamics

In order to describe the critical dynamics of ^{3}He–^{4}He mixtures along the λ line, it is necessary to consider two conserved densities, in addition to the superfluid order parameter ψ (Khalatnikov, 1965). We choose these to be the ^{3}He "concentration fluctuation" $c(\mathbf{x})$ and the "thermal fluctuation" $q(\mathbf{x})$, which were defined in Sec.

V.E, with the property that the equal-time cross-correlations between $c(\mathbf{x})$, $q(\mathbf{x})$, and the pressure vanish at long wavelengths.

For concentrations less than the tricritical concentration, the susceptibility χ_c behaves like the specific heat of pure ^{4}He, i.e., $\chi_c \propto \text{const} + \xi^{\alpha/\nu}$, which is finite at T_λ, since α is slightly negative, for $d=3$. The peak value of χ_c increases as the concentration approaches its tricritical value, and for higher concentrations the superfluid transition is first order. At the tricritical concentration, χ_c is predicted to diverge as ξ^{4-d}, for $d \geq 3$, with $\xi \propto (T - T_t)^{1/(2-d)}$, where T_t is the tricritical temperature. [We neglect the logarithmic corrections which occur in static properties at the tricritical point, in three dimensions (Wegner and Riedel, 1973).] The susceptibility χ_q remains finite along the λ line and at the tricritical point.

As at the consolute point of a classical binary fluid, one eigenvalue of the transport matrix λ diverges at the superfluid transition, while the other remains finite. Now, however, the divergent eigenfunction (which we identify with the variable m) is a linear combination of q and c, and the divergent part of λ does not commute with χ. A mode-coupling calculation of λ and of the order parameter kinetic coefficient Γ, in which only the nondissipative couplings are included, leads to the prediction

$$\lambda_{qq} \propto \lambda_{cc} \propto \lambda_{cq} \propto \xi^{x_\lambda}, \tag{6.39}$$

$$\Gamma \propto \xi^{4-d-x_\lambda} \approx \xi^{x_\Gamma}, \tag{6.40}$$

with $x_\lambda = x_\Gamma = (4-d)/2$, both along the λ line and at the tricritical concentration (Kawasaki and Gunton, 1972; Grover and Swift, 1973). Note that the macroscopically measured thermal conductivity is proportional to $\det\lambda/\lambda_{cc}$, which remains finite, for finite concentrations (see Ahlers, 1976). The divergence of the thermal conductivity along the λ line as one approaches pure ^{4}He has been studied both experimentally and theoretically, but the present situation is still controversial (see Tanaka *et al.*, 1977; Siggia, 1977).

It was pointed out by Siggia and Nelson (1977) that the dissipative coupling between ψ and c, which is ignored in the mode-coupling approach, should be quite important near the tricritical point. These authors have written down recursion relations which take both dissipative and nondissipative couplings into account, and have obtained for the tricritical behavior:

$$x_\lambda = (2\epsilon/3) + O(\epsilon^2), \tag{6.41}$$

$$x_\Gamma = \epsilon - x_\lambda = (\epsilon/3) + O(\epsilon^2), \tag{6.42}$$

where $\epsilon = 4-d$. As was noted by Siggia and Nelson, however, these results must be viewed with considerable caution. The slower of the two diffusive modes has a relaxation rate $Dk^2 \propto \chi_c^{-1}k^2$, which is slow on the scale of the order parameter relaxation $\omega_\psi \propto \xi^{x_\Gamma - 2}$. The dissipative coupling between ψ and this slow mode leads to serious difficulties in the justification of the recursion relations, similar to the problems encountered in the case of model C, with $2<n<4$ and $\epsilon \to 0$ (see Sec. IV.C.3). The tricritical point of ^{3}He–^{4}He mixtures thus seems to be an appropriate system in which to study these subtle effects, since they may be experimentally accessible. Recent measurements of the attenuation of first sound in ^{3}He–^{4}He mixtures (Roe *et al.*, 1977) support the predictions (6.41) and (6.42).

G. Two-dimensional superfluid films

A problem of considerable theoretical and experimental interest is the transition between the normal and superfluid states in two dimensional helium films (i.e., films with thickness much smaller than the correlation length. It is known rigorously that such films cannot exhibit true long range order in the boson operator ψ, at any temperature $T>0$ (Hohenberg, 1967). Nevertheless, it is believed that there is a well-defined superfluid transition temperature T_c, and some progress has been made towards understanding the nature of the transition. (See Kosterlitz and Thouless, 1977; and references therein; José *et al.*, 1977; Luther and Scalapino, 1977). The superfluid order parameter exhibits some characteristics of a critical point at all temperatures below T_c. Critical behavior in the dynamics has been considered by a number of authors. (See Langer, 1968, Villain, 1973, Blank *et al.*, 1974, and Nelson and Fisher, 1977.) Needless to say, expansions in $4-d$ are not particularly useful in this case.

VII. HEISENBERG MAGNETS

A. Antiferromagnet

1. Model G

The isotropic antiferromagnet is another system in which reversible mode-coupling terms are important. The simplest model which represents this system (model G) consists of two densities, a nonconserved order parameter $\boldsymbol{\psi}$ which is a three-component vector representing the staggered magnetization, and a conserved density $\mathbf{m}$, also a three-component vector, representing the total magnetization of the system. The equations of motion are given by

$$\frac{\partial \boldsymbol{\psi}}{\partial t} = -\Gamma_0 \frac{\delta F}{\delta \boldsymbol{\psi}} + g_0 \boldsymbol{\psi} \times \frac{\delta F}{\delta \mathbf{m}} + \theta, \tag{7.1a}$$

$$\frac{\partial \mathbf{m}}{\partial t} = \lambda_0 \nabla^2 \frac{\delta F}{\delta \mathbf{m}} + g_0 \boldsymbol{\psi} \times \frac{\delta F}{\partial \boldsymbol{\psi}} + g_0 \mathbf{m} \times \frac{\delta F}{\delta \mathbf{m}} + \boldsymbol{\zeta}, \tag{7.1b}$$

$$F[\boldsymbol{\psi}, \mathbf{m}] = F_0 - \int d^d x \{\mathbf{h}(\mathbf{x}, t) \cdot \boldsymbol{\psi} + \mathbf{h}_m(\mathbf{x}, t) \cdot \mathbf{m}\}, \tag{7.1c}$$

$$F_0 = \int d^d x \{\tfrac{1}{2} r_0 \psi^2 + \tfrac{1}{2} |\nabla \boldsymbol{\psi}|^2 + u_0 \psi^4 + \tfrac{1}{2} (\chi_0^m)^{-1} m^2\}, \tag{7.1d}$$

where θ and $\boldsymbol{\zeta}$ are Langevin noise sources. The nondissipative terms involving $g_0\psi$ in (7.1a) and (7.1b) reflect the Poisson-bracket relation between $\boldsymbol{\psi}$ and $\mathbf{M} \equiv \int d^d x \mathbf{m}(\mathbf{x})$,

$$\{\psi_\alpha, M_\beta\} = g_0 \epsilon_{\alpha\beta\gamma} \psi_\gamma \tag{7.2}$$

($\epsilon_{\alpha\beta\gamma}$ is the antisymmetric unit tensor), while the third term on the right-hand side of (7.1b) reflects the Poisson-bracket relation

$$\{M_\alpha, M_\beta\} = g_0 \epsilon_{\alpha\beta\gamma} M_\gamma. \tag{7.3}$$

The coupling constant g_0 is equal to unity if the magnetization $\mathbf{M}$ is measured in units of angular momentum,

and $k_B T_c = 1$. Equations (7.2) and (7.3) in turn express the fact that the magnetization is the infinitesimal generator of rotations of ψ and $\mathbf{M}$, a transformation which leaves the free-energy functional F_0 invariant.

The lowest coupling between $\mathbf{M}$ and ψ in F_0 will be of the form $M^2\psi^2$ or $(\mathbf{M}\cdot\psi)^2$, both of which are irrelevant at the critical point. As in the planar ferromagnet or liquid helium, the Poisson-bracket relations have important consequences for the dynamics of the system. In particular one can prove a Larmor precession theorem for the system (see Sec. VI.A). The fact that the Larmor precession theorem holds exactly for this model means that the coupling constant g_0 is not renormalized by the interactions, i.e., the frequency scale is set by g_0 (Halperin *et al.*, 1976b).

The hydrodynamic modes of the antiferromagnet are similar to those of liquid helium with a propagating (spin-wave) mode below T_c, and decaying modes above (Halperin and Hohenberg, 1969b). Since the density $\mathbf{m}$ is a three-component vector, it has both longitudinal and transverse components below T_c, of which only the latter participate in the spin-wave modes. The longitudinal components of both $\mathbf{m}$ and ψ appear to have rather complicated behavior even away from T_c, as indicated from low-order mode-coupling and ϵ-expansion calculations (Villain, 1970; Michel and Schwabl, 1970; Mazenko *et al.*, 1977; Sasvári and Szépfalusy, 1977b).

2. Critical behavior

Early applications of dynamic scaling and mode-coupling methods led to the prediction that the transport coefficient λ_m and the kinetic coefficient Γ will diverge as $T \to T_c^+$ (see Halperin and Hohenberg 1969a; Kawasaki, 1976). The exponents turn out to be the same as in the symmetric planar-spin model (E), treated in Sec. VI.B, for which

$$z = d/2 . \tag{7.4}$$

One again finds a scaling relation for x_λ leading to the expressions

$$\lambda_m = R_\lambda g_0 \xi^{\epsilon/2} \chi_m^{1/2} , \tag{7.5}$$

$$\Gamma = R_\Gamma g_0 \xi^{-d/2} \chi_m^{-1/2} \chi_\psi k_B T_c , \tag{7.6}$$

where R_λ and R_Γ are universal constants, which may be estimated from the ϵ expansion (Halperin *et al.*, 1976b), or from self-consistent mode-coupling calculations. The scaling functions for finite $\mathbf{k}$ and ω may also be expressed in the form of Eq. (3.25), and evaluated in various approximations (see Kawasaki, 1976, Sec. VIII.A.2, and references therein; Freedman and Mazenko, 1975, 1976). An example is shown in Fig. 8.

3. Couplings to other fields and effects of anisotropy

If one considers a spin system, defined by an exchange Hamiltonian, which orders antiferromagnetically, then in addition to the densities $\mathbf{m}$ and ψ present in model G, the hydrodynamics also contains the energy density $e(\mathbf{x}, t)$ (Halperin and Hohenberg, 1969b). When the dissipative coupling between the e and ψ fields is considered, energy conservation can in principle modify the critical behavior of the order parameter, if the specific heat

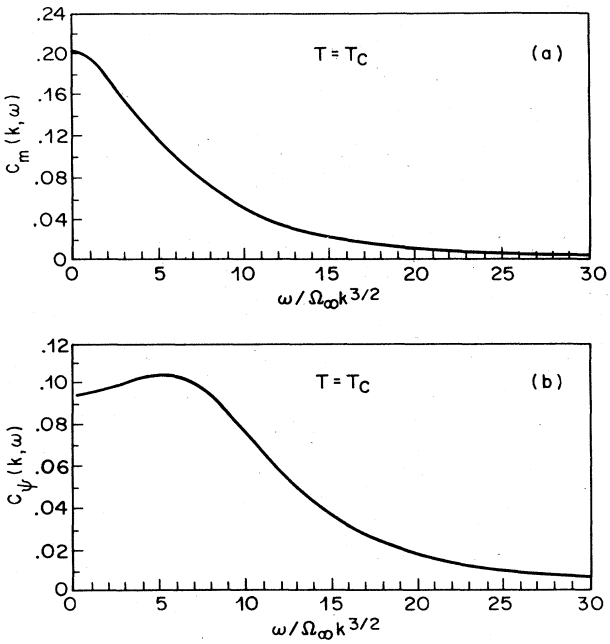

FIG. 8. The correlation functions $C_\psi(\mathbf{k},\omega)$ and $C_m(\mathbf{k},\omega)$ for the staggered and total magnetizations, respectively, in the Heisenberg antiferromagnet at T_c, plotted as a function of the reduced frequency $\omega/\Omega_\infty k^{3/2}$. The calculations (Wegner, 1969) were performed using a self-consistent mode-coupling approximation analogous to the one depicted in Fig. 3. Note the remnant of spin-wave propagation at T_c in the staggered magnetization, predicted in this approximation. After Wegner (1969).

exponent is positive, as is the case for d near 4. In three dimensions, however, the exponent α is expected to be negative for $n = 3$, and only the corrections to scaling will be affected (Halperin *et al.*, 1976b). In real systems, of course, the remarks made in Sec. IV.D on the role of spin-lattice coupling near T_c will once again apply, and it is not clear *a priori* whether a model with or without energy conservation is more appropriate.

It is possible to add to the free-energy functional (7.1d) a wide variety of anisotropic terms, and thus to study the crossover from the isotropic antiferromagnetic fixed point to other types of critical behavior (Fisher, 1974). Of greatest practical interest is the crossover to a uniaxial system (represented by models A or C), due to the presence of anisotropy. The crossover of the isotropic antiferromagnet to relaxational behavior of type A may also occur as a result of the addition of terms with cubic anisotropy, which destroy the conservation of $\mathbf{m}$. Another interesting effect comes from the application of a uniform magnetic field which causes crossover to behavior characteristic of the asymmetric x-y model (model F) (Halperin and Hohenberg 1969b; Fisher and Nelson, 1974). See also the discussion on multicritical points in magnetic systems in Sec. VIII.D.3, below.

4. Experimental studies

The antiferromagnet is a favorable case for experimental study since the dynamic correlation functions for both the staggered magnetization ψ and the total magnet-

ization $\mathbf{m}$ can be measured by the technique of inelastic neutron scattering, as a function of $\mathbf{k}$, ω, and temperature, on a variety of isotropic and anisotropic materials. (See Als-Nielsen 1976; Tucciarone *et al.*, 1971, Hohenberg, 1971a). Unfortunately, however, the presently attainable accuracy of such experiments is considerably poorer than for macroscopic or light scattering measurements, and it has not yet proved possible to test the quantitative details of the theories described above. Moreover, there are inherent limitations to the accuracy of experiments near T_c, due to the presence, in real solids, of impurities, crystal imperfections, complicated interactions with lattice degrees of freedom, and small anisotropy fields, of unknown magnitude. Nevertheless, the main predictions of dynamic scaling and mode coupling have been verified at least semiquantitatively for the isotropic magnet $RbMnF_3$, and for a number of anisotropic systems, such as MnF_2 or FeF_2 (see Als-Nielsen 1976; Kawasaki, 1976). For instance, the dynamic exponent $z=\frac{3}{2}$, Eq. (7.4), was shown to be consistent with data above and at T_c in $RbMnF_3$ (see Fig. 9), and the scaling function $\Omega(k\xi)$, Eq. (3.27), was measured and compared with theory (see Tucciarone *et al.*, 1971; Als-Nielsen, 1976). In the limit $k\xi \ll 1$ the behavior in Eqs. (7.5)–(7.6) was found, with $R_\lambda^{\text{expt}}=0.17$ and $R_\Gamma^{\text{expt}}=0.23$, which compares favorably with the ϵ expansion estimates $R_\lambda=0.16$ and $R_\Gamma=0.16$ (Halperin *et al.*, 1976b). The crossover from isotropic to uniaxial behavior was observed in both FeF_2 and MnF_2 (Schulhof *et al.*, 1970) in reasonable agreement with mode-coupling calculations (see Kawasaki, 1976; Bagnuls and Joukoff-Piette, 1975).

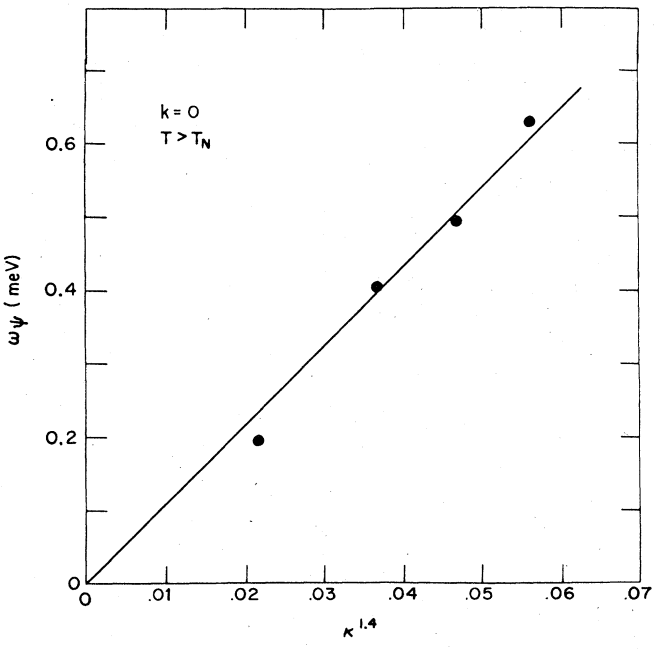

FIG. 9. Critical slowing down of the characteristic frequency $\omega_\psi(\mathbf{k}=0)$ for the staggered magnetization of the Heisenberg antiferromagnet $RbMnF_3$ for $T>T_N$ as measured by inelastic neutron scattering, plotted as a function of $\kappa^{1.4}\approx 0.35\,(T-T_N)/T_N$. The quantity κ is the inverse correlation length, measured in Å^{-1}. After Tucciarone *et al.* (1971).

B. Isotropic ferromagnet

1. Model J

The simplest model for an isotropic ferromagnet (model J), has a conserved three-component vector field whose components are coupled according to the Larmor precession law (Ma and Mazenko, 1975)

$$\frac{\partial \boldsymbol{\psi}}{\partial t}=\lambda_0\nabla^2\frac{\delta F}{\partial \boldsymbol{\psi}}+g\boldsymbol{\psi}\times\frac{\delta F}{\delta \boldsymbol{\psi}}+\theta \tag{7.7a}$$

$$F=F_0-\int d^dx\,\{\mathbf{h}(\mathbf{x},t)\cdot\boldsymbol{\psi}\}\,, \tag{7.7b}$$

$$F_0=\int d^dx\,\{\tfrac{1}{2}r_0\psi^2+\tfrac{1}{2}|\nabla\boldsymbol{\psi}|^2+u_0\psi^4\}\,. \tag{7.7c}$$

This model represents a Heisenberg ferromagnet, without energy conservation. In the paramagnetic phase ($T>T_c$) the order parameter has diffusive behavior with frequency

$$\omega_\psi(\mathbf{k})=(\lambda/\chi_\psi)k^2\,. \tag{7.8}$$

In the ordered phase ($T<T_c$) the components of $\boldsymbol{\psi}$ transverse to $\langle\boldsymbol{\psi}\rangle$ have propagating spin-waves

$$\omega_\psi(\mathbf{k})=\zeta k^2\,, \tag{7.9}$$

with

$$\zeta=g_0\rho_s/|\langle\psi\rangle|\,, \tag{7.10}$$

where ρ_s is the stiffness constant defined by an equation analogous to (6.8). The longitudinal component of $\boldsymbol{\psi}$ is expected to relax to its average value $\langle\boldsymbol{\psi}\rangle$, but the precise form of this relaxation is not known; it is probably singular in the limit $k\to 0, \omega\to 0$, due to nonlinear interactions among the spin-wave modes (see Villain, 1971a, 1971b; Sasvári, 1977).

2. Dynamic scaling and mode coupling

At the phase transition, it is possible to relate the dynamic exponent z to static exponents, by arguments similar to those employed for liquid helium or the antiferromagnet. Using Eqs. (7.9)–(7.10) and the static scaling relations, one thus finds (Halperin and Hohenberg, 1969a; Wagner, 1970)

$$z=d-\beta/\nu=\tfrac{1}{2}(d+2-\eta)\,. \tag{7.11}$$

Recalling that $\eta=0$ for $d\geq 4$, we note that the dynamic exponent reaches its conventional value $z=4-\eta$ for $d=6$, in contrast to the antiferromagnet, Eq. (7.4), where the conventional value $z=2-\eta$ is reached for $d=4$ (Villain, 1968; Kawasaki, 1968b; Hohenberg *et al.*, 1973). Applying the dynamic scaling assumption to Eq. (7.8) we find that the transport coefficient λ diverges as[35]

$$\lambda\approx\xi_+^{(6-d-\eta)/2}\,, \tag{7.12}$$

which again shows that the Van Hove theory breaks down for $d<6$.

[35] In what seems to be the first example of a self-consistent calculation of a divergent transport coefficient at T_c, Bennett and Martin (1965) obtained the result $\lambda\propto\chi^{3/4}$, which corresponds precisely to (7.12), in three dimensions and with an Ornstein-Zernike approximation ($\eta=0, \chi\propto\xi_+^2$). See also Kawasaki (1967).

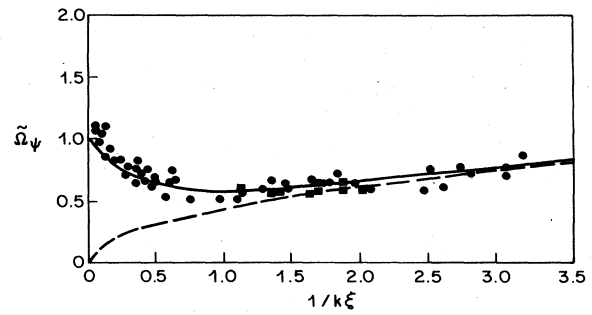

FIG. 10. The scaling function $\tilde{\Omega}_\psi$ [Eq. (3.27)] for the characteristic frequency of the order parameter in Fe above T_c, plotted as a function of the dimensionless parameter $(k\xi)^{-1}$. The dots are from the neutron scattering data of Parette and Kahn (1971), and the solid line is the mode-coupling calculation of Résibois and Piette (1970). The dashed line shows the hydrodynamic behavior $\tilde{\Omega}_\psi(k\xi) \propto (k\xi)^{-1/2+\eta/2}$, which is correct for $k\xi \ll 1$ (leading to $\omega_\psi(k) \propto \xi^{-1/2+\eta/2}k^2$), but which breaks down for $k\xi > 1$ ($T \to T_c$). After Parette and Kahn (1971).

The dynamic scaling predictions (7.11) and (7.12) are easily obtained from a self-consistent mode-coupling calculation based on the lowest-order graphs, as discussed for the fluid in Sec. V.B. This calculation may also be generalized to finite $\mathbf{k}$ and ω, and the scaling function $\tilde{\Omega}(x)$ and $Y(w,x)$ [cf. Eqs. (3.27) and (3.25)] obtained numerically. An example of the function $\tilde{\Omega}(x)$ calculated by Résibois and Piette (1970) is shown in Fig. 10 for the case $T \geq T_c$, and compared to data on Fe.

3. Renormalization group

Recursion relations may be obtained for the quantities λ_l and g_l, to linear order in $\epsilon = 6 - d$, and a finite nonzero fixed point found for the quantity $f_l = g_l/\lambda_l$, for $d < 6$ (Ma and Mazenko, 1975; Dohm, 1976; Nolan and Mazenko, 1977). This result leads to the scaling law (7.11), which may be reexpressed in the form

$$\lambda = R_\lambda g_0 \chi_\psi^{1/2} \xi_+^{(4-d)/2}, \tag{7.13}$$

where R_λ is a universal amplitude. Just as in the previous cases, it may be shown (Kawasaki, 1975; Bausch *et al.*, 1976) that the scaling relations (7.11) will hold to all orders in ϵ. Of course there will be new nonanalyticities at $d = 4$, where the static behavior changes. Thus to reach $d = 3$, formally, one must make a double power series expansion in $6 - d$ and $4 - d$.

4. Comparison with experiment

As in the case of the antiferromagnets, inelastic neutron scattering provides the most complete method of investigating the critical dynamics of ferromagnets. Experiments on the metals Fe, Co, and Ni, and the anisotropic insulators $CrBr_3$ and MnP are consistent with the theory outlined above, but the experiments do not provide a stringent test of that theory (see Kawasaki, 1976; Als-Nielsen, 1976, and references therein). Recently, an extensive set of measurements were performed on the isotropic insulator EuO, and quantitative comparison was made with dynamic scaling and mode-coupling predictions (Dietrich *et al.*, 1976). The general agreement was found to be satisfactory, but specific deviations were observed in both the power laws and the magnitudes of the scaling functions. The universal amplitude for $\omega_\psi(\mathbf{k})$ at T_c, analogous to R_λ in (7.13), was found to deviate substantially from mode-coupling estimates for the Heisenberg system, but the scaling function $\tilde{\Omega}(x)$ [see Eq. (3.27)] showed reasonable agreement. Some of the deviations could be interpreted in terms of dipolar forces, which have an important effect on the dynamics of Heisenberg systems at long wavelengths, since these forces destroy the conservation of total spin. Calculations of these effects have been carried out by Villain (1971b), Maleev (1974, 1975), by Raghavan and Huber (1976), and by Finger (1977), using mode-coupling theory to describe the crossover between Heisenberg and dipolar behavior.

VIII. MISCELLANEOUS TOPICS

A. Other dynamic properties

1. Sound propagation, EPR, and NMR

In our previous discussions, emphasis was placed on the relaxation rate of the order parameter, or of certain other conserved densities, which couple to the order parameter at long wavelengths through dissipative and nondissipative interactions. In many other experimental situations one is interested in the relaxation of a nonconserved variable, which contains a part proportional to the square of the order parameter ψ. An example of such an operator is the "energy density"

$$e(\mathbf{x}, t) = (dr_0/dT)\psi^2(\mathbf{x}, t), \tag{8.1}$$

for model A, or any other model in which energy is not conserved. A second example is the quadrupole operator

$$Q_{\alpha\beta} = \psi_\alpha\psi_\beta - n^{-1}\psi^2\delta_{\alpha\beta}, \tag{8.2}$$

for an isotropic n-component model such as the Heisenberg ferromagnet or antiferromagnet, or for a relaxational model with $n > 1$. The correlation function $C_e(\mathbf{k}, \omega)$ plays an essential role in calculations of the sound attenuation and velocity shift near a magnetic critical point, whereas both $C_e(\mathbf{k}, \omega)$ and $C_Q(\mathbf{k}, \omega)$ may be important for sound propagation near a structural transition with a multicomponent order parameter, such as is found in the perovskites (Murata, 1976b). The correlation function $C_Q(\mathbf{k}, \omega)$ is important for electron-paramagnetic-resonance (EPR) in the limit of weak dipolar or anisotropy fields (Huber, 1971; Kawasaki, 1968, 1976).

According to the scaling hypothesis for static critical phenomena, the equal-time correlation functions $C_e(\mathbf{k})$ and $C_Q(\mathbf{k})$ behave as

$$C_e(k) \approx \text{const} + \kappa^{-\alpha/\nu} X_e(k/\kappa), \tag{8.3}$$

$$C_Q(k) \approx \kappa^{-x} X_Q(k/\kappa), \tag{8.4}$$

where α is the specific heat exponent, and the exponent x is related to the "crossover exponent" ϕ for axial anisotropy via

$$x = \frac{2\phi}{\nu} - d = \frac{n+1}{n+2}\epsilon + \frac{2n^2 - 9n + 7}{8(n+2)^3}\epsilon^2 + O(\epsilon^3) \tag{8.5}$$

(Wilson, 1972; Fisher, 1974). If we apply the dynamic scaling hypothesis (3.25) to $C_Q(\mathbf{k}, \omega)$ and to the singular

part of $C_e(\mathbf{k},\omega)$, we predict that these functions diverge as

$$C_e(\mathbf{k},\omega) \approx \kappa^{-z-\alpha/\nu} Y_e(\omega/\kappa^z, k/\kappa)\,, \tag{8.6}$$

$$C_Q(\mathbf{k},\omega) \approx \kappa^{-x-z} Y_Q(\omega/\kappa^z, k/\kappa)\,, \tag{8.7}$$

where z is the dynamic scaling exponent for ψ. The functions Y_e and Y_Q are expected to be finite as ω/k^z and $k/\kappa \to 0$. The behavior of $C_e(\mathbf{k},\omega)$ for the relaxational models (A-D), near $d=4$, has been studied using renormalization group techniques by Halperin *et al.* (1976a) and Suzuki (1973b). The results for models A and B, where the energy is not conserved, are indeed in agreement with (8.6). The scaling form (8.6) is also found to be correct for model C (where energy is conserved) in that region of the $d-n$ plane where $z = 2+\alpha/\nu$. In model D,[21] and in model C when $z > 2+\alpha/\nu$, Eq. (8.6) is correct only when k/κ is not too small.

The analysis of Halperin *et al.* (1976a) may readily be extended to the function $C_Q(\mathbf{k},\omega)$ for the relaxational models with $n>1$, as well as to the functions $C_e(\mathbf{k},\omega)$ and $C_Q(\mathbf{k},\omega)$ for a variety of other systems, including model G, the Heisenberg antiferromagnet without energy conservation. In each of these cases the dynamic scaling results are found to hold, at least near $d=4$.

It is interesting to compare (8.6) and (8.7) with the decoupling approximation

$$\langle \psi_\alpha(\mathbf{x},t)\psi_\beta(\mathbf{x},t)\psi_\alpha(0,0)\psi_\beta(0,0)\rangle \approx (1+\delta_{\alpha\beta})\langle \psi_\alpha(\mathbf{x},t)\psi_\alpha(0,0)\rangle^2\,, \tag{8.8}$$

which has often been made in the literature (see Huber, 1971; Kawasaki, 1976). This approximation leads to the prediction

$$C_e(\mathbf{k},\omega) \propto C_Q(\mathbf{k},\omega) \propto \kappa^{d-2(2-\eta)-z} f(\omega/k^z, k/\kappa)\,. \tag{8.9}$$

For the three-dimensional Heisenberg model, the exponent x has a value ≈ 0.8, according to Eq. (8.5) or $x \approx 0.51$, according to the high-temperature series expansions of Pfeuty *et al.* (1974). The difference between the exponents in (8.7) and (8.9) is numerically small (0.1-0.4), and the decoupling approximation is probably quite reasonable for $C_Q(\mathbf{k},\omega)$. On the other hand, this approximation overestimates the divergence of $C_e(\mathbf{k},\omega)$ by a more considerable margin.

There exists an approximate decoupling formula for $C_e(\mathbf{k},\omega)$, based on mode-coupling ideas, which preserves the correct scaling behavior at long wavelengths, and is therefore more accurate than (8.8) (see Kadanoff and Swift, 1968a; Kawasaki, 1976, and references therein). In terms of the correlation functions $C_e(\mathbf{k},t)$ and $C_\psi(\mathbf{k},t)$, this approximation may be written as

$$C_e(\mathbf{k}=0,t) \approx \int \frac{d^dp}{(2\pi)^d}\left[\frac{d\ln\chi_\psi^{-1}(\mathbf{p})}{dT}\right]^2 [C_\psi(\mathbf{p},t)]^2\,. \tag{8.10}$$

Analysis of sound propagation experiments in magnetic systems is complicated by the occurrence of several different regimes, depending on whether or not energy exchange between the spins and the phonon system is important at the frequency of interest. The correlation function of the "magnetic shear tensor,"

$$T_{ij} = (\nabla_i\psi)(\nabla_j\psi) - d^{-1}\delta_{ij}|\nabla\psi|^2$$

may also be important in some cases. Reviews of the extensive experimental and theoretical literature in the field may be found in Kawasaki (1976), Garland (1970), the Lüthi *et al.* (1970).

Measurements of EPR linewidths near the critical point of Heisenberg-like systems have typically shown divergences that are considerably weaker than would be predicted from theories using either (8.7) or (8.8). The origin of these discrepancies is not understood at the present time (see Kawasaki, 1976).

The critical behavior of an NMR line near a magnetic transition depends on the symmetry of the nuclear sites, and on the form of the coupling between the nucleus and the magnetic order parameter. When linear coupling is permitted, as in the case of FeF_2, cited in Sec. IV.D.2, the NMR linewidth is proportional to

$$\int \frac{d^dk}{(2\pi)^d}\, C_\psi(\mathbf{k},\omega=0) \approx \kappa^{d-(2-\eta)-z}\,, \tag{8.11}$$

(Heller, 1966; Halperin and Hohenberg, 1969a; Gottlieb and Heller, 1971). Measurements of perturbed angular correlation of γ rays give information which is similar in principle to the NMR measurements (Gottlieb and Hohenemser, 1973).

Along with NMR techniques, EPR studies of paramagnetic impurities provide a tool for examining order parameter relaxation at a *structural phase transition.* Here, again, the results obtained depend on the symmetry of the spin site, the nature of the coupling to the order parameter, the direction of the applied magnetic field, and the ratio of the order parameter relaxation rate to the resonance frequency shifts in the low-temperature phase. (See, for example, Müller and Rigamonti, 1976; Petschek and Halperin, 1977.)

Anomalies in sound propagation have also been observed at the gas-liquid and binary-fluid critical points and discussed using mode-coupling ideas (see Kawasaki, 1976). An approximation analogous to (8.10) leads to an expression for the complex sound attenuation at long wavelengths of the form

$$\hat{\alpha}(\omega) = \frac{\omega^2}{2c^3}\,\frac{k_B T^3}{\rho^3 C_v^2}\left(\frac{\partial P}{\partial T}\right)_V^2 \int \frac{d^dp}{(2\pi)^d}\left(\frac{\partial \ln\chi_\psi(\mathbf{p})}{\partial T}\right)_S^2 \times \int_0^\infty dt e^{i\omega t}\left(\frac{C_\psi(\mathbf{p},t)}{C_\psi(\mathbf{p})}\right)^2, \tag{8.12}$$

from which the velocity $c(\omega)$ and attenuation $\alpha(\omega)$ follow, as

$$\frac{c(\omega)-c(0)}{c(\omega)} = \frac{c(\omega)}{2\omega}\,\mathrm{Im}\hat{\alpha}(\omega)\,, \tag{8.13}$$

$$\alpha(\omega) = \mathrm{Re}\hat{\alpha}(\omega)\,. \tag{8.14}$$

Equation (8.12) can be compared to experiments using calculated forms for χ_ψ and C_ψ, without any adjustable parameters. The agreement obtained is quite good, in both temperature and frequency dependence, except at the highest frequencies $[\omega \gtrsim 10\,\omega_\psi(k=\kappa)]$. It is to be remarked that in this frequency range the theory becomes very sensitive to the precise behavior of $\chi_\psi(\mathbf{p})$ for $p\xi > 1$, and a more careful discussion of Eq. (8.12) is necessary (Kawasaki, 1976).

In helium just above the lambda point there is a fluctuation contribution to $\hat{\alpha}(\omega)$ analogous to the one in Eq.

(8.12). Below T_λ, the theory becomes more complicated, as C_e contains terms such as $\psi_0^2\langle\delta\psi_r(t)\delta\psi_r(0)\rangle$ and $\psi_0\langle\delta\psi_r(t)\delta|\psi^2|(0)\rangle$, where $\psi_0 \equiv \langle\psi\rangle$ is assumed to be real, $\delta|\psi|^2 \equiv |\psi|^2 - \psi_0^2$, and $\delta\psi_r \equiv \mathrm{Re}\psi - \psi_0$. The term $\psi_0^2\langle\delta\psi_r\delta\psi_r\rangle$, which is the only one that contributes to the sound absorption in mean field theory, was first considered by Landau and Khalatnikov (1954), and has been discussed by many authors. In the scaling region there will be cancellations between the various contributions, but no detailed analysis has yet been presented which satisfactorily explains all the experimental data (see Williams and Rudnick, 1970; Khalatnikov, 1969; Hohenberg, 1971a; Kawasaki, 1976; Ginzburg and Sobyanin, 1976; Kroll and Kawasaki, 1977; Tozaki and Ikushima, 1977). We remark, finally, that the Landau–Khalatnikov process will also contribute in magnetic systems, but, as mentioned earlier, the interpretation of experiments is more complicated in this case (see Kawasaki, 1976).

2. Electrical conductivity

There has been considerable experimental and theoretical interest in the temperature dependence of the electrical conductivity near the critical point of a ferromagnet or antiferromagnet, or near the order-disorder transition of various metallic binary alloys. It is well known that the conductivity is finite at T_c (conventional theory applies) and interest centers on the temperature derivative $d\sigma/dT$. Most of the theoretical investigations have been based on the formula

$$\sigma = (ne^2/m^*)\tau\,, \tag{8.15}$$

where n is the carrier density, m^* is the electronic effective mass, and τ is the transport lifetime for scattering of the conduction electrons. The scattering rate, in turn, is assumed to be of the form (de Gennes and Friedel, 1958)

$$\tau^{-1} \propto V^2 \int d^3q f(\mathbf{q}) C_s(\mathbf{q})\,, \tag{8.16}$$

where $C_s(\mathbf{q})$ is the equal-time correlation function for the spins, V is the scattering matrix element for a single spin, and

$$f(\mathbf{q}) = q^3, \quad \text{for} \quad q < 2k_F$$
$$= 0, \quad \text{for} \quad q > 2k_F\,. \tag{8.17}$$

(For an order-disorder transition, one considers scattering from pseudospins which are equal to ±1, according to which atomic species occupies a given lattice site.) It was pointed out by Mannari (1968) and Fisher and Langer (1968), for the case of a ferromagnet, that the main contribution to the temperature dependence of Eq. (8.16) comes from wave vectors of order $2k_F$, and that $dC_s(2k_F)/dT$ is expected to be positive and to have the same singularity as the specific heat, for $T \to T_c^+$. It follows that $d\sigma/dT$ should be negative and proportional to the specific heat, in this limit.[36,37] A somewhat more detailed analysis (e.g., Kasuya and Kondo, 1974a) of Eq. (8.16) leads to the same conclusion for the ferromagnet below T_c. For the antiferromagnet or the order-disorder transition, Eq. (8.16) predicts that $d\sigma/dT$ will be positive in most cases, but again proportional to the specific heat, above and below T_c, provided that one includes the contribution of the Bragg peak in the correlation function C_s below T_c (Suezaki and Mori, 1968; Kasuya and Kondo, 1974b).

In a magnetic semiconductor, the temperature dependence of the carrier density is of great importance in determining the behavior of $d\sigma/dT$. Once again, however, one is led to a specific heat singularity in $d\sigma/dT$. Changes in carrier density may also become important in metallic antiferromagnets a short distance below T_c, and may lead to deviations from specific heat-like behavior in this case (Suezaki and Mori, 1969).

Alexander *et al.* (1976) have been able to interpret the qualitative behavior of the electrical conductivity in a variety of ferromagnetic and antiferromagnetic metals and semiconductors in terms of the formulas (8.15)–(8.17). The asymptotic critical behavior of $C_s(\mathbf{k})$ is not necessarily reached in the temperature range of interest, and Alexander *et al.* find it necessary to consider the crossover from scaling behavior to a normalized Ornstein–Zernike form for $C_s(\mathbf{k})$.

Despite these successes, it should be borne in mind that the starting equations (8.15)–(8.16) are based on assumptions which may not be reasonable very close to T_c. The assumptions behind (8.15) and (8.16) are:

(i) that the relaxation rate for fluctuations in the spin system is small compared to $k_B T_c/\hbar$,

(ii) that the scattering may be treated in the lowest Born approximation, and

(iii) that the relaxation of the current can be characterized by a single relaxation rate.

Assumption (i) is probably not too severe; qualitatively similar behavior would be expected even if finite frequencies were taken into account. Assumption (ii) is exact in the limit where V is small, at least for a fixed correlation function $C_s(\mathbf{q})$. The Born approximation must be used with some care, however, when the electron mean free path is shorter than the correlation length for the spins.

Assumption (iii) is exact for a spherical Fermi surface when $k_B T_c \ll E_F$, provided that $C_s(\mathbf{q})$ depends only on the magnitude of $\mathbf{q}$ and not on its direction. This last assumption is particularly unreasonable in the case of an antiferromagnet or order-disorder transition, since $C_s(\mathbf{q})$ is then divergent at points which occur only in certain discrete directions in the Brillouin zone. In this more general case, different regions of the Fermi surface may have different scattering rates, and it is necessary in principle to solve an *integral equation* (Boltzmann equation) for the relaxation of the current, even when the Born approximation is valid (Ziman, 1960).

[36] It was independently pointed out by Nabutovskii and Patashinskii (1968) that there is a contribution to $d\sigma/dT$ proportional to C_p, arising from changes in the electron-phonon scattering. This contribution is additional to the one arising from the change in magnetic scattering.

[37] In their original paper, deGennes and Friedel (1958) employed the Ornstein-Zernike form for $C_s(\mathbf{q})$, which is not sufficiently accurate at large q, and leads to qualitatively incorrect behavior of $d\sigma/dT$.

Solution of this integral equation may lead to a singularity in $d\sigma/dT$ which differs from the behavior of an averaged relaxation rate, calculated according to (8.15)–(8.17) (Halperin and Thomas, 1972).

Sufficiently close to T_c, when the correlation length for the spins becomes large compared to the electronic mean free path, one may take a different point of view, which leads to the result $d\sigma/dT \propto C_p$, under conditions more general than the assumptions behind (8.15) and (8.16). In this case one may assume that in a region whose diameter is large compared to the mean free path but small compared to the correlation length, one has a local value of the conductivity, of the form

$$\sigma(\mathbf{x}) \approx \sigma_0 + \sigma_1 \bar{\psi}(\mathbf{x})^2 , \tag{8.18}$$

where σ_0 and σ_1 are constants, and $\bar{\psi}(x)$ is the average value of the order parameter over the region in question, at a particular moment of time. The average conductivity of the sample will then be

$$\sigma = \sigma_0 + \sigma_1 \langle \bar{\psi}(\mathbf{x})^2 \rangle , \tag{8.19}$$

and the singular part of $d\sigma/dT$ is proportional to the specific heat, both above and below T_c.

For a review of the experimental and theoretical situation as of 1971, the reader is referred to Parks (1972). Some more recent references may be found in Alexander *et al.* (1976), and Shacklette (1974). See also Simons and Salamon (1971), and Takada (1971). Some closely related properties, the critical behavior of the thermopower and electronic thermal conductivity, and of the spin-flip scattering lifetime in metallic magnetic systems, have been discussed by Zorić *et al.* (1973); Thomas *et al.* (1972); and Entin-Wohlman *et al.* (1975, 1976). The critical behavior of the electrical resistivity of a ferroelectric semiconductor in which dipolar forces are important has been discussed by Binder *et al.* (1976).

Experimental measurements and theoretical predictions for the ionic conductivity of binary fluid mixtures near their consolute points have been reviewed by Shaw and Goldberg (1976). The theory is more complicated than for solid-state transitions, because there is a contribution to the local conductivity *linear* in the order parameter ψ. Similar complications arise in the electrical conductivity of metallic vapors and of metal-ammonia solutions near their critical points. (See, for example, Cohen and Jortner, 1975).

B. Nonlinear relaxation

The characteristic frequencies (3.16) or (3.19) are appropriate for the *linear* response function $\chi_\psi(\mathbf{k}, \omega)$, defined in the limit of zero external field. More generally, in the presence of a finite field $h_\psi(\mathbf{x}, t)$, one can consider the response $\langle \psi(\mathbf{x}, t) \rangle_{h_\psi(\mathbf{x},t)}$ as a functional of h_ψ. Let us for simplicity assume that h_ψ is nonzero and constant for $t < 0$, and $h_\psi = 0$ for $t \geq 0$. Then it is possible to show from general renormalization group arguments (Bausch and Janssen, 1976; Suzuki, 1976a; Suzuki and Ikeda, 1976) that for $t > 0$ the average order parameter takes the form

$$\langle \psi(t) \rangle_{h_\psi} = (\Delta T)^\beta f_1[t(\Delta T)^{\nu z}, \psi_0 (\Delta T)^{-\beta}] , \tag{8.20}$$

where $\psi_0 = \langle \psi(t=0) \rangle_{h_\psi}$. A relaxation time

$$\tau = \int_0^\infty dt \, \frac{\langle \psi(t) \rangle_{h_\psi}}{\psi_0} \tag{8.21}$$

may be defined (Suzuki, 1971) and written as

$$\tau = (\Delta T)^{-\nu z} f_2[\psi_0 (\Delta T)^{-\beta}] . \tag{8.22}$$

In the case $\psi_0 \ll (\Delta T)^\beta$, Eq. (8.22) yields the linear relaxation time with exponent $\Delta^{(l)} = \nu z$, but in the opposite case, $\psi_0 \gg (\Delta T)^\beta$, one obtains the *nonlinear* relaxation time whose exponent satisfies

$$\Delta^{(nl)} = \Delta^{(l)} - \beta . \tag{8.23}$$

Equation (8.23) was first proposed by Rácz (1975, 1976) on the basis of mean-field theory, and then obtained by Fisher and Rácz (1976) from more general phenomenological arguments. [See also the series analysis by Rácz and Collins (1976), and by Ikeda (1976)]. The recent renormalization group calculation by Bausch and Janssen (1976) for model A, evaluated the function f_1 in (8.20) explicitly to lowest order in ϵ. This function shows nonexponential relaxation and memory effects, and permits the calculation of nonlinear response functions. A more phenomenological calculation, valid in three dimensions, was performed by Binder *et al.* (1975) and Kretschmer *et al.* (1976), based on a cluster dynamics treatment, and small departures from exponential behavior were found. In contrast to the work of Fisher and Rácz (1976) and of Suzuki (1976a), the work of Bausch and Janssen (1976) and Binder *et al.* (1975) takes into account the fact that the equation of motion for $\langle \psi(t) \rangle$ is nonlocal in time. Other calculations of nonlinear relaxation have been performed by Saito and Kubo (1976), using the Bethe approximation in the Ising model, and by Rácz and Tél (1977) in the spherical model.

C. Other methods of calculation

1. Series expansions and computer modeling

For systems with simple relaxational dynamics, such as the kinetic Ising model, critical dynamics has been studied using high-temperature series expansions (Suzuki, 1970; Yahata and Suzuki, 1969; Yahata, 1971; Rácz and Collins, 1976; Ikeda, 1976) and Monte Carlo methods (Schneider *et al.*, 1972; Stoll *et al.*, 1973; Binder, 1976). Generally speaking, the results are consistent in two or three dimensions with those obtained from the renormalization group, within the limited accuracy of the various estimates.

Computer modeling for nonstochastic models has also been carried out on certain classical spin or lattice-dynamical systems, by solving the equations of motion of the constituent particles and carrying out the necessary ensemble averages [Schneider and Stoll, 1973, and references therein]. Due to practical limitations, however, these "molecular dynamics" calculations are restricted to one- and two-dimensional systems, and give little information on the asymptotic critical behavior.

2. Renormalization group on a lattice

Attempts to solve the static renormalization group equations using approximate truncation methods on a discrete lattice have yielded rather accurate results in two and three dimensions [see Niemeijer and van Leeuwen,

1974, 1976; Kadanoff, 1975]. The corresponding dynamical problem may also be amenable to solution, but no progress has yet been reported. An interesting combination of renormalization group and Monte-Carlo methods for both statics and dynamics was recently developed by Ma (1976b), but it is too early to say whether this scheme can yield reliable quantitative information.

3. Cluster dynamics and nucleation theory

A phenomenological theory which leads to quantitative predictions for the response functions both near and far from equilibrium has been developed by Binder, Stauffer, and Müller-Krumbhaar (1975). This theory is based on the dynamics of clusters, and can be considered a generalization of the droplet model to dynamic phenomena. As mentioned in Sec. VIII.B, the response functions obtained from this theory near equilibrium are consistent with scaling; moreover, the detailed form of the spectrum agrees quantitatively with Monte-Carlo results on the kinetic Ising model.

Far from equilibrium, the theory may be applied to a study of nucleation, for which scaling laws can also be formulated near the critical point. Let us suppose that the system is at a temperature $T_c - \Delta T < T_c$ with a small positive field, and the field is changed to a value $-h < 0$ at time $t = 0$. There will be a nucleation of the opposite phase, with a rate $J(h, \Delta T)$ which was expressed by Binder and Stauffer (1976) in the form

$$J(h, \Delta T) = (\Delta T)^j J_0 \tilde{J}(h/\Delta T^{\beta\delta}) , \tag{8.24}$$

where

$$j = 2 - \alpha + z\nu , \tag{8.25}$$

and $\tilde{J}(x)$ is a universal function, if suitable scales are chosen for the variables h and ΔT. An analogous function may be defined for fluids, where it is possible to extract j and $\tilde{J}(x)$ from experiments, but little quantitative information is available at present. A related problem is that of spinodal decomposition in alloys and glasses, which may also be studied near T_c, either experimentally or with Monte Carlo calculations. For recent work on nucleation phenomena near phase transitions the reader is referred to Riste (1975).

D. Other systems

In this section we shall mention, but not discuss in detail, a number of other systems whose dynamic critical behavior has been studied.

1. Forces of long but finite range

It is well known that the static properties of systems with infinite-range forces agree with mean-field theory for $d < 4$ as well as $d > 4$ (Kac, 1968). If the force range R is finite but large, the reduced temperature region where fluctuations cause departures from mean-field behavior vanishes as $R^{-2d/(4-d)}$ (Ginzburg, 1960; Hohenberg, 1968; Amit *et al.*, 1973).

The dynamic properties of such systems will be quite different for the relaxational and Heisenberg cases. In the former, the long-range forces suppress fluctuations, and the system follows time-dependent mean-field theory, i.e., conventional behavior, except for the temperature region close to T_c where mean-field theory breaks down in the statics. For Heisenberg dynamics, the conventional theory does not hold even away from T_c in three dimensions, since mode coupling is not suppressed by long-range forces (Résibois and De Leener, 1969). The criterion for the breakdown of conventional theory is $d < 4$ for the antiferromagnet, and $d < 6$ for the ferromagnet, whether or not the static exponents are mean-field-like (Hohenberg *et al.*, 1973; Kawasaki, 1968b; Villain, 1968).

2. Quantum effects and the case $T_c = 0$

When the transition temperature of a system is sufficiently low, then beside the classical scaling region[8] $(\Delta T) \ll T_c$ there will be a temperature domain where quantum effects play an important role, and new power laws may appear. One may also have a phase transition at $T = 0$, as a function of a parameter such as the pressure, the magnetic field, or the chemical composition, and in that case quantum effects must dominate (see, for instance, Pfeuty and Elliott, 1971). The separation we have made between statics and dynamics is then no longer possible, since even the equilibrium properties will depend in detail on the dynamics of the system. In some cases, where the elementary excitation spectrum is linear in k, the frequency ω plays the role of an additional component of the wave vector, and the correlation functions behave like the static correlation functions at a classical critical point in $(d+1)$ dimensions. The connection between a quantum field theory in d dimensions and a corresponding classical model in $d+1$ dimensions has been known for some time (see Kac, 1959; Symanzik, 1969; Pfeuty and Elliott, 1971; Osterwalder and Schrader, 1973; and Suzuki, 1976b). This connection was exploited by Rechester (1971) for a structural transition in three dimensions, with a one-component order parameter. The ensuing four-dimensional Ising-like model was solved by Rechester, using the methods of Larkin and Khmel'nitzkii (1969), which lead to mean-field behavior with logarithmic corrections. More general situations, in which the system does not necessarily behave as the corresponding $(d+1)$-dimensional model, have been investigated by Béal-Monod (1974); Béal-Monod and Maki (1975); Young (1976); Hertz (1976); and Schneider *et al.* (1976).

In *one* dimension, there are no transitions at finite temperature, but one-dimensional quantum systems at $T = 0$, or classical systems in the limit $T \to 0$, can legitimately be considered to display critical behavior. We shall not, however, discuss these systems here [see, for instance, Lurie *et al.* (1974); Luther and Peschel (1975); Efetov and Larkin (1975); Steiner *et al.*, (1976); Nelson and Fisher (1977); and Reiter and Sjölander (1977)].

It is also believed that two-dimensional classical systems with component number $n > 2$, such as the classical Heisenberg model, have a phase transition only at $T = 0$. (See, for example, Fisher and Nelson, 1977, and references therein.) The dynamics of these systems, at long wavelengths and low temperatures, have not been explored in detail. However, as mentioned above in Sec.

IV.A.5, DeDominicis *et al.* (1977) have explored the critical dynamics of the relaxational model A for $n>2$ and $d-2\to 0^+$.

An interesting case of a phase transition at $T=0$ is a *percolation transition*, which arises when magnetic ions on a lattice with nearest-neighbor interactions are randomly replaced by nonmagnetic impurities, until T_c is reduced to zero. The impurity concentration at which this occurs is known as the critical concentration p_c, for percolation of the magnetic sites. The behavior of the system as a function of concentration p and temperature T, near $T=0$ and $p=p_c$, must be interpreted in terms of crossover between the various regimes [Stanley *et al.* (1976); Lubensky (1977); Stauffer (1975a, 1976)]. Dynamics near the percolation threshold has been discussed by Stauffer (1975b) and Harris and Kirkpatrick (1977).

3. Multicritical points in magnetic systems

A number of multicritical points exist in magnetic systems, which have interesting dynamic properties.

When a uniform magnetic field H is applied to certain Ising-like antiferromagnets (e.g., $FeCl_2$), the transition to the paramagnetic state becomes first order, for H greater than a minimum value, the tricritical field H_t. Dynamics near a magnetic tricritical point has been studied with renormalization group techniques by Siggia and Nelson (1977). [See also Huber (1974a)]. If energy is conserved and the system is described by the two-field relaxational dynamics of model C (Sec. IV.C), with a free-energy F_0 appropriate to the tricritical point, then strong deviations from the conventional theory are predicted. In particular, at the tricritical point the relaxation rate $\omega_\psi(\mathbf{k})$ goes to zero as k^z, with

$$z=2+\alpha_t/\nu_t\,, \tag{8.26}$$

where α_t and ν_t are the exponents of the specific heat and correlation length, respectively, at the tricritical point, and $\alpha_t/\nu_t=1$, at $d=3$. Note: If the total magnetization M_z is conserved, as well as the energy E, then the auxiliary field m of model C is to be interpreted as that linear combination of energy and magnetization which appears in the slowest diffusive mode.

If energy and total magnetization are not conserved in the dynamics of the spin system, because of strong coupling to phonons with a high thermal conductivity, then the system is represented by a relaxational model of type A, with no conserved quantities. The dynamic tricritical exponent in this case takes its conventional value, $z=2$, at $d=3$.

The dynamics of relaxational models of type A and of type B (conserved order parameter), at a tricritical point and at critical points of higher order, have also been investigated by Prodnikov and Teitel'baum (1976).

Another multicritical point with interesting dynamics is the spin-flop bicritical point that can be reached in certain easy-axis antiferromagnets, when a uniform magnetic field is applied parallel to the easy axis. The bicritical point is the point at which the x-y like spin-flopped phase meets simultaneously the paramagnetic and the easy-axis antiferromagnetic phases. Although static properties at the bicritical point are the same as for the Heisenberg model, the dynamic properties differ from those of the Heisenberg antiferromagnet (model G) for several reasons. In the first place, only the component of total magnetization parallel to the easy axis m_z is conserved in the uniaxial system. Moreover, the susceptibility χ_{m_z} is divergent at the bicritical point. Huber and Raghavan (1976) have predicted that m_z and the components of the staggered magnetization perpendicular to $\hat{z}$ have a dynamic critical exponent which may be written in the form

$$z_{\psi_\perp}=z_{m_z}=\phi/\nu\approx 1.78\,, \tag{8.27}$$

where ϕ is the uniaxial crossover exponent and ν the correlation length exponent appropriate to the Heisenberg fixed point at $d=3$ [see also Huber (1974b)]. On the other hand, the staggered magnetization parallel to the easy axis has a different dynamic critical exponent, $z_{\psi_z}>2-\eta$.

A special kind of phase transition known as the Lifshitz point occurs at the intersection of the ferromagnetic–(or antiferromagnetic) paramagnetic and helicoidal–paramagnetic phase boundaries (Hornreich *et al.*, 1976). The scaling theory for such transitions involves two different correlation lengths, which reflect an anisotropy in k space at the transition. The dynamic exponents were recently obtained using mode-coupling methods by Huber (1976), who found generalizations of the scaling relations appropriate to isotropic Heisenberg systems.

4. Multicomponent Bose fluid

The hydrodynamic properties of an m-component Bose fluid with $m\geq 2$ differ in significant respects from those of superfluid helium ($m=n/2=1$), since the multicomponent system is invariant under the group $U(m)$ of unitary transformations. This system has $(m-1)$ new propagating modes with $\omega\propto k^2$ in the superfluid phase, and (m^2-1) new diffusive modes in the normal phase. An analysis of the critical dynamics of this model using scaling and mode-coupling theory was carried out by Halperin (1975). The dynamic critical exponent in this case is predicted to be $z=\phi/\nu$, where ϕ is the crossover exponent for a symmetry breaking perturbation of the axial type. A renormalization group analysis near $d=4$ indicates that the scaling prediction is valid only for m larger than a critical value $m_c(d)$, which is given by $m_c\approx 2.213$ for $d\to 4$.

A number of authors have studied the dynamic critical behavior of the multicomponent boson system in the limit $m\to\infty$. [See Ma and Senbetu (1974); Abe and Hikami (1974); Suzuki and Tanaka (1974b); Kondor and Szépfalusy (1974); Sak (1976); Oppermann (1976); Grest (1977)]. A satisfactory renormalization group analysis to order $1/m$ has not yet been done, however, and the behavior in this limit is not completely understood.

5. Superconductors

Because the transition temperature in all known superconductors is very small compared to the Fermi energy E_F, it turns out that critical fluctuations have a very small effect on the free energy, and the temperature region where asymptotic critical behavior should occur is predicted to be unattainably small in any bulk sample

(Ginzburg, 1960; Hohenberg, 1968).[38] In thin films and whisker samples, the critical fluctuations are enhanced, and it has been possible to observe a fluctuation contribution to the conductivity in the normal phase, sufficiently close to T_c. The explanation of this enhancement given by Aslamazov and Larkin (1968), using the microscopic theory of superconductivity, is very similar in its physical origins to the divergence of the transport coefficient λ_m near the λ point of helium, or near the Néel point of an antiferromagnet, as discussed in Secs. VI and VII above. In the superconductor one is primarily interested in temperatures outside the asymptotic critical region, however, so that the relaxation of the order parameter is of the simple time-dependent Ginzburg–Landau form (i.e., model A), with a temperature-independent kinetic coefficient Γ and classical Ornstein–Zernike behavior for the susceptibility $\chi_\psi(\mathbf{k})$. It was realized subsequently, however, that there are other contributions to the electrical conductivity in the temperature range of interest, in addition to the one proposed by Aslamazov and Larkin. These contributions may be dominant when the spin-flip scattering rate is small compared to the order parameter relaxation rate at the temperature of interest. A review of the experimental and theoretical situation may be found in Parks (1971), Hohenberg (1971b), and Skocpol and Tinkham (1975).

6. Liquid crystals

The most interesting transitions in liquid crystals, from the point of view of critical phenomena, are the transitions from nematic to smectic A, from nematic to smectic C, and smectic A to smectic C. All of these have the possibility of being second order, according to the Landau theory (de Gennes, 1974; Stephen and Straley, 1974), although the first two cases have been predicted to be at least weakly first order, because of fluctuation effects (Halperin *et al.*, 1974c; Halperin and Lubensky, 1974).

The possible effects of fluctuations on the critical dynamics in liquid crystals have been investigated theoretically, with a variety of coupled-mode and dynamic scaling approaches by Brochard (1973, 1976), Jähnig and Brochard (1974), and Shiwa (1976).

The experimental situation for the nematic–smectic A transition is very puzzling, however. For example, measurements of the elastic properties and of the smectic order parameter in the material *p*-cyanobenzylidene-*p*-*n*-octyloxyaniline (CBOOA), strongly indicate a critical point transition, while specific heat and volume measurements suggest a latent heat and volume jump. For an entry into the experimental literature of this system, see Clark (1976), Chu and McMillan (1975), and Als-Nielsen *et al.*, 1977.

[38] When one takes into account interactions with thermal fluctuations of the electromagnetic field, it appears that the superconducting transition should actually be very weakly first order, in a bulk sample. The predicted latent heat is too small to have been observed in existing experiments, however. See Halperin *et al.* (1974c).

7. Polymer solutions

The theory of stochastic motions of flexible polymer chains in solution bears many formal relationships to thermodynamic phase transitions (de Gennes, 1972; des Cloiseaux, 1975). In the polymer system, the correlation length diverges when the chain length goes to infinity and the concentration goes to zero. A scaling theory can be developed, for both statics and dynamics, which predicts critical exponents for the viscosity and relaxation time (de Gennes, 1976a, 1976b; Daoud *et al.*, 1975; deGennes *et al.*, 1976).

8. "Phase transitions" far from equilibrium

There exist a number of examples of systems in a steady state far from equilibrium, which undergo instabilities under the influence of an external force. These instabilities have some elements in common with equilibrium phase transitions, and it is interesting to ask whether scaling concepts will also apply to that case. At present, most of the analogies have been drawn at the level of mean field theory, and it is not clear whether fluctuations can be treated in the same way as for phase transitions [see Riste (1975); Haken (1975); Swift and Hohenberg (1977)].

IX. CONCLUSION

A. Summary of results

Let us summarize the principal accomplishments of the modern theory of critical dynamics.

(1) The singularities which occur in transport coefficients and time-dependent correlations near the critical point have been explained in terms of the long-wavelength fluctuations of slow variables in a system. The concepts of scaling and universality, which are generalizations of the corresponding ideas applicable to static properties, provide a framework for understanding a large number of dynamic phenomena.

(2) The mode-coupling method enables one to make semiquantitative calculations on simple models, and to understand the breakdown of the conventional theory of critical slowing down, which occurs in many cases.

(3) The renormalization group method may be applied to dynamic models, where it provides the mathematical mechanism for scaling and universality, in complete analogy to the static situation. The dynamic universality classes are smaller than the static ones, however, due to the relevance of conservation laws and Poisson-bracket relations in the dynamic case. The renormalization group method provides a justification for the earlier mode-coupling theories, and permits one to see which aspects of these theories are exact, and which are approximate.

(4) A large number of different systems have been analyzed and quantitative results obtained, many of which are in agreement with experiment. The principal dynamic universality classes are listed in Table I, and the most important results of the renormalization group analysis are summarized in Table II.

(5) Corrections to the leading asymptotic behavior have been estimated for a number of dynamic quantities. In

certain cases these corrections play a much more important role than for static properties, either because the correction exponent is small, or because the leading singularity is weak. Under such circumstances the region of universal scaling behavior is severely reduced, and further elaborations of the theory are necessary, in order to obtain predictions which are experimentally verifiable.

(6) The connection between a fully microscopic description and the stochastic models which are the basis of the renormalization group, has not been established in full detail. A suggestion for how such a derivation would proceed has been outlined in Sec. IV.D, for a system undergoing a structural phase transition. Similar arguments can be given to "derive" the stochastic models for gas-liquid and binary-fluid critical points, or for superfluid helium and various magnetic systems.

B. Problems and prospects

In order to test the theory of critical phenomena it is important to have accurate experiments on well characterized systems, very close to the critical point. Such experiments exist primarily in fluids, and it may be said that the theory has been tested successfully at the 10%–15% level on a number of pure fluids and binary-fluid mixtures. For the superfluid transition in helium, on the other hand, where accurate experiments are also available, there are a number of striking quantitative disagreements with the theory, which pose a major challenge at this time. Apart from these cases, there is relatively little experimental information of high accuracy available on systems which are well enough characterized to make the theory unambiguous.

With regard to the theory itself, it must be said that the methods presently available for making predictions in three dimensions are extremely crude. In essence what has been used in the renormalization group framework is an extrapolation from four dimensions via the ϵ expansion, or lowest-order perturbation theory in the mode coupling directly at $d = 3$. It would be interesting to extend to dynamics some of the other methods which have been applied successfully to the static theory in three dimensions, such as the finite-lattice renormalization group. Apart from calculations of exponents and scaling functions, it is also necessary to develop techniques for obtaining the corrections to the asymptotic critical behavior, in terms of a small number of nonuniversal parameters which can be fit to experiments on different materials. In this way, it is hoped that a more rigorous confrontation between experiment and theory can be achieved.

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