The anomalous magnetic moment of the muon: A review of the theoretical contributions

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We review the possible contributions to the muon g-factor anomaly, $\alpha_\mu \equiv 1/2(g_\mu - 2)$ which are potentially significant for a comparison between theory and experiment at present. This includes purely quantum electrodynamic (QED) perturbation theory effects up to eighth order; hadronic effects at fourth and sixth order; and weak interaction effects. From these sources we get a total theoretical contribution α_μ(theory) $= (1\ 165\ 920.6 \pm 12.9) \times 10^{-9}$, to be compared with the latest CERN experimental result (Bailey *et al.*, 1975), α_μ(experiment) $= (1\ 165\ 895 \pm 27) \times 10^{-9}$.

CONTENTS

I. INTRODUCTION

The latest published value of the anomalous magnetic moment of the muon measured by the CERN Muon Storage Ring Collaboration (Bailey *et al.*, 1975) is

$$a_\mu \equiv \tfrac{1}{2}(g_\mu - 2) = (1\,165\,895 \pm 27) \times 10^{-9}. \tag{1.1}$$

The accuracy of this measurement represents an improvement by an order of magnitude with respect to the previous result.[1] Since then, the CERN Muon Storage Ring has been in operation for a certain time and a still further improvement in accuracy is soon expected. At this level of precision there are contributions to a_μ, other than those governed by the quantum electrodynamics of muons and electrons (QED), like hadronic effects and possibly weak interaction effects, which play an important role.[2]

Progress in obtaining a theoretical value for a_μ worthy of the improving experimental accuracy has been achieved in four directions:

(i) A more accurate determination of some of the sixth-order QED contributions has been made, either by analytical calculations in some cases (Barbieri and Remiddi, 1975a; Levine *et al.*, 1976), or by improved numerical techniques in others (Calmet and Peterman, 1973; 1975b; Cvitanović and Kinoshita, 1974; Samuel and Chlouber, 1976).

(ii) It has been shown (Kinoshita, 1967; Lautrup and de Rafael, 1974) that certain classes of Feynman diagrams contributing to a_μ are governed by a simple renormalization group equation. As a result, it has been possible to evaluate a large set of eighth-order contributions to a_μ with powers of $\ln(m_\mu/m_e)$ factors (Lautrup and de Rafael, 1974). Many of the remaining eighth-order contributions with potentially important $\ln(m_\mu/m_e)$ factors have also been estimated numerically (Calmet and Peterman, 1975a).

(iii) Hadronic contributions to a_μ which take into account the new structures in the hadronic e^+e^- annihilation cross section $\sigma_H(t)$ have also been calculated (Bailey *et al.*, 1975; Barger *et al.*, 1975; Calmet *et al.*, 1976). More recently, the hadronic contributions of order $(\alpha/\pi)^3$ have been estimated (Calmet *et al.*, 1976), partly from the experimental information on $\sigma_H(t)$, partly in a model-dependent way. An estimate of the induced contribution to a_μ from the asymptotic behavior of $\sigma_H(\sqrt{t} > 7.4$ GeV) based on asymptotic freedom[3] has also been incorporated (Calmet *et al.*, 1976).

(iv) The contribution to a_μ from weak interactions of leptons are now unambiguously calculable within the framework of gauge theory models with spontaneous symmetry breakdown (Bardeen *et al.*, 1972; Fujikawa *et al.*, 1972; Primack and Quinn, 1972).

All these contributions will be reviewed in the following sections. Section II is devoted to QED contributions up to sixth order; Sec. III to the eighth-order QED contributions; Sec. IV to the hadronic contributions; and Sec. V to the weak interaction contributions. The final results with comments upon their significance in the comparison between theory and experiment are given in Sec. VI.

II. QUANTUM ELECTRODYNAMIC CONTRIBUTIONS UP TO SIXTH ORDER

The latest WQED value of α ($\alpha^{-1} = 137.035987$ (29); see Olsen and Williams, 1975) leads to the following value for the second-order Schwinger term

$$a_\mu^{(2)} = a_e^{(2)} = \alpha/2\pi = (1\,161\,409.835 \pm 0.246) \times 10^{-9}. \tag{2.1}$$

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[1]For a review of previous measurements of $g-2$ see the review articles by Combley and Picasso (1974) and Farley (1975).

[2]For a discussion of the different theoretical contributions, where earlier references can be found, as well as for a review of the comparison between theory and experiment prior to August 1971, see Lautrup *et al.* (1972).

[3]For a clear exposition of asymptotic freedom, where earlier references can be found, see Gross and Wilczek (1973) and Politzer (1974).

The error quoted here represents an improvement by an order of magnitude with respect to the previous one.[4] Note that the value of $a_\mu^{(2)}$ has changed by about one unit of 10^{-9}.

The fourth-order contributions (α^2 contributions) are also well-known (Lautrup *et al.*, 1972). They include the Peterman, Sommerfield term plus mass ratio corrections due to vacuum polarization

$$a_e^{(4)} = \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{197}{144} + \frac{\pi^2}{12} + \frac{3}{4}\zeta(3) - \frac{\pi^2}{2}\ln 2 + \frac{1}{45}\left(\frac{m_e}{m_\mu}\right)^2 + O\left\{\left(\frac{m_e}{m_\mu}\right)^4 \ln\frac{m_\mu}{m_e}\right\}\right];$$

and

$$(a_\mu - a_e)^{(4)} = \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{1}{3}\ln\frac{m_\mu}{m_e} - \frac{25}{36} + \frac{\pi^2}{4}\frac{m_e}{m_\mu} - 4\left(\frac{m_e}{m_\mu}\right)^2 \ln\frac{m_\mu}{m_e} + \frac{134}{45}\left(\frac{m_e}{m_\mu}\right)^2 + O\left\{\left(\frac{m_e}{m_\mu}\right)^3\right\}\right],$$

where $\zeta(3) = 1.202056903$ is the Riemann ζ function of argument 3. From these analytic expressions we get

$$a_e^{(4)} = (\alpha/\pi)^2(-0.328\,478\,445) = -1772.303(1) \times 10^{-9}; \tag{2.2}$$

$$(a_\mu - a_e)^{(4)} = (\alpha/\pi)^2(1.094\,260\,675) = 5904.074(2) \times 10^{-9}. \tag{2.3}$$

The errors in Eqs. (2.2) and (2.3) come from the uncertainty on α^{-1}. The contribution from the mass correction term $\frac{1}{45}(m_e/m_\mu)^2$ in Eq. (2.2) is larger than the uncertainty due to α^{-1}. Numerically[5]

$$\left(\frac{\alpha}{\pi}\right)^2 \frac{1}{45}\left(\frac{m_e}{m_\mu}\right)^2 \cong 0.003 \times 10^{-9}.$$

In general, the α^2 contribution to the muon anomaly from a heavy lepton of mass M ($M \gg m_\mu$) (see Fig. 1) is (Lautrup and de Rafael, 1968)

$$a_\mu\,(\text{heavy lepton}) = \left(\frac{\alpha}{\pi}\right)^2 \left\{\frac{1}{45}\left(\frac{m_\mu}{M}\right)^2 + O\left[\left(\frac{m_\mu}{M}\right)^4 \ln\frac{M}{m_\mu}\right]\right\}. \tag{2.4}$$

If the anomalous $e\mu$ events observed in the e^+e^- annihilation at SLAC (Perl *et al.*, 1975; Feldman, 1975) are due to the production and decay of a M^+M^- pair with mass $M \cong 1.8$ GeV/c^2, then we have

$$a_\mu(M \cong 1.8 \text{ GeV}) = 0.4 \times 10^{-9}. \tag{2.5}$$

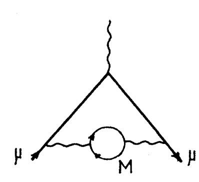


FIG. 1. α^2 contribution to the muon anomaly from a heavy lepton of mass M ($M \gg m_\mu$).

[4]See for example Lautrup *et al.* (1972).

[5]The mass ratio m_μ/m_e is also known with an amazing precision (Casperson *et al.*, 1975): $m_\mu/m_e = 206.769\,27\,(17)$ (0.8 ppm).

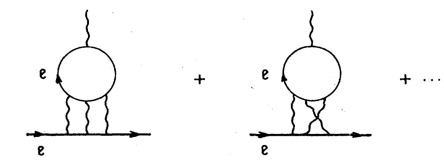


FIG. 2. Two of the six graphs of class I contributing to $a_{e,\,\mathrm{I}}^{(6)}$.

In order to describe the sixth-order contributions (α^3 contributions) we shall use the same notation as Lautrup *et al.* (1972).

(a) Class I (Figs. 2 and 3): 12 graphs involving scattering of light by light subgraphs. Their contributions are, respectively,[6]

$$a_{e,\,\mathrm{I}}^{(6)} = (0.368 \pm 0.010)(\alpha/\pi)^3; \tag{2.6}$$

and (Samuel and Chlouber, 1976)[7]

$$(a_\mu - a_e)_{\mathrm{I}}^{(6)} = (21.32 \pm 0.05)(\alpha/\pi)^3. \tag{2.7}$$

In both cases, the errors come from the integration procedure.

(b) Class II + Class III [Figs. 4(a), (b) and 5(a), (b)]: they include second-order electron vacuum polarization subgraphs and fourth-order electron vacuum polarization subgraphs. These contributions are known analytically (Kinoshita, 1967; Lautrup and de Rafael, 1968; Barbieri and Remiddi, 1975a). Their numerical contribution is

$$a_{e,\,\mathrm{II+III}}^{(6)} = -0.094\,74(\alpha/\pi)^3; \tag{2.8}$$

$$(a_\mu - a_e)_{\mathrm{II+III}}^{(6)} = 1.944\,04(\alpha/\pi)^3. \tag{2.9}$$

(c) Class IV + Class V + Class VI (Figs. 6–8): three across-type photons; two across-type photons; one across-type photon. These graphs do not contribute to $(a_\mu - a_e)$, and the more precise evaluation we have gives (Levine *et al.*, 1976)[8]

$$a_{e,\,\mathrm{IV+V+VI}}^{(6)} = (0.915 \pm 0.015)(\alpha/\pi)^3. \tag{2.10}$$

Adding Eqs. (2.6), (2.8), and (2.10) we obtain the

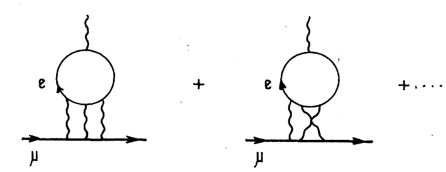


FIG. 3. Two of the six graphs of class I contributing to $(a_\mu - a_e)_{\mathrm{I}}^{(6)}$.

[6]This value has been obtained as a weighted average of results given in Calmet and Peterman (1973), Aldins *et al.* (1970), and Chang and Levine (unpublished).

[7]Previously published values for this contribution are $(18.4 \pm 1.1)(\alpha/\pi)^3$ [Aldins *et al.* (1970)] and $(19.79 \pm 0.16)(\alpha/\pi)^3$ [Calmet and Peterman (1975)]. The three calculations agree on the algebraic part of the calculation. The differences are due to the numerical integration procedure: different choice of integration variables; differences in the integration subroutines and machine time.

[8]The result quoted in Eq. (2.10) comes from Levine *et al.* (1976). It has been obtained by combining known analytic values with numerical values for graphs not yet known analytically. See Levine *et al.* (1976) and references therein.

a) b)

FIG. 4. (a) One of the twelve graphs of class II contributing to $a_{e,\mathrm{II}}^{(6)}$. (b) One of the four graphs of class III contributing to $a_{e,\mathrm{III}}^{(6)}$.

sixth-order contribution to the electron anomaly[9]

$$a_e^{(6)} = (1.188 \pm 0.025)(\alpha/\pi)^3. \tag{2.11}$$

This, together with the second- and fourth-order contributions leads to the result

$$a_e^{\mathrm{QED}} \equiv a_e^{(2)} + a_e^{(4)} + a_e^{(6)} = (1\,159\,652.4 \pm 0.6) \times 10^{-9}, \tag{2.12}$$

to be compared with the experimental result (Wesley and Rich, 1971)

$$a_e^{\mathrm{exp}} = (1\,159\,656.7 \pm 3.5) \times 10^{-9}.$$

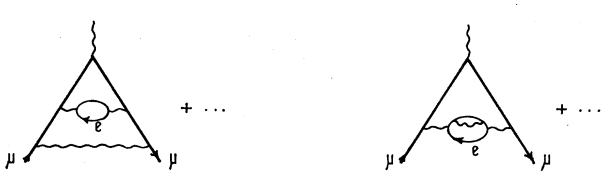
The corresponding value of the muon anomaly obtained from the previous results is

$$a_\mu^{(2)} + a_\mu^{(4)} + a_\mu^{(6)} = (1\,165\,848.1 \pm 1.2) \times 10^{-9}. \tag{2.13}$$

III. EIGHTH-ORDER QUANTUM ELECTRODYNAMIC CONTRIBUTIONS

The eighth-order contribution to the muon anomaly may be potentially significant for a comparison between theory and experiment because of the presence of powers of $\ln(m_\mu/m_e)$ terms which in spite of the over-all $(\alpha/\pi)^4$ factor may lead to large numerical values. Since $\ln(m_\mu/m_e)$ terms can only appear in the difference $(a_\mu - a_e)^{(8)}$ it will be sufficient to limit our discussion to the class of diagrams which contribute to this difference.

Altogether, there are 469 Feynman diagrams which, up to terms which vanish as $m_e/m_\mu \to 0$, may contribute to $(a_\mu - a_e)^{(8)}$. They all have characteristic subgraphs of the vacuum polarization type and/or of the light by light scattering type with electron-type loops. Of these 469 diagrams, there are 304 which are governed by a simple renormalization group equation of the Callan–Symanzik type (Callan, 1970; Symanzik, 1970; 1971). They are schematically represented in Fig. 9. Let us



a) b)

FIG. 5. (a) One of the fourteen graphs of class II contributing to $(a_\mu - a_e)_{\mathrm{II}}^{(6)}$. (b) One of the four graphs of class III contributing to $(a_\mu - a_e)_{\mathrm{III}}^{(6)}$.

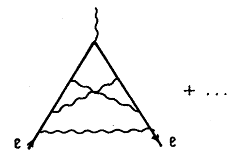


FIG. 6. One of the six graphs of class IV contributing to $a_{e,\mathrm{IV}}^{(6)}$.

denote by $A(m_\mu/m_e, \alpha)$ the contribution to the muon anomaly from the class of diagrams obtained by replacing all internal photon lines in a renormalized muon vertex by dressed photon propagators with fermion lines of the electron type only. It has been shown by Lautrup and de Rafael (1974) that at the limit $m_\mu/m_e \to \infty$ the corresponding asymptotic expansion of $A(m_\mu/m_e, \alpha)$ which we call $A^\infty(m_\mu/m_e, \alpha)$ obeys, order by order in perturbation theory, the differential equation

$$\left(m_e \frac{\partial}{\partial m_e} + \beta(\alpha)\alpha\frac{\partial}{\partial\alpha}\right) A^\infty\left(\frac{m_\mu}{m_e}, \alpha\right) = 0. \tag{3.1}$$

Here $\beta(\alpha)$ is the Callan–Symanzik function, known in perturbation theory up to sixth order (de Rafael and Rosner, 1974)[10]

$$\beta(\alpha) = \left(\beta_1 = \frac{2}{3}\right)\frac{\alpha}{\pi} + \left(\beta_2 = \frac{1}{2}\right)\left(\frac{\alpha}{\pi}\right)^2 + \left(\beta_3 = -\frac{121}{144}\right)\left(\frac{\alpha}{\pi}\right)^3 + \cdots. \tag{3.2}$$

Equation (3.1) summarizes the constraints on the muon anomaly due to the renormalization group property to all orders in α.[11] In particular, it predicts the complete $\ln(m_\mu/m_e)$ structure of the eighth-order contribution to $A^\infty(m_\mu/m_e, \alpha)$, which we denote $A_{(8)}^\infty(m_\mu/m_e)$, in terms of the lower-order constant terms

$$B_i = A_{(2i)}^\infty(m_\mu/m_e = 1) \quad i = 1, 2, 3; \tag{3.3}$$

and the coefficients $\beta_1, \beta_2, \beta_3$. More precisely,

$$A_{(8)}^\infty\left(\frac{m_\mu}{m_e}\right) = \left(\frac{\alpha}{\pi}\right)^4 \left(B_4 + C_4 \ln\frac{m_\mu}{m_e} + D_4 \ln^2\frac{m_\mu}{m_e} + E_4 \ln^3\frac{m_\mu}{m_e}\right), \tag{3.4}$$

with

$$E_4 = \beta_1^3 B_1, \tag{3.5a}$$

$$D_4 = \tfrac{5}{2}\beta_1\beta_2 B_1 + 3\beta_1^2 B_2, \tag{3.5b}$$

$$C_4 = \beta_3 B_1 + 2\beta_2 B_2 + 3\beta_1 B_3. \tag{3.5c}$$

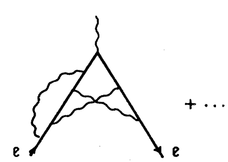


FIG. 7. One of the twenty graphs of class V contributing to $a_{e,\mathrm{V}}^{(6)}$.

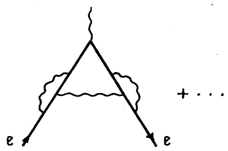
[9]The error quoted in Eq. (2.11) is the sum of the errors in (2.6) and (2.10). The error quoted by Levine *et al.* (1976) is calculated quadratically.

[10]In fact, the eighth-order contribution to $\beta(\alpha)$ from fourth-order vacuum polarization insertions is also known (Calmet and de Rafael, 1975):

$$\left(-\frac{61}{1296} + \frac{2}{27}\pi^2 + \frac{1}{2}\zeta(3)\right)\left(\frac{\alpha}{\pi}\right)^4 = 1.285\left(\frac{\alpha}{\pi}\right)^4.$$

[11]The fourth- and sixth-order constraints on the muon anomaly due to the renormalization group property were first pointed out by Kinoshita (1967). For a discussion of various choices of effective coupling constants relevant to the muon anomaly problem see Barbieri and Remiddi (1975b).

FIG. 8. One of the twenty-four graphs of class VI contributing to $a_{e,\mathrm{VI}}^{(6)}$.



The explicit expressions for the constant terms B_i $i=1,2,3$ are [$a_e^{(2)}$, $a_e^{(4)}$, and $a_e^{(6)}$, being the second-, fourth- and sixth-order contributions to the electron anomaly given in (2.1), (2.2), and (2.11)]

$$B_1 = (\pi/\alpha)a_e^{(2)} = 1/2; \tag{3.6a}$$

$$B_2 = (\pi/\alpha)^2\, a_e^{(4)} - 25/36 = -1.02292; \tag{3.6b}$$

and, from the analytic calculations of Kinoshita (1967), Lautrup and de Rafael (1968), and Barbieri and Remiddi (1975a)

$$B_3 = (\pi/\alpha)^3 a_e^{(6)} + 1.566\,02 = 2.754 \pm 0.025. \tag{3.6c}$$

Altogether, we find that the total numerical contribution from the $\ln(m_\mu/m_e)$ terms of the 304 diagrams represented in Fig. 9 is

$$\left(C_4 \ln\frac{m_\mu}{m_e} + D_4 \ln^2\frac{m_\mu}{m_e} + E_4 \ln^3\frac{m_\mu}{m_e}\right)\left(\frac{\alpha}{\pi}\right)^4 = (17.20 \pm .27)\left(\frac{\alpha}{\pi}\right)^4. \tag{3.7}$$

Let us next consider the remaining 165 diagrams which may contribute to powers of $\ln(m_\mu/m_e)$ terms in $(a_\mu - a_e)^{(8)}$. They are schematically represented in Figs. 10 and 11. The three diagrams in Fig. 10 correspond to a special kind of mixed vacuum polarization insertions. Using a combination of the techniques developed by de Rafael and Rosner (1974) and Kinoshita's theorem on mass singularities (1972) it can be shown (de Rafael, unpublished) that these diagrams do not give $\ln(m_\mu/m_e)$ terms.

The eighteen diagrams in Fig. 11(a) have been evaluated numerically by Calmet and Peterman (1975a). As expected,[12] they give a large contribution

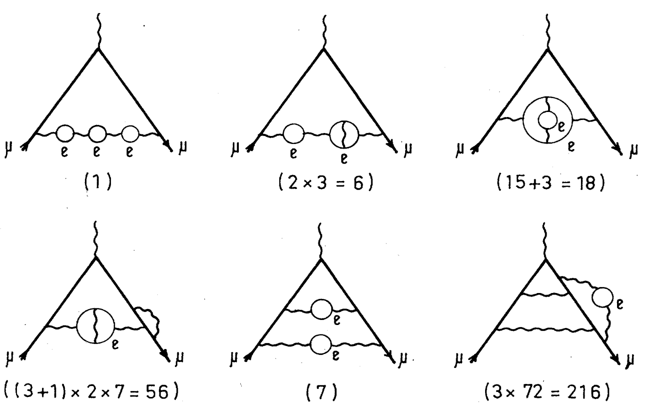
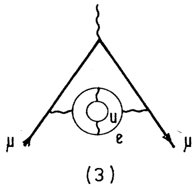


FIG. 9. The 304 diagrams governed by a simple renormalization group equation of the Callan–Symanzik-type. There are six classes, and the number of diagrams in each class is indicated under a typical diagram of the class.

[12]The importance of this class of diagrams was first pointed out by Lautrup (1972).

FIG. 10. One of the three diagrams with a muon loop inside an electron loop.



$$a_\mu^{(8)}[\text{Fig. 11(a)}] = (111.1 \pm 8.1)(\alpha/\pi)^4. \tag{3.8}$$

These diagrams give rise to $\ln^2(m_\mu/m_e)$, $\ln(m_\mu/m_e)$ and constant terms. In fact, the $\ln^2(m_\mu/m_e)$ contribution can be calculated using renormalization group techniques

$$a_\mu^{(8)}[\text{Fig. 11(a)}] = \left(\frac{\alpha}{\pi}\right)^4 \left\{ 3\,\frac{2}{3}(A = 6.290 \pm .056)\ln^2\frac{m_\mu}{m_e} + \text{“unpredicted } \ln(m_\mu/m_e) \text{ terms”} + \mathrm{Cte} \right\}, \tag{3.9}$$

where A is the coefficient of $(\alpha/\pi)^3 \ln(m_\mu/m_e)$ in the contribution to a_μ from the diagrams in Fig. 3. A comparison between Eqs. (3.8) and (3.9) shows that the size of "unpredicted $\ln(m_\mu/m_e)$ terms" + Cte is rather large.

The diagrams from Figs. 11(b), (c), (d) can give at most $\ln(m_\mu/m_e)$ terms. They have not as yet been calculated.

A beautiful application of Kinoshita's theorem on mass singularities (1972) shows that the six diagrams of Fig. 11(e) give no $\ln(m_\mu/m_e)$ terms. The same theorem can be applied (Lautrup, private communication), with some care because of vertex renormalization, to the diagrams in Fig. 11(f). Again no $\ln(m_\mu/m_e)$ terms arise from these 12 diagrams.

All the $\ln^3(m_\mu/m_e)$ and $\ln^2(m_\mu/m_e)$ terms of the

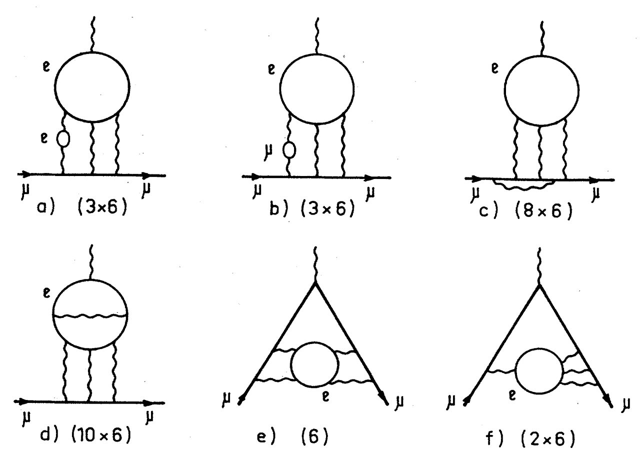


FIG. 11. (a) One of the eighteen diagrams obtained by inserting a single electron loop in the photon–photon scattering (with electron loop) sixth-order graphs. (b) One of the eighteen diagrams obtained by inserting a single muon loop in the photon–photon scattering (with electron loop) sixth-order graphs. (c) and (d) Two of the 108 diagrams obtained by attaching a single virtual photon line to the photon–photon scattering (with electron loop) sixth-order graphs. (e) and (f) Two of the eighteen diagrams of the following type: photon–photon scattering graphs in which all four vertices on the single electron loop loop involve a virtual photon.

eighth-order contribution to the muon anomaly are therefore known. The $\ln(m_\mu/m_e)$ terms are also known except for those which may arise from the light by light scattering corrections shown in Figs. 11(b), (c), and (d).

As a limit of error due to the lack of knowledge of these contributions we suggest

$$\pm 3\times\frac{\alpha}{\pi}\times 21\left(\frac{\alpha}{\pi}\right)^3 = \pm 63\left(\frac{\alpha}{\pi}\right)^4; \tag{3.10}$$

i.e., we have taken the value of the sixth-order contribution to the muon anomaly from the diagrams in Fig. 3 [see Eq. (2.7)] times α/π for the radiative correction, times a factor of 3 since there are three independent classes of diagrams. The over-all eighth-order contribution thus obtained is

$$(\alpha/\pi)^4(128.3\pm 8.4\pm 63) = (3.7\pm 2.1)\times 10^{-9}. \tag{3.11}$$

For a comparison we give the results of previous educated guesses by Lautrup (1972)

$$(100\text{–}200)(\alpha/\pi)^4; \tag{3.12}$$

and by Samuel (1974)

$$230(\alpha/\pi)^4. \tag{3.13}$$

IV. HADRONIC CONTRIBUTIONS

The dominant hadronic corrections to the muon anomaly are due to hadronic vacuum polarization as shown in Fig. 12. They have been discussed in detail elsewhere [see, for example, Lautrup *et al.* (1972)]. Here we shall limit ourselves to a review of their most recent numerical evaluation. Their contribution to the muon anomaly is most conveniently expressed in the form

$$a_\mu(\text{Fig. 12}) = \frac{1}{4\pi^3}\int_{4m_\pi^2}^{\infty} dt\,\sigma_H(t)\int_0^1 dx\,\frac{x^2(1-x)}{x^2+(1-x)(t/m_\mu^2)}, \tag{4.1}$$

where $\sigma_H(t)$ is the total one photon e^+e^- annihilation cross section into hadrons. Clearly their contribution to the anomaly is positive definite.

Three recent evaluations of the integral in Eq. (4.1) give

$$(73\pm 10)\times 10^{-9} \quad (\text{Bailey } et\ al.,\ 1975); \tag{4.2a}$$

$$(66\pm 10)\times 10^{-9} \quad (\text{Barger } et\ al.,\ 1975); \tag{4.2b}$$

$$(70.2\pm 8.0)\times 10^{-9} \quad (\text{Calmet } et\ al.,\ 1976;\ \text{Narison, 1976}). \tag{4.2c}$$

The differences between these results are due to different evaluations of the low-energy region as well as to different appreciations of the background contributions and different estimates of the contribution from the asymptotic region ($\sqrt{t} > 7.4$ GeV at present).

In Calmet *et al.* (1976) and Narison (1976), the con-

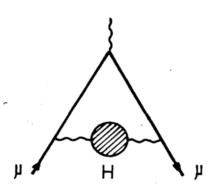


FIG. 12. Hadronic vacuum polarization correction to lowest order contribution to a_μ.

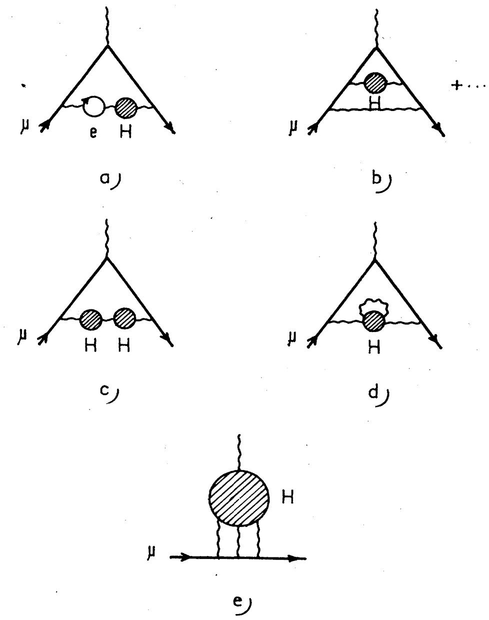


FIG. 13. Hadronic contributions to a_μ of order $(\alpha/\pi)^3$. (a) Hadronic vacuum polarization corrections to diagrams with two fermion lines μ and e. There are two diagrams of this type. (b) Example of a correction to a fourth-order diagram with only muon lines. Altogether, there are fourteen diagrams in this class. (c) Improper fourth-order hadronic vacuum polarization corrections to the second-order QED diagram. (d) Internal radiative corrections to the usual hadronic vacuum polarization correction. (e) Hadronic light by light scattering contributions.

tribution to a_μ (Fig. 12) from the asymptotic region was obtained using a parametrization for the total hadronic cross section as predicted by gauge theories with asymptotic freedom (Gross and Wilczek, 1973; Politzer, 1974):

$$\sigma_H^\infty(t) = \frac{4}{3}\pi\alpha^2\frac{1}{t}\left(3\sum_i Q_i^2\right)\left(1+\delta\frac{1}{\log(t/\mu^2)}\right), \tag{4.3}$$

where Q_i denotes the electric charge of the relevant constituent quark in units of e, and δ is a numerical factor depending on the choice of gauge group. For $SU(3)_{\text{color}}\times SU(N)_{\text{flavor}}$

$$\delta = 4/(11-\tfrac{2}{3}N). \tag{4.4}$$

The parameter μ^2 in Eq. (4.3) is an arbitrary mass scale. It corresponds to the subtraction point chosen to renormalize the theory. This expression for the asymptotic cross section induces the following contribution to the muon anomaly

$$a_\mu^\infty = \left(\frac{\alpha}{\pi}\right)^2\frac{1}{9}\left(3\sum_i Q_i^2\right)\frac{m_\mu^2}{t_>}\left(1-\delta\frac{li(\mu^2/t_>)}{\mu^2/t_>}\right), \tag{4.5}$$

where $t_>$ is the largest energy for which $\sigma_H(t)$ is experimentally known ($\sqrt{t_>} = 7.4$ GeV at present), and $li(x) = \int_0^x dx/\log x$. If the mass scale μ in Eq. (4.3) is fixed in such a way that $\sigma_H(t_>) = \sigma_H^{\exp}(t_>)$, which corresponds to $\mu = 5.4\pm 0.6$ GeV, one obtains for $SU(4)_{\text{flavor}}$

$$a_\mu^\infty(\sqrt{t} > 7.4\ \text{GeV}) = (0.55\pm 0.05)\times 10^{-9}. \tag{4.6}$$

Higher-order virtual hadronic effects have also been recently estimated (Calmet *et al.*, 1976; Narison, 1976). These hadronic effects give contributions of order $(\alpha/\pi)^3$ and can be cast in five classes which are shown diagrammatically in Fig. 13. The contribution from class (a) (two diagrams) is exactly given by the expression

$$a_\mu[\text{class (a)}] = -\frac{1}{2\pi^3}\int_{4m_\pi^2}^{\infty} dt\, \sigma_H(t) \int_0^1 dx \frac{x^2(1-x)}{x^2+(1-x)(t/m_\mu^2)} \times \Pi^{(e)}\left(-\frac{x^2}{1-x}\frac{m_\mu^2}{m_e^2}\right), \tag{4.7}$$

where

$$\Pi^{(e)}\left(\frac{k^2}{m_e^2}\right) = -\frac{\alpha}{\pi}\int_0^1 dy\, y(1-y)\ \ln\left[1-\frac{k^2}{m_e^2}y(1-y)\right]^2. \tag{4.8}$$

From these expressions, it is clear that the contribution to a_μ from class (a) must be positive.

The contribution from class (b) (fourteen diagrams) can be written in a form analogous to the well-known fourth-order expression

$$a_\mu[\text{class (b)}] = \frac{1}{4\pi^2\alpha}\int_{4m_\pi^2}^{\infty} dt\ \sigma_H(t) K^{(4)}\left(\frac{t}{m_\mu^2}\right), \tag{4.9}$$

where $K^{(4)}(t/m_\mu^2)$ is a QED function which has been calculated analytically, by Barbieri and Remiddi (1975a), as an intermediate step in the evaluation of some of the purely QED contributions to a_μ. Its asymptotic behavior for $t \gg m_\mu^2$ is as follows:

$$K^{(4)}\left(\frac{t}{m_\mu^2} \gg 1\right) = \left(\frac{\alpha}{\pi}\right)^2\left(-2\frac{m_\mu^2}{t}\right)\left\{\frac{23}{36}\ln\frac{t}{m_\mu^2}+\frac{\pi^2}{3}-\frac{223}{54} + \Theta\left(\frac{m_\mu^2}{t}\ln\frac{t}{m_\mu^2}\right)\right\}. \tag{4.10}$$

In fact, $K^{(4)}(t/m_\mu^2)$ is negative definite throughout the integration region in Eq. (4.9) and the contribution from class (b) must therefore be negative.

The contribution from diagram (c) can also be written in a closed form in terms of $\sigma_H(t)$,

$$a_\mu[\text{class (c)}] = \frac{1}{16\pi^5\alpha}\int_{4m_\pi^2}^{\infty} dt \int_{4m_\pi^2}^{\infty} dt'\sigma_H(t)\sigma_H(t') \times \int_0^1 dx \frac{x^4(1-x)}{[x^2+(1-x)(t/m_\mu^2)][x^2+(1-x)(t'/m_\mu^2)]} \tag{4.11}$$

which clearly exhibits the positivity property.

Numerical estimates of these contributions [Eqs. (4.7), (4.9), and (4.11)] have been made using the present experimental information on the $e^+e^- \to$ hadrons cross section. The results can be found in Table I. There is an important cancellation between the contributions from classes (a) and (b); the contribution from class (c) is negligible.

Diagrams of classes (d) and (e) are topologically different from those of the other classes. They involve the hadronic light by light scattering amplitude for which no empirical information is available at present. One is obliged to make a model-dependent estimate of these contributions. For this purpose, it has been assumed that the bulk of the hadronic effects in diagrams of the type (d) and (e) is effectively characterized by the sum of quarklike loop contributions of different colors and flavors (Fritzsch and Gell-Mann, 1972). It is also assumed that the effective quark masses M_i are larger than the muon mass m_μ so that an asymptotic expansion in m_μ/M_i is justified. The contribution to a_μ from class (d) is then governed by the low k^2 behavior of the vacuum polarization quark-loop with an internal photon correction; k is the energy momentum of the virtual photon attached to the external muon line. For three colored quarks with flavors: $i=u,d,s,c,\ldots$ one gets

TABLE I. Contribution to the muon-g factor anomaly $a_\mu \equiv \frac{1}{2}(g_\mu - 2)$ from the five classes of higher-order virtual hadronic effects represented in Fig. 13.

Contribution	Result
Eq. (4.7); Fig. 13(a); 2-diagrams	$(1.10\pm 0.14)\times 10^{-9}$
Eq. (4.9); Fig. 13(b); 14-diagrams	$(-2.07\pm 0.29)\times 10^{-9}$
Eq. (4.11); Fig. 13(c); 1-diagram	0.02×10^{-9}
Eq. (4.12); Fig. 13(d); 3-diagrams	0.09×10^{-9}
Eq. (4.13); Fig. 13(e); 6-diagrams	$(-2.6\pm 1.0)\times 10^{-9}$
Estimated total	$(-3.5\pm 1.4)\times 10^{-9}$

$$a_\mu[\text{class (d)}] \cong \left(\frac{\alpha}{\pi}\right)^3 \frac{1}{3} m_\mu^2\left(3\sum_i \frac{Q_i^4}{M_i^2}\right)\frac{41}{62}. \tag{4.12}$$

Similarly, the contribution to a_μ from class (e) is governed by the low energy behavior of the virtual light by light scattering amplitude with a quark-loop. A numerical evaluation of this contribution gives the result

$$a_\mu[\text{class (e)}] \cong \left(\frac{\alpha}{\pi}\right)^3 m_\mu^2\left(3\sum_i \frac{Q_i^4}{M_i^2}\right)(-2.5\pm 1.0), \tag{4.13}$$

where the error is due to the numerical integration procedure. Using the SU(4) flavor scheme and typical[13] quark-mass values $M_u \cong M_d = 0.3$ GeV; $M_s = 0.5$ GeV; and $M_c = 1.5$ GeV, one obtains the results shown in Table I. These results do not change significantly if new quark flavors with higher and higher masses are added; however, they are rather sensitive to the u and d quark masses.

From all these calculations we consider that the best estimate at present of the hadronic contributions is

$$a_\mu(\text{hadrons}) = (66.7\pm 9.4)\times 10^{-9}. \tag{4.14}$$

It corresponds to the sum of Eq. (4.2c) plus the total of Table I.

V. WEAK INTERACTION CONTRIBUTIONS

With the advent of renormalizable spontaneously broken gauge theories, it is now possible to calculate unambiguously the contribution to the muon anomaly from the weak interactions. In fact, many calculations have already been done in various models (Bardeen *et al.*, 1972; Fujikawa *et al.*, 1972; Primack and Quinn, 1972). It turns out that one predictive model, in the sense that it gives finite upper and lower bounds to the muon

[13]These are typical mass values used in quark model calculations of weak decays. For a comprehensive review see Gaillard (1975).

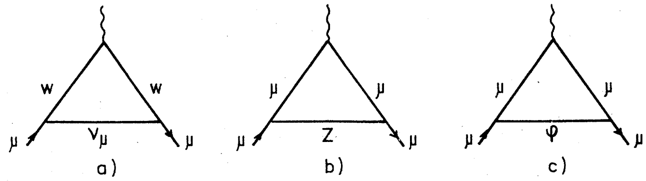


FIG. 14. (a) Neutrino exchange diagram contributing to a_μ in in the Salam–Weinberg model. (b) Z-boson exchange contributing to a_μ in the same model. (c) Higgs scalar exchange diagram contributing to a_μ in the same model.

anomaly, for any values of the parameters, is the popular Weinberg–Salam model (Weinberg, 1967; Salam, 1968). In the other models so far studied [Georgi–Glashow (1972), Lee–Prentki–Zumino (Lee, 1972; Prentki and Zumino, 1972)] the contributions can be extremely large for particular values of the parameters. In these models, it is better to use the muon anomaly to put bounds to some of their parameters.

We shall review in this section the results for the models mentioned above.

(i) *Weinberg–Salam model*

We have three Feynman diagrams [see Figs. 14(a), (b), and (c)] which give the following contributions (Bardeen *et al.*, 1972):

$$a_\mu[\text{Fig. 14(a)}] \equiv a_\mu^\nu = \frac{Gm_\mu^2}{8\pi^2\sqrt{2}}\frac{10}{3}, \tag{5.1}$$

$$a_\mu[\text{Fig. 14(b)}] \equiv a_\mu^z = -\frac{\alpha}{\pi}\frac{1}{3}\left(\frac{m_\mu}{M_z}\right)^2 + \frac{Gm_\mu^2}{8\pi^2\sqrt{2}}\frac{4}{3}\left(1-\frac{2}{R}\right), \tag{5.2}$$

$$a_\mu[\text{Fig. 14(c)}] \equiv a_\mu^\varphi = \frac{Gm_\mu^2}{8\pi^2\sqrt{2}}2\int_0^1 dx\frac{x^2(2-x)}{x^2+r(1-x)}, \tag{5.3}$$

where

$$R = \left(\frac{M_z}{M_W}\right)^2, \quad r = \left(\frac{m_\varphi}{m_\mu}\right)^2.$$

The contribution $a_\mu^\nu + a_\mu^z$ can be written in terms of the Weinberg angle only. More precisely, we have a quadratic form in $\cos^2\theta_w$. It is represented in Fig. 15. Figure 16 shows the contribution a_μ^φ as a function of r. Altogether, $a_\mu^w \equiv a_\mu^\nu + a_\mu^z + a_\mu^\varphi$ has the following upper and lower bounds:

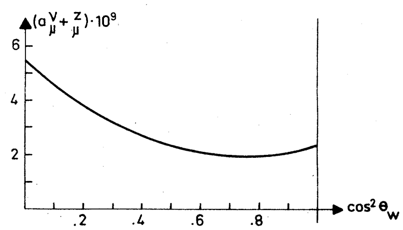


FIG. 15. The contribution $a_\mu^\nu + a_\mu^z$ as a function of $\cos^2\theta_w$ (Salam–Weinberg model).

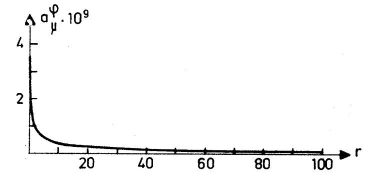


FIG. 16. The contribution a_μ^φ as a function of the ratio $r = (m_\varphi/m_\mu)^2$ (Salam–Weinberg model).

$$1.9\times10^{-9} \leq a_\mu^w \leq 8.9\times10^{-9}. \tag{5.4}$$

If we now take into account the experimental limits on $\sin^2\theta_w$ (Morfin, 1975)[14] and if one accepts Weinberg's argument on the mass of the Higgs boson (Weinberg, 1975), then a_μ^φ is quite negligible and the bounds on a_μ^w become

$$1.9\times10^{-9} \leq a_\mu^w \leq 2.3\times10^{-9}. \tag{5.5}$$

(ii) *Georgi–Glashow model.*

We have also three Feynman diagrams [Figs. 17(a), (b), and (c)] which give the following contributions (Primack and Quinn, 1972):

$$a_\mu[\text{Fig. 17(a)}] \equiv a_\mu^\nu = \frac{\alpha\sin^2\beta}{8\pi}\frac{m_\mu^2}{M_W^2}\frac{10}{3}, \tag{5.6}$$

$$a_\mu[\text{Fig. 17(b)}] \equiv a_\mu^{y^0} = -\frac{\alpha}{8\pi}\frac{m_\mu(m_{y^+}+m_\mu)}{M_W^2} \times\left\{\frac{3}{(1-\rho)^2}\left(1-3\rho-\frac{2\rho^2}{1-\rho}\ln\rho\right)+1\right\}, \tag{5.7}$$

$$a_\mu[\text{Fig. 17(c)}] \equiv a_\mu^\varphi = \frac{\alpha}{8\pi}\frac{(m_{y^+}-m_\mu)^2}{M_W^2}\int_0^1 dx\frac{x^2(2-x)}{x^2+r(1-x)}, \tag{5.8}$$

where

$$\rho = \left(\frac{m_{y^0}}{M_W}\right)^2, \quad r = \left(\frac{m_\varphi}{m_\mu}\right)^2.$$

We have in this model

$$\frac{\pi\alpha\sin^2\beta}{M_W^2} = \frac{G}{\sqrt{2}}$$

so that the contribution a_μ^ν is the same as in the Weinberg–Salam model. The neutral heavy lepton mass m_{y^0} is related to the charged heavy lepton mass m_{y^+}

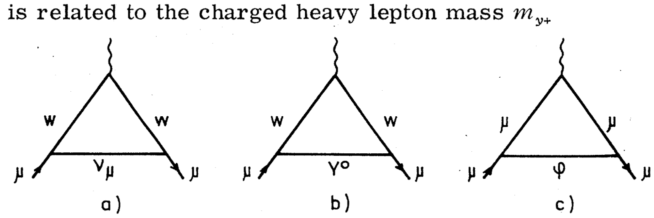


FIG. 17. (a) Neutrino exchange diagram contributing to a_μ in the Georgi–Glashow model. (b) Neutral heavy lepton exchange diagram contributing to a_μ in the same model. (c) Higgs scalar exchange diagram contributing to a_μ in the same model.

[14]We have taken from Morfin (1975)

$$0.1 \leq \sin^2\theta_w \leq 0.5\,.$$

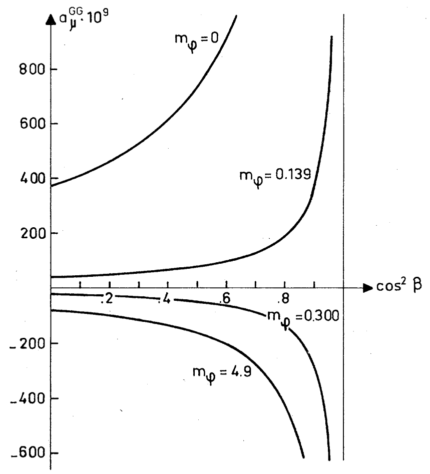


FIG. 18. The contribution a_μ^{GG} as a function of $\cos^2\beta$ for various values of $r=(m_\varphi/m_\mu)^2$, at the mass $m_{y^+}=1.8$ GeV/c^2. In fact, we have quoted m_φ (in GeV/c^2) rather than r.

$$2m_{y^0}\cos\beta = m_{y^+} + m_\mu$$

and this allows one to express $a_\mu^{\mathrm{GG}} \equiv a_\mu^{\nu} + a_\mu^{y^0} + a_\mu^{\varphi}$ as a function of $\cos^2\beta$, for various values of r, at fixed charged heavy lepton mass m_{y^+} (Fig. 18). It can be seen that for r small the contribution $a_\mu^{\varphi} > 0$ dominates, whereas for r large, a_μ^{φ} becomes negligible and $a_\mu^{y^0} < 0$ dominates. It is clear that nothing can be said about the magnitude of a_μ^{GG} so long as no constraints are imposed on the parameters $\cos^2\beta$, m_{y^+}, and m_φ.

(iii) *Lee–Prentki–Zumino model*

The contribution to the muon anomaly in this model can be obtained from the contribution calculated in the Salam–Weinberg model by simply redefining the coupling constants (Fujikawa *et al.*, 1972); we get

$$a_\mu^{\mathrm{LPZ}} = \frac{Gm_\mu^2}{8\pi^2\sqrt{2}}\left[\frac{10}{3} - \frac{4}{3}\,\frac{u^2+v^2}{u^2}\,(1+\sin^2\theta\cos^2\theta)\right]. \tag{5.9}$$

With (see Lee, 1972)

$$\frac{G}{\sqrt{2}} = \frac{1}{4(u^2+v^2)}$$

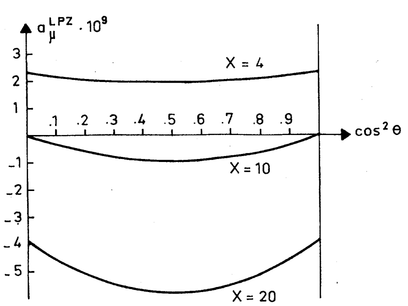


FIG. 19. The contribution a_μ^{LPZ} as a function of $\cos^2\theta$ for various values of the vacuum expectation value u. In fact, we have quoted $x=\sqrt{2}/uG$ rather than u.

a_μ^{LPZ} depends only on $\cos^2\theta$ and u.[15] Thus, putting

$$x \equiv \sqrt{2}/u^2G$$

and with the constraint $x \geq 4$, we have

$$a_\mu^{\mathrm{LPZ}} = \frac{Gm_\mu^2}{8\pi^2\sqrt{2}}\left[\frac{10}{3} - \frac{x}{3}(1+\sin^2\theta\cos^2\theta)\right]. \tag{5.10}$$

In Fig. 19, we have plotted a_μ^{LPZ} as a function of $\cos^2\theta$, for various values of x, that is of u. We can see that the upper curve ($x=4$) provides an upper bound for the contribution to the muon anomaly

$$a_\mu^{\mathrm{LPZ}} \leq 2.3\times10^{-9}. \tag{5.11}$$

There is no lower bound unless some of the parameters are fixed.

VI. FINAL RESULTS

Let us summarize the various theoretical contributions to the muon anomaly, discussed in the previous sections.

QED contributions including all the second-, fourth- and sixth-order diagrams give [see Eq. (2.13)]

$$a_\mu(\text{QED "2"+"4"+"6"}) = (1\,165\,848.1 \pm 1.2)\times10^{-9}.$$

The error is mostly due to the numerical method of evaluation of some of the sixth-order diagrams. The contribution to the error from the present knowledge on the fine structure constant is $\pm0.2\times10^{-9}$.

From the eighth-order contributions we get [see Eq. (3.11)]

$$a_\mu(\text{8th order}) = (3.7\pm2.1)\times10^{-9}.$$

All eighth-order diagrams which contribute to $\ln^3(m_\mu/m_e)$ and/or $\ln^2(m_\mu/m_e)$ terms have been evaluated. The error $\pm2.1\times10^{-9}$ is due partly to the numerical method of evaluation of the calculated diagrams [$\pm8.4(\alpha/\pi)^4$ in Eq. (3.11)]; partly to an estimated limit of error of the uncalculated diagrams [$\pm63(\alpha/\pi)^4$ in Eq. (3.11)].

Our estimate of the hadronic contributions is [see Eq. (4.14)]

$$a_\mu(\text{hadrons}) = (66.7\pm9.4)\times10^{-9}.$$

This includes fourth- and sixth-order contributions. The result is partly model dependent because on the one hand we have used asymptotic freedom to evaluate the contribution from the high-energy behavior of the $e^+e^- \to$ hadrons cross section; on the other hand we have used a simple quark model to evaluate some of the sixth-order contributions.

The weak interaction contributions calculated within the Weinberg–Salam model give lower and upper bounds which we shall adopt as characteristic of weak contributions [see Eq. (5.5)]

$$a_\mu(\text{Weinberg–Salam}) = (2.1\pm0.2)\times10^{-9}.$$

The total of the theoretical contributions is therefore

$$a_\mu = (1\,165\,920.6\pm12.9)\times10^{-9}.$$

The comparison between theory and experiment can be

[15]Here u and v are the vacuum expectation values of Higgs scalars in this model.

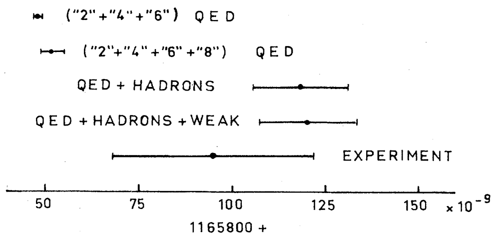


FIG. 20. Final comparison between the theoretical contributions and experiment.

best seen in Fig. 20. It is clear that the hadronic corrections are required for an agreement between theory and experiment.

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