Phonon–phonon interactions in liquid helium*

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This is a short review of recent experimental and theoretical studies of phonon interactions in superfluid helium. Phonon–phonon scattering in helium is strongly influenced by the conservation laws and by the peculiarities of the phonon dispersion relation. Theory predicts that because of these factors the spontaneous decay rate varies dramatically with phonon energy, and experiments have confirmed this prediction. The conservation laws can also be used to show that phonon–phonon collisions are predominantly small-angle scattering. We discuss the influence of this special property of phonon collisions on the hydrodynamics of the phonon gas, and we consider the solutions of the hydrodynamic equations in several situations of experimental interest.

CONTENTS

I. INTRODUCTION

In this article we will review recent work on phonons and their interactions in superfluid helium-4. To appreciate the reasons for the current interest in this subject, one has to look at the history of phonons. Although the term phonon was apparently not introduced until about 1930, the importance of quantizing the energy of the mechanical waves in a crystal was, of course, appreciated much earlier. Einstein (1907), Debye (1912), and Born and von Karman (1912, 1913) had succeeded in developing a successful theory of the specific heat of solids. In modern language one says that these theories calculate the vibrational energy of a solid as the sum of the energies of a number of noninteracting phonons. The number $n(\epsilon)$ of phonons of a particular type of energy ϵ is given by the Bose–Einstein distribution function:

$$n(\epsilon) = (e^{\epsilon/k_B T} - 1)^{-1}, \tag{1}$$

where T is the temperature of the solid. The specific heat is then obtained as the derivative of the vibrational energy with respect to temperature. It was realized by Debye (1914) that although equilibrium properties such as the specific heat can be calculated to a reasonable accuracy by a theory in which interactions between phonons are neglected, it was essential to include these interactions in order to calculate thermal conductivity. Microscopically, a temperature gradient inside a solid is equivalent to a gradient in the number density of phonons. If phonons travel freely without scattering, any temperature gradient will disappear within a time τ equal to the time it takes a phonon to travel the distance from the hot region to the cold. In a solid with linear dimensions of a few centimeters τ is of the order of 10^{-5} sec. In fact, at room temperature a temperature gradient in a typical solid disappears much more slowly than this. Debye realized that this must be because phonons are frequently scattered. Thus, phonons actually move through a solid in a slow, diffusive way, rather than propagating directly from one end of a solid to the other. Debye also realized that a significant contribution to the scattering comes from the nonlinearity of the interatomic forces, and he derived an approximate expression for the phonon scattering rate due to this mechanism.

The theory of phonon scattering and its relation to thermal conductivity was developed in a more rigorous and systematic way by Peierls (1929). He calculated the thermal conductivity using the phonon Boltzmann equation and succeeded in explaining in a satisfactory way the approximately exponential increase in the conductivity observed in many crystals as the temperature is lowered below room temperature. Much theoretical work has been carried out since then to develop Peierls' approach. This work has included more accurate calculations of the scattering rates for phonons, better schemes for solving the Boltzmann equation, and discussions of the validity of the Boltzmann equation as applied to phonon transport. For reviews of these topics see Klemens (1956), Callaway (1959), Carruthers (1961), and Horie and Krumhansl (1964).

Experimental studies of phonon scattering remained in the dark ages until comparatively recently. The traditional experimental approach was to measure the thermal conductivity as a function of temperature and to try to work backwards to obtain information about phonon scattering rates. This is a very difficult and uncertain procedure. The phonons in a crystal have a variety of polarizations, energies, velocities, etc., and consequently have widely differing scattering cross sections. It is as if one tried to determine the lifetimes of the elementary particles by using an accelerator that produced every kind of particle, a target that contained a little bit of everything, and a detector that had no energy, momentum, or even time resolution!

In the last ten years the experimental situation has improved considerably. A major advance was the introduction of the heat-pulse technique (von Gutfeld and

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Nethercot, 1964, 1966; von Gutfeld, 1968). In these experiments a dielectric crystal has a "generator" film of constantan evaporated onto one surface. This can be heated for a short time (typically 10^{-7} sec) by passing a current through it. The hot film cools itself by radiating a pulse of phonons into the interior of the crystal. On the opposite face of the crystal is a thin film of a superconductor that is at a temperature close to its resistive transition. This film acts as a detector because phonons reaching it raise the temperature slightly and cause a change in the film's resistance. The heat-pulse method represents an enormous advance over traditional thermal conductivity studies as a way to study phonon interactions. Phonons of different velocity can now be studied separately. In addition, by using small generator and detector films, one can restrict attention to phonons traveling in a fairly well-defined direction.

In the original heat-pulse experiments the constantan generators probably produced phonons with a broad distribution of energies of the order of $k_B T_G$ (T_G =generator temperature). The detector was also sensitive to phonons having a wide range of energies. More recently, several schemes have been developed for producing and detecting pulses of phonons that are either monoenergetic or that have an energy spectrum with a cutoff above or below some definite threshold. These schemes have included the generation of phonons by the relaxation of paramagnetic spin systems, by superconducting tunnel junctions, by optical excitation, and by selective filtering of an initially broad energy spectrum. References to much of this work may be found in the proceedings of recent conferences at Ste. Maxime in 1972[1] and Nottingham in 1975 (see Challis *et al.*, 1975).

While most of these techniques can be applied to the study of phonons in any material in which phonons propagate, studies in superfluid helium are particularly interesting. Some of the reasons for this are the following:

(a) Many of the techniques for generating monochromatic phonons that have been developed so far can be used to produce phonons of energy up to only 20 or 30 °K. In a dielectric solid the sound velocity is typically about 5×10^5 cm sec^{-1}, and so a 20 °K phonon has a wavelength of around 100 Å. Thus, in the case of solids, one is restricted to the study of only those phonons having wavelengths much larger than the lattice parameter. In helium, on the other hand, the velocity of sound is only 2.4×10^4 cm sec^{-1}, and the entire spectrum of the phonons falls in the energy range below 20 °K.

(b) Experiments in solids tend to be affected by impurities and defects of various kinds. Apart from a very small amount of ^{3}He, all impurities freeze out of liquid ^{4}He. In addition, there are no structural defects, apart from vortices, which can usually be neglected.

(c) Helium is isotropic whereas crystals are not.

(d) The properties of excitations in helium can be changed considerably by the application of pressure.

In this review we describe the experiments that have been performed to study the scattering of phonons in helium, and we give a qualitative description of the associated theory. Our subject divides neatly into two parts. The first of these is the study of the scattering of one phonon by another and the spontaneous decay of phonons (Sec. II). The information obtained from this study is then applied to the investigation of collective effects that involve macroscopic motions of the phonon gas (Sec. III).

II. SCATTERING AND DECAY OF PHONONS

A. The dispersion curve

The dispersion curve for elementary excitations in superfluid ^{4}He is shown in Fig. 1. These data were obtained from neutron scattering measurements (Woods and Cowley, 1970; Cowley and Woods, 1971). Although the energy is actually a continuous function of the momentum, the spectrum of excitations is traditionally divided into two parts. Excitations with momenta less than 1 Å^{-1} are phonons, and those with momenta greater than 1 Å^{-1} are called rotons. (By 1 Å^{-1} we mean that the momentum is $\hbar Q$, where Q is 10^8 cm^{-1}.) Of particular importance for the subsequent discussion is the form of the dispersion curve for phonons of fairly small momentum. This is presented most conveniently by plotting the momentum dependence of the phonon phase velocity $c(p)$ and the group velocity $v(p)$ defined by

$$c(p) \equiv \epsilon/p\,, \tag{2}$$

$$v(p) \equiv \partial\epsilon/\partial p\,. \tag{3}$$

One estimate of these velocities is shown in Fig. 2. This is based on the functional form (Maris, 1973c)

$$\epsilon = c_0 p\left[1 + \gamma p^2 \frac{1 - p^2/p_A^2}{1 + p^2/p_B^2}\right], \tag{4}$$

with

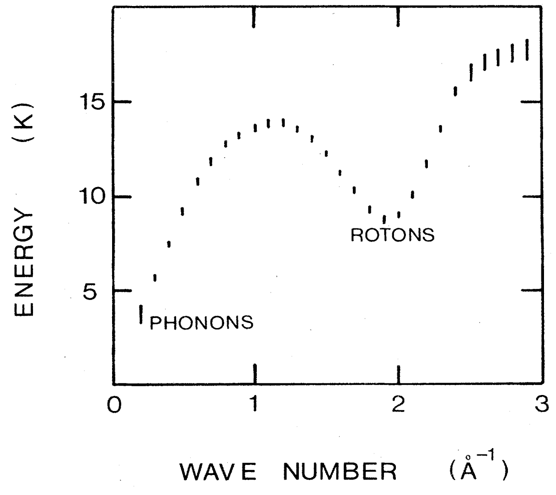

FIG. 1. Dispersion curve for elementary excitations in superfluid helium as measured by neutron scattering (Woods and Cowley, 1970; Cowley and Woods, 1971).

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[1] The papers presented at this conference are collected in J. Phys. (Paris) **33**, C-4, 1972.

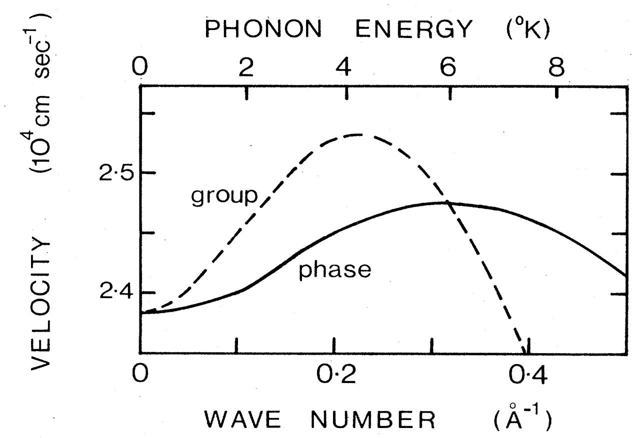

FIG. 2. Phase and group velocity of low-energy phonons as a function of phonon momentum.

$$\gamma = 1.112\ \text{Å}^2,$$
$$p_A = 0.5418\ \text{Å}^{-1},$$
$$p_B = 0.3322\ \text{Å}^{-1},$$
$$c_0 = 2.383\times 10^4\ \text{cm sec}^{-1}.$$

In the limit of very small momentum, both the phase and the group velocity are equal to c_0. This is just the sound velocity, which can be measured in an ultrasonic experiment. For higher momentum, the phase and group velocity at first increase. In most crystalline solids the opposite is observed, i.e., as the momentum increases, a monotonic decrease of c and v occurs. It is for this reason that the behavior in helium is referred to as anomalous dispersion.[2,3] At present there is no rigorous first-principles theory of the excitation spectrum of liquid helium. For reviews of this topic, see Feenberg (1969, 1970), Reatto (1975), and Woods and Cowley (1973). Recent theoretical work that is especially relevant to the problem of anomalous dispersion is described in the papers by Pines and Woo (1970), Bhatt and McMillan (1974), Lin-Liu and Woo (1974), Hastings and Halley (1975), and Aldrich, Pethick, and Pines (1976).

According to Fig. 2, the phase velocity has a maximum at about 0.3 Å^{-1} and at this maximum the velocity is about 4% above c_0. Above the maximum, the phase velocity decreases rapidly and becomes less than c_0 when p is greater than p_A. Although there is an uncertainty of possibly ±30% in the magnitude of the dispersion at the maximum, the qualitative features as illustrated by Fig. 2 are definitely correct. The uncertainty arises because it is very difficult to use neutron scattering to measure the dispersion curve with good accuracy for phonons of such low energy. (For a recent attempt, see Svensson *et al.*, 1975). Consequently, many alternative schemes have been devised to investigate the dispersion. We describe these measurements in the Appendix. It is appropriate to emphasize at this stage that Eq. (4) has no theoretical basis and is just a convenient functional form to fit the dispersion curve. Several other forms for the dispersion curve have been proposed, and these are also discussed in the Appendix.

B. The interaction between phonons

To understand the source of the interaction between phonons, it helps to use a classical picture. In a liquid the velocity of sound is

$$c = (B/\rho)^{1/2}, \tag{5}$$

where B is the bulk modulus, defined as

$$B = \rho\, \partial P/\partial \rho\,.$$

(We are ignoring subtleties about derivatives at constant entropy or temperature.) The velocity of sound thus depends upon density because B increases with increasing density, and also because the density appears explicitly in the formula for the velocity. For liquid helium it is known from the measurements by Abraham *et al.* (1970) that

$$\frac{dc}{d\rho} = 2.84\frac{c}{\rho_0}, \tag{6}$$

where ρ_0 is the density of helium at zero applied pressure. Consider now what happens when a liquid contains two sound waves with wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, frequencies ω_1 and ω_2, and amplitudes $\delta\rho_1$ and $\delta\rho_2$. The wave $\mathbf{k}_1$ produces a local density variation

$$\delta\rho_1(\mathbf{r}t) = \delta\rho_1 \exp[i(\mathbf{k}_1\cdot\mathbf{r} - \omega_1 t)],$$

which gives a local variation in the velocity of sound

$$\frac{\delta c(\mathbf{r}t)}{c} = 2.84\frac{\delta\rho_1}{\rho_0}\exp[i(\mathbf{k}_1\cdot\mathbf{r} - \omega_1 t)]. \tag{7}$$

A change in velocity is equivalent to a change in refractive index. Thus, the wave $\mathbf{k}_1$ may be considered to produce a diffraction grating. This grating has a spacing and orientation fixed by $\mathbf{k}_1$ and is moving with the phase velocity of the wave, i.e.,

$$\mathbf{c}_1 \equiv \frac{\omega}{|\mathbf{k}_1|}\hat{k}_1 \tag{8}$$

($\hat{k}_1$ is a unit vector in the direction of $\mathbf{k}_1$). The wave $\mathbf{k}_2$ will be scattered by this moving grating, and the most intense scattered waves will be those with wave vectors $\mathbf{k}_2 \pm \mathbf{k}_1$ (Fig. 3). In a frame of reference moving at the same velocity as the grating, the scattering must appear to be elastic. But in this frame, the wave $\mathbf{k}_2$ has a Doppler-shifted frequency

$$\omega_2' = \omega_2 - \mathbf{k}_2\cdot\mathbf{c}_1\,.$$

Now if the frequency of a wave with wave vector $\mathbf{k}_3$ equal to $\mathbf{k}_2 + \mathbf{k}_1$ is ω_3 in the rest frame, then in the moving frame the frequency will be

$$\omega_3' = \omega_3 - (\mathbf{k}_2 + \mathbf{k}_1)\cdot\mathbf{c}_1$$
$$= \omega_3 - \mathbf{k}_2\cdot\mathbf{c}_1 - \omega_1\,.$$

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[2] Note that in optics the *increase* of the velocity of a photon with increasing momentum is called *normal* dispersion.

[3] Until about 1970 it was generally assumed that the dispersion of phonons in helium was normal. This assumption led to a theory of phonon–phonon interactions which was radically different from the present theory. Much of this older work is summarized by Khalatnikov (1965). References to theoretical developments between 1960 and 1970 are listed in Maris (1973c, 1973d, 1974).

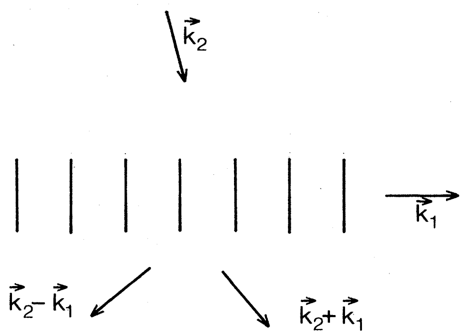

FIG. 3. Classical interpretation of phonon–phonon scattering.

Since ω_2' and ω_3' must be equal, we get the scattering conditions

$$\text{(a)}\quad \omega_3 = \omega_2 + \omega_1\,, \qquad \mathbf{k}_3 = \mathbf{k}_2 + \mathbf{k}_1\,.$$

Similarly, by considering the other scattered wave, we find the conditions

$$\text{(b)}\quad \omega_3 = \omega_2 - \omega_1\,, \qquad \mathbf{k}_3 = \mathbf{k}_2 - \mathbf{k}_1\,.$$

These conditions are simply the laws of conservation of energy and momentum for phonon scattering. In (a) phonon 2 absorbs phonon 1, producing phonon 3. Thus,

$$\text{(a)}\quad \epsilon_3 = \epsilon_1 + \epsilon_2\,, \qquad \mathbf{p}_3 = \mathbf{p}_1 + \mathbf{p}_2\,. \tag{9}$$

For (b) phonon 2 decays into phonons 1 and 3. Thus,

$$\text{(b)}\quad \epsilon_3 = \epsilon_2 - \epsilon_1\,, \qquad \mathbf{p}_3 = \mathbf{p}_2 - \mathbf{p}_1\,. \tag{10}$$

Note that, classically, the difference between these two processes is that in (a) the moving grating does work on the wave 1, whereas in (b) the grating absorbs energy from the wave.

It is clear from this physical argument that the matrix element for phonon–phonon scattering is proportional to $dc/d\rho$. There is also a dependence on the energy of the phonons involved and on the relative directions of their wave vectors. Without working out the details, one can see that the matrix element must increase with increasing phonon energy. Classically, the scattering from a grating is proportional to the square of the variation of the refractive index. Thus, in the present case, the scattering should be proportional to

$$(\delta\rho_1/\rho_0)^2\,.$$

For a classical sound wave this quantity is proportional to the energy of the wave. Thus, a higher-energy phonon will set up a diffraction grating which will scatter more strongly.

An important point to note is that a phonon can decay even if it is the only phonon present in the liquid. This is because the zero-point motion is always present, leading to spontaneous decay processes of type (b).

Higher-order interactions can also occur. Classically, this is the result of higher-order diffraction from one grating, or multiple scattering from two or more gratings. These processes can also be divided into scattering processes, for example,

$$1 + 2 \rightarrow 3 + 4\,,$$

or decay process such as

$$1 \rightarrow 2 + 3 + 4\,.$$

C. The spontaneous decay of phonons

We begin with a discussion of what happens to a phonon of energy ϵ_1 which is excited in liquid helium at zero temperature. Under these conditions there will be no other phonons present. Thus, we can neglect scattering and just consider the lifetime of the phonon due to the spontaneous decay process. Then the conditions

$$\epsilon_1 = \epsilon_2 + \epsilon_3\,, \qquad \mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_3$$

must be satisfied. We would like to investigate these conditions, using the dispersion law for phonons in helium. Consider what would happen if the dispersion law were exactly linear, i.e., if

$$\epsilon = c_0 p$$

for all p. Then the conditions become

$$p_1 = p_2 + p_3\,, \qquad \mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_3\,.$$

These equations obviously require that $\mathbf{p}_2$ and $\mathbf{p}_3$ be exactly parallel to $\mathbf{p}_1$. Consider now the more general case when the velocities of the three phonons are c_1, c_2, and c_3, and these differ from each other. Then,

$$c_1 p_1 = c_2 p_2 + c_3 p_3$$

and so

$$p_1 = p_2 + p_3 + \frac{c_2 - c_1}{c_1} p_2 + \frac{c_3 - c_1}{c_1} p_3\,.$$

If c_2 and c_3 are both larger than c_1, then

$$p_1 > p_2 + p_3\,.$$

But this is inconsistent with the momentum equation, since this obviously requires

$$p_1 \leq p_2 + p_3\,.$$

Hence, if c_2 and c_3 are both greater than c_1, the phonon 1 is stable against decay into two particles. It is easy to generalize this result to show that decay into any number of particles is also impossible if all of these particles have velocity greater than the velocity of particle 1. Thus for a general dispersion law when a phonon decays, at least one of the phonons produced must have a lower velocity than the original phonon.

For low-energy phonons ($\epsilon < 5$ °K), the velocity increases with increasing energy. These phonons therefore always have a velocity greater than the velocity of all phonons of lower energy, and hence the decay process is possible. The differences in velocity are small, however. Suppose, for example, that

$$c(p)=c_0(1+\gamma p^2),$$

where γp^2 is much less than 1. Then one can show that the angle between $\mathbf{p}_2$ and $\mathbf{p}_1$ is

$$\theta_2 \approx (6\gamma p_3^2)^{1/2} \tag{11a}$$

and the angle between $\mathbf{p}_3$ and $\mathbf{p}_1$ is

$$\theta_3 \approx (6\gamma p_2^2)^{1/2}. \tag{11b}$$

For p_2, p_3 around 0.1 Å^{-1} these angles are about 10°, and so the product phonons are still propagating in nearly the same direction as the original phonon. This has interesting consequences for the transport properties of a gas of phonons (see Sec. III).

For phonons of higher energy, the velocity starts to decrease (see Fig. 2), and some of the lower-energy phonons have a higher velocity. The decay possibilities then become restricted, and above a certain critical energy ϵ_c phonons become totally stable against decay. The precise way in which this last step occurs may depend to some extent on the details of the dispersion relation and has led to some confusion in the literature [for a discussion, see Pitayevskii and Levinson (1976)]. It is clear that just below the energy at which stability occurs, the phonon decays only into particles with momenta nearly parallel to its momentum. Thus, for decay into two particles, we consider the largest value of p_1 for which there exists a value of p_2 such that

$$\epsilon(p_1)=\epsilon(p_2)+\epsilon(p_1-p_2).$$

We consider this condition for the dispersion relation already described [Eq. (4)]. This leads to a largest value of p_1 given by

$$p_1^2=\frac{8p_A^2}{[25+16p_A^2/p_B^2]^{1/2}+5}. \tag{12}$$

Using the previously estimated values of p_A and p_B, we find $p_1=0.4215$ Å^{-1}. The critical energy for stability against decay into two phonons is thus

$$\epsilon(0.4215)=7.90\ ^\circ\mathrm{K}\equiv\epsilon_c^{(2)}.$$

Just below this energy it is possible for a phonon to decay into two phonons of equal energy. Above $\epsilon_c^{(2)}$ decay into two phonons is not possible, but the phonon may still be able to decay into three or more phonons. One can show that stability against possible decay into n particles is reached when

$$p_1^2=\frac{2n^2p_A^2}{[(n^2+1)^2+4n^2p_A^2/p_B^2]^{1/2}+(n^2+1)}. \tag{13}$$

For $n=3$ $\epsilon_c^{(3)}=8.71$ °K, for $n=4$ $\epsilon_c^{(4)}=9.12$ °K, and for large n

$$\epsilon_c^{(\infty)}\approx c_0p_A=9.86\ ^\circ\mathrm{K}.$$

The net effect of these results can be summarized as follows. A low-energy phonon is unstable against decay into any number of lower-energy particles. The decay rate into two particles can be shown to be (Beliaev, 1958; Maris, 1974)

$$\Gamma(\epsilon)=A\epsilon^5, \tag{14}$$

where

$$A=\frac{(u_0+1)^2}{240\pi\rho\hbar^4c_0^5} \tag{15}$$

and

$$u_0\equiv\frac{\rho_0}{c_0}\frac{dc_0}{d\rho}=2.84. \tag{16}$$

If ϵ is measured in °K, the constant A has the value 7.12×10^5 sec^{-1}. Thus, at the cutoff energy $\epsilon_c^{(2)}$ for decay into two phonons, the lifetime is 4.6×10^{-11} sec and the mean free path is about 100 Å. The rates for decay into three or more particles have not been calculated, but presumably they go as a higher power of ϵ. They are therefore not important at low energies, but might become significant in the range between $\epsilon_c^{(2)}$(7.90 °K) and $\epsilon_c^{(\infty)}$(9.86 °K). Figure 4 shows the estimated mean free path for spontaneous decay.

Before discussing the experiments related to this theory, it should be mentioned that there is some theoretical controversy about the mean free path for phonons of energy above $\epsilon_c^{(\infty)}$. On the basis of the theory we have just described, a phonon with energy greater than $\epsilon_c^{(\infty)}$ is completely stable against decay, and we expect that the mean free path measured experimentally will be limited only by extraneous effects, such as scattering from ^{3}He impurities. Sluckin and Bowley (1974) have performed a more elaborate calculation, however, and obtained a finite result for the scattering rate. They find, for example, that at around 1.5 or 2 times $\epsilon_c^{(\infty)}$ the decay rate has dropped less than an order of magnitude from its value just below $\epsilon_c^{(\infty)}$. Their theory is based

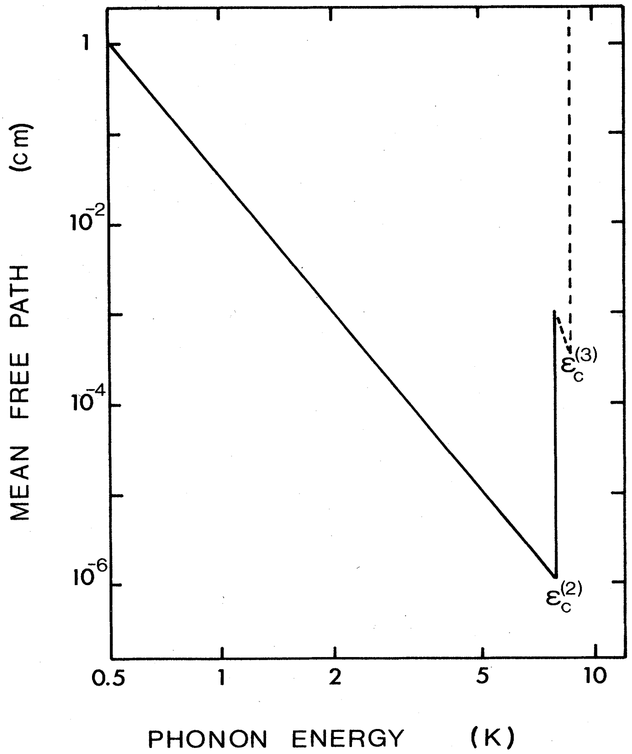

FIG. 4. Phonon mean free path as a function of energy. The dashed part of the graph is only qualitative because the decay rate into three or more phonons is not known.

upon the following physical picture. Phonons below $\epsilon_c^{(\infty)}$ are able to decay. Thus they have an uncertainty in their energy, which Sluckin and Bowley represent mathematically by approximating their spectral functions by Lorentzians. Phonons of energy above $\epsilon_c^{(\infty)}$ now have some finite probability of decay because the conservation of energy rule

$$\epsilon_1 = \epsilon_2 + \epsilon_3$$

is relaxed slightly by the uncertainties in the energies ϵ_2 and ϵ_3. The dangerous step in the argument is the assumption that the spectral functions are Lorentzians. Bhatt and McMillan (1974) have independently performed a calculation that is rather similar to Sluckin's and Bowley's, but which allows for a general spectral function that is obtained self-consistently. They find highly non-Lorentzian spectral functions and conclude that there is, in fact, a critical energy above which phonons are stable. As far as we can see, the Bhatt–McMillan conclusion is correct, but further theoretical study would be worthwhile.

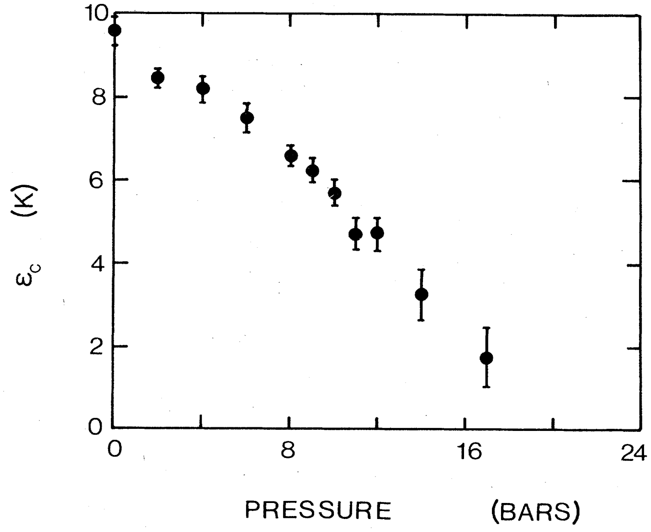

FIG. 6. Critical energy above which phonons are stable as a function of pressure (Dynes and Narayanamurti, 1975).

The existence of stable phonons with energies above $\epsilon_c^{(\infty)}$ has been confirmed by a series of elegant experiments carried out by groups at Bell Laboratories and at Nottingham University. In these experiments phonons of a definite energy ϵ were generated at some point in the liquid and a detector sensitive only to phonons of this energy was placed a few millimeters away. The number of phonons reaching the detector was measured as a function of ϵ. From the theoretical results shown in Fig. 4, one can calculate the fraction of phonons that are expected to reach a detector a given distance away. This fraction is shown in Fig. 5 for a propagation distance of 0.2 cm. It can be seen that a large increase in detector signal should be observed at an energy ϵ_c somewhere between $\epsilon_c^{(2)}$, where the mean free path is 100 Å, and $\epsilon_c^{(\infty)}$, where phonons are completely stable. The precise energy at which this increase occurs is not known definitely because the decay rates into three or more particles have not been calculated. The initial experiments gave confusing and conflicting results (Narayanamurti, Andres, and Dynes, 1973; Lockerbie, Wyatt, and Sherlock, 1974). However, later experiments in which generators and detectors having better understood characteristics were used gave clear evidence of ϵ_c and also resolved the earlier difficulties (Dynes and Narayanamurti, 1974, 1975; Wyatt, Lockerbie, and Sherlock, 1974). Dynes and Narayanamurti (1975) find that ϵ_c for liquid helium at zero pressure is 9.5 °K, and that this energy decreases with increasing pressure, as shown in Fig. 6. The result that ϵ_c decreases with increasing pressure and becomes zero implies that the dispersion curve must vary with pressure as shown qualitatively in Fig. 7, i.e., above some pressure around 20 bar the dispersion is normal for all phonons. [The discovery that the dispersion becomes normal under pressure was not a new result, since it had been indicated by the earlier specific heat measurements of Phillips *et al.* (1970).] As expected, the experimental result for the critical energy at zero pressure (9.5 °K)

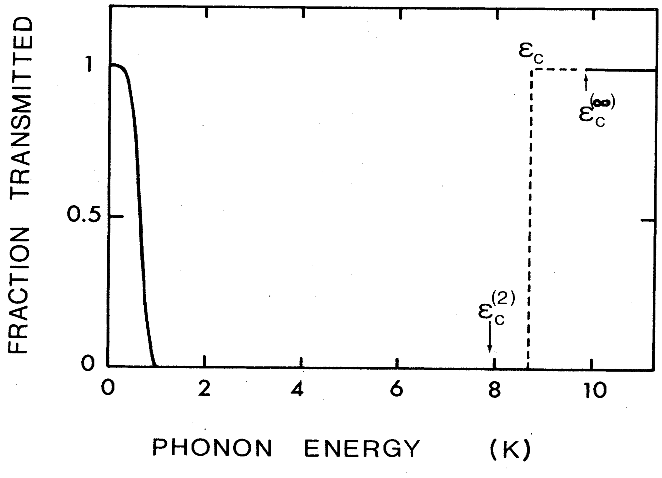

FIG. 5. The fraction of phonons that travel a distance 0.2 cm without decaying. The part of the graph between $\epsilon_c^{(2)}$ and $\epsilon_c^{(\infty)}$ is only qualitatively correct.

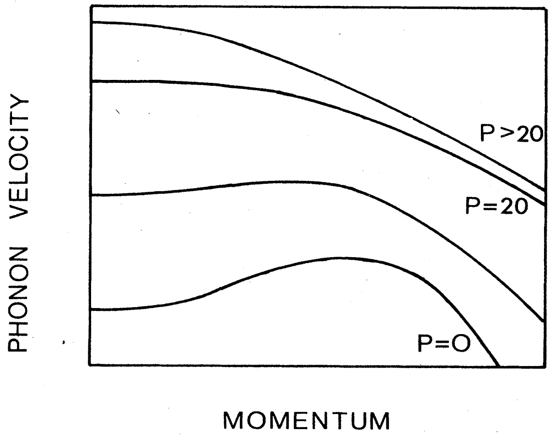

FIG. 7. Qualitative form of the variation of phonon velocity with momentum for various pressures. For a pressure greater than about 20 bar all phonons have velocity less than the velocity of low-energy phonons.

does indeed lie between $\epsilon_c^{(2)}$ and $\epsilon_c^{(\infty)}$.

The theoretical prediction that the decay products have momenta close to the direction of propagation of the original phonon has also been confirmed experimentally (Mills, Sherlock, Wyatt, 1974; and Sherlock, Mills, and Wyatt, 1975). The experimental arrangement is shown schematically in Fig. 8(a). A source of phonons (a Joule-heated film) was placed behind a slit. The angular distribution of phonons passing through the slit was measured using a detector 1 or 2 cm away. The detector was sensitive to phonons of any energy. At a pressure of 24 bar all phonons are stable, and so the signal at the detector should be a measure of the angular resolution of the apparatus. At lower pressures, decays can occur, and a wider angular distribution is observed than at 24 bar [see Fig. 8(b)]. It is difficult to make a quantitative comparison between theory and this experiment. However, the spreading angle observed does seem to be roughly consistent with expectations. Consider what happens when, for example, a phonon of energy 6 °K is excited in the liquid. The mean free path for a phonon of this energy is only 400 Å. The two phonons produced in the decay typically have energy roughly half the original energy and have a much longer mean free path because of the ϵ^5 dependence of the decay rate [see Eq. (14)]. However, the mean free path is still very small on the macroscopic scale and so further splitting processes occur. After each splitting, the mean free path is longer and the angle of scattering is reduced [see Eq. (11)]. We show this schematically in Fig. 9. In this figure each decay product has been assumed to decay after one mean free path. The angles between the phonons produced in each decay step have been calculated using the dispersion curve[4] and should be fairly realistic. In the experiment those secondary phonons that are products of decays occurring immediately in front of the generator cannot be distinguished by the detector from primary phonons which have propagated directly from the generator without decay. Thus, the increase in the angular width observed at the detector must be caused by phonons which have undergone a decay process a significant distance from the generator [at point A in Fig. 8(a), for example]. It can be seen from Fig. 9 that by the time phonons have reached a distance of 0.5 mm from the generator the energy has been reduced to 1 °K or less. For a 1 °K phonon decaying into two 0.5 °K phonons, the decay angle is 4°. This is comparable to the increase in angular spread that is observed.

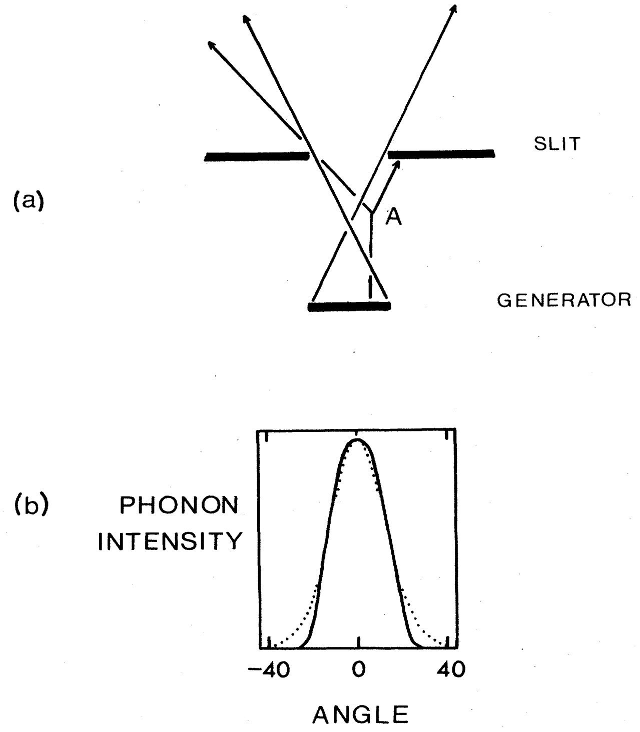

FIG. 8. The experiment of Mills, Sherlock, and Wyatt (1974) to study the angular spreading of phonon "showers" (a) Schematic diagram of the apparatus. (b) Phonon intensity as a function of angular position of the detector. Solid line shows data at 24 bar pressure, and dotted line is for zero applied pressure.

The above discussion ignores interactions between different decay products of one primary phonon and also neglects interactions between the large number of primary phonons generated in each pulse. It is not clear at the moment whether these approximations are valid under the experimental conditions of Mills *et al.* It seems to be difficult to devise a simple experiment which will study decay angles in a more qualitative way.

Note added in proof. The spreading of a phonon beam in helium at a finite temperature has been considered very recently by B. D. Laikhtman and A. V. Lomakin (Pis'ma Zh. Eksp. Teor. Fiz. **23**, 624 (1976) [Sov. Phys.-JETP Lett. **23**, 572 (1976)]).

D. Scattering of phonons by phonons

The group of experiments described in the last section all involve the study of what happens when one phonon is excited into liquid helium, and this phonon decays. Throughout our discussion we assumed implicitly that any other phonons in the liquid had an insignificant effect. It turns out that if the liquid is at a temperature below about 0.1 °K, the number of phonons thermally excited is so small that this assumption is reasonable. We now describe a different class of experiments,

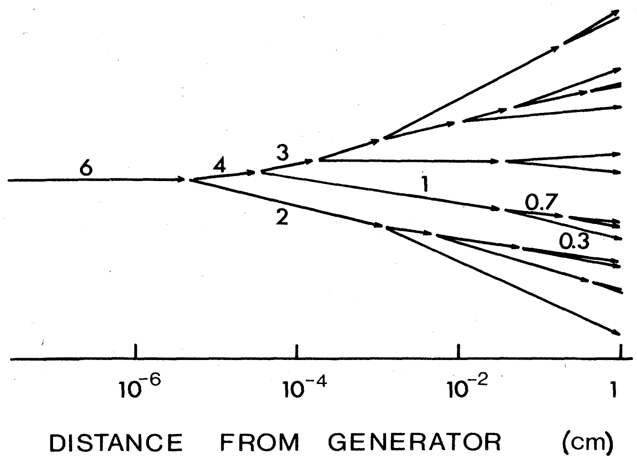

FIG. 9. The decay of a 6 °K phonon.

namely, those in which scattering occurs. The simplest process is one in which two phonons collide and one phonon is produced. Thus,

$$\epsilon_1 + \epsilon_2 = \epsilon_3 ,$$

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 .$$

The most complete study of scattering would involve an investigation of the cross section σ as a function of $\mathbf{p}_1$ and $\mathbf{p}_2$. This sort of experiment would require two monoenergetic sources of phonons, together with a suitable detector system. The experiments actually performed so far have been much less ambitious. A phonon generator produces a monoenergetic beam which propagates through liquid at a temperature T. The incident beam is scattered by the thermal phonons in the liquid and the attenuation length is measured. This gives a value of the scattering cross section averaged over the whole distribution of momenta of the thermal phonons. Despite this averaging, interesting results have been obtained. Quantitative experiments have been performed so far only for an incident beam of very low energy ($\epsilon_1 < 0.05$ °K).[4] For these phonons we may assume that the energy of all the thermal phonons is much larger; thus

$$\epsilon_2 \gg \epsilon_1 ,$$

$$p_2 \gg p_1 .$$

Then

$$\begin{aligned} \epsilon_3 &\equiv \epsilon(\mathbf{p}_1 + \mathbf{p}_2) \\ &\approx \epsilon(\mathbf{p}_2) + \frac{\partial \epsilon}{\partial \mathbf{p}_2} \cdot \mathbf{p}_1 \\ &= \epsilon_2 + \mathbf{v}_2 \cdot \mathbf{p}_1 . \end{aligned}$$

Thus, the conservation of energy condition becomes

$$\epsilon_1 = \mathbf{v}_2 \cdot \mathbf{p}_1 .$$

But the velocity of a low-energy phonon is very close to c_0, and so

$$c_0 p_1 = \mathbf{v}_2 \cdot \mathbf{p}_1 . \tag{17}$$

Thus the component of the group velocity of phonon 2 in the direction of $\mathbf{p}_1$ has to be equal to the velocity of sound c_0. Now, if v_2 is larger than c_0, there will always be some angle θ_2 between $\mathbf{p}_1$ and $\mathbf{p}_2$ such that Eq. (17) is satisfied. However, if v_2 is less than c_0, there is no solution for θ_2. Thus, the conditions of conservation of energy and momentum have the effect that a low-energy phonon can be absorbed only by high-energy phonons that have a higher velocity. The scattering rate should therefore be proportional to the number of these fast phonons and to the average scattering cross section. As discussed in Sec. II.B, the scattering cross section is proportional to phonon energy.

For the dispersion curve we have been using [Eq. (4)], all phonons of energy up to a critical momentum $\tilde{p}_c$ have velocity greater than c_0. This momentum is given by

$$\tilde{p}_c^2 = \frac{6 p_A^2 p_B^2}{[(5p_B^2 - p_A^2)^2 + 36 p_A^2 p_B^2]^{1/2} + 5p_B^2 - p_A^2} . \tag{18}$$

For the values of p_A and p_B given previously, $\tilde{p}_c$ comes out to be 0.3769 Å^{-1}, and the energy of a phonon of this momentum is

$$\tilde{\epsilon}_c = 7.10\ ^\circ\mathrm{K} . \tag{19}$$

Note that this cutoff energy is not the same as the energy $\epsilon_c^{(2)}$ involved in the *decay* of a phonon at zero temperature.[5] This was pointed out by Dynes and Narayanamurti (1975). In general, $\tilde{\epsilon}_c$ is less than $\epsilon_c^{(2)}$ but the difference between the two cutoffs depends to some extent on the precise form assumed for the dispersion relation. Thus, for example, Jäckle and Kehr (1971) considered the dispersion relation

$$\epsilon = c_0 p (1 + \gamma p^2 - \delta p^4)$$

and found that

$$\tilde{p}_c^2 = 3\gamma / 5\delta .$$

Dynes and Narayanamurti (1975) showed that for the same dispersion relation the momentum $p_c^{(2)}$ corresponding to the energy $\epsilon_c^{(2)}$ is given by

$$p_c^{(2)\,2} = 4\gamma / 5$$

and so

$$\epsilon_c^{(2)} / \tilde{\epsilon}_c \approx (\tfrac{4}{3})^{1/2} .$$

Thus, this dispersion relation leads to a slightly larger difference between $\epsilon_c^{(2)}$ and $\tilde{\epsilon}_c$ than the dispersion relation (4) that we have been using.

At temperatures below 0.5 °K, essentially all thermal phonons will have momenta less than $\tilde{p}_c$. The number of fast phonons is then simply the total number of phonons, which is proportional to T^3. The average energy of these fast phonons is of the order of $k_B T$, and so the average scattering cross section is also proportional to T. Thus, the total scattering rate for a low-energy phonon should be proportional to T^4. Roach *et al.* (1970, 1972) have measured the scattering of low-energy phonons as a function of temperature and pressure (see Fig. 10). At zero pressure they find an attenuation of the phonons which is proportional to T^4, as expected from the above argument. At higher pressures the scattering decreases and varies more slowly with temperature. This was first explained by Jäckle and Kehr (1971). As pressure is increased, the dispersion curve is modified as shown in Fig. 7, and $\tilde{\epsilon}_c$ decreases. When $\tilde{\epsilon}_c$ is comparable to $k_B T$, not all of the thermal phonons are fast phonons and the scattering is decreased. The situation is particularly simple when

$$\tilde{\epsilon}_c < k_B T .$$

The number of fast phonons is then proportional to T, and their average energy is independent of temperature. Thus, the attenuation of low-energy phonons should vary as T, as is observed experimentally at $P = 19$ bar around $T = 0.5$ °K, for example. Above 20 bar the dis-

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[4] The discussion given is valid for phonons of low energy, but not too low! We consider here the scattering of a low-energy phonon by high-energy phonons when complications arising from interactions *between* the high-energy phonons can be ignored. This is a good approximation if $\epsilon_1 \gg \hbar/\tau$ (τ = mean collision time for a high-energy phonon).

[5] Wyatt *et al.* (1974) have incorrectly assumed that the two cutoffs are equal.

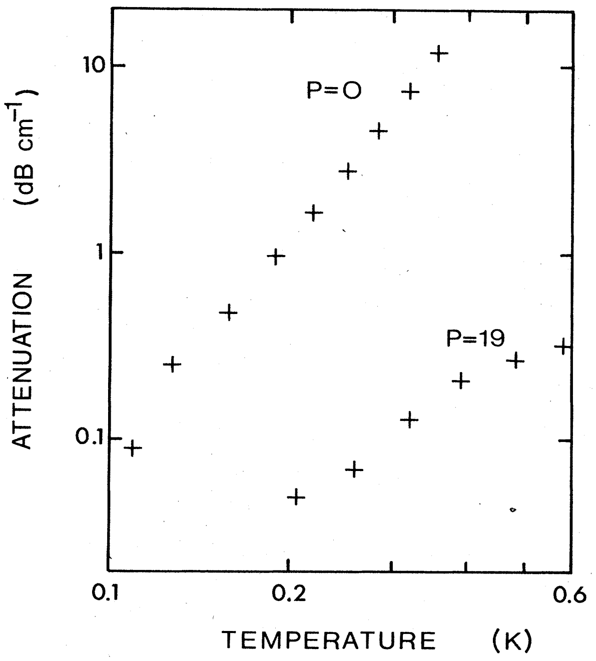


FIG. 10. Attenuation of low-energy phonons as a function of temperature at zero pressure and at 19 bar. The phonon frequency is 105 MHz corresponding to an energy of 0.005 °K. These data are measurements of Roach *et al.* (1970, 1972).

persion curve is completely normal; there are no fast thermal phonons; and there should be no scattering of low-energy phonons by the three-phonon process. The experimental results confirm this.

III. COLLECTIVE EFFECTS AND HYDRODYNAMICS

At a nonzero temperature helium contains a gas of thermally excited phonons, which have many properties similar to those of ordinary gases. It is this gas that is the normal fluid of the well-known two-fluid model of superfluid helium [for a discussion, see Wilks (1967)]. The equilibrium properties of the gas may be calculated using conventional results from the theory of the statistical mechanics of ideal gases. To calculate the transport properties, however, requires a kinetic theory in which the details of the collisions between phonons are correctly taken into account. It turns out that because phonon collisions are predominantly of small angle, the gas has several interesting and unique transport properties. We now discuss these.

A. Viscosity of the phonon gas

The viscosity of an ordinary gas of particles is

$$\eta = \tfrac{1}{3} nmv\Lambda \,, \tag{20}$$

where n is the number density, m the mass, v the mean particle velocity, and Λ the mean free path. This formula also applies to a phonon gas in helium if we replace nm by the normal fluid density ρ_n, v by the mean phonon velocity, and Λ by the phonon mean free path. Thus,

$$\eta = \tfrac{1}{3}\rho_n c\Lambda \,. \tag{21}$$

The number of phonons per unit volume at a temperature T is

$$n = 8\pi\zeta(3)(k_B T/hc)^3 \tag{22}$$

where $\zeta(3) = 1.202$. This assumes that all phonons have the same velocity c. Note that the typical wavelength of a phonon is

$$\lambda \sim hc/k_B T \,,$$

and so there is roughly one phonon per volume λ^3. The normal fluid density is (Wilks 1967)

$$\rho_n = \frac{16\pi^5}{45}\,\frac{(k_B T)^4}{h^3 c^5} \,, \tag{23}$$

implying that the average "mass" of a phonon is

$$m = \frac{2\pi^4}{45\zeta(3)}\,\frac{k_B T}{c^2} = 3.60\frac{k_B T}{c^2} \,, \tag{24}$$

i.e., roughly the average energy divided by c^2.

The problem that remains is to calculate the mean free path Λ. The appropriate mean free path to be used for the viscosity is the mean free path for large-angle scattering. However, we have already noted that there are no large-angle scattering processes involving only three phonons. Hence it is necessary to build up large-angle collisions out of many small-angle collisions. Thus, if $\tau_{\parallel}$ and $\tau_{\perp}$ are the small- and large-angle scattering times, respectively,

$$\tau_{\perp} = N\tau_{\parallel} \,, \tag{25}$$

where N is the number of small-angle scatterings necessary to produce the effect of one large-angle scattering. N is some function of the collision angle α. We now determine this function.

For a gas in which small-angle collisions are occurring very rapidly, we may assume that phonons with momenta of different magnitude, but lying in the same direction, will always be in equilibrium. Thus the state of the gas can be specified by giving the total number $n(\theta, \phi)$ of phonons having a given direction of $\mathbf{p}$. An equivalent description can be made through the introduction of an effective temperature $T(\theta, \phi)$, which again depends on the direction θ, ϕ of $\mathbf{p}$. Thus, the number of phonons with momentum $\mathbf{p}$ and energy ϵ_p is

$$n_p = \frac{1}{\exp[\epsilon_p / k_B T(\theta, \phi)] - 1} \,. \tag{26}$$

Consider now a phonon gas which is in a homogeneous state, i.e., $T(\theta, \phi)$ is independent of position. After a long time compared to $\tau_{\perp}$, the gas will reach complete equilibrium and T will be a constant value independent of θ and ϕ. We would like to construct an equation for the time variation of $T(\theta, \phi)$. To do this, we note that the effect of small-angle collisions is to make the direction of the momentum of a phonon undergo a sort of random walk. Hence, we expect that the equation of motion of $T(\theta, \phi)$ may be similar to the equation governing diffusion on the surface of a sphere, i.e.,

$$\frac{\partial T(\theta, \phi)}{\partial t} = A\nabla^2 T(\theta, \phi) + B\nabla^4 T(\theta, \phi) + \cdots \,, \tag{27}$$

where

$$\nabla^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,. \tag{28}$$

Equation (27) has the appearance of a simple power series in ∇^2. One might expect that this series is an expansion in the parameter

$$\alpha\nabla\,,$$

where α is the collision angle. Thus we guess

$$A = O(\alpha^2)\,, \tag{29}$$

$$B = O(\alpha^4)\,. \tag{30}$$

This is consistent with the idea of diffusion, since the ordinary first-order diffusion coefficient is proportional to the square of the random-walk step length.[6] However, if we continue with this view, we are led to an inconsistency. Suppose that at time $t = 0$

$$T(\theta,\phi) = T_0 + a_l P_l(\theta)\,, \tag{31}$$

where T_0 and a_l are independent of θ and ϕ, and $p_l(\theta)$ is a Legendre polynomial and a_l is small. Then at later times, Eq. (27) has the solution

$$T(\theta,\phi) = T_0 + a_l P_l(\theta) e^{-\lambda_l t}\,, \tag{32}$$

where

$$\lambda_l = A l(l+1) + B l^2(l+1)^2 + \cdots\,. \tag{33}$$

If we set $l = 1$,

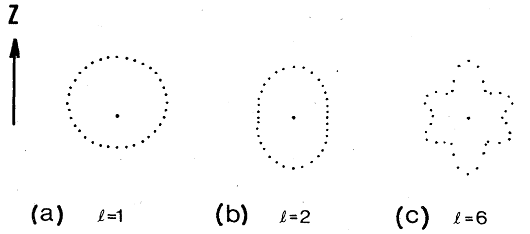
$$T(\theta,\phi) = T_0 + a_1 (3/4\pi)^{1/2} \cos\theta\, e^{-(2A+4B+\cdots)t}\,. \tag{34}$$

In the situation that we are considering, T varies with θ as shown schematically in Fig. 11(a). At $t = 0$ there are more phonons traveling in the positive z direction than in the negative z direction, and the phonon gas therefore has a net momentum. Collisions do not change the total momentum and so this momentum should persist as time develops. This is inconsistent with Eq. (34), which implies that the momentum disappears in a time of the order of $(2A + 4B + \cdots)^{-1}$. To resolve the problem we have to assume that

$$2A + 4B + \cdots = 0\,.$$

This means that Eq. (29) must be incorrect and that A actually contains no term of $O(\alpha^2)$. As a result, we must have to an accuracy of $O(\alpha^4)$:

$$A = -2B\,,$$



(a) l=1 (b) l=2 (c) l=6

FIG. 11. Polar plots of the phonon temperature.

and so

$$\lambda_l = B[-2l(l+1) + l^2(l+1)^2] + \cdots$$
$$= B(l-1)l(l+1)(l+2) + O(\alpha^6)\,. \tag{35}$$

The reason for this difficulty can be understood more directly by considering the decay of a single phonon p_1. The directions of the momenta p_2, p_3 of the phonons produced in the decay are random within some solid angle. However, because of conservation of momentum, there is complete correlation between p_2 and p_3, and this means that the process is more complicated than simple angular diffusion (Maris, 1973c). A detailed discussion of this process has been given very recently by Gurevich and Laikhtman (1975), who call it "superdiffusion."

The rapid increase of λ_l with increasing l predicted by Eq. (35) has a simple explanation. λ_l is the rate at which collisions relax a temperature distribution of the type given by Eq. (31). As l increases, the maxima and minima in $T(\theta)$ move closer together (see Fig. 11) and the phonons have to diffuse a much shorter distance in order to make the temperature uniform. Detailed calculations of λ_l at various temperatures have been made by Maris (1973a, 1974) and by Benin (1975). Results at 0.25 °K are shown in Fig. 12. These results were obtained by direct numerical solution of the phonon Boltzmann equation and do not rely on a power series expansion like Eq. (27). For large l it is found that λ_l tends to a constant value. This happens because as l increases, the maxima and minima in $P_l(\theta)$ get so close together that one small-angle collision is able to take a phonon from a "hot" direction to a "cold" one. Then the temperature distribution relaxes back to equilibrium in a time of the order of $\tau_\parallel$, and so

$$\lim_{l\to\infty} \lambda_l \approx \tau_\parallel^{-1}\,. \tag{36}$$

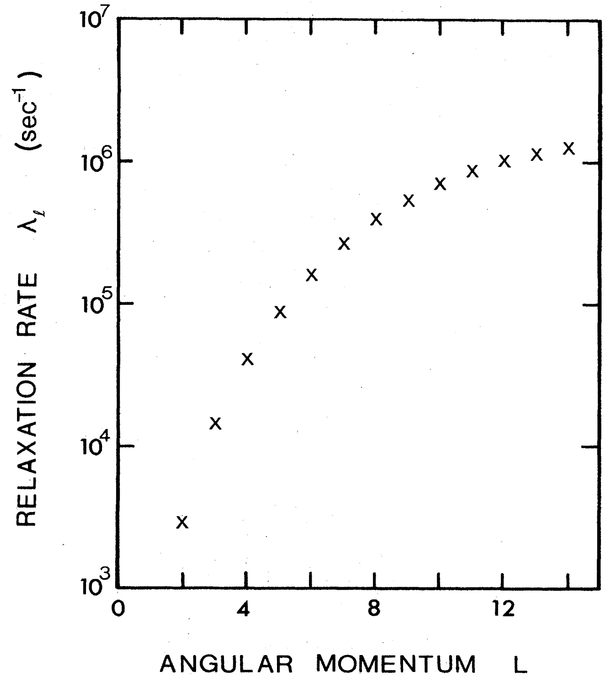


FIG. 12. Relaxation rates of the phonon gas at 0.25 °K.

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[6] In his calculation of the viscosity Matveev (1973) arrives by a different method at the result that would be obtained by believing the guess (29). For the reasons given below it appears that his result must be incorrect.

For $l=2$ the temperature of the phonons is slowly varying with direction, being a maximum near $\theta=0$ and π, and a minimum at $\pi/2$ [see Fig. 11(b)]. Thus, one expects that the effective large-angle scattering rate $\tau_\perp^{-1}$ is proportional to λ_2, and this idea can be shown to be correct by a more detailed calculation (Maris, 1973b, c; Benin 1975). Now

$$\begin{aligned}\lambda_2 &= 6A + 36B + O(\alpha^6)\\ &= 24B + O(\alpha^6),\end{aligned} \tag{37}$$

and so $\tau_\perp$ is proportional to α^{-4} (α = collision angle). Thus, the number N of small-angle collisions needed to make up one large-scale scattering event is of the order of α^{-4}. The viscosity mean free path calculated in this way is shown in Fig. 13 together with the experimental results of Whitworth (1958). The agreement between theory and experiment is very good. Jäckle and Kehr (1974) have described how the viscosity mean free path is expected to vary with pressure, but no experimental results are available yet.

This calculation has assumed that repeated three-phonon collisions make the largest contribution to the effective large-angle scattering rate. This is not *a priori* obvious, since four-phonon collisions, although they occur at a much slower rate, are not restricted to being of small angle and, thus, might be important. An estimate of the relative importance of three- and four-phonon processes has been made by Gurevich and Laikhtman (1975). They show that the assumption we have made is valid, at least for helium under no applied pressure. The situation under pressure is more complicated because four-phonon processes become relatively more important, since the three-phonon collision angle decreases.

B. Waves in the phonon gas

To be able to consider time-dependent phenomena such as the propagation of waves we need to derive an equation of motion for the phonon gas. At the microscopic level, of course, the equation of motion is the phonon Boltzmann equation

$$\frac{\partial n_p}{\partial t} = -\mathbf{v}_p \cdot \nabla n_p + \left(\frac{\partial n_p}{\partial t}\right)_{\text{coll}}, \tag{38}$$

where n_p is the phonon distribution function, $\mathbf{v}_p$ is the group velocity of a phonon with momentum p, and $(\partial n_p/\partial t)_{\text{coll}}$ is the contribution to the rate of change of n_p arising from collisions between phonons. While in principle the Boltzmann equation does provide a means to calculate the rate of change of the state of a gas, it is usually more convenient to work with simpler equations, even if these are of restricted validity. As an example, consider the propagation of a low-frequency sound wave in a gas of atoms obeying Fermi–Dirac statistics. For a low-frequency disturbance of the gas, the collisions between the atoms are so frequent that the gas is always in local equilibrium. This means that the distribution function for the atom at position $\mathbf{r}$ and time t is given by

$$n_p(\mathbf{r},t) = \frac{1}{\exp\{[\epsilon_p - \mathbf{p}\cdot\mathbf{V}(\mathbf{r},t) - \mu(\mathbf{r},t)]/k_B T(\mathbf{r},t)\}+1}, \tag{39}$$

where

$\mu(\mathbf{r},t)$ = local chemical potential,

$V(\mathbf{r},t)$ = local drift velocity,

$T(\mathbf{r},t)$ = local temperature.

Thus the distribution is completely determined by the five hydrodynamic variables μ, $\mathbf{V}$, and T. A hydrodynamic description of this type is valid when

$$\Omega\tau \ll 1, \tag{40}$$

where Ω is the frequency of the disturbance and τ is the collision time.

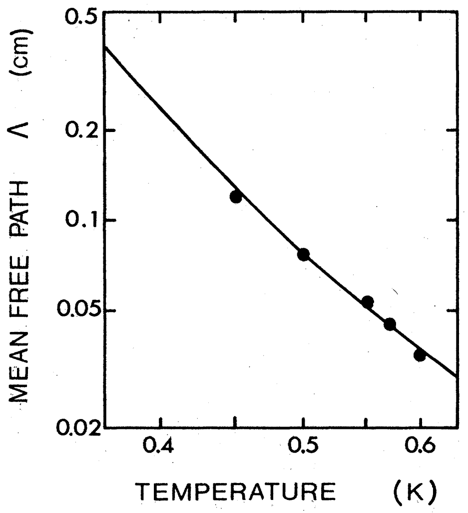

FIG. 13. Viscosity mean free path. The solid circles are the experimental results of Whitworth (1958). The theoretical curve is from Maris (1973c).

The hydrodynamic variables and the equations they satisfy are intimately connected with conservation laws. For each microscopic quantity, which is conserved in collisions between the elementary units of the system, there is a hydrodynamic variable. In the above example, collisions between gas atoms conserve energy, momentum (three components), and the number of atoms. Since there are five conserved quantities, there are five hydrodynamic variables. We would now like to derive a hydrodynamic theory for the phonon gas in liquid helium. The first problem is to decide on the list of conserved quantities. Energy and momentum are conserved, but the total number of particles is clearly not conserved in a three-phonon process. Thus, there are only four rigorously conserved quantities. However, phonon collisions have the additional special property that they are all of small angle. Thus, in a loose way, one can say that "direction" is an approximately conserved quantity, and so there should somehow be extra hydrodynamic variables associated with this. The difficulty is to formulate this quantitatively. If the collisions were actually all of zero angle, the number of phonons traveling in each direction in space would be conserved. Then, since there are an infinite number of directions in space, there would be an infinite number of conserved quantities, an infinite number of hydrodynamical variables, etc. A hydrodynamical theory with infinitely many parameters would be of similar complexity to the Boltzmann equation itself, and so would not be very useful.

One can get around these problems and construct a useful hydrodynamic theory for phonons in helium in the following way. Suppose we are interested in a theory to describe waves in the phonon gas that have oscillation periods much larger than the small-angle collision time $\tau_{\parallel}$. Then

$$\Omega\tau_{\parallel} \ll 1 ,$$

where Ω is the wave frequency. Then we can make the same approximation that was made in calculating the viscosity; i.e., we can assume that the frequent small-angle collisions keep phonons with momenta in the same direction in equilibrium with each other. Thus, the gas can again be described by giving the "temperature" $T(\theta, \phi, \mathbf{r})$ of phonons having a given direction of momentum at each point $\mathbf{r}$ in the liquid. The Boltzmann equation now becomes

$$\frac{\partial T(\theta, \phi, \mathbf{r})}{\partial t} = -\langle \mathbf{v}(\theta, \phi)\rangle \cdot \nabla T(\theta, \phi, \mathbf{r}) + \left(\frac{\partial T(\theta, \phi, \mathbf{r})}{\partial t}\right)_{\text{coll}} , \tag{41}$$

where $\langle \mathbf{v}(\theta, \phi)\rangle$ is the average group velocity of phonons with momenta in the direction θ, ϕ. For simplicity, let us consider what happens when T is independent of ϕ, and ∇T is in the z direction (i.e., T is independent of x and y). Then Eq. (41) becomes

$$\frac{\partial T(\theta, z)}{\partial t} = -\langle v\rangle \cos\theta \frac{\partial T(\theta, z)}{\partial z} + \left(\frac{\partial T(\theta, z)}{\partial t}\right)_{\text{coll}} . \tag{42}$$

We now expand $T(\theta, z)$ in terms of Legendre polynomials:

$$T(\theta, z) = T_0 + \sum_{l} a_l(z) P_l(\theta) . \tag{43}$$

T_0 is the average temperature of the gas. But

$$\cos\theta P_l(\theta) = \frac{l+1}{2l+1} P_{l+1}(\theta) + \frac{l}{2l+1} P_{l-1}(\theta)$$

and so we obtain an equation of motion for $a_l(z)$ in the form

$$\frac{\partial a_l(z)}{\partial t} = -\langle v\rangle \left[\frac{l}{2l-1} \frac{\partial a_{l-1}(z)}{\partial z} + \frac{l+1}{2l+3} \frac{\partial a_{l+1}(z)}{\partial z}\right] + \left(\frac{\partial a_l}{\partial t}\right)_{\text{coll}} . \tag{44}$$

But from Eq. (32),

$$\left(\frac{\partial a_l}{\partial t}\right)_{\text{coll}} = -\lambda_l a_l , \tag{45}$$

where λ_l is given by Eq. (33).

We have thus obtained a set of equations for the hydrodynamic variables $\{a_l\}$. Because λ_l increases rapidly as l increases, the $\{a_l\}$ for large l are heavily damped and will always be small. This means that if we are interested in solving Eq. (44) for a wave of frequency Ω, we need only worry about values of l for which

$$\lambda_l \lesssim \Omega . \tag{46}$$

As an example of this, one can consider the limiting case of very small Ω such that

$$\lambda_l > \Omega \tag{47}$$

for all $l \geq 2$. Then we need only retain a_0 and a_1 and so Eq. (44) reduces to

$$\frac{\partial a_0}{\partial t} = -\frac{\langle v\rangle}{3} \frac{\partial a_1}{\partial z} , \tag{48}$$

$$\frac{\partial a_1}{\partial t} = -\langle v\rangle \frac{\partial a_0}{\partial z} . \tag{49}$$

These equations have a simple physical interpretation. The energy density E is proportional to the phonon temperature averaged over all angles θ, ϕ. From Eq. (43) it can be seen that this is proportional to a_0. In a similar way the energy flux in the z direction, Q_z, is proportional to a_1. Thus, Eqs. (48) and (49) become

$$\frac{\partial E}{\partial t} = -\text{const} \frac{\partial Q_z}{\partial z} , \tag{50}$$

$$\frac{\partial Q_z}{\partial t} = -\text{const} \frac{\partial T}{\partial z} . \tag{51}$$

The first of these is just the condition for conservation of energy, i.e., the condition that the local rate of increase of the energy density be the divergence of the energy flux. The second equation says that the rate of change of the energy flux is proportional to the temperature gradient. At first sight, this is rather surprising, since in normal materials (i.e., nonsuperfluid) the energy flux itself is proportional to the temperature gradient (Fourier's law). In a thermal superfluid there is a different relation between temperature gradient and energy flux because the field equations must have the consequence that no temperature gradients can exist in bulk liquid in the steady state.

Now consider a wave of the form

$$a_l = a_l^{(0)} e^{i(Kz - \Omega t)} . \tag{52}$$

Then

$$\Omega = \langle v\rangle K/\sqrt{3} . \tag{53}$$

Thus, the velocity of very-low-frequency waves in the phonon gas is

$$c_2(0) = \langle v\rangle/\sqrt{3} . \tag{54}$$

These waves are called second sound, and this result for the velocity was first derived by Landau (1941). Second sound is conventionally viewed as "essentially an entropy or temperature wave" (Wilks, 1967, p. 50). Thus one imagines that at some time a spatially periodic temperature variation is set up [Fig. 14(a)]. The phonon gas accelerates, causing a heat flux that at some later time makes the temperature uniform [Fig. 14(b)]. However, although there is now no temperature gradient, the heat continues to flow. Eventually, a temperature distribution is set up that decelerates the heat current [Fig. 14(c)], and then the process repeats itself. An alternative view of second sound has been given by Maris (1976). This view stresses the similarity between second sound and an ordinary pressure wave in a gas of atoms. The phonon gas exerts a pressure P, and so a bulk modulus can be defined in the usual way as

$$B = -V\, \partial P/\partial V .$$

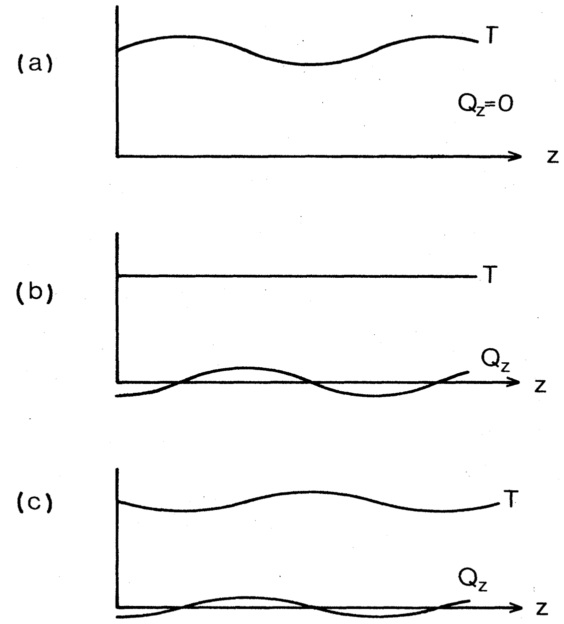

FIG. 14. Temperature and heat flux associated with a second sound wave.

Then the velocity of second sound can be shown to be given by the expression

$$c_2(0) = (B/\rho_n)^{1/2}\,. \tag{55}$$

This is the usual formula for the velocity of sound in a fluid of density ρ_n. According to this view, second sound is a pressure wave in the phonon gas. The temperature fluctuations associated with the wave occur because the temperature increases when a phonon gas is compressed at constant entropy.

To calculate the velocity of second sound for higher frequencies such that Eq. (47) does not hold, one has to allow the coefficients $a_2, a_3, \ldots,$ etc., to be nonzero. If Eq. (52) is substituted into Eqs. (44) and (45), one obtains the set of equations

$$(\Omega + i\lambda_l)a_l = K\langle v\rangle\left(\frac{l}{2l-1}\,a_{l-1} + \frac{l+1}{2l+3}\,a_{l+1}\right). \tag{56}$$

These equations must now be solved numerically. Calculations of this type have been performed by Maris (1973a, 1974) and Benin (1976). The velocity of the wave increases as the frequency goes up, and for a real wave frequency Ω the wave number K is complex, indicating that the wave is attenuated. Results for the velocity and attenuation at 0.25 °K are shown in Fig. 15. The velocity makes a very gradual increase from the low-frequency limiting value $\langle v\rangle/\sqrt{3}$ to a high-frequency limit close to $\langle v\rangle$. One can understand this velocity increase by looking at the phonon distribution function associated with these waves. Figure 16 is a polar plot of the phonon temperature $T(\theta)$ at different phases of the wave. For a low-frequency wave [Fig. 16(a)] $T(\theta)$ is slowly varying and the oscillation involves all of the phonons. As the frequency increases, the wave becomes more localized, and only involves phonons with momenta lying in directions close to the direction of propagation [Fig. 16(b)]. These phonons have a greater component of velocity in the direction of the wave propagation, and so the wave travels faster.

The theory just described is valid only for $\Omega\tau_{\|} < 1$. At first sight it would seem that for

$$\Omega\tau_{\|} \gtrsim 1 \tag{57}$$

no collective mode should exist, since the usual condition for the existence of a collective excitation is that there be many collisions during one period of the wave. Benin (1976) has constructed a more general theory which should be at least qualitatively correct for arbitrary $\Omega\tau_{\|}$. He obtains the remarkable result that a col-

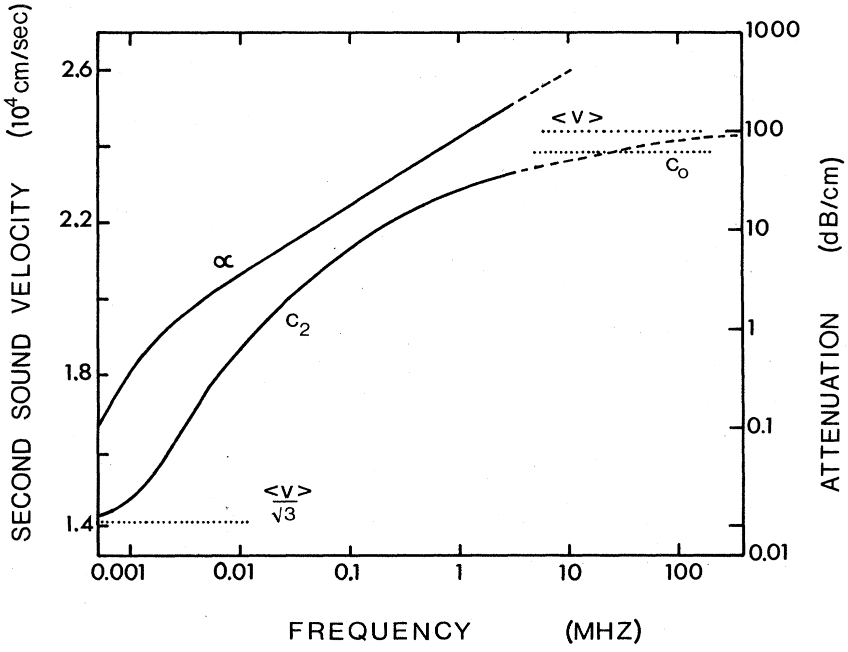

FIG. 15. Velocity c_2 and attenuation α of second sound as a function of frequency (Maris, 1973c). The low-frequency limit of c_2 is $\langle v\rangle/\sqrt{3}$. The dashed curves are a qualitative indication of the behavior expected in the frequency region $\Omega\tau_{\|} \gtrsim 1$ (Benin, 1976).

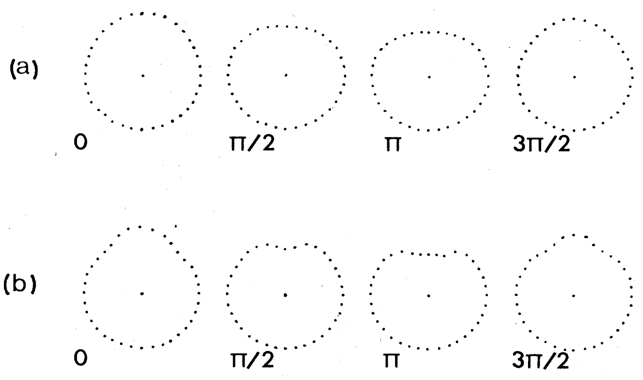

FIG. 16. Polar plots of the direction-dependent temperature $T(\theta)$ for (a) a 5 kHz second sound wave and (b) a 200 kHz second sound wave (Maris, 1974). The mean temperature is 0.25 °K. The temperature is plotted at various phases of the oscillation as indicated.

lective mode exists far into the $\Omega\tau_{\parallel}>1$ regime, and he provides a simple explanation of this effect. Consider again a wave propagating in the z direction. A phonon p has a component of group velocity in the direction of wave propagation

$$v_z = \mathbf{v}_p \cdot \hat{z} \equiv v_p \cos\theta_p \,. \tag{58}$$

Different phonons will have different values of v_z and so will gradually get out of phase with the wave. The velocity of the wave is equal to some suitably weighted average $\langle v_z\rangle$ over the v_z for those phonons involved in the wave. After a time t, a phonon with a large value of v_z, i.e.,

$$v_z = \langle v_z\rangle + \delta v_z \,, \tag{59}$$

will have traveled a distance

$$\delta z = t\delta v_z$$

further than the wave. Thus, this phonon will become 1 rad out of phase with the wave after a time τ_ϕ, such that

$$\lambda/2\pi = \delta z$$

where λ is the wavelength of the wave. But

$$\lambda/2\pi \approx \langle v_z\rangle/\Omega$$

and so the dephasing time is

$$\tau_\phi \approx \langle v_z\rangle/\Omega\delta v_z \,.$$

Benin points out that the phonon will remain coupled to the wave as long as it makes at least one collision within the time τ_ϕ. Thus, the condition for a collective mode to exist is

$$\tau_{\parallel} < \tau_\phi \,,$$

or

$$\Omega\tau_{\parallel} < \langle v_z\rangle/\delta v_z \,. \tag{60}$$

For $\Omega\tau_{\parallel} \gtrsim 1$ the only phonons which participate in the wave motion are those that have wave vectors lying within an angle σ of the z direction. The angle σ is roughly equal to the scattering angle in a small-angle collision, and so the range of values of $\cos\theta_p$ is very small. Thus, the spread in δv_z for the thermal phonons is small compared to the average value $\langle v_z\rangle$, and a collective mode continues to exist far into the $\Omega\tau_{\parallel}>1$ regime.

Quantitative measurements of the velocity and attenuation of second sound as a function of frequency and temperature have not yet been made. Most investigations of second sound at low temperature ($T<0.6$ °K) have involved the propagation of pulses having a broad frequency spectrum. The analysis of these experiments is very difficult [for a discussion see Maris (1974)]. Experiments with continuous waves would be very worthwhile and should be possible. The strongest confirmation of the theory at the moment has been obtained indirectly. Very accurate measurements of the velocity of *first* sound (i.e., ordinary sound) as a function of frequency[7] have been made by Abraham *et al.* (1967, 1969), Roach *et al.* (1972a, 1973), and Junker and Elbaum (1974, 1977). These show a peculiar behavior (Fig. 17). This was explained by Maris (1973d) and Wehner (1974) by postulating a resonant interaction between first and second sound. The proposal was that for frequencies in the range $\Omega\tau_{\parallel}>1$ the velocity of second sound continues to increase and eventually becomes greater than the velocity c_0 of first sound. This is shown qualitatively by the dashed line in Fig. 15. The dispersion relations of first and second sound thus have the qualitative form shown in Fig. 18. A weak coupling between first and second sound then causes a small correction to the first

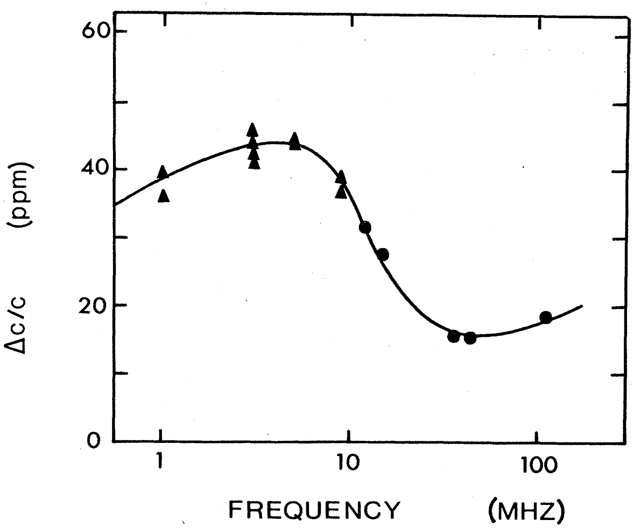

FIG. 17. Frequency dependence of the velocity of first sound at 0.35 °K. The solid line is a smooth curve through the experimental points.

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[7] To be precise, these experiments measure the frequency dependence of the temperature-dependent contribution to the velocity of sound! Thus the points shown in Fig. 17 are obtained by measuring the difference between the velocity $c(\Omega, T)$ at frequency Ω and temperature T and the velocity $c(\Omega, 0)$ at the same frequency and zero temperature. Any frequency dependence of the velocity at zero temperature is not detected by these experiments.

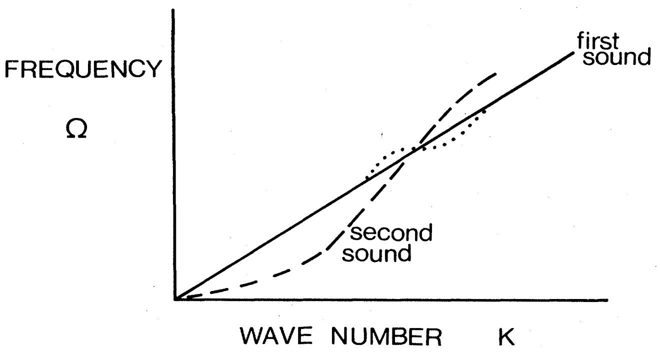

FIG. 18. Qualitative form of the dispersion relations of first and second sound. The dotted curve shows the effect on the first sound dispersion relation of a small coupling between first and second sound.

sound dispersion relation as shown. This interaction increases slightly the velocity of first sound below the resonance and decreases it above the resonance, giving a frequency dependence to the velocity in agreement with the experimental results. The proposal that the second sound velocity becomes greater than c_0 has now been confirmed by Benin (1976), using the theory he has developed for second sound propagation in the $\Omega\tau_{\parallel} > 1$ regime. In addition, quantitative calculations of the correction to the velocity of first sound have been performed (Maris 1972, 1973d; Meier and Beck 1973; Meier, Beck, and Beck 1975), and these are in good agreement with the experimental results.

Another interesting prediction of the basic theory is that at high frequencies there should be more than one type of second sound (Maris, 1973a, 1974). These extra modes are obtained in a straightforward way by solving the set of Eqs. (56). For very small real K, the only solutions of these equations are the second sound mode already discussed and a set of completely damped modes having purely imaginary frequency. As the wave number is increased, the frequency of some of the damped modes changes from pure imaginary to complex. The real part of the frequency then increases, and when $\Omega\tau_{\perp} \gg 1$, there are several modes which are "propagating modes" in the sense that

$$\mathrm{Re}\Omega \gg \mathrm{Im}\Omega\,. \tag{61}$$

These modes are propagating in the sense that they are attenuated by only a small amount for each wavelength that they travel. They have smaller velocity than the conventional second sound. The various modes have been named second second sound, third second sound, etc., in order of decreasing velocity. The velocity and attenuation of second second sound at 0.25 °K is shown in Figs. 19 and 20. The attenuation of the various modes increases rapidly as one goes to higher order. The angular dependence of the phonon temperature oscillations is more complicated for the higher-order second sounds. It is shown for second second sound in Fig. 21.

These higher-order waves all have a nonzero value of a_0 associated with them. Thus, the phonon temperature averaged over all angles is nonzero, and so the waves can be generated by a heater immersed in the liquid and may be detected by a suitable bolometer. They have not been observed experimentally yet. For a discussion of some of the considerations relating to their possible observation, see Maris (1974).

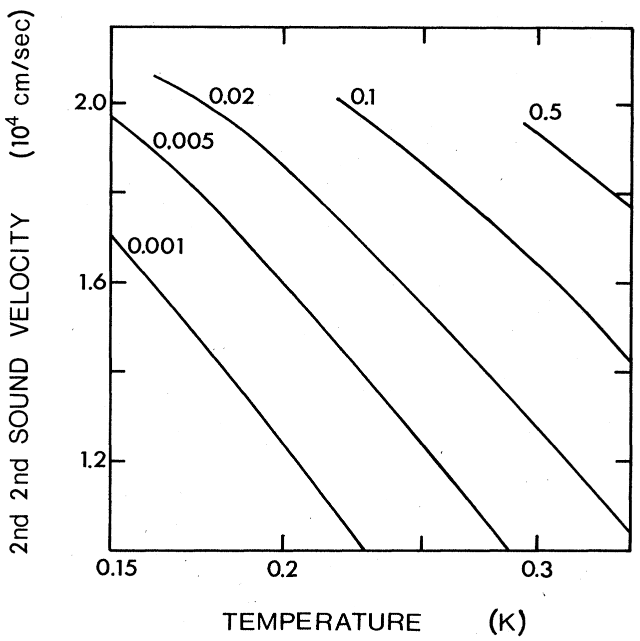

FIG. 19. Velocity of second sound as a function of temperature. The curves are labeled by the frequency in MHz.

Another effect related to these higher-order modes has been discussed by Saslow (1974). He has looked for *time-independent* solutions of the hydrodynamic equations of the phonon gas. Thus he sets

$$a_l = a_l(0)e^{-Kz}\,. \tag{62}$$

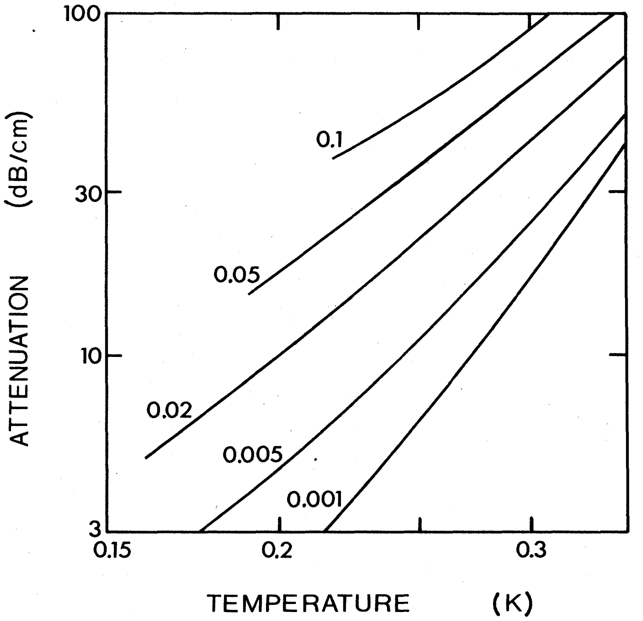

FIG. 20. Attenuation of second sound as a function of temperature. The curves are labeled by the frequency in MHz.

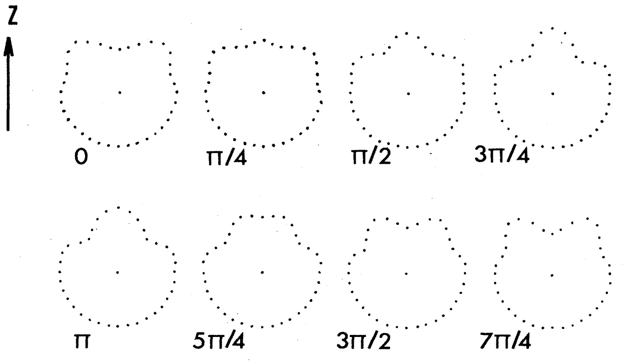

FIG. 21. Polar plot of the direction-dependent phonon temperature for a second sound wave of frequency of 50 kHz at 0.25 °K (Maris, 1974). The temperature is plotted at various phases of the oscillation as indicated.

There exist trivial solutions with $K=0$ which correspond to a homogeneous increase in the temperature of the liquid ($a_l=0$ for $l\neq 0$), or to a uniform flow of phonons ($a_l=0$ for $l\neq 1$). However, in addition there are solutions with $K\neq 0$ for which a_0 is nonzero. This means that over distances of the order of K^{-1}, it is possible to have temperature gradients in the liquid. Saslow suggests that these might be observable near a surface where heat is entering the liquid.

A weakly interacting Bose gas has a phonon spectrum that also exhibits anomalous dispersion. Ma (1972) has considered the properties of transverse waves in this phonon gas. He finds that above a critical frequency there should be propagating modes. This should also be true for transverse waves in liquid helium, but since these waves have no temperature fluctuation associated with them, they would be difficult to observe.

IV. SUMMARY AND OUTLOOK

The experiments that we have described provide a convincing demonstration of the application of the conservation laws governing phonon–phonon scattering. For the reasons given in the Introduction, superfluid helium-4 is in many ways an ideal material in which to study phonon interactions. While many predictions of the theory of phonon scattering in helium have been confirmed, there are still interesting experiments to perform. Particularly valuable would be any experiments to test the hydrodynamic theory (Sec. III.B). The extra second sound waves predicted by the theory have not yet been observed, and even the dispersion relation for ordinary second sound has been verified only indirectly and incompletely.

The success that has been achieved in the study of phonon–phonon scattering should stimulate renewed interest in roton scattering. The matrix element for phonon–phonon scattering can be related by theory to experimentally known quantities; for roton–roton scattering this is not possible. However, quantitative investigations of roton–roton scattering could be analyzed to yield the roton–roton matrix element and its momentum dependence, etc. A knowledge of how rotons interact could help answer the perennially awkward question, "What is a roton?", and might also settle recent arguments about the existence or nonexistence of two-roton bound states [for one view of this topic, see Woods and Cowley (1973)].

Finally, we note that these phonon scattering experiments have led us to obtain quite detailed information about the phonon dispersion for small momentum. The development of an accurate microscopic theory of the dispersion curve to explain these experimental results is a very challenging problem, which will doubtless receive considerable attention during the next few years.

APPENDIX: THE DISPERSION CURVE FOR SMALL MOMENTUM

This is a complicated topic that could almost be the subject of a review article itself. In large part the complications are historical, and so it is simplest to describe the work that has been done in roughly chronological order. In their original paper, Landau and Khalatnikov (1949) determined the small-momentum spectrum in the following way. They *assumed* that the whole dispersion curve including the roton region could be expressed by the formula

$$\epsilon^2 = A_1 p^2 + A_2 p^2 + A_3 p^6 + A_4 p^8 . \tag{63}$$

They then found $A_1, \ldots, A_4$ by imposing the conditions

$$\left(\frac{\partial \epsilon}{\partial p}\right)_{p=0} = c_0 , \tag{64}$$

$$\left(\frac{\partial \epsilon}{\partial p}\right)_{p=p_0} = 0 , \tag{65}$$

$$\left(\frac{\partial^2 \epsilon}{\partial p^2}\right)_{p=p_0} = \frac{1}{\mu} , \tag{66}$$

$$\epsilon(p=p_0) = \Delta , \tag{67}$$

where Δ and p_0 are the energy and momentum at the roton minimum, and μ is the roton effective mass. For small p Eq. (63) becomes

$$\epsilon \approx c_0 p (1 + \gamma p^2 \cdots) \tag{68}$$

with

$$c_0 = A_1^{1/2} . \tag{69}$$

Using known values for c_0, p_0, Δ, and μ gave

$$\gamma = -2.8 \times 10^{37}\ \mathrm{g}^{-2}\,\mathrm{cm}^{-2}\,\mathrm{sec}^2 \tag{70}$$

$$= -0.311\ \text{Å}^2 . \tag{71}$$

Note that there is no *a priori* theoretical basis for Eq. (63). This is an important point because three of the four fitting conditions [Eqs. (65), (66), and (67)] relate to the roton part of the spectrum. Thus, this procedure really determines the details of the small-momentum dispersion from the form of the dispersion in a totally different momentum range. If there is no strong reason to believe the analytic form to which the fit is made, this is clearly a dangerous thing to do.

Landau and Khalatnikov calculated the normal fluid viscosity using this value of γ. The later experimental measurements of the viscosity by Whitworth (1956) were not even in qualitative agreement with their theory. In addition, Khalatnikov (1963) had to assume a different value, i.e.,

$$\gamma = -0.0017\ \text{Å}^2, \tag{72}$$

to get agreement between his theory of the absorption of sound and the experimental results of Chase and Herlin (1953). Further problems arose when more extensive measurements of the attenuation of sound and the temperature dependence of the sound velocity were made by Abraham *et al.* (1967, 1969). Their results were not compatible with theoretical predictions (Khalatnikov and Chernikova, 1965, 1966) even when the magnitude of γ was treated as an adjustable parameter. To resolve these difficulties Maris and Massey (1970) proposed that γ as defined by Eq. (68) was actually positive. This idea soon received support from specific heat measurements by Phillips *et al.* (1970). If the phonon phase velocity is independent of momentum, the phonon contribution to the specific heat C should be exactly proportional to T^3. Phillips *et al.* were able to detect a small deviation of C from a T^3 law. While the sign of this deviation definitely showed that the dispersion was anomalous ($\gamma > 0$), these data do not give a precise value for γ. For example, Phillips *et al.* treated γ *and* c_0 as adjustable parameters and found

$$\gamma = 0.378\ \text{Å}^2, \tag{73}$$

whereas in a later analysis Maris (1972) used the value of c_0 determined independently by ultrasonics and found a best fit with

$$\gamma = 0.720\ \text{Å}^2. \tag{74}$$

These values of γ initially appeared to be in conflict with the results of neutron scattering experiments by Woods and Cowley (1970). Their data could be fit by the expression

$$\epsilon = c_0 p(1 + \gamma p^2 - \delta p^4) \tag{75}$$

with

$$\gamma = 0 \pm 0.02\ \text{Å}^2, \tag{76}$$

$$\delta = 0.30 \pm 0.02\ \text{Å}^4. \tag{77}$$

Thus, the neutron scattering indicated normal dispersion. The conflict disappears when it is realized that the specific heat experiment probes phonons of energy 1 to 2 °K, whereas the neutron data are quite inaccurate for energies below about 9 °K. However, this comparison of the two experiments does show clearly that a low-order polynomial of the form of Eq. (75) cannot describe the spectrum over an extended range of momentum such as from zero to 1.0 Å^{-1}. Thus one must either choose a more complicated form for the dispersion curve, or alternatively use different polynomial expansions for different ranges of momentum. Equation (4), which we have used for most of the discussion in this article, may be regarded as an interpolation scheme between two polynomials. For p much less than p_A and p_B, it becomes

$$\epsilon \approx c_0 p(1 + \gamma p^2) \tag{78}$$

and for p much greater than p_A and p_B

$$\epsilon \approx c_0 p\left[(1 + \gamma p_B^2) - (\gamma p_B^2/p_A^2)p^2\right]. \tag{79}$$

It is possible to obtain a good fit to both the specific heat and the neutron data in this way (Maris, 1973c). Another approach was adopted by Brooks and Donnelly (1973), who showed that good agreement with experiment could also be obtained by taking

$$\epsilon = c_0 p + a_3 p^3 + a_4 p^4 + a_5 p^5 + a_6 p^6 + a_7 p^7 + a_8 p^8. \tag{80}$$

This expression, which contains more parameters, gives in addition a good fit to the entire dispersion relation including the roton region. A fit to the spectrum has also been made by Iachello, Rasetti, and Rasetti (1973).

For the Landau–Khalatnikov dispersion curve [Eq. (63)] and the dispersion curve defined by Eq. (4) the energy is expressed as a series containing only odd powers of the momentum p. In the Brooks–Donnelly series all powers up to p^8 are included except the quadratic term. There has been considerable discussion about which powers of p should actually be present in the expansion. Kemoklidze and Pitaevskii (1970) and Feenberg (1971) have shown that because of the long-range van der Waals potential between helium atoms there is a term in ϵ proportional to p^4. The magnitude of this term can be calculated from first principles and is

$$-\pi^2 \phi\rho/24M^2 c_0^2, \tag{81}$$

where the van der Waals potential between two helium atoms is $-\phi/R^6$, and M is the mass of an atom. This gives a correction to the phase velocity of the form

$$\Delta c/c = \alpha_3 p^3 \tag{82}$$

with

$$\alpha_3 = -3.34\ \text{Å}^3. \tag{83}$$

It is possible that this is the *only* contribution to the term in p^4, but this has not been proved rigorously.

The possibility of a p^2 term has been considered by Molinari and Regge (1971). They claimed that a p^2 term cannot be excluded on theoretical grounds, and they were able to get a good fit to the specific heat and neutron scattering data by the formula

$$\epsilon = c_0 p(1 + 0.5465p - 1.3529p^2 + 0.2595p^3 + 0.1860p^4 - 0.0522p^5)^{1/2}. \tag{84}$$

For small p this becomes

$$\epsilon = c_0 p(1 + \alpha_1 p + \alpha_2 p^2 + \cdots \quad \cdots) \tag{85}$$

with

$$\alpha_1 = 0.2733\ \text{Å}, \tag{86}$$

$$\alpha_2 = -0.7138\ \text{Å}^2. \tag{87}$$

Roach *et al.* (1972b) have looked for a p^2 term in ϵ by measuring the velocity of sound at 30 and 90 MHz with high accuracy. They found that

$$|\alpha_1| \lesssim 0.01\ \text{Å}, \tag{88}$$

in strong disagreement with the Molinari–Regge fit. This low limit on the magnitude of α_1 clearly suggests that there is no p^2 term in the dispersion relation. For further discussion of this point see Zasada and Pathria (1972).

Additional information about the dispersion has been obtained from the following experiments. Note that the result obtained for the dispersion curve in each case depends upon the analytic form that is assumed.

(1) Normal-fluid viscosity. Measurements have been made by Whitworth (1956), and their relation to the dispersion curve has been considered by Maris (1973b, c) and by Benin (1975).

(2) Ultrasonic velocity as a function of frequency and temperature. Junker and Elbaum (1977) have analyzed their experimental measurements and the earlier data of Abraham *et al.* (1967, 1969) and Roach *et al.* (1972a, 1973) to obtain information about the phonon dispersion curve.

(3) Thermal expansion. Measurements have been made by Van Degrift (1974) and by Berthold *et al.* (1976). The thermal expansion at temperatures less than 0.5 °K is found to show a small deviation from a T^3 law. This deviation can be analyzed to determine the phonon dispersion.

(4) Direct measurement of phonon velocity. Anderson and Sabisky (1972) have used a thin-film acoustic-resonance technique to measure the phase velocity for phonons of energy in the range 1–3 °K. Neutron scattering measurements have been made by Svensson, Martell, and Woods (1975) and by Copley (1976). Measurements of the group velocity of phonons with energies above the stability energy $\epsilon_c^{(2)}$ have been made by Dynes and Narayanamurti (1975) using tunnel junctions.

(5) Analysis of critical energies. Jäckle and Kehr (1971) analyzed the ultrasonic attenuation measurements of Roach *et al.* (1970, 1972c) to determine the cutoff energy $\tilde{\epsilon}_c$ (see discussion in Sec. II.D). They were then able to relate this to the dispersion curve. Dynes and Narayanamurti (1975) used tunnel junctions to determine the critical energy at which phonons became stable at a low temperature (around 0.1 °K). They assumed that energy was $\epsilon_c^{(2)}$, and then used their results for this quantity, together with their group velocity measurements [see (4) above], to estimate the dispersion relation. They used the simple-polynomial dispersion relation (75) and found two surprising results. At zero pressure their result for γ was 0.12 Å^2, which is much smaller than most other estimates. Secondly, γ *increased* with increasing pressure, in contrast with the generally accepted idea that the dispersion becomes normal at a pressure around 20 bar. The *decrease* in $\epsilon_c^{(2)}$ with increasing pressure was attributed to a very large increase in δ (by a factor of 22 between $P=0$ and $P=10$ bar). It is possible that the peculiar results obtained for γ are just a consequence of the overly simplified form for the phonon dispersion, and so an alternative analysis would be interesting.

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