End-loss processes from mirror machines*

D. E. Baldwin

Lawrence Livermore Laboratory, University of California, Livermore, California 94550

The processes leading to end loss of ions from a mirror machine are reviewed. These include breakdown of adiabaticity, scattering and energy drag by classical collisions, and scattering by unstable fluctuations. Described are the linear theory of those modes thought to be of significance in present and reactor-size plasmas, and those features known of their nonlinear saturation.

CONTENTS

I. INTRODUCTION

Devices for the magnetic confinement of plasma are conveniently classified as having either closed or open magnetic lines of force. As candidates for a fusion reactor, each type of device possesses a different mix of advantages and disadvantages; these have recently been reviewed by Ribe (1975) and Post (1976). Among the several necessary qualities is a containment time for energetic ions that is sufficient to allow thermonuclear reactions. Just as the fundamental mode of containment differs for the two types of systems, so do the mechanisms for particle and energy loss.

In closed-line systems, the magnetic lines of force do not leave the plasma volume, but rather are confined to a family of nested, topologically toroidal, flux surfaces. Plasma pressure is more or less constant along a line, and containment is achieved because the nested set of plasma-loaded flux surfaces is isolated from material walls. The mechanisms of particle or energy loss are those which transport these quantities between flux surfaces, across magnetic lines. Such devices generally make inefficient use of relatively large volumes of magnetic field, being limited to the range of 10% in β, which is the ratio of plasma-energy density to magnetic-energy density. However, because loss is associated with transport in space, the lifetime of an MHD-stable equilibrium increases with physical dimensions. Furthermore and most importantly, the local particle–velocity distributions are essentially Maxwellian, eliminating a large class of instabilities driven by non-Maxwellian, distributions. The principal such toroidal containment device now under investigation is the Tokamak, the current state of which has most recently been reviewed by Furth (1975). Transitory high β can be achieved in a torus such as Scyllac (Ribe, 1975), wherein the toroidal plasma is compressed and heated by a rapidly rising magnetic field.

Open systems, on the other hand, present an almost complementary picture. Lines leave the plasma volume to pass through material walls. The ion axial confinement is then achieved either because the ions are mirror-trapped in regions of local minima in the magnetic field strength by the adiabatic invariance of the magnetic moment or, in the case of long, straight reactor concepts, because of simple time of flight (e.g., Post *et al.*, 1973). Because the mirror effect depends on the pitch angle of a particle with respect to the magnetic field B, confinement is velocity-space selective, implying that there will exist void regions in the velocity space of trapped particles. The existence of the voids, called loss cones, creates a source of free energy that can drive a variety of instabilities (Fowler, 1968) as discussed in Sec. V. The lifetime of a particle in a mirror trap is limited by the time for velocity-space scattering from regions where a particle is trapped to one where it is not; therefore, no direct lifetime advantage is to be gained from large physical dimensions.

For the energy release rate from thermonuclear reactions to exceed the power input to any reactor requires a product of density times confinement time of several times 10^{13} s/cm^3 (Lawson, 1957; see Ribe, 1975). To ensure adequate values of this product in a mirror reactor requires ion energies of order 100 keV, implying that ion mean-free paths must be very long, of the order 10^5 to 10^6 times the machine length. Such a reactor would be relatively compact in size, with a length-to-radius ratio in the range 2 to 4. The magnetic field would be efficiently used, with values of β less than but approaching unity. Many general features of these devices have been reviewed by Fowler (1969).

The adiabatic invariant upon which mirror confinement of ions is based is given, to lowest order in the ratio of Larmor radius to magnetic scale length, by $\mu = \frac{1}{2}w^2/B$, where w is the component of velocity **v** locally perpendicular to **B** (Northrup, 1963; Bernstein, 1971). This and the conservation of energy, $E = \frac{1}{2}v^2 + (q/m)\Phi$, where Φ is an electrostatic (ambipolar) potential which (for reasons described later is positive with respect to ground) give rise to the mirror effect: that the component of velocity parallel to **B**,

$$u = \pm\sqrt{2}(E - \mu B - (q/m)\Phi]^{1/2}, \tag{1.1}$$

will vanish as a particle guiding center moves into a sufficiently strongly increasing magnetic field. Particles with sufficient μ will be trapped in a field having a local

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minimum along B in the intensity $|\mathbf{B}|$. The ratio of adjacent maxima of $|\mathbf{B}|$ to the minimum value is defined as the mirror ratio R_m. From Eq. (1.1), particles are trapped provided u vanishes inside those points where the effective potential $q\Phi + m\mu B$ has maxima. Under the most common mirror machine conditions, Φ decreases monotonically from the center to the outside of the mirror trap, so as to expel ions. When Φ may be neglected, an ion is trapped provided $E < \mu B_{\max}$. Introducing the components of velocity at the field minimum, u_0, w_0, the pitch angle $\theta_0 \equiv \tan^{-1} w_0/u_0$ of trapped particles satisfies $\cos^2\theta_0 < 1 - R_m^{-1}$. The cone of untrapped particles is the loss cone. Inclusion of nonvanishing Φ introduces an energy dependence to the loss boundary. For example, if the axial dependence of Φ is such that it decreases approximately linearly in B (a surprisingly common occurrence), ions are trapped provided $E < \mu B_{\max} + (q/m)\Phi(B_{\max})$, where $\Phi(B_{\max}) > 0$ is the potential where B is maximum. The corresponding loss boundary at the field minimum is hyperbolic, and the region of confined particles has $u_0^2 > w_0^2(R_m - 1) - 2(q/m)\Delta\Phi$, where $\Delta\Phi > 0$ is the drop of potential from the point of maximum density to the point of maximum B (see Fig. 1).

Ions are untrapped by any process that changes E/μ to exceed $B_{\max}$; when this happens, they are lost from the trap along the magnetic lines. Those processes that can lead to changes in pitch angle are (i) nonadiabaticity of the ion motion in the confining field, (ii) two-body collisions, and (iii) diffusion in velocity space due to the existence of electric field fluctuations (rf) within the plasma. Provided containment is good (meaning that changes in E, μ occur on a time scale long compared to the bounce time of the ions in the trap), all of these processes can be included in a generalized Fokker–Planck equation that describes the slow diffusive evolution of a distribution of trapped ions in the space of E, μ or equivalently v_0, θ_0.

End loss is the crucial issue for a mirror reactor. In the following sections, each of the three contributory mechanisms is discussed at a level which attempts to introduce the important considerations and describe how they are treated mathematically. A general review of mirror physics is not intended, and we omit discussion of a number of other topics. In particular, guiding-center equilibrium and stability are assumed, and questions relating to these nontrivial subjects are not discussed (see e.g., Hall and McNamara, 1975). Also, we refer to the previously cited review (Ribe, 1975) for a discussion of the reactor prospects and requirements of a mirror machine. Because of the free energy associated with an ion loss cone distribution, it can drive a variety of modes of rf fluctuation. The topic of instabilities in mirror machines must therefore dominate a description of end-loss processes in much the same way that it forms a major part of theoretical mirror research.

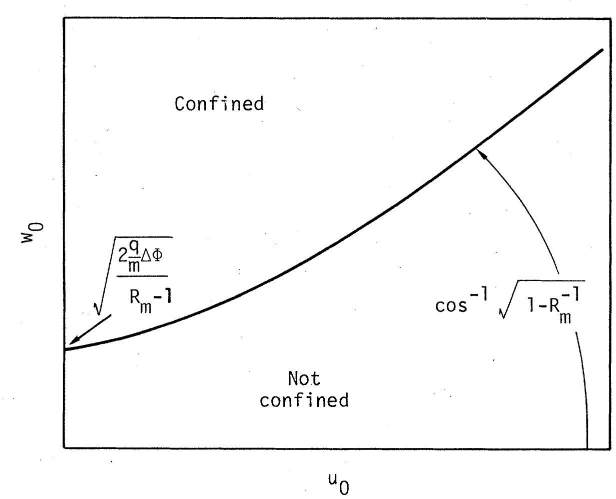

FIG. 1. Ion velocity space showing loss of low energy confinement induced by drop in ambipolar potential to the mirror throat $\Delta\Phi$.

It is useful to set the stage for a summary of this kind by introducing a number of parameters that characterize the operating range of a mirror machine and that provide values that are typical for present-day experiments and foreseen possible reactor regimes.

The electron density n_e is conveniently measured in units of electron plasma frequency $\omega_{pe} = (4\pi n_e e^2/m_e)^{1/2}$; the range of values extends up to 10^{14} cm^{-3}. Defining the electron-cyclotron frequency as $\Omega_e = |e|B/m_e c$, reactors are seen to operate with $\omega_{pe}^2/\Omega_e^2 \gtrsim 1$, while present experiments operate with this parameter from this range downward. For the same density, the corresponding ionic ratio is larger by a mass ratio m_i/m_e.

If $\bar{v}_i$ is defined as a typical ion velocity, the average ion Larmor radius $a_i = \bar{v}_i/\Omega_i$ is typically a centimeter at a few kilovolts energy and a few kilogauss field strength. (Temperatures will be cited directly in such units of energy.) Under reactor conditions of greater than 100 keV and perhaps 20 kG field, the ion Larmor radius would be 2 to 4 cm. For reasons of stability discussed below, a reactor is thought to have an upper limit on length and a lower limit on radius, the estimates of which change with time and theoretical models and are still under study. For purposes here, it is sufficient to envision a reactor as having a mirror-to-mirror length of perhaps 200 ion Larmor radii, and a radius of 50 radii. Present machines are smaller in this dimensionless length by about a factor of 2, but are smaller in the dimensionless radius by a factor 10 to 20.

The vacuum mirror ratio R_{vac} lies in the range 2 to 3, set in part by requirements of adiabaticity and economics. With the inclusion of the central field reduction due to the contained diamagnetic plasma, total mirror ratios R_m for reactors are anticipated to lie in the range 7 to 10.

Plasma pressure is measured by β, the ratio of the energy density of the plasma to that of the magnetic field. Several definitions of β are used and have importance for different purposes. Comparison of the plasma pressure with the vacuum magnetic field pressure, a ratio denoted by β_{vac}, is a common means of presenting experimental data. The limiting β_{vac} which a magnetic field will hold stably against the MHD mirror mode depends on the velocity-space pitch-angle distribution of trapped ions, but has a practical limit of about 0.7 to 0.8 (Hall, 1972; Hall *et al.*, 1975). However, the plasma pressure measured relative to the diamagnetically depressed local $B^2/8\pi$ gives rise to a larger ratio which we denote by β_{loc}. The degree of the depression depends on the shape of the diamagnetic plasma; for the long, thin plasma model described below, the relation between these two β's is $\beta_{1oc} \approx \beta_{vac}/(1-\beta_{vac})$. In describing waves inside the plasma and the effects of finite β on them, it is clearly β_{1oc} that is significant; this quantity may realize values of 2 to 4.

One very probable feature of a mirror reactor, and one common to most present experiments, is that by proper choice of coil design the plasma is contained in a region of minimum in $|\mathbf{B}|$ transverse as well as along $\mathbf{B}$. Although not uniquely, this feature immediately assures stability on virtually all low-frequency ($\omega \ll \Omega_i$) time scales, including MHD, up to β limits such as that imposed by the mirror mode mentioned above (Taylor, 1963, 1964; review by Jukes, 1967). An example of an idealized current configuration generating such a field is one having the shape of a baseball seam as shown in Fig. 2. A price paid for this shape is the analytic complexity introduced by the three-dimensional shape of the flux surfaces. For example, a flux surface that is circular at the midplane has high ellipticity at the mirrors (the points along each line where $|\mathbf{B}|$ is maximum—roughly at the radius of the conductors), and the major axes of the ellipses at either end of such a flux tube are rotated 90°. An analytic example is that of a long, thin vacuum field (Furth and Rosenbluth, 1964; Cordey and Watson, 1969) for which the scalar potential, when expanded near the axis of symmetry, has the form

$$\chi=\int_0^z dz f(z)-\frac{f'(z)}{4}(x^2+y^2)+\frac{g(z)}{2}(x^2-y^2)+O(x^4,y^4), \tag{1.2}$$

where f and g are arbitrary even functions of z. The modulus of $\mathbf{B}=\nabla\chi$,

$$B^2=f^2+(x^2+y^2)\left(\frac{f'^2}{4}+g^2-\frac{ff''}{2}\right)+(x^2-y^2)(g'f-f'g), \tag{1.3}$$

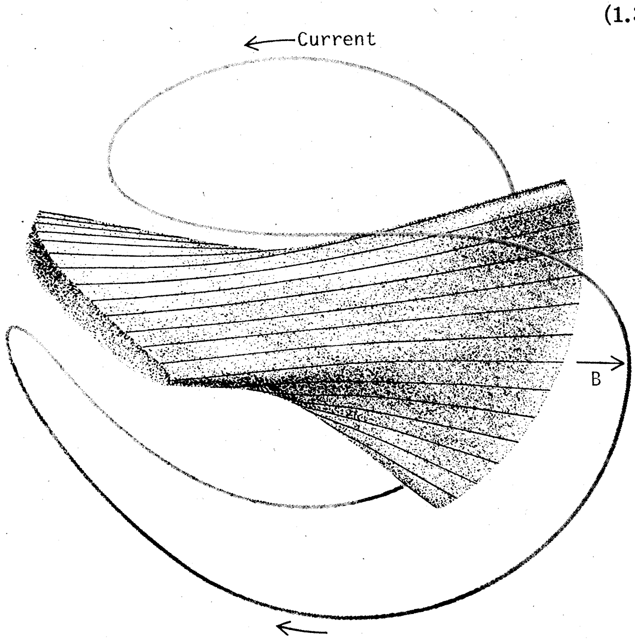

FIG. 2. Typical shape of plasma in the minimum-B magnetic field of a baseball-seam current winding.

has a minimum near $z=0$ provided

$$g^2(0)>\left.\frac{ff''}{2}\right|_0 . \tag{1.4}$$

The field components may be obtained from Eq. (1.2), and the magnetic line coordinates are given by

$$x(z)=x(0)\left(\frac{B(0)}{B(z)}\right)^{1/2}\exp\left(\int_0^z\frac{dzq}{f}\right), \tag{1.5}$$

and

$$y(z)=y(0)\left(\frac{B(0)}{B(z)}\right)^{1/2}\exp\left(-\int_0^z\frac{dzq}{f}\right), \tag{1.6}$$

where $x(0), y(0)$ are the line coordinates at the midplane. The function $f(z)$ determines the principal mirror field. It is generally parabolic for small z and maximizes at the mirror throats. The function g, which is in general independent of f, determines the fanning of the flux surfaces. Because a baseball-shaped coil has only one set of windings, g is related to f. For such windings, to a fair approximation, g/f is a constant; therefore in such fields fanning is roughly exponential. Typical ellipticities at the mirror points of flux surfaces that are circular at the midplane are 15 to 30. The effect of finite-β on minimum-B vacuum fields is a subject of current investigations (see Hall and McNamara, 1975). Even for a long, thin equilibrium, there exist no simple finite β analytic descriptions of the field shape comparable to Eqs. (1.5) and (1.6). The variation of the $|\mathbf{B}|$ for finite β can, however, be obtained by simple pressure balance in the perpendicular direction, neglecting line curvature,

$$p_\perp+\frac{B^2}{8\pi}=\frac{B^2_{vac}}{8\pi}. \tag{1.7}$$

Defining $\beta_{vac}\equiv 8\pi p_\perp/B^2_{vac}$, Eq. (1.7) immediately gives

$$B/B_{vac}=\sqrt{1-\beta_{vac}} \tag{1.8}$$

and the relation between β_{1oc} and β_{vac} cited above. Note that, to the order of accuracy for which Eq. (1.7) is valid, B_{vac} must be considered to be only a function of z alone; i.e., it is given by $f(z)$ in Eq. (1.3).

II. NONADIABATIC EFFECTS

The constancy of the magnetic moment upon which mirror containment is based is an asymptotic result stemming from the slow variation of the magnetic field in the frame of the particle's motion. Assign the small dimensionless parameter $\delta\equiv\max\{|\Omega^{-2}d\Omega/dt|;\ |\Omega^{-3}d^2\Omega/dt^2|\}$, where Ω is the particle's cyclotron frequency and d/dt is the convective derivative. The quantity usually termed the magnetic moment, $\mu=\frac{1}{2}w^2/B$, where w is the component of velocity locally perpendicular to B, is but the first term of a series asymptotic in powers of δ; the entire series is more accurately termed the adiabatic invariant. However, even the quantity represented by a finite number of terms of this series will experience changes in time, occurring as rapid changes (jumps) localized near minima of B, that can be larger than the error entailed in truncating the series. The magnitude of these jumps is not expandable in powers of δ, occurring in the form $\exp(-\alpha\delta^{-1})$. These properties of the time history of the magnetic moment of a particle in a magnetic field have been discussed for multipole and cusp fields (Howard, 1971; Hastie *et al.*, 1968) and for mirror fields, with particular emphasis on effects of finite plasma pressure (Rowlands and Cohen, 1975; Cohen *et al.*, 1976). The mathematical description of these jumps is very similar to the calculation of the reflection of a wave from an under-dense medium in the WKB limit (Heading, 1962).

To demonstrate how the above-described changes in μ occur, consider Newton's equations for a charged particle in a nonuniform, static, magnetic field wherein the particle velocity is described by the variables $\mu=\frac{1}{2}w^2/B$, $E=\frac{1}{2}v^2$, and an azimuthal angle θ about the direction of the local magnetic field $\hat{\mathbf{b}}$

$$\frac{d\mu}{dt}=-\frac{\mathbf{w}}{B}\left(\mu\nabla B+u^2\frac{\partial\hat{\mathbf{b}}}{\partial s}\right)+\mu u(I_\perp-2\hat{\mathbf{w}}\hat{\mathbf{w}}):\nabla\hat{\mathbf{b}}, \tag{2.1}$$

$$\frac{dE}{dt}=0, \tag{2.2}$$

and

$$\frac{d\theta}{dt}=-\Omega-\hat{\mathbf{e}}_2\cdot(\mathbf{v}\cdot\nabla)\hat{\mathbf{e}}_1-\frac{u}{w}\mathbf{b}\times\hat{\mathbf{w}}\cdot(\mathbf{v}\cdot\nabla)\hat{\mathbf{b}}. \tag{2.3}$$

In these equations, $u^2=2(E-\mu B)$ is the component of velocity parallel to B; and $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ form with $\mathbf{b}$ an orthonormal set, $\mathbf{I}_\perp=\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1+\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2$. Throughout, the hat (^) denotes unit vectors. [To obtain Eq. (2.1), use must be made of $\nabla\cdot\mathbf{B}=0$, a point which can be significant when integrating single-particle equations of motion in magnetic fields which are known only numerically on a finite grid.] The particle position vector is approximately

$$d\mathbf{r}/dt=\mathbf{v}\approx u\hat{\mathbf{b}}+\sqrt{2\mu B}(\hat{\mathbf{e}}_1\cos\theta+\hat{\mathbf{e}}_2\sin\theta). \tag{2.4}$$

Equations (2.1) through (2.3) have the structure that time derivatives of quantities normally taken to be nonoscillatory are either manifestly oscillatory or of order δ, except for $\Omega(\mathbf{r})$ in Eq. (2.3). The time integral of these oscillatory terms will be nonvanishing due to the slow variation of Ω along a particle (or guiding-center) orbit. From Eq. (2.4), $\mathbf{r}$ changes principally due to motion parallel to $\hat{\mathbf{b}}$, the arc length of which we denote by s. Thus, to lowest order in δ

$$\theta(s)=-\int^s\frac{ds'\Omega'}{u'}, \tag{2.5}$$

where $\Omega'=\Omega(s')$ and $u'=\sqrt{2}[E-\mu B(s')]^{1/2}$, and

$$\mu(s_2)-\mu(s_1)=-\int_{s_1}^{s_2}\frac{ds'}{u'}\left[\left(\frac{2\mu}{B'}\right)^{1/2}\left(\mu\nabla' B'+u'^2\frac{\partial\hat{\mathbf{b}}'}{\partial s'}\right)\cdot\hat{\mathbf{w}}'-\mu u'(\mathbf{I}_\perp-2\hat{\mathbf{w}}'\hat{\mathbf{w}}'):\nabla'\hat{\mathbf{b}}'\right]. \tag{2.6}$$

The integrand in Eq. (2.6) is dominated by the rapid phase dependence of $\hat{\mathbf{w}}$. Evaluation of the integral is most easily carried out by continuation of the integrand into the complex s plane. The result is integrals of the form

$$\int_{s_1}^{s_2}ds\,A(s)\exp[\pm i\theta(s)],$$

where $A(s)$ is a slowly varying function relative to $\theta(s)$. Given a specific s dependence of B, the s integral can be deformed to lie along the direction of most rapid descent of $\exp(\pm i\theta)$, starting from s_1 and s_2. The resulting integrals ending at s_1 and s_2 will either be connected at infinity or be connected by integration through one or more saddle points where $d\theta/ds=0$, i.e., $\Omega=0$. The latter situation prevails when s_1 and s_2 lie on opposite sides of the minimum in B. The indefinite integrals terminating at s_1 and s_2 can be performed by integration by parts with the results combined, respectively with $\mu(s_1)$ and $\mu(s_2)$ to give a quantity which is the first-order correction to the zero-order adiabatic invariant. Said differently, the quantity

$$\mu_1=\frac{w^2}{2B}-\frac{1}{\Omega B}\left(\frac{w^2}{2B}\nabla B+u^2\frac{\partial\hat{\mathbf{b}}}{\partial s}\right)\cdot\hat{\mathbf{b}}\times\mathbf{w}-\frac{1}{\Omega}\frac{w^2u}{4B}[(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2)\sin2\theta-(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2+\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1)\cos2\theta] \tag{2.7}$$

is the adiabatic invariant correct to first order in δ.

The jump, $\Delta\mu$, resulting from the saddle-point integration contains two terms, one of single and one of double frequency; only the former will be significant. For symmetric, analytically simple fields (e.g., parabolic) having a single upper half-plane saddle point, is_0, nearest the real s axis, the result of evaluating the saddle-point integral gives

$$\frac{\Delta\mu}{\mu}=\pi\delta^{-1/8}\left(\frac{E}{\mu B}\right)^{1/2}\exp\left[-\int_0^{s_0}\sqrt{2}\frac{d\sigma\,\Omega(i\sigma)}{(E-\mu B(i\sigma))^{1/2}}\right]\cos\theta_0, \tag{2.8}$$

where θ_0 is the phase before entering the midplane region.

The magnitude of $\Delta\mu/\mu$ is clearly of order $\exp(-\alpha\delta^{-1})$, where α is of order unity in δ, but becomes large for velocities nearly normal to B. The details depend on the functional form for $B(s)$ continued to the complex plane. In particular, depression of the central field and the reduced axial scale length introduced by finite β will have a detrimental effect. The model of a parabolic pressure profile in a parabolic vacuum magnetic well very roughly introduces the factor $\exp(-3\beta_{\text{vac}})$ into the exponent of Eq. (2.8) when the equilibrium can be described in the long, thin approximation (Cohen *et al.*, 1976).

The accumulation of $\Delta\mu$ on successive passes of the midplane will be stochastic provided θ_0 may be treated as a random variable. In this event, the effect of small $\Delta\mu$ induced by weak breakdown of adiabaticity may be included in a generalized Fokker–Planck equation describing the diffusion of particles in μ space (Bernstein and Rowlands, 1976). For specialized parameters, when successive values of θ_0 are not random, particle orbits in μ, θ_0 space can be closed, leading to the situation where, although μ changes, particles are confined in regions of μ space, a phenomenon termed superadiabaticity (Jaeger *et al.*, 1972).

III. EQUILIBRIUM ELECTRON PROPERTIES

In certain respects, the electrons in a mirror machine act as a benign neutralizing fluid for the ions. Having such a high collision frequency, they are constrained by a positive ambipolar potential to a particle loss rate equal to that of the ions. Because this mode of containment produces a distribution function which is nearly Maxwellian except for energies above the ambipolar potential, it is virtually completely stable to high-frequency instability. However, the electrons do play a central role in the energetics picture of the confined ions.

For reasons discussed later, a mirror reactor would be fueled by the injection of energetic neutrals. Ionization of the neutrals produces cold-electron and hot-ion pairs which then begin to equilibrate via the ion-electron energy transfer process. Because their confinement is electrostatic, it is always the most energetic electrons that are lost (at a rate which must be the same as that of the ions). The combination of injection below the average energy and escape above, plus the fact that the lifetime of an ion–electron pair in the trap is at best only their energy exchange time, means that the electron energy is low compared to that of the ions, with a steady-state ratio having an upper limit of about 0.1. This heat loss to the electrons is the principal classical limit on the energy containment time of trapped ions. The rate of energy transfer increases with decreasing electron temperature; therefore, any additional energy loss process through the electron channel is of direct significance to the ion energy confinement time.

Due to their Maxwellian distributions, the electrons could be described either by a temperature T_e or by an average energy $E_e = \frac{3}{2}T_e$. However, because of their loss-cone distribution, the ions are most conveniently described directly by their average energy. The literature is inconsistent on this matter, it being common to quote both an electron temperature and an average ion energy E_i.

To illustrate the energy bookkeeping for electrons, consider the model wherein the energy input to the electrons from the hot ions competes with that lost by escaping electrons

$$\frac{d}{dt}\left(\tfrac{3}{2}nT_e\right) = \frac{n(E_i - 3T_e/2)}{\tau_d} - \eta T_e J_{\text{ion}}, \tag{3.1}$$

where T_e is the electron temperature in keV; E_i is the average energy of the non-Maxwellian ions in keV; n is the common density; τ_d is the ion–electron energy exchange time, $n\tau_d = 10^{12}\sqrt{M}T_e^{3/2}$ for an ion of atomic number M; J_{ion} is the net flux per unit volume of ions (and therefore electrons) through the machine; and η is a parameter measuring the energy (in units of T_e) expended per electron. This model assumes that the electrons are injected with negligible energy. Because the electrons are contained by the ambipolar potential Φ, η will be at least Φ/T_e plus any energy loss associated with the two degrees of freedom perpendicular to **B**. Values such as $\eta = 5.5$ are a typical result of loss based strictly on classical collisional processes. Any other energy loss processes on the electrons, such as ionization of background gas or the injection of (colder) secondary electrons from ion bombardment of the walls, will contribute to a further energy drain on the electrons. It is customary to include these effects in the definition of η, making that a phenomenological constant. In 2XII, under conditions of good vacuum, $\eta \approx 8$ gave a good description of the electron-temperature history (Coensgen *et al.*, 1974) when J_{ion} was determined by the measured escape rate of trapped ions. If the ion lifetime for scattering by classical processes is used (see Sec. IV), J_{ion} is about $n^2[2.4\times 10^{10}E_i^{3/2}\log_{10}R_m]^{-1}$. From Eq. (3.1) may be obtained the steady-state relationship $E_e = \frac{3}{2}T_e = 0.1E_i$ for an $R_m = 2$ classical machine having no electron-cooling mechanism other than the loss of energetic electrons.

The ambipolar potential which contains electrons also ejects ions with insufficient magnetic moment to allow magnetic trapping, causing the ion loss boundary to be modified from the strict loss cone of magnetic confinement. However, because only ions with perpendicular energies $\lesssim q\Phi/(R_m - 1)$ are thus not confined, this effect is usually a less important factor in the ion lifetime than energy degradation by electron drag. Both the smallness of $q\Phi$ compared to E_i and the $R_m - 1$ factor for $R_m > 2$ act to reduce its importance. The low-energy hole thus created does, however, have an important implication on the stability of the resulting ion distribution function to ion cyclotron fluctuations, a matter discussed in Sec. V.

To determine both the magnitude of the ambipolar potential and its spatial distribution, it is necessary to investigate the velocity-space behavior of the electrons. The average electron–electron collision rate ν_{ee} exceeds the ion–ion rate by roughly the ratio $(E_i^3 m_i/E_e^3 m_e)^{1/2}$. Electrons are rapidly scattered into their magnetic loss cone, defeating magnetic containment, and the ambipolar potential is set up, equating the electron and ion loss rates. For an ion lifetime $\tau_I = n/J_{\text{ion}}$, the magnitude of this ambipolar potential is roughly the solution of the transcendental equation

$$\nu_{ee}\frac{T_e}{\Phi_{\max}}e^{-|e|\Phi_{\max}/T_e} \sim \frac{1}{\tau_I}$$

or

$$|e|\Phi_{\max} \sim T_e \ln(\nu_{ee}\tau_I T_e/\Phi_{\max}), \tag{3.2}$$

(Pastukhov, 1974; Ben Daniel, 1961), where $\Phi_{\max}$ is the maximum potential relative to grounded walls and occurs at the point of maximum ion density. Electrons with energies $E < |e|\Phi_{\max}/m_e$ are electrostatically trapped and will have distributions close to Maxwellian. Higher-energy electrons are trapped by the magnetic field provided $E < \mu B_m$. If we use the classical values of $n\tau_I$, the T_e/E_i ratio, and $n\tau_{ee} = 6\times 10^8 T_e^{3/2}$(keV) in Eq. (3.2), we obtain $\Phi_{\max}/T_e \sim 4.7$, which is essentially the typical Fokker–Planck computed value.

The spatial distribution of Φ is more complicated. With variation along **B**, quasineutrality holds until the density becomes so low that the Debye length is comparable with the scale length; i.e., throughout the machine $n_e = n_i$. To the extent that the electrons are Maxwellian, the axial dependence of the potential is then given by

$$\Phi_{\max} - \Phi(s) = -T_e \ln[n_i(s)/n_i(0)], \tag{3.3}$$

which remains a rough guide over most of the ion density variation. More precise determination of the axial distribution of the ambipolar potential must be obtained from a better model for the electron distribution than the simple Maxwellian (Yushmanov, 1966; Pastukhov, 1974). This distribution is affected in part by the ion density in and beyond the mirror throat (Guillory and Kunkel, 1970). For example, consider the case when a low ion density extends beyond the mirror, either because of an external plasma or because of a density due directly to ions' escaping from the mirror. The drop in potential from the center to the mirror throat will then not be $\Phi_{\max}$ as given by Eq. (3.2), but will be better approximated by Eq. (3.3) evaluated at the mirror point s_m. The confined region of electron phase space of midplane variables u_0, w_0 will be divided into two regions, shown in Fig. 3. Region I has a hyperbolic boundary fixed by the mirror ratio and the potential at the mirror throat Φ_m.

$$u_0^2 = w_0^2(R_m - 1) + 2\frac{|e|}{m_e}(\Phi_{\max} - \Phi_m). \tag{3.4}$$

These electrons do not escape beyond the mirrors. Region II describes electrons that escape beyond the mirror throat but are trapped by the remaining ambipolar potential relative to ground. These electrons neutralize the ion density external to the mirrors and pass freely between this external region and the region of the mirror-confined plasma. Clearly, their characteristic energy will be that of the electrons trapped between the mirrors. The boundaries of this second region are given by Eq. (3.4) and

$$u_0^2 = w_0^2(R_w - 1) + 2\frac{|e|}{m}\Phi_{\max}, \tag{3.5}$$

where R_w is B(wall)/B(midplane). It is the net flux of electrons caused by collisions across the composite boundary formed by the greater value of $u_0^2 + w_0^2$ as given by Eqs. (3.4) or (3.5) which must equal the net ion loss rate. This required equality fixed $\Phi_{\max}$ originally; it is the velocity-space refinement of Eq. (3.2). If the electron density is calculated from a distribution satisfying these boundary conditions and the axial distribution of Φ is calculated from quasi-neutrality, at low density there will be a deviation from the Maxwellian result [Eq. (3.3)]. For example, at the mirror point there are no electrons from Region I of Fig. 3. If $R_w \ll 1$ and the el-

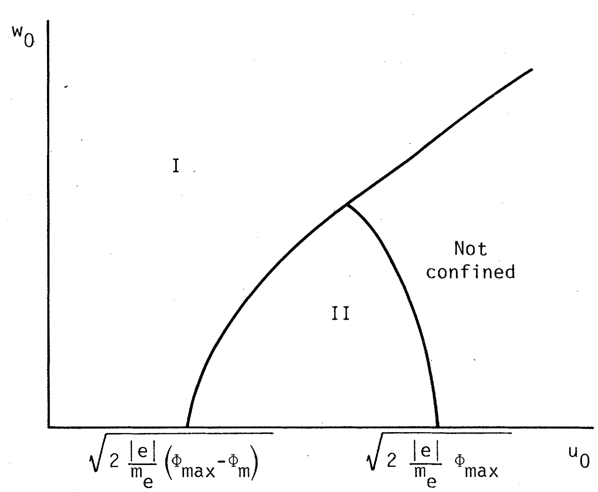

FIG. 3. Electron velocity space. Electrons in Region I are confined between the mirrors and those in II by the wall sheaths. High u_0 electrons are not confined.

ectron distribution in Region II is taken as a Maxwellian which is truncated at $\Phi_{\max}$, then the electron density at the mirror is proportional to

$$n_e(s_m) \propto \exp\left(-\frac{\Phi_{\max} - \Phi_m}{T_e/|e|}\right)\int_0^{\Phi_m} dE\, E^{1/2}\exp\left(-\frac{E}{T_e}\right). \tag{3.6}$$

The integral factor that multiplies the exponential implies that, for a given ion density ratio $n_i(s_m)/n_i(0)$, the drop in potential $\Phi_{\max} - \Phi_m$ will be less than that given by Eq. (3.3). Clearly, such details of the potential in the mirror throat depend critically upon the electron distribution at high energy.

The distribution of ambipolar potential *across* field lines is not nearly so clear. It is affected by particle drifts, finite Larmor radius, cross-field transport, variation of T_e across B, and the mechanism of plasma injection. Until a clearer picture emerges, it is probably safest to assume that the potential distribution along each line is determined as described above, but it is important to emphasize that at this stage this is little more than an assumption.

We have now described part of the hot-ion energy being transferred to the electrons. Upon escape, electrons carry away ηT_e per particle in energy; assuming no other loss processes, the ions escape with an energy below the injected energy by the same amount. However, because the plasma has a positive potential relative to its environment, on leaving the plasma the ions gain $\Phi_{\max}$ in kinetic energy and the electron lose the same amount. This has the effect of converting back to the ions much of the energy originally lost to the electrons. The final kinetic energy carried away per electron is $(\eta T_e - \Phi_{\max})$, which ideally is only of order T_e. The directed, escaping, ion kinetic energy can be converted to electrical current (Post, 1969) by decelerating the ions in externally imposed electrostatic fields (see Sec. IV). To the extent that this process is efficient, the energy originally lost to the electrons can be regained. The significance of the transfer process occurs within the plasma; the average ion energy is caused to be below that injected.

IV. CLASSICAL COLLISIONAL LOSSES

The unavoidable loss of ions is that due to scattering from other ions, and this loss determines the ultimate limit of mirror confinement. The multiple weak interaction of ions of similar mass results in a diffusion in velocity space. Because of their small mass, the electrons affect the ions not by a scattering *per se*, but rather by a polarization friction, or drag, force. When $E_e \approx 0.1 E_i$, the time scale for energy degradation of ions having low atomic mass is about the same as that for 90° deflection by ion–ion scattering. As discussed in Sec. III, this energy ratio is the normal one for mirror confinement, so that ion lifetimes are also significantly affected by electron drag. In fact, the loss process in phase space of an ion injected at high energy is first to be dragged down in energy at constant pitch angle and then to be scattered into the loss cone by ion–ion collisions. Electron-to-ion energy ratios less than this classical value only accentuate the dominance of electron drag over ion scattering.

Because of the long range of the Coulomb cross section, classical collisions at the densities and energies of interest entail the simultaneous weak interaction of a large number of particles. This many-body collision is conveniently described by a renormalization whereby each particle is "dressed" by a cloud of other particles (principally electrons) that acts to shield the field of the original particles over distances greater than the order of the Debye length, $\lambda_D = (T_e/4\pi n e^2)^{1/2}$. The external magnetic field influences the collision only when the particle Larmor radius is less than the interaction distance (i.e., for electrons when $\Omega_e > \omega_{pe}$); even then the effect is weak (Baldwin and Watson, 1975). Thus, to an accuracy of the reciprocal of the number of particles in a Debye sphere, the dressed particles may be treated as being statistically independent and interacting through their shielded potentials (Rostoker, 1964).

To the extent that the resultant interaction is weak, the effect of the multiple, cumulative, small-angle deflections can be described by a Fokker–Planck operator in velocity space (Chandrasekhar, 1943):

$$\left.\frac{\partial f}{\partial t}\right|_{\text{collisions}} = \frac{\partial}{\partial \mathbf{v}} \cdot \left[-\left\langle\frac{\Delta \mathbf{v}}{\tau}\right\rangle f + \frac{1}{2}\frac{\partial}{\partial \mathbf{v}} \cdot \left(\left\langle\frac{\Delta\mathbf{v}\Delta\mathbf{v}}{\tau}\right\rangle f\right)\right]$$
$$\equiv -\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{J}, \tag{4.1}$$

where $\langle \Delta \mathbf{v}/\tau\rangle$ and $\langle \Delta\mathbf{v}\Delta\mathbf{v}/\tau\rangle$ are the deflections averaged over times τ that are short on the time scale of f and long on the scale of a collision duration. Similarly, f is assumed to be slowly varying on the scale of a typical Δv. The form of Eq. (4.1) follows by a Taylor series expansion in Δv from a more complete one involving transition probabilities. The neglected higher derivatives are smaller than those retained, as discussed later. The resulting current in velocity space has two equivalent for forms. The first, due to Landau (1936), is

$$\mathbf{J} = \frac{2\pi q^2 \ln\Lambda}{m} \sum_\sigma q_\sigma^2 \int d^3v' \frac{g^2\mathbf{I} - \mathbf{gg}}{g^3}\left[\frac{1}{m_\sigma} f(\mathbf{v}) \frac{\partial f_\sigma(\mathbf{v}')}{\partial \mathbf{v}'} - \frac{1}{m} f_\sigma(\mathbf{v}') \frac{\partial f(\mathbf{v})}{\partial \mathbf{v}}\right], \tag{4.2}$$

where the summation is over species σ and $\mathbf{g} = \mathbf{v} - \mathbf{v}'$. The second form introduces the Rosenbluth potentials (Rosenbluth *et al.*, 1957),

$$G_\sigma(\mathbf{v}) = \frac{4\pi q^2 \ln\Lambda}{m^2} \int d^3v' \,|\mathbf{v} - \mathbf{v}'|\, f_\sigma(\mathbf{v}') \tag{4.3}$$

and

$$H_\sigma(\mathbf{v}) = \frac{4\pi q^2 \ln\Lambda}{m^2}\left(1 + \frac{m}{m_\sigma}\right) \int d^3v' \,|\mathbf{v} - \mathbf{v}'|^{-1} f_\sigma(\mathbf{v}'), \tag{4.4}$$

for which the current becomes

$$\mathbf{J} = \sum_\sigma \left[f \frac{\partial H_\sigma}{\partial \mathbf{v}} - \frac{1}{2}\frac{\partial}{\partial \mathbf{v}} \cdot \left(f \frac{\partial^2 G_\sigma}{\partial \mathbf{v}\partial \mathbf{v}}\right)\right]. \tag{4.5}$$

The quantity $\ln\Lambda$ appearing in Eqs. (4.2) to (4.5), called the Coulomb logarithm, represents a number of physical effects upon which the net result is only weakly dependent. By screening the interacting particles, the characteristic logarithmic divergence of the integrated Coulomb cross section has been removed. However, the assumption of weak interactions leading to the Fokker–Planck equation is violated at small impact parameters, introducing a second logarithmic divergence. Strictly speaking, such large angle scattering properly necessitates a Boltzmann collision integral description (Baldwin, 1962; Bernstein and Ahearne, 1968). It is customary to simply cut off the cross section at the classical distance of closest approach, in which case the argument of the logarithm becomes the ratio of λ_D to this distance (Spitzer, 1962)

$$\ln\Lambda = \ln(\lambda_D E/q^2), \tag{4.6}$$

where E in this expression is, strictly speaking, the energy in the center-of-mass. The λ_D appears naturally in the theory, being the upper limit of the range of the screened interactions. Under reactor conditions, $\ln\Lambda$ is about 20. This number is modified slightly by refinements such as allowing a species dependence to the distance of closest approach, in which case $\ln\Lambda$ becomes σ dependent in Eqs. (4.3) and (4.4). However, corrections of this type usually are of the same order as terms already neglected in formulating the scattering as a Fokker–Planck equation. The coefficients of the neglected higher derivatives in Eq. (4.1) are smaller than those retained only by the absence of a $\ln\Lambda$ factor. Physically, those few particles suffering large angle collisions cannot be described by a diffusive process; by neglecting them, the Fokker–Planck description has an inherent inaccuracy of order $(\ln\Lambda)^{-1}$.

Before reviewing solutions of the Fokker–Planck equation, it is useful to note the time scales for the important processes in forms which may be used for scaling laws and even for factor-of-two estimates (Spitzer, 1962). The fastest process is that of electron–electron collisions with an average 90° scattering time

$$\tau_{ee} = 1.1 \times 10^{10}[T_e^{3/2}(\text{keV})/n\ln\Lambda], \tag{4.7}$$

where n is the number of electrons/cm^3. Because the ion–electron relative speed is fixed by the electrons, the time for electron scattering from ions is essentially the same as that for scattering from electrons. The corresponding ion–ion 90° scattering time is longer than τ_{ee} by roughly $Z^{-4}(E_i^3 m_i/E_e^3 m_e)^{1/2}$, or

$$\tau_{ii} = 2.5 \times 10^{11}[E_i^{3/2}(\text{keV})M^{1/2}/Z^4 n\ln\Lambda], \tag{4.8}$$

where M and Z are the atomic and charge numbers, respectively. When the electron thermal velocity is large compared to the ion thermal velocity, the time for energy exchange between ions and electrons is basically the electron–ion collision time times a mass ratio, or

$$\tau_d = 1.0 \times 10^{13}\frac{MT_e^{3/2}(\text{keV})}{Z^2 n\ln\Lambda}. \tag{4.9}$$

Here τ_d is often referred to as the electron drag time for the ions.

Embodied in these simple fomulae are the important classical scaling laws for mirror machines. From Eq. (4.8) the $n\tau_{ii}$ value increases with $E_i^{3/2}$, suggesting a requirement of high ion energy; and from Eq. (4.9), $n\tau_d$

has a $T_e^{3/2}$ dependence which, when compared to Eq. (4.8), sets a limit $T_e \gtrsim E_i/15$ in order for drag not to dominate collisions. A characteristic of all classical scattering is that the $n\tau$ product is a function only of energy and particle quantities such as mass and charge, but is independent of density. Properly speaking, the product is a functional of particle distribution functions, which have been taken to be Maxwellian in the above estimates. For related quantities in a mirror machine, one would expect the electron based quantities τ_{ee} and τ_d to be unchanged. However, there would be some expected alteration of τ_{ii} due to the non-Maxwellian character of the ion distribution. It is found that for mirror distributions the absence of low-energy ions gives an average $n\tau_{ii}$ larger by a factor of 1.5 to 2.0 when measured against the average ion energy as in Eq. (4.8).

Because the classical collisions occur on a time scale long compared to virtually all other processes, an accurate calculation of their effect is a complicated matter, and is in the stages of continuing development. Even in present experiments, an ion lifetime by classical scattering is of order 10^3 axial bounce periods; in a reactor it would be 10^5 to 10^6. In this latter case, at least, even the particle-drift time around the machine is shorter than the collision time. In principle then, a proper procedure to describe scattering on the slow time scale is to average the original velocity-space Fokker-Planck equation over the successively faster time scales, cyclotron motion, bounce motion, and (if required) drift motion. The original Fokker-Planck equation describes diffusion in velocity space, and the cyclotron-averaged equation describes diffusion in the space of the constants of the cyclotron motion, μ, E. Similarly, averaging this equation over the rapid bounce motion further reduces the dimensionality of the problem by restricting particle motion in the absence of collisions to surfaces of constant $J=\oint u ds$; collisions induce transition between such surfaces.

The calculations which have been carried out to date are approximations to the spatial aspects of these general notions. The model used almost universally is that of a magnetic square well; i.e., B is constant with a step rise by the mirror ratio at the mirrors (Ben Daniel and Allis, 1961). The bounce average is then trivial, as is the cyclotron average. The distribution depends only on u, w, t, has a time rate of change given solely by Eqs. (4.1) to (4.4), and vanishes on the loss boundary where

$$u^2 = w^2(R_m - 1) - 2(q/m)\Phi . \tag{4.10}$$

The ambipolar potential appearing in Eq. (4.10) is related to, and is often set equal to, the total potential containing electrons $\Phi_{\max}$. Loss is calculated flux tube by flux tube, and any effects of drifts are neglected. Even with these assumptions, solution for a number of species entails a set of coupled, nonlinear, integro-differential equations in two dimensions plus time. The numerical techniques employed will not be reviewed here (see Killeen *et al.*, 1968, 1976); only the results will be summarized. The series of investigations treating this problem have involved a number of further approximations, the accuracies of which have been cross-checked in time, so that there is now a fair level of confidence in the results. The early treatments have been reviewed and compared by Kuo-Petravic *et al.* (1969).

The set of coupled Fokker-Planck equations includes an equation for the electron distribution and contributions to the ion scattering due to the electrons. However, the high scattering rate of the electrons and their electrostatic mode of confinement described in Sec. III assure that their distribution will be closely Maxwellian for energies below that of the confining potential. If the electron contribution to the ion collision rate is expanded in the ratio of electron to ion mass, the significant collisional effect of electrons on ions is then through the polarization, or frictional, drag. Because the drag is due to those electrons traveling slower than the ions, it is insensitive to the non-Maxwellian features of the electron distribution and depends only on the temperature of the electrons. The other electron-related quantity affecting the ions is the ambipolar potential. This enters in two ways: (i) the drop in potential to the mirror throat enters the ion loss boundary, Eq. (4.10); and (ii) becasue those electrons being lost are the most energetic, the maximum drop of Φ to the wall determines the loss rate of energy by the electrons. [The latter effect gave rise to the dependence of η in Eq. (3.1) on $\Phi_{\max}$]. Both the total confining potential and the axial variation near the mirrors are sensitive functionals of the departure of the electron distribution from Maxwellian at high energy. The electron distribution at high energy may be obtained from the full electron Fokker-Planck equation by a linearization wherein the relatively few high-energy electrons are scattered by the more numerous low-energy ones. This linear equation is solved subject to the vanishing of the distribution on the loss boundary and its matching to a Maxwellian at energies comparable to T_e. The problem has been solved *approximately* by Pastukhov (1974) for the case of a hyperbolic loss boundary. The result of all such calculations, be they the full electron Fokker-Planck equation or its linearized approximations, is to give the result relating $|e|\Phi_{\max}/T_e$ to the ratio of electron scattering rate to ion lifetime, for which Eq. (3.2) is an approximation. The drop in potential from midplane to mirror throat is even more model dependent; Eq. (3.6) is an example. In the past, the drop has either been taken as $\Phi_{\max}$ itself, or as $\Phi_{\max}$ reduced by an arbitrary amount of order 20% to test sensitivity.

There are really two distinct reasons for carrying out detailed Fokker-Planck solutions. The first is to quantify the classical confinement picture for the purpose of evaluating a mirror machine as a potential reactor. For this purpose, the picture described above, with the electron physics modeled by Eqs. (3.1) and (3.2) involving only heat input by the hot ions and loss by the loss of energetic electrons giving rise to the ratio $E_e/E_i \approx 0.1$, represents an optimum of the classical picture. A second application of Fokker-Planck studies is to provide a standard for current experiments against which nonclassical behavior can be compared. In these experiments, there may be (and usually are) additional energy loss processes through the electron channel, such as ionization of background gas or emission of secondary electrons from walls. The result is that the electron-to-ion energy ratio is considerably below the nominal 0.1, sometimes by as much as another order of magnitude. The accompanying dominance of electron drag over ion–ion collisions means that the ion lifetime is reduced by a process that is "classical" in the sense usually used in mirror physics; i.e., it does not entail anomalous scattering due to electric field fluctuations. Experimentally, such situations will still bear the hallmark which is characteristic of all classical processes: $n\tau$ that is independent of density. Whether in such situations one can replace Eqs. (3.1) and (3.2) with a set which correctly describes the electron temperature history in a given experiment is an important consideration in its own right, but that question should properly be separated from that of the ion lifetime in the presence of a reduced electron temperature. For comparing experiment to classical prediction, provided the independent measurements exist, the time dependence of T_e should be used as an *input* to a Fokker–Planck set describing ions alone (Hall, 1975).

All early numerical treatments of the ion equations reduced the calculations to one dimension in energy by assuming that the pitch-angle distribution was in "lowest" normal mode" or a "collisional distribution", i.e., that it had the smoothest shape consistent with the requirement that f vanish on the loss-cone boundary. There was some weak justification for this assumption. The Rosenbluth potentials G, H in Eq. (4.3) were shown to vary in pitch angle somewhat less rapidly than f itself (Ben Daniel and Allis, 1962); and, assuming G, H to be independent of pitch angle, the Fokker–Planck operator admits to separable solutions. An approximate functional form for the angular part of the collisional distribution, assuming separability in pitch angle and energy and neglecting ambipolar potential, may be obtained by combining solutions valid for low and high mirror ratios (Holdren, 1972)

$$f(E, \mu) = \left[\frac{\mu B_{\max}}{E} - 1 + (2R_m - 3)\ln\frac{\mu B_{\max}}{E}\right] f(E), \quad (4.11)$$

where $f(E)$ is the solution to the one-dimensional energy transport equation.

A limitation on the assumption of separability and the reduction to one dimensionality, even for distributions which are well spread, is that sources resulting from injected ions must also be assumed to have the lowest normal mode angular distribution. Because such an assumption means that some particles are injected near the loss cone and thus are poorly trapped, this treatment of the source at best opens a question of proper normalization. Recently, two-dimensional codes have been developed to test the validity of the single normal mode approximation for well spread distributions, to study the effect of varying the angle of injection, and to study properties of angular distributions which are clearly not collisional (Marx, 1970; Mirin, 1975; Killeen *et al.*, 1976). For well spread distributions decaying freely without sources and for well spread sources, the one- and two-dimensional codes give the same lifetimes. For $R_m > 2$ and injection angles peaked in the range 70° to 90° to the magnetic field, a 20 to 40% improvement in confinement is found (Rensink *et al.*, 1975; Rensink, 1975).

Certain of the approximations described earlier have also been avoided both by developing codes which numerically performs the average over the axial bounce motion (Marx, 1970; Rensink and Cutler, 1976) and by allowing for a drop in ambipolar potential to the mirror throat which is less than $\Phi_{\max}$ by an arbitrary chosen 20%. The corrections in each case are generally only a few percent.

For evaluating the potential of a mirror machine as a fusion reactor, a commonly used figure of merit is the ratio Q of the nuclear power produced to the injected power required. The power produced depends of course upon the reacting species. For example, consider the reaction giving the highest yield at the lowest ion energy, deuterium-tritium (D–T), because this would certainly be the first (and likely the only) fuel used in a mirror reactor. Reactors, including alternate fuels, cross sections, energy release, power balance, etc., were reviewed by Ribe (1975).

The direct nuclear products of a D–T reaction are a 14.1-MeV neutron and 3.5-MeV alpha particle

$$D^+ + T^+ \to n^0 + He^{++} + 17.6\ \text{MeV}.$$

The reaction rate σv, where σ is the cross section and v the relative velocity, rises sharply in the range of 50 to 70 keV, has a maximum just below 100 keV, and falls as v^{-1} for higher energy. When averaged over typical distribution functions, the maximum $\langle\sigma v\rangle_{DT}$ is about $9 \times 10^{-16}\,\text{cm}^3/\text{s}$, although the exact value depends upon the detailed shape of the ion distributions.

A mirror reactor would most certainly be fueled by the injection of energetic neutral beams (Hovingh and Moir, 1973; Hamilton and Osher, 1974). The past 10 to 15 years have seen such rapid development in this area that feasible effective neutral currents have gone from milliamps to kiloamps, with the present focused current densities about 0.5 A/cm^2 at 40 keV and the prospect of another factor of 3 increase in energy in the next few years. If we assume neutral injection to be the only source of power to the plasma, for D and T injection currents per unit volume I_D and I_T at energies $E_{D\text{inj}}$ and $E_{T\text{inj}}$, the power injected per unit volume is

$$P_{\text{inj}} = I_D E_{D\text{inj}} + I_T E_{T\text{inj}}.$$

Each current is related to the species density by a lifetime

$$I_{D,T} = n_{D,T}/\tau_{D,T}.$$

If we assume equal injection energies $E_{D,\text{inj}} = E_{T,\text{inj}} = E_0$ and neglect the mass difference so that the lifetimes are equal, $\tau_D = \tau_T = \tau$, the expression for Q becomes

$$Q = \frac{n_D n_T \tau \langle\sigma v\rangle E_{\text{nucl}}}{(n_D + n_T)E_{0\text{inj}}}.$$

This is maximized at equal densities, $n_D = n_T = n/2$, giving

$$Q = \frac{n\tau\langle\sigma v\rangle E_{\text{nucl}}}{4E_0}, \quad (4.12)$$

so that Q is directly proportional to the $n\tau$ product, which classically is independent of density. Q's for nonuniform plasmas necessitate that Eq. (4.12) be replaced by the ratio of appropriate averages.

The form of $n\tau$ would be expected to be similar to the $n\tau_{ii}$ for ion–ion collisions given by Eq. (4.8) except for modifications due to the non-Maxwellian nature of the distribution, the existence of the loss cone, and effects due to electron drag. For purposes of comparison, it has become standard practice to express $n\tau$ in terms of the injection energy E_0 rather than the average energy E_i appearing in Eq. (4.8). This practice makes application of $n\tau$ in Eq. (4.12) straightforward; and, because of the sensitivity to the ion distribution function, the relationship of $n\tau$ to E_i is no more fundamental than that to E_0. In the absence of an ambipolar potential the effect on $n\tau$ of the mirror ratios R_m in the range 2 to 10 can be shown analytically and numerically to appear as a multiplicative factor $\log R_m$. The β-depressed actual mirror ratio is significant, not the vacuum mirror ratio. (The factor $\ln R_m^{1/2}$ appears in the analysis, but this is replaced by $\log R_m$ with the near unity proportionality constant included in the coefficient discussed below.) The effect of a nonvanishing ambipolar potential Φ can be estimated by replacing R_m by an effective ratio

$$R_{\rm eff} = R_m/[1 + (q\Phi/mE_i)]\,. \qquad (4.13)$$

It is also usual practice to evaluate the mass as a D–T mean and $\ln\Lambda$, which we will take as equalling 20 for reactor conditions. With these conventions, differing Fokker–Planck calculations of $n\tau$ may be compared through the proportionality constant κ.

$$n\tau = \kappa \times 10^{10} E_0^{3/2}\ (\mathrm{keV})\ \log R_{\rm eff}\,. \qquad (4.14)$$

When a Fokker–Planck code is run without the effects of electrons, the average energy rises considerably above the energy of injection, so that the nominal κ employed in Eq. (4.14) is quite high, about 17. When this result for $n\tau$ was used for early calculations of Q (Post, 1962), the very favorable value of $8 \log R_m$ was obtained. Inclusion of electron drag and the ambipolar potential greatly reduced this value (Fowler and Rankin, 1966). The presently accepted range of κ in Eq. (4.14) for injection at 90° to the magnetic field including electron effects is 2.4 to 2.8 (Killeen *et al.*, 1975), and the average ion energy is about equal to the injection energy. The difference between this value and the 1.7 which would be inferred from Eq. (4.8) is due to the loss cone distribution. The lack of low-energy ions gives a somewhat higher value of κ, and thus for $n\tau$, for a given average energy. The presently accepted values of Q are in the range $(1.0 \text{ to } 1.3)\times \log R_m$, with the variation due to optimization strategies such as energies and distributions of injection.

The entire mirror reactor picture and illustrative calculations of the Q values required for economic operation were included in the reviews by Ribe (1975) and will not be repeated here. The net result is that a simple, thermally converted D–T mirror reactor would not be economic due to the order-of-unity value of Q resulting from the classical effects just described. Because of the short energy-confinement time, there results a large recirculating power which must be reinjected with high efficiency in order for the reactor to produce net power; the thermal conversion fails on the measure of the required efficiency. The concept of a mirror machine as a reactor becomes viable only when it is combined with both high-efficiency neutral injection and direct conversion of the energy of escaping ions to electricity (Moir *et al.*, 1976).

Certain features of neutral beams as a means of injecting energetic ions into a magnetic well have been mentioned earlier this section. An additional feature of this mode of injection is that, for energies above 150 keV, overall efficiencies in the range 80 to 85% are anticipated (Hovingh and Moir, 1973). These require the acceleration of negative ions in order that the conversion of the energetic ions to neutrals be efficient. Such beams are still in the developmental stage; however, there are good reasons to believe that the calculated efficiencies can be realized.

The notion of direct conversion of the end-loss ions first entails expanding the magnetic lines, and thus the escaping density, to a sufficiently low value that the ions and electrons can be separated. The electrons, whose energies are low, are diverted magnetically; the ions are decelerated in a set of charged electrodes which collects them at low kinetic energy. The high kinetic energy of the escaping ions is thus converted to dc power which may be used to directly power the grids of the neutral injectors. The theoretical efficiency of such converters can be very high for systems of many electrodes at small voltage intervals, and experimental efficiencies of about 85% have been achieved (Moir *et al.*, 1972). In current conceptual reactor designs, it is considered more cost effective to employ less complex direct converter system operating in the 65% efficiency range (Carlson and Moir, 1975).

V. FLUCTUATIONS

A. Linear theory

The third source of diffusion in velocity-space is the existence of fluctuating electric fields. While in a general sense Coulomb collisions themselves can be considered to be due to fluctuations with wave number $k > k_{De}$, the inverse electron Debye length, it is usual practice to treat wave numbers above and below this value separately. Because plasma cannot support waves with $k > k_{De}$, all such fluctuations are single particle in origin. Wavelengths longer than these entail the collective motion of the plasma and necessarily involve consideration of the various waves that the plasma will support. Of particular importance are those wavelengths which are unstable due to their coupling to sources of free energy such as non-Maxwellian energy distributions or the pressure gradients of confined plasmas.

The beginning of a survey of loss processes due to fluctuations must therefore begin with a review of the linear stability theory of mirror-contained plasma. However, finding instability is not enough; its effect on particle lifetime is not determined until both the nonlinear limit and its efficacy in scattering particles are determined. These calculations, especially the nonlinear ones, can be very much complicated by various effects of plasma nonuniformity. Accordingly, while linear stability, at least in the electrostatic limit, may be considered to be well in hand, the theory of the nonlinear saturation of the various modes is in a state of relative infancy. In fact, with some notable exceptions, there has historically been what amounts to a reluctance to consider such problems. This attitude has its origins in the belief that, because a mirror reactor would be economical only in the absence of scattering in excess of classical, the major direction of mirror microstability theory should be searching parameter space for a stable region. While a successful result of such a search would be highly desirable and might be realizable, knowledge of the nonlinear properties is clearly important in order to obtain a less demanding parameter evaluation, to interpret experiments not lying in the favored corner of parameter space, and to devise means of building a plasma up to the stable parameters.

The list of linear electrostatic instabilities of an infinite uniform plasma possessing a loss-cone distribution seems almost endless. A large number of these were discussed by Hall *et al.* (1965). Various aspects of the nonuniformity of the plasma and/or the magnetic field serve to reduce this list at the price of introducing others through the free energy associated with diamagnetic currents (Mikhailovskii, 1974; Jukes, 1968). We will review only those modes that are thought capable of existing in mirror machines of present or reactor sizes, although mentioning in passing those effects that stabilize some of the infinite-medium modes.

It is best to deal from the outset with space-varying equilibrium parameters, although the relative scale lengths often allow an eikonal description. In order to tap the free energy of inverted ion energy distributions, almost all modes of interest have wavelengths perpendicular to the magnetic field on the scale of the ion Larmor radius, a scale that is in turn shorter than that of the variation of the equilibrium parameters. On the other hand, the high mobility of electron motion parallel to $\mathbf{B}$ forces the fluctuating potential to be nearly constant along $\mathbf{B}$, meaning that variation parallel to $\mathbf{B}$ can occur on a length scale comparable to that of the equilibrium. This disparity in length scales may be gainfully employed to reduce the dimensions of the problem from the original three to one along the magnetic field (Baldwin, 1974). To do so, one introduces the Clebsch coordinates α, β that are constant along a field line, defining the magnetic field $\mathbf{B} = \nabla\alpha \times \nabla\beta$, and an arbitrary third variable ξ that is independent of α, β which in effect measures distance along $\mathbf{B}$. For example, if $\mathbf{B}$ is a vacuum field, $\mathbf{B} = \nabla\chi$, then χ itself is a useful variable, with $\nabla\chi \cdot \nabla\boldsymbol{\alpha} = \nabla\chi \cdot \nabla\beta = 0$, although in general $\nabla\boldsymbol{\alpha} \cdot \nabla\beta \neq 0$. Any function constant along B is a function of α, β alone. In analogy to plane-wave solutions in uniform plasma, solutions for the fluctuating electric field of the form

$$\mathbf{E}_1(\alpha, \beta, \xi) = \hat{\mathbf{E}}_1(\alpha, \beta, \xi) \exp[i\psi(\alpha, \beta)] \tag{5.1}$$

are sought where $|\nabla\psi|^2 \gg |\nabla\nabla\psi|, |\nabla \ln|\hat{\mathbf{E}}_1||^2$. This eikonal solution perpendicular to $\mathbf{B}$ introduces a perpendicular wave number

$$\mathbf{k}_\perp \equiv \nabla\psi = \nabla\alpha\partial\psi/\partial\alpha + \nabla\beta\partial\psi/\partial\beta. \tag{5.2}$$

Even though ψ is constant along $\mathbf{B}$, $\nabla\psi$ is not, due to its dependence on $\nabla\alpha, \nabla\beta$. As noted above, the variation of $\mathbf{E}_1$ along $\mathbf{B}$ often cannot be treated in the short wavelength limit, and plasma currents and/or charge density must be related to the fields by appropriate differentials or nonlocal integrals along $\mathbf{B}$.

Descriptions of microinstabilities stay as close to an infinite-medium description as the physics of the situation permits. Because the background evolves on a time scale long compared to that of the instabilities, it is treated as constant, and frequencies of the system are sought taking all time-varying quantities to vary as $\exp(-i\omega t)$. The eikonal solution mentioned above introduces an effective $\mathbf{k}_\perp(s)$ that is a function of distance s along a field line, with its two components at the midplane as free parameters. Determination of ω is then fixed by solving a one-dimensional eigenvalue equation in s. When the wavelength along $\mathbf{B}$ is comparable to the machine length, this solution is of the form of a low-lying eigenvalue. When the parallel wavelength is much smaller than the axial scale length, so that an eikonal description in this dimension $[\sim \exp(i\int^s k_\parallel ds)]$ is also possible, there results a dispersion relation obtained by treating the plasma as locally uniform, and it is important to distinguish whether the mode is absolute or convective (Briggs, 1965). Absolute instabilities are characterized by zero or small group velocities, so that a disturbance grows in time at a fixed point in space. Convective instabilities have sufficiently high group velocities that time growth is observed only in a moving frame; in a fixed frame, the wave is seen to amplify in space. In this case, it is not so meaningful to discuss roots of ω with positive imaginary parts for real $k_\parallel$, but rather roots of $k_\parallel$ with negative imaginary parts for real frequency. The distinction becomes important in consideration of waves in finite plasmas. The localized growth of an absolute instability can only be saturated by a nonlinear limit; for this reason, absolute instabilities are considered the more severe of the two types. A convectively unstable wave can be either reflected or absorbed by propagating into a region of changing parameters near a plasma boundary. Sufficient reflection leads to repeated stages of amplification as the wave passes through the unstable region, so that again saturation is achieved only nonlinearly. On the other hand, if such a wave is absorbed at the boundary, its maximum amplitude becomes a function of the plasma size, and for sufficiently short plasmas it is limited to low amplitude by strictly linear processes. Depending on one's preferences, one may say that the unstable, infinite-medium mode has thus been stabilized by the finite geometry; however, it is important to note that an enhanced fluctuation level will result. The extensive literature on this subject includes a variety of techniques for distinguishing between the two types (see Briggs, 1965).

Because the electrons are substantially Maxwellian and any instability must tap some source of free energy, one does not anticipate instabilities in the range typical of electron frequencies. Therefore, all waves of interest have time and length scales that are long compared to the electron–cyclotron period and radius, so that the electron response normal to $\mathbf{B}$ may justifiably be treated in the guiding-center approximation. Furthermore, the magnetic scale lengths are sufficiently long and the electron temperature sufficiently low that the electron particle drifts may also be neglected. Motion parallel to $\mathbf{B}$ is not quite so easily dispensed with. The mirror-machine length scaling that we shall obtain later $(L/a_i \sim \sqrt{m_i/m_e})$ implies that the electron bounce frequency in the trap is only 0.2 to 0.3 of the ion-cyclotron frequency, and therefore axial bounce motion should properly be included on the time scale of the instabilities. However, this effect is often not central to the existence of a mode, and a simpler description of the electrons is then possible.

We first obtain the linear current response of an arbitrary species to an electric field when the field variation is eikonal along $\mathbf{B}_0$ as well as transverse. The result is expressed by a dyadic $\mathsf{Q}(\omega, k_\parallel, \mathbf{k}_\perp, s)$ by

$$\mathbf{J} = -(4\pi i\omega)^{-1}\mathsf{Q}\cdot\mathbf{E}. \tag{5.3}$$

Equation (5.3) will yield a result directly comparable to the infinite medium result (see e.g., Baldwin *et al*., 1969), although the $k_\parallel$ and $\mathbf{k}_\perp$ thus introduced are to be treated as functions of s (as discussed above) and our formulation will include the effect of diamagnetic drifts. The perturbed distribution f_1 satisfies the equation

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\cdot\nabla f_1 + \Omega\mathbf{v}\times\mathbf{b}\cdot\frac{\partial f_1}{\partial \mathbf{v}} = -\frac{q}{m}\left(\mathbf{E}_1 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_1\right)\cdot\frac{\partial f_0}{\partial \mathbf{v}}, \tag{5.4}$$

where $\mathbf{B}_1 = c\mathbf{k}\times\mathbf{E}_1/\omega$. Equation (5.4) may be solved by integration along characteristics defined by the left-hand side, that are the particle trajectories in the unperturbed field. If $\mathbf{r}'(\tau), \mathbf{v}'(\tau)$ are defined by

$$\frac{\partial \mathbf{r}'}{\partial\tau} = \mathbf{v}'; \quad \frac{\partial \mathbf{v}'}{\partial \tau} = -\Omega\mathbf{b}\times\mathbf{v}'; \quad \mathbf{r}'(0) = \mathbf{r}; \quad \mathbf{v}'(0) = \mathbf{v},$$

where $\Omega = qB_0/mc$, when the eikonal assumption is introduced the solution to Eq. (5.4) is

$$f_1 = -\frac{q}{m}\int_{-\infty}^{0} d\tau \exp -i[\omega - \mathbf{k}\cdot(\mathbf{r}'(\tau) - \mathbf{r})] \times \left(\mathbf{E}_1 + \frac{1}{\omega}\mathbf{v}\times(\mathbf{k}\times\mathbf{E}_1)\right)\cdot\frac{\partial f_0'}{\partial \mathbf{v}'}. \tag{5.5}$$

Convergence of the integral is assured by the causal requirement that $\mathrm{Im}\,\omega$ be taken >0, and the result for $\mathrm{Im}\,\omega \leqslant 0$ is then obtained by analytic continuation.

In Eq. (5.5), the equilibrium distribution f_0 satisfies an equation of the form

$$\Omega\mathbf{b}\times\mathbf{v}\cdot\frac{\partial f_0}{\partial \mathbf{v}} = \mathbf{v}\cdot\nabla f_0 + \ldots, \tag{5.6}$$

where the dots ... represent terms such as collisions and the equilibrium electric field acceleration, all of which are assumed small compared to the terms explicitly given. By treating Ω as a large parameter, this equation may be solved by perturbation, and one obtains to first order

$$f_0 = f_0(w, u, \mathbf{r}) + \frac{1}{\Omega}\mathbf{v}\times\mathbf{b}\cdot\nabla f_0(w, u, \mathbf{r}), \tag{5.7}$$

where w, u are the components of $\mathbf{v}$ perpendicular and parallel to $\mathbf{b}$. The first-order contribution in Eq. (5.7) allows for the effects of the diamagnetic drifts. Thus in Eq. (5.5),

$$\frac{\partial f_0'}{\partial \mathbf{v}'} = \left(1 + \frac{\mathbf{v}'\times\mathbf{b}}{\Omega}\cdot\nabla\right)\left[\mathbf{v}'\frac{\partial f}{w\partial w} + \mathbf{b}\frac{1}{w}\left(w\frac{\partial f_0}{\partial u} - u\frac{\partial f_0}{\partial w}\right)\right] + \frac{1}{\Omega}\mathbf{b}\times\nabla f_0, \tag{5.8}$$

where all unprimed quantities are constants of the τ integration. The contribution $w\partial f_0/\partial u - u\partial f_0/\partial w$ vanishes when f_0 is isotopic, i.e., when it is a function solely of $v = (w^2 + u^2)^{1/2}$.

From Eq. (5.5), the dyadic $\mathbf{Q}$ in Eq. (5.3) becomes

$$\mathbf{Q} = \frac{4\pi q^2 i}{m}\int d^2v\mathbf{v}\int_{-\infty}^{0} d\tau\, e^{-i\omega\tau + i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})}\left\{\left[\left(1 + \frac{\mathbf{v}'\times\mathbf{b}}{\Omega}\cdot\nabla\right)\left(\frac{\omega}{w}\frac{\partial f_0}{\partial w} + \frac{k_\parallel}{w}\left(w\frac{\partial f_0}{\partial u} - u\frac{\partial f_0}{\partial w}\right)\right) + \frac{1}{\Omega}\mathbf{k}\cdot\mathbf{b}\times\nabla f_0\right]\mathbf{v}' + (\omega - \mathbf{k}\cdot\mathbf{v}')\left[\left(1 + \frac{\mathbf{v}'\times\mathbf{b}}{\Omega}\cdot\nabla\right)\frac{\mathbf{b}}{w}\left(w\frac{\partial f_0}{\partial u} - u\frac{\partial f_0}{\partial w}\right) + \frac{1}{\Omega}\mathbf{b}\times\nabla f_0\right]\right\}. \tag{5.9}$$

For mirror machines in which the plasma radius R_p is small compared to the radius of curvature of the magnetic field R_c, $\mathbf{v}'$, and $\mathbf{r}'$ in Eq. (5.9) may be computed neglecting the gradient B and curvature drifts

$$\mathbf{v}_D = \hat{\mathbf{b}}\times\left(\frac{w^2}{2}\frac{\nabla B}{B} + u^2\mathbf{b}\cdot\nabla\mathbf{b}\right)\frac{1}{\Omega}. \tag{5.10}$$

The ∇f_0 terms in Eq. (5.9) lead to the diamagnetic current

$$\mathbf{j}_d = q\mathbf{b}\times\nabla\cdot\mathbb{P}/\Omega,$$

where $\mathbb{P}$ is the pressure tensor. The latter drift velocity is of order $\bar{v}_i a_i/R_p$, and is therefore larger than v_D which is of order $\bar{v}_i a_i/R_c$.

Treating the electrons in the zero Larmor radius limit, the electron contribution $\mathbf{Q}_e$ is particularly easy to evaluate. Assuming isotropy of f_{0e}, one obtains

$$\mathbf{Q}_e \approx \frac{4\pi e^2\omega i}{m_e}\int d^3u\mathbf{u}\frac{1}{w}\frac{\partial f_{0e}}{\partial w}\int_{-\infty}^{0} d\tau e^{-i\omega\tau}\mathbf{v}' \xrightarrow[\omega\ll\Omega_e]{} -\frac{4\pi e^2\omega}{m_e}\mathbf{bb}\int d^3v\frac{u\partial f_{0e}/\partial u}{\omega - k_\parallel u} - \frac{\omega^2\omega_{pe}^2}{\Omega_e^2}\left(\mathbf{I} - \mathbf{bb} - i\mathbf{b}\times\mathbf{I}\frac{\Omega_e}{\omega}\right), \tag{5.11}$$

where $\mathbf{I}$ is the unit dyadic.

Because the general expression for $\mathbf{Q}_i$ involves many components, it is convenient to determine from Maxwell's equations those that are necessary to describe the modes considered. In the limit of β small compared to m_e/m_i, ion-cyclotron modes exist for which $\nabla\times\mathbf{E}$ is negligible. However, β as large as even this small value introduces certain electromagnetic effects (Callen and Guest, 1971, 1973). To determine these, introduce scalar and vector potentials, $\mathbf{E}=-i[\mathbf{k}\phi-(\omega/c)\mathbf{A}]$, giving in the Coulomb gauge the four equations

$$(k^2c^2-\omega^2)\mathbf{A}=\frac{c}{\omega}\Sigma\mathbf{Q}\cdot\left(\mathbf{k}\phi-\frac{\omega}{c}\mathbf{A}\right)-\omega c\mathbf{k}\phi, \quad (5.12)$$

$$\mathbf{k}\cdot\mathbf{A}=0, \quad (5.13)$$

where ω^2 may be neglected compared to k^2c^2 and Σ refers to summation over species. The equation for ϕ follows from the divergence of Eq. (5.12)

$$\left[k^2-\frac{1}{\omega^2}\mathbf{k}\cdot\Sigma\mathbf{Q}\cdot\mathbf{k}\right]\phi=-\frac{1}{\omega c}\ \mathbf{k}\cdot\Sigma\mathbf{Q}\cdot\mathbf{A}\ . \quad (5.14)$$

We first consider modes which are strictly longitudinal in the limit $c\to\infty$ and eliminate $\mathbf{A}$ from Eqs. (5.12) through (5.14). An examination of Eqs. (5.10) and (5.11) reveals that $|\mathbf{b}\cdot\Sigma\mathbf{Q}\cdot\mathbf{b}|$ is of order ω_{pe}^2, and so is dominated by the electrons; whereas for $\omega\sim\Omega_i$ and $\omega_{pe}^2\sim\Omega_e^2$, all perpendicular components are of order ω_{pi}^2, so that both ions and electrons can contribute. The scalar product of Eq. (5.12) with $\mathbf{b}$ gives

$$\frac{\omega}{c}\mathbf{b}\cdot\mathbf{A}=\frac{\mathbf{b}\cdot\mathbf{Q}_e\cdot\mathbf{b}}{[k^2c^2+\mathbf{b}\cdot\mathbf{Q}_e\cdot\mathbf{b}]}k_\parallel\phi$$

$$=\frac{\omega_{pe}^2}{k^2c^2+\omega_{pe}^2}k_\parallel\phi \quad \text{when } \omega\gg k_\parallel\bar{v}_e\,. \quad (5.15)$$

Because

$$\frac{\omega_{pe}^2}{k_\perp^2c^2}=\frac{m_i}{m_e}\frac{\beta}{k^2a_i^2} \quad (5.16)$$

and $k_\perp a_i\gtrsim 1$, the measure of β is m_e/m_i, at which value the inductive electric field parallel to $\mathbf{b}$ can balance that component of the longitudinal field. From Eq. (5.13), $\mathbf{k}_\perp\cdot\mathbf{A}=-k_\parallel A$, and so $(\omega/c)\mathbf{k}_\perp\cdot\mathbf{A}$ may be neglected compared to $k_\perp^2\phi$ in computing $\mathbf{k}_\perp\cdot\mathbf{E}$ as long as $k_\parallel^2\ll k_\perp^2$. When $\omega_{pi}^2/k_\perp^2c^2<1$, the $\mathbf{e}_2=\hat{\mathbf{b}}\times\hat{\mathbf{k}}$ component of Eq. (5.12) gives

$$\frac{\omega A_2}{c}=\frac{1}{k_\perp^2c^2}\mathbf{e}_2\cdot\Sigma\mathbf{Q}\cdot\mathbf{e}_1k_\perp\phi. \quad (5.17)$$

The scalar product of Eq. (5.12) with $\mathbf{k}_T$, using Eq. (5.15) and (5.17), gives equation for ϕ alone

$$\left(\omega^2-\frac{\mathbf{b}\cdot\mathbf{Q}_e\cdot\mathbf{b}k_\parallel^2c^2}{k_\perp^2c^2+\mathbf{b}\cdot\mathbf{Q}_e\cdot\mathbf{b}}-\mathbf{e}_1\cdot\Sigma\mathbf{Q}\cdot\mathbf{e}_1+\frac{\mathbf{e}_1\cdot\Sigma\mathbf{Q}\cdot\mathbf{e}_2\mathbf{e}_2\cdot\Sigma\mathbf{Q}\cdot\mathbf{e}_1}{k^2c^2}\right)\phi=0, \quad (5.18)$$

where $\hat{\mathbf{e}}_1=\hat{\mathbf{k}}_\perp$. For the $k_\parallel^2$ term not to dominate this equation, $k_\parallel^2$ must satisfy

$$\frac{k_\parallel^2}{k_\perp^2}=0\left(\frac{\omega_{pi}^2}{k_\perp^2c^2}\frac{k_\perp^2c^2+\omega_{pe}^2}{\omega_{pe}^2}\right)$$

$$\le 0\left(\frac{\beta}{k_\perp^2a_i^2}\right),$$

which is the basis of neglecting $k_\parallel^2$ compared to $k_\perp^2$ used above. This same condition may be used to neglect $k_\parallel$ in the computation of $\mathbf{Q}_i$; however, such an approximation neglects the possibility of anisotropy-driven modes occurring at higher β as the allowed $k_\parallel$ increases. Although the ion contribution to $\mathbf{k}\times\mathbf{b}\cdot\mathbf{A}$ in Eq. (5.17) is formally of the same order as that of the electrons and actually cancels it when $k_\perp a_i\ll 1$ and $\omega\ll\Omega_i$, it tends to be small when $\omega\sim\Omega_i$ and $k_\perp a_i\gtrsim 1$, and so it is usually neglected. It can be shown that Eq. (5.18) remains valid for local $\beta\approx 1$ and $\omega\approx\Omega_i$ except for the addition of ∇B drifts in the denominators of $\mathbf{Q}_i$ which act to broaden the $\omega=n\Omega_i$ resonances (Dominguez and Berk, 1976) obtained below.

In carrying out the orbit integrals in Eq. (5.9), it is convenient to Fourier analyze the sinusoidal exponents by means of the Bessel identity

$$e^{iz\sin\phi}=\sum_{n=-\infty}^{+\infty}J_n(z)e^{in\phi},$$

after which the angular and τ integrations are trivial.

The final dispersion relation for quasi-longitudinal ion-cyclotron frequency waves in plasmas with $\beta>m_e/m_i$ is

$$\frac{k_\parallel^2c^2}{(\omega_{pe}^2+k_\perp^2c^2)}=\frac{\omega^2}{\omega_{pe}^2}\left\{1+\frac{\omega_{pe}^2}{\Omega_e^2}+\frac{\omega_{pe}^4}{\Omega_e^2k_\perp^2c^2}+\frac{\mathbf{k}\cdot\mathbf{b}\times\nabla\omega_{pe}^2}{\omega\Omega_e^2k_\perp^2}+\frac{1}{k_\perp^2}\sum_i\frac{4\pi q_i^2}{m_i}\left[-\int d^3v\frac{\partial f_{0i}}{w\,\partial w}+\sum_i\left(\frac{\omega}{w}\frac{\partial f_{0i}}{\partial w}+\frac{\mathbf{k}\cdot\mathbf{b}\times\nabla f_{0i}}{\Omega_i}\right)\sum_{n=-\infty}^{\infty}\frac{J_n^2(k_\perp w/\Omega_i)}{\omega-n\Omega_i}\right]\right\}$$

$$\equiv D(\omega,k_\perp,s). \quad (5.19)$$

When $\omega\sim k_\parallel\bar{v}_e$, so that electron Landau damping is significant, ω_{pe}^2 on the left side is to be replaced by

$$-\frac{4\pi e^2}{m_e}\omega\int d^3v\frac{u\partial f_{0e}/\partial u}{\omega-k_\parallel u}\,.$$

When the parallel wavelength is of the order of the axial scale length, so that the eikonal parallel to B is not justified, Eq. (5.19) is to be replaced by a differential equation along $\mathbf{b}$

$$\frac{B}{\omega_{pe}^2}\frac{d}{ds}\left[\frac{1}{B}\left(\frac{\omega_{pe}^2c^2k_\perp^2}{(\omega_{pe}^2+c^2k_\perp^2)}-\omega^2\right)\frac{d\phi}{ds}\right]+k_\perp^2D(\omega,\mathbf{k}_\perp,s)\phi=0, \quad (5.20)$$

where D is the right side of Eq. (5.19). When ω is of the order of the electron axial bounce frequency, neither of these local descriptions is sufficient. The left side of Eq. (5.19) must be replaced by an orbit integral over the bounce motion, converting that equation to an integral equation. The question of the approximation of this integral operator by the differential operator in Eq. (5.20) has been discussed by Berk and Pearlstein (1971). Under low-density conditions, $\omega_{pi}\sim\Omega_i$, the ion cyclotron motion can actually couple to the electron-bounce motion. Sharp *et al.* (1976) have suggested such a mechanism to explain the observed beam-driven instability in Baseball II (Anderson *et al.*, 1975; see also Beasley *et al.*, 1974). In the local eikonal forms, Eq. (5.19), the differential equation form, Eq. (5.20), or the integral equation form, $\mathbf{k}_\perp$ is a function of position as given by Eq. (5.2) with its two components at one point, say the midplane, as independent parameters.

To Eq. (5.20) must be added boundary conditions at the ends of the machine. These will be strongly influenced by low density plasma lying beyond the mirror throat; for as long as $\omega_{pe}(s) > \omega \sim \Omega_i(0)$, Eq. (5.20) continues to yield propagating waves. Because in the density range of interest $\Omega_i(0)/\omega_{pe}(0) = 0(m_e^2/m_i^2)$, an external density of 10^{-6} times the central density is sufficient to allow waves to propagate out to a region where they finally become absorbed by electron Landau damping, i.e., where locally $\omega \approx \bar{v}_e k_\parallel$. An allowed boundary condition on Eq. (5.20) is that of outgoing waves at the mirror throat, with the assumed lack of reflection accounting for the Landau damping. For the longest parallel wavelength mode inside the mirrors, an approximate eigenvalue condition may be obtained by noting that, formally speaking, D is a small number, so that $d\phi/ds$ is small. Writing $\phi(s) = \phi_0 + \phi_1(s)$, with ϕ_0 constant, the equation for ϕ_1 becomes approximately

$$\frac{B}{\omega_{pe}^2}\frac{d}{ds}\left[\frac{1}{B}\frac{\omega_{pe}^2 c^2 k_\perp^2}{(\omega_{pe}^2 + c^2k_\perp^2)}\frac{d\phi_1}{ds}\right] + k_\perp^2 D\phi_0 = 0.$$

Integrating and assuming an eigenfunction even about $s=0$, this becomes

$$\phi_0\int_0^{s_m}\frac{ds}{B(s)}\omega_{pe}^2 k_\perp^2 D = -\left.\frac{\omega_{pe}^2}{B}\frac{d\phi_1}{ds}\right|_{s_m}.$$

Provided the density is low at the mirror throat s_m, the right-hand side may be neglected, and there results a condition for ω which is the appropriate line average of the local dispersion relation

$$\int_0^{s_m}\frac{ds}{B(s)}\omega_{pe}^2 k_\perp^2 D(\omega, k_\perp, s) = 0. \qquad (5.21)$$

This flute approximation to the eigenvalue condition has proved very satisfactory when its results are compared to solutions of the full differential equation Eq. (5.20) (Baldwin *et al.*, 1971; Pearlstein, 1975). It has the conceptual advantage that the eigenvalue condition is cast in a form very similar to that which would obtain with $k_\parallel = 0$ in an infinite medium.

The question of the assumed ion distribution function is important. Differing model distributions possessing empty loss cones all predict instability but yield quantitatively different value for growth rates, stability boundaries, etc. Post (1967) pointed out that partial filling of the loss cone has a strong stabilizing effect, and this has been incorporated into several nonlinear descriptions of these instabilities (see below). To establish the types of instabilities which might be anticipated, we will first describe the situation which prevails when the loss cone is taken as completely empty. Because $k_\parallel$ is neglected in the ion current response, only the moment $\int_{-\infty}^{\infty} du f_i$ enters the dispersion relation, and thus only a function of w need be modeled. Two popular examples which allow analytic integrals of the form $\int_0^\infty \partial f_i/\partial w^2 J_n^2(k_\perp w/\Omega_i)dw^2$ are

$$f_i \propto w^2\exp(-\alpha w^2), \qquad (5.22)$$

and

$$f_i \propto \exp(-\alpha w^2) - \exp(-\beta w^2),$$

with $\beta > \alpha$. Both of these distributions have the hole at low energy characteristic of loss-cone distributions, but do not have the finite ambipolar cutoff at $\Phi/(R_m - 1)$ resulting from the complete lack of trapping of low-energy ions. No model distributions having such a cutoff allow analytic evaluation of the Bessel integrals.

The presence of the magnetic field introduces, for given $\mathbf{k}_\perp$, an infinite spectrum of roots of Eq. (5.19) between multiples of the ion-cyclotron frequency. For effective coupling, the Bessel functions must be near their peaks, so that oscillation near $n\Omega_i$ requires $k_\perp a_i \gtrsim n$. The ion loss cone provides the drive for most of these instabilities by causing some or many of these harmonic waves to become negative energy (Sturrock, 1960; Kadomtsev *et al.*, 1964; Hall and Heckrotte, 1966; Bers and Gruber, 1965). These negative-energy ion waves can be driven unstable either by coupling to positive energy waves or by electron dissipation. In practice, the search for unstable modes entails finding conditions where one or the other of these mechanisms is at work.

It is instructive to first consider plots of $D(\omega, \mathbf{k}_\perp)$ versus ω as shown in Fig. 4 for a stable, uniform Maxwellian. Because of the variation of parameters and particularly of Ω_i in a realistic geometry, such constructions are strictly inappropriate. However, for many purposes they remain useful and qualitatively correct when Ω_i is taken as the midplane value where $d\Omega_i/ds = 0$. As might be anticipated by the flute average form of Eq. (5.21), this value plays much the same role as the constant value in a uniform field.

A root of Eq. (5.19) lies where each curve crosses the line $k_\parallel^2 c^2/(\omega_{pe}^2 + k_\perp^2 c^2)$. The positive slope of the curves at these intercepts implies that these modes are positive energy. If f_{0i} is nonmonotonic in w, certain of the coefficients $\int d^3v\, J_n^2 \partial f_{0i}/w\partial w$ can be positive for ranges of $k_\perp$, and the associated mode then becomes negative energy. For f_{0i} sharply peaked at nonzero energy, in uniform plasmas there can occur isolated sign changes of these coefficients for narrow bands of $k_\perp$. The resulting coupling of adjacent positive and negative energy modes leads to the Dory-Guest-Harris (1965) instability. However, the variation of $|\mathbf{k}_\perp|$ induced by the fanning in a realistic machine [see Eq. (5.2)] smears the arguments of the Bessel function over sufficient range to completely stabilize all such modes that are resonant in $\mathbf{k}_\perp$ when a line average such as that appearing in Eq. (5.21) is performed (Baldwin, 1974).

Generally, a band of coefficients will change sign, leading to a set of stable negative-energy waves (see

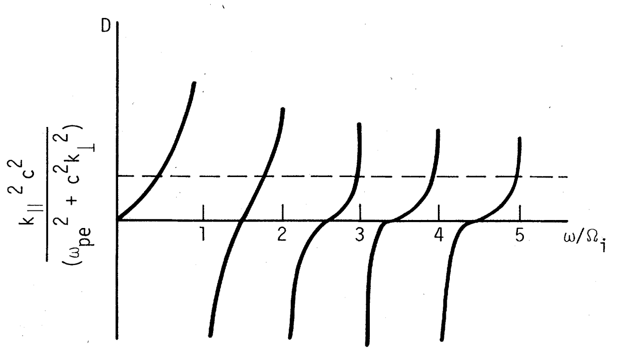

FIG. 4. Frequency dependence of $D(\omega, k_\perp)$ for a Maxwellian distribution; intercept with dashed line determines root.

Fig. 5) which can be destabilized by electron dissipation. In particular, these have been shown (Berk *et al.*, 1969; Berk *et al.*, 1972) to be destabilized in a realistic geometry by plasma waves outgoing from the low-density ends of the machine, with the transport of energy away from the hot plasma representing the dissipation process. The calculation begins with the differential form Eq. (5.20); that equation is integrated to the ends of the machine where the density is low. The calculation is made complicated by the series of singular points where $\omega = m\Omega_i$, but in principle it is similar to quantum-mechanical calculations of metastable states wherein real eigenvalues are driven complex by being coupled to outgoing waves. However, the negative-energy character of the originally real-frequency eigenmodes gives a positive imaginary part to the complex eigenvalues. The intercepts of the negative-energy curves in Fig. 5 with the axis are the $k_\parallel = 0$ roots, denoting zero axial group velocity or, equivalently, WKB turning points. Berk *et al.* find standing waves between these turning points. The waves are first destabilized with decreasing magnetic axial scale length L_m, but then all but the lowest mode are stabilized by an axial scale length of the magnetic field

$$L_m/a_i \approx (m_i/m_e)^{1/2}, \tag{5.23}$$

where L_m is defined as the length for magnetic intensity doubling along B.

The lowest mode, which is flute in character and occurs for $\omega < \Omega_i$, requires a mechanism for stabilization that is not included in Eq. (5.20). It will be appreciated that when the finite-β term

$$\omega_{pe}^4/\Omega_i^2 k^2 c^2 = 2\beta_{1\mathrm{oc}} m_i \omega_{pe}^2 / m_e \Omega_e^2 k_\perp^2 a_i^2,$$

is large, this mode is driven to resonance; i.e., goes to $\Omega_i(0)$. For roots for which $\delta\omega = |\omega - \Omega_i(0)|$ is small, effects previously neglected in obtaining Eq. (5.19) become significant. In particular, Berk *et al.* (1972) show than when the axial scale is sufficiently short, the transit time of an ion through the resonance region limits the strength of the ion–wave interaction, and stability is restored provided $\delta\omega \lesssim \Omega_i (a_i/L_m)^{2/3}$ (Baldwin *et al.*, 1971). Note that, although finite axial scale length stabilizes this mode, finite β is required to drive it to resonance so that the ion transit time becomes significant.

The growth rates of these negative-energy modes are predicted to be small, with γ of order $0.02\Omega_i(0)$ for the flute mode, and 2 to 4 times as large for the most unstable higher modes. 2XII and 2XIIB were designed to roughly satisfy Eq. (5.23) so that these higher modes would not be expected; and the lowest, unstabilized mode has not been identified. Whether this mode has been masked by the more violent instabilities described below or is stabilized by a weak effect not included in the theory is not clear. The stabilizing mechanism for the drift-cone mode described in Sec. V.B also stabilizes the flutelike negative energy mode.

As finite plasma radius is introduced by the $\nabla\omega_{pe}^2$ in Eq. (5.19), curves such as b in Fig. 6 develop due to the additional term which is linear in ω and negative for the proper sign of $k_\perp$. For a sufficiently large such term, a positive root of $k_\parallel = 0$ disappears with the formation of curves of the type c, leading to the instability

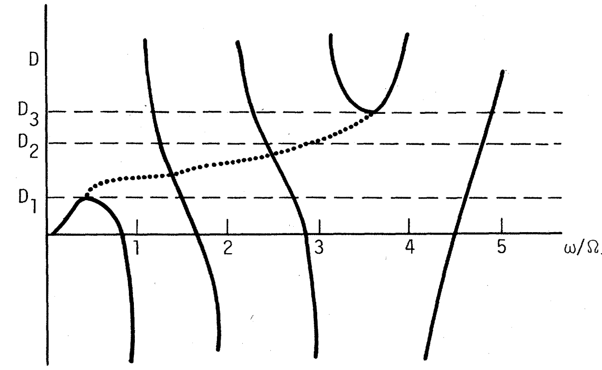

FIG. 5. Frequency dependence of $D(\omega, k)$ for typical loss-cone distribution showing negative energy structure; intercepts between D_1 and D_3 have complex roots, with the dotted curve indicating the real part.

known as the drift-cyclotron loss-cone mode (DCLC) (Post and Rosenbluth, 1966). This mode, originating due to a coupling of the positive-energy electron drift wave in the direction of the ion diamagnetic drift and the negative-energy ion-cyclotron wave, has formed the basis of considerable theoretical investigation the last few years (Baldwin *et al.*, 1971; Tang *et al.*, 1972; Lindgren *et al.*, 1976). Although theory predicted it would occur in all present-day experiments with broad-band frequency and wave-number spectra, it either did not occur, or it did so with properties quite different from those expected from linear theory. The final resolution has apparently required a description of its nonlinear properties as discussed in Sec. V.B.

The DCLC mode is also stabilized by finite β in a manner similar to the flute-negative energy mode described above, except that in this case the magnitude of the $\omega_{pe}^4/\Omega_e^2 k_\perp^2 c^2$ term in Eq. (5.19) for sufficiently large plasma radius R_p/a_i causes the dispersion curve of Fig. 6 to revert character from type c to type b. Specifically,

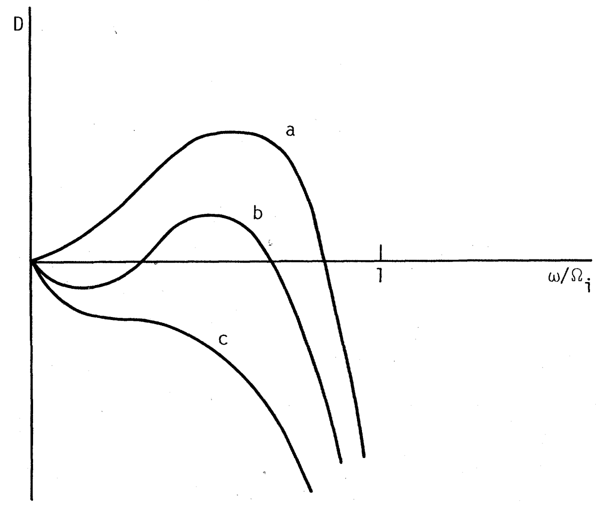

FIG. 6. Detail of low-frequency dependence of $D(\omega, k)$ for loss-cone distribution with negative energy structure and various radial density gradients; passage from "b" to "c" produces two complex roots.

Tang *et al.* (1972) find that for $\beta > (m_e/m_i + \Omega_i^2/\omega_{pi}^2)^{1/3}$, the DCLC mode is stable for $R_p/a_i > \frac{1}{2}\beta^{-1/2}(m_e/m_i + \Omega_i^2/\omega_{pi}^2)^{-1/2}$. (Throughout, R_p is to be interpreted as a local radial density scale length, and not as an actual plasma radius.)

Both the negative-energy mode and the drift-cone mode can be stabilized by the addition of warm plasma. Direct axial injection of warm plasma as a means of stabilization has been proposed by Post (1967) and demonstrated by Baiborodov *et al.* (1973), Ioffe *et al.* (1975), and Coensgen *et al.* (1975). The addition of a warm, untrapped plasma component to a hot, trapped plasma discussed above has two stabilizing effects. The cyclotron waves can be driven to positive energy by the coefficients

$$\int d^3v\, J_n^2\left(\frac{k_\perp w}{\Omega}\right)\frac{1}{w}\frac{\partial f_0}{\partial w}$$

becoming negative. In addition, the nonvanishing of $\int d^3v\, \partial f_{0i}/\partial w$ gives to the right-hand side of Eq. (5.20) a positive term that adds to and plays the same role as the finite-β term described above. Because hot (warm) density and temperature of n_H, T_H (n_w, T_w), the quantity thus added to β is $n_w T_H/n_H T_w$, warm plasma densities of only a few percent can be significant. However, too low a ratio T_w/T_H gives rise to another class of instability due to the double-humped nature of the resulting distribution (Pearlstein, Rosenbluth, and Chang, 1966) when

$$n_w/n_H \gtrsim (T_w/T_H)^{3/2}.$$

The limit additive to β is therefore about $(T_w/T_H)^{1/2}$. The summary of the R_p/a_i versus Ω_i^2/ω_{pi}^2 at 50-keV ion energy for various n_w/n_H and optimized T_w/T_H and the loss cone modeled by Eq. (5.22) are shown in Fig. 7 (Baldwin *et al.*, 1975, 1976). This generalizes the result of Tang *et al.* (1972) to nonzero n_w/n_H and includes the effect of axial nonuniformity. The reduced R_p/a_i at low Ω_i^2/ω_{pi}^2 is the finite-β stabilization effect described above (Tang *et al.*, 1972). The effect of small amounts of warm plasma at these high densities is pronounced. The reason for this sensitivity at large R_p/a_i is that, because unstable waves have phase velocities small compared to velocities typical of E_h, the optimum temperature T_w is likewise small compared to E_h.

The third mode described by Eq. (5.21) which is thought to be relevant to realistic-geometry mirror machines is the high-frequency, convective-loss-cone mode (HFCLC) (Rosenbluth and Post, 1965, 1966). In terms of the dispersion diagram, Fig. 5, this mode corresponds to the continuum of complex roots lying between D_1 and D_3. It requires $k_\parallel^2 \neq 0$ and does not require a radial gradient. Because the growth rates are found to be large compared to Ω_{ci}, and $k_\perp a_i \gg 1$, it is possible

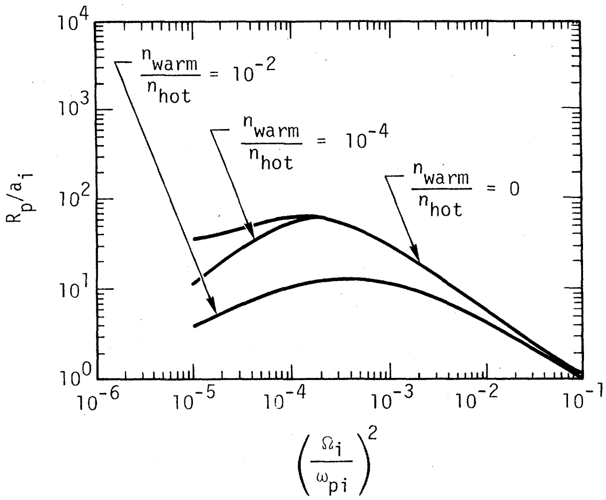

FIG. 7. Marginally stable radial scale lengths of drift-cyclotron loss-cone mode with addition of warm plasma; parameters are defined in the text.

to describe the mode by a reduced description for the ions obtained by neglecting the finite-β terms and the magnetic field in describing the ion motion. Since the mode is convective along the magnetic field, it is convenient to solve for $k_\parallel$

$$k_\parallel^2 = k_\perp^2 \frac{\omega^2}{\omega_{pe}^2}\left(1 + \frac{\omega_{pe}^2}{\Omega_e^2} + \frac{\omega_{pi}^2}{k_\perp^2}\int d^3v \frac{k_\perp \cdot \partial f_{0i}/\partial v}{\omega - k_\perp \cdot v}\right). \tag{5.24}$$

The real part of $k_\parallel$ is given principally by the electrons; however, the ions contribute when $k_\perp^2 \lesssim k_{Di}^2$. Twice the imaginary part of $k_\parallel$ gives the amplification rate of the fluctuation energy along the magnetic field. When $\omega/\mathrm{Re}k_\parallel \gtrsim 3\bar{v}_e$, the neglect of electron Landau damping implicit in Eq. (5.24) is justified, and

$$2\,\mathrm{Im}\,k_\parallel = \frac{1}{a_i}\frac{\omega_{pe}/\Omega_e}{(1+\omega_{pe}^2/\Omega_e^2)^{1/2}}\alpha\,, \tag{5.25}$$

where α is a constant dependent on the ion-distribution function, typically taking value $0.1(R_m+1)^{-1/2}$ for reasonably well-spread distributions. Horton (1967) has shown that finite electron temperature enters principally through dispersion rather than damping. Ratios of $T_e/T_i \approx 0.1$ reduce α by about 40%. The coefficient of α in $\mathrm{Im}\,k_\parallel$ is evaluated at midplane values, including the field depression effects of finite β.

A mirror machine of sufficient length to stabilize the negative-energy modes, given by Eq. (5.22), may still be subject to the high-frequency fluctuations of the convective mode that have been amplified several e foldings. The scattering rate to be expected from this instability has been calculated for a slab plasma (Baldwin and Callen, 1972; Baldwin, 1975)

$$\left(\frac{\partial f_i}{\partial t}\right)_{\substack{\text{convective}\\ \text{mode}}} = \frac{16\pi^3 q_i}{m_i^2}\frac{\partial}{\partial \mathbf{v}}\cdot\sum_j n_j q_j^2\int\frac{d^2k_\perp}{(2\pi)^2}\frac{\mathbf{k}_\perp\mathbf{k}_\perp}{k_\perp^4}\int d^3v'\int dz'\int d\omega\,\delta(\omega-\mathbf{k}_\perp\cdot\mathbf{v})\,f_j(\mathbf{v}',z')\,\delta(\omega-\mathbf{k}_\perp\cdot\mathbf{v}')$$
$$\times\frac{\exp\left[-2\,\mathrm{sgn}(z-z')\int_{z'}^{z}\mathrm{Im}\,k_\parallel\,dz''\right]}{\partial\epsilon/\partial k_\parallel|_z\,\partial\epsilon/\partial k_\parallel|_{z'}}\cdot\frac{\partial f_i}{\partial \mathbf{v}}\,, \tag{5.26}$$

from which one obtains the rough estimate

$$\frac{1}{f_i}\left(\frac{\partial f_i}{\partial t}\right)_{\substack{\text{convective}\\ \text{mode}}} \approx \frac{0.1}{\tau_{\text{drag}}}\frac{\exp(-2\int \operatorname{Im} k_{\parallel}\, dz)}{(1+\omega_{pe}^2/\Omega_e^2)^2}, \tag{5.27}$$

where the integral is over the length of the plasma and τ_{drag} is the classical electron drag time given by Eq. (4.9). For this scattering not to dominate the drag, ω_{pe}^2/Ω_e^2 must be in excess of 3 or 4 and the exponent limited by about 5. In a reactor, higher values of ω_{pe}^2/Ω_e^2 are precluded by the total pressure which can be held by the magnetic field, although for the purpose of Eq. (5.27), the β-reduced midplane value may be used. The resulting limit on the axial scale length obtained from Eqs. (5.25) and (5.27) is about 50 ion Larmor radii. A more detailed study of the effects of this mode requires the incorporation of a scattering term similar to Eq. (5.26) into the Fokker–Planck equation described in Sec. II. A beginning in this direction has been made by Fader (1975), but further work is required because his formulation did not provide conservation of energy between particles and waves. Low-energy (~1 keV) nonclassical plasma lifetimes in 2XII displayed a density dependence of their decay rates that was consistent with Eq. (5.27) (Coensgen *et al.*, 1972); however, its presence has never been directly verified.

The electromagnetic modes thought to be a potential problem in mirror machines form a class of Alfvén-related modes that are driven unstable either by resonant particles or by the ion isotropy in the fluid limit $\omega/k_{\parallel} \gg \bar{v}_i$. In both cases, these modes are most easily described with $|\mathbf{k}_{\perp}|=0$ and with $E_{\parallel}=\phi=0$, thus avoiding the electron parallel conductivity. Only the $\hat{\mathbf{k}}\times\hat{\mathbf{b}}$ component of $\mathbf{A}$ is nonvanishing and the general electromagnetic wave equation [Eq. (5.12)] immediately yields the dispersion relation

$$0=\omega^2-k_{\parallel}^2c^2-\omega_{pe}^2\frac{\omega}{\Omega_e}-\tfrac{1}{2}\omega_{pi}^2\sum_{+,-}\int d^3v\,\frac{k_{\parallel}w^2\partial f_{0i}/\partial u \pm 2\Omega_i f_{0i}}{\omega-k_{\parallel}u_{\parallel}\pm\Omega_i}. \tag{5.28}$$

The anisotropy can be modeled using bi-Maxwellian distributions with differing $T_{\perp}$ and $T_{\parallel}$. For small anisotropy, $T_{\perp}-T_{\parallel}$, instabilities are weakly growing, due to a small number of resonant particles. Two sets of authors (Rosenbluth, 1960; Sagdeev and Shafranov, 1960) find growth rates

$$\gamma \sim \Omega_i \exp -\left[\beta^{-1}\left(\frac{T_{\perp}}{T_{\parallel}}-1\right)^{-2}\right], \tag{5.29}$$

whereas Cordey and Hastie (1972) consider model collisional distributions to find

$$\gamma\sim\Omega_i \exp[-\beta^{-1}\log\beta R] \tag{5.30}$$

for $\beta R>1$, where $\beta\equiv 8\pi n T_i/B^2$. When the anisotropy is severe, Davidson and Ogden (1975) find a mode for which the growth was algebraic in β. In the extreme of $T_{\parallel}=0$, it becomes

$$\gamma\sim\Omega_i(\beta_{\perp}/2)^{1/2}. \tag{5.31}$$

Because of the nonzero value of $k_{\parallel}$, for weak anisotropy these modes are convective along the magnetic field, and so they lead to a length restriction that is weak, at least in the case of the resonant modes. The nonresonant mode of Davidson and Ogden can be convective or absolute, depending on the degree of anisotropy (Pearlstein and Watson, 1975), and its effect on mirror-contained plasmas is still under study. At time of writing this mode appears to place a restriction on the limiting β for a mirror machine, although little is known of its nonlinear properties. It has not been observed in 2XIIB for β_{vac} up to about 1.5 (Logan *et al.*, 1976); however, in that experiment the theory may not be applicable because of the small number of Larmor radii in both radius and axial scale lengths.

B. Nonlinear theory

It is a common experimental observation that, though mirror machines are susceptible to microinstability driven by the inverted ion distribution, the saturation level of fluctuations is considerably less than might be expected. Typically, a confined plasma with ions in the kilovolt range exhibits fluctuations in the range of a few tens of volts. Although these fluctuations are sufficient to greatly decrease particle lifetimes to a hundred or so bounce times, the fact that they saturate at such low levels implies that the nonlinear effect on the ions must be effective at low amplitude. This reduces the number of possible nonlinear mechanisms which might be considered as limits to the unstable growth. Because calculated growth times are short compared to almost any time of observation, such a plasma must always be observed at a nonlinear steady state in which the governing instability has saturated.

The most frontal theoretical attack on the nonlinear aspects of microinstability is that of particle simulation. In numerical codes of this type, the equations of motion of a large number of particles are solved in the presence of their self-consistent fields. Because of the large number of particles required to minimize collisional effects, concessions must be made on the dimensionality of the model. Early codes followed motion in only one dimension perpendicular to $\mathbf{B}$ with no variation along $\mathbf{B}$ and treated initial-value problems of highly peaked distributions. Roughly speaking, linear growth was seen until the wave energy reached nearly the ion kinetic energy; at this time the wave growth ceased and the ions were seen to scatter violently in phase space (Byers and Grewal, 1970; Byers *et al.*, 1971). Similar results have been seen when motion in two dimensions perpendicular to $\mathbf{B}$ is allowed (Birdsall *et al.*, 1974). Clearly, such simulations omit some physics of the experiments which is vital to the saturation of the wave growth.

In analytic approaches, as in most nonlinear plasma physics theory, the subject divides into two parts: in the first, a single (or predominant) wave is assumed; in the second, the fluctuations are assumed to exist in the form of a spectrum of uncorrelated waves. The results of both models will probably be useful ultimately, because in many experimental situations only a narrow spectrum is observed when a broad spectrum would be expected on the basis of linear theory. Presumably, an originally broad spectrum narrows as the wave saturates, and this latter case is the marginally stable one.

Conceptually, by far the simplest piece of the problem is to study the influence on the particles by a given wave or spectrum, and only later to make the fields self-consistent with the distribution. Because of the influence of the containing mirror field, even the linear aspect of this problem has its complications (Timofeev, 1973). Eqs. (2.1) to (2.3) are amended to include a fluctuating electric field (here we will neglect any fluctuating magnetic field). If the induced changes in E, μ, ϕ are small, these equations may be solved by perturbation, substituting on the right-hand side only equilibrium quantities. When the field is composed of one or more real frequencies, beating occurs unless $\Omega=\omega$, in which case resonant transfer of energy develops. A particle interacting with such a wave while moving along a nonuniform magnetic field is affected irreversibly only locally while in resonance, and this interaction is strongest if it occurs at a point where $\partial\Omega/\partial s=0$. In the absence of such resonance, a particle in the presence of a simple wave having no variation parallel to $\mathbf{B}$ possesses a new adiabatic invariant which explicitly depends on the wave amplitudes (Aamodt, 1971; Aamodt and Byers, 1972); there then would not be any systematic loss of particles. If resonance does occur, there are changes in E and μ, the sign of which varies with the angular phase of the particle before entering the region of resonance. The calculation and its results are similar to those for the $\Delta\mu$ introduced by the partial breakdown of adiabaticity in Sec. II. If upon successive passes through the resonant region this phase is random, the uncorrelated $\Delta\mu, \Delta E$ will describe a diffusion in phase space. Because the phases on successive passes through resonance are related by the axial bounce motion, changes induced by weak fields are not random, and periodic motion in $\Delta\mu$ may exist (Rosenbluth, 1972), leading to a superadiabaticity. The fluctuating potentials $\tilde{\phi}$ satisfying this condition have $\tilde{\phi}/T_i \lesssim (a_i/L_m)^{5/3}$, where L_m is the axial magnetic scale length. At very strong fields, $\tilde{\phi}/T_i \gg (a_i/L_m)^{2/3}$, particles become trapped in the troughs of the fluctuations, and again limited excursions in E, μ exist. These regimes of extended adiabaticity have been reviewed by Timofeev (1974). The circumstance described by Rosenbluth of good particle containment in the presence of a weak, coherent fluctuation has been observed in PR-7 (Gott *et al.*, 1974); whether this was a demonstration of superadiabaticity is not clear. Similarly, at a ratio $L_m/a_i \approx 22$, the 5- to 25-V fluctuations observed in a 13-keV ion plasma in 2XIIB (Coensgen *et al.*, 1975) lie within but near the limit of the range that superadiabatic effects would be significant.

A finite $k_\parallel$ wave in a magnetized plasma has a set of resonant velocities $u=(\omega-n\Omega_i)/k_\parallel$. Recently, Smith and Kaufman (1975) have shown that, provided the wave amplitude is sufficient for the trapping width $2|e\tilde{\phi}J_n(k_\perp a_i)/m_i|^{1/2}$ to exceed the separation in resonant velocities, it is possible for a particle to skip diffusively in u in a stochastic manner and so gain energy from the fluctuations. However, as with most nonlinear ion dynamic effects, this process requires fluctuations approaching the average ion energy.

In considering nonlinear mechanisms of species other than the hot ions, several authors have investigated mechanisms whereby electron motion might be responsible for saturating linear growth. Such a mechanism is particularly attractive for the drift-cone mode, because in this mode specific electron response is required to generate the instability. Baldwin *et al.* (1972) investigated the mechanism of nonlinear line tying (i.e., the influence of axial end currents to bounding walls), and (Baldwin *et al.*, 1974) the possibility that electron instabilities would induce enhanced electron crossfield transport. Both of these effects prove insufficient in a quantitative comparison with experiment. Aamodt (1975) investigated the effect of eddying in the electron response, specifically the contribution of the $\mathbf{v}\cdot\nabla\mathbf{v}$ term in the fluid response. This mechanism is found to limit the unstable wave amplitude through the establishment of convective cells. The theory generates a number of predictions which can be checked experimentally; but until they have been verified, it is not clear if or when the mechanism he describes is actually the limiting one.

Another electron nonlinear effect is the parametric decay of an ion cyclotron mode into lower frequency electron drift-waves obtained from Eq. (5.19) by neglecting all ion terms. An ion wave labeled by $(\omega_0, \mathbf{k}_0)$ can decay into two such waves labeled by $(\omega_1, \mathbf{k}_1)$ and $(\omega_2, \mathbf{k}_2)$ provided $\omega_0=\omega_1+\omega_2$ and $\mathbf{k}_0=\mathbf{k}_1+\mathbf{k}_2$ can be satisfied. Pastukhov (1975) showed that with certain constraints such conditions could be satisfied. Liu and Aamodt (1975) showed that, when the ion wave was negative energy as with DCLC and the decay products were positive energy, the process was explosive; i.e., all amplitudes grew in time (Byers *et al.*, 1971). Both sets of authors show that the process is greatly inhibited by azimuthal asymmetry. The mechanism is suggested as the cause of subharmonic fluctuation often observed in unstable mirror machines. Beyond this specific application, these investigations do raise the interesting question as to the existence of other parametric, and particularly explosive, instabilities in mirror machines which are driven by the negative energy character of the loss-cone distribution.

The evolution of a distribution under the action of a spectrum of uncorrelated waves has been studied in a few examples relevant to mirror machines. These formulations begin by expressing the fluctuation-induced rate of change of f_0 by a quasi-linear approximation

$$\left.\frac{\partial f_0}{\partial t}\right|_{\text{fluctuatios}} = \frac{q}{m}\left\langle \nabla\phi_1\cdot\frac{\partial f_1}{\partial \mathbf{v}}\right\rangle. \tag{5.32}$$

The average is over the uncorrelated modes; f_1 is given by the linear response to ϕ_1, Eq. (5.4), and ϕ_1 evolves according to the instantaneous form of f_0. For a justification for this approximation, see Sagdeev and Galeev (1969). Briefly, it requires that a state of marginal stability exist to which the operator can drive f_0 with fluctuations small enough to justify the linear treatment of f_1.

The determination of such a marginally stable state for modes that are driven by the ion loss cone requires a mechanism whereby the loss cone can be partially filled in. For, if the loss cone is allowed to exist as a void in phase space, then there always exists a drive for instability, and fluctuations will grow to a large level. The notion of an empty loss cone is an approximation based on the smallness of the ratio of the transit

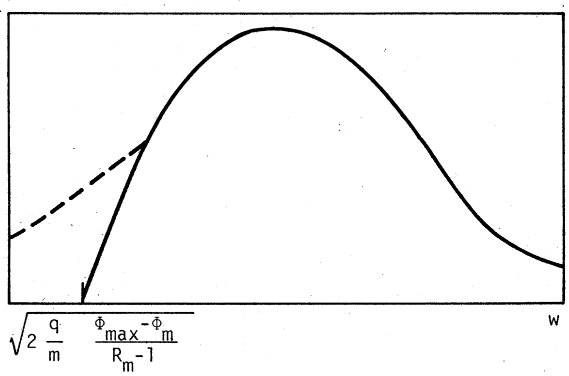

FIG. 8. Typical distribution $F(w)$ showing empty and partially filled loss cones.

time of an untrapped ion out of the machine to the time for its becoming untrapped. The reintroduction of a finite transit time into the theory allows for the existence in the machine of particles lying in the untrapped region of phase spaces, i.e., for the partial filling of the loss cone by particles whose lifetime is an axial transit time. For small $k_\parallel$ modes, the significant quantity for the instability is $F(w) = \int du\, f_{0i}$, which for an empty loss cone has the forms shown by the solid line in Fig. 8. Define $E_{\perp i}$ as the $w^2/2$ average of $F(w)$ and E_h as the equivalent energy parameter measuring the size of the loss-cone hole in the distribution. Then a partial filling of the loss cone, such as that resulting in the dotted curve in Fig. 8, gives rise to a density of untrapped ions

$$n_h = \alpha n_i (E_h / E_{\perp i}), \tag{5.33}$$

where n_i is the trapped ion density and $\alpha < 1$ is a measure of the degree to which the loss cone is filled in. For marginal stability of the DCLC mode, the magnitude of α as a function of R_p/a_i becomes small at large R_p/a_i, as may be inferred from Fig. 7. If the untrapped particles have a transit time (and thus a lifetime) τ_h, they represent a loss flux n_h/τ_h per unit volume which must be supplied by the trapped ion flux to maintain steady state. Thus, whatever the mechanism of the fluctuation level which is saturated by this partial filling of the loss cone, the trapped ion lifetime τ_i must be

$$\tau_i = \tau_h (E_{\perp i} / \alpha E_h). \tag{5.34}$$

Galeev (1966) has considered the particular examples of losses due to the Post–Rosenbluth convective mode for well spread distributions of mirror ratio of order 1.5 to 2, for which E_h is approximately $E_{\perp i}$ and τ_h is the transit time of a thermal ion. His resulting calculated lifetime was very short, as may be seen by the above scaling law, of the order a few ion transit times. Chu *et al.* (1969) apply similar considerations to the DCLC mode with similar results.

Recently, Baldwin *et al.* (1976b) have applied this idea to the saturation of the DCLC mode for distributions that are highly peaked in pitch angle, such as those created by neutral injection normal to B in 2XIIB (Coensgen *et al.*, 1975). Because E_h is then of the order of the drop in ambipolar potential from midplane to mirror, $3T_e$, and $T_e \approx 0.02 E_{\perp i}$ in this machine, and because the untrapped ions escape at only the ion sound speed, the lifetime they obtain is of order 100 ion bounce times. They extend this theory to offer an explanation for the observed stabilization by injection of cold plasma of what is thought to be the drift-cone mode in 2XIIB (see Coengsen *et al.*). Since there is then a source of particles in the loss cone, the trapped plasma lifetime is no longer restricted by the *particle flux* requirement, Eq. (5.34); rather it is fixed by the requirement that the *power flux* lost by the untrapped plasma be supplied by the trapped component. This leads to the lifetime estimate

$$\tau_i \sim \tau_h (E_{\perp i}^2 / \alpha E_h^2). \tag{5.35}$$

The price paid for the introduction of the stream to fill the loss cone is a reduction of the electron temperature so that, in fact, the energy lifetime becomes dominated by the drag time τ_d, given by Eq. (4.9).

Based upon these ideas, a quasi-linear code has been developed which takes advantage of the special properties of the 2XIIB plasma (Berk and Steward, 1975; Baldwin *et al.*, 1976a). In that machine, ions are injected at 90° to the magnetic field, and the resultant flute fluctuations ($E_\parallel = 0$) and electron drag act to preserve the highly peaked pitch angle distribution. The original calculation was reduced to one velocity dimension by assuming scattering and drag only in w, the component of velocity normal to B. The same calculation extended to two velocity dimensions has confirmed the validity of the original model. In these calculations, injected ions migrate in velocity space under the action of diffusion due to the fluctuating fields and electron drag. The rapid loss of nominally untrapped plasma is allowed for by ascribing to such particles a loss rate given by the inverse axial transit time of an ion at the energy of the ambipolar potential. An important quantity is the electron temperature, both because it fixes the drag rate and because it fixes the ambipolar potential defining the energy below which there is a high probability of loss. In the model, T_e is taken as determined by Eq. (3.1); i.e., the electrons are heated by the hot ions and cooled by the ion flux dominated by n_h/τ_h. The equation governing the distribution $F(w,t) = \int du\, f_{0i}(u, w, t)$ used is

$$\frac{\partial F}{\partial t} = \frac{1}{w}\frac{\partial}{\partial w} w \left[\frac{wF}{2\tau_d} + D(w,t)\frac{\partial F}{\partial w}\right] - \nu_{\text{loss}}(w) F + S(w,t), \tag{5.36}$$

where τ_d is the electron drag time given by Eq. (4.9), and $D(w,t)$ is the quasilinear diffusion coefficient described below. Here ν_{loss} is the loss rate modeling the transit time loss of untrapped particles; specifically, the model used was

$$\nu_{\text{loss}} = \begin{cases} \dfrac{w_h}{L} & \text{for } w < w_h \\ 0 & \text{for } w > w_h \end{cases}, \tag{5.37}$$

where L is the plasma length, and $w_h^2 = q\Phi/m_i \approx 3T_e/m_i$ is is the ion velocity characteristic of the ambipolar energy. $S(w,t)$ is the source of particles, at high energies due to the injected neutral beams, and at very low energy due to the plasma stream, if present. For longitudinal flute fluctuations, the quasi-linear velocity diffusion coefficient takes the form (Davidson, 1972)

$$D(w,t)=\frac{q^2}{m_i w^2}\sum_n n^2\Omega_i^2\int\frac{d^2k_\perp}{(2\pi)^2}\,|\tilde{\phi}_k|^2\times J_n^2\left(\frac{k_\perp w}{\Omega_i}\right)\frac{\Delta\omega_k}{(\omega_k-n\Omega_i)^2+\Delta\omega_k^2},\quad(5.38)$$

where $\Delta\omega_k$ is the inverse autocorrelation time of the fluctuations. The fluctuating potentials satisfy

$$\frac{\partial|\phi_k|^2}{\partial t}=2\,\mathrm{Im}\omega_k|\phi_k|^2+\alpha_k,\quad(5.39)$$

where ω_k is the unstable root of the dispersion relation of the unstable modes, and α_k is a thermal source of noise. Because unstable waves will have phase velocities of the order of the hole velocity w_h, where the slope of $F(w)$ is positive, it may be seen from Eq. (5.38) that a diffusion coefficient caused by such fluctuations at Ω_i will be peaked at $w\lesssim w_h$ and will fall as w^{-3} at large velocities. Because of this preferential scattering of low velocity ions, the diffusion coefficient in Eq. (5.36) can balance the large loss time ν_{loss} for $w<w_h$, and so cause the hole to fill, and still not severely cause diffusion of ions with $w\gg w_h$. The model is, therefore, capable of achieving the quasi-linear steady state described earlier. The results of this code predict in detail the history of the 2XIIB plasma, including buildup, beam sustenance, and decay after beam turn-off with and without the stabilizing stream. Time histories of such quantities as particle densities, energies, and lifetimes, as well as the fluctuation level and spectra, are in good agreement with experiment (Berk *et al.*, 1976).

VI. SUMMARY

The foregoing can be summarized into a fairly concise present-day picture of end-loss processes from a mirror-confined plasma and their implications for its reactor potentialities.

As a single particle phenomenon, adiabaticity and its limits for a given magnetic configuration have long been understood. As a constraint upon vacuum magnetic fields, these considerations do not usually place serious limitations upon mirror machine design. The more difficult question, and an area presently being explored, is that of the detailed shape of the magnetic field in the presence of self-consistent high-β equilibria. The ultimate loss of adiabaticity sets a limit on the pressure that a given vacuum field can hold, although that limit depends upon the pitch angle distribution of confined ions. Recently it has been shown that, with increasing β and fixed pitch angle distribution, loss of adiabaticity always precedes breakdown of a long, thin equilibrium due to the onset of the mirror mode (Cohen and Hall, 1976).

Microstability questions have long dominated mirror research; the recent coalescence of theory and experiment have increased confidence in the ability of the former to understand and predict machine behavior for loss cone driven modes such as the DCLC. The principal result appears to be that stability of such modes for small radius plasmas does not require a full filling-in of the loss cone, but only of the low-energy portion, so that the quantity $\int_{-\infty}^{+\infty}du\,f_{0i}$ has not too deep a hole at low velocity. The predicted geometric stabilization of the DCLC mode in plasmas with radii large compared to ion Larmor radii, as opposed to stabilization by partially filling the loss cone, has not been demonstrated experimentally. Such a demonstration must await the next generation of larger machines.

As currently conceived, the high-frequency convective mode sets a limit on axial length for a mirror machine not to suffer enhanced scattering. However, the calculated amplification of this mode is greatly diminished by partial filling of the loss cone; its role, and even occurrence, in experiment has never been documented beyond that inferred in 2XII.

A plasma property central to the notion of mirror confinement is that of pressure anisotropy. Therefore, a particularly important conclusion from the foregoing loss-cone mode stabilization picture is that, at least for $\beta_{\mathrm{vac}}\lesssim 0.2$, the existence of anisotropy is both theoretically and experimentally compatible with operation characterized by very low fluctuation level. For higher β the agreement between theory and experiment is less clear. With cold plasma stabilization, 2XIIB has demonstrated such operation with β_{vac} up to and exceeding unity. In this range of β_{vac}, WKB modified infinite medium theory predicts instability to the anisotropy driven Alfvén ion cyclotron mode which is unaffected by cold plasma. It remains to be seen whether the apparent inadequacy of the theory is due to the comparable size of the 2XIIB plasma to the instability wavelength. For the moment, this mode must be considered a possible constraint on the achievable β in a mirror machine and one which is driven directly by the anisotropy.

The electrons play a pivotal role in classical ion confinement. The poor energy confinement of electrostatically contained electrons and the consequent $E_e/E_i\approx 0.1$ ratio act to limit the ion lifetime both by energy degradation and ion loss boundary modification. [Artificially increasing the electron temperature, e.g., by external wave heating, would indeed reduce the energy drag, but at the expense of a larger circulating power and, unless the ambipolor potential drop to the mirrors could be simultaneously limited, by a de-trapping of a larger portion of the low-energy ion distribution (Gormezano *et al.*, 1976).] The order of unity resultant energy multiplication Q would require a reactor depending on high technology and efficiency for economic operation and begs for an enhancement, if only by a factor 2 or 3. Such a Q-enhancement effort forms a current thrust of the mirror-confinement program. What is required is a modification of the concept of an open-ended minimum-B system in a way which enhances ion confinement while not sacrificing the essential ingredient of a localized high-β plasma.

In conclusion, the role played by the ambipolar potential cannot be overstressed. The total drop to conducting walls is vital to axial electron confinement; however, it implies a large energy loss through the electron channel. Furthermore, the existence of a potential drop to the mirror throat has a deleterious effect on ion confinement, both by the classical de-trapping of low-energy ions and by the generation of loss-cone instabilities feeding on the resultant free energy of the inverted ion distribution. It seems likely that any successful Q-enhancement scheme must successfully counter the negative features of the ambipolar potential in ion confinement while not sacrificing good electron confinement. One such idea has been presented by Kelley (1967) in a scheme employing three linked mirrors to transfer the drop in ambipolar potential outside the mirrors of a dominant central cell. Ideas involving toroidal linking of mirror machines in which ion density, but not pressure, is constant along field lines might accomplish the same end (see e.g., Cordey and Watson, 1975).

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