The application of renormalization group techniques to quarks and strings

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Recent work by K. G. Wilson, A. A. Migdal and others has led to a statistical mechanical treatment of systems of interaction quarks and strings. This work is summarized here. The major topics discussed include boson and fermion variables in statistical mechanics; descriptions of local and gauge symmetries; exact solutions of one-dimensional problems with nearest-neighbor interactions; exact solutions of two dimensional problems with plaquette interactions; Wilson's model of quarks and strings; asymptotic freedom and trapping for this model; the effect of a phase transition in this system; approximate recursion relations of the Migdal form. Finally, all this is put together to give a partial argument for the simultaneous existence of asymptotic freedom and trapping O_2 in the quark-string case. Arguments are developed which distinguish this case from the superficially analogous example of quantum electrodynamics.

CONTENTS

INTRODUCTION

This paper describes how modern renormalization techniques may be applied to models of elementary particle phenomena in which quarks and strings are placed upon a lattice.

We concentrate on two points: (1) how the theory may be made consistent with the apparently contradictory concepts of asymptotic freedom and quark trapping, and (2) how one might begin to approach actual renormalization calculations for these problems.

Section I describes the formulation of lattice problems in statistical physics. Some statistical variables are described, including the standard Gaussian "boson" random variables and anticommuting "fermion" random variables. The discussion of variable types is continued in Sec. II, which is essentially a description of the relationship between symmetries and variable types. Local and global symmetries and the roles of statistical variables as bases for representations of the symmetry are discussed. Finally, K. G. Wilson's model (1974, 1975a, 1976a) of quarks and strings on a lattice is explicitly written down.

In Sec. III renormalization group techniques are introduced as a method of solving one-dimensional problems. Particular attention is given to the fermion variables, which can serve as a representation of quarks, and to the case in which the statistical variables define a homogeneous space for the symmetry group. Following Migdal (1975a), we describe how the solution of the one-dimensional problem for this case can be converted into a solution of a two-dimensional problem with a local (gauge) symmetry. In this section we also describe how the renormalization method can be used to calculate Green's functions.

Section IV describes the qualitative properties of Wilson's lattice model. The quarks are represented by fermion random variables and the strings by homogeneous variables. This formulation can include asymptotic freedom and the unobservability of free quarks. However, Lorentz invariance is not an automatic conse-

quence of the theory. Instead, it can only arise because the theory is near a critical point. The renormalization group technique is then suggested as a natural way of attacking and viewing these near-critical problems. Furthermore, the simultaneous existence of asymptotic freedom and trapped quarks is explained as a consequence of the nonexistence of a phase transition in the string variables.

Section V introduces approximation techniques for attacking lattice problems. The potential moving method of Kadanoff (1975) and Kadanoff, Houghton, and Yalabik (1976) is used to "derive" Migdal's (1975a, 1975b) approximate recursion relations.

Section VI discusses the phenomenology of fixed points with particular reference to the critical dimensionalities at which phase transitions become unstable or disappear. Finally, Section VII applies the Migdal approximation to the quark–string model.

I. STATISTICAL SUMS AND PARTICLE PHYSICS

A. Introduction

In recent years, a whole body of knowledge has been developed about the connection between problems in quantum field theory and those in classical statistical physics. A first point of contact is the similarity between the diagrammatic expansions employed in the two areas. The mathematical similarity between these areas was then further exploited by K. G. Wilson's [(1970, 1971, 1972) see also Wilson and Fisher (1972) and the review of Wilson and Kogut (1974)] wedding of the renormalization ideas in particle physics (Gell-Mann and Low, 1954) with the concepts of universality and scaling which were current in phase transition physics. This extended view of the renormalization group then provided a microscopic theory of phase transitions as well as many insights into the structure of particle theory.

Constructive field theorists have deepened our understanding of these heuristically derived connections by showing how problems in "classical" (i.e., commuting-variable) statistical physics in Euclidean space of dimensionality d could be connected to quantum mechanics problems in $d-1$ spatial dimensions. [See, for example, Osterwalder and Schrader (1973a, b, 1975), Nelson (1973), Simon (1974), and Jaffe and Glimm (1976).] Roughly speaking, the two kinds of problems can be linked together by comparing their Green's functions. One goes from one kind of problem to the other by making an analytic continuation of one of the coordinates—the "time"—according to

$$t \to \pm it \tag{1.1}$$

This "Wick rotation" of the coordinate can, in practice, usually be carried out very simply. Imagine that the problem in statistical physics were described in terms of field variables $\sigma(x)$ and $\bar{\sigma}(x)$ and that the simplest correlation function constructed from these was the one-particle propagator

$$G(x,x') = \langle \sigma(x)\bar{\sigma}(x') \rangle \quad .$$

Here $\langle\ \rangle$ means some sort of statistical average, and x and x' are points in a d-dimensional Euclidean space. In the most elementary example, $G(x,x')$ will have a Fourier transform with a simple pole, i.e.,

$$G(x,x') = \int \frac{dp}{(2\pi)^d} \frac{e^{ip(x-x')}}{p^2+m^2} \quad . \tag{1.2}$$

The "momentum" integral, $\int dp$, is an integral over a Euclidean momentum vector. The net result is that $G(x,x')$ will depend upon a distance in Euclidean space

$$r = \left(\sum_{\alpha=0}^{d-1} (x_\alpha - x'_\alpha)^2 \right)^{1/2}$$

and will have an asymptotic form for large separations,

$$G(r) \sim e^{-mr} \quad .$$

Under the Wick rotation, x_0 and x'_0 become pure imaginary. Thence r can become real for spacelike separations and imaginary for timelike separations. This analytic continuation of $G(r)$ yields the typical time-ordered Green's functions of field theory. In particular, the correlation function (1.2) gives, when continued, the usual propagator for a noninteracting spin-zero particle, including the typical oscillatory structure for timelike separations. However, in one sense, the whole continuation is quite unnecessary since whenever we see a structure like (1.2) we can simply recognize it as the Euclidean reflection of a spin-zero particle with mass m.

Thus it is formally possible to move back and forth between quantum field theory and classical problems involving statistical fields. However, the classical statistical mechanics of fields is itself a subtle subject, involving all kinds of potential divergences. To eliminate the ultraviolet (short-distance) divergences, one can replace the continuum problem by a lattice problem in which the basic "fields" are only defined at the set of lattice points

$$x = (x_1, x_2, \ldots, x_d) = a_0(n_1, n_2, \ldots, n_d) \quad .$$

Here the n's are integers, and the length a_0 is called a lattice constant. The entire ultraviolet divergence problem is then reduced to defining a suitable limit, $a_0 \to 0$. This limit should, for example, yield a Euclidean rotational invariance of the statistical problem so that the related quantum field theory can have a Lorentz invariant structure.

This limiting process must be rather sophisticated. For example, the basic Green's function $G(x,x')$ may be considered to depend upon the separation vector $\Delta x = x - x'$ the lattice constant a_0, and a set of coupling parameters $\mathbf{K}$ which appear in the Hamiltonian for the statistical problem. Hence we write $G(x,x')$ as $G(\Delta x, a_0, \mathbf{K})$. The limit $a_0 \to 0$ is defined first of all by keeping the coordinates x, x', and Δx fixed. To get a physically meaningful result, one must then vary the coupling parameters $\mathbf{K}$ with the cutoff a_0 so as to keep physical quantities like the mass of Eq. (1.2) fixed. The renormalization group (Gell-Mann and Low, 1954; Callen, 1972; Symanzik, 1971) is exactly a method of discussing how physical parameters vary with the change in couplings and cutoffs. Hence the renormalization group is an absolutely essential ingredient in

deriving physical information from a lattice theory.

Alternatively, one can make contact with the standard theory of critical phenomena on a lattice by saying that the mass m is exactly the reverse coherence length ξ^{-1}. For fixed couplings K the coherence length is a fixed multiple M^{-1} of the lattice constant a_0. Therefore we can write

$$m = (a_0/\xi)a_0^{-1} = M(\mathrm{K})a_0^{-1} \ . \tag{1.3}$$

As $a_0 \to 0$ with fixed couplings, $m \to \infty$. In order to get finite masses, one must adjust the couplings K as a function of a_0 so that $M(\mathrm{K})$ decreases linearly with a_0. A decreasing dimensionless mass M is evidence that the problem is coming closer and closer to a critical point. Thus the physical limit of a lattice theory is necessarily one in which the lattice problem shows near-critical behavior.

B. Formulation of lattice problems in statistical physics

To describe a problem in $d-1$ dimensions of space and one time dimension, define a d-dimensional hypercubic lattice with the set of lattice points separated by the distance a_0. On each lattice point define a statistical variable $\sigma(x)$. Since there may indeed be several variables at each point, $\sigma(x)$ can be considered to be a vector of variables $\sigma_i(x)$, with an internal index $i = 1, 2, \ldots$. One then defines a statistical problem by giving a meaning to statistical sums and averages.

The basic summation operation is a sum over a single variable at a particular point in space x, which we denote by $\mathrm{tr}_{\sigma(x)}$. The full statistical sum is then written as

$$\mathrm{Tr}_\sigma = \prod_x \mathrm{tr}_{\sigma(x)} \ . \tag{1.4}$$

For example, in the Ising model, each $\sigma(x)$ takes on the values ± 1 and

$$\mathrm{tr}_\sigma f(\sigma) = f(1) + f(-1) \ .$$

Next, one weighs the sum (1.4) with a factor

$$\rho(\sigma) = (\exp A[K,\sigma]) \div Z[K] \ . \tag{1.5}$$

Here $A[K,\sigma]$ is an action which depends on the variables σ and a set of coupling functions or parameters K. (In standard statistical physics this action is replaced by minus the Hamiltonian divided by Boltzmann constant times temperature.) For example, a nearest-neighbor coupling problem is described by a coupling function $K(\sigma,\sigma')$ and has

$$A[K,\sigma] = \sum_{\langle xx' \rangle} K(\sigma(x),\sigma(x')) \ ,$$

where $\langle xx' \rangle$ is used as a notation for a sum over all nearest-neighbor pairs on the lattice.

Finally, the partition function $Z[K]$ is defined to give a proper normalization to the probability function (1.5). It is

$$Z[K] = \mathrm{Tr}_\sigma e^{A[K,\sigma]} \ . \tag{1.6}$$

The density matrix defined by Eq. (1.5) is used to define all the statistical averages. Given any function of the statistical variables $Q(\sigma)$, the average of Q is defined to be

$$\langle Q \rangle = \mathrm{Tr}_\sigma(\sigma) Q(\sigma) \ . \tag{1.7}$$

In particular, then, the basic Green's function is defined as an average $\langle \sigma(x)\bar{\sigma}(x') \rangle$, where $\bar{\sigma}(x)$ is a variable conjugate to $\sigma(x)$. In the spin-zero case, $\sigma(x)$ may be complex, and $\bar{\sigma}$ will then be σ^*; for the spin one-half case $\sigma(x)$ will become a spinor $\psi(x)$, and $\bar{\sigma}(x)$ will be

$$\bar{\psi}(x) = \psi^\dagger(x)\gamma_0 \ ,$$

where γ_0 is one of the usual gamma matrices.

C. Variable types

In this paper we shall use local spinor variables $\psi_i(x)$ and $\bar{\psi}_i(x)$ to represent quarks. The strings will, however, have a slightly more complex representation via statistical variables $U_{ij}(x,x')$ with two spatial indices and two internal indices. The basic set of U's are defined when x and x' are nearest neighbors. In this case, we shall say that $U_{ij}(x,x')$ describes a *string bit*. To handle a finite length of string one forms the matrix product of a succession of connected string bits. Thus, if $x_0, x_1, x_2, \ldots, x_n$ are a succession of points such that x_i and x_{i+1} are always nearest neighbors, a piece of string extending from x_0 to x_n is described by a variable

$$U_\gamma(x_0,x_n) = U(x_0,x_1)U(x_1,x_2)\cdots U(x_{n-2},x_{n-1})U(x_{n-1},x_n) \ . \tag{1.8a}$$

In this definition, matrix multiplication over the internal indices of the U's is implied. The subscript γ on $U_\gamma(x_0,x_n)$ is intended as a reminder that this product variable depends upon the path $\gamma = (x_0,x_1,x_2,\ldots,x_n)$ followed by the string bits in going from x_0 to x_n.

The case $x_0 = x_n$ is especially important. In this case, the piece of string is closed. This string loop is represented here by writing a capital Γ instead of a small γ, to remind ourselves that the path is closed, and taking

$$U_\Gamma = U_\gamma(x_0,x_0) \ . \tag{1.8b}$$

The string loop variable (1.8b) has two internal indices like any other string variable. For a depiction of these different variable types, see Fig. 1.

For the moment, let us put these string variables aside and focus our attention upon the more usual vari-

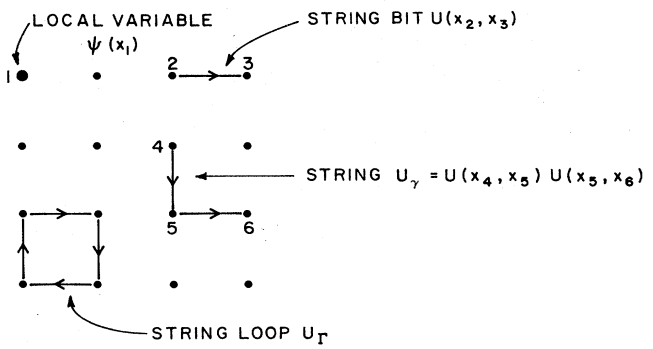

FIG. 1. Type of variables.

ables which have but a single spatial index. The simplest statistical variable is an Ising variable for which $\sigma(x)$ takes on the values ± 1. These Ising variables may then be fully defined by two algebraic statements. The first is a definition of the square of the variables

$$[\sigma(x)]^2 = 1 \ . \tag{1.9a}$$

The second is a definition of the basic trace operation for a single Ising variable

$$\begin{aligned} &\mathrm{tr}_\sigma 1 = 1 \\ &\mathrm{tr}_\sigma \sigma = 0 \ . \end{aligned} \tag{1.9b}$$

In general, we can define the behavior of statistical variables $\sigma(x)$ by giving:

(1) algebraic rules for adding and multiplying the variables and,

(2) a definition of the fundamental statistical sum tr_σ. We shall use these algebraic rules for constructing variables to represent bosons and fermions. In either case, let (1) stand for the statistical variable at the space-time point x_1 with internal index i_1. For the boson case, $\sigma(1)$ is a complex number $\phi(1)$ and the conjugate variable $\bar{\sigma}(1)$ is just the complex conjugate of $\varphi(1)$, i.e., $\varphi^*(1)$. Thus the boson variables are added and multiplied as complex numbers. The basic trace operation is

$$\mathrm{tr}_\varphi = \int \frac{d(\mathrm{Re}\varphi)d(\mathrm{Im}\varphi)}{2\pi} \ .$$

Fermions have a more complex statistical representation [see Berezin (1966) and Abers and Lee (1973)]. The basic objects are statistical variables $\psi(1)$ and $\bar{\psi}(1)$ which have typical fermion anticommutation properties

$$\{\psi(1), \psi(2)\} = \{\bar{\psi}(1), \bar{\psi}(2)\} = \{\psi(1), \bar{\psi}(2)\} = 0 \ . \tag{1.10}$$

Therefore, for a particular position and value of the internal indices, there are only four possible quantities which can be formed

$$1, \psi(1), \bar{\psi}(1), \bar{\psi}(1)\psi(1) = -\psi(1)\bar{\psi}(1) \ .$$

The basic trace operation is defined by giving the trace of these objects according to

$$\mathrm{tr}_\psi 1 = \mathrm{tr}_\psi \psi = \mathrm{tr}_\psi \bar{\psi} = 0 \tag{1.11a}$$

but

$$\mathrm{tr}_\psi \bar{\psi}\psi = -\mathrm{tr}_\psi \psi\bar{\psi} = 1 \ . \tag{1.11b}$$

Thus the only terms which contribute to $\mathrm{Tr}e^{A[\psi]}$ are then products over all spatial indices and internal indices of $\bar{\psi}(1)\psi(1)$. We shall see that this rather peculiar representation generates the standard Fermi Green's functions.

Notice that the summation operations described by Eqs. (1.11) are not representable as sums with positive weights. For this reason, some of the standard theorems of statistical mechanics will fail for the ψ's.

D. Free bosons and fermions

To handle both bosons and fermions at one time, we introduce the pair of variables $\sigma(1)$ and $\bar{\sigma}(1)$ to stand for $\psi(1)$ and $\bar{\psi}(1)$, in the Fermi case, and $\phi(1)$ and $\phi^*(1)$ in the Bose. The action for a non-interacting system may be written as

$$A[\sigma] = -\bar{\sigma}(\bar{1})\sigma(\bar{1}) + \bar{\sigma}(\bar{1})\Sigma(\bar{1}, \bar{2})\sigma(\bar{2}) \ , \tag{1.12}$$

where we employ the convention that one should sum over any repeated barred index. We wish to calculate the partition function and the set of Green's functions

$$G(12\cdots; 1'2'\cdots) = \langle \sigma(1)\sigma(2)\cdots\bar{\sigma}(2')\bar{\sigma}(1')\rangle \ . \tag{1.13}$$

To calculate this sum, we isolate the part of the action which depends upon the variables $\sigma(1)$ and $\bar{\sigma}(1)$ as

$$A_1[\sigma] = -\bar{\sigma}(1)\sigma(1) + \bar{\sigma}(1)\eta(1) + \bar{\eta}(1)\sigma(1) \tag{1.14}$$

with

$$\begin{aligned} \eta(1) &= \Sigma(1, \bar{2})\sigma(\bar{2}) \\ \bar{\eta}(1) &= \bar{\sigma}(\bar{2})\Sigma(\bar{2}, 1) \end{aligned} \tag{1.15}$$

[we take $\Sigma(1, 1)$ to be zero]. This part of the action may be used to analyze the behavior of averages of product terms like $\sigma(1)$ or $\bar{\sigma}(1)$ or $\sigma(1)\bar{\sigma}(1)$ inside an average such as the one defined by Eq. (1.13).

The calculation starts from the fact that we can calculate the trace of $A_1[\sigma]$ as

$$z = \mathrm{tr}_{\sigma(1)} e^{A_1[\sigma]} = \pm e^{\bar{\eta}(1)\eta(1)} \ . \tag{1.16}$$

Here the upper sign refers to bosons, the lower to fermions. For the Fermi case, the result (1.16) is derived by using the fact that

$$[\sigma(1)] = [\bar{\sigma}(1)]^2 = 0$$

so that e^{A_1} may be written

$$\begin{aligned} e^{A_1[\sigma]} = 1 &- \bar{\sigma}(1)\sigma(1) + \bar{\sigma}(1)\eta(1) \\ &+ \bar{\eta}(1)\sigma(1) + \bar{\sigma}(1)\eta(1)\bar{\eta}(1)\sigma(1) \ . \end{aligned}$$

Then Eq. (1.16) follows directly from Eq. (1.11). The calculation then proceeds by noting that

$$\frac{1}{z}\,\mathrm{tr}_{\sigma(1)} e^{A_1[\sigma]}\sigma(1) = \eta(1) \tag{1.17}$$

$$\frac{1}{z}\,\mathrm{tr}_{\sigma(1)} e^{A_1[\sigma]}\sigma(1)\bar{\sigma}(1) = \eta(1)\bar{\eta}(1) + 1 \ . \tag{1.18}$$

Equation (1.17) serves as an equation of motion for σ since it says that if the index value 1 appears only once in $G(12\cdots; 1'2'\cdots)$ one can make the replacement

$$\sigma(1) \to \eta(1) = \Sigma(1, \bar{2})\sigma(\bar{2})$$

under the average (1.13). On the other hand, Eq. (1.18) describes the modification in this replacement which is required if a σ and $\bar{\sigma}$ have the same index. This modification is, then, essentially identical with a Wick contraction.

When applied to $G(1, 1')$ these two rules give

$$G(1, 1') = \delta(1, 1') + \Sigma(1, \bar{1})G(\bar{1}, 1') \ , \tag{1.19}$$

which is the standard Dyson equation for the one-particle Green's function. For the two-particle function, we find

$$G(12; 1'2') = G(1, 1')G(2, 2') \pm G(1, 2)G(2, 1') \ . \tag{1.20}$$

Therefore this and all higher-order Green's functions appear in the standard form appropriate for noninteracting bosons and fermions.

Finally, we write the partition function. As $\Sigma(1,2)$ is changed infinitesimally, the partition function changes according to

$$\delta \ln Z = \pm G(\bar{1},\bar{2})\delta\Sigma(\bar{2},\bar{1}) \ .$$

If we imagine that $G(1,2)$ is a matrix in its indices so that we can write a formal solution to Eq. (1.19) as

$$G = 1/(1-\Sigma) \ ,$$

then the partition function obeys

$$\delta(\ln Z) = \delta[\pm \operatorname{trace} \ln(1-\Sigma)] \ .$$

Here trace is a diagonal sum over the indices of G. This equation can then be integrated to read

$$\ln Z = \pm \operatorname{trace} \ln(\pm G) \ . \tag{1.21}$$

E. Specific examples

To see the possible relationship between these Green's functions and the propagators of high-energy physics, we shall look at two specific examples. Start from a "scalar" example in which the fields have no internal indices. Then a nearest-neighbor interaction with coupling constant K can be represented by choosing

$$\Sigma(x_1,x_2) = \sum_\alpha K[\delta(x_1-x_2-\hat{e}_\alpha a_0)+\delta(x_1-x_2+\hat{e}_\alpha a_0)] \ . \tag{1.22}$$

Here $\hat{e}_\alpha$ are the set of d lattice vectors

$$\begin{aligned} \hat{e}_1 &= (1,0,0,\ldots) \ , \\ \hat{e}_2 &= (0,1,0,\ldots) \ , \\ \hat{e}_3 &= (0,0,1,\ldots) \ . \end{aligned} \tag{1.23}$$

Then Eq. (1.19) is immediately solved by Fourier transformation, which gives

$$G(p) = 1/[1-\Sigma(p)] \tag{1.24}$$

with

$$\begin{aligned} \Sigma(p) &= K\sum_{\hat{e}} e^{ip\cdot\hat{e}a_0} \\ &= 2K\sum_{\alpha=1}^{d} \cos p_\alpha a_0 \ . \end{aligned} \tag{1.25}$$

In turn then the coordinate space Green's function takes the form

$$G(x,x') = \int \frac{dp\, a_0^d}{(2\pi)^d} \frac{e^{ip\cdot(x-x')}}{1-2K\sum_\alpha \cos p_\alpha a_0} \ . \tag{1.26}$$

We wish to focus on the limit in which $a_0 \to 0$ while $x-x'$ remains fixed. In order that G not vanish in this limit, we require that $1-2dK$ be very small. In particular, we write

$$1-2dK = \frac{a_0^2}{2d} m^2 \ll 1 \ . \tag{1.27}$$

Then the denominator in Eq. (1.26) may be expanded in a power series in a_0 to give

$$G(x,x') = \int \frac{dp}{(2\pi)^d} \frac{e^{ip\cdot(x-x')}}{m^2+p^2} \frac{2d}{a_0^2} a_0^d \ . \tag{1.28}$$

Except for the multiplicative factor $2da_0^{d-2}$, Eq. (1.28) is almost the same as the boson propagator in field theory. The major difference is between the Euclidean nature of the metric in Eq. (1.28) and the Minkowski metric in high-energy physics. However, the difference may be eliminated by an appropriate analytic continuation.

The second example involves $\psi(1)$ and $\bar{\psi}(1)$ with spinor internal indices. These indices appear in matrix multiplication of γ matrices γ_α with $\alpha = 1,2,\ldots,d$. These matrices obey the Euclidean version of the standard anticommutation relations, i.e.,

$$\{\gamma_\alpha,\gamma_\beta\} = 2\delta_{\alpha,\beta} \ . \tag{1.29}$$

We then follow Wilson (1975a) and choose an interaction which, instead of (1.22), has the structure

$$\begin{aligned} \Sigma(x_1,x_2) = \sum_\alpha K[&(1-\hat{e}_\alpha\gamma)\delta(x_1-x_2-\hat{e}_\alpha a_0) \\ &+(1+\hat{e}_\alpha\gamma)\delta(x_1-x_2+\hat{e}_\alpha a_0)] \ , \end{aligned} \tag{1.30}$$

and find that Eq. (1.24) holds true once more but, instead of (1.25), $\Sigma(p)$ obeys

$$\Sigma(p) = 2K\sum_\alpha [\cos p_\alpha a_0 - i\gamma_\alpha \sin p_\alpha a_0] \ . \tag{1.31}$$

Once again we focus on the possibility of having a finite $G(x,x')$ in the limit $a_0 \to 0$. To achieve this result, we pick K to be very close to $(2d)^{-1}$ and define a mass m by

$$1-2dK = ma_0/d \ . \tag{1.32}$$

Thence, by the same line of reasoning as before, we find the (almost) standard result for a spin-one-half particle

$$\begin{aligned} G(x,x') &= \int \frac{dp}{(2\pi)^d} \frac{e^{ip(x-x')}}{m-i\gamma p} da_0^{d-1} \\ &= \int \frac{dp}{(2\pi)^d} \frac{e^{ip\cdot(x-x')}}{m^2+p^2} (m+i\gamma p) da_0^{d-1} \ . \end{aligned} \tag{1.33}$$

II. SYMMETRIES

In setting up a quark–string model we shall eventually choose an action $A(\sigma)$ which depends upon two kinds of variables, quark creation and annihilation variables $\bar{\psi}(1)$ and $\psi(1)$, and string variables $U(1,2)$. These statistical quantities will contain internal indices reflecting the basic symmetries of the problem. The quark variables contain spinor indices, color indices, and flavor indices, while $U(1,2)$ is a matrix in its color indices. The color symmetry needed for this problem is of a very special nature. It is a gauge symmetry in which there is an independent symmetry operation at each point in space. But, before discussing gauge symmetries, let us look at the simpler case of a global symmetry in which there is a single set of symmetry operations for the entire action.

A. Global symmetry

Consider an action A which depends on two sets of variables $\sigma_i(x)$ and $\bar{\sigma}_i(x)$. Here $\sigma(x)$ and $\bar{\sigma}(x)$ may be identical, as in the Ising model, or different as in the case in which σ and $\bar{\sigma}$ are ψ and $\bar{\psi}$. We say that σ and $\bar{\sigma}$ form a basis for a representation of the symmetry group $\{G\}$ if there exists a set of matrices G^α (with the different matrices labeled by the index α) such that

$$A[\sigma,\bar{\sigma}] = A[\sigma',\bar{\sigma}'] \tag{2.1}$$

whenever the new variables σ' are defined as linear combinations of the old variables by

$$\begin{aligned}\sigma_i'(x) &= \sum_j G_{ij}^{\alpha}\sigma_j(x)\\ \bar{\sigma}_i'(x) &= \sum_j (G_{ij}^{\alpha})^*\bar{\sigma}_j(x)\,.\end{aligned} \tag{2.2}$$

We limit ourselves to unitary representations of the group, i.e., those which obey

$$\sum_j G_{ij}^{\alpha}(G_{kj}^{\alpha})^* = \delta_{ik}\,.$$

For example, if we take a nearest-neighbor interaction

$$A[\sigma] = \sum_{\langle xx'\rangle} K(\sigma(x),\sigma(x'))\,,$$

Eq. (2.1) is equivalent to

$$K(\sigma_1,\sigma_2) = K(G^{\alpha}\sigma_1, G^{\alpha}\sigma_2)\,.$$

Of course, Eq. (2.1) is not sufficient to guarantee the invariance of the final problem. We must have the basic statistical sum Tr also be invariant under the change of variables (2.2). This additional condition will be met if tr_σ is an invariant sum; i.e., if it obeys

$$\mathrm{tr}_\sigma f(\sigma) = \mathrm{tr}_\sigma f(G^{\alpha}\sigma) \tag{2.3}$$

for all group elements G^{α} and all possible functions f. Then, tr_σ is said to be an *invariant sum*.

Equations (2.1) and (2.2) are the condition that the action be invariant under group operations, while Eq. (2.3) is the requirement that the basic sums be so invariant. The action will certainly be invariant if $A\{\sigma\}$ is a function of scalars like

$$\sum_i \bar{\sigma}_i(x)\sigma_i(x') = \bar{\sigma}(x)\sigma(x)\,.$$

Thus, for example, the conventional two-dimensional rotation symmetry of the XY model is obtained by taking $\sigma_i(x)$ to be a two-component vector and by choosing the coupling function K to depend upon the combination

$$\sigma_1(x)\sigma_1(x') + \sigma_2(x)\sigma_2(x') = \boldsymbol{\sigma}(x)\cdot\boldsymbol{\sigma}(x')\,.$$

This action will then be invariant under two-dimensional rotations with the rotation matrix

$$G^{\alpha} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

The full statistical problem will be invariant if $\mathrm{tr}_{\sigma(x)}$ takes the form

$$\mathrm{tr}_{\sigma(x)} = \int d\sigma_1 d\sigma_2\, w(\sigma_1^2+\sigma_2^2) \tag{2.4}$$

where w is any weight function.

B. Transitive representations

In this way, we can make the σ's a basis for a representation of a symmetry group. Let us give a few examples. For the Ising case, $\sigma(x) = \pm 1$, an action which is bilinear in the σ's, e.g.,

$$K\sum_{\langle xx'\rangle}\sigma(x)\sigma(x')$$

has two symmetry operations,

$$\sigma(x)\to\sigma(x)$$

and

$$\sigma(x)\to -\sigma(x)\,. \tag{2.5}$$

Hence the representation has two elements $G^0 = 1$ and $G' = -1$. The abstract group so represented is called Z_2.

Another rather analogous case is one in which $\sigma(x)$ runs over all real values between $-\infty$ and ∞. Then

$$\mathrm{tr}_\sigma = \int_{-\infty}^{\infty} d\sigma\,.$$

If the action is even in σ, the transformations (2.5) are once again the basic symmetry operations. But there is a significant difference between the two cases. In the Ising case, the symmetry operations

$$\sigma\to G^{\alpha}\sigma$$

define all the possible variations in the basic variables. Thus the basic Ising sum can be written as

$$\mathrm{tr}_\sigma f(\sigma) = \sum_{\alpha} f(G^{\alpha}\sigma)\,. \tag{2.6}$$

If G^{α} are the two transformations of Z_2, Eq. (2.6) gives a correct representation of the Ising sum, but certainly not of a sum over all real numbers. Whenever Eq. (2.6) is satisfied we say that $\sigma(x)$ forms a homogeneous space for the symmetry group or that $\sigma(x)$ is a transitive representation of the symmetry. We shall find that transitive representations are particularly simple and useful representations of the symmetry, since for these representations the group theory will give all possible information about the variation of $\sigma(x)$.

For the XY model described above, the $\sigma_i(x)$ form a transitive representation only if the weight $w(\sigma_1^2+\sigma_2^2)$ in Eq. (2.4) is a delta function which limits the magnitude of $\sigma_1^2+\sigma_2^2$ to a particular value (say 1).

The group SU_n can be represented by choosing $\sigma(x)$ to be an n-component complex vector of unit length

$$\sum_i \sigma_i(x)\sigma_i^*(x) = 1\,.$$

The basic symmetry is the transformation

$$\sigma\to\sigma' = u\sigma\,,$$

where the u is a unitary matrix ($uu^{\dagger}=1$) which is required to have a unit determinant. This representation is also transitive if we choose tr_σ to be a proper invariant sum. Thus

$$\mathrm{tr}_\sigma f(\sigma) = \int_u f(u\sigma)\,.$$

Here the invariant sum $\int_u$ can be defined in the following way (Murnaghan, 1938). Since u is an n by n complex matrix it has n^2 different complex components. Call the integral over the complex plane of all these components $\int du$. The conditions on u are then inserted as simple delta function conditions under the integral.

Hence

$$\int_u = \int du\, \delta(\det u - 1)\, \delta(1 - u^\dagger u)\,. \tag{2.7}$$

This invariant integral obeys

$$\int_u f(u) = \int_u f(uu_1) \tag{2.8}$$

for any unitary u_1 with determinant equal to 1.

We shall not make direct use of Eq. (2.7) for finding averages over SU_3 transformations. Instead, we shall simply list some averages defined by

$$\langle x \rangle = \int_u x(u) \Big/ \int_u 1\,. \tag{2.9}$$

We have

$$\begin{aligned}
&\langle 1 \rangle = 1\,,\\
&\langle u_{ij} \rangle = \langle u^\dagger_{ij} \rangle = 0\,,\\
&\langle u_{ij}\, u_{kl} \rangle = \langle u^\dagger_{ij}\, u^\dagger_{kl} \rangle = 0\,,\\
&\langle u_{ij} (u^\dagger)_{kl} \rangle = \tfrac{1}{3} \delta_{il} \delta_{jk}\,,\\
&\langle u_{ij}\, u_{jk}\, u^\dagger_{mn} \rangle = 0\,,\\
&\langle u_{ij}\, u_{kl}\, u_{mn} \rangle = \tfrac{1}{6} \epsilon_{ikm} \epsilon_{jln}\,.
\end{aligned} \tag{2.10}$$

Here ϵ_{ijk} is the completely antisymmetric tensor of rank 3.

For instance $\langle u_{ij} \rangle = 0$ can be proved as follows: $\int_u$ is invariant under the transformation $u \to u' = ue^{\pm 2\pi i/3}$, because of Eq. (2.8). Then $\langle u_{ij} \rangle$ is an average over the three cubic roots of unity, and hence it must vanish.

Later on there will be considerable use made of various transitive representations.

C. String variables and local symmetries

For the global symmetry, the entire action $A\{\sigma\}$ is transformed with the aid of the same matrix G^α_{ij}. In particular, the new variables are

$$\begin{aligned}
&\psi'_i(x) = \sum_j G^\alpha_{ij}\, \psi_j(x)\,,\\
&\bar\psi'_i(x) = \sum_j (G^\alpha)^*_{ij}\, \bar\psi_j(x)\,.
\end{aligned}$$

If there is a string variable $U_{ij}(x,x')$ defined for each ordered pair of nearest neighbors xx', it transforms according to

$$U'(x,x') = G^\alpha U(x,x') (G^\alpha)^\dagger\,.$$

The color symmetry assumed in high-energy physics is a far richer symmetry than this. Under transformations, $A\{\psi, \bar\psi, U\}$ is required to be invariant even when the transformation is different at every point in space. Thus, if $u(x)$ is a color transformation matrix which depends upon the space–time point x, A is required to be unchanged under the transformation,

$$\begin{aligned}
&\psi(x) \to \psi'(x) = u(x)\, \psi(x)\,,\\
&\psi(x) \to \bar\psi'(x) = \bar\psi(x)\, [u(x)]^\dagger\,,\\
&U(x,x') \to U'(x,x') = u(x)\, U(x,x')\, [u(x')]^\dagger\,.
\end{aligned} \tag{2.11}$$

(We omit here the ij indices.)

This new kind of invariance is easily constructed. Just let A depend upon terms like

$$\sum_{ij} \bar\psi_i(x)\, U_{ij}(x,x')\, \psi_j(x') = \bar\psi(x)\, U(x,x')\, \psi(x')\,. \tag{2.12a}$$

This kind of term is certainly invariant under the local color symmetry. It represents the motion of a quark from x' to x, with the aid of the bit of string $U_{ij}(x,x')$. Other invariant terms which involve no motion are

$$\sum_i \bar\psi_i(x)\, \psi_i(x) = \bar\psi(x)\, \psi(x) \tag{2.12b}$$

and

$$\sum_{ij} U_{ij}(x,x')\, U_{ji}(x',x)\,. \tag{2.12c}$$

One additional term is needed: A term which provides string–string interactions. Imagine that we construct a closed path Γ from x_1 to x_2 to $..x_n$ to x_1 and construct the combination U_Γ which is a product of string bits along the path as in Eq. (1.8). Then any trace in the 3 by 3 color space of a power of U_Γ

$$\text{trace}(U_\Gamma)^p \tag{2.12d}$$

is also a scalar. We now construct an action from these invariant pieces.

D. The Wilson model

Now, we can put together the pieces and write the Wilson model (1975a, 1976a) of quarks and strings. Each $\psi(1)$ and $\bar\psi(1)$ is a four-component spinor with an additional flavor index f and a color index $i = 1, 2, 3$. The color symmetry is a SU_3 symmetry and is an exact local symmetry. The flavor index takes on the values f = up, down, strange, and (perhaps) charmed. The first three values represent the broken SU_3 symmetry of Gell-Mann and Ne'eman. Thus we write $\bar\psi_{i,f}(x)$ and $\psi_{i,f}(x)$. The spinor indices are never written explicitly but represented by the γ matrices of Eq. (1.29).

Each ordered pair of nearest-neighbor sites on the lattice defines a string bit variable $U_{ij}(x, x')$. Here i and j are color indices. They run from 1 to 3. We make $U_{ij}(x, x')$ itself a unitary matrix with unit determinant. To limit the number of variables, we choose

$$U(x',x) = [U(x,x')]^\dagger = [U(x,x')]^{-1}\,. \tag{2.13}$$

For each x, x' and j, $U_{ij}(x,x')$ is a vector (with index i) which is a basis function for a transitive representation of the SU_3 symmetry, so that, again, the trace will be a sum over group elements.

Now we will set up an action which includes all the symmetries mentioned so far. This action is a sum of two terms

$$A = A_\psi + A_U\,. \tag{2.14}$$

Here A_U depends only on the string variables, while A_ψ depends on both quark and string variables. Visualize a d-dimensional simple "cubic" lattice. On this lattice the nearest neighbors are connected by vectors $\pm \hat e_\mu a_0$ where $\hat e_\mu$ are the lattice vectors given by Eq. (1.23). These $\hat e_\mu$ enable one to construct a closed path over a basic square (called a plaquette). The path starts at x, proceeds to $x + \hat e_\mu a_0$, then to $x + (\hat e_\mu + \hat e_\nu) a_0$, to $x + \hat e_\nu a_0$,

and finally returns to x. This closed path is denoted by $\Gamma_{\mu\nu}(x)$. Then, from Eq. (1.8b), one defines a product of string bits around this path as

$$U_{\Gamma_{\mu\nu}(x)} = U(x, x+\hat{e}_\mu a_0)\, U(x+\hat{e}_\mu a_0, x+\hat{e}_\mu a_0+\hat{e}_\nu a_0) \times U(x+\hat{e}_\mu a_0+\hat{e}_\nu a_0, x+\hat{e}_\nu a_0)\, U(x+\hat{e}_\nu a_0, x)\,. \tag{2.15}$$

The basic string–string coupling in the lattice can be written in terms of these closed loop variables as

$$A_U = \sum_{x,\mu,\nu} J(U_{\Gamma_{\mu\nu}}(x))\,. \tag{2.16}$$

This structure will be color symmetric if the coupling function $J(U)$ is only a function of the trace of U and powers of U. A structure of this kind was employed by Wilson (1975a, 1976b) as a generalization of an action first used by Wegner (1971a) and then analyzed by Balian, Drouffe, and Itzykson (1974, 1975a, 1975b). This form is

$$J(U) = \tfrac{1}{2} J \,\mathrm{trace}(U+U^\dagger)\,. \tag{2.17}$$

Here J is the statistical version of the string–string coupling constant.

The action (2.17) is the simplest color symmetric structure which can be constructed from the U's. Notice that terms like (2.12c) cannot usefully be included because Eq. (2.13) implies that these terms are simply unity.

Next consider A_ψ. We would like to make this part of the action as close as possible to the form of the free-fermion action (1.12). To do this we write

$$A_\psi = -\sum_{x,i,f} \bar{\psi}_{if}(x)\,\psi_{if}(x) + \sum_{\langle xx'\rangle}\sum_{fij} \bar{\psi}_{if}(x)\,\Sigma_{0f}(x,x')\,U_{ij}(x,x')\,\psi_{jf}(x')\,. \tag{2.18}$$

The first term simply sets the normalization of the quark variables; the second is a nearest-neighbor sum which describes the hopping of quarks from one site to neighboring sites with the aid of the bit of string U and the hopping amplitude Σ_0. This nearest-neighbor hopping amplitude is written exactly as in Eq. (1.30) as

$$\Sigma_{0f}(x,x') = K_f\left(1-\gamma\cdot\frac{x-x'}{a_0}\right). \tag{2.19}$$

This action maintains the basic color symmetry (2.11). Notice that the action could contain additional color singlet terms which could be built from the color singlet combinations

$$\sum_i \psi_{if}(x)\,\bar{\psi}_{if'}(x)\,, \tag{2.20a}$$

$$\sum_{ijk} \epsilon_{ijk}\,\psi_{if_1}(x)\,\psi_{jf_2}(x)\,\psi_{kf_3}(x)\,. \tag{2.20b}$$

The combination (2.20a) can be connected with a SU_6 meson multiplet of dimension 35, and an SU_6 singlet. The combination (2.20b) is connected with a 56-dimensional representation of SU_6 including the nucleons. These physical objects do not explicitly appear in our starting action but they should be generated by the action of the renormalization group. As we shall see, the exclusion of these terms from our action in the limit $a_0 \to 0$ is closely related to the free quark behavior generated by the theory in this limit.

E. Connection with field theory

In the limit $a_0 \to 0$, the theory just outlined should reduce to the field theory description of a set of Fermi particles interacting with a gauge field. To make this connection, we write $U(x,x')$ in terms of a line integral of a vector potential $A_\mu(x)$ as

$$U(x,x') = \exp\left[ig\int_x^{x'} d\mathbf{x}''\cdot\mathbf{A}(x'')\right]. \tag{2.21}$$

Here g will turn out to be the coupling constant of the gauge field theory. Since x and x' will be separated by one lattice constant, one can expand the exponential in Eq. (2.21) in a power series in a_0. When this expansion is applied to the right-hand side of Eq. (2.16) one finds a part of the action

$$A_u = -\tfrac{1}{2}\,\mathrm{trace}\sum_{\mu\nu}\sum_x a_0^4\, F_{\mu\nu}F_{\mu\nu}g^2 J_{\mu\nu}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig\,[A_\mu(x), A_\nu(x)]\,. \tag{2.22}$$

The standard coupling term (cf. Abers and Lee, 1973) is then recovered if we replace the sum over x by an integral and choose $J_{\mu\nu}$ as

$$J_{\mu\nu} = 1/2g^2\,. \tag{2.23}$$

Then A_U becomes simply

$$A_U = -\frac{1}{4}\sum_{\mu\nu}\int dx\, F_{\mu\nu}F_{\mu\nu}\,. \tag{2.24}$$

Notice one very important point. To the statistical mechanic, weak coupling means $J \to 0$; to the field theorist, weak coupling means $g \to 0$. Therefore, according to Eq. (2.23), the field theorist and statistical mechanic have exactly opposite views of weak and strong coupling.

The fermion term A_ψ can be handled in a similar fashion. Take the coupling in Eq. (2.19) to be of the form

$$K_F = (1/2d)(1 - m_f a_0/d)\,, \tag{2.25}$$

where m_f is physically the mass of a particle with flavor f. Then an expansion of Eq. (2.18) in a_0 yields

$$A_\psi = \frac{1}{a_0^3}\int dx \sum_f \bar{\psi}_f\,[m_f - \gamma_\mu(\partial_\mu - igA_\mu)]\,\psi_f\,. \tag{2.26}$$

This is then the standard structure of a fermion term in a gauge field theory. [See, for example, Abers and Lee (1973).]

III. RENORMALIZATION THEORY: A POINT OF VIEW AND A CALCULATIONAL METHOD

The quark–string theory involves a formulation in which the variables $\psi(1)$ and $U(x,x')$ appear on a lattice with lattice constant much smaller than one fermi. Clearly this lattice is only a formulational and calculational tool. It must disappear from all the final results of the theory.

The renormalization theory can be viewed [see Wilson (1976b)] as a description of how we can simultaneously change the lattice constant and the basic action but nonetheless leave all physical results of the theory entirely unchanged. This approach, then, permits us to visualize how it might be possible for the original lattice to drop out of any physical end results of the theory.

At the same time as the renormalization method provides an important insight into the formulation of the problem, it can also provide a very useful calculational tool. In statistical physics, a variety of problems which do not yield to perturbation theory or any other "classical" analytic tool were attacked with considerable success by approximate renormalization methods.

In this section, we describe the general formulation of these methods and their particular application to one- and two-dimensional problems.

A. Formulation

Given a set of statistical variables σ on a lattice with lattice constant a_0 and an action $A[K,\sigma]$ which depends upon a set of coupling constants or coupling functions K, we can, in principle, compute the partition function and all the correlation functions via Eqs. (1.4)–(1.7). Now imagine that we have another set of variables, μ, on a lattice embedded in the original lattice (see Fig. 2). This new lattice has lattice constant λa_0. We view these new variables as providing an alternative description of the original problem.

To make the conversion from one description to the other, we define a function $T(\mu,\sigma)$. At the start, T is arbitrary; in the end, we shall make very specific choices of T to achieve calculational convenience. This T is used to define a new effective action via

$$e^{A'(\mu)} = \mathrm{Tr}_\sigma e^{T(\mu,\sigma)+A[K,\sigma]} . \tag{3.1}$$

We demand the one condition that the transformation leave the partition function invariant, i.e., that

$$\mathrm{Tr}_\mu e^{T(\mu,\sigma)} = 1 . \tag{3.2}$$

Then the partition function generated by $A'(\mu)$, i.e.,

$$Z' = \mathrm{Tr}_\mu e^{A'(\mu)}$$

will be identical with the partition function generated by $A[K,\sigma]$.

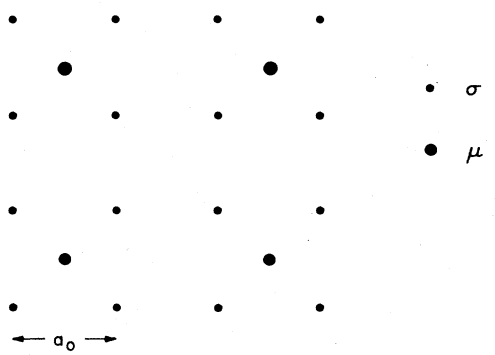

FIG. 2. A renormalization transform with $\lambda = 2$.

The new action $A'(\mu)$ will, in general, be very complex. If the transformation obeys translational (Galilean) invariance, $A'(\mu)$ can be characterized by the same kind of coupling functions which might appear in a very general $A[K,\sigma]$. These coupling functions, K, would describe two-body, three-body, ... interactions. Thus, we can write $A'(\mu)$ as $A[K',\mu]$, where K' is the set of new couplings generated by the transformation.

In summary the change of description $\sigma \to \mu$ has two effects. The lattice constant changes from a_0 to λa_0; the couplings change from K to K'. If we could calculate the sum in Eq. (3.1), we could find out how the new couplings K' depend upon the old. In general, we write this dependence as

$$K' = R^\lambda[K] . \tag{3.3}$$

Equation (3.3) defines a renormalization transformation. It is also possible to define a composition rule for these transforms. If R^{λ_1} represents the change $a_0 \to \lambda_1 a_0$, and R^{λ_2} describes the change $a_0 \to \lambda_2 a_0$, then the function R^λ, defined by

$$K' = R^\lambda[K] = R^{\lambda_2}[R^{\lambda_1}[K]] ,$$

defines a new transformation with a change in lattice constant

$$\lambda = \lambda_1 \lambda_2 .$$

We can express this type of relation formally as the composition rule

$$R^{\lambda^n} = \{R^\lambda\}^n . \tag{3.4}$$

This renormalization transform has several important invariance properties. Since the partition function is left invariant, we see that

$$Z[K] = Z[R^\lambda[K]] . \tag{3.5}$$

Equation (3.5) is very useful for describing the "phases" of the statistical system. The different phases of the system are regions in the space of possible couplings distinguished by different singularity structures in $Z[K]$. For example, Fig. 3 shows a piece of the coupling space defined by an action for an Ising system $[\sigma(x) = \pm 1]$

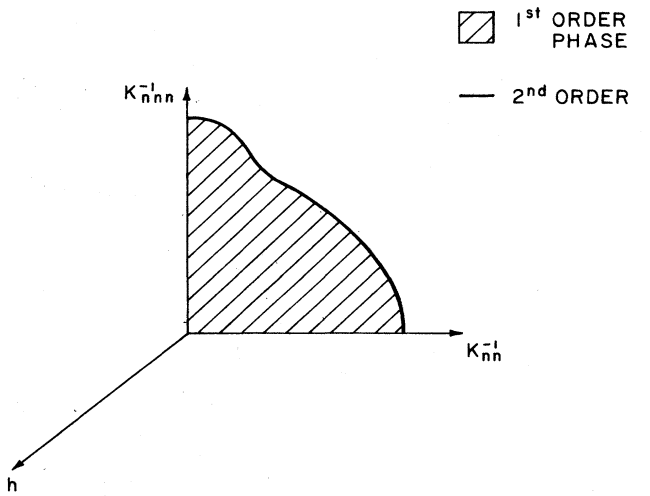

FIG. 3. A simplified phase diagram for the Ising model.

with nearest-neighbor couplings K_{nn} and next-neighbor couplings K_{nnn}, together with a magnetic field h. The action is then

$$A[K,\sigma]=\sum_x h\sigma(x)+\sum_{\langle xx'\rangle} K_{nn}\sigma(x)\sigma(x')+\sum_{nnn} K_{nnn}\sigma(x)\sigma(x')\,.$$

We depict three different phases:

1. A first-order region at $h=0$ and sufficiently strong K_{nn} and K_{nnn}. In this region $\partial(\ln Z)/\partial h$, the magnetization, has a discontinuity at $h=0$.
2. A second-order line at the boundary of the first-order region. On this line, $\partial^2(\ln Z)/\partial h^2$ is infinite and the "mass" $M[K]$ goes to infinity.
3. The remainder of the space shown, where the system is in the high-temperature phase, which is characterized by having no singularities at all.

The renormalization transform (3.3) is assumed to have the property that $R^\lambda[K]$ is a totally nonsingular function of K. If Eq. (3.4) is to hold the singularities on both sides of the equation must come totally from Z. Hence we conclude that, *when K′ and K are connected by a renormalization transformation, they lie in the very same phase.*

One can make much more quantitative statements too. We assume that $\exp T[\mu,\sigma]$ is a short-ranged function which induces correlations between $\mu(x)$ and the $\sigma(x')$ lying near x. Then the range, ξ, of the $\mu-\mu$ and the $\sigma-\sigma$ correlation function must be the same. From Eq. (1.3) we find that if $m=\xi^{-1}$ is to remain invariant, the dimensionless mass $M[K]=\xi/a_0$ must obey

$$M[R^\lambda[K]]=\lambda M[K]\,. \tag{3.6}$$

Now let us follow the consequences of this point of view. Imagine that we started with couplings K_0 and constructed the couplings

$$K_\alpha=R^\lambda[K_{\alpha-1}]$$

for $\lambda>1$ and $\alpha=1,2,\ldots$. If we could arrange for the coupling to stay in the space of K_{nn} and K_{nnn}, we might imagine a picture of successive transformations like those shown in Fig. 4. The flow lines shown depict how different couplings may be connected while always flowing within the same phase.

Notice how the flow lines converge on fixed points. These points are not special parts of the phase diagram, but are instead determined by the particular renormalization transformation. Nonetheless, it is very helpful to analyze the flow patterns with the aid of these fixed points. To do this, turn to Fig. 5a, where we draw a slice of Fig. 4 including all three fixed points.

The flow lines leave the critical fixed point and flow toward the other two. As we shall see this instability (flow away from the fixed point) is a necessary concomitant of a critical point characterized by an infinite correlation length, i.e., $M[K]=0$. All fixed points are by definition places where there is scale invariance; but this invariance is possible for $M=0$ and for $M=\infty$. The critical fixed points are those with zero mass. Since we wish $m=M/a_0$ to approach finite values as $a_0\to 0$, we are aiming at a theory which approaches such a zero-mass fixed point.

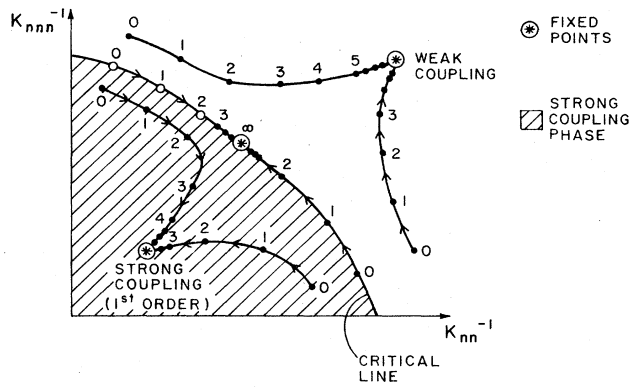

FIG. 4. Structure of the fixed points for the Ising model.

The discussion in this section is a very simplified picture of the fixed point theory introduced by Wilson [(1970); see also Wilson and Kogut (1974)] and then given further mathematical form by Wegner (1972 and 1973). In Sec. VI we shall further develop this discussion. For now, however, let us turn to specific examples.

B. Example: Decimation in one dimension

All of this can be made very explicit in a one-dimensional example. [See Houghton and Kadanoff (1973) and Nelson and Fisher (1975).] Physically, the one dimension will become time after the Wick rotation. Hence, for the particle physicist, our "one-dimensional" example includes no spatial variable whatsoever.

Let the x in $\sigma(x)$ be simply na_0, where n is an integer. Let the basic action be

$$A[\sigma,K]=\sum_n K\left(\sigma(na_0),\,\sigma((n+1)a_0)\right)\,. \tag{3.7}$$

Let the new variables $\mu(x)$ be defined at the points shown in Fig. 6

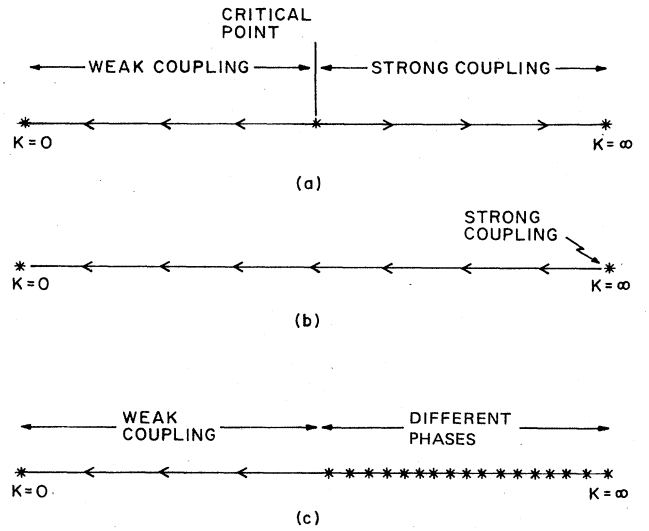

FIG. 5. Different phase diagrams. The stars are fixed points; the arrowheads in the diagrams show the direction of flow of the couplings as the lattice constant is increased.

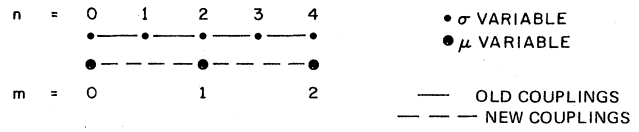

FIG. 6. Decimation applied to a one-dimensional problem.

$$X_m = \lambda m a_0 = 2 m a_0 .$$

Hence we have a change in lattice constant by a factor of 2. We choose these new variables to be essentially identical with the old ones at the same point by writing

$$e^{T(\mu,\sigma)} = \prod_m \delta(\mu(X_m) - \sigma(X_m)) \tag{3.8}$$

where δ is the delta function. The net result of the transformation function (3.1) is to decimate or thin out the statistical variables, leaving us with as many variables in the new problem as in the old. Each new variable is equal to an old variable.

In one dimension the new action $A'(\mu)$ is easy to calculate from Eq. (3.1). The result is exactly of the nearest-neighbor form (3.7),

$$A'(\mu) = \sum_m K'\left(\mu(m a_0 \lambda),\ \mu((m+1) a_0 \lambda)\right) \tag{3.9}$$

with a new coupling function K', which is given by

$$e^{K'(\mu,\mu')} = \mathrm{tr}_\sigma e^{K(\mu,\sigma)+K(\sigma,\mu')} . \tag{3.10}$$

Equation (3.10) then serves as an explicit construction of the dependence of K' upon K. This construction gives an explicit definition of the function $R_0^2[K]$. Here the superscript "2" indicates a change in lattice constant by a factor of 2, while the subscript "0" indicates that we have done the renormalization in the trivial, one-dimensional, example.

One has found $R_0^2[K]$ via Eq. (3.10); one can apply Eq. (3.4) to find R_0^λ via

$$R_0^\lambda = \{R_0^2\}^{\log_2 \lambda} .$$

After that, one can rather easily find all the correlation functions for the problem. For example consider the correlation function

$$G(x - x', K, a_0) = \langle \sigma(x) \bar{\sigma}(x') \rangle . \tag{3.11}$$

Since $\mu(x)$ is $\sigma(x)$, this correlation function can just as easily be calculated in terms of an average of μ's, at least for the case in which $(x - x')/a_0$ is an even integer. Thence we find the identity, applicable to the decimation transform

$$G(x - x', R_0^2[K], 2a_0) = G(x - x', K, a_0) .$$

The successive application of this rule gives

$$G(x - x', K, a_0) = G(x - x', R_0^\lambda[K], \lambda a_0) . \tag{3.12}$$

C. Fermions in one dimension

A simple example of this analysis is the fermion case for which $K(\psi, \psi')$ has the form

$$K(\psi, \psi') = -\tfrac{1}{2}\bar{\psi}\psi - \tfrac{1}{2}\bar{\psi}'\psi' + K\bar{\psi}(1-\gamma_1)\psi' + K\bar{\psi}'(1+\gamma_1)\psi, \tag{3.13}$$

including one coupling constant, K. In this case, Eq. (1.24) implies that

$$G(p) = \frac{1}{1 - 2Ke^{-ip\gamma_1 a_0}} .$$

The inversion of this Fourier transform gives, for $x \neq x'$,

$$G(x - x') = \tfrac{1}{2}[\,\mathrm{sgn}(1-2K) + \gamma_1 \,\mathrm{sgn}(x - x')]\, e^{-m|x-x'|} . \tag{3.14}$$

In Eq. (3.14) sgn is the sign function, which is plus or minus one, depending upon the sign of its argument. The mass m appears in the form (1.3), i.e., $m = M/a_0$, with the dimensionless mass being

$$M(K) = |\ln 2K| . \tag{3.15}$$

Notice the singularities in (3.14) and (3.15) which appear at

$$K = K^* = \tfrac{1}{2} .$$

At this critical point, the mass passes through zero and the roles of holes and particles are interchanged.

The trace (3.10) is calculated in Appendix A. There it is shown that the new coupling has the same form as (3.13) with the parametric change

$$K \to K' = \tfrac{1}{2}(2K)^2 = R_0^2(K).$$

Equation (3.4) then implies that the recursion relation for general λ is given by

$$K \to K' = R_0^\lambda(K) = \tfrac{1}{2}(2K)^\lambda . \tag{3.16}$$

Equation (3.16) implies that there are three fixed points: a weak coupling point $K^* = 0$; a strong coupling point $K^* = \infty$; and a critical point $K^* = \frac{1}{2}$. At these fixed points the mass is infinite ($K^* = 0, \infty$) or zero ($K^* = \frac{1}{2}$). Hence these fixed points are—like all fixed points—places where the theory is scale invariant. This series of fixed points appears in the same form as shown in Fig. 5a. Notice also the pattern of flows. As λ increases, K' moves away from the fixed point at $K^* = \frac{1}{2}$ and toward the other two fixed points. Hence the direction of the arrows in Fig. 5a.

In general, to give a quantitative meaning to this flow away from the fixed point, one can follow Wegner (1972, 1973) and define a variable $h(K)$ which is an analytic function of K near the fixed point, vanishes at the fixed point, and has the simple recursion relation

$$h[K'] = \lambda^y h[K] . \tag{3.17}$$

Then y is termed a scaling index (or critical index) for h. Notice that the scaling property (3.6) of the dimensionless mass implies that

$$M(h) \sim h^\nu , \tag{3.18a}$$

where the coherence length index ν is given by

$$\nu = 1/y . \tag{3.18b}$$

In this fermion example, the Wegner variable is

$$h = -\ln 2K \tag{3.19a}$$

with associated index

$$y = 1 . \tag{3.19b}$$

Notice that this variable is essentially the same as the dimensionless mass, M. The scale invariant quantity $|h/a_0|$ is the physical mass, m.

We can use the recursion relations to calculate the Green's functions if we happen to know them anywhere in a given phase. For example, at small K, first-order perturbation theory gives the nearest-neighbor Green's function as

$$G(x-x'=\pm a_0, K, a_0)=(1\pm\gamma_1)K. \tag{3.20}$$

We apply the recursion formula (3.12) for the Green's function to the case $\lambda=|x-x'|a_0$. Then Eq. (3.12) reads

$$G(x-x', K, a_0)=G(\mathrm{sgn}(x-x')\lambda a_0, R^\lambda(K), \lambda a_0). \tag{3.21}$$

For $K'=R^\lambda[K]\ll 1$, we can evaluate the Green's function on the right-hand side of (3.21) as the nearest-neighbor result (3.20) to find

$$G(x-x', K, a_0)=[1+\mathrm{sgn}(x-x')\gamma_1]R^\lambda(K).$$

If we substitute the appropriate values of λ and R^λ we find

$$G(x-x', K, a_0)=\frac{1+\mathrm{sgn}(x-x')\gamma_1}{2}(2K)^{|x-x'|/a_0} \tag{3.22}$$

which is the exact Green's function in the weak coupling phase. Hence recursion calculations can indeed give us Green's functions and the values of masses.

In the end, we are interested in interacting quarks and strings. The analysis of this section can be extended to the case in which string variables are included in the basic coupling in a term of the form (2.12a). Then the basic nearest-neighbor coupling takes the form

$$K(\psi, \psi', U)=-\tfrac{1}{2}\bar{\psi}\psi-\tfrac{1}{2}\bar{\psi}'\psi' + K\bar{\psi}U(1-\gamma_1)\psi'+K\bar{\psi}'U^\dagger(1+\gamma_1)\psi$$

instead of (3.13). Here we have assumed that $\psi'=\psi(x')$, $\psi=\psi(x)$, $U=U(x,x')$, and $x'>x$. Once again we calculate the trace (3.9) as in Appendix A. Just as before, we find a new coupling function of exactly the same form as the old one, with a new coupling parameter given by Eq. (3.16). The only difference is that the new coupling invokes a new string variable which is the matrix product of the old string variables. Thus if X and X' are nearest-neighbor sites on the new lattice, and x is the point between them, the new string variable is

$$U'(X,X')$$

with

$$U'_{ij}(X,X')=\sum_k U_{ik}(X,x)U_{kj}(x,X'). \tag{3.23}$$

Thus the one-dimensional recursion relation (3.16) will remain valid in the presence of strings. However, the Green's functions will be changed by the strings. In fact, the nearest-neighbor Green's function (3.20) will be proportional to an average of $U(x,x')$ and this average will vanish. For this reason

$$G(x-x', K, a_0)=0 \quad \text{for } x\neq x'$$

in the presence of strings.

D. Other examples

The Ising case has a variable $\sigma=\pm 1$ and a coupling of the form

$$K(\sigma,\sigma')=K_0+K\sigma\sigma'. \tag{3.24}$$

The basic sum (3.10) can be calculated quite easily if we write

$$e^{K(\sigma,\sigma')}\sim 1+\sigma\sigma'\tanh K.$$

Then we find a new coupling of the form (3.24) with a new coupling parameter K' given by

$$\tanh K'=(\tanh K)^2. \tag{3.25a}$$

Equation (3.25a) represents the effect of changing the lattice constant by a factor of 2. More generally, if the change is by a factor of λ, the new coupling is $K(\lambda a_0)$ given by

$$\tanh K(\lambda a_0)=[\tanh K(a_0)]^\lambda. \tag{3.25b}$$

As λ increases $K(\lambda a_0)$ decreases. This situation is represented by the phase diagram shown in Fig. 5b. There are two fixed points at $K^*=0$ and $K^*=\infty$. This phase diagram shows no critical point for any finite value of K. There is, after all, no phase transition for the one-dimensional Ising model. However, as K goes to infinity the model does show a quasicritical structure. [See Houghton and Kadanoff (1973) and Nelson and Fisher (1975).] For example, for large K, the recursion relation (3.25b) reads

$$K(\lambda a_0)=K(a_0)-\frac{1}{2}\ln\lambda \tag{3.26}$$

so that K is close to a fixed point, in that it is changing only logarithmically slowly with λ. Alternatively one can look at the correlation function, which has the form

$$\langle\sigma(x)\sigma(x')\rangle=(\tanh K)^{|x-x'|/a_0},$$

and see that the mass is given by

$$m=-(\ln\tanh K)/a_0=M(K)/a_0.$$

For large K, the dimensionless mass M becomes very small, i.e.,

$$M(K)\approx 2e^{-2K}. \tag{3.27}$$

But even though this mass can be small, the system shows no phase transition and all its properties are analytic functions in the region $-\infty<K<\infty$.

Most of the systems analyzed in statistical physics show phase diagrams which bear a qualitative similarity to either Fig. 5a or 5b. We could hope that the systems of interest for particle physics might fall into one of these two classes. However, more complex situations are conceivable. For example, let

$$K'=1+(K-1)[1-\mathrm{Re}\sqrt{1-K}\,]^{\lambda-1}. \tag{3.28}$$

Then all points for $K<1$ are in a weak coupling phase while each point for $K>1$ is a fixed point. Hence each of these $K>1$ points is a separate phase. We believe that such a possible existence of an infinite number of different phases is characteristic of the Baxter [see Baxter (1971 and 1972) and also Kadanoff and Wegner (1971)] model for $d=2$. It is also probably the kind of

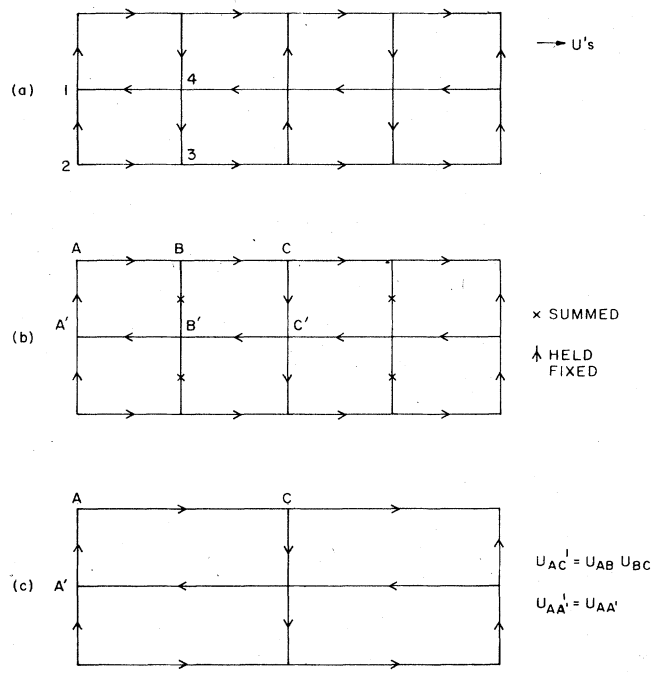

FIG. 7. Decimation on a two-dimensional gauge lattice.

behavior shown by the XY model at $d=2$. This phase diagram is shown in Fig. 5c.

E. Recursion relations for gauge symmetries at $d = 2$

Consider a two-dimensional problem on the square lattice drawn in Fig. 7a. Each arrow represents a string bit $U(x, x')$ labeled by the coordinates of its end points. Each $U(x, x')$ is an n by n matrix which obeys

$$U(x, x') = [U(x', x)]^{-1} = [U(x', x)]^{\dagger}.$$

The gauge invariance is represented by the transform

$$U(x, x') \to u(x)U(x, x')u^{-1}(x').$$

Here $U(x)$ is a matrix in the representation of the symmetry group. In principle, then, U and u are entirely different kinds of n by n matrices. However, if we demand that U forms a basis for a transitive representation of the symmetry, then we can always pick the set of possible U's to be identical with the set of transformation matrices, u.

In this case of a transitive representation, one can find the recursion relations for a gauge interaction on the basic plaquettes. We assume a gauge invariant action which is of the form of a sum over squares, i.e.,

$$A[U] = \sum_{\Gamma} J(U_{\Gamma}), \tag{3.29}$$

where the string loop U_{Γ} is defined as in Eq. (1.8). For example, the numbered square in Fig. 7a has

$$U_{\Gamma} = U_{\Gamma_{1234}} = U(x_1, x_2)U(x_2, x_3)U(x_3, x_4)U(x_4, x_1). \tag{3.30}$$

If we insist that J obeys

$$J(U) = J(U^{-1}) \tag{3.31a}$$

as well as the gauge invariance condition

$$J(U) = J(uUu^{-1}), \tag{3.31b}$$

then $J(U_{\Gamma})$ will be independent of the particular method of tracing the path Γ. In addition, the basic sum obeys the invariance conditions

$$\mathrm{tr}_u f(U) = \mathrm{tr}_u f(Uu) = \mathrm{tr}_u f(uU) \tag{3.32}$$

for any function, f. Since U and u are exactly the same kinds of matrices, $\mathrm{tr}_u f(uU)$ will have exactly the same meaning as $\mathrm{tr}_U f(U)$.

To find a recursion relation, sum over the string bits defined by the crosses in Fig. 7b. This sum will lead us to a new problem on the lattice shown in Fig. 7c. Since the latter has a lattice constant in the x-direction which is twice as large as the former, we describe the summation indicated as an x decimation. If we call the summation variables U_x, the new action is given by

$$e^{A'[U]} = \mathrm{Tr}_{U_x} e^{A[U]}. \tag{3.34}$$

Notice that each summation is completely independent of all of the others. Thus we can look at a basic pair of squares like the one with the labeled vertices in Fig. 7b and calculate the summation as

$$e^{J'[U]} = \mathrm{tr}_{U_{BB'}} \exp[J(U_{\Gamma_{ABB'A'}}) + J(U_{\Gamma_{BB'C'C}})] \tag{3.35}$$

with the U_{Γ}'s defined as in Eq. (3.30).

Migdal (1975a) made two important observations regarding the sum (3.35). The first is that since $J'[U]$ is a gauge invariant quantity, it can only depend upon the one gauge invariant product that one can form from the U's in the boxes. After $U_{BB'}$ is integrated out, this one product is

$$U_{\Gamma_{ABCC'B'A'}} = U_{\Gamma}.$$

Therefore J' must depend upon this product alone. Without loss of generality, we may evaluate (3.33) by setting all U's except U_{AB} and $U_{BB'}$ equal to the unit matrix. Thence we get a relatively simple recursion relation

$$e^{J'(U_{AB})} = \mathrm{tr}_{U_{BB'}} e^{J(U_{AB}U_{BB'})} e^{J(U_{BB'})}. \tag{3.36}$$

Migdal's second observation involved the connection between Eq. (3.36) and a special form of the one-dimensional recursion relation. Imagine a nearest-neighbor problem at $d=1$ with the matrices U as the basic statistical variables. Pick a coupling with the special form

$$K(U_1, U_2) = J(U_1 U_2^{\dagger}) \tag{3.37}$$

and a $J(U)$ which obeys Eqs. (3.31). For this problem, the nearest-neighbor recursion relation Eq. (3.10) will read

$$e^{K'(U_1, U_2)} = \mathrm{tr}_U e^{J(U_1 U) + J(U_2 U)}. \tag{3.38}$$

By changing the variable of integration in (3.38) according to

$$U \to U_2^{-1} U$$

we can see that K' is also of the form (3.37). Therefore we can set $U_2 = 1$ and write the nearest-neighbor recursion relation as

$$e^{J'(U_1)} = \mathrm{tr}_U e^{J(U_1 U) + J(U)}. \tag{3.39}$$

But, Eq. (3.39) is identical to the gauge theory recursion relation of Eq. (3.36)!

We conclude that *the one-dimensional recursion relation applies equally well to the two-dimensional gauge case, if we specialize the one-dimensional theory to couplings which obey Eqs. (3.37) and (3.31).* [For work which led up to Migdal's and included the special case $U=\pm 1$, i.e., the symmetry group Z_2, see Balian *et al.* (1974, 1975a, 1975b) and also Wegner (1971a).] Thus we can say that a change of lattice constant in the x-direction leads to a new coupling which is determined by exactly our previous recursion function

$$J' = R_0^{\lambda}[J]. \tag{3.40}$$

If we change the lattice constant in both directions we must compose two such transforms. According to Eq. (3.4) this composition is given by

$$J(\lambda a_0) = R_0^{\lambda^2}[J(a_0)]. \tag{3.41}$$

Equation (3.41) represents a complete formal solution of the two-dimensional gauge problem. Since no one-dimensional nearest-neighbor problems involving bounded variables and couplings have a phase transition, none of the $d=2$ gauge problems for compact symmetry groups will show a phase transition. They will all have a phase diagram like Fig. 5b.

For future reference, we list two results for the case in which the basic symmetry group is Z_2. In that case $U(x,x')$ is the Ising variable $U=\pm 1$ and the coupling takes the form (3.24)

$$J(U) = J_0 + JU. \tag{3.42}$$

Then, according to Eq. (3.25), the recursion relation takes the form

$$J(\lambda a_0) = (\tanh^{-1})[\tanh J(a_0)]^{\lambda^2} \tag{3.43}$$

which for large J reduces to

$$J(\lambda a_0) = J(a_0) - \ln\lambda. \tag{3.44}$$

Finally, if U_Γ is a product of U's which surround an area A, the average of U_Γ is

$$\langle U_\Gamma \rangle = \exp(-CA) \tag{3.45}$$

where the constant is

$$C = |\ln\tanh J|/a_0^2. \tag{3.46}$$

This rule that the average of U_Γ decreases exponentially with the area will be very important in what follows.

IV. PROPERTIES OF THE QUARK–STRING MODEL

This section is devoted to describing how the quark–string theory can lead to two different pictures of elementary particle phenomena, both of which seem to have a good experimental basis. These pictures are:

1. *Asymptotic Freedom.* This picture arose in some measure from the parton concept (Feynman, 1970; and Bjorken and Paschos, 1969) in which the properties of elementary particles are explained by treating them as weakly interacting Han–Nambu (1965) quarks. It was extended to the suggestion (Gross and Wilczek, 1973a; Politzer, 1973) that in the high-momentum transfer limit quark–quark interactions renormalized to zero.

2. *The infrared trap.* The experimental point is the fact that no free quarks have ever been observed. A picture which explains this fact is that quarks are bound, with infinite binding energy, into color singlet combinations [see 't Hooft (1972, 1974); Gross and Wilczek (1973b); Weinberg (1973)].

The renormalization group point of view suggests a way out of this dilemma presented by the apparent incompatibility of these two statements. How can quarks be at once weakly interacting and unobservable? They can be so if the qualitative nature of the couplings change in the different energy ranges (length scales) so that a free quark picture, which is asymptotically correct for small distances, becomes vastly wrong for large distances.

Thus our physical discussion will proceed in three stages. First, the trapping will be derived in the $J\to 0$ limit. Next, the freedom will be shown to be a consequence of the theory in the $J\to\infty$ limit. Finally, we shall argue that renormalizations can effectively connect these two limiting cases if only there is no critical fixed point for $0 \leq J < \infty$.

A. Arguments for trapping

All quark Green's functions must be invariant under gauge transformations. Thus, for example, the one-quark propagator $G(ifx;jf'x')$ must be unchanged under the transformation

$$G \to u_{ik}(x) G(kfx; k', f', x')(u^{-1})_{k'j}. \tag{4.1}$$

This invariance will only be possible if G is diagonal in its color indices and in x, i.e.,

$$G(ifx;jf'x') = \delta_{x,x'}\delta_{i,j}F(f,f'). \tag{4.2}$$

Hence this quark "propagator" in essence says that free quarks will not propagate, but instead will just disappear an instant after one tries to create them.

Similarly the gauge invariance implies that the two- and three-quark propagators only describe the motion of color singlet combinations in the form described, respectively, by Eqs. (2.20a) and (2.20b). For example, $G_3(123;1'2'3')$ has (for x_1 unequal to x_1' or x_2' or x_3') a piece which describes the propagation of the singlet combination which has the quantum numbers of the baryons. This piece only appears for

$$\begin{aligned} x_1 &= x_2 = x_3, \\ x_{1'} &= x_{2'} = x_{3'}, \end{aligned} \tag{4.3}$$

and has the form

$$G(123;1'2'3') = \epsilon_{i_1,i_2,i_3}\epsilon_{i_1',i_2',i_3'} \times F(f_1f_2f_3; f_1'f_2'f_3'; x_1 - x_{1'}) \tag{4.4}$$

In fact, the flavor indices were introduced with exactly the purpose of allowing such baryon propagation.

It is tempting, but wrong, to assert that results like (4.2) and (4.4) mean that free quarks are unobservable. A separated quark–antiquark pair might be observed if our measuring apparatus detected something different from these Green's functions. A more careful argument would have to be built upon the possibility of observing the energy carried by a free quark with some local de-

tector even if its color was not observable. The necessity for constructing this more careful argument becomes obvious if we notice that the same argument which gives the vanishing of $G(x_1, x_1')$ for $x_1 \neq x_1'$ can also be applied to electrodynamics for which G is the electron propagator. Hence if we take the above argument really literally, we would conclude that electrons were not observable either!

To make a more careful analysis, we consider a process in which a quark–antiquark pair is produced at the space–time point $x_0 = (0, 0, 0, 0)$ annihilated at $x = (0, 0, 0, t)$. In the meantime a quark is observed at the spatial point $r = (0, 0, \frac{1}{2}z)$ and an antiquark at $r = (0, 0, -\frac{1}{2}z)$. A heuristic picture of such a process is shown in Fig. 8a and is redrawn on the lattice in Fig. 8b.

These pictures suggest that it is not impossible to observe a separated quark and antiquark. But, is it likely? We follow Wilson (1975a, 1976a) in showing that for small J and K this process is so unlikely as to be unobservable when the quarks are separated over an appreciable distance in space (z) and time (t).

Let us examine the probability for the process in question. Each step in the path involves moving the quark via a term $K(1 - e\gamma)U$. Therefore to lowest order in K the probability takes the form

$$\text{Prob} \sim \frac{\text{Tr}_U e^{A_U} U_\Gamma(K)^{(2z+2t)/a_0}}{\text{Tr}_U e^{A_U}} . \tag{4.5}$$

Here U_Γ is the product of the U's over the closed path shown in Fig. 8. Since the average of each individual U vanishes

$$\text{Tr}_{U(x,x')} U(x, x') = 0,$$

one must expand A_U in Eq. (4.5) in order to get a nonzero result. The nonzero terms arise from structures like $\text{tr}_U UU^\dagger$ which are indeed nonzero.

The first nonzero result in perturbation theory arises

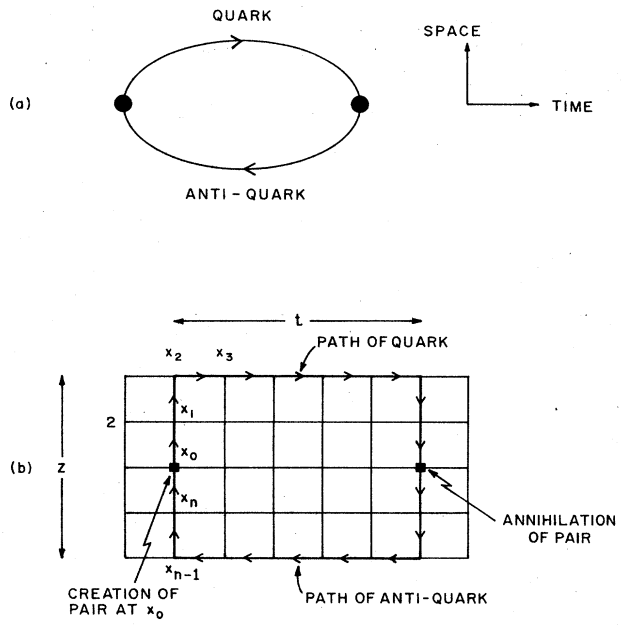

FIG. 8. Separation of a quark–antiquark pair.

when one takes a term in perturbation theory

$$JU^\dagger(1, 2)U^\dagger(2, 3)U^\dagger(3, 4)U^\dagger(4, 1)$$

for each and every square contained in the path of Fig. 8. Thence, for small J, we estimate the probability of this kind of process as

$$\text{Prob} \sim J^{zt/a_0^2} = e^{-(zt/a_0^2)|\ln J|} . \tag{4.6}$$

To interpret Eq. (4.6), remember that in the Euclidean space the probability of a process is proportional to the exponential of $-\Delta E \times \Delta t$, where ΔE is the energy of the process and Δt is the time during which that energy is available. We then see from Eq. (4.6) that the energy of separating two quarks by a distance z is proportional to the separation distance

$$\Delta E \sim z|\ln J|/a_0^2 . \tag{4.7}$$

If the energy grows linearly with the separation distance then clearly the quarks cannot become unbound.

Thus we have established the infinitely strong binding of quarks in lowest-order perturbation theory. It has arisen because $\langle U_\Gamma \rangle$, where U_Γ is defined as a product over a closed loop, is of the form

$$\ln \langle U_\Gamma \rangle \sim -\text{Area of loop} \tag{4.8}$$

for large loops.

But, Eq. (4.8) is only known to be true in perturbation theory. If Eq. (4.8) remains true in the exact theory then quarks will be trapped with an infinite binding energy. If this result of perturbation theory disappears in the exact theory quarks can become unbound. What will happen?

Our experience in statistical mechanics indicates that qualitative results of perturbation theory, like Eq. (4.8), will remain true for some range of couplings J, whenever J is too weak to produce a phase transition. Thus, if J is lower than some critical value J_c, we might expect Eq. (4.8) to remain true so that free quarks will be unobservable. We follow W. Bardeen and call this situation a baryon phase. In particular, in the baryon phase for large loops of linear dimension L (see Balian *et al.*, 1975a)

$$\ln \langle U_\Gamma \rangle \sim -L^2 \quad \text{(baryon phase)} . \tag{4.9a}$$

In the opposite limit of large J one can do an expansion in $1/J$ and find, to lowest order, that

$$\ln \langle U_\Gamma \rangle \sim -L \quad \text{(quark phase)} . \tag{4.9b}$$

As we shall see, in this situation, free quarks are definitely observable. More generally, one can imagine that

$$\ln \langle U_\Gamma \rangle \sim -L^{\delta(J)} \quad \text{(complex phase)} \tag{4.9c}$$

where perhaps δ depends continuously upon J.

To study the observability of quarks, we are then impelled to understand the phase transitions of the system as a function of J. [See Migdal (1975a).]

B. Asymptotic freedom

There is a host of theoretical work in which the behavior of elementary particles at high energy is de-

scribed by assuming that particles are made up of non-interacting or weakly interacting quarks. The renormalization group point of view and the quark–string model provide a beautiful description of how this might occur.

In the renormalization group picture, one can have different forms of the action to describe different energy ranges. In particular, many renormalizations are required to move from high-energy phenomena to lower-energy phenomena. As the renormalization proceeds, the effective interaction can change. Let us assume that the effective interaction which describes high-energy phenomena includes a very strong four-string interaction J. This strong interaction will tend to suppress fluctuations in $U(x,x')$. If U cannot fluctuate, then the quark interaction A_ψ is a pure two-body term. There is no higher-order quark interaction. Hence the quarks behave as free particles.

Wilson (1975a) has shown how to make this conceptual framework more explicit. The string–string interaction is

$$A_U = \frac{1}{2}\sum_{\text{plaquettes}} J\,\text{trace}(U_\Gamma + U_\Gamma^\dagger)\,. \tag{4.10}$$

For very large positive J, we would like to make the trace as large as possible. Since U_Γ is a unitary matrix, the largest possible value of the trace is achieved when U_Γ equals the unit matrix. A general form of $U(x,x')$ which will lead to $U_\Gamma = 1$ is

$$U(x,x') = u(x)u^{-1}(x') \tag{4.11}$$

for arbitrary gauge matrices $u(x)$. When J is large, this form of U might be expected to describe all short-ranged correlation phenomena reasonably well. However, we cannot expect to apply (4.11) to long-ranged correlations, e.g., to closed-path U_Γ's in which there are a very large number of links in the path. According to (4.11), U_Γ will be the unit matrix for all paths. But, in each step of the product, there will be some error arising from the imperfection in the approximation (4.11). If there is no phase transition, (that is, if we are not in the quark phase) these errors will accumulate after a large number of steps. Thus we expect to find that there is a characteristic distance ξ_0 and a characteristic number of steps n such that

$$\begin{aligned} U_\Gamma &\approx 1 \quad \text{for } na_0 \ll \xi_0\,, \\ &\approx 0 \quad \text{for } na_0 \gg \xi_0\,. \end{aligned} \tag{4.12}$$

For larger distances than ξ_0, fluctuations in U_Γ will be very important; for smaller distances they will be unimportant. For the small distances, we can expect to use a theory based upon Eq. (4.11).

This theory then has as its action

$$\begin{aligned} A = A_\infty = &-\sum_{xf} \bar\psi_f(x)\psi_f(x) \\ &+ \sum_{\substack{\langle xx'\rangle \\ f}} \bar\psi_f(x)u(x)\Sigma_{0f}(x,x')u^{-1}(x')\psi_f(x') \end{aligned} \tag{4.13}$$

where Σ_{0f} is given by Eq. (2.19). The sum over $U(x,x')$'s has been performed by setting them equal to the special values (4.11), but a sum over all u's, i.e., all gauges, and a sum over all ψ's remain to be calculated. Hence the remaining sums can be represented by

$$\text{Tr}_U \text{Tr}_\psi \to \text{Tr}_u \text{Tr}_\psi\,. \tag{4.14}$$

However, the remaining sums are trivial. They all can be performed by making the gauge transformation

$$\begin{aligned} &\psi(x) \to \psi'(x) = u(x)\psi(x)\,, \\ &\text{Tr}_\psi = \text{Tr}_{\psi'}\,. \end{aligned} \tag{4.15}$$

This transformation completely eliminates the u's from the action (4.13) and leaves one with a pure free fermion calculation. Hence the partition function and all other gauge invariant averages are exactly the same in the limit $J \to \infty$ in the free fermion case. In short for all gauge invariant quantities the quarks behave as free particles.

What about gauge dependent quantities, e.g., $G(1,1')$, which is, according to Eq. (4.13)

$$G(1,1') = \frac{\text{Tr}_u \text{Tr}_\psi e^{A_\infty[\psi,u]}\psi(1)\bar\psi(1')}{\text{Tr}_u \text{Tr}_\psi e^{A_\infty[\psi,u]}}\,. \tag{4.16}$$

After the transformation (4.15), we find

$$G(xf,x'f') = \frac{\text{Tr}_u \text{Tr}_\psi e^{A_\infty[\psi,1]}u(x)\psi_f(x)\bar\psi_f(x')u^\dagger(x')}{\text{Tr}_u \text{Tr}_\psi e^{A_\infty[\psi,1]}}\,. \tag{4.17}$$

The remaining trace over u vanishes unless $x = x'$. Thence only the gauge invariant piece at $x = x'$ is left in G, and this—once again—can be evaluated by the free fermion theory.

The end result is that the limit $J \to \infty$ gives the average over gauges of free fermion behavior.

C. Renormalization effects

So far this chapter has made two essential arguments:

(1) If J is sufficiently small, i.e., J is smaller than a critical coupling at which a phase transition occurs, then quarks will be bound together with an infinite binding energy. Call this critical value of the coupling J^*.

(2) In the limit $J \to \infty$, the quarks will show almost free particle behavior, except when they are separated by a very large distance. The larger the value of J, the greater the distance (measured in lattice constants) over which free particle behavior will be seen.

In short, trapping is characteristic of a theory with sufficiently weak couplings; freedom is characteristic of a theory with very strong couplings J. In nature, freedom and trapping are both observed, but in different regions of energy, i.e., on different distance scales.

These contrasting observations can be made to agree within the context of the very simplest renormalization group point of view. Imagine that under successive increases of lattice constant $a_0 \to \lambda a_0$, J continually decreases, and that after many renormalizations J approaches zero. Thus, no matter how large J is initially, a sufficient number of renormalizations will bring it close to zero. This kind of behavior is characteristic of systems which show no phase transition. (See Fig. 5b.) Then, for small distance scales, we can have an action with very large J—i.e., asymptotic freedom—while we always remain in the baryon "phase." There is no quark "phase," so unbound quarks cannot be observed.

Hence our contrasting observations of freedom and trapping will be consistent if there is no phase transition in the four-dimensional system of quarks and strings, no matter how large J might be.

These observations can be expressed graphically by redoing Fig. 5 as in Fig. 9. In the latter, we have indicated the physical nature of the different phases which arise. The arrows on the lines show the directions of charge of the couplings when a_0 *decreases*. An arrow going toward $g=0$ (or $J=\infty$) shows asymptotic freedom. Only the diagram without a phase transition (9b) is consistent with asymptotic freedom. In this diagram all values of the coupling (save $g=0$) put the system in the baryon phase, and therefore show quark trapping. In the other two cases, the existence of quark trapping is a function of coupling and only occurs for sufficiently small J.

If the gauge theory has the same kind of recursion equations as the Z_2 theory in two dimensions, then both freedom and trapping would be possible. Equation (3.43) implies that the phase diagram is the same as in Fig. 9b while Eq. (3.45) shows a correlation function like Eq. (4.9a) and hence trapping. In fact, Gross and Wilczek (1973) and Politzer (1973) have analyzed the renormalization structure of the coupled quark–string theory. They used the continuum form described in Sec. II.E and concluded that at $d=4$ if there were not too many quark flavors (fewer than 17) there would be no phase transition near $J=\infty$.

We shall try to follow a similar line of argument for the lattice version of the theory.

V. APPROXIMATION METHODS FOR LATTICE SYSTEMS

To make further progress, one needs approximation techniques. In this section, we will describe some approximation techniques borrowed from statistical mechanics.[1] They are all designed to produce effective calculations of a "free energy" $F[A]$ or partition function $Z[A]$ defined via

$$-F[A]=\ln Z[A]=\ln \mathrm{Tr}_\sigma e^{A[K,\sigma]} \ . \tag{5.1}$$

In statistical applications, one finds that the approximations described here are especially useful and accurate in calculations of critical indices. They have not yet been extensively employed for correlation functions. Hence we have no experience which would inform us about how accurate mass calculations might be.

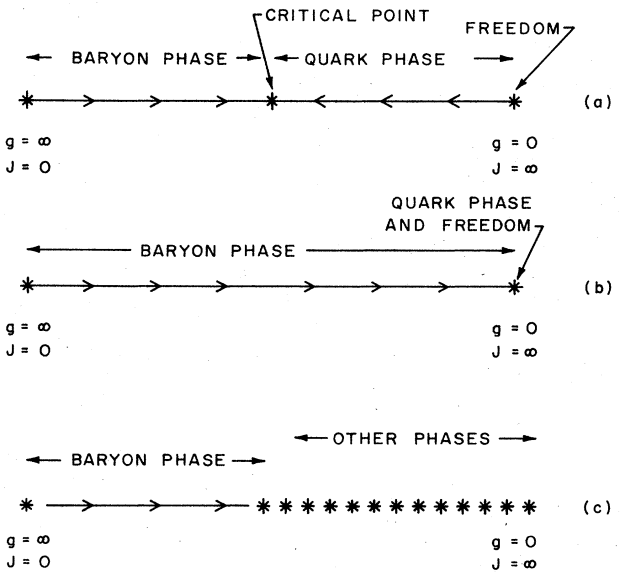

FIG. 9. Possible structures of phases for the string system. The arrows show the flow of couplings for decreasing lattice constant. We hope case *b* appears in the quark–string theory.

A. Lower bound approximations

Start from the exact recursion relation for the action defined by

$$e^{A'(\mu)}=e^{A[K',\mu]}=\mathrm{Tr}_\sigma e^{T(\mu,\sigma)+A[K,\sigma]} \ . \tag{5.2}$$

Here, $T(\mu,\sigma)$ is normalized so that

$$\mathrm{Tr}_\mu e^{T(\mu,\sigma)}=1 \ . \tag{5.3}$$

Hence the free energy or partition function defined from A' is identical to that defined from A, i.e.,

$$F[A']=F[A] \ . \tag{5.4}$$

Unfortunately, one cannot calculate the sum in Eq. (5.2). To circumvent this difficulty, we define an approximate calculation that we can indeed perform. We add to the exponent in (5.2) an error term $\Delta(\mu,\sigma)$ which makes the sum calculable. Then we find an approximate recursion equation

$$A_a'(\mu)=\ln \mathrm{Tr}_\sigma e^{T(\mu,\sigma)+A[K,\sigma]}e^{\Delta(\mu,\sigma)} \ . \tag{5.5}$$

This approximate action can be described in terms of some new coupling functions K', which are some functional of the coupling functions. This form of the approximate recursion relation is then written

$$K'=R_a[K] \ . \tag{5.6}$$

How do we choose a good approximation of this nature? More specifically, given several possible choices of $\Delta(\mu,\sigma)$, how can we choose the one which is "smallest" and thus generates the smallest possible error?

One guide comes from a variational principle—or rather an inequality. This inequality requires the following conditions:

(a) $\mathrm{Tr}_\sigma e^{T(\mu,\sigma)+A[K,\sigma]}$ is a sum with positive semidefinite weights.

(b) $\Delta(\mu,\sigma)$ is real.

(c) The average of Δ is zero, i.e.,

$$\mathrm{Tr}_\mu \mathrm{Tr}_\sigma e^{T(\mu,\sigma)+A[K,\sigma]}\Delta(\mu,\sigma)=0 \ . \tag{5.7}$$

Under these conditions, the free energy generated from A_a' is smaller than the true free energy, i.e., instead of

[1] The particular style of lattice renormalization approximation which we shall describe in this chapter, the potential-moving method, was developed by Kadanoff (1975) and Kadanoff, Houghton, and Yalibik (1976). Earlier lattice calculations include Niemeijer and van Leeuwen (1973 and 1974), Wilson (1975b), Nauenberg and Nienhuis (1974, 1975), and Houghten and Kadanoff (1973).

Eq. (5.4) we have

$$F[A_a] < F[A'] = F[A] \ . \tag{5.8}$$

To prove Eq. (5.8), one defines

$$F(\lambda) = -\ln \mathrm{Tr}_\mu \ \mathrm{Tr}_\sigma e^{T(\mu,\sigma)+A[K,\sigma]+\lambda\Delta(\mu,\sigma)} \ . \tag{5.9}$$

Then $F(0)$ is the exact free energy, $F(1) = F[A'_a]$. By virtue of Eq. (5.7)

$$\frac{dF}{d\lambda}(\lambda)\bigg|_{\lambda=0} = 0 \ . \tag{5.10}$$

Also,

$$\frac{d^2F}{d\lambda^2} = -\langle(\Delta - \langle\Delta\rangle_\lambda)^2\rangle_\lambda \tag{5.11}$$

where $\langle\ \rangle_\lambda$ is an average with weight $\exp(T + A + \lambda\Delta)$. If the weight is positive, the second derivative is negative and the theorem is proved.

Thus, from all possible Δ's, we choose that Δ which maximizes the approximately calculated free energy and we then get the "best" possible answer. In the meantime, the average of the squared fluctuations in Δ, as defined by integrating Eq. (5.11) over λ between 0 and 1, has been minimized.

The first problem is to find a Δ which obeys Eq. (5.7). To do this imagine any set of local variables $a_i(x)$. For example, $a_i(x)$ might be $\sigma(x)\sigma(x + \hat{e}_1 a_0)$. The labels i on $a_i(x)$ distinguish among different kinds of inequivalent variables; the labels x describe equivalent variables at different positions. Then if

$$\Delta(\mu,\sigma) = \sum_i c_i(x) a_i(x) \ , \tag{5.12}$$

where $c_i(x)$ is some set of coefficients independent of μ and σ which obeys

$$\sum_x c_i(x) = 0 \quad \text{for all } i \ , \tag{5.13}$$

then Eq. (5.10) will certainly be satisfied because, at $\lambda = 0$, the average of $a_i(x)$ is independent of x.

If we consider $a_i(x)$ to be in effect bits and pieces of the action $A[K,\sigma]$, then the net effect of $\Delta(\mu,\sigma)$ is to add something to the action at some points and subtract something at others. The condition (5.13) says that we are allowed to add and subtract such couplings within the variational constraint if we just demand that for every bit of strength we add at one set of points we make sure we subtract an equivalent total strength at other points.

More simply stated: the variational principle allows us to move potential terms from one set of bonds in the lattice to equivalent bonds but not to increase or decrease the total amount of any type of bond.

This potential moving method will permit us to conduct approximate recursion calculations in a controllable fashion. The basic technique is to use the potential moving to move hard-to-handle bonds into a location where their effect may be taken into account.

B. Migdal approximation

To see this approximation technique in its simplest form, we follow Kadanoff (1976) and consider the derivation of an approximation similar to that employed by Migdal (1975a, 1975b). We start with variables $\sigma(x)$ and nearest-neighbor bonds in the $x, y, z, \ldots$ directions. Thus the action is

$$A[K,\sigma] = \sum_{x\alpha} K_\alpha(\sigma(x), \sigma(x + \hat{e}_\alpha a_0)) \ .$$

The label α on K_α distinguishes the different bonds in the different directions.

Now we employ a recursion calculation in which the new variables $\mu(x)$ are defined to be exactly the same as the old variables $\sigma(x)$ on a fraction $1/\lambda$ of the lattice sites; i.e.,

$$\mu(x) = \sigma(x) \quad \text{for } x = (\lambda n_1, n_2, n_3, \ldots)a_0 \ .$$

The remaining σ's are summation variables. (See Figure 10.)

To make the summation possible we pick $\Delta[\sigma]$ to be

$$\Delta(\sigma) = \sum_x a_\alpha(x) K(\sigma(x), \sigma(x + \hat{e}_\alpha a_0 1)) \tag{5.14}$$

and take a_α to be exactly zero if $\alpha = 1$. For the remaining bonds, we choose

$$a_\alpha(x) = -1 \tag{5.15a}$$

when $\sigma(x)$ and $\sigma(x + \hat{e}_\alpha a_0)$ are summation variables and

$$a_\alpha(x) = (\lambda - 1) \tag{5.15b}$$

for the bonds which connect two μ variables. The net effect of (5.15) is to move $(\lambda - 1)K_\alpha$ bonds ($\alpha = 2, 3, \ldots$) from summation bonds to the bonds between two μ variables.

Now the summation over σ is easy to perform. We sum as before and find a new x coupling

$$\mathrm{K}'_1 = R_0^\lambda(\mathrm{K}_1) \ . \tag{5.16a}$$

The other bonds are the sum of the bond that was always present and the $\lambda - 1$ bonds that were moved

$$\mathrm{K}'_\alpha = \lambda K_\alpha, \quad \alpha = 2, 3, \ldots \ . \tag{5.16b}$$

Migdal's result now emerges if we consider the effect of successive $x, y, z, \ldots$ decimations. All these decimations together change the lattice constant from a_0 to λa_0. Successive applications of Eqs. (5.16) imply that the x coupling constant after the change is

$$\mathrm{K}_1(\lambda a_0) = \lambda^{d-1} R_0^\lambda(\mathrm{K}_1(a_0)) \tag{5.17a}$$

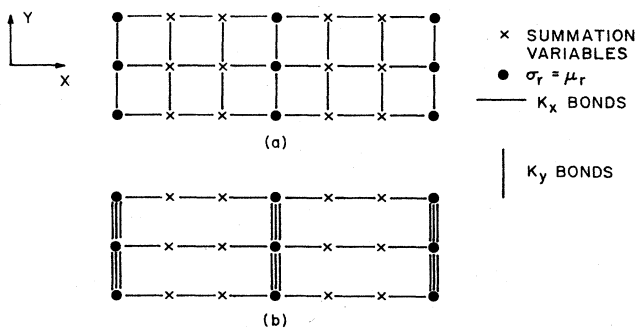

FIG. 10. Potential moving in the Migdal approximation depicted for $d = 2$ and $\lambda = 3$. Part (a), before the potential moving; part (b), afterward.

while all the other coupling constants obey

$$K_\alpha(\lambda a_0) = \lambda^{d-\alpha} R_0^\lambda(\lambda^{\alpha-1} K_\alpha(a_0)) \quad . \tag{5.17b}$$

Here $\alpha = 2, 3, 4\ldots$ is the index which describes the coupling constant in the $y, z, t, \ldots$ direction.

Equation (5.17a) is exactly the same as Migdal's result. We shall discuss the consequences of Eq. (5.17) below.

The recursions (5.16) and (5.17) will generate a lower bound on the free energy for all problems with positive semidefinite statistical weights. Unfortunately, we need to apply them to fermion systems in which the positivity is lost. Hence one of our main controls on the accuracy of the approximation is lost too.

When is this kind of approximation likely to be accurate? There are three limits in which we might expect reasonable results from Eqs. (5.16):

(1) As $d \to 1$. It is exact at $d = 1$ and the number of bonds to be moved goes to zero as $d \to 1$.

(2) For weak couplings $K_\alpha(\sigma, \sigma')$ for all $\alpha > 1$. Then the errors in Z must be of order K_α^2.

(3) For very strong couplings $K(\sigma, \sigma')$ which force $\sigma \approx \sigma'$ with a very high probability. Then the donor and the recipient sites are likely to have very closely the same values of σ and σ' so that the effect of the motion is quite small.

C. Migdal approximations for the gauge system

The same general scheme can be applied to the gauge system, with interaction $J(U_\Gamma)$ on square plaquettes. A three-dimensional version of this situation is shown in Fig. 11. The recursion is a decimation in which the lattice constant in the "1" direction changes by a factor of $\lambda = 3$. The new variables μ are shown on the figure.

By using the same calculational method as in the two-dimensional case, we could do the sums if only the couplings J_{12} and J_{13} were present. However, couplings J_{23}, like those in plaquette D, prevent us from calculating the correlated summations of the neighboring variables. We therefore move all couplings like D which link together summation variables to plaquettes like C where they give no difficulty. The net result is a new coupling on the plaquettes C of the form

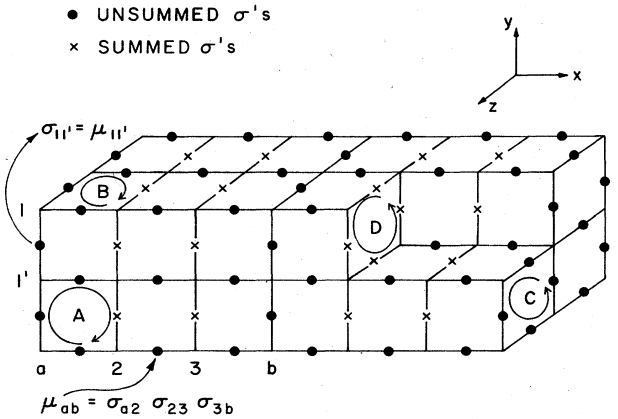

FIG. 11. The three-dimensional Migdal recursion for a gauge system.

$$J'_{\alpha\beta} = \lambda J_{\alpha\beta} \quad \text{for } \alpha \neq 1 \text{ and } \beta \neq 1 \quad . \tag{5.18a}$$

Now we are able to do the summations. Just as in two dimensions these sums can be calculated as a special case of the one-dimensional nearest-neighbor recursion. In direct analogy to Eq. (3.40), we find

$$J'_{\alpha\beta} = R_0^\lambda[J_{\alpha\beta}] \quad \text{for } \alpha = 1 \text{ or } \beta = 1 \quad . \tag{5.18b}$$

Once again, the recursions may be presumed to be accurate if the couplings (5.18a) are weak, if the couplings (5.18b) are strong, or if $d \to 2$.

To derive the full consequences of this approximation, consider the effect of successive decimations in the directions $1, 2, \ldots, d$ upon $J_{\alpha\beta}$. Take the spatial indices $\alpha\beta$ to be such that $\alpha < \beta$ and $\beta = 2, 3, \ldots, d$. Then Eqs. (5.18) imply the Migdal-style recursion relations

$$J_{\alpha\beta}(\lambda a_0) = \lambda^{d-\beta} R_0^\lambda[\lambda^{\beta-\alpha-1} R_0^\lambda[\lambda^{\alpha-1} J_{\alpha\beta}(a_0)]] \quad . \tag{5.19}$$

An especially interesting example of this recursion occurs if we take $\alpha = 1, \beta = d/2 + 1$. In that case, the recursion (5.19) is a composition of two identical steps. Each step can be described by an effective recursion

$$J \to J' = R_1^\lambda[J] = \lambda^{d/2-1} R_0^\lambda[J] \quad . \tag{5.20}$$

In terms of this effective recursion function R_1^λ, Eq. (5.19) may be written as

$$J_{\alpha\beta}(\lambda a_0) = R_1^\lambda[R_1^\lambda[J_{\alpha\beta}(a_0)]] . \tag{5.21}$$

But, if we view the change in lattice constant $a_0 \to \lambda a_0$ as taking place in two steps

$$a_0 \to \sqrt{\lambda} a_0 \to \sqrt{\lambda}\,(\sqrt{\lambda} a_0) \quad ,$$

then we can consider R_1^λ to be the recursion function for a single step. In this way, we reinterpret (5.21) as

$$\begin{aligned} J_{\alpha\beta}(\sqrt{\lambda} a_0) &= R_1^\lambda[J_{\alpha\beta}(a_0)] \\ &= \lambda^{d/2-1} R_0^\lambda[J_{\alpha\beta}(a_0)] \quad . \end{aligned} \tag{5.22}$$

The net result of this argument is that a particular coupling function $J_{1, d/2+1}$ in the gauge case obeys exactly the same style of recursion as Eq. (5.17a)—which describes the nearest-neighbor case. Thus, for these particular couplings, *the recursion in the d-dimensional gauge case is just the same as the recursion in the d/2-dimensional nearest-neighbor situation* (Migdal, 1975a). For example, the four-dimensional gauge case has some recursions which are exactly the same (in the Migdal approximation) as the two-dimensional nearest-neighbor situation. Thus, if the Migdal approximation is accurate, the four-dimensional gauge case can be understood in terms of the much simpler problem of nearest-neighbor interactions in two dimensions.

But the Migdal approximation *is* accurate when the couplings are strong. And, it is exactly this strong-interaction limit which is significant for discussing whether or not the quark–string system has a phase transition. Therefore the Migdal approach can be very useful for determining whether the Wilson model does in fact give both asymptotic freedom and quark trapping.

D. The Ising example (Migdal, 1975b)

To illustrate the considerations of this chapter, consider a nearest-neighbor Ising model. In this case, the

couplings K obey the one-dimensional recursion relation

$$K' = R_0^{\lambda}(K) \ ,$$

where, according to Eq. (3.25),

$$\tanh K' = (\tanh K)^{\lambda} \ .$$

Therefore, from Eq. (5.17a), the x-direction coupling for the system with nearest-neighbor interactions obeys

$$K_x(\lambda a_0) = \lambda^{d-1} \tanh^{-1}[\tanh K_x(a_0)]^{\lambda} \ . \tag{5.23}$$

Equation (5.23) expresses the result of one recursion in which the lattice constant increases by a factor of λ.

The case $d \to 1$ is especially interesting. In this situation, there is a fixed point for large values of K_x. When $K_x \gg 1$, Eq. (5.23) implies

$$K_x(\lambda a_0) = \lambda^{d-1}[K_x(a_0) - \ln\lambda/2] \ . \tag{5.24}$$

Then $d \to 1$, there is a fixed point at $K_x = K^*$, where K^* goes to infinity as $d \to 1$, in the form

$$2K^* = 1/(d-1) \ . \tag{5.25}$$

Notice also that the recursion relation (5.24) has a critical index

$$y = d - 1 \tag{5.26}$$

which goes to zero.

Clearly one dimension is a very special limit of the Ising model. In this limit, the critical couplings go to infinity. For $d < 1$, the critical point disappears entirely. We describe a value of the dimension at which the critical couplings go to infinity and then the phase transition disappears as a *lower critical dimension* d_L^*. For the nearest-neighbor Ising model $d_L^* = 1$.

A very similar analysis can be applied to the gauge-style coupling. In this case, for Ising variables $U(x,x') = \pm 1$, the basic coupling on a plaquette takes the form (3.42). For each pair of spatial indices (α,β) there is a single coupling $J_{\alpha\beta}$, which is directly analogous to the K_α described above.

Let us apply Eq. (5.22) to the case in which $J_{\alpha\beta}$ is very large. Then by using the same calculation which led from Eq. (5.23) to Eq. (5.24), we find a recursion relation for the gauge case

$$J_{\alpha\beta}(\lambda a_0) = \lambda^{d-2} J_{\alpha\beta}(a_0) - \tfrac{1}{2} \ln(\lambda^{d-\beta} + \lambda^{d-\alpha-1}) \ . \tag{5.27}$$

This recursion relation shows a lower critical dimensionality d_L^* equal to 2. Near the lower critical dimensionality there is a fixed point at very strong values of the coupling,

$$J_{\alpha\beta}^* = \frac{\lambda^{2-\beta} + \lambda^{1-\alpha}}{2(d-2)} \ , \tag{5.28}$$

and a critical index

$$y = d - 2 \ . \tag{5.29}$$

We now make an analogy between these results for the group Z_2 near $d = 2$ and the desired results for the group SU_3 at $d = 4$. Assume for a moment that nature had one space and one time dimension and had a "color" symmetry Z_2. For $d \approx 2$, and strong coupling, the Migdal recursion relations of the section would be reliable. They would show a phase transition, i.e., the structure of Fig. 9a, for $d = 2 + \epsilon$, with $\epsilon > 0$. However, at $\epsilon = 0$ there would be no phase transition and the phase diagram would look like Fig. 9b. Hence the theory would show both quark trapping and asymptotic freedom. The theory, however, would be far from trivial since a perturbation theory in g would only work in the quark phase. Since this phase only exists at $g = 0$, it is very likely that the radius of convergence of this perturbation theory would be zero.

Two dimensions is special for a Z_2 gauge theory because $d_L^* = 2$ for this theory. According to the Migdal approximation, this value of the lower critical dimensionality is in turn derivable from the fact that $d_L^* = 1$ for the Z_2 nearest-neighbor coupling. In fact, the general result is that for a given representation of a given symmetry, the lower critical dimensionality of the gauge theory, (d_L^*) gauge, is related to the lower critical dimensionality of the corresponding nearest-neighbor theory, $(d_L^*)_{\text{global}}$:

$$(d_L^*)_{\text{gauge}} = 2(d_L^*)_{\text{global}} \ . \tag{5.30}$$

Now let us turn to the consideration of the interesting case, one in which $U(x,x')$ is the fundamental representation of a particular Lie group. According to Migdal (1975a) and to the more accurate calculations of Brezin and Zinn-Justin (1976) and of Polyakov (1975), the lower critical dimension for the global symmetry in this situation is $(d_L^*)_{\text{global}} = 2$. Therefore the gauge case shows a lower critical dimensionality at $d = 4$!

In fact, one can make a slightly (but crucially) stronger statement. For all the regular representations of compact semisimple Lie groups (e.g., SU_3), according to calculational methods of Brezin and Zinn-Justin (1976), there will be no phase transition in the $d = 2$ nearest-neighbor case and consequently no phase transition in the $d = 4$ gauge case. Therefore, if the string–string interactions dominated the quark–string interaction, the strings would show no phase transition. In this case, we would indeed obtain the desired phase diagram, i.e., Fig. 9b. We would then have a theory containing as we wish both asymptotic freedom and trapping. It would be near-critical (since $d_L^* = 4$), so it might even be Lorentz invariant. What could be more satisfying!

But there is more. Quantum electrodynamics can be expressed in this same language, with a symmetry group U_1, an Abelian group. The corresponding nearest-neighbor problem is called the XY model. For this special case, all proofs of the non-existence (Polyakov, 1975; Brezin and Zinn-Justin, 1976) of a phase transition at d_L^* fail. In fact, there are plausible arguments which suggest the existence of a phase transition at d_L^*, perhaps of the nasty nature shown by the phase diagram 9c. But this phase transition is quite desirable from the point of view of experiment. It permits the observation of electrons and positrons and also permits the theory to not be asymptotically free.

However, the reader should be aware that all of the analysis of this section depends on the idea that the strings determine their own interactions with no help from the quarks. In the Migdal approximation, this

idea is valid. Different couplings can be moved independently of one another in this approximation. But, in the real world, the quarks may enter into the string recursion relations in an essential way and thereby invalidate the reasoning outlined here. In fact, the analyses of Gross and Wilczek (1973) and Politzer (1973) show that as the number of flavors increases, the quarks in fact do produce a phase transition in the string system.

VI. FIXED POINT PHENOMENOLOGY

Before analyzing the Migdal-style recursion relations, we go through a digression in which we discuss the physical interpretation of recursion equations and their fixed points. We are particularly interested in surveying the meaning of a lower critical dimensionality and of a marginal critical index, i.e., an index y which goes to zero. The basis for this section is, of course, Wilson's (1970) recognition of the importance of fixed points and his concern (1975b) with marginal variables. The first explicit treatment of such variables is in Kadanoff and Wegner (1971). The beautiful mathematical formulation of fixed point behavior is largely due to Wegner (1972 and 1973).

A. A first example

For a system with the Ising model symmetry $\sigma_r \to -\sigma_r$ one can follow Wilson and Fisher (1972) and draw a plot of critical indices versus dimensionality. For example, of one looks at the critical index y_1 which appears in the recursion relation for the deviation of the coupling strength from its critical value

$$(K_1 - K^*)' = \lambda^{y_1}(K_1 - K^*) \ ,$$

one can draw a picture like Fig. 12. Here the physical quantity of interest is not really y_1 but instead $\nu = 1/y_1$ which is defined by the statement that the inverse coherence length (i.e., mass) behaves near critically as

$$\xi^{-1} = m \sim (K_1 - K^*)^{\nu} \ . \tag{6.1}$$

Figure 12 contains a description of two different fixed points. (There are actually many, many more). There are the nontrivial Ising-like fixed point and the Gaussian fixed point. Notice that these cross at an upper "critical" dimensionality (namely $d = 4$) and that the Ising fixed point disappears at the lower "critical" dimensionality ($d = 1$).

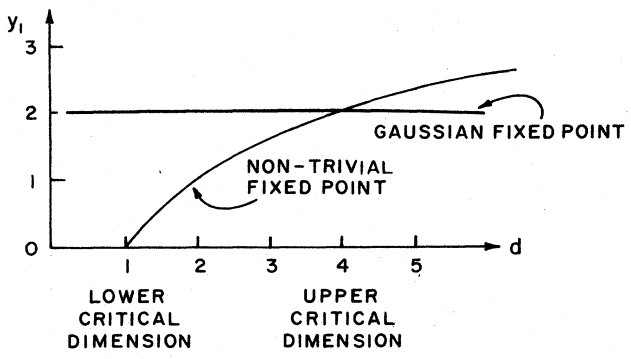

FIG. 12. Critical index y_1, plotted against dimensionality.

One of the major purposes of a phenomenology of fixed points is to explain and describe the behavior in the neighborhood of these critical dimensionalities.

B. Wegner variables, free energy, and scaling

To form this explanation, we begin from a recursion relation

$$K' = R^{\lambda}[K] \ .$$

This expresses the new coupling functions K' as a function of the old ones K, when the lattice constant is changed by a factor of λ, i.e.,

$$a_0 \to \lambda a_0 \ . \tag{6.2}$$

If we express the coupling functions K in terms of a set of coupling parameters $\mathbf{K}$ then the basic recursion for the parameters may be written as

$$\mathbf{K}' = R^{\lambda}(\mathbf{K}) \ . \tag{6.3}$$

Once we have picked the form of the transformation function $T(\mu,\sigma)$ the recursion relation (6.3) is well-defined. It may have one or several or a continuum of fixed points. We pick a given fixed point K^*, which obeys

$$K^* = R^{\lambda}(K^*) \ . \tag{6.4}$$

We expand about this fixed point by writing $K_\alpha = K_\alpha^* + h_\alpha$ with $h_\alpha \ll 1$. Then, to first order in h, Eq. (6.3) may be expanded as

$$h'_\alpha = \sum_\beta B_{\alpha\beta} h_\beta \tag{6.5}$$

where

$$B_{\alpha\beta} = \frac{\partial R_\alpha^{\lambda}(K)}{\partial h_\beta}\bigg|_{K=K^*} \ . \tag{6.6}$$

Then we form the eigenvectors of Eq. (6.5) by writing the linear combinations

$$h'_i = \sum_\alpha U_{i\alpha} h_\alpha \tag{6.7}$$

which diagonalize $B_{\alpha\beta}$. These linear combinations then obey

$$h'_i = \lambda^{y_i} h_i \ . \tag{6.8}$$

Here we have written the eigenvalue of $B_{\alpha\beta}$ as λ^{y_i}. This approach then provides a full description of all first-order deviations from criticality and from the fixed point. But one can go further. One can construct the Wegner (1972) functions

$$h_i = h_i(\mathbf{K} - \mathbf{K}^*) \ ,$$

which have a power series expansion in the form

$$h_i = \sum_\alpha U_{i\alpha} h_\alpha + \sum_{\alpha\beta} U^2_{i\alpha\beta} h_\alpha h_\beta + \cdots$$

when $U, U^2, \ldots$ are chosen correctly the Wegner functions obey Eq. (6.8) for all values of h_α. Wegner has described how to perform this construction. It will work whenever the y_i are incommensurate, i.e., whenever

$$\sum_i y_i m_i \neq 0 \tag{6.9}$$

for any set of positive or negative integers m_i.

The most significant kind of failure of Eq. (6.9) occurs when one of the y_i equals zero. But, for the moment, let us ignore this case of a marginal variable and all other failures of Eq. (6.9) and assume that Eq. (6.8) is true arbitrarily far away from the critical point. This assumption enables us to analyze the behavior of any quantity with a simple scale dependence. Consider, for example, an inverse coherence length or mass defined by

$$\xi^{-1} = m = M(K)/a_0 \ . \tag{6.10}$$

Express M as a function of the Wegner variables h_i. Then the scale invariance of the physical mass implies that $M(h)$ obeys

$$M(\lambda^{y_1}h_1, \lambda^{y_2}h_2, \dots) = \lambda M(h_1, h_2, \dots) \ . \tag{6.11}$$

Therefore M must obey the homogeneity requirement

$$M(h_1, h_2, \dots) = h_1^{1/y_1}\bar{M}(h_2/h_1^{\delta_2}, h_3/h_1^{\delta_3}, \dots) \tag{6.12a}$$

with

$$\delta_i = y_i/y_1 \ . \tag{6.12b}$$

Here $\bar{M}$ is called a scaling function.

Some of the h_i's satisfy Eq. (6.12) in a very simple way: they do not appear in this or any other quantity describing the critical behavior. These redundant variables may be safely ignored.

The remaining variables may be arranged into three categories according to the sign of y_i. For the three possible cases we say that

$$\text{if} \begin{cases} y_i > 0 \\ y_i = 0 \\ y_i < 0 \end{cases} \text{ then } h_i \text{ is } \begin{cases} \text{relevant} \\ \text{marginal} \\ \text{irrelevant} \end{cases} .$$

For most purposes one can neglect the irrelevant variables. In statistical physics they are neglected because one is interested in long-ranged correlations which may be studied as $\lambda \to \infty$ and in this limit they vanish. In particle physics they are set equal to zero so that the action may have a well-defined limit as $\lambda \to 0$, i.e., for zero lattice constant.

If we throw away these variables, then the mass M depends upon only a few different variables, the small class for which $y_i \geq 0$. This mass is then a universal function of any of a few variables. This universality is a justification for studying model problems since these models can well be sufficient for establishing all the important functional dependence of such quantities as $\bar{M}$.

C. Universality failure and marginal variables

The analysis of the previous section fails whenever Eq. (6.9) fails. The most interesting case of this kind occurs when we have a marginal variable, i.e., one with $y_i = 0$.

This marginality may take several forms. The simplest case conceptually is the one in which the marginal Wegner variable, $h_{\rm mar}$ simply obeys

$$h'_{\rm mar} = h_{\rm mar} \tag{6.13}$$

for a whole range of values of $h_{\rm mar}$. Then we have a case in which the fixed point K^*, the scaling function $\bar{M}$, and all the other critical indices y_i can be continuous functions of this one (or perhaps several) marginal variables. This case is depicted in Fig. 3.4c and is realized in Baxter's $d=2$ solution of the 8-vertex model. Luther and Scalapino (1976) have suggested that there are two marginal variables of this type in the two-dimensional XY model.

Another kind of failure occurs when the recursion equation takes the form

$$h'_{\rm mar} = h_{\rm mar} + b \ .$$

In one sense, this equation says that there is no fixed point. In another sense, we "almost" have a fixed point. If, for example, $\lambda = 2$ then a transition through n steps changes a_0 into $2^n a_0$ and gives

$$h_{\rm mar}(2^n a_0) = h_{\rm mar}(a_0) + bn$$

or

$$h_{\rm mar}(\lambda a_0) = h_{\rm mar}(a_0) + b \log_2 \lambda \ . \tag{6.14}$$

This very slow motion away from a fixed point is characteristic of the behavior at the lower critical dimensionality at which the phase transition just barely disappears.

Notice that Eq. (6.14) exactly describes the Ising model recursion at its lower critical dimension, i.e., Eq. (3.26).

A third conceivable behavior is given by the recursion relation

$$h' = h + bh^2 \ .$$

If h is small, and the basic step is 2, this equation may be written as

$$h(2^{n+1}a_0) = h(2^n a_0) + b[h(2^n a_0)]^2 \ .$$

Let $h_n = h(2^{n+1}a_0)$ and assume h varies slowly with n. Then

$$\frac{dh_n}{dn} = bh_n^2 \ ,$$

so that

$$h_n = \frac{1}{h_0^{-1} - bn} \ ,$$

or

$$h(\lambda a_0) = \frac{h(a_0)}{1 - h(a_0) b \log_2 \lambda} \ . \tag{6.15}$$

If $bh(a_0)$ is negative, then after many iterations $h(\lambda a_0)$ will very slowly approach zero. If, on the other hand, $bh(a_0)$ is positive, after a large number of iterations $h(\lambda a_0)$ will get very far from zero, and we will approach an entirely different coupling structure.

This type of behavior is characteristic of a situation in which there are two fixed points with very similar physical behavior which approach one another. For example, at dimensionality 4, the Gaussian fixed point and the Ising-type fixed point become essentially identical in all of the critical indices and critical behavior.

The variable which takes you from one of these fixed points to the other is h and its slow charge reflects the degeneracy between these two solutions. The logarithm in Eq. (6.15) is reflected in logarithms which show up in the thermodynamic behavior at these "upper" critical dimensionalities.

D. Stability

To see this behavior in more detail, we plot in analogy to Figure 12 the second largest critical index for a variable which is even in σ. This plot is Fig. 13.

From this figure it follows that the nontrivial fixed point has an extra thermodynamically relevant variable h_2 at dimensionalities above four, while the Gaussian fixed point has this extra variable at dimensionalities below four.

To see the consequences of these extra variables, we should turn to an examination of the standard relevant variables. These include:

(1) h_0, which is just a constant term in the action of the Hamiltonian and uninteresting.

(2) h_1, which physically represents a deviation of the coupling from its critical strength, i.e., is $T-T_c$ or $-(K-K^*)$. As a_0 grows to λa_0 the action for the system is pushed further and further from its critical point so that the coherence length—which is measured in units of a_0—may remain fixed. Thus renormalization increases the value of this and every other thermodynamically relevant variable.

(3) h_σ, a relevant variable, which is odd in the spin and represents a symmetry breaking term like a magnetic field. This symmetry breaking term also grows and forces one away from the critical point as one does successive renormalizations.

To get to the critical point, the experimentalist adjusts the temperature to set $h_1=0$. Either nature—or the experimentalist—adjusts the symmetry breaking term to zero. Then what happens to h_2? Assume that we are "near" a Gaussian fixed point at say $d=3$. Assume h_2 is small but not zero. Nonetheless $h_2\neq 0$ means we are not at the Gaussian fixed point. How can we calcalculate where we are? We observe the system on a large distance scale. To describe this observation, the theorist renormalizes with $\lambda>1$. In this renormalization h_2 grows and grows and pushes the system toward the competitive fixed point which is the nontrivial fixed point.

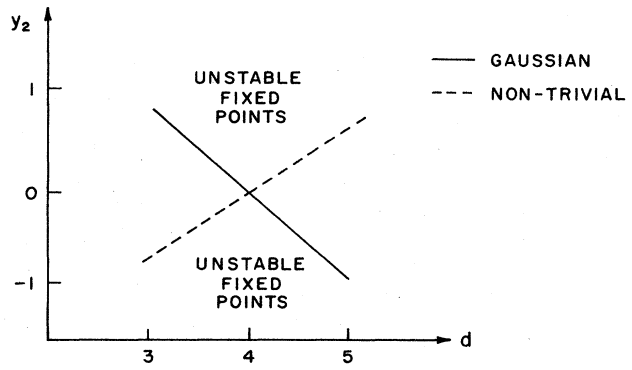

FIG. 13. Relative stability of two fixed points.

We conclude that in statistical systems we will naturally observe only that fixed point with the fewest relevant variables. These fixed points are termed the "most stable" fixed points. For $d<4$ the nontrivial fixed point is most stable. For $d>4$ the Gaussian fixed point is most stable.

VII. THE MIGDAL RECURSION FORMULAE APPLIED TO THE QUARK-STRING SYSTEM

The Migdal-style recursion formulae can be applied quite directly to the Wilson action defined by Eqs. (2.17), (2.18), and (2.19). The pieces of this calculation are already before us. The quarks have a nearest-neighbor coupling and can be attacked via the method of Sec. V.B—and especially with the aid of the recursion formula (5.17b). The strings have a gauge-style coupling and can be attacked via the method of Sec. V.C—with the key equation being (5.22) for this case.

Notice that both types of potential-moving approximation can be applied simultaneously. In the motion, the different types of potential bonds do not interfere with each other. Thus, in this first analysis, we can handle the different parts of the recursion problem quite independently of one another. [For descriptions of perturbation theories on lattices see Wilson (1975a) and Baluni and Willemsen (1976).]

A. Recursion relation for quarks

To handle the quarks, we write the quark part of the action in the form of a sum of nearest-neighbor coupling terms. In particular, we write the A_ψ of Eq. (2.18) as

$$A_\psi=\sum_{x,\mu} K_{\mu f}(\psi(x),\psi(x+\hat{e}_\mu a_0),U(x,x+\hat{e}_\mu a_0)). \tag{7.1}$$

To fit the form (2.18), we take the coupling function to be

$$K_{\mu f}(\psi,\psi',U)=-K^0_{\mu f}\sum_i(\bar{\psi}_{if}\psi_{if}+\bar{\psi}'_{if}\psi'_{if})$$
$$+K_{\mu f}\sum_{ij}[\bar{\psi}_{if}(1-\gamma_\mu)U_{ij}\psi'_{jf}$$
$$+\bar{\psi}'_{if}(1+\gamma_\mu)(U^\dagger)_{ij}\psi_{jf}]. \tag{7.2}$$

Expression (7.2) contains two coupling constants: The hopping parameter $K_{\mu f}$, and a normalization parameter $K^0_{\mu f}$. We arrange these in a vector

$$\mathbf{K}_{\mu f}=(K^0_{\mu f},K_{\mu f}). \tag{7.3}$$

Our starting point is a symmetrical situation in which $\mathbf{K}_{\mu f}$ is independent of μ. To make the calculation as symmetrical as possible we also choose $K^0_{\mu f}$ to be μ independent. Then, to match Eq. (2.26), we must choose the normalization parameter to be

$$K^0_{\mu f}=1/2d. \tag{7.4}$$

Thus we begin from

$$\mathbf{K}_{\mu f}=(1/2d,K_f). \tag{7.5}$$

Now we apply a decimation in the x direction, $\mu=1$.

The couplings in the other directions simply increase by a factor of λ as in Eq. (5.16b), i.e.,

$$\mathbf{K}'_{\mu f} = \lambda(K^0_{\mu f}, K_{\mu f}) \quad \text{for } \mu > 1 . \tag{7.6}$$

On the other hand, the new coupling in the "1" direction is given by

$$e^{K'_{1f}(\psi,\psi',U')} = \mathrm{tr}_\eta e^{K_{1f}(\psi,\eta,U_1)+K_{1f}(\eta,\psi',U_2)}$$

for the particular case in which $\lambda = 2$. Except for an additive constant, $K'_{1f}(\psi, \psi', U')$ turns out to be exactly of the form (7.2) with a set of new couplings

$$\mathbf{K}'_{1f} = (K^0_{1f}, (K_{1f})^2/K^0_{1f}) , \tag{7.7}$$

and a new U which is

$$U_{ij} = \sum_k (U_1)_{ik}(U_2)_{kj} . \tag{7.8}$$

Thus, as we would wish, the longer string is represented by a variable U' which is simply the matrix product of the variables representing its component parts.

Equation (7.7) describes the case $\lambda = 2$. By composing $n = \log_2\lambda$ transformations of the form (7.7), we find the more general result

$$\mathbf{K}'_{1f} = (K^0_{1f}, K^\lambda_{1f}/(K^0_{1f})^{\lambda-1}) \tag{7.9}$$

which represents the lattice constant in the x direction by a factor of λ.

Now imagine changing successively the lattice constants in the $1, 2, \ldots, d$ direction by a factor of λ. Then we would have to apply $d-1$ transforms of the form (7.6) and one transform of the form (7.9). The net result is that for all μ the new coupling is

$$\mathbf{K}'_{\mu f} = \lambda^{d-1}(K^0_{\mu f}, K^\lambda_{\mu f}/(K^0_{\mu f})^{\lambda-1}) .$$

Given the starting point (7.5) we have

$$\mathbf{K}'_{\mu f} = \lambda^{d-1}(1/2d, (2dK_f)^\lambda/2d) . \tag{7.10}$$

Equation (7.10) has one satisfactory feature and one unsatisfactory feature. The new couplings are independent of direction, as we would like. However, the normalization term K^0 is no longer given by Eq. (7.4), but instead by

$$K^0_{\mu f} = \lambda^{d-1}/2d .$$

We would like to recover the structure of our original action in which K^0 is $1/2d$. Fortunately we can very easily redefine the size of our couplings by making the replacement

$$\psi(1) \to Z_3\psi(1) . \tag{7.11}$$

This transformation changes no correlation functions or equations of motion; it merely changes a constant additive term in the action. Under this new "renormalization" transformation all the couplings $K_{\mu f}$ change according to

$$\mathbf{K}_{\mu f} \to \mathbf{K}_{\mu f} Z_3^2 .$$

Hence we pick $Z_3 = \lambda^{-(d-1)/2}$ and derive from (7.10) the new coupling

$$\mathbf{K}'_{\mu f} = (1/2d, (2dK_f)^\lambda/2d) .$$

We have now constructed the renormalization so that the normalization term $K^0_{\mu f}$ is left invariant by the transformation, the only change being in K_f which is seen to obey the recursion relation

$$K_f(\lambda a_0) = [2dK_f(a_0)]^\lambda/2d . \tag{7.12}$$

Equation (7.12) is our desired result, a simple recursion relation for the hopping constant obtained from the Migdal scheme.

B. Interpretation of the quark recursion formula

Equation (7.16) has exactly the same structure as the one-dimensional recursion formula, (3.16)

$$K_f(\lambda a_0) = \tfrac{1}{2}[2K_f(a_0)]^\lambda .$$

This latter formula was derived for the special case $U(x, x') = 1$, but it applies equally for arbitrary U. Hence Eq. (7.12) is exact, as it must be, when $d = 1$.

Our new recursion formula has two major consequences: First, there is a fixed point at

$$K^*_f = 1/2d . \tag{7.13}$$

Secondly, at this fixed point, there is a critical index

$$y = 1 \tag{7.14}$$

just as in one dimension.

We can check these results against the solvable special case $U(x, x') = 1$. In that case, the fixed point and critical index are exactly those of Eq. (7.13) and (7.14). Hence the consequences of Eq. (7.12) are right at least in this special limit. Since this limit is essentially the asymptotic freedom limit, we have set up a formalism which at least generates the known answers in that asymptotic situation.

But this result is certainly not new to us. Consider our discussion of the limit $a_0 \to 0$ in Sec. II.E. Then we wrote

$$K_f = (1/2d)(1 - m_f a_0/d) \tag{2.25}$$

and considered the mass m_f to be a scale invariant quantity. Equation (2.25) has an $a_0 \to 0$ limit (a fixed point) at exactly the value (7.13). The correct term scales linearly with a_0, which is the statement (7.14).

Thus we have learned that the Migdal approach generates a simple recursion formula for K_f which is at least right in the limit $a_0 \to 0$.

C. Recursion relation for the strings: One-dimensional recursion

The gauge theory is represented in terms of matrix variables U which form a homogeneous space for the symmetry group in question. In particular, each plaquette contains a coupling of the form $J(U_c)$, where U_c is the product of U's around the square. We choose $J(U_c)$ real and demand that J be parity independent, i.e.,

$$J(U_c) = J(U_c^\dagger) . \tag{7.15}$$

In addition, the symmetry requires that

$$J(uU_cu^\dagger) = J(U_c) \tag{7.16}$$

for all u which are matrix representation of the symmetry group. The transformation matrix u will be of exactly the same type as the variables U and their product U_c. Finally, the homogeneity of the space demands that

for any two matrices U and U' we can construct a u which converts U into U' via

$$U' = uU . \tag{7.17}$$

This gauge theory maps into a one-dimensional nearest-neighbor theory in which $K(U_1, U_2)$ has the structure

$$K(U_1, U_2) = J(U_1 U_2^\dagger) .$$

Hence the one-dimensional recursion formula can be written for $\lambda = 2$, in the form

$$e^{J'(U_1 U_2^\dagger)} = \mathrm{tr}_U e^{J(U_1 U^\dagger) + J(U U_2^\dagger)} . \tag{7.18}$$

The most convenient way of writing Eq. (7.18) comes from the choice of U_1 and U_2

$$U_1 = U_2^\dagger = U_T^{1/2} .$$

Then (7.18) takes the form

$$e^{J'(U_T)} = \mathrm{tr}_U e^{J(U_T^{1/2} U^\dagger) + J(U U_T^{-1/2})} \tag{7.19}$$

since U_T is unitary.

The usual way of parametrizing U starts from the generators Λ_a of the group representation. Then, since U is unitary, it may be written as

$$U = \exp i\theta_a \Lambda_a = \exp iX . \tag{7.20}$$

Here there is an implied sum over a. The parameters θ_a are all real.

We are interested in analyzing Eq. (7.19) in the strong coupling limit. In this limit, $J(U)$ is largest for small values of θ_a. Then it is reasonable to expand $J(U)$ in the form

$$J(U) = J_0 - \frac{J}{2!}\,\mathrm{trace}\, X^2 + \frac{J_4}{4!}\mathrm{trace}\, X^4 + \cdots . \tag{7.21}$$

For most of our purposes, we need not worry about the constant term J_0, while the higher-order term J_4 will prove to be negligible, when J is sufficiently large. The trace in Eq. (7.21) is, of course, a diagonal sum over the indices of the representation matrix X.

We shall need two properties of the generators Λ_a: their commutators which can be written in the form

$$[\Lambda_a, \Lambda_b] = i f_{abc} \Lambda_c \tag{7.22}$$

and the trace of their product, which we shall assume to be

$$\mathrm{trace} \Lambda_a \Lambda_b = \delta_{ab} C . \tag{7.23}$$

We take C to be positive so that the Lie group is compact. Finally, since θ_a is required to be small by the condition that the coupling constant J is large, one can replace the invariant sum over U in Eq. (7.19) by a simple integral over the θ_a, i.e.,

$$\mathrm{tr}_U \sim \int \prod_a d\theta_a = \int d\theta . \tag{7.24}$$

Now the calculation is all set up. We write

$$U_T = \exp(i\varphi_a \Lambda_a) = \exp iX_T .$$

Then Eq. (7.19) reads

$$\exp[-\tfrac{1}{2} J' C \varphi_a \varphi_a] \sim \int d\theta \exp[Y(\theta, \varphi)] = \int d\theta \exp[J(U_T^{1/2} U^\dagger) + J(U U_T^{-1/2})] . \tag{7.25}$$

We expand the exponent on the right-hand side of Eq. (7.25) in a power series in φ_a and θ_a. If we employ only the coupling J and not J_4, we discover—after some algebra—that

$$Y(\theta, \varphi) = -\frac{JC}{2}\left(\frac{\varphi_a \varphi_a}{2} + 2\theta_a \theta_a\right) + \frac{JC}{48} \varphi_a \theta_b \varphi_{a'} \theta_{b'} f_{abc} f_{a'b'c'} .$$

The integrals in Eq. (7.25) can now be performed directly to give the result

$$e^{-J' C \varphi_a \varphi_a/2} \sim e^{-(JC/2)\varphi_a \varphi_a/2}\, e^{\varphi_a \varphi_{a'} f_{abc} f_{a'bc}/24} . \tag{7.26}$$

In one sense, Eq. (7.26) is a very satisfactory result. We made the assumption that we need include only the Gaussian term in J but not the term in J_4 in Eq. (7.21). Now (7.26) shows that only bilinear terms in φ_a emerge after the recursion. This result is then consistent in that sense. But the quadratic form on the left-hand side of (7.26) is diagonal and a-independent, while the right-hand side does not seem to be diagonal. This appears inconsistent. However, if the Lie group is compact, we can always choose the Λ_a so that the terms in f come to a diagonal form

$$\sum_{bc} f_{abc} f_{a'bc} = C_f \delta_{aa'}$$

with C_f being a positive constant independent of a. In this way we derive the one-dimensional strong-coupling recursion relation

$$J' = \frac{J}{2} - \frac{C_f}{C}\frac{1}{12}$$

for $\lambda = 2$. By doing $n = \log_2 \lambda$ such recursions, we find the general strong-coupling recursion formula

$$J' = \frac{J}{\lambda} - \frac{C_f}{C}\frac{(\lambda - 1)}{6\lambda} . \tag{7.27}$$

D. Nearest-neighbor interactions in 2 + ϵ dimensions

Equation (7.27) can be applied to describe the phase transition behavior for a nearest-neighbor coupling of variables which are the regular representation of the symmetry group. However, notice that most often in statistical physics we do not deal with this particular representation. For example, when Brezin and Zinn-Justin (1976) attacked the group of rotations in n-dimensional Euclidean space their variables were not rotation matrices—which is the case we just analyzed—but instead n-component unit vectors.

The only regular representation of a Lie group conventionally treated in statistical physics is the representation of U1. This Abelian group has $U = e^{i\varphi}$. Then one can take a nearest-neighbor coupling of the form

$$K(U_1, U_2) = J(e^{i(\varphi_1 - \varphi_2)}) .$$

In the particular case in which $J(e^{i\varphi})$ is $J\cos\varphi$, then the model under consideration is called the XY or planar model. It is equivalent to the interaction of two-component unit vectors. In this special case, $C_f = 0$.

Let us analyze the structure of the recursions for the nearest-neighbor case in d dimensions via the Migdal

approximation. From Eqs. (7.27) and (5.17a) one finds that after a change in lattice constant by a factor of λ there is a new coupling in the x direction, $J_x(\lambda a_0)$, which obeys

$$J_x(\lambda a_0) = \lambda^{d-2} J_x(a_0) - \frac{C_f}{C}\frac{(\lambda-1)}{6}. \tag{7.28}$$

Consequently, there is a fixed point at

$$J_x^* = \frac{C_f}{6C}\frac{\lambda-1}{(d-2)\ln\lambda} \tag{7.29}$$

for d approximately equal to 2. Equation (7.28) also implies a critical index

$$y = d-2 = \nu^{-1}. \tag{7.30}$$

Notice that as $d \to 2$, y becomes marginal. When $C_f > 0$, i.e., when the group is non-Abelian, the recursion equation has the structure (6.14) at two dimensions. Then, step by step, each recursion slowly weakens the coupling. Hence in this case there is no phase transition at $d=2$. There is a phase transition for $d>2$. Thence $d_L^* = 2$ for these examples of non-Abelian nearest-neighbor interactions.

But, for the elementary particle application, the most important fact is that at the lower critical dimensionality the phase transition disappears or is rather pushed to zero temperature.

The U1 case is different. Here the group is Abelian so that C_f vanishes. Then at two dimensions the recursion relation is one again marginal but is apparently of the form (6.13). [This form has been more carefully demonstrated by Zittartz (1976) and the correlation functions calculated by Berezinski (1971).] Thus we have very strong indications that at $d=2$ the XY problem has a line of fixed points and shows the structure plotted in Fig. 5c.[2]

E. Gauge interactions in 4 + ϵ dimensions

The Migdal-style analysis of the string interactions in $4+\epsilon$ dimensions is precisely similar to the line of argument we have just carried out for the nearest-neighbor case.

The one-dimensional recursion formula (7.27) can be written

$$R_0^\lambda(J) = \frac{1}{\lambda}\left(J - \frac{C_f}{C}\frac{\lambda-1}{6}\right). \tag{7.31}$$

From this formula and Eq. (5.22) we derive the Migdal-style recursion formula for the coupling $J_{\alpha\beta}$, which describes gauge plaquettes in the α-β plane. This recursion formula is

$$J_{\alpha\beta}(\lambda a_0) = \lambda^{d-4}\left[J_{\alpha\beta}(a_0) - \frac{C_f}{C}\frac{\lambda-1}{6}(\lambda^{1-\alpha}+\lambda^{3-\beta})\right]. \tag{7.32}$$

Consequently the fixed point is given by

$$J_{\alpha\beta}^* = \frac{C_f}{6C}\frac{\lambda-1}{d-4}\frac{\lambda^{1-\alpha}+\lambda^{3-\beta}}{\ln\lambda} \tag{7.33}$$

for dimensionality near 4, while the critical index is

$$y = d-4. \tag{7.34}$$

Now four dimensions is the lower critical dimensionality for this phase transition. Just as in the nearest-neighbor situation we must distinguish between two cases: The Abelian situation (the group U1) in which $C_f = 0$ and the non-Abelian case when $C_f > 0$.

The color carrying strings are representations of SU_3 symmetry and fall therefore into the non-Abelian case. For this situation, there is no phase transition at $d=4$. Of course, this is exactly the result we would have wanted since then the situation is described by the phase diagram of Fig. 9b. As indicated in Sec. IV, this phase diagram implies both asymptotic freedom and quark trapping.

On the other hand, electromagnetism is represented by strings which have Abelian U1 symmetry. Our line of argument implies the existence of a whole line of fixed points for this case, as depicted in Fig. 9c. There are two closely related ways of forming this conclusion. First, one can directly analyze the four-dimensional gauge theory as was done by Polyakov (1975). Alternatively, one can follow Migdal's (1975a, 1975b) line of argument to map the gauge problem into the XY model and then employ perturbation theory about $J=\infty$ to find [see Zittantz (1976), Berezenski (1971), and Wegner (1971b)] a line of fixed points. Either line of argument leaves gaps, so that we should not consider the picture of Fig. 9c to be necessarily meaningful. On the other hand, one can correctly say that *the lines of argument which imply that quarks are at once trapped and asymptotically free fail to go through for electrons coupled by electromagnetism*. Therefore, quantum electrodynamics does *not* provide a counter argument to the line of reasoning that we have applied to quarks.

F. An assessment

Are these conclusions about the behavior of a quark–string system reliable? How far can we trust this kind of application of the Migdal approximation? In my view, some parts of the argument are quite trustworthy, while other parts hide some very real opportunities for substantial errors.

The Migdal-style argument seems to me to be most reliable when it is used to assess the possibility of a phase transition for a pure system of interacting strings at four dimensions and when these strings are describable as representations of a compact, non-Abelian group. These arguments basically describe the sign of the β-function (i.e., the direction of the arrows in Fig. 9) near zero and infinite couplings. These signs can be accurately calculated, I believe, for this situation. Then if the values of the β-function at these points have the same sign (i.e., if the arrows point in the same direction) there will be an even number of fixed points in the interval between zero and infinite coupling. If they have opposite signs, there will be an odd number of fixed points. We have concluded that in the non-Abelian

[2]See Zittartz (1976), Jose (1976), Berezinskii (1971, 1972), Wegner (1971b), Kosterlitz and Thouless (1973), Kosterlitz (1974), and Luther and Scalapino (1976), who discuss the two-dimensional case. Adler (1972) suggests a rather similar picture for the U1 gauge theory at $d=4$.

situation the signs are the same. Hence there are an even number of fixed points.

We hope that this even number is zero. We can make some physical arguments [see Eqs. (4.9)] about one kind of phase transition which might occur, producing a change in correlation structure from Eq. (4.9a) to Eq. (4.9b). But, it is harder to imagine how two or four or six phase transitions might take place. Thus we do have some foundation for our hope that in the non-Abelian case there might be no phase transition at all.

On the other hand, in the Abelian four-dimensional case, the β function as defined by

$$\beta(J) = \frac{\partial}{\partial \ln \lambda} \ln J(\lambda a_0)\Big|_{\lambda=1}$$

vanishes at $J=\infty$. In my view, one cannot form a reliable conclusion about the phase transition behavior from this fact alone.

The quarks further complicate this story. The analysis of this paper has been carried out as if the quarks did not play an essential role in the phase transition. They were merely probes which enabled us to observe this transition or its absence. However, even though this point of view is consistent with the Migdal approximation scheme, it has a rather doubtful validity. Notice that the quarks are also undergoing a near phase transition as $a_0 \to 0$. When two phase transitions occur together, they may interfere in a very subtle and complex manner. Notice, for example, that the Baxter (1971, 1972) phase transition may be viewed as the result of the interaction between two Ising models which are both going critical at the same time (Kadanoff and Wegner, 1971). And the Baxter solution is much richer and more complex than the Onsager solution of the Ising model. Similarly, the quark–string phase transition may be much richer than the phase transition of either quarks or strings alone.

An additional weakness arises because the quarks are fermions. Our potential moving argument is partially predicted upon a "variational principle." The "variational principle" in turn gives lower bounds on the free energy whenever the basic probability function Tr is a sum with positive semidefinite weights. This positivity fails for fermion variables. The best we can say, then, is that our error is second order in the moved potential, but we cannot assess the sign of the error.

This problem is made even more serious because the fermion problem does not have a "strong" interaction. In this case one-half of our argument for the Migdal approximation—that part based upon the strength of the interaction—is absent. Thus our argument about phase transitions in the coupled quark–string system is far from watertight. In fact it is wrong when there are too many different flavors. To make the situation even worse, we have no experience with any analogous renormalization calculations in any statistical mechanical calculation involving fermions on lattices.

On the other hand, there is some calculational evidence that we are moving in the right direction. In recent calculations, W. Bardeen and R. B. Pearson (1976) and also Kogut and Suskind (1975) have formulated partial lattice versions of the theory outlined here. These groups, respectively, have 2 and 1 continuum coordinates and consequently $d-2$ and $d-1$ dimensional lattices. These partial lattice theories have the advantage that they can include chiral invariance ($\psi \to \gamma_5 \psi$; $\bar{\psi} \to -\bar{\psi}\gamma_5$) in a convenient manner. These groups then attempt calculations with a fair success in giving, respectively, the leading Regge trajectories, the structure of the two-dimensional theory, and the lowest hadron masses.[3]

G. For the future

It is likely that further developments from the point of view presented in this paper would do well to include chiral symmetry and perhaps to allow the theory to spontaneously break that symmetry. Thus the underlying Lagrangian used in this paper may well have to be modified in some crucial respects in order to make proper contact with the physics.

In addition some kinds of technical progress will be necessary in order to make any kind of lattice theory into a useful tool. At the moment, we have only a very slight acquaintance with renormalization calculations involving fermions. It would be very helpful if this experience were extended through approximate renormalization calculations (perhaps on lattices) of such problems as the Schwinger (1962) model, the massive Thirring (1958) model [see Glaser (1958), Johnson (1961), Summerfield (1963), and Luther (1976)], and the backward scattering model [see Luther and Emery (1974)]. Indeed even some experience with purely Gaussian models with a bilinear action would be useful.

Notice that most approximate renormalization calculations performed to date describe thermodynamic functions rather than correlation functions. For the elementary particle example, the correlation functions are crucial since these functions define both the masses and the scattering amplitudes. It would be very useful if the relevant approximations could be extended to the calculation of correlation functions and the results checked against known accurate calculations.

Kogut and Suskind have pointed the way to a worthwhile parallel development: the use of perturbation theory about strong coupling in conjunction with series expansion techniques. One can hope that this method will yield expressions for many of the quantities of physical interest.

The problems of quarks and strings are indeed more complex than those problems which have been attacked in statistical physics. Nonetheless, we can hope and expect that the basic methods of modern statistical renormalization theory can be applied to gain some qualitative and indeed quantitative picture of the consequences of a quark–string model of elementary particles.

APPENDIX A. RECURSION RELATION FOR QUARKS IN ONE DIMENSION

We would like to calculate the trace in Eq. (3.10) for the case in which the coupling function involves fermion

[3]See Bardeen and Pearson (1976), Kogut and Suskind (1974, 1975, 1976), Carrol *et al.* (1975), Banks *et al.* (1976).

variables and is of the form (3.13). The quantity to be calculated is

$$e^{K'(\psi,\psi')} = \mathrm{tr}_\sigma\, e^{-1/2\bar\psi\psi - 1/2\bar\psi'\psi' - \bar\sigma\sigma}\, e^{K[\bar\psi(1-\gamma_1)\sigma + \bar\sigma(1-\gamma_1)\psi']} \times e^{K[\bar\psi'(1+\gamma_1)\sigma + \bar\sigma(1+\gamma_1)\psi]}. \tag{A1}$$

Here σ, ψ, and ψ' are anticommuting fermion variables, and γ_1 is a matrix with eigenvalues ± 1.

To calculate the trace (A1), we decompose the spinor variable σ into parts which are eigenvectors of γ_1 with eigenvalue ± 1. In particular, we write

$$\sigma_\pm = \tfrac{1}{2}(1\pm\gamma_1)\sigma\,, \qquad \bar\sigma_\pm = \tfrac{1}{2}\bar\sigma(1\pm\gamma_1) \tag{A2}$$

and notice, then, that the trace over σ can be decomposed into separate traces over σ_+ and σ_-:

$$\mathrm{tr}_\sigma = \mathrm{tr}_{\sigma+}\,\mathrm{tr}_{\sigma-}\,. \tag{A3}$$

Thus Eq. (A1) implies

$$K'(\psi,\psi') = -\tfrac{1}{2}\bar\psi\psi - \tfrac{1}{2}\bar\psi'\psi' + a_+ + a_-\,, \tag{A4}$$

where

$$a_+ = \ln \mathrm{tr}_{\sigma+}\, e^{-\bar\sigma_+\sigma_+}\, e^{K\bar\psi'(1+\gamma_1)\sigma_+}\, e^{K\bar\sigma_+(1+\gamma_1)\psi} \tag{A5}$$

and

$$a_- = \ln \mathrm{tr}_{\sigma-}\, e^{-\bar\sigma_-\sigma_-}\, e^{K\bar\psi(1-\gamma_1)\sigma_-}\, e^{K\bar\sigma_-(1-\gamma_1)}\,. \tag{A6}$$

Because the fermion variables have squares which vanish

$$\psi^2 = \bar\psi^2 = 0 \tag{A7}$$

[see Eq. (1.10)], any exponential or logarithm of an expression involving these variables is very simple. In particular, for any pair of components of the spinors $\bar\psi_\alpha, \psi_\beta$

$$e^{K\bar\psi_\alpha\psi_\beta} = 1 + K\bar\psi_\alpha\psi_\beta\,, \tag{A8}$$

$$\ln(a + b\bar\psi_\alpha\psi_\beta) = \ln a + b/a\,\bar\psi_\alpha\psi_\beta\,. \tag{A9}$$

Thus, if the projection operation (A2) leaves only a single component in σ_+ and in σ_-, expression (A5) may be expanded as

$$a_+ = \ln \mathrm{tr}_{\sigma_+}(1-\bar\sigma_+\sigma_+)[1 + K\bar\psi'(1+\gamma_1)\sigma_+] \times [1 + K\bar\sigma_+(1+\gamma_1)\psi]\,. \tag{A10}$$

Because of (A7), this may be further simplified to

$$a_+ = \ln \mathrm{tr}_{\sigma_+}[1 - \bar\sigma_+\sigma_+ + K\bar\psi'(1+\gamma_1)\sigma_+ + K\bar\sigma_+(1+\gamma_1)\psi + K^2\bar\psi'(1+\gamma_1)\sigma_+\bar\sigma_+(1+\gamma_1)\psi]\,. \tag{A11}$$

Then Eq. (1.11) permits the evaluation of the trace in the form

$$a_+ = \ln - 1 - K^2\bar\psi'(1+\gamma_1)^2\psi$$

so that Eq. (A9) implies

$$a_+ = (\ln - 1) + 2K^2\bar\psi'(1+\gamma_1)\psi\,. \tag{A12}$$

If the projection (A2) left n independent components in σ_+, the result for a_+ would be just the same, except that $\ln - 1$ would be replaced by $n\ln - 1$. When a_+ and a_- are added together as in Eq. (A4), this trivial term cancels out leaving

$$K'(\psi,\psi') = -\tfrac{1}{2}\bar\psi\psi - \tfrac{1}{2}\bar\psi'\psi' + 2K^2\bar\psi'(1+\gamma_1)\psi + 2K^2\bar\psi(1-\gamma_1)\psi'\,. \tag{A13}$$

Eq. (A13) is the result discussed in Sec. III.C of the text.

APPENDIX B. SOLVING THE TWO-DIMENSIONAL GAUGE THEORY VIA A GAUGE TRANSFORM

It is instructive to find the solution of the two-dimensional gauge theory by using a gauge transformation argument. The basic goal is to reduce the two-dimensional gauge problem to a nearest-neighbor problem with the aid of a more physical argument than that employed in Sec. III.E.

Start with the fragment of lattice shown in Fig. 14a. The lattice sites are labeled (n,m) with n and m being the x and y coordinates of the sites in units of the lattice constant. The basic action is expressed in terms of the variables $U_{n,m;\,n'm'}$ $[=(U_{n',m';\,nm})^{-1}]$ where (n,m) and (n',m') are nearest-neighbor sites. In particular it is a function of the product variables

$$U_\Gamma(n,m) = U_{n,m;\,n,m+1}\,U_{n,m+1;\,n+1,m+1} \times U_{n+1,m+1;\,n+1,m}\,U_{m+1,m;\,n,m}\,. \tag{B1}$$

In terms of these product variables, the action takes the form

$$A[U] = \sum_{n,m} J(U_\Gamma(n,m))\,. \tag{B2}$$

Let U_Γ be representations of SU_n, specifically unit determinant unitary matrices. Then, in Eq. (B1), we imply matrix multiplication over the internal indices of the matrices. The trace is a diagonal sum over these matrix indices. Now, imagine a calculation in which we find the partition function

$$Z = \mathrm{Tr}_U\, e^{AU} \tag{B3}$$

by doing an invariant sum over all these SU_n matrices. This sum is invariant under the replacement of all the U's according to $U \to U'$ with

$$U'_{nm;\,n'm'} = u(n,m)\,U_{nm;\,n'm'}[u(n',m')]^{-1}\,. \tag{B4}$$

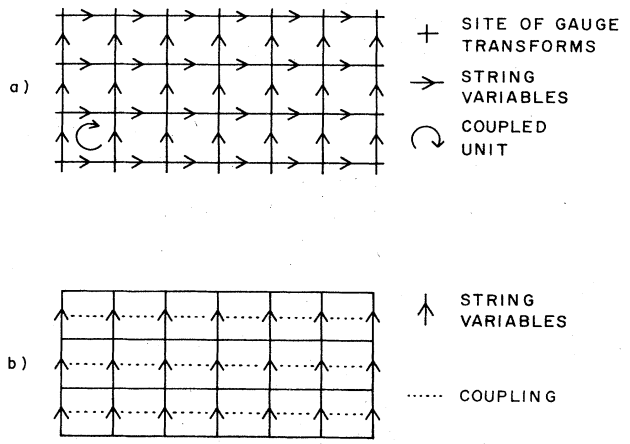

FIG. 14. Two-dimensional gauge problem before (a) and after (b) the choice of a special gauge.

In Eq. (B4), $u(n,m)$ is an n by n special unitary matrix, and matrix multiplication over the internal indices is understood.

For simplicity, assume that the summation in Eq. (B2) is over m from $-\infty$ to ∞ but only covers $n=1$ to ∞. Now choose a special gauge to perform the sum (B3). Basically, we wish to choose a gauge in which the vector potential in the x direction vanishes. To do this we take

$$u(1,m)=\underline{1} \tag{B5}$$

where $\underline{1}$ stands for the unit matrix, and

$$u(n+1,m)=[u(n,m)\,U_{n,m;\,n+1,m}]^{-1}. \tag{B6}$$

Given this special choice of u, the variables in the new gauge are very simple for all strings stretched in the x direction. In fact, according to Eqs. (B4) and (B6)

$$U'_{n,m;\,n+1,m}=\underline{1}. \tag{B7}$$

In this way we have "gauged away" one-half of the original variables. If we then choose the particular normalization

$$\mathrm{tr}_U 1=1,$$

the sum (B3) may be simplified to the form

$$Z=\mathrm{Tr}'_U e^{A[U']}. \tag{B8}$$

Here the prime on Tr indicates that we sum over only y-direction strings. According to Eqs. (B1), (B2), and (B7), this new action still takes the form (B2)

$$A[U']=\sum_{n,m} J(U'_\Gamma(n,m)), \tag{B9}$$

but now the basic argument of the function J is

$$U'_\Gamma(n,m)=(U_{n,m;\,n,m+1}U_{n+1,m+1;\,n+1,m}). \tag{B10}$$

Notice that all strings point in the y direction. Nearest-neighbor bits of string separated by one unit in the x direction are coupled by Eqs. (B9) and (B10) *but* there is no coupling between strings with $U_{nm;\,nm'}$ with different values of their central point $(m+m')/2$. Then the summation (B8) decomposes into a group of uncorrelated individual sums over variables with different $(m+m')/2$ indices. The $d=2$ gauge problem is thereby reduced to a set of uncoupled $d=1$ nearest-neighbor problems QED.

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