On understanding spin-flip synchrotron radiation and the transverse polarization of electrons in storage rings*

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A mainly didactic discussion is given of the mechanism for the gradual build up of transverse polarization of electrons and positrons in storage rings. The history and basic results are reviewed briefly. Then an intuitive explanation of the polarization in terms of spontaneous emission via a nonrelativistic magnetic dipole transition in a moving inertial frame is presented and criticized. This simple treatment contains a large part of the essential physics, but not all. It is surprisingly successful for electrons and positrons ($g=2$), becomes exact for large g factors of either sign, but fails badly for particles with g factors of the range $0 \lesssim g \lesssim 1.5$. The failure occurs because here the spin-magnetic-moment system cannot be treated even approximately in isolation from the orbital motion. A correct semiclassical description of radiation by a spin system is then given, in direct analogy with semiclassical radiation theory for charged particles ignoring spin. The classical equation of motion for a spin in relativistic motion, derived originally by Thomas, is used to obtain an effective Hamiltonian of interaction of a spin with electromagnetic fields. Emission and absorption of radiation is then described by replacing the classical electromagnetic fields with the appropriately normalized photon fields. The resulting formulas are applicable to charged particles of arbitrary g factor. Expressions are given for the differential spectra in angle and in frequency for numbers of photons and for radiated power, as well as the previously known results for the total transition rates. These results seem of physical interest only for $g=2$ but serve useful pedagogic purposes, refuting some of the expectations of the naive explanation. The various differential spectra are treated in detail for $g=2$ and compared with the corresponding spectra for ordinary synchrotron radiation.

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I. INTRODUCTION

The emission of synchrotron radiation by a relativistic charged particle subject to transverse acceleration is a much studied and much used phenomenon. Its history as a theoretical possibility extends back at least to before 1900 with the relativistic generalization of the Larmor power formula by Liénard and others. For a charge in uniform, circular motion, the detailed harmonic content and angular distributions for each harmonic were calculated in 1907 and appear in an Adams Prize Essay by Schott (1912), but they remained an exercise in mathematical physics until the 1940's when the first electron synchrotrons were constructed and synchrotron light was observed. The names of Pomeranchuk, Schiff, and Schwinger are among those who gave modern theoretical discussions of the phenomena in published and unpublished work, with the paper of Schwinger (1949) containing the various theoretical results in their most tractable form. The essentials now occur in numerous advanced texts.[1]

More recent, and somewhat less well known, is the realization of a gradual polarization of electrons and positrons as they experience a sustained transverse acceleration while orbiting in a storage ring. The mechanism is the emission of spin-flip synchrotron radiation, as first pointed out by Ternov, Loskutov, and Korovina (1961). For initially unpolarized electrons or positrons of charge e, mass m, energy $E=\gamma mc^2$ in uniform motion in a circle of radius ρ, there is a gradual buildup of transverse polarization according to

$$P(t)=P_0(1-e^{-t/\tau_0}), \tag{1a}$$

[1]See, for example, Jackson (1975), Sec. 14.6, or Landau and Lifshitz (1971), Sec. 74.

where the maximum polarization is

$$P_0 = 8/5\sqrt{3} = 0.9238, \tag{1b}$$

and the characteristic time τ_0 is

$$\tau_0 = \left[\frac{5\sqrt{3}}{8}\,\frac{e^2\hbar\gamma^5}{m^2c^2\rho^3}\right]^{-1} \tag{1c}$$

(Sokolov and Ternov, 1963). The polarization is perpendicular to both velocity and acceleration, that is, along the direction of the magnetic field responsible for the bending. Positrons are polarized parallel to the magnetic field, electrons antiparallel.

The original work of Sokolov, Ternov, and collaborators was done with exact solutions for a relativistic Dirac electron in a uniform magnetic field. Subsequently, Baier and Katkov generalized the results to motion in inhomogeneous fields. For the spin-flip radiation by relativistic electrons or positrons, they obtained (Baier and Katkov, 1967a; Baier, 1971a, b) the transition probability per unit time,

$$w = \frac{5\sqrt{3}}{16}\,\frac{e^2\hbar}{m^2c^5}\gamma^5|\dot{\vec{\beta}}|^3\left[1 - \tfrac{2}{9}(\vec{\zeta}\cdot\hat{\beta})^2 + \frac{8\sqrt{3}}{15}\vec{\zeta}\cdot(\hat{\beta}\times\hat{\dot{\beta}})\right], \tag{2}$$

where the unit axial vector $\vec{\zeta}$ specifies the initial spin direction in the electron's rest frame and, $\hat{\beta}$ and $\hat{\dot{\beta}}$ are unit vectors in the directions of the local velocity and acceleration, respectively. For a circular orbit with $|\dot{\vec{\beta}}| = c/\rho$, Eq. (2) leads to results (1) with the senses of polarization for electrons and positrons already stated.

The amount of spin-flip radiation is extremely small compared to the ordinary (nonflip) synchrotron radiation. The ratio of the powers radiated is (Ternov, Loskutov, and Korovina, 1961; Sokolov and Ternov, 1963)

$$\frac{\mathcal{P}_{\text{spin-flip}}}{\mathcal{P}_{\text{ordinary}}} = 3\left(\frac{\hbar\gamma^2}{mc\rho}\right)^2\left(1 \pm \frac{35\sqrt{3}}{64}\right), \tag{3}$$

where the choice of sign depends on the initial spin state of the particle. Only when γ approaches the critical value,

$$\gamma_c = \left(\frac{mc\rho}{\hbar}\right)^{1/2}, \tag{4}$$

will the amount of spin-flip radiation be comparable to the ordinary synchrotron radiation. At present, a typical bending radius for an electron storage ring is $\rho \simeq 13$ m. Hence $\gamma_c \simeq 6\times 10^6$, while $\gamma < 10^4$, showing that the ratio (3) is of the order of 10^{-11}. The smallness of this ratio is reflected in the relative largeness of the buildup time τ_0.

In practice one must distinguish the ring's effective bending radius ρ from the average orbit radius R, defined is the circumference of the orbit divided by 2π. Let the s be the length along the actual orbit in the storage ring, and $\rho(s)$ be the radius of curvature of the orbit at each point. Then by consideration of the accumulation of probabilities it is easy to show that the effective radius of curvature ρ to be inserted in (1c) is

$$\rho^{-3} = \oint [\rho(s)]^{-3}ds \Big/ \oint ds. \tag{5}$$

This formula is valid even if $\rho(s)$ changes sign locally around the orbit as would occur with the so-called wiggler magnets, suggested as a means of controlling the characteristic time τ_0 (Paterson, Rees, and Wiedemann, 1975). For a storage ring consisting of a set of identical bending magnets of bending radius ρ and straight sections combining to an orbit of circumference $2\pi R$, the right-hand side of (5) is equal to $(\rho/R)\rho^{-3}$. In practical units the time constant τ_0 is then

$$\tau_0(\text{sec}) = 98.66\,\frac{[\rho(\text{m})]^3}{[E(\text{GeV})]^5}\,\frac{R}{\rho}. \tag{6}$$

For SPEAR, the storage ring at the Stanford Linear Accelerator Center, $\rho = 12.7$ m, $R = 37.3$ m. At 2 GeV per beam the buildup time is roughly 5 h, while at 4 GeV per beam it is about 10 min. The strong dependence on energy means that the polarization can be utilized as an effective physics tool only in the upper energy range of existing storage rings (SPEAR and DORIS, at Hamburg).

Indications of a buildup of the polarization in a single circulating beam were first reported in 1968 by the Orsay group (Belbeoch *et al.*, 1970), with unambiguous evidence from both Novosibirsk and Orsay in 1971.[2] The first observations on polarization with two beams, under conditions similar to actual running for physics, were made at Orsay and presented by LeDuff *et al.* (1973). More recently observations have been made at SPEAR on the polarization of a single stored beam with $E = 2.4$ GeV (Camerini *et al.*, 1975) and for colliding beams at $E = 3.7$ GeV in the reactions $e^+e^- \to e^+e^-$ and $e^+e^- \to \mu^+\mu^-$ (Learned, Resvanis, and Spencer, 1975) and in $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to$ hadrons (Schwitters *et al.*, 1975). Contemporaneously, polarization measurements in the reaction $e^+e^- \to \mu^+\mu^-$ at 0.5–0.7 GeV per beam have been communicated from Novosibirsk by Kurdadze *et al.* (1975). Since the observed azimuthal asymmetries for the QED processes $e^+e^- \to e^+e^-$, $\mu^+\mu^-$ are entirely consistent with the expectations of theory, these data are used to determine the time constant τ_0 and the degree of polarization P_0. At SPEAR the observed value of polarization was $P_0 = 0.76 \pm 0.05$, while the time constant was in rough accord with Eq. (6) (Learned, Resvanis, and Spencer, 1975). It is presumed that depolarizing effects in the storage ring are responsible for the failure to achieve the anticipated $P_0 = 0.924$. The azimuthal asymmetries of the inclusive hadronic cross sections elucidate the dynamics of the production of hadrons in e^+e^- annihilation. The same information is in principle available from the polar angular distributions with unpolarized beams, but the peculiarities of the experimental arrangement make the azimuthal information more reliable (Schwitters *et al.*, 1975).

For all practical purposes the works of Sokolov and Ternov and of Baier and Katkov, especially the review by Baier (1971b) with its discussion of both theoretical and practical problems, are more than adequate to describe the radiative polarization of beams in storage rings. Nevertheless, it seems that there is the need for an *anschaulich*, didac-

[2]The results from VEPP-2 at Novosibirsk are summarized in Sec. 6 of Baier (1971b) which is a slightly updated version of Baier (1971a), with the addition of these experimental observations. The results of the Orsay storage ring group are contained in the report by Potaux (1971) to the accelerator conference in Geneva.

tic discussion of the subject. After all, Schwinger (1954) demonstrated clearly that ordinary synchrotron radiation is an entirely classical phenomenon. He showed that the orbit is classical provided $(\hbar c/E\rho)\ll 1$, where E is the total energy of the particle and ρ is the orbit radius of curvature, and that the first order quantum-mechanical corrections were obtained by replacement of $\omega \to \omega(1+\hbar\omega/E)$ in the differential transition probability. It follows that for relativistic particles with $1\ll\gamma\ll\gamma_c$, the orbit can be treated classically, and recoil effects can be neglected. This regime of approximation is the basis of the treatment of the spin-flip synchrotron radiation and similar problems by Baier and Katkov (1967a, b; 1968).[3] The works of Schwinger and of Baier and Katkov are important in seeking as classical an understanding as possible of the phenomenon. We focus on the spin itself and seek in its dynamics a simple physical basis for the spin-flip radiation. The words "spin-flip" warn, of course, that the treatment cannot be completely classical—the electron spin must be treated quantum mechanically—but otherwise it is reasonable to expect that one can obtain an understanding of the phenomenon in simple intuitive terms. It turns out that there are subtleties that prevent the realization of this expectation in its naivest form, but a satisfying elementary explanation can be obtained nevertheless.

The plan of the paper is as follows. First, the simple description is presented. It has direct intuitive appeal and does surprisingly well for electrons and positrons. The circumstances in which the elementary treatment is exact or approximately correct are then described, as well as the reasons for its failure in general. Next, the familiar semiclassical treatment of emission of radiation found in texts on quantum mechanics is outlined and generalized via the classical relativistic equation of motion of spin to include spin-flip radiation. The effective Hamiltonian so obtained serves as the basis of a semiclassical treatment of the radiative polarization of a particle of charge e and arbitrary g factor. Differential spectra in frequency and angle are presented, as well as the total transition rate, lifetime, and asymptotic polarization. The virtue of a treatment for arbitrary g factor, seemingly only an academic curiosity, is in its ability to confound some of the "common sense" notions of the simple discussion. A final section presents in detail various spectra of the spin-flip radiation for electrons and positrons ($g=2$). These are of pedagogical, if not practical, value. Some mathematical details appear in an Appendix.

II. INTUITIVE TREATMENT, ITS SUCCESSES AND SHORTCOMINGS

A. Elementary description[4]

The physicist's appetite for an elementary description of radiative polarization is whetted by the following facts:

(1) The effect involves spin-flip.

(2) The rate is very slow, as befits a magnetic dipole transition between states with a small energy difference.

(3) The electrons and positrons are polarized with their magnetic moments parallel to the magnetic field, corresponding to the state of lowest energy of an isolated spin system.

(4) Formulas (1c) or (2) smack of magnetic dipole, with $|\dot{\vec\beta}|^3$ providing the factor of ω^3 and $|\vec\mu|^2/\hbar$ visible in the product of fundamental constants.

Obviously, he says, go to the rest frame of the orbiting electron and consider a simple $M1$ transition from the upper energy level to the lower. We follow his prescription.

Though we know that for relativistic particles all that affects the character of the radiation is a segment of trajectory of length $d\sim\rho/\gamma$, corresponding to an angular deflection $\Delta\theta\sim 1/\gamma$, for simplicity we consider a particle of charge e and mass m moving at constant speed $v=c\beta$ in a circular orbit of radius ρ in a uniform static magnetic field B. The orbital frequency is $\omega_0=v/\rho=\omega_B/\gamma$, where $\omega_B=eB/mc$ is the nonrelativistic cyclotron frequency. We now consider the fields in an instantaneously comoving inertial frame K' moving with speed $v=c\beta$ tangent to the circle. The magnetic field B appears in this frame as a magnetic field $B'=\gamma B$ in the same direction as $\vec{B}$ and an electric field $E'=\gamma\beta B$ in the direction $\vec{v}\times\vec{B}$ as shown in Fig. 1. Suppose that the spin degree of freedom can be treated nonrelativistically in this frame. With magnetic moment

$$\vec\mu=\frac{g}{2}\cdot\frac{e\hbar}{2mc}\vec\sigma, \tag{7}$$

the spin system has two energy levels in K' with frequency difference,

$$\omega'_{12}=\left|\frac{g}{2}\right|\cdot\frac{eB'}{mc}=\left|\frac{g}{2}\right|\gamma^2\omega_0. \tag{8}$$

The transition probability per unit time for a spontaneous magnetic dipole transition from the upper state to the lower is

$$w'=\frac{4}{3\hbar}\left(\frac{\omega'_{12}}{c}\right)^3|\langle 2|\vec\mu|1\rangle|^2. \tag{9}$$

With (7) and (8) this becomes

$$w'=\frac{2}{3}\left|\frac{g}{2}\right|^5\frac{e^2\hbar}{m^2c^5}\gamma^6\omega_0^3. \tag{10}$$

Time dilatation gives a laboratory transition rate reduced by one power of γ. With $\omega_0=c/\rho$ for a relativistic particle, (10) then leads to a characteristic time

$$\tau_{\text{naive}}=\left[\frac{2}{3}\left|\frac{g}{2}\right|^5\frac{e^2\hbar\gamma^5}{m^2c^2\rho^3}\right]^{-1} \tag{11}$$

[3]A summary of the work of Baier and Katkov on the classical regime and lowest order quantum corrections for ordinary and spin-flip synchrotron radiation can be found in Sec. 59 of Berestetskii, Lifshitz, and Pitaevskii (1971), written in collaboration with Baier.

[4]This simple approach was first published by Lyboshitz (1966) who used it to discuss the radiative polarization of neutrons in magnetic and electric fields, a phenomenon previously treated in less transparent fashion by Ternov, Bagrov and Khapaev (1965). For such uncharged particles, the method is essentially exact.

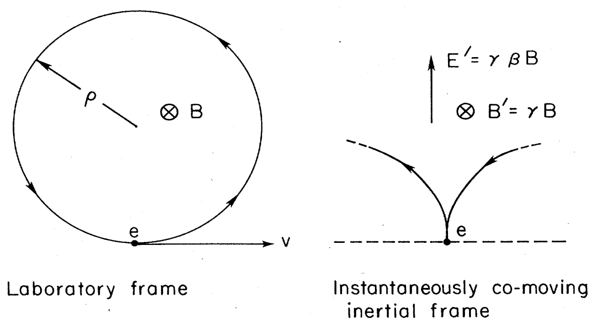

FIG. 1. Orbit of a positively charged particle with a uniform magnetic field B into the page is a circular path of radius ρ traversed at constant speed v. In the frame moving with velocity $\vec{v}$ to the right the orbit is retrograde, caused by a magnetic field $B' = \gamma B$ and a crossed electric field $E' = \gamma\beta B$ with directions as shown on the right.

to be compared with (1c).

For $|g| = 2$, Eq. (11) agrees with (1c) to within a factor of order unity. Furthermore, spontaneous emission from the "upper" to "lower" energy level leads trivially to 100% polarization with the correct senses for electrons and positrons. Comparison with Eq. (2), with its ratio of approximately 25 for the "downwards" transition rate compared to the "upwards" one and its ultimate polarization of 92.4%, indicates that all the essentials are given qualitatively, and even semiquantitatively, by the naive argument. Not bad! The physicist then waves his hands expressively and remarks that of course the spin is not exactly at rest all the time in the moving frame and such neglected refinements can explain away the remaining small discrepancies. The phenomenon is "understood."

The physicist might then proceed to consider how the phenomenon appears in detail in the laboratory. Of interest are the differential spectra in angle and in frequency. In the moving frame, the radiation is monochromatic (the linewidth being negligible) with frequency ω'_{12} and has a $(1 + \cos^2\xi')$ angular distribution, where ξ' is the polar angle relative to the direction of the magnetic field. The differential transition probability is thus

$$\frac{d^2w'}{d\Omega'\, d\omega'} = \frac{\gamma}{\tau}\delta\left(\omega' - \left|\frac{g}{2}\right|\gamma^2\omega_0\right) \cdot \frac{3}{16\pi}(1 + \cos^2\xi'), \qquad (12)$$

where τ is the laboratory lifetime (11). The differential transition probability in the laboratory can be obtained by a Lorentz transformation, using the fact that the number of photons in an element of invariant phase space d^3k/k_0 is a Lorentz invariant quantity. This means that

$$\frac{d^2w}{d\Omega\, d\omega} = \frac{\omega}{\gamma\omega'}\frac{d^2w'}{d\Omega'\, d\omega'}.$$

Here unprimed quantities are those in the laboratory, and the right-hand side of the equation is to be expressed in terms of those quantities through the Lorentz transformation. Using the notation of Fig. 3 below and assuming $\gamma \gg 1$, the physicist finds straightforwardly,

$$\tau\frac{d^2w}{d\Omega\, d\omega} = \frac{3\gamma^2}{4\pi\omega}\frac{[(1+\gamma^2\theta^2)^2 + 4\gamma^2\theta^2\sin^2\varphi]}{(1+\gamma^2\theta^2)^3} \times \delta(1 + \gamma^2\theta^2 - |g|\gamma^3\omega_0/\omega) \qquad (13)$$

as the doubly differential transition probability in laboratory frequency and laboratory angle. The unique correlation between angle and frequency implied by the delta function is a direct consequence of the monochromatic nature of the frequency spectrum in the moving inertial frame. The well-known forward peaking of radiation from rapidly moving sources is evident; the radiation is confined to angles of the order of $1/\gamma$ around the direction of motion.

Integration over angles yields the differential frequency spectrum

$$\tau\frac{dw}{d\omega} = \frac{3}{4\omega_{\max}}\left[1 + 2\left(\frac{\omega}{\omega_{\max}}\right) - 2\left(\frac{\omega}{\omega_{\max}}\right)^2\right], \qquad (14)$$

where $\omega_{\max} = |g|\gamma^3\omega_0$ is the frequency at which the spectrum falls abruptly to zero. This spectrum is displayed below in Fig. 4, where it is compared with the correct results for various g factors. It is a simple parabolic continuum with relative values 1.0 at each end, and 1.5 at the center.

B. Comments and criticism

If the reader has only a passing interest in the process of radiative polarization of the beams in electron storage rings, he or she may stop here and be content with the above simple discussion. It is qualitatively correct for electrons and positrons and contains almost, but not quite, all of the physics. But if it nags that the polarization is only 92.4% instead of 100% and that the lifetime τ contains a curious $5\sqrt{3}/8$ instead of the expected $\frac{2}{3}$ of Eq. (11), then he or she must read on. First we establish when the elementary argument holds without question, when it should hold approximately, and when it might be expected to fail.

For neutral particles possessing magnetic moments, the simple argument is exact, as was pointed out by Lyboshitz (1966). [For this situation we must eliminate g, e, and ρ from Eq. (11) in favor of the magnetic field B and the magnetic moment μ by means of $ge = 4\mu mc/\hbar$, $e\rho = \gamma mc^2/B$, and put $\omega_{\max} = 4|\mu|B\gamma^2/\hbar$ in Eq. (14).] This is because a neutral particle experiences no Lorentz force in the presence of electromagnetic fields, and the forces of translation arising from the coupling of its magnetic moment to the fields are generally negligible. The only significant effect of electromagnetic fields is the precession of the magnetic moment by the magnetic field B' in the rest frame (which is now our inertial frame). The above results, (11)–(14), apply to a neutral particle moving perpendicular to the direction of the magnetic field, but it is easy to generalize them to an arbitrary angle between the particle's velocity and the field and also to include the effects of an electric field in the laboratory (Lyboshitz, 1966; Ternov, Bagrov and Khapaev, 1965).

Though correct for neutral particles with magnetic moments, the simple arguments are only approximate for charged particles, failing badly in detail in some circumstances. A clue to when they might be expected to hold to high accuracy can be found by considering the neutral particle with a given magnetic moment as the limit of a particle with infinitesimally small charge. Since the g factor of a particle of spin s possessing both charge e and magnetic moment μ is defined as

$$g=\mu(2mc/e\hbar s),$$

the neutral particle can be thought of as having an infinite g factor. As a consequence we suspect that the elementary results (11)–(14) should hold with reasonable accuracy for charged particles possessing large g factors, and become exact in the limit $|g|\to\infty$.

The relative reliability of the simple arguments for large $|g|$ and their unreliability for small $|g|$ can be understood by considering the general features of the spin motion and the mechanical motion of the charged particle. It is well known (Bargmann, Michel and Telegdi, 1959) that the magnetic moment of a particle with $g=2$ precesses in a uniform magnetic field at exactly the orbital frequency ω_0. For $g\neq 2$, the laboratory precessional frequency is (see Sec. IV.B below)

$$\Omega=\left[1+\gamma\left(\frac{g-2}{2}\right)\right]\omega_0 .$$

For a g factor appreciably different from 2, Ω becomes very large compared to ω_0 for extreme relativistic motion. More relevant is the number of precessions during the short time $\Delta t\sim 1/\gamma\omega_0$ it takes the particle to trace out a segment of path that subtends an angle $\Delta\theta\sim 1/\gamma$ at the center of the orbit, for it is this time interval that is germane to the emission of relativistic synchrotron radiation in any given direction. For $\gamma\gg 1$, the number of precessions in Δt is evidently $\sim(g-2)/4\pi$.

For large $|g|$ the magnetic moment thus precesses many complete revolutions during the characteristic time interval Δt. This rapidly spinning system has ample time, in effect, to establish the two-level system described above and to undergo its simple magnetic dipole transition (with very small probability), without being influenced appreciably by the orbital motion. Said another way, the instantaneously comoving inertial frame closely approximates the particle's rest frame for long enough, namely for times of order $\Delta t_g\sim 1/\gamma|g|\omega_0$, that the simple nonrelativistic arguments apply there (Derbenev and Kondratenko, 1973).

For g factors of order unity, however, the magnetic moment does not precess rapidly enough to ignore the coupling between the spin system and the orbital motion. A proper calculation shows that the lifetime τ has a complicated dependence on g, approaching the $|g|^5$ behavior of Eq. (11) only for large $|g|$ (Derbenev and Kondratenko, 1973). Furthermore, the frequency spectrum differs significantly from Eq. (14) unless $|g|$ is large—see Eq. (58) and Fig. 4 below. The final and most dramatic shortcoming of the simple treatment is that the actual degree and sense of the polarization depends sensitively on the value of g and is such that for the range $0<g<1.2$ the "upper" energy level is populated preferentially over the "lower" one!—see Fig. 6 below.

C. Energy levels and the classical limit

Some further understanding of the failure of the idea of spontaneous emission from an "upper" to a "lower" energy level of the magnetic-moment-spin system can be found in consideration of the energy levels of the orbital motion. At the same time, an elementary justification for the use of the classical trajectory and soft-photon limit can be obtained. For relativistic circular orbits in the laboratory, Bohr's quantization rule for angular momentum gives the orbital quantum number as

$$n=\gamma mc\rho/\hbar=\gamma\gamma_c^2 \tag{15}$$

where γ_c is the critical value, (4). The spacing between adjacent orbital energy levels is

$$\Delta E=\hbar\omega_0, \tag{16}$$

where $\omega_0=c/\rho$ is the orbital frequency. For highly relativistic particles, this spacing is very small compared to $\hbar\omega_{12}'$ given by (8), or, more properly for considerations in the laboratory, $\gamma\hbar\omega_{12}'\sim\gamma^3\hbar\omega_0$, where we are assuming $|g|=O(1)$. With the spacing between orbital levels very small compared to the transition energy, that transition will inevitably involve some changes in orbital quantum number. In other words, there will occur exchanges of energy between spin and orbital degrees of freedom. There is then little significance in the concept of "upper" and "lower" energy states for the spin system alone.

Another way to reach the same conclusion is to consider the conservation of momentum during the emission of a typical "spin-flip" photon. For ordinary synchrotron radiation, the photons emerge within angles of the order of $\Delta\theta\sim\gamma^{-1}$ of the path of the particle and possess a broad spectrum of energies up to $\gamma^3\hbar\omega_0$ and somewhat beyond. The same will be demonstrated below for the spin-flip synchrotron radiation. With emission essentially parallel to the particle's direction and a typical momentum of the order of $\gamma^3\hbar\omega_0/c$, the photon will cause the particle's momentum to decrease by an amount

$$\Delta p\simeq\gamma^3\hbar\omega_0/c=\gamma^3\hbar/\rho. \tag{17}$$

This corresponds to a fractional change in orbital quantum number,

$$\frac{\Delta n}{n}\simeq\frac{\Delta p}{p}\simeq\frac{\gamma^3\hbar}{\gamma mc\rho}=(\gamma/\gamma_c)^2, \tag{18}$$

and, using (15), to a value of Δn itself of the order of

$$\Delta n\simeq\gamma^3 .$$

This demonstrates that the changes in orbital quantum number from recoil are enormous. With 2.5 GeV electrons, $\gamma\simeq 5\times 10^3$ and $\Delta n\simeq 10^{11}$. At the quantum level the orbital motion is evidently disturbed by the emission act! The disturbance is nevertheless totally negligible to the orbit and its classical description provided $\gamma\ll\gamma_c$. For the typical conditions of $\rho\simeq 13$ m and $\gamma\simeq 5\times 10^3$, Eq. (15) yields $n\simeq 2\times 10^{17}$ and (18), $\Delta n/n\simeq 5\times 10^{-7}$. The astronomical value of n shows how classical the orbit is; the minute value of $\Delta n/n$ shows how small the perturbation of the orbit. Note from (15) and (17) that $\Delta n/n$ is just the fractional change in the energy of the particle as it emits the photon. These considerations provide justification for a classical treatment of the problem (given classical trajectory and soft-photon limit).

D. Particle motion in the laboratory and in the moving inertial frame

An assumption of the elementary treatment is that in the instantaneously comoving inertial frame the effects of the electric field E' can be ignored, that is, that the motion of the particle in the moving frame can be neglected for the time intervals of interest. When the g factor is large and the relevant time interval is $\Delta t_g \sim 1/|g|\gamma\omega_0$, this is valid, as already observed, but for $g=O(1)$ it is not. In terms of the instantaneous radius of curvature ρ of a particle's arbitrary trajectory, the usual synchrotron radiation time interval is $\Delta t \sim \rho/\gamma v$, corresponding to a change in direction by an angle $\Delta\theta \sim 1/\gamma$. In practical circumstances this time interval is so short that the radius of curvature and the speed can be treated as constants during it. The arbitrary trajectory can thus be approximated locally as a circular path of radius ρ along which the particle moves at constant speed $v=\beta c$ or angular velocity $\omega_0=\beta c/\rho$. A suitable choice of coordinates in the laboratory is shown in Fig. 2. The zero of time is chosen when the particle is at the origin. For a horizontal storage ring the guiding magnetic field is in the vertical (z) direction, in or out of the page, the velocity at $t=0$ is in the x direction, and the acceleration at that instant in the y direction. The instantaneously comoving inertial frame is defined by a boost in the positive x direction with speed βc. Denoting coordinates in the moving frame with primes, we have the orbit described parametrically in the two frames by

$$\left.\begin{aligned} x &= \rho\sin\omega_0 t,\\ y &= \rho(1-\cos\omega_0 t),\\ z &= 0, \end{aligned}\right\}\ \text{laboratory} \tag{19a}$$

$$\left.\begin{aligned} x' &= \gamma\rho(\sin\omega_0 t-\omega_0 t),\\ y' &= \rho(1-\cos\omega_0 t),\\ z' &= 0 \end{aligned}\right\}\ \text{moving frame.} \tag{19b}$$

The time coordinate in the moving frame is

$$\omega_0 t' = \gamma(\omega_0 t - \beta^2\sin\omega_0 t). \tag{20}$$

For laboratory times such that $\gamma\omega_0|t|=O(1)$, the orbit equations (19b) and (20) can be approximated as

$$\left.\begin{aligned} x' &\simeq -(\rho/6\gamma^2)(\gamma\omega_0 t)^3,\\ y' &\simeq (\rho/2\gamma^2)(\gamma\omega_0 t)^2,\\ \omega_0 t' &\simeq (\omega_0 t/\gamma)(1+\gamma^2\omega_0^2 t^2/6). \end{aligned}\right\} \tag{21}$$

The equation of the orbit is thus

$$y' \simeq (\rho/2\gamma^2)(6\gamma^2|x'|/\rho)^{2/3}. \tag{22}$$

This path is shown on the right-hand side of Fig. 2. Note that the unit of length is ρ/γ^2, so the scale is greatly enlarged compared to the laboratory figure. Values of the parameter $\gamma\omega_0 t$ are indicated along the path to show the correspondence with points on the circular arc in the laboratory. In terms of this parameter the components of the velocity and acceleration of the particle in the moving frame are

$$\begin{aligned} \beta_x' &\simeq -\gamma^2\omega_0^2 t^2/2\gamma',\\ \beta_y' &\simeq \gamma\omega_0 t/\gamma',\\ \dot\beta_x' &\simeq -\gamma^2\omega_0(\gamma\omega_0 t)/\gamma'^3,\\ \dot\beta_y' &\simeq \gamma^2\omega_0(1-\gamma^2\omega_0^2 t^2/2)/\gamma'^3, \end{aligned} \tag{23}$$

where

$$\gamma' = 1+\gamma^2\omega_0^2 t^2/2 \tag{24}$$

is the ratio of energy to rest energy for the particle in the moving frame.

Since the relevant range of $\gamma\omega_0 t$ is of order unity, (23) and (24) tell us that the particle, while instantaneously at rest in the moving frame at $t=0$, soon attains speed close to that of light. It is admittedly not ultrarelativistic in the contributing time interval, but is changing its state of motion rapidly and is certainly not even approximately at rest for purpose of calculating the radiation.

Two comments in passing:

(1) The path in the moving frame can be thought of as being produced by the combined action of a magnetic field in the z' direction and an electric field in the y' direction. The scale of curvature of the path is ρ/γ^2, as shown in Fig. 2. This means that, although the speed is not constant in this frame, the characteristic orbital angular frequency is $\omega_0' \sim \gamma^2\omega_0$, of the same order of magnitude as Eq. (8), the frequency associated with intrinsic spin, provided $|g|=O(1)$.

(2) It is amusing to verify the Lorentz invariance of total radiated power by calculating in the moving frame with Liénard's generalization of the Larmor power formula,

$$\mathcal{P}' = \frac{2e^2\gamma'^6}{3c}[(\dot{\vec\beta}')^2 - (\vec\beta'\times\dot{\vec\beta}')^2]. \tag{25}$$

Substitution from (23) leads to the familiar result,

$$\mathcal{P}' = 2e^2\omega_0^2\gamma^4/3c = 2e^2c\gamma^4\beta^4/3\rho^2, \tag{26}$$

independent of time, even though the components of ve-

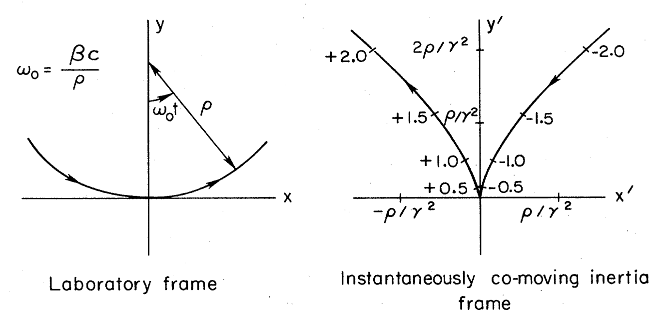

FIG. 2. Segment of particle orbit as seen in the laboratory and in the instantaneously comoving inertial frame. In the laboratory the path is the arc of a circle of radius ρ, traversed at constant angular speed ω_0. In the moving frame it has a cusp at the origin. The tick marks and numbers along the path give the values of the *laboratory* time parameter, $\gamma\omega_0 t$. Note that the length scale in the moving frame is ρ/γ^2.

locity and acceleration are time dependent in the moving frame.

It is hoped that by now the reader is persuaded that the naive consideration of the electron's spin as an isolated, nonrelativistic system in the moving frame is unjustified. Because of ease of exchange of energy between mechanical and spin degrees of freedom, no significance can be attached to the labels "upper" and "lower" energy levels for the magnetic moment interaction. Since the motion in the instantaneously comoving inertial frame becomes somewhat relativistic in the time interval of interest, there is no compelling reason for considering the phenomenon in that frame. The laboratory serves as well and is more familiar. We now proceed to a discussion of a semiclassical derivation of the correct results.

It may be objected that the business of the instantaneously comoving inertial frame is a straw man, that there is a frame where the spin is always at rest, namely the exactly comoving Lorentz frame obtained by a boost with the instantaneous velocity $\vec{v}(t)$. The difficulty with such an approach is that discussion of frequency spectra and transition probabilities inevitably requires consideration of nonvanishing time intervals. A time-dependent Lorentz transformation to a noninertial frame seems to present insurmountable problems, and is not *anschaulich*, to say the least. The relativistic effects of acceleration, i.e., the Thomas precession, are included automatically in the derivation that follows.

III. SEMICLASSICAL DESCRIPTION

A. Semiclassical radiation theory for charge

The time honored elementary treatment of spontaneous emission proceeds as follows. First consider a nonrelativistic charged particle of mass m and charge e interacting with an *external* classical electromagnetic field described by scalar and vector potentials $(\Phi, \vec{A})$ and also with another given interaction potential U. Its motion is described quantum mechanically by the Schrödinger equation with a Hamiltonian

$$H = (1/2m)(\vec{p} - e\vec{A}/c)^2 + e\Phi + U. \tag{27}$$

Commonly (e.g., in atomic physics) the potential U is absent and the scalar potential is the Coulomb potential of the fixed nuclei. If the vector potential is treated as a perturbation, the Hamiltonian is written as a zeroth order term

$$H_0 = (\vec{p})^2/2m + e\Phi + U$$

plus a small interaction term

$$H_{\text{int}} = -e\vec{A}\cdot\vec{\beta}, \tag{28}$$

where the velocity operator is $\vec{\beta} = (-i\hbar/mc)\vec{\nabla}$ and the potentials are in the radiation gauge with $\vec{\nabla}\cdot\vec{A} = 0$. The term in A^2 has been neglected. Effects of weak external fields are examined by use of perturbation theory with the states of the unperturbed Hamiltonian H_0 as the basis. Phenomena like the Zeeman effect involve static external fields, but one can also treat time-varying applied fields and discuss transitions between different energy levels of the unperturbed system.

It is then an easy step to consider $\vec{A}$ in (28) as the vector potential of a plane electromagnetic wave incident on the unperturbed system

$$\vec{A}(\vec{r}, t) = \vec{\epsilon} A_0 \exp(i\vec{k}\cdot\vec{r} - i\omega t) + \text{c.c.}\,. \tag{29}$$

The constant A_0 is initially arbitrary, but is soon chosen to have the value

$$A_0 = (2\pi\hbar c/k)^{1/2}, \tag{30}$$

corresponding to one photon of energy $\hbar\omega$ per unit volume in the incident beam, computed by equating the classical time-averaged Poynting vector to $\hbar c^2\vec{k}$. The substitution of the vector potential (29) into the interaction Hamiltonian (28), followed by a treatment of time-dependent perturbation theory using the method of variation of parameters of Dirac, and leading to a discussion of the photoelectric effect or other transitions involving the absorption of photons, can be traced in almost any book on quantum mechanics.

The derivation involves at some step a resonant enhancement (conservation of energy!) arising from the time integral of the product of two exponentials

$$e^{i(E_f - E_i)t/\hbar} \cdot e^{-i\omega t}.$$

The first factor comes from the time dependences of the initial and final unperturbed states, and the second from the first term in Eq. (29). Since $E_f > E_i$ by assumption, the second (complex conjugate) term in (29) gives no contribution to the time integral. However, it takes no prodding to convince the student to consider the opposite situation where $E_i > E_f$. He or she is thus led smoothly to spontaneous emission where the second (complex conjugate) piece in (29) is operative. It is plausible in considering a transition with the emission of a single photon of wave number $\vec{k}$ that the same normalization constant (30) enters the vector potential here as for absorption.

For our purposes the "golden-rule" result for the transition probability is not as appropriate as an expression for the differential probability at time t for the emission of a photon of polarization $\vec{\epsilon}$ and wave number $\vec{k}$ in an elemental volume d^3k

$$dp(t) = \left|\frac{1}{i\hbar}\int_{-\infty}^{t} \langle\Psi_f(t')|H_{\text{int}}(t')|\Psi_i(t')\rangle dt'\right|^2 \frac{d^3k}{(2\pi)^3}. \tag{31}$$

It is customary to extract the time dependence of the initial and final states and so obtain the exponential factor discussed above, but because of our transition to a classical orbit following Schwinger, Baier, and Katkov we treat the states and operators in the Heisenberg picture. In the limit as $t \to +\infty$, Eq. (31) is the probability of photon emission into d^3k. The energy radiated can be obtained by multiplying by $\hbar\omega$. We are thus led to a result with a classical counterpart, the differential intensity of energy radiated with polarization $\vec{\epsilon}$ per unit solid angle and per unit frequency interval

$$\frac{d^2I}{d\Omega d\omega} = \hbar\omega\frac{dp(\infty)}{d\Omega d\omega}.$$

With the second term in Eq. (29) operative in Eq. (28), the interaction becomes

$$(H_{\text{int}})_{\text{emission}} = -e\left(\frac{2\pi\hbar c}{k}\right)^{1/2}\vec{\epsilon}^{\,*}\cdot\vec{\beta}\,e^{i\omega t - i\vec{k}\cdot\vec{r}}. \tag{32}$$

This gives

$$\frac{d^2I}{d\Omega d\omega}=\frac{e^2\omega^2}{4\pi^2c}\left|\int_{-\infty}^{\infty}\langle\Psi_f(t)|\vec{\epsilon}^*\cdot\vec{\beta}(t)e^{i\omega t-i\vec{k}\cdot\vec{r}(t)}|\Psi_i(t)\rangle dt\right|^2. \tag{33}$$

Here the velocity $\vec{\beta}(t)$ and the coordinate $\vec{r}(t)$ are Heisenberg operators. Equation (33) can be compared with its classical analog.[5] The transition to the classical limit is evidently achieved by the replacement,

$$\langle\Psi_f(t)|\vec{\epsilon}^*\cdot\vec{\beta}(t)e^{i\omega t-i\vec{k}\cdot\vec{r}(t)}|\Psi_i(t)\rangle\rightarrow\vec{\epsilon}^*\cdot\vec{\beta}(t)e^{i\omega t-i\vec{k}\cdot\vec{r}(t)} \tag{34}$$

where now $\vec{\beta}(t)$ and $\vec{r}(t)$ are given classical quantities. This is just the result of Schwinger (1954) and Baier and Katkov (1967b) in the limit that the orbit is classical (the wave functions localized tightly around the orbit) and the energy of the emitted photon is very small compared to the energy of the particle (the noncommutativity of the various Heisenberg operators can then be neglected).

The result (33) with the replacement (34) can form a starting point for the derivation of the classic results of Schwinger (1949) and others for ordinary synchrotron radiation.[1] The alert reader may have noticed that we began with the nonrelativistic Schrödinger equation and are now discussing extreme relativistic motion! The reason this is permitted is that to the neglect of spin the interaction Hamiltonian (28) is correct relativistically with a suitable velocity operator. In the classical limit, the velocity operator is replaced by the classical velocity. The result is therefore generally applicable for arbitrary speeds provided the trajectory is classical and $\gamma\ll\gamma_c$.

B. Semiclassical radiation theory for spin

1. Nonrelativistic spin system

A semiclassical treatment of emission and absorption of radiation by a spin system in motion parallels the discussion of the last section. For orientation we first consider a spin $\hbar\vec{s}$ with associated magnetic moment $\vec{\mu}_0=ge\hbar\vec{s}/2mc$ at rest in interaction with an external magnetic field $\vec{B}$. The Hamiltonian of interaction is

$$H_{\rm int}=-\vec{\mu}_0\cdot\vec{B}=-\left(\frac{ge\hbar}{2mc}\right)\vec{s}\cdot\vec{B}. \tag{35}$$

The corresponding Heisenberg equation of motion is the familiar result

$$\frac{d\vec{s}}{dt}=\frac{i}{\hbar}[H_{\rm int},\vec{s}]=\left(\frac{g}{2}\right)\frac{e}{mc}\vec{s}\times\vec{B}. \tag{36}$$

The interaction Hamiltonian (35) can be used to discuss the effects of static or time-varying magnetic fields on the energy levels and transitions of the spin system in isolation or perhaps with coupling to other (orbital) degrees of freedom. Spontaneous emission can be treated by the *ansatz* of the previous section—the emitted photon is described by the second term of the vector potential (29) with strength A_0 given by (30). The electric and magnetic fields of the emitted photon are thus

$$\vec{E}(\vec{r},t)=-i\sqrt{2\pi\hbar\omega}\,\vec{\epsilon}^*e^{i\omega t-i\vec{k}\cdot\vec{r}},$$
$$\vec{B}(\vec{r},t)=-i\sqrt{2\pi\hbar\omega}\,(\vec{n}\times\vec{\epsilon}^*)e^{i\omega t-i\vec{k}\cdot\vec{r}}, \tag{37}$$

where $\vec{n}$ is a unit vector in the direction of $\vec{k}$. With this magnetic field inserted into (35), standard lowest order perturbation theory leads, in the long wavelength limit, to the magnetic dipole transition rate (9).

2. Relativistic spin system

In order to describe radiation by a spin system in relativistic motion we must obtain suitable generalizations of (35) and (36). The relativistic equation of motion for spin is by now relatively well known. It was first derived by Thomas (1927) in his detailed paper on what is called the Thomas precession, was discussed in a particle physics context by Bargmann, Michel, and Telegdi (1959), and is now standard textbook fare.[6] The Thomas–BMT equation of motion for the spin $\vec{s}$ of a particle of charge e, mass m, and rest-frame magnetic moment $\vec{\mu}_0=ge\hbar\vec{s}/2mc$, in motion with velocity $\vec{v}=\vec{\beta}c$ in external electromagnetic fields $\vec{E},\vec{B}$ can be written in Thomas's original form,

$$\frac{d\vec{s}}{dt}=\frac{e}{mc}\vec{s}\times\left[\left(a+\frac{1}{\gamma}\right)\vec{B}-\frac{a\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})-\left(a+\frac{1}{\gamma+1}\right)\vec{\beta}\times\vec{E}\right], \tag{38}$$

where a is called the *magnetic moment anomaly* and is defined by

$$a=(g-2)/2. \tag{39}$$

The spin vector $\vec{s}$ describes the spin in its rest system (just as does the Pauli $\vec{\sigma}/2$ and the Pauli spinors in the 2-component reduction of the 4-component Dirac spinor), but the time rate of change in (38) is with respect to laboratory time.

Equation (38) is the relativistic generalization of (36). Strictly speaking, it holds only for spatially uniform fields, but is an adequate description for sufficiently slowly varying fields or for weak fields, whatever their space and time variation. The Thomas–BMT equation can be thought of as following from an *effective Hamiltonian* in the same way as (36) follows as a Heisenberg equation of motion from (35). Evidently this effective Hamiltonian is

$$H_{\rm int}^{(\rm eff)}=-\frac{e\hbar}{mc}\vec{s}\cdot\left[\left(a+\frac{1}{\gamma}\right)\vec{B}-\frac{a\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})-\left(a+\frac{1}{\gamma+1}\right)\vec{\beta}\times\vec{E}\right]. \tag{40}$$

Although (40) is explicit and the most useful form for calculation, the terms in the square bracket can be rearranged into a more intuitive, if implicit, form. First we define the magnetic field $\vec{B}'$ in the rest frame of the spin

$$\vec{B}'=\gamma(\vec{B}-\vec{\beta}\times\vec{E})-\frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B}). \tag{41}$$

Then we introduce the Thomas precession angular ve-

[5] Jackson (1975), Eq. (14.67).

[6] See, for example, Barut (1964), Sec. II.4; Hagedorn (1963), Chap. 9; Jackson (1975), Sec. 11.11; Sard (1970), Sec. 5.4.

locity vector $\vec{\omega}_T$

$$\vec{\omega}_T=\frac{\gamma^2}{\gamma+1}(\dot{\vec{\beta}}\times\vec{\beta})=\frac{e}{mc}\cdot\frac{\gamma}{\gamma+1}[\beta^2\vec{B}-\vec{\beta}(\vec{\beta}\cdot\vec{B})-\vec{\beta}\times\vec{E}]. \quad (42)$$

In terms of $\vec{B}'$ and $\vec{\omega}_T$ the effective Hamiltonian (40) can be written

$$H_{\rm int}^{(\rm eff)}=-\frac{1}{\gamma}\vec{\mu}_0\cdot\vec{B}'+\hbar\vec{\omega}_T\cdot\vec{s}. \quad (43)$$

The two terms in (43) have immediate physical interpretations. The first is the expected rest-frame coupling between magnetic moment and magnetic field in that frame, diminished by a factor γ^{-1} to account for the time dilatation seen in the laboratory [remember that (38) is a laboratory equation of motion, even though $\vec{s}$ is the rest-frame spin vector]. The second term is the contribution to the energy from the relativistic Thomas precession of axes in the accelerated rest frame.

3. Radiation formula

The semiclassical description of radiation by the spin $\vec{s}$ proceeds with the replacement of the classical external fields $\vec{E}, \vec{B}$ in (40) by the fields (37) of the emitted photon. The effective Hamiltonian for emission then becomes

$$[H_{\rm int}^{(\rm eff)}]_{\rm emission}=i\sqrt{2\pi\hbar\omega}\,\frac{e\hbar}{mc}\vec{s}\cdot\vec{V}e^{i\omega t-i\vec{k}\cdot\vec{r}}, \quad (44)$$

where

$$\vec{V}=\left(a+\frac{1}{\gamma}\right)\vec{n}\times\vec{\epsilon}^*-\frac{a\gamma}{\gamma+1}\vec{\beta}\vec{\beta}\cdot(\vec{n}\times\vec{\epsilon}^*)-\left(a+\frac{\gamma}{\gamma+1}\right)\vec{\beta}\times\vec{\epsilon}^*. \quad (45)$$

The matrix element of (44) between particle states (of spin and spatial coordinates) can be used straightforwardly to discuss transitions between states of different spin orientation. For the present purposes we consider the classical limit of the *orbital* motion, as in going from (31) to (33) and (34). Comparison of the Hamiltonian (32) for the emission of radiation by a charge e with (44) shows that the formula at the end of the last section can be transcribed with the substitution,

$$\vec{\epsilon}^*\cdot\vec{\beta}(t)\to -i\frac{\hbar k}{2mc}\vec{s}(t)\cdot\vec{V}(t). \quad (46)$$

Now the only quantum-mechanical aspect is the spin vector. The spin analog of (33) and (34) is

$$\frac{d^2I_{\rm spin}}{d\Omega d\omega}=\frac{e^2\hbar^2\omega^4}{4\pi^2m^2c^5}\left|\int_{-\infty}^{\infty}dt\langle f|\vec{s}(t)|i\rangle\cdot\vec{V}(t)e^{i\omega t-i\vec{k}\cdot\vec{r}(t)}\right|^2, \quad (47)$$

with $\vec{V}(t)$ given by (45). The radiation is emitted in the course of a transition from initial spin state i to final spin state f, both states specified in the rest frame of the particle. While in principle nonflip, as well as spin-flip, transitions contribute to the radiation, the nonflip transitions are dominated overwhelmingly by the ordinary charge radiation [see Eq. (3)]. Thus only spin-flip transitions need concern us.

It should be remarked at this point that the author's idea of the effective semiclassical Hamiltonian, Eqs. (40), (44), and (45), was anticipated by Derbenev and Kondratenko (1973). They point out that, for spin $\frac{1}{2}$ particles, it follows to first order in $\hbar$ from a Foldy–Wouthuysen reduction of the Dirac equation. The author's justification was via a Pauli reduction of the momentum-space matrix element of the Dirac–Pauli current (with $\sigma_{\mu\nu}$ coupling for the anomalous magnetic moment) in the soft-photon limit.

IV. SPIN-FLIP SYNCHROTRON RADIATION FOR ARBITRARY g FACTOR

A. Definitions of differential energy, photon number, and transition rates

We now apply (47) to a calculation of the radiation emitted by a relativistic spin-$\frac{1}{2}$ particle of charge e and arbitrary g factor in a spin-flip transition while moving at velocity $\vec{\beta}c$ in an instantaneously circular arc of radius ρ. Defining the time integral in (47) to be

$$\tfrac{1}{2}\mathfrak{M}=\int dt\,e^{i\omega t-i\vec{k}\cdot\vec{r}(t)}\vec{V}(t)\cdot\langle f|\vec{s}(t)|i\rangle \quad (48)$$

we have the intensity of energy radiated per unit solid angle and per unit frequency interval with polarization $\vec{\epsilon}$ in a single passage along the arc

$$\frac{d^2I}{d\Omega d\omega}=\frac{e^2\hbar^2\omega^4}{16\pi^2m^2c^5}|\mathfrak{M}|^2. \quad (49)$$

The *number* of photons emitted per unit solid angle, etc., is obtained by dividing by $\hbar\omega$

$$\frac{d^2N}{d\Omega d\omega}=\frac{e^2\hbar\omega^3}{16\pi^2m^2c^5}|\mathfrak{M}|^2. \quad (50)$$

The differential *transition rate* follows from (50) with multiplication by $\omega_0/2\pi$, where $\omega_0=\beta c/\rho$

$$\frac{d^2w}{d\Omega d\omega}=\frac{e^2\hbar\omega^3\omega_0}{32\pi^3m^2c^5}|\mathfrak{M}|^2. \quad (51)$$

This last result rigorously depends on the assumption of continuous motion at constant speed in a circular orbit, but in practice holds provided the speed and radius of curvature are sensibly constant over a reasonable segment of path. The modifications for storage-ring orbits with bending sections and straight sections are almost self-evident. For the total rate they have been incorporated in (5) and (6).

B. Nonradiative motion of the spin and $\langle f|\vec{s}(t)|i\rangle$

The spin operator $\vec{s}(t)$ in (47) and (48) is a Heisenberg, time-dependent spin operator whose motion is described by the Thomas–BMT equation, (38). In the absence of an electric field and the approximation of a uniform static magnetic field $\vec{B}$ and motion of the particle perpendicular to the field, Eq. (38) reduces to

$$d\vec{s}/dt=\vec{s}\times(1+\gamma a)\vec{\omega}_0, \quad (52)$$

where

$$\vec{\omega}_0=e\vec{B}/\gamma mc.$$

This equation describes the precession of $\vec{s}$ around the direction of $\vec{B}$ with angular velocity,

$$\Omega = (1+\gamma a)\omega_0$$

with respect to axes fixed in the laboratory. (With respect to axes rotating in the plane of the orbit so that they are fixed in orientation relative to the particle's velocity, the precessional frequency is only $\Omega' = \Omega - \omega_0 = \gamma a \omega_0$.)

With the coordinate axes chosen as shown in Fig. 3 and the magnetic field in the negative z direction for a positive charge e, the solutions of the Eqs. (52) can be written in terms of spin operators at some initial time $t = t_0$ as

$$\left.\begin{aligned} s_x(t) &= [s_+(t_0)e^{i\Omega(t-t_0)} + s_-(t_0)e^{-i\Omega(t-t_0)}],\\ s_y(t) &= \frac{1}{i}[s_+(t_0)e^{i\Omega(t-t_0)} - s_-(t_0)e^{-i\Omega(t-t_0)}],\\ s_z(t) &= s_z(t_0). \end{aligned}\right\} \tag{53}$$

The constant operators $s_+(t_0)$, $s_-(t_0)$, $s_z(t_0)$ are

$$s_\pm(t_0) = \tfrac{1}{4}(\sigma_x \pm i\sigma_y),$$
$$s_z(t_0) = \tfrac{1}{2}\sigma_z,$$

where σ_x, σ_y, σ_z are the familiar Pauli spin operators.

We choose the initial spin direction to be along a unit vector $\vec{\zeta}$ in the rest frame and consider a radiative transition in which the spin direction changes from $\vec{\zeta}$ to $-\vec{\zeta}$. The spherical angles of $\vec{\zeta}$ are (θ_0, φ_0) with respect to the z axis of Fig. 3 (not to be confused with the θ and φ of the photon shown there). In Eq. (48) we are therefore interested in the matrix element of $\vec{s}(t)$ between the initial state $|i\rangle = R|\frac{1}{2}, \frac{1}{2}\rangle$ and the final state $|f\rangle = R|\frac{1}{2}, -\frac{1}{2}\rangle$, where R is a rotation operator that rotates the state from z direction to the $\vec{\zeta}$ direction. With the choice $(\varphi_0, \theta_0, 0)$ of the Euler angles and the customary phases for the rotation matrices, we find

$$\langle f|s_+(t_0)|i\rangle = -\tfrac{1}{4}e^{i\varphi_0}(1-\cos\theta_0),$$
$$\langle f|s_-(t_0)|i\rangle = \tfrac{1}{4}e^{-i\varphi_0}(1+\cos\theta_0),$$
$$\langle f|s_z(t_0)|i\rangle = -\tfrac{1}{2}\sin\theta_0.$$

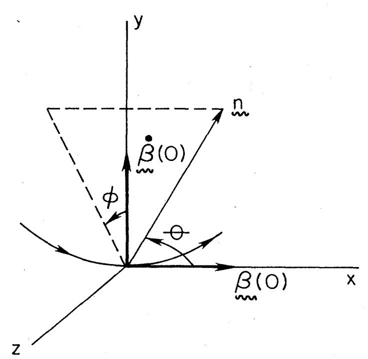

FIG. 3. Coordinate system used in the calculations. The orbit lies in the x-y plane with x and y axes defined by the directions of $\vec{\beta}$ and $\dot{\vec{\beta}}$ at $t=0$. The unit vector $\vec{n}$ specifies the direction of the photon wave vector $\vec{k}$.

These can be inserted into the matrix elements of Eqs. (53) to obtain $\langle f|\vec{s}(t)|i\rangle$.

C. Matrix element

The matrix element $\mathfrak{M}$, defined by Eq. (48), can be written in the form,

$$\begin{aligned}\mathfrak{M} = &-\sin\theta_0 U_1 - \tfrac{1}{2}(1-\cos\theta_0)e^{i(\varphi_0 - \Omega t_0)}U_2\\ &+ \tfrac{1}{2}(1+\cos\theta_0)e^{-i(\varphi_0-\Omega t_0)}U_3,\end{aligned} \tag{54}$$

where the three integrals are

$$\left.\begin{aligned} U_1(\omega, \vec{n}, \vec{\epsilon}) &= \int dt\, V_z(t) e^{i\omega t - i\vec{k}\cdot\vec{r}(t)},\\ U_2(\omega, \vec{n}, \vec{\epsilon}) &= \int dt\, (V_x(t) - iV_y(t)) e^{i(\omega+\Omega)t - i\vec{k}\cdot\vec{r}(t)},\\ U_3(\omega, \vec{n}, \vec{\epsilon}) &= \int dt\, (V_x(t) + iV_y(t)) e^{i(\omega-\Omega)t - i\vec{k}\cdot\vec{r}(t)}. \end{aligned}\right\} \tag{55}$$

When the absolute square of (54) is taken it is only necessary to keep the sum of the absolute squares of the three separate terms. The interference terms involve sinusoidal terms in Ωt_0 or $2\Omega t_0$. For beams in storage rings the physical situation corresponds to random initial times t_0; the interference terms average to zero.

What remains now is a calculation of U_i for $\gamma \gg 1$ with $\vec{V}(t)$ given by (45) and $\vec{\beta}(t)$ and $\vec{r}(t)$ found from the orbit equations (19a) suitably approximated for $\omega_0|t| = O(\gamma^{-1})$. The approximations are essentially the same as for ordinary synchrotron radiation,[1] and the integrals encountered the same. The relative complexity of $\vec{V}(t)$ compared with $\vec{\beta}(t)$ of Eq. (33), especially for $a \neq 0$, and the presence of the factor $\exp(\pm i\Omega t)$ in U_2 and U_3 makes the calculation at least an order of magnitude more cumbersome and not very illuminating. We merely quote results. Some of the mathematical details are given in the Appendix.

D. Doubly differential spectrum in frequency and angle

For the sake of compactness in the relatively unwieldy formulas, we introduce some notation. The angles of emission of the photon are shown in Fig. 3. We define the following variables:

$$\left.\begin{aligned} t &= \gamma\theta\sin\varphi,\\ \nu &= 2\omega/3\gamma^3\omega_0,\\ z_0 &= (3\nu/4)^{2/3}(1+t^2),\\ z_\pm &= (3\nu/4)^{2/3}(1+t^2 \pm 4a/3\nu). \end{aligned}\right\} \tag{56}$$

The dimensionless frequency variable ν is the standard synchrotron radiation variable, called ξ by Schwinger (1949). In terms of these variables and τ_0 defined by Eq. (1c), the differential transition probability (51), summed over photon polarizations, is

$$\frac{d^2w}{d\Omega\, d\omega}=\frac{8(3\nu/4)^{7/3}}{5\pi\sqrt{3}\,\tau_0\gamma^2\omega_0}\Big\{\sin^2\theta_0\Big[(1-at^2)^2|\mathrm{Ai}(z_0)|^2+a^2t^2\Big(\frac{4}{3\nu}\Big)^{2/3}|\mathrm{Ai}'(z_0)|^2\Big]$$
$$+\tfrac{1}{4}(1-\cos\theta_0)^2\Big[t^2\Big|(1+a)\mathrm{Ai}(z_+)+a\Big(\frac{4}{3\nu}\Big)^{1/3}\mathrm{Ai}'(z_+)\Big|^2+\Big|\Big[1+a\Big(1+t^2+\frac{2a}{3\nu}\Big)\Big]\mathrm{Ai}(z_+)+(1+a)\Big(\frac{4}{3\nu}\Big)^{1/3}\mathrm{Ai}'(z_+)\Big|^2\Big]$$
$$+\tfrac{1}{4}(1+\cos\theta_0)^2\Big[t^2\Big|(1+a)\mathrm{Ai}(z_-)-a\Big(\frac{4}{3\nu}\Big)^{1/3}\mathrm{Ai}'(z_-)\Big|^2+\Big|\Big[1+a\Big(1+t^2-\frac{2a}{3\nu}\Big)\Big]\mathrm{Ai}(z_-)-(1+a)\Big(\frac{4}{3\nu}\Big)^{1/3}\mathrm{Ai}'(z_-)\Big|^2\Big]\Big\}. \quad (57)$$

The functions $\mathrm{Ai}(x), \mathrm{Ai}'(x)$ are the Airy function and its derivative, as defined in Abramowitz and Stegun (1964) and in Eq. (A1) of the Appendix.

This rather formidable expression gives the differential rate of emission in angle and frequency for arbitrary a and arbitrary direction of spin flip. For $a=0$ (electrons and positrons) it simplifies drastically. This limit is discussed in detail separately in the following section. There the Airy integrals are replaced by the perhaps more familiar modified Bessel functions, $K_{1/3}$ and $K_{2/3}$, as was done by Schwinger for ordinary synchrotron radiation. Here the Airy functions are retained because the arguments $z_\pm$ can be positive or negative, depending on the sign of a and the value of ν. With Bessel functions it is necessary to treat separately the "exponential" domain ($z>0$) and the "oscillatory" domain ($z<0$). A single definition of the Airy function suffices for the whole range of z.

The functional dependence on angle and frequency is sufficiently complicated that general discussion is not profitable, but some remarks on the limit of large $|a|$ are in order. For definiteness, assume $a>0$. (The opposite choice follows very similarly.) Also consider $\theta_0=0$ or $\theta_0=\pi$, i.e., spin-flip along the magnetic field direction. Since the minimum value of z_+ is $3(a/2)^{2/3}(1+t^2)^{1/3}$ and the Airy functions decrease exponentially for large positive argument, the second group of terms in (57) will vanish rapidly for large a. Only the third group survive, corresponding to the transition from "upper" to "lower" energy level in the intuitive description. Furthermore, z_- is large positively or negatively, except for a very small range of t^2 for a given ν and a. The angular distribution is therefore sharply peaked in angle, just as in Airy's original application to the rainbow, for a given frequency. The result is in complete correspondence with the simple delta function dependence of Eq. (13).

E. Differential frequency spectrum

Integration of (57) over angles leads to the frequency spectrum of spin-flip synchrotron radiation. This is accomplished by noting from Fig. 3 that for small θ, $\theta\sin\varphi\simeq\sin\theta\sin\varphi=\cos\theta'$, where θ' is a polar angle measured from the z axis. Introducing a corresponding azimuthal angle φ' and noting that the distribution (57) is confined to a range of angles $\theta'=\pi/2\pm O(\gamma^{-1})$, we can write the solid angle element as

$$d\Omega=d\varphi'\, d(\cos\theta')\simeq(1/\gamma)d\varphi'\, dt,$$

with the range of t effectively $(-\infty,\infty)$ for $\gamma\gg1$. The integration involving squares of Airy functions can be performed by means of formulas derived by Aspnes (1966) in another connection. The relevant formulas are given in the Appendix. The result for the differential frequency spectrum times $(5\tau_0/\sqrt{3})$ is

$$\frac{5\tau_0}{\sqrt{3}}\frac{dw}{d\nu}=\sin^2\theta_0\Big\{a^2y_0\mathrm{Ai}(y_0)+\Big(a+\frac{a^2}{2}\Big)y_0^2\mathrm{Ai}'(y_0)+\Big(1+a+\frac{a^2}{2}\Big)y_0^3\mathrm{Ai}_1(y_0)\Big\}$$
$$+\tfrac{1}{4}(1-\cos\theta_0)^2\Big\{-[a^2+2a^3+3(1+a)^2\nu]y_0\mathrm{Ai}(y_+)-2\Big(1+\frac{5a}{2}+\frac{5a^2}{4}\Big)y_0^2\mathrm{Ai}'(y_+)$$
$$+\Big[a^4+\tfrac{9}{2}(1+a)\nu^2-\frac{9}{2}\Big(1+\frac{3a}{2}+\frac{a^2}{4}\Big)\Big(\nu^2+\frac{4a\nu}{3}\Big)\Big]\mathrm{Ai}_1(y_+)\Big\}$$
$$+\tfrac{1}{4}(1+\cos\theta_0)^2\Big\{-[a^2+2a^3-3(1+a)^2\nu]y_0\mathrm{Ai}(y_-)-2\Big(1+\frac{5a}{2}+\frac{5a^2}{4}\Big)y_0^2\mathrm{Ai}'(y_-)$$
$$+\Big[a^4+\tfrac{9}{2}(1+a)\nu^2-\frac{9}{2}\Big(1+\frac{3a}{2}+\frac{a^2}{4}\Big)\Big(\nu^2-\frac{4a\nu}{3}\Big)\Big]\mathrm{Ai}_1(y_-)\Big\}. \quad (58)$$

In Eq. (58) the variables $y_0, y_\pm$ are

$$\left.\begin{aligned} y_0&=(3\nu/2)^{2/3},\\ y_\pm&=(3\nu/2)^{2/3}[1\pm(4a/3\nu)],\end{aligned}\right\} \quad (59)$$

and the function $\mathrm{Ai}_1(x)=\int_x^\infty \mathrm{Ai}(x')\,dx'$.

This formula, like its predecessor, is a formidable function of ν, a, and θ_0. For $a=0$, it simplifies greatly—see Eq. (72) below. In the opposite limit of $|a|\gg1$ it approaches the expression (14). This can be seen as follows. Choose $a>0$ and $\theta_0=0$ or $\theta_0=\pi$, just as below Eq. (57). For large a, y_+ is always large and positive; the Airy functions of this argument are negligible; only the last group of terms in (58) are important. When $\nu>4a/3$, $y_->0$. Then for large a, even these Airy functions decrease rapidly as ν goes above $4a/3$. The spectrum thus cuts off effectively at $\nu=\nu_{\max}=4a/3$. This is equivalent to $\omega=\omega_{max}$ in Eq. (14). For ν values significantly less than ν_{max}, the argument y_- is large and negative. The functions $\mathrm{Ai}(y_-)$ and $\mathrm{Ai}'(y_-)$ are rapidly oscillating and damped by a power, while $\mathrm{Ai}_1(y_-)\simeq 1+$ (damped oscillatory terms). The frequency spectrum is given by the coefficient of $\mathrm{Ai}_1(y_-)$ in (58). For large a, this is exactly Eq. (14).

The changes in the *shape* of the frequency spectrum as a function of a are illustrated in Fig. 4. Only the "down" transition ($\theta_0=0$) is displayed, but the other transition has qualitatively similar spectra (with $a\to -a$). Because the frequency spectrum extends over the range $0<\omega<|a|\gamma^3\omega_0$ for large $|a|$, the abscissa in Fig. 4 has been scaled according to $x=\nu(1+16a^2/81)^{-1/2}$. To facilitate further the comparison of the shapes for different a, the ordinates have been scaled so that the curves have unit area. The asymptotic shape of Eq. (14) is then $(1+2x/3-2x^2/9)/4$, shown as the dashed curve in Fig. 4. For negative a, for which the "down" transition is unfavored, and even for $a=0$, the frequency spectrum is far from the simple parabolic shape. For $a>0$, the behavior described in the preceding paragraph is evident. For large a, the spectrum is given for $x<3$ by the asymptotic shape plus oscillatory terms decreasing as $(2\pi a)^{-1/2}$, and for $x>3$ by $(2\pi a)^{-1/2}$ times exponentially decreasing terms.

F. Total rate, characteristic time and polarization

The total transition rate is obtained from Eq. (58) by integration over frequencies. For the Airy functions of argument y_0 the integrals are straightforward, but for those of argument $y_\pm$ they are less so. Again the relevant integrals are discussed in the Appendix. The result for the total transition probability for a spin-flip transition is

$$w=\frac{1}{2\tau_0}\left\{\left[F_1(a)e^{-\sqrt{12}|a|}+\frac{a}{|a|}F_2(a)\right]\left(\frac{1+\cos^2\theta_0}{2}\right)+F_3(a)\sin^2\theta_0+F_2(a)\cos\theta_0\right\}, \tag{60}$$

where

$$\left.\begin{aligned}F_1(a)&=(1+\tfrac{41}{45}a-\tfrac{23}{18}a^2-\tfrac{8}{15}a^3+\tfrac{14}{15}a^4)\\&\quad-\frac{8}{5\sqrt{3}}\cdot\frac{a}{|a|}(1+\tfrac{11}{12}a-\tfrac{17}{12}a^2-\tfrac{13}{24}a^3+a^4),\\F_2(a)&=\frac{8}{5\sqrt{3}}(1+\tfrac{14}{3}a+8a^2+\tfrac{23}{3}a^3+\tfrac{10}{3}a^4+\tfrac{2}{3}a^5),\\F_3(a)&=\tfrac{1}{18}(7-2a+\tfrac{13}{5}a^2).\end{aligned}\right\} \tag{61}$$

Equation (60) is a generalization to arbitrary g factor of the result (2) of Baier and Katkov (1967a). It was first obtained (explicitly for $\theta_0=0,\pi$) by Derbenev and Kondratenko (1973), using a standard method that bypasses the differential spectra in angle and frequency, and goes directly to the total transition rate.

The polarization of an initially unpolarized beam grows in time according to Eq. (1), but with a mean life τ obtained by summing the rates for $\theta_0=0$ and $\theta_0=\pi$.[7] This yields a characteristic time,

$$\tau=\tau_0\left[F_1(a)e^{-\sqrt{12}|a|}+\frac{a}{|a|}F_2(a)\right]^{-1}. \tag{62}$$

The asymptotic polarization (in the negative z direction in Fig. 3, or in the direction of $\dot{\hat\beta}\times\hat\beta$) is

$$P=F_2(a)\Big/\left[F_1(a)e^{-\sqrt{12}|a|}+\frac{a}{|a|}F_2(a)\right]. \tag{63}$$

The growth time τ in units of τ_0 is shown as a function of a or g in Fig. 5. It decreases as $|a|^{-5}$ for large $|a|$, in conformity with the elementary result Eq. (11), but exhibits a maximum at $a=-0.498$ ($g=1.004$) where $\tau/\tau_0=4.76$. The polarization P as a function of a or g is shown in Fig. 6. For large $|a|$, the polarization approaches $+1$ or -1, in accord with the ideas of a spontaneous transition from "upper" to "lower" energy level of the spin system. But for modest $|g|$, the interplay of orbital and spin motion causes drastic departures. At $g=0$, for example, where there is no splitting between the hypothetical levels, $\tau\simeq 1.58\tau_0$ and $P\simeq -0.98$. In fact, for the range of g factors, $0<g<1.2$, the "wrong" spin energy level is preferentially populated.

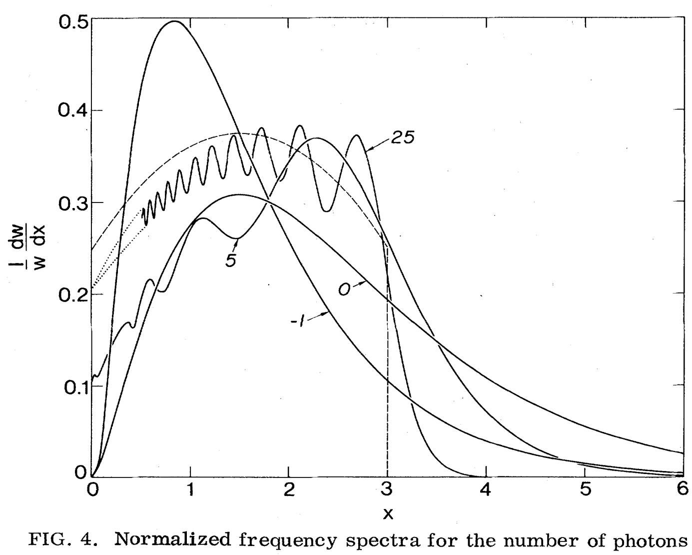

FIG. 4. Normalized frequency spectra for the number of photons per unit frequency interval of the "down" transition [Eq. (58) with $\theta_0=0$] for different values of the anomaly, a. The abscissa variable is a scaled frequency, $x=\nu(1+16a^2/81)^{-1/2}$. The numbers adjacent to the curves are the values of a. The total transition probabilities are very different for different values of a; the curves have been normalized to unit area to facilitate comparison of their shapes. The dashed curve is the asymptotic spectrum ($|a|\to\infty$) of Eq. (14).

V. ANGULAR AND FREQUENCY DISTRIBUTIONS FOR $g=2$

The only physically relevant g factor is $g=2$, appropriate for electrons and positrons. The total transition rate for spin-flip synchrotron radiation has been discussed in the Introduction. Here we examine the angular and frequency distributions of the radiation. These

[7]See the solutions for the temporal behaviors of the components of the polarization vector given by Baier (1971a, b), Sec. 3, especially Eq. (3.23) ff.

are of academic interest only because, as we observed in connection with Eq. (3), the energy radiated in the spin-flip transitions is negligible compared with the ordinary synchrotron radiation provided $\gamma \ll \gamma_c$.

A. Angular distributions of photons and of radiated power

The starting point is Eq. (57), specialized to $a=0$, for the doubly differential transition rate in angle and frequency

$$\frac{d^2w}{d\Omega\, d\omega} = \frac{3\sqrt{3}}{40\pi^3}\,\frac{\nu^3(1+t^2)}{\tau_0\gamma^2\omega_0} \times \{\sin^2\theta_0 K_{1/3}^2(\eta) + \tfrac{1}{2}(1+\cos^2\theta_0)(1+t^2)[K_{1/3}^2(\eta)+K_{2/3}^2(\eta)] + 2\cos\theta_0\sqrt{1+t^2}\,K_{1/3}(\eta)K_{2/3}(\eta)\}. \quad (64)$$

Here we have changed from Airy functions to modified Bessel functions. The argument of the latter is

$$\eta = \frac{2}{3}(z_0)^{3/2} = \frac{\nu}{2}(1+t^2)^{3/2}.$$

The angular distribution of photons (number of photons per unit time per unit solid angle[8]) is obtained by integration over frequencies. Making use of formula 6.576.4, p. 693, of Gradshteyn and Ryzhik (1965), we find

$$\frac{dw}{d\Omega} = \frac{16\gamma}{45\pi^2\tau_0}(1+t^2)^{-5}\left\{\sin^2\theta_0 + \tfrac{9}{8}(1+\cos^2\theta_0)(1+t^2) + \frac{105\sqrt{3}\,\pi}{256}\sqrt{1+t^2}\cos\theta_0\right\}. \quad (65)$$

Recall that the angle variable t is, for small θ, γ times the *latitude* with respect to the z axis of Fig. 3, that is, the angle between the direction of emission and the instantaneous plane of the orbit. It is the traditional synchrotron radiation angle, called ψ by Schwinger (1949) and θ by Jackson (1975).

The angular distribution of *radiated power* (energy per unit time per unit solid angle) is obtained by multiplying (64) by $\hbar\omega$ and then integrating over frequencies. The result is[8]

$$\frac{d\mathcal{P}}{d\Omega} = \frac{77\sqrt{3}}{256\pi}\,\frac{\gamma^4\hbar\omega_0}{\tau_0}(1+t^2)^{-13/2}\left\{\sin^2\theta_0 + \tfrac{12}{11}(1+\cos^2\theta_0)(1+t^2) + \frac{2^{13}\sqrt{3}}{3^3\cdot 77\pi}\sqrt{1+t^2}\cos\theta_0\right\}. \quad (66)$$

These angular distributions can be compared with the angular distribution of radiated power for the ordinary (nonflip) synchrotron radiation,

$$\frac{d\mathcal{P}_{\text{ordinary}}}{d\Omega} = \frac{\gamma^5}{32\pi}\left(\frac{e^2c}{\rho^2}\right)\cdot\frac{(7+12t^2)}{(1+t^2)^{7/2}}. \quad (67)$$

We see that in the relativistic domain all the angular distributions are confined to angles of the order of γ^{-1} away from the instantaneous orbital plane, with $t=\gamma\psi$ as the natural variable. The spin-flip angular distributions

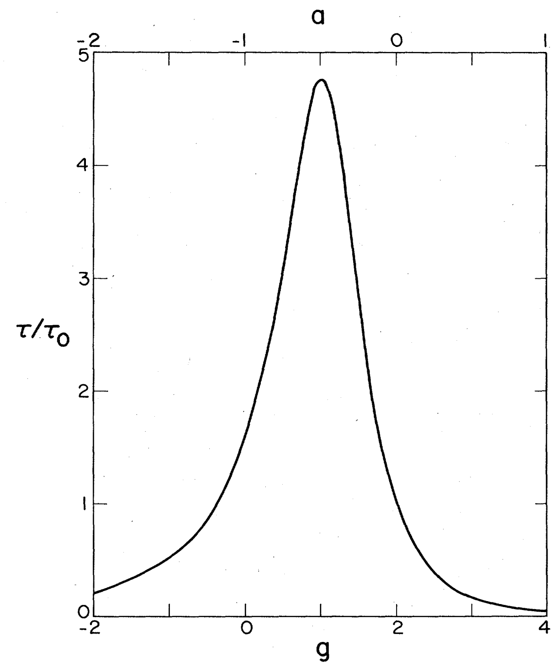

FIG. 5. Characteristic time τ for growth of transverse polarization in units of the electron-positron time τ_0, Eq. (1c), as a function of anomaly a (top scale) or g factor (bottom scale).

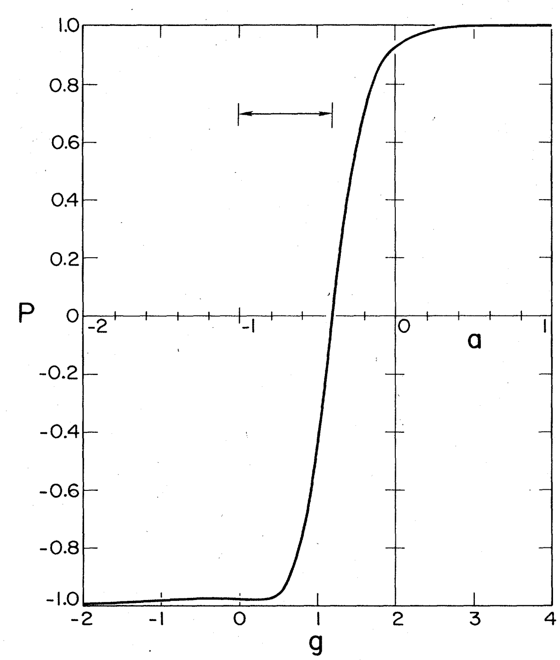

FIG. 6. Asymptotic transverse polarization P as a function of the anomaly a or g factor. Positive values of P correspond to a preponderance of spins in the direction of $\dot{\beta}\times\beta$ (the direction of the guiding magnetic field for $e>0$). For $0<g<1.2$, the particles' magnetic moments end up preferentially opposite to the magnetic field, contrary to naive expectations. This range is indicated by the horizontal arrow.

[8]Strictly, the number of photons per unit time is not an instantaneous rate but actually the number of photons per passage of the particle times the repetition rate $\omega_0/2\pi$. Similarly, the radiated power is energy per passage times $\omega_0/2\pi$.

are somewhat narrower than the nonflip, the power decreasing as $|t|^{-11}$ compared to $|t|^{-5}$ at large $|t|$. This is a reflection of the harder photon spectrum of the spin-flip, magnetic radiation with an overall additional factor of ω^2 in its frequency spectrum relative to that emitted by a charge. Similarly, the difference in t dependence between the *number* distribution (65) and the *energy* distribution (66) is explained by the fact that the softer photons have a broader distribution in angle than the harder ones.

B. Total transition rate and total spin-flip power radiated

The total transition rate of Baier and Katkov is obtained by specialization of (60) to $a=0$ or integration of (65) over angles with $d\Omega=\gamma^{-1}\,dt\,d\phi'$. The result is Eq. (2), averaged over the azimuth of $\vec{\zeta}$, which in the present notation is

$$w=\frac{1}{2\tau_0}\left[1-\tfrac{1}{9}\sin^2\theta_0+\frac{8}{5\sqrt{3}}\cos\theta_0\right]. \tag{68}$$

The total spin-flip power, from (66), is

$$\mathcal{P}_{\text{spin-flip}}=\frac{16}{5\sqrt{3}}\,\frac{\gamma^3\hbar\omega_0}{\tau_0}\left[1-\tfrac{1}{12}\sin^2\theta_0+\frac{35\sqrt{3}}{64}\cos\theta_0\right]. \tag{69}$$

The ordinary radiated power is

$$\mathcal{P}_{\text{ordinary}}=\frac{2}{3}\left(\frac{e^2}{\hbar\rho}\right)\gamma^4\hbar\omega_0. \tag{70}$$

This leads to a ratio of spin-flip to ordinary power of

$$\frac{\mathcal{P}_{\text{spin-slip}}}{\mathcal{P}_{\text{ordinary}}}=3\left(\frac{\hbar\gamma^2}{mc\rho}\right)^2\left[1-\tfrac{1}{12}\sin^2\theta_0+\frac{35\sqrt{3}}{64}\cos\theta_0\right] \tag{71}$$

in agreement with (3) for $\cos\theta_0=\pm1$.

C. Frequency distributions

The frequency spectrum can be found by integrating (64) over angles. The necessary integrals are those of Aspnes (1966). Since these are given in terms of Airy functions in the Appendix, it is appropriate to abstract the result from the general expression, Eq. (58). Upon converting the Airy functions to Bessel functions, we find

$$\frac{dw}{d\nu}=\frac{9}{10\pi}\,\frac{\nu^2}{\tau_0}\left[\tfrac{1}{2}\sin^2\theta_0\int_\nu^\infty K_{1/3}(s)\,ds+\tfrac{1}{2}(1+\cos^2\theta_0)K_{2/3}(\nu)+\cos\theta_0K_{1/3}(\nu)\right]. \tag{72}$$

The corresponding expression for the spin-flip power radiated per unit interval in ν is

$$\frac{d\mathcal{P}_{\text{spin-flip}}}{d\nu}=\frac{27}{20\pi}\left(\frac{\gamma^3\hbar\omega_0}{\tau_0}\right)\nu^3\left[\tfrac{1}{2}\sin^2\theta_0\int_\nu^\infty K_{1/3}(s)\,ds+\tfrac{1}{2}(1+\cos^2\theta_0)K_{2/3}(\nu)+\cos\theta_0K_{1/3}(\nu)\right]. \tag{73}$$

This can be compared with the frequency spectrum of the ordinary synchrotron radiation,

$$\frac{d\mathcal{P}_{\text{ordinary}}}{d\nu}=\mathcal{P}_{\text{ordinary}}\left[\frac{9\sqrt{3}}{8\pi}\,\nu\int_\nu^\infty K_{5/3}(s)\,ds\right] \tag{74}$$

with the total power given by (70).

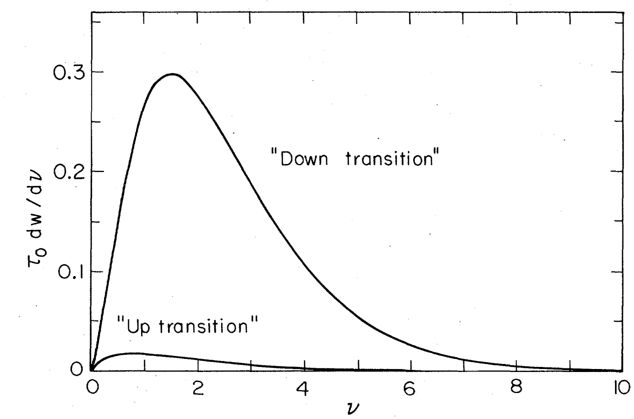

FIG. 7. Normalized frequency spectra $\tau_0 dw/d\nu$ for the *number of photons* emitted per unit interval in the dimensionless frequency variable $\nu=2\omega/3\gamma^3\omega_0$. The dominant "down" transition corresponds to a spin-flip from $\cos\theta_0=+1$ to $\cos\theta_0=-1$ (spin finally in the direction opposite to $\hat{\beta}\times\dot{\hat{\beta}}$). The small "up" transition is in the reverse direction.

The normalized frequency distributions of the *number* of photons emitted per unit time in spin-flip transitions are shown in Fig. 7 for the "down" transition ($\cos\theta_0=+1$) and the "up" transition ($\cos\theta_0=-1$). The spectrum for the predominant "down" transition peaks around $\nu\simeq1.5$ and extends to well beyond $\nu=4$. The weaker "up" transition consists of somewhat softer photons, with a maximum at $\nu\simeq0.7$. The areas are, respectively, 0.962 and 0.038, the "down" transition being 25.25 times as probable as the "up".

A graphical comparison of the *separately* normalized power spectra for the spin-flip and the nonflip synchrotron radiations is given in Fig. 8. For the ordinary radiation the quantity plotted is the coefficient of $\mathcal{P}_{\text{ordinary}}$ in

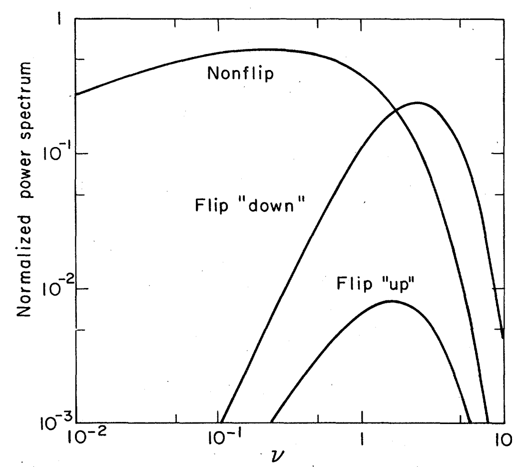

FIG. 8. Log–log plot of separately normalized ordinary (nonflip) and spin-flip *power* frequency spectra as functions of the dimensionless variable $\nu=2\omega/3\gamma^3\omega_0$. The actual spin-flip power is much smaller than the ordinary power provided $\gamma\ll\gamma_c$ [see Eq. (3) or (71)]. At low frequencies ($\nu\ll1$), the nonflip distribution varies as $\nu^{1/3}$, while the spin-flip distributions vary as $\nu^{7/3}$. At high frequencies ($\nu\gg1$) all spectra vanish exponentially (times different powers).

(74). For the spin-flip radiation it is $27\sqrt{3}\,\nu^3/128\pi$ times the square-bracket in (73) with $\cos\theta_0=\pm1$. All the power spectra fall exponentially for large ν, but for $\nu\lesssim1$ their behaviors are very different. The ordinary synchrotron radiation spectrum is proportional to $\nu^{1/3}$ for small ν, while the spin-flip spectra vary as $\nu^{7/3}$. The spin-flip radiation involves harder photons, as already mentioned in discussion of the angular distributions. The presence of an extra factor of ω^2 in the frequency distribution of radiation arising from a magnetic moment in motion as compared to that for a charge in motion is a general feature, classically and quantum mechanically.

VI. SUMMARY

The primary purpose of this paper is didactic: to present as intuitive an interpretation as possible of the gradual transverse polarization of electron and positron beams as they orbit in storage rings. A simple intuitive description of the process, utilizing a moving inertial frame, is shown to be deficient, even though it appears superficially to give roughly correct answers for electrons and positrons. The basic reason for its failure (and hence the absence of a truly simple description) is that the spin system cannot be treated in isolation because it is imbedded in a virtual continuum of states associated with the mechanical motion of the particle. Only for large g factors is the spin precession rapid enough that the magnetic-moment-spin system effectively decouples from the orbital motion. Then the simple treatment becomes valid.

A semiclassical description of the radiative process is given by analogy with the well-known semiclassical treatment of radiation by a charged particle. The classical relativistic equation of motion for a spin in arbitrary motion in electromagnetic fields (the Thomas–BMT equation) yields an effective Hamiltonian for the coupling of a spin to electromagnetic fields. In analogy with the substitution

$$e\vec{\beta}\cdot\vec{A}_{\text{external}}\rightarrow e\vec{\beta}\cdot\vec{A}_{\text{photon}}\ ,$$

in the conventional transition to emission process in the interaction Hamiltonian for a charged particle, we replace the external $\vec{E}$ and $\vec{B}$ fields in the Thomas-BMT effective Hamiltonian with the corresponding fields for a photon. Perturbation theory then yields an essentially classical expression for the transition probability with quantum mechanics entering only via the matrix element of the particle's spin operator.

Some new results are derived concerning the differential angular and frequency distributions of the spin-flip synchrotron radiation. The results of Derbenev and Kondratenko (1973) for the characteristic time τ and the asymptotic polarization P for a charged particle of spin-$\frac{1}{2}$ and arbitrary g factor are confirmed. Since electrons and positrons are the only particles likely to show detectable polarizations by this mechanism, these results are of no practical interest. They serve a pedagogic purpose, however, since they permit the upsetting of one of the key concepts of the naive description, namely, that the polarization arises from spontaneous emission as the spin moves from its "upper" to its "lower" state in the magnetic field. It is found that for $0<g<1.2$ the opposite is true.

The angular and frequency distributions of numbers of photons and of radiated power are presented for the physically interesting circumstance of $g=2$. They are compared with the corresponding spectra for the ordinary synchrotron radiation. This again is of limited practical value because of the minuteness of the spin-flip radiation, but may serve a pedagogic end.

The reader may, with justification, feel that the author has wandered endlessly in a labyrinth of Airy functions without coming to grips with the minotaur, the mysterious and peculiar $8/5\sqrt{3}$! Why *is* the polarization for electrons so large, and yet not complete? I have no compelling answer. Inspection of Fig. 6 shows that $8/5\sqrt{3}$ is only one from a continuum of possible values, depending upon a. Since there is nothing special about $a=0$ in the effective Hamiltonian (40) that serves as the basis for the calculation, there seems no reason to put special emphasis on the particular value of P that emerges when $a=0$. Admittedly, the Thomas–BMT equation and the effective Hamiltonian *are* simpler when $a=0$. To those who focus on that fact let me observe that ordinary synchrotron radiation, as well as spin-flip, abounds in square roots of 3. They can be traced to the Airy integral, Eq. (A1) whose 3 in the exponent can be attributed to the expansion of $\sin\theta\simeq\theta-\theta^3/6+\cdots$, that is, the approximation of a small segment of the trajectory by the arc of a circle! I personally believe that at least the $\sqrt{3}$ in $8/5\sqrt{3}$ has no more mysterious origin. Prove me wrong!

Finally, we note that our concern has been with the basic phenomenon and mechanism of transverse polarization by spin-flip synchrotron radiation. Important practical aspects of the secular motion of spins in e^+e^- storage rings and of various mechanisms of detection of the transverse polarization can be found in the papers by by Baier (1971a, b), Derbenev and Kondratenko (1972, 1973), Derbenev, Kondratenko, and Skrinskii (1971), Ford, Mann and Ling (1972), Schwitters (1974), and the references cited therein.

ACKNOWLEDGMENTS

The author is indebted to R. J. Riddell, Jr. for the computations of the spectra in Fig. 4. He is also indebted to Ya. S. Derbenev and A. M. Kondratenko for drawing his attention to their prior work on this topic, and more importantly, for pointing out a serious error in an earlier version of this paper.

APPENDIX

The Airy functions $\mathrm{Ai}(x)$, $\mathrm{Ai}'(x)$, and $\mathrm{Ai}_1(x)$ are defined by

$$\left.\begin{aligned}\mathrm{Ai}(x)&=\frac{1}{\pi}\int_0^\infty\cos\left(\frac{t^3}{3}+xt\right)dt\\&=\frac{1}{2\pi}\int_{-\infty}^\infty e^{it^3/3+ixt}\,dt\end{aligned}\right\}\qquad\text{(A1a)}$$

and

$$\mathrm{Ai}'(x)=d\,\mathrm{Ai}(x)/dx\qquad\text{(A1b)}$$

and

$$\mathrm{Ai}_1(x)=\int_x^\infty \mathrm{Ai}(x')\,dx' \ . \tag{A1c}$$

The Airy function Ai(x) satisfies the differential equation

$$y''-xy=0 \ . \tag{A2}$$

Various formulas for Airy functions, including asymptotic expansions for large positive and negative argument, can be found in Abramowitz and Stegun (1964), p. 446 ff.

In the integration of Eq. (57) over angles the following classes of integrals arise:

$$F_p(x)=\int_0^\infty \frac{du}{\sqrt{u}}u^p[\mathrm{Ai}(u+x)]^2 \ , \tag{A3a}$$

$$G_p(x)=\int_0^\infty \frac{du}{\sqrt{u}}u^p[\mathrm{Ai}'(u+x)]^2 \ , \tag{A3b}$$

$$H_p(x)=\int_0^\infty \frac{du}{\sqrt{u}}u^p\mathrm{Ai}(u+x)\mathrm{Ai}'(u+x) \ , \tag{A3c}$$

where $p=0,1,2,\ldots$. This type of integral has been evaluated by Aspnes (1966). Although he gives recursion relations, for convenience we exhibit explicitly the relevant results.

$$\left.\begin{aligned}
F_0(x)&=\tfrac{1}{2}\mathrm{Ai}_1(y) \ ,\\
F_1(x)&=-\frac{2^{1/3}}{8}[\mathrm{Ai}'(y)+y\mathrm{Ai}_1(y)] \ ,\\
F_2(x)&=\frac{3.2^{2/3}}{64}[\mathrm{Ai}(y)+y\mathrm{Ai}'(y)+y^2\mathrm{Ai}_1(y)] \ ,\\
G_0(x)&=-\frac{2^{1/3}}{8}[3\mathrm{Ai}'(y)+y\mathrm{Ai}_1(y)] \ ,\\
G_1(x)&=\frac{2^{2/3}}{64}[5\mathrm{Ai}(y)+y\mathrm{Ai}'(y)+y^2\mathrm{Ai}_1(y)] \ ,\\
H_0(x)&=-\frac{2^{2/3}}{4}\mathrm{Ai}(y) \ ,\\
H_1(x)&=-\tfrac{1}{8}\mathrm{Ai}_1(y) \ .
\end{aligned}\right\} \tag{A4}$$

Here $y=2^{2/3}x$.

In the integration of Eq. (58) over frequencies, after introduction of a new variable $s=(3\nu/2)^{1/3}$ and some integration by parts, the remaining integrals are of the form

$$I_n^{(\pm)}(a)=\int_0^\infty ds\, s^{3n+1}\mathrm{Ai}\left(s^2\pm\frac{2a}{s}\right) \ , \tag{A5}$$

with $n=0,1,2,3$. For definiteness, we consider $a>0$. The results for negative a can be obtained by interchange of the roles of $I_n^{(+)}$ and $I_n^{(-)}$. By use of the differential equation (A2) it is possible to show that the I_n satisfy a recursion formula

$$I_{n+1}^{(\pm)}=\pm\left[\frac{a}{4}\frac{d^2}{da^2}-\frac{3n+1}{4}\frac{d}{da}-2a\right]I_n^{(\pm)}(a) \ . \tag{A6}$$

Thus, only $I_0^{(\pm)}(a)$ need be evaluated. From (A1a) the Airy function in (A5) can be written

$$\mathrm{Ai}\left(s^2\pm\frac{2a}{s}\right)=\frac{1}{2\pi}\int_{-\infty}^\infty dt\,\exp i\left[\frac{t^3}{3}+\left(s^2\pm\frac{2a}{s}\right)t\right] .$$

A change of variable, $t=sz$, leads to

$$\mathrm{Ai}\left(s^2\pm\frac{2a}{s}\right)=\frac{s}{2\pi}\int_{-\infty}^\infty dz\,\exp i\left[s^3\left(\frac{z^3}{3}+z\right)\pm 2az\right] . \tag{A7}$$

If this now inserted into (A5) for $n=0$, the integration variable s is replaced by $u=s^3$, and the orders of integration are interchanged, we obtain

$$I_0^{(\pm)}(a)=\frac{1}{6\pi}\int_{-\infty}^\infty dz\, e^{\pm 2iaz}\int_0^\infty du\,\exp i\left[\left(\frac{z^3}{3}+z\right)u\right] . \tag{A8}$$

Here it is necessary to give z a small positive imaginary part in order to give meaning to the u integral at its upper limit. In terms of a contour integral in z it means that the contour goes above the origin.

The u integral in Eq. (A8) is elementary and leads to

$$I_0^{(\pm)}(a)=\frac{i}{2\pi}\int_{-\infty}^\infty dz\frac{e^{\pm 2iaz}}{(z+i\epsilon)(z^2+3)} \ , \tag{A9}$$

where ϵ is a positive infinitesimal. This remaining integral can be done by contour integration. The denominator has simple zeros at $z_1=-i\epsilon, z_2=i\sqrt{3}, z_3=-i\sqrt{3}$. For the positive sign (and $a>0$) the contour is closed with impunity in the upper half plane, enclosing the pole at $z=i\sqrt{3}$. The result is

$$I_0^{(+)}(a)=\tfrac{1}{6}\,e^{-\sqrt{12}a} . \tag{A10}$$

Similarly, for the negative sign, the contour is closed in the lower half plane, with residues at $z=-i\epsilon$ and $z=-i\sqrt{3}$, yielding

$$I_0^{(-)}(a)=\tfrac{1}{3}-\tfrac{1}{6}\,e^{-\sqrt{12}a} . \tag{A11}$$

With (A10) or (A11) and the recursion formula (A6), the necessary values of $I_n^{(\pm)}(a)$ can be generated. For negative a, we have

$$I_n^{(\pm)}(-|a|)=I_n^{(\mp)}(|a|) . \tag{A12}$$

For positive x, the Airy function and its derivative are related to the modified Bessel functions according to

$$\left.\begin{aligned}
\mathrm{Ai}(x)&=\frac{1}{\pi}\left(\frac{x}{3}\right)^{1/2}K_{1/3}(\eta) \ ,\\
\mathrm{Ai}'(x)&=\frac{-1}{\pi}\frac{x}{\sqrt{3}}K_{2/3}(\eta)
\end{aligned}\right\} \tag{A13}$$

where

$$\eta=\tfrac{2}{3}x^{3/2} \ . \tag{A14}$$

We also have, for $x>0$,

$$\mathrm{Ai}_1(x)=\frac{1}{\pi\sqrt{3}}\int_\eta^\infty K_{1/3}(\eta')\,d\eta' \ . \tag{A15}$$

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