# Introduction to the technique of dimensional regularization*

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**The purpose of this review article is to explain and illustrate in detail the technique of dimensional regularization, which is a major mathematical tool in the renormalization program of gauge theories. The most important single feature of the new technique is the concept of analytic continuation in the number of space–time dimensions 2ω, where the regulating parameter ω is complex in general, and ω = 2 corresponds to four-dimensional space–time. The technique of dimensional regularization preserves the local gauge symmetry of the underlying Lagrangian and thereby permits a consistent gauge-invariant treatment of divergent Feynman integrals to all orders in perturbation theory. The method can thus be applied—as demonstrated in this article—not only to Abelian gauge models, but more importantly to non-Abelian theories such as Yang–Mills fields and quantum gravity, to which the majority of conventional regularization procedures is inapplicable. We illustrate both the advantages and the limitation of dimensional regularization, as well as its extension to massless particles.**

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## I. INTRODUCTION

### A. Ultraviolet infinities in quantum field theory

#### 1. Preliminary remarks

The treatment of ultraviolet infinities continues to be one of the most challenging and tenacious problems in relativistic quantum field theory. Only in quantum electrodynamics has it been possible to eliminate these infinities consistently and in a physically meaningful manner by absorbing them into the bare, i.e., unobserved, charge and mass of the electron. The result is the successful renormalization program of Dyson (1949a,b) and Salam (1951a,b), in which the renormalized quantities, mass and charge, correspond precisely to the observed mass and charge of the electron. Although many questions remain to be answered for either the strong, gravitational, or weak interactions, there are indications now that perhaps one or even two of these interactions might be renormalized in the not-too-distant future. The reasons for this cautious optimism are (i) the recent progress in the unification of electromagnetic and weak interactions, (ii) the breakthrough in the renormalizability of certain non-

  

Abelian gauge field theories, and (iii) a new powerful regularization method known as the technique of *dimensional regularization*. It is the purpose of this review article to explain in some detail the general features of dimensional regularization and to illustrate it with examples drawn from both Abelian and non-Abelian gauge theories.

## 2. Previous regularization techniques

### (a) *Pauli–Villars regularization*

A regularization technique is any mathematical prescription which renders divergent Feynman amplitudes finite by means of a specific cutoff procedure. The technique is then said to *regularize* the divergent integrals. One of the earliest regularization procedures is due to Pauli and Villars (1949), who introduced massive auxiliary fields called *regulators* in order to eliminate the singularities from propagators and other ill-defined field functions (Bogoliubov and Shirkov, 1959). The Pauli–Villars technique gives a precise prescription for the use of these regulators in quantum electrodynamics in such a way that the theory remains gauge-invariant at each order of perturbation theory. Since the auxiliary fields do not admit a physical interpretation, the Pauli–Villars method must be regarded as a purely formal cutoff procedure.

### (b) *Analytic regularization*

The method of analytic regularization differs completely from that of Pauli and Villars, in that it exploits for the first time the concept of *analytic continuation* in some complex parameter $\alpha$. To gain an overview of this method, let us consider the Feynman propagator $(p^2+m^2-i\epsilon)^{-1}$ for a scalar particle of mass $m$ and four-momentum $p_\mu$. The crucial step in the prescription is to replace the above propagator by $(p^2+m^2-i\epsilon)^{-\alpha}$, where the regulating parameter $\alpha$ may be complex. The result of such a replacement is to transform originally divergent integrals into well behaved *analytic* functions of $\alpha$. Since the integrals are now convergent, they can be evaluated unambiguously by performing the usual operations of integration by parts, symmetric integration, etc. Once these formal manipulations have been executed one can continue the resulting expressions analytically to $\alpha=1$; the original ultraviolet divergences then reassert themselves as *poles* at $\alpha=1$. Subtraction of these poles at the end of the calculation yields the desired finite portion of the integral. It turns out that the notion of analytic continuation, first exploited by Speer, Bollini, and others is also an essential ingredient of the technique of dimensional regularization.

### (c) *Speer's analytic renormalization and the BPH approach*

Analytic renormalization has been popularized by several authors (Riesz, 1949; Gel'fand and Shilov, 1964; Bollini *et al.*, 1964; Güttinger, 1966; Caianiello, 1973), but especially by Speer (1968, 1969). In order to place Speer's work in proper perspective with earlier renormalization techniques, especially with the Bogoliubov–Parasiuk–Hepp (BPH) theory, we shall briefly review the principal features of Speer's approach.

Consider a perturbative quantum field theory in which a Feynman amplitude $\mathfrak{F}$ corresponds to the connected graph $G(V,\mathfrak{L})$ with vertices $V_i\in V$, $i=1,2,\ldots,n$ and internal lines $l\in\mathfrak{L}$, $l=1,2,\ldots,m$. The graph $G(V,\mathfrak{L})$ gives in general rise to ill-defined products of Green's functions of the form

$$\prod_{l\in\mathfrak{L}}\Delta_F{}^l(x_{i_l}-x_{f_l}),\tag{1.1}$$

where $x_{i_l}$ and $x_{f_l}$ denote, respectively, the initial and final coordinates of the $l$th line (Speer, 1968, 1969; Breitenlohner and Mitter, 1968). Following Speer (1968), we write the causal propagator in momentum space for a spinless particle of mass $m_l>0$ as

$$\tilde{\Delta}_F{}^l(p)=iZ_l(p)(p^2+m_l{}^2-i0)^{-1},\tag{1.2}$$

where $Z_l(p)$ is an arbitrary but regular polynomial of degree $r_l$. The first significant step in Speer's prescription is to replace[1] the propagator (1.2) by the *generalized propagator* (Gel'fand and Shilov, 1964; Speer, 1968)

$$\tilde{\Delta}_{\lambda_l}{}^l(p)=Z_l(p)e^{(1/2)i\pi\lambda_l}(p^2+m_l{}^2-i0)^{-\lambda_l},\tag{1.3}$$

where the regulating parameters $\lambda_l$, $l=1,2,\ldots,m$ are complex. It is essential in this kind of analytic renormalization that we assign a *different* $\lambda_l$ to each different internal line. The generalized propagator $\tilde{\Delta}_{\lambda_l}{}^l$ may then, with the help of Hepp's regularization (Hepp, 1966), be expressed as

$$\tilde{\Delta}_{\lambda_l}{}^l=\lim_{\epsilon\to0^+}\lim_{r\to0^+}\tilde{\Delta}_{\lambda_l,\epsilon,r}{}^l,\tag{1.4}$$

the distribution (Schwartz, 1966)

$$\tilde{\Delta}_{\lambda_l,\epsilon,r}{}^l(p)=\frac{Z_l(p)}{\Gamma(\lambda_l)}\int_r^\infty d\alpha_l\,\alpha_l{}^{\lambda_l-1}$$
$$\times\exp[i\alpha_l(p^2+m_l{}^2-i\epsilon)]\tag{1.5}$$

being an entire function of $(\lambda_1,\lambda_2,\ldots,\lambda_m)\equiv\boldsymbol{\lambda}$. For $\mathrm{Re}(\lambda_l)$ sufficiently large, the corresponding momentum integrals converge and can be evaluated with the aid of the integral representation (1.5). These operations—including integration over the auxiliary variables $\alpha_l$—lead eventually to the *generalized* Feynman amplitude $\mathfrak{F}(\lambda_l)$, which is an analytic function of the $\lambda$'s. If we were to continue the integrals in $\mathfrak{F}(\lambda_l)$ to the "physical" value $\boldsymbol{\lambda}^0=(1,1,\ldots,1)$, we would find that the original ultraviolet divergences reassert themselves either as poles of $\gamma$ functions or in some other singular form. In order to remove these singularities, Speer applies a certain operator $\mathcal{E}_{\lambda_l}$, called an *evaluator*, which effectively "regularizes" the generalized amplitude $\mathfrak{F}(\lambda_l)$. Analytic renormalization in the sense of Speer (1971, 1972) corresponds, therefore, basically to the extraction of the *regularized* part $\mathfrak{F}_{\mathrm{Reg}}$ at the physical point $\boldsymbol{\lambda}^0$:

$$\mathfrak{F}_{\mathrm{Reg}}=\mathcal{E}_{\lambda_l}\mathfrak{F}(\lambda_l).\tag{1.6}$$

In the simple case where $\mathfrak{F}(\lambda_l)$ contains only poles, the operator $\mathcal{E}_{\lambda_l}$ first symmetrizes the amplitude with respect to the $\lambda$'s and then extracts from $\mathfrak{F}(\lambda_l)$ the regular part of the Laurent expansion (Speer, 1972).

Using his method of evaluators, Speer was able to prove the equivalence between analytic renormalization and the additive renormalization procedure of Bogoliubov, Parasiuk, and Hepp.[2] The proof consists of demonstrating that analytic

[1] See the discussion in Sec. 2(b) above.

[2] The converse of this statement has been proven by Hepp (1971).

renormalization is additive in structure and that it can be implemented by the addition of counterterms to the original Lagrangian (Speer, 1968, 1969, 1971).

Although Speer's analytic renormalization is *not* gauge-invariant—the generalized propagators such as (1.3) violate the Slavnov–Taylor identities (Slavnov, 1972; Taylor, 1971)—his work is nevertheless of considerable importance, since it provides a close link with earlier renormalization proofs such as the BPH approach. Almost 20 years have elapsed since Bogoliubov and Parasiuk developed their *additive* renormalization scheme for the perturbation series in Lagrangian quantum field theory. The basic mathematical ingredient of that scheme is a recursive subtraction procedure known as Bogoliubov's $R$ operation (Bogoliubov and Parasiuk, 1957; Bogoliubov and Shirkov, 1959; Parasiuk, 1960). Bogoliubov's rules for the renormalized Feynman integrals were originally formulated in coordinate space and are equivalent to the addition of infinite counterterms to the Lagrangian (Bogoliubov and Parasiuk, 1957; Bogoliubov and Shirkov, 1959; Hepp, 1966, 1969a,b; Speer, 1969). The work of Bogoliubov and Parasiuk was subsequently refined by Hepp, who made several significant contributions to the theory of renormalization (Hepp, 1965, 1966, 1969a,b). He proved, among other things, that any two renormalizations—for example, the analytic and additive renormalizations of Speer and BPH—differ only by a finite renormalization (Hepp, 1971; Speer, 1972). Hepp (1971) also gave an axiomatic treatment of renormalization and showed that Speer's theory satisfies the given axioms.

There have been many other contributions to the theory of renormalization in recent years, notably by Zimmermann (1967, 1968, 1969, 1970) and Epstein and Glaser (1971, 1973). A detailed discussion of their powerful and mathematically rigorous expositions as well as those of other authors lies, however, outside the scope of this introductory review of dimensional regularization and we refer the reader to the fairly extensive literature on this subject (Westwater, 1969; Symanzik, 1969, 1970b; Guerra, 1971; Schroer, 1972; Lowenstein, 1971a,b, 1972; Stora, 1973; Lowenstein, Rouet, Stora, and Zimmermann, 1973; Bergere and Zuber, 1973).

## 3. Abelian and non-Abelian gauge theories

The purpose of this section is to summarize several useful definitions which are currently in vogue in connection with the renormalization of gauge theories ('t Hooft, 1971a,b; Lee, 1972; Lee and Zinn-Justin, 1972, 1973; 't Hooft and Veltman, 1972a,b). Of particular interest to us are the concepts of Abelian and non-Abelian gauge fields which will be used repeatedly in our subsequent illustrations of the continuous dimension method (Secs. IV–VII). For a detailed account of this fascinating subject of gauge theories we refer the reader to some excellent review articles, for example by Veltman (1974) and Abers and Lee (1973).

Consider a Lagrangian density $\mathcal{L}$ which is a function of $n$ fields $\phi_i(x)$, $i = 1, 2, \ldots, n$, and let us construct a Lie group $G$ of gauge transformations on these fields (Abers and Lee, 1973):

$$\phi \to \phi' = U(\Lambda)\phi(x), \quad U(\Lambda) = \exp[-iL\cdot\Lambda], \tag{1.7}$$

where $\phi(x)$ is a column vector, $\Lambda^j$ are the gauge parameters, and $L^j$ the representation matrices associated with the group generators $T_i$, $i = 1, 2, \ldots, n$. The operators $T_i$ satisfy the commutation relations

$$[T_i, T_j] \equiv T_iT_j - T_jT_i = f_i{}^k{}_j T_k, \quad i, j, k = 1, 2, \ldots, n; \tag{1.8}$$

the numerical constants $f_i{}^k{}_j$ are called structure constants and are totally antisymmetric.

If the generators $T_i$ commute,

$$[T_i, T_j] = 0, \tag{1.9}$$

we call $G$ an *Abelian* Lie group and (1.7) an Abelian gauge transformation. Examples of Abelian gauge theories are quantum electrodynamics and the $\lambda\phi^3$ and $\lambda\phi^4$ theories. On the other hand, if the structure constants in (1.8) differ from zero, so that the generators $T_i$ do not commute, then $G$ is called a *non-Abelian* Lie group. A field associated with the latter group structure is called a non-Abelian gauge field. Yang–Mills fields and quantum gravity are examples of such non-Abelian gauge fields (Yang and Mills, 1954; Utiyama, 1956).

The gauge parameter $\Lambda$ associated with the transformation (1.7) may or may not be a function of space-time. If $\Lambda$ is a function of the space-time variable $x$, the gauge group is called *local*; otherwise we speak of a *global* gauge group.

Let us illustrate some of these concepts in the case of quantum electrodynamics. To say that quantum electrodynamics is an Abelian gauge theory means that the Lagrangian density

$$\begin{aligned} \mathcal{L}_{\text{QED}} &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma\cdot\partial - m + e\gamma\cdot A)\psi, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \end{aligned} \tag{1.10}$$

remains invariant under the Abelian gauge transformation

$$\begin{aligned} \psi(x) &\to \psi(x)\exp[ie\Lambda(x)], \\ A_\mu(x) &\to A_\mu(x) + \partial_\mu\Lambda(x), \\ \bar{\psi}(x) &\to \bar{\psi}(x)\exp[-ie\Lambda(x)], \end{aligned} \tag{1.11}$$

where $e$ and $m$ denote, respectively, the charge and mass of the electron. It is easy to see that in this case the gauge field is just the photon field $A_\mu(x)$.

Ever since the pioneering work of Yang and Mills (1954) and Shaw (1954) nearly 20 years ago, the subject of gauge theories has been under constant investigation by many theorists, for example Bludman (1958), Gell-Mann (1960, 1961), Salam and Ward (1961), Feynman (1963), Faddeev and Popov (1967), and others. One of the questions asked by these authors was whether or not theories of the Yang–Mills type are renormalizable. The answer was provided by 't Hooft (1971a,b) in a series of fundamental papers which demonstrated the renormalizability of both massless and massive Yang–Mills theories in the context of spontaneous symmetry breaking (Englert and Brout, 1964; Higgs, 1964; Guralnik, Hagen, and Kibble, 1964; Kibble, 1967). The work of 't Hooft hinged decisively on finding a suitable cutoff procedure that would preserve the gauge symmetry of the underlying Lagrangian. Such a procedure was subsequently developed by 't Hooft himself, in collaboration with Veltman ('t Hooft and Veltman, 1972a), by

Bollini and Giambiagi (1972), and by Ashmore (1972, 1973), and became known as the *technique of dimensional regularization.*

## B. Concept of dimensional regularization

### 1. General idea

The technique of dimensional regularization, also called the *continuous dimension method*, is probably the best regularization procedure on the market.[3] To understand the basic motivation behind this technique let us consider the four-dimensional integral

$$I(4) \equiv \int \frac{d^4k}{(2\pi)^4 k^2[(k-p)^2+m^2]},$$

defined over Euclidean momentum space. In the limit of large momenta, $k^2 \to +\infty$, $I(4)$ *diverges*, whereas in three dimensions the corresponding integral

$$(2\pi)^{-3} \int d^3k \{k^2[(k-p)^2+m^2]\}^{-1}$$

is finite as $k^2 \to +\infty$. We see that a reduction in the number of dimensions from four to three makes the original integral $I(4)$ *convergent.* The idea, therefore, is first to generalize the dimensionality of the space from *four* to $n$, where $n = 0, 1, 2, \ldots$, and then to go one step further and replace $n$ by a complex number[4] $2\omega$ which is called the regulating parameter. Symbolically,

$$I(4) \equiv \int \frac{d^4k}{(2\pi)^4} J(k^2, k\cdot p) \Rightarrow \int \frac{d^nk}{(2\pi)^n} J(n, k^2, k\cdot p)$$

$$\Rightarrow \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} J(2\omega, k^2, k\cdot p) \equiv I(2\omega). \tag{1.12}$$

Whereas $I(4)$ does *not* exist, the integral $I(2\omega)$ may be defined in such a way that it is an analytic function of $\omega$ which can in principle be evaluated explicitly. Once all formal manipulations involving integrals, such as symmetric integration, shift of integration variable, and integration by parts, have been completed, we can invoke the principle of analytic continuation to return to four-dimensional space ($2\omega = 4$). The concept of analytic continuation in the number of space-time dimensions is the most important single feature in the technique of dimensional regularization.

This may be an appropriate place to ask what meaning, if any, we can attach to a space of complex dimensions. Although we are unable to give at present a *physical* interpretation of such a space, it may be worthwhile remembering that the abstraction from a discrete to a continuous parameter space is by no means uncommon in physics. We recall, for example, in classical mechanics the transition from particle theory to field theory, where the generalized coordinates $q_j(t)$ are replaced by the real space-time function $\psi_\sigma(\mathbf{x},t)$. Here $\sigma = 0, 1, 2, \ldots$ labels the field components, while the index $j$, $j = 0, 1, 2, \ldots, N$, counts the number of degrees of freedom of the system. In the transition to classical field theory, $q_j(t) \to \psi_\sigma(\mathbf{x},t)$, the discrete index $j$ is formally "replaced" by both the *discrete index* $\sigma$ *and the continuous variable* $\mathbf{x}$. Expressed in mathematical language, the above transition describes the continuation from a discrete to a continuous parameter space. There exist, no doubt, other examples which are based on the same mathematical principle.

It is also worth noting that, inasmuch as physical results are expressed in terms of inproducts between vectors, no objection can be raised against the assumption that space is of noninteger dimensions. An experimental test in this regard, suggested by J. S. Bell, is the decrease in the intensity of light, emitted from a point source, as a function of the distance from that source. Since the intensity behaves as $1/r^{2\omega-2}$, the integral over a sphere remains constant. The experimental observation that the gravitational potential behaves very nearly as $1/r$ is likewise related to the fact that the dimensionality of space-time is almost equal to 4.[5]

### 2. Usefulness

Since dimensional regularization preserves the local symmetries in the Lagrangian such as gauge invariance, the technique is eminently well suited for treating gauge field theories in general. In addition, the continuous dimension method is not only simpler than that of Pauli and Villars, but also more elegant and intuitive. Whereas the Pauli–Villars regulators are completely devoid of any physical meaning,[6] the regulating parameter $\omega$ admits—at least for some of its values—a realistic interpretation, for example when $2\omega = 1, 2, 3, \ldots$. In summary, dimensional regularization permits a consistent gauge-invariant treatment of divergent Feynman integrals to all orders in perturbation theory. The method can be applied not only to Abelian gauge models, but more importantly to non-Abelian theories such as Yang–Mills fields and quantum gravity, to which the majority of conventional regularization procedures is inapplicable.

## C. Outline

Section II begins with the central theorem on analytic continuation, which is subsequently applied to Euler's representation of the $\gamma$ function. We then summarize the principal features of Ashmore's approach to dimensional regularization (Ashmore, 1972, 1973) and comment briefly on the generalization of four-dimensional combinatorics to complex-dimensional space. A rather vital formula is the trace relation $\delta_{\mu\mu} = 2\omega$. A second approach to dimensional regularization, developed by 't Hooft and Veltman, is discussed near the end of Sec. II.

In the first half of Sec. III we apply the technique to quantum electrodynamics, an Abelian gauge theory, by regularizing both the electron self-energy and the vacuum polarization tensor. These calculations are carried out to lowest order in $e$ ($e^2/4\pi = \alpha$ is the fine-structure constant). It is reassuring that the new technique reproduces the gauge-invariant factor $(p^2\eta_{\mu\nu} - p_\mu p_\nu)$, which is so characteristic of

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[3] See also Cicuta and Montaldi (1972), Geist, Kühnelt, and Lang (1973), and more recently Butera, Cicuta, and Montaldi (1974), as well as Speer (1974a,b).

[4] We find it extremely convenient to use $2\omega$ as our regulating parameter rather than $\omega$.

[5] The author is indebted to Professor M. Veltman for bringing Bell's work to his attention.

[6] Pauli–Villars particles could become physical, however, if made unstable. This possibility has been investigated in recent years by T. D. Lee.

the four-dimensional photon self-energy. The limitation of the continuous dimension method is examined at the end of Sec. III. The technique must be used judiciously *whenever the final Slavnov–Taylor identities* (Slavnov, 1972; Taylor, 1971) *contain factors of* $\gamma^5$ or equivalently, of the totally antisymmetric tensor $\epsilon_{\alpha\beta\mu\nu}$, since the $\epsilon_{\alpha\beta\mu\nu}$ symbol is only defined in *four* dimensions, whereas the Slavnov–Taylor identities must hold in any dimension, whether integral or complex.

The procedure of dimensional regularization outlined in the second section and illustrated in the third works remarkably well for massive particles, but requires modification in the case of massless theories, such as quantum gravity, in order to cope with genuine infrared divergences. One such modification is presented in Sec. IV (Leibbrandt and Capper, 1974a); it involves a redefinition of the $2\omega$-dimensional Gaussian integral in such a way that both massless tadpoles and $\delta^4(0)$ terms can be treated consistently. The new definition [see Eq. (4.9)] further permits proof of a 't Hooft–Veltman conjecture which asserts that integrals over a polynomial vanish in dimensional regularization. In Sec. V we illustrate the continuous dimension method for Yang–Mills fields by calculating the muon $(g-2)$ factor in the Georgi–Glashow model.

In Secs. VI and VII we apply the new method to non-Abelian gauge theories to which the majority of conventional regularization procedures is inapplicable. Thus in Sec. VI we regularize pure quantum gravity to lowest order in the gravitational coefficient $\kappa^2 = 32\pi G$, $G$ being Newton's constant. It is specifically demonstrated that the total amplitude $T_{\alpha\beta\alpha'\beta'}(p)$ for the graviton self-energy respects the crucial Slavnov–Taylor identities [Eqs. (6.17) and (6.18)] thereby verifying that dimensional regularization is indeed a gauge-invariant procedure. The invariant operator $T_{\alpha\beta\alpha'\beta'}(p)$ includes all contributions from fictitious particles which are needed in order to restore the unitarity of the scattering matrix $S$ and the transversality of the scattering amplitudes. $T_{\alpha\beta\alpha'\beta'}(p)$ is then used to construct the connected Green's function $Q_{\nu\sigma\mu\lambda}{}^{(g)}(p)$ which satisfies another Slavnov–Taylor identity [Eq. (6.16)]. The latter is independent of $\omega$ and holds therefore in any dimension. Expanding $Q_{\nu\sigma\mu\lambda}{}^{(g)}(p)$ about $\omega = 2$ and continuing analytically from Euclidean to Minkowski space, we can separate $Q_{\nu\sigma\mu\lambda}{}^{(g)}$ into a pole term and the finite portion of the connected Green's function [Eqs. (6.22) and (6.21), respectively]. In view of the modified regularization technique employed, there are no infrared-divergence problems. The second part of Sec. VI deals with neutrino and photon corrections to the graviton propagator. The treatment is analogous to the pure graviton case except for one important difference. In order to preserve both conformal invariance and gravitational gauge invariance it is essential that the counterterm Lagrangian be itself a function of the regulating parameter $\omega$ (Capper and Duff, 1974b). Finally, at the end of Sec. VI, we address ourselves once more to the question of renormalizing quantum gravity.

In Sec. VII we apply our technique to higher-order diagrams such as the triangle graph and multiple-loop diagrams. It is found that for multiple-loop diagrams the *same* $\omega$ must be employed for each closed loop. The article concludes in Sec. VIII with a brief summary of the essential aspects of dimensional regularization. A list of useful $2\omega$-dimensional integrals is given in the Appendix.

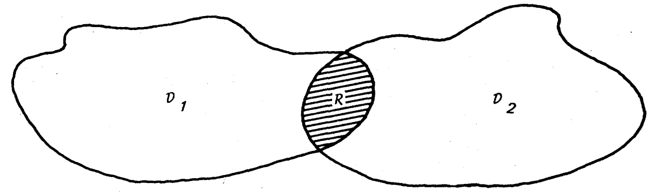



FIG. 1. The subregion $\mathcal{R}$ is common to the two regions $\mathcal{D}_1$ and $\mathcal{D}_2$.

Unless otherwise specified, our notation corresponds to that of Bjorken and Drell (1964) with $\hbar = c = 1$. The metric is in most sections Euclidean. Only in Secs. III and V do we work partially in Minkowski space.

## II. THE TECHNIQUE OF DIMENSIONAL REGULARIZATION

### A. Mathematical tools

To help clarify the meaning of analytic continuation, we first discuss the basic theorem on analytic continuation and then apply it to Euler's $\gamma$ function. The principle of analytic continuation is expressed by the following theorem (Knopp, 1945).

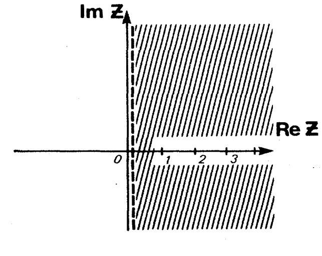
**Theorem.** Let an analytic function $g_1(z)$ be defined in a region $\mathcal{D}_1$ and let $\mathcal{D}_2$ be another region which has a certain subregion $\mathcal{R}$, but only this one, in common with $\mathcal{D}_1$. Then if a function $g_2(z)$ exists which is analytic in $\mathcal{D}_2$ and coincides with $g_1(z)$ in $\mathcal{R}$, there can only be one such function. We call $g_1(z)$ and $g_2(z)$ *analytic continuations* of each other. (See Fig. 1.)

The above theorem asserts that $g_2(z)$ is *unique* provided $\mathcal{R}$ is not the empty set, $\mathcal{R} = \mathcal{D}_1 \cap \mathcal{D}_2 \neq \phi$ ($\mathcal{R}$ contains infinitely many points). It further implies that the representations of $g_1(z)$ and $g_2(z)$ are equal in the subregion $\mathcal{R}$. Outside $\mathcal{R}$, the functions $g_1$ and $g_2$ possess, of course, different representations.

The $\gamma$ function $\Gamma(z)$, $z$ complex, arises naturally in the continuous dimension method from the computation of Gaussian momentum integrals. Although there exist many representations of $\Gamma(z)$ in the mathematical literature, only two representations, those of Euler and Weierstrass, appear suitable in the context of dimensional regularization. Let us first consider Euler's $\gamma$ function $\Gamma_E(z)$ (Magnus and Oberhettinger, 1954),

$$\Gamma_E(z) = \int_0^\infty dt\, t^{z-1}e^{-t}, \quad \mathrm{Re}z > 0 \tag{2.1}$$

FIG. 2. The shaded region Re$z$ $> 0$ shows the domain of analyticity of Euler's $\gamma$ function $\Gamma_E(z)$.



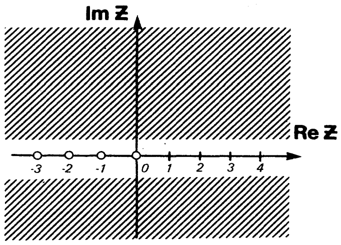



FIG. 3. Weierstrass's representation of the $\gamma$ function $\Gamma_W(z)$ is analytic everywhere except at the points $z = 0, -1, -2, -3, \ldots$.

which is analytic in the shaded area depicted in Fig. 2. In order to discuss points lying *outside* this region, for example in the left-hand $z$ plane, it is mandatory to find first an analytic continuation of $\Gamma_E(z)$ which is valid in that region. Such a continuation is precisely Weierstrass's partial fraction expansion (Markushevich, 1965)

$$\Gamma_W(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z)} + \int_1^\infty dt\, t^{z-1}e^{-t}, \tag{2.2}$$

which is analytic in the entire $z$ plane, *except* at the points $z = 0, -1, -2, -3, \ldots$ (Fig. 3). The representation $\Gamma_W$ is a *unique* analytic continuation of $\Gamma_E$, since its domain of definition clearly overlaps that of $\Gamma_E$ [cf. Figs. 2 and 3].

## B. The technique of dimensional regularization

### 1. Prescription

Let us assume that the four-dimensional integral

$$I(p) = \int \frac{d^4k}{(2\pi)^4} J(k^2, k\cdot p)$$

is ultraviolet divergent. (To simplify matters, we work in Euclidean space.) For *massive* fields, the basic steps in the method of dimensional regularization then are (Ashmore, 1972, 1973):

(i) Define all inner vector products over a complex $2\omega$-dimensional space.

(ii) Parametrize all momentum-space propagators according to

$$\frac{1}{k^2+m^2} = \int_0^\infty d\alpha \exp[-\alpha(k^2+m^2)], \quad m^2 \neq 0. \tag{2.3}$$

Since we work in Euclidean space, there is no need for an $i\epsilon$ term in the propagator.

(iii) Integrate over momentum space by means of the generalized Gaussian integral

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \exp(-xk^2 + 2k\cdot b) = \frac{(\pi/x)^\omega}{(2\pi)^{2\omega}} \exp\frac{b^2}{x},$$
$$x > 0. \tag{2.4}$$

For integer or half-integer values of $\omega$, $2\omega = 1, 2, 3, \ldots$, formula (2.4) reduces to the standard Gaussian formula, but for complex values of $\omega$, the right-hand side of (2.4) must be taken as the definition of the integral on the left.

(iv) The resulting amplitude is now well defined as a function in a finite domain of the complex $\omega$ plane. Outside this domain the amplitude is defined as the analytic continuation of the amplitude inside. The domain is in practice determined by the convergence of the Feynman parameter integrations. Because of infrared problems, this domain may in certain special instances shrink to zero as, for example, in the case

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}k^2[(k-p)^2+m^2]}$$
$$= \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \left(\frac{1}{2\pi}\right)^{2\omega} \left(\frac{\pi}{\alpha_1+\alpha_2}\right)^\omega$$
$$\times \exp\left[+\frac{\alpha_2{}^2p^2}{\alpha_1+\alpha_2} - \alpha_2(p^2+m^2)\right],$$

where the domain becomes vanishingly small as $p$ approaches zero.

(v) Integration over Feynman parameters leads, in the region where the integrals exist, to $\gamma$ functions. The analytic continuation defined in step (iii) above is then implemented by using for these $\gamma$ functions the Weierstrass representation

$$\Gamma_W(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z)} + \int_1^\infty dt\, t^{z-1}e^{-t}. \tag{2.5}$$

The subscript $W$ on the $\gamma$ function will, in the future, not be written explicitly.

(vi) Expand all $\omega$-dependent quantities in a Laurent series about the point $\omega = 2$, so that the integral $I(p)$ becomes

$$I(p) = G(p^2)/(\omega-2) + F(p^2) + O(\omega-2). \tag{2.6a}$$

We see that the original ultraviolet infinities manifest themselves as poles at the "physical" value $\omega = 2$. (We recall that $2\omega = 4$ corresponds to four-dimensional space.)

(vii) Cancel the pole term $G(p^2)/(\omega-2)$ in Eq. (2.6a) by adding appropriate counterterms to the original interaction Lagrangian in which case the *regularized* integral finally reads

$$I_{\text{Reg}}(p) = F(p^2) + O(\omega-2). \tag{2.6b}$$

(viii) Analytically continue the right-hand side of (2.6b) to four-dimensional space, i.e., take the limit[7] $\omega \to 2^+$ so that the *value* of the integral is given by the finite portion $F(p^2)$ of the Laurent expansion, properly continued to Minkowski space ($p^2 = p_0{}^2 - \mathbf{p}^2$).

The prescription (i)–(viii) is sufficient to regularize integrals associated with massive fields, provided there appear no anomalies from partially conserved axial-vector currents which would imply the presence of $\gamma^5$ terms (cf. Sec. III.C). The above prescription may break down for certain values of the external momenta if there are infrared divergence

[7] It is convenient in algebraic calculations to state explicitly whether $\omega$ approaches 2 from above, $\omega \to 2^+$, or from below, $\omega \to 2^-$. We shall consistently use the notation "$\omega \to 2^+$".

problems. If the momenta are off-mass-shell, however, as well as nonzero, as in the example in step (iv), then there are no difficulties. Nor are there any problems in theories where the infrared difficulties eventually cancel out, provided one works on-mass-shell and considers only the *sum* of the various contributions. In these theories it is necessary to continue the external momenta analytically outside four dimensions, since the infrared cancellations occur specifically between internal and external momentum integrations (Gastmans and Meuldermans, 1973). We intend to deal with the infrared divergences in a different manner (see particularly Secs. IV and VI) which is especially well suited for vacuum polarization diagrams of the type discussed in point (iv) above. To what extent our approach coincides with the usual results, and indeed maintains the cancellations of infrared divergences of loop integrals and external phase space integrations, will not be discussed here.

### 2. Combinatorics and gauge invariance

The computation of Feynman integrals involves a certain amount of combinatorics between vectors $p_\mu$ and the generalized Kronecker $\delta$ symbol $\delta_{\mu\nu}$. The procedure outlined in Sec. II.B.1 must therefore be supplemented by the following rules valid in $2\omega$-dimensional space ('t Hooft and Veltman, 1972a, 1973).

$$\delta_{\mu\nu}p_\nu = p_\mu, \quad p_\mu p_\mu = p^2, \quad \delta_{\mu\nu}\delta_{\nu\alpha} = \delta_{\mu\alpha}, \tag{2.7}$$

and

$$\delta_{\mu\nu}\delta_{\mu\nu} = 2\omega, \tag{2.8a}$$

$$\delta_{\mu\mu} = 2\omega, \tag{2.8b}$$

$$p_\mu p_\nu \rightarrow (p^2/2\omega)\delta_{\mu\nu}. \tag{2.8c}$$

Whereas the operations in (2.7) are formally the same as in four-dimensional space, it is highly significant according to (2.8b) that the *trace* of $\delta_{\mu\nu}$ is no longer four but $2\omega$. This amazing result leads to the following observation: In the framework of dimensional regularization, the four-dimensional Feynman rules must in general be replaced by $2\omega$-dimensional rules. We shall have occasion to demonstrate this explicitly in Sec. VI in the case of quantum gravity.

For Feynman diagrams involving spin one-half particles and $\gamma$ matrices we have these additional rules ('t Hooft and Veltman, 1972a, 1973; Gastmans and Meuldermans, 1973),

$$\begin{aligned} &\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}I, \quad I \equiv \text{unit matrix} \\ &\text{Trace }(\gamma_\mu\gamma_\nu) = 2^\omega\delta_{\mu\nu}, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\gamma_\mu \not{p}\gamma^\mu = 2(1-\omega)\not{p}, \quad \not{p} \equiv \gamma\cdot p \\ &\gamma_\mu \not{p}\not{q}\gamma^\mu = 4p\cdot q + 2(\omega-2)\not{p}\not{q}. \end{aligned} \tag{2.10}$$

Another useful feature of dimensional regularization concerns the shift of origin in momentum space. Once a divergent integral has been regularized by defining it over a space of $2\omega$ dimensions, the integration variable $k_\mu$ can be shifted to $k_\mu + bp_\mu$, regardless of the original degree of divergence. In four dimensions, such a shift is only allowed for convergent and logarithmically divergent integrals.

Formulas (2.7)–(2.10) are essential in proving that the continuous dimension method preserves the gauge invariance of the $S$ matrix to all orders in perturbation theory. To say that dimensional regularization is gauge-invariant simply means that it respects certain identities such as the Slavnov–Taylor identities which depend, according to (2.8) and (2.10), on the dimensionality of the space. Hence if gauge invariance is to hold, these identities must either be *independent of* $\omega$ or they must be *satisfied identically for all values of* $\omega$, especially for $\omega = 2$ (which corresponds to four-dimensional space-time). The validity of the Slavnov–Taylor identities or their counterparts is essential in proving both the unitarity and the causality of the $S$ matrix ('t Hooft and Veltman, 1972a,b).

### 3. Infrared vs ultraviolet divergences

The primary objective of most regularization procedures is to tackle either the ultraviolet divergences or the less severe infrared divergences, but not both. It is therefore natural to inquire whether the technique of dimensional regularization is perhaps capable of dealing with both types of infinities. Since we already know how to attack the ultraviolet problem, the question really is: Can the method also cope with the infrared infinities which are specifically connected with zero-mass particles? A prescription for dealing with these divergences has recently been proposed by Leibbrandt and Capper (1974a,b) and is analyzed in Sec. IV.

## C. Other techniques

### 1. The 't Hooft–Veltman approach

As noted earlier, the renormalizability of both Abelian and non-Abelian gauge field theories depends crucially upon the existence of a regularization procedure that preserves the gauge symmetry of the underlying Lagrangian. 't Hooft and Veltman (1972a) were among those who succeeded in finding such a procedure—the technique of dimensional regularization. Since their method is more intuitive than that of either Ashmore (1972, 1973) or Bollini *et al.* (1972), we shall highlight here the principal features of the 't Hooft–Veltman approach.

Consider the photon self-energy loop carrying momentum $k$. 't Hooft and Veltman (1972a) separate the $2\omega$-dimensional momentum space over $k$ into a *physical* four-dimensional space, characterized by the four-vector $\hat{k}$, plus a $(2\omega-4)$-dimensional space over the vector $K$. Symbolically,

$$I(2\omega) \equiv \int d^{2\omega}k\, f(k) \Rightarrow \int d^4\hat{k} \int d^{2\omega-4}K\, g(\hat{k},K^2), \tag{2.11}$$

where $k^2 = \hat{k}^2 + K^2$. Working in polar coordinate space and defining the length of the vector $K$ by $x$, 't Hooft and Veltman obtain via (2.11)

$$I(2\omega) = \frac{2\pi^{\omega-2}}{\Gamma(\omega-2)} \int d^4\hat{k} \int_0^\infty dx\, x^{2\omega-5} g(\hat{k},x^2). \tag{2.12}$$

At this stage of the procedure, $I(2\omega)$ contains both infrared and ultraviolet divergences. To eliminate the infrared divergences these authors employ partial differentiation to obtain a domain of analyticity for $I(2\omega)$. They specifically integrate over $x$ partially $\lambda$ times so that (2.12) becomes

$$I(2\omega,\lambda) = \frac{2\pi^{\omega-2}}{\Gamma(\omega - 2 + \lambda)} \int d^4\hat{k} \int_0^\infty dx\, x^{2\omega-5+2\lambda} \times\left(-\frac{\partial}{\partial x^2}\right)^\lambda g(\hat{k},x^2). \tag{2.13}$$

By choosing the parameter $\lambda$ large enough, the infrared problem can readily be eliminated. Of course (2.13) remains ultraviolet divergent, but at least there is now a region in $\omega$ space in which the $x$ integral converges. A more challenging task is to find a domain in which the ultraviolet integrals converge as well. To achieve this 't Hooft and Veltman employ integration by parts in order to continue the representation (2.13) analytically in the complex $\omega$ plane. The result is an enlarged domain of definition for the $x$ integral in Eq. (2.13). For example, if the original integral holds only for $\mathrm{Re}\omega > \frac{5}{2}$, the continued integral can be made to hold for $\mathrm{Re}\omega > \frac{3}{2}$. By choosing $\lambda$ sufficiently large, one can effectively replace all ultraviolet integrals by convergent ones.

This completes our discussion of the one-loop photon self-energy. The procedure just described is also applicable to diagrams containing two or more closed loops, although the amount of algebra increases rapidly now with each new loop.

The 't Hooft–Veltman approach[8] respects the structure of the Slavnov–Taylor identities which play such a fundamental rôle in discussions of gauge invariance; moreover it preserves the usual operations of partial differentiation, integration by parts, and shifting the origin in momentum space.

## 2. Other one-loop techniques

To round out this section, we review two other approaches to dimensional regularization: the five-dimensional procedure of 't Hooft (1971a) and the intermediate method of Brown (1973).

### (a) 't Hooft's method

It was originally introduced by 't Hooft in connection with Slavnov–Taylor identities (Slavnov, 1972; Taylor, 1971) in his work on the renormalization of massless Yang–Mills fields. Observing that certain Slavnov–Taylor identities hold not only in Minkowski space but also in a space of five dimensions, 't Hooft proceeds to add a fixed fifth component to all four-momenta occurring inside closed loops; the Yang–Mills fields $W_\mu{}^a$ become then 15-component objects. Thus

$$k_\mu = (\hat{k},M), \qquad \mu = 1, 2, \ldots, 5 \tag{2.14a}$$

$$W_\mu{}^a = (\hat{W}^a,W_5{}^a), \quad a = 1, 2, 3 \tag{2.14b}$$

where $\hat{k}$ and $\hat{W}^a$ denote (Minkowski) *four*-vectors, and where momentum conservation demands that the regulator mass $M$ be the *same* for each propagator of a closed loop. All *external* momenta are strictly defined over four-space. The fifth component $W_5{}^a$ is treated as a new particle; its Feynman rules for the propagator and vertices are determined by the $(4 + 1)$-dimensional Yang–Mills Lagrangian.

Introduction of the parameter $M$ leads to convergent integrals that can be evaluated by standard techniques. After all integrations have been completed, the regulator mass $M$ is allowed to approach infinity. The above technique yields gauge-invariant amplitudes, but may only be applied, as 't Hooft (1971a) points out, to one-loop diagrams.

### (b) Brown's method

It lies somewhere between 't Hooft's five-dimensional regulator approach and the $2\omega$-dimensional technique described at the beginning of Sec. II. Brown's prescription may be summarized as follows (Brown, 1973):

*Rule a*: Let the momentum $k$ inside a closed loop have $(n + m)$ components, the first $n$ components being an extension from four to $n$ dimensions. The $(n + 1)$th component is fixed and of length $M$, as in 't Hooft's case, while the remaining $(m - 1)$ components of $k$ are of magnitude zero:

$$k^{(n+m)} = (k^{(n)},M,O^{(m-1)}). \tag{2.15}$$

*Rule b*: In evaluating Feynman integrals, we must adhere to the rules

(i) $\delta_{\mu\mu} = n + m$ [cf. Eq. (2.8b)];
(ii) if $k$ is a loop momentum, its magnitude equals

$$k^2 = k^{2(n)} + M^2,$$

whereas if $k$ is an *external* momentum,

$$k^2 = k^{2(n)}.$$

*Rule c*: The third and final step in Brown's approach is to introduce *different* regulator masses $M_i$ by replacing terms such as $G(M^2)/(k^2 + M^2)$ by

$$\frac{G(M^2)}{k^2 + M^2} \Longrightarrow \sum_i e_i \frac{G(M_i{}^2)}{k^2 + M_i{}^2}, \tag{2.16}$$

where the signs $e_i$ and masses $M_i$ are selected according to the following scheme[9]:

$$e_0 = 1, \quad M_0 = 0, \tag{2.17a}$$

$$\sum_i e_i = 0, \quad \sum_i e_i M_i{}^2 = 0, \quad \sum_i e_i M_i{}^4 = 0, \tag{2.17b}$$

$$\sum_{i\neq 0} e_i \ln M_i{}^2 = A, \quad \sum_{i\neq 0} e_i M_i{}^2 \ln M_i{}^2 = B, \quad \sum_{i\neq 0} e_i M_i{}^4 \ln M_i{}^2 = C. \tag{2.17c}$$

Near the end of the calculation, i.e., after the residue at $n = 4$ has already been extracted from the integral under consideration, the limit $M_i \to \infty$, $i \neq 0$, is finally taken.

Let us illustrate Brown's method for the integral (factors of $2\pi$ are omitted)

$$I_4 = \int \frac{d^4k}{k^2(k + p)^2} k_\mu k_\nu f(n = 4) G(n = 4; p), \quad \nu, \mu = 1, 2, 3, 4 \tag{2.18}$$

where the factor $f(n = 4)$ arises from the rule $\delta_{\mu\mu} = 4$ and

[8] For a simpler procedure see Sec. VII.

[9] See also 't Hooft (1971a), Eqs. (5.2) to (5.3).

$\mu$, $\nu$ are external indices. Rules a and b transform the integral (2.18) into

$$I_{n+m} = \int \frac{d^n k\, k_\nu{}^{(n)} k_\mu{}^{(n)} f(n+m) G(n;p;M^2)}{(k^2+M^2)[(k+p)^2+M^2]}. \tag{2.19}$$

Making the replacement (2.16) and integrating over $k$, we obtain

$$I_{n+m} = \pi^{n/2}\Gamma(2-\tfrac{1}{2}n)\sum_i e_i \times \int_0^1 \frac{dx\, G(x;n;p;M_i{}^2) f(n+m)}{[M_i{}^2 + p^2 x(1-x)]^{2-n/2}}, \tag{2.20}$$

whose pole part is equal to

$$I_{n+m}{}^{\text{pole}} = \frac{2\pi^2 f(4+m)}{4-n} \sum_i e_i \int_0^1 dx\, G(x;4;p;M_i{}^2). \tag{2.21}$$

Employing the expansion

$$\begin{aligned}[M_i{}^2 + p^2x(1-x)]^{-(2-n/2)} &= \exp\{-(2-\tfrac{1}{2}n)\ln[M_i{}^2 + p^2x(1-x)]\} \\ &= 1-(2-\tfrac{1}{2}n)\ln[M_i{}^2+p^2x(1-x)] \\ &\quad + O((2-\tfrac{1}{2}n)^2),\end{aligned} \tag{2.22}$$

we can readily calculate the expression

$$\lim_{n\to 4}(I_{n+m} - I_{n+m}{}^{\text{pole}}) \equiv I_{4+m}{}^{\text{Reg}}.$$

As $M_i \to \infty$, $i \neq 0$, we find that the only nonvanishing term is given by

$$I_{4+m}{}^{\text{Reg}} = -\pi^2 f(4+m)\sum_i e_i \int_0^1 dx\, G(n=4;x;p;M_i{}^2) \times \ln[M_i{}^2 + p^2x(1-x)]. \tag{2.23}$$

It is significant to note that $m$ occurs only in $f(4+m)$, and not in its derivatives. Since the Slavnov–Taylor identities are either independent of the dimensionality of the space or must hold for all values of $(n+m)$, we are at liberty to choose a convenient value for $m$[10]; for example, we may choose $m=0$ and still preserve gauge invariance.

The above variant to dimensional regularization was originally designed to regularize in a gauge-invariant manner divergent one-loop Feynman integrals in pure quantum gravity. The procedure has, to the best of our knowledge, only been tested at the one-loop level

## III. ILLUSTRATIONS FROM QUANTUM ELECTRODYNAMICS

We illustrate the technique of dimensional regularization in the case of quantum electrodynamics—an Abelian gauge theory—by regularizing first the electron self-energy and then the vacuum polarization tensor. We only work to second order in the coupling constant[11] $e$ ($\alpha = e^2/4\pi$). The electron self-energy calculation is presented in considerable detail, since it contains several features common to many of our subsequent illustrations.

### A. Electron self-energy

We start with the free Dirac equation in momentum space

$$(i\gamma\cdot p + m)\psi(p) = 0 \tag{3.1}$$

and take

$$ie\bar{\psi}\gamma_\mu\psi A^\mu \tag{3.2}$$

as the interaction between the spinor field $\psi$ and the electromagnetic field $A^\mu$. In four-dimensional space, the amplitude for the proper electron self-energy may then be expressed as (see Fig. 4)

$$\Sigma(p) = +ie^2 \int \frac{d^4k\, \gamma_\mu[i(\gamma\cdot p - \gamma\cdot k) - m]\gamma^\mu}{(2\pi)^4(k^2 - i\epsilon)[(p-k)^2 + m^2 - i\epsilon]}. \tag{3.3}$$

This typical four-dimensional integral is plagued by both ultraviolet infinities (large $k$-behavior) and infrared infinities (zero photon mass). Since the ultraviolet problem is appreciably harder to solve than the infrared one, we shall devote our attention almost exclusively to the former. The infrared problem can be rectified, for instance, by assigning to the photon a fictitious mass $\mu$.

Let us return to Eq. (3.3). Under the assumption that both (3.1) and the interaction term (3.2) remain valid in a complex space of $2\omega$ dimensions, we can generalize $\Sigma(p)$ to read (Bollini and Giambiagi, 1972)

$$\Sigma(p,2\omega) = ie^2 \int \frac{d^{2\omega}k\, \gamma_\mu[i(p\cdot\gamma - k\cdot\gamma) - m]\gamma^\mu}{(2\pi)^{2\omega}(k^2 - i\epsilon)[(p-k)^2 + m^2 - i\epsilon]}. \tag{3.4}$$

Reducing the numerator with the help of Eq. (2.10) and combining the propagators by means of the standard formula

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2}, \tag{3.5}$$

we obtain

$$\Sigma(p,2\omega) = ie^2 \int_0^1 dx \times \int \frac{d^{2\omega}k\{2i(1-\omega)[(1-x)p\cdot\gamma - k\cdot\gamma] - 2m\omega\}}{(2\pi)^{2\omega}[k^2 + H_0]^2} \tag{3.6}$$

with

$$H_0 = (p^2 + m^2)x(1-x) + m^2x^2 - i\epsilon, \tag{3.7}$$

where the $-i\epsilon$ term will henceforth be omitted. The transition from (3.4) to (3.6) involves a shift of the integration variable, which is permissible here since the integral (3.4) is known to *converge* for complex $\omega$. We should also stress

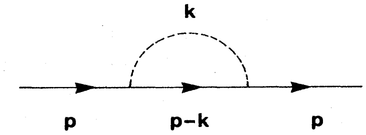



FIG. 4. The electron self-energy.

[10] See the discussion in Sec. VI.A.3.
[11] Natural units $\hbar = c = 1$ are employed throughout this paper.

that Feynman's formula (3.5) works perfectly well for one-loop diagrams, but that it must be used cautiously in the case of multiple-loop diagrams, as first emphasized by 't Hooft and Veltman (1972a). Their argument is that it is possible—in the case of two- and three-loop diagrams—for the ultraviolet divergences to become *transferred* to the parameter integrations, in which case the infrared and ultraviolet infinities get mixed up.

To evaluate the $k$-space integral in (3.6) we use [cf. Eq. (A1)]

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}(k^2+H_0)^\alpha} = \frac{i\pi^\omega H_0{}^{\omega-\alpha}\Gamma(\alpha-\omega)}{(2\pi)^{2\omega}\Gamma(\alpha)}, \tag{3.8}$$

together with

$$\int \frac{d^{2\omega}k\, k_\mu}{(2\pi)^{2\omega}(k^2+H_0)^\alpha} = 0, \tag{3.9}$$

where (3.9) is the analog of symmetric integration in four-space. Application of Eqs. (3.8) and (3.9) to (3.6) yields

$$\Sigma(p,2\omega) = -\frac{e^2\Gamma(2-\omega)}{(4\pi)^\omega}\int_0^1 dx$$
$$\times[2i(1-\omega)(1-x)p\cdot\gamma - 2m\omega]$$
$$\times[(p^2+m^2)x(1-x)+m^2x^2]^{\omega-2} \tag{3.10}$$

or, with $(m^2+p^2)/m^2 \equiv \rho$,

$$\Sigma(p,2\omega) = -\frac{e^2\Gamma(2-\omega)(\rho m^2)^{\omega-2}}{(4\pi)^\omega}$$
$$\times\int_0^1 dx\, x^{\omega-2}\left(1+\frac{1-\rho}{\rho}x\right)^{\omega-2}$$
$$\times[(ip\cdot\gamma+m)(2-2\omega)(1-x)$$
$$+mx(2-2\omega)-2m]. \tag{3.11}$$

Since the last integral is a well defined analytic function of $\omega$, it can be readily computed using the formula [see p. 284 of Gradshteyn and Ryzhik (1965)]

$$\int_0^1 \frac{dx\, x^{\mu-1}}{(1+\beta x)} = \frac{1}{\mu}\,{}_2F_1(1,\mu;\mu+1;-\beta),$$
$$|\arg(1+\beta)| < \pi, \qquad \mathrm{Re}\,\mu > 0 \tag{3.12a}$$

where the hypergeometric function ${}_2F_1$ has the infinite series representation (Magnus and Oberhettinger, 1954)

$${}_2F_1(a,b;c;z) = 1+\frac{ab}{c}\frac{z}{1!}$$
$$+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\cdots, \tag{3.12b}$$

which is absolutely convergent for $|z|<1$ and divergent for $|z|>1$. For $|z|=1$, the series converges absolutely if $\mathrm{Re}(a+b-c)<0$. A convenient integral representation for $|z|<1$ is

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}$$
$$\times(1-tz)^{-a}, \qquad \mathrm{Re}(c)>\mathrm{Re}(b)>0. \tag{3.12c}$$

For $|z|>1$, the hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by analytic continuation. Hence the electron self-energy becomes

$$\Sigma(p,2\omega) = -\frac{2e^2\Gamma(2-\omega)(\rho m^2)^{\omega-2}}{(4\pi)^\omega}$$
$$\cdot\left\{\frac{1}{\omega-1}[(ip\cdot\gamma+m)(1-\omega)-m]\right.$$
$$\times{}_2F_1\left(2-\omega,\omega-1;\omega;\frac{\rho-1}{\rho}\right)$$
$$+\frac{1-\omega}{\omega}[m-(ip\cdot\gamma+m)]$$
$$\left.\times{}_2F_1\left(2-\omega,\omega;\omega+1;\frac{\rho-1}{\rho}\right)\right\}. \tag{3.13}$$

It is customary in the renormalization program of quantum electrodynamics (Jauch and Rohrlich, 1959) to express $\Sigma(p,2\omega)$ with the help of the identity

$$(ip\cdot\gamma+m)^2 = 2m(ip\cdot\gamma+m)-(p^2+m^2) \tag{3.14}$$

in the convenient form

$$\Sigma(p,2\omega) = A(\omega)+(ip\cdot\gamma+m)B(\omega)$$
$$+(ip\cdot\gamma+m)^2\Sigma_f(p,2\omega), \tag{3.15}$$

where $A(\omega)$ and $B(\omega)$ must be independent of the external momentum $p$. The coefficients $A(\omega)$ and $B(\omega)$ contain the virulent ultraviolet divergences, while $\Sigma_f(p,2\omega)$ is finite apart from a harmless infrared term. It is significant to realize that all three coefficients depend on the dimensionality of the space. The determination of $A(\omega)$, $B(\omega)$, and $\Sigma_f(p,2\omega)$ is completely analogous to that in four dimensions (Jauch and Rohrlich, 1959) and will not be repeated here. The general idea is to express the electron self-energy (3.13) as a second-degree Taylor polynomial about $ip\cdot\gamma = -m$. To determine $A(\omega)$, for example, we replace $ip\cdot\gamma$ in Eqs. (3.13) and (3.15) by $-m$, which leads to the answer

$$A(\omega) = \frac{e^2(2\omega-1)m^{2\omega-3}}{(2\omega-3)(4\pi)^\omega}\Gamma(2-\omega). \tag{3.16}$$

$B(\omega)$ is derived by first differentiating Eqs. (3.13) and (3.15) partially with respect to $ip\cdot\gamma$ and then substituting $-m$ for $ip\cdot\gamma$. The result,

$$B(\omega) = \frac{e^2(2\omega-1)m^{2\omega-4}}{(2\omega-3)(4\pi)^\omega}\Gamma(2-\omega)+B_{\mathrm{ir}}(\omega), \tag{3.17}$$

consists of an ultraviolet first part and an infrared term $B_{\mathrm{ir}}(\omega)$, whose origin, as noted at the beginning of this section, can be traced back directly to the use of a vanishing photon mass in Eq. (3.3) (Ahmed and Qadir, 1974). The *finite* portion of $\Sigma(p,2\omega)$ reads

$$\Sigma_f(p,2\omega) = -e^2(4\pi)^{-\omega}m^{2\omega-4}\Gamma(2-\omega)$$
$$\cdot\left\{\frac{1}{m\rho}\left[\frac{2}{\omega-1}\,{}_2F_1(2-\omega,1;\omega;1-\rho)\right.\right.$$
$$\left.-\frac{2(1-\omega)}{\omega}\,{}_2F_1(2-\omega,1;\omega+1;1-\rho)-\frac{2\omega-1}{2\omega-3}\right]$$
$$+\frac{m-ip\cdot\gamma}{\rho m^2}\left[-2\,{}_2F_1(2-\omega,1;\omega;1-\rho)\right.$$
$$\left.-\frac{2(1-\omega)}{\omega}\,{}_2F_1(2-\omega,1;\omega+1;1-\rho)+1\right]$$
$$+\frac{m-ip\cdot\gamma}{\rho^2m^2}\left[\frac{4}{1-\omega}\,{}_2F_1(2-\omega,1;\omega;1-\rho)\right.$$
$$+\frac{4(1-\omega)}{\omega}\,{}_2F_1(2-\omega,1;\omega+1;1-\rho)$$
$$\left.\left.+\frac{2(2\omega-1)}{2\omega-3}\right]\right\}+\Sigma_{\rm ir}(p,2\omega). \tag{3.18}$$

The subscript $f$ in $\Sigma_f(p,2\omega)$ is short for "finite with respect to *ultraviolet* divergences." The infrared divergences of $\Sigma_f$ are contained in the additive term $\Sigma_{\rm ir}(p,2\omega)$ and can be shown to cancel to order $e^2$. The behavior of $A$, $B$, and $\Sigma_f$ at $\omega=2$ depends decisively on the analytic structure of the $\gamma$ functions. Expansion of $\Gamma(2-\omega)$ about $\omega=2$, for example, yields [cf. Eq. (2.5)]

$$\Gamma(2-\omega)=1/(2-\omega)+\psi(1)$$
$$+\tfrac{1}{2}(2-\omega)[\tfrac{1}{3}\pi^2+\psi^2(1)-\psi'(1)]$$
$$+O((2-\omega)^2), \tag{3.19}$$

with

$$\psi(\omega)=(d/d\omega)\ln\Gamma(\omega). \tag{3.20}$$

Returning to Eqs. (3.16) and (3.17) and keeping (3.19) in mind, we see that the lethal ultraviolet infinities in $A(\omega)$ and $B(\omega)$ manifest themselves as poles of $\Gamma(2-\omega)$ at the "physical" value $\omega=2$. The third coefficient $\Sigma_f(p,2\omega)$ is finite, however, since the ultraviolet pole terms vanish in the limit $\omega\to 2^+$, as may be verified by using Eq. (3.19) together with the infinite series representations for ${}_2F_1(2-\omega,1;\omega;1-\rho)$ and ${}_2F_1(2-\omega,1;\omega+1;1-\rho)$, respectively:

$${}_2F_1(2-\omega,1;\omega;1-\rho)=1+\frac{2-\omega}{\omega}\frac{1-\rho}{1!}$$
$$+\frac{(2-\omega)(3-\omega)2!}{\omega(\omega+1)}\frac{(1-\rho)^2}{2!}+\cdots$$

and

$${}_2F_1(2-\omega,1;\omega+1;1-\rho)=1+\frac{2-\omega}{\omega+1}\frac{1-\rho}{1!}$$
$$+\frac{(2-\omega)(3-\omega)2!}{(\omega+1)(\omega+2)}\frac{(1-\rho)^2}{2!}+\cdots.$$

The next step in the renormalization program of quantum electrodynamics is to absorb the troublesome coefficients $A(\omega)$ and $B(\omega)$ into the bare charge and mass of the electron, which brings us to the *finite* portion $\Sigma_f(p,2\omega)$. Continuing the latter analytically to four-dimensional Minkowski space ($p^2=p_0^2-\mathbf{p}^2$), we finally obtain the regularized expression for the electron self-energy:

$$\lim_{\omega\to 2^+}\Sigma_f(p,2\omega)\equiv\Sigma_{\rm Reg}(p,\,2\omega=4).$$

This completes our first illustration of the continuous dimension method.

## B. Vacuum polarization

The purpose of this section is to summarize the major steps in the dimensional regularization of the photon self-energy. The reason for choosing this particular example is purely pedagogical: We want to demonstrate that the continuous dimension method does indeed preserve the gauge symmetry inherent in the polarization tensor.

The integral associated with the photon self-energy depicted in Fig. 5,

$$\Pi_{\mu\nu}(p)=ie^2$$
$$\times{\rm Tr}\int\frac{d^4k\,\gamma_\mu(i\gamma\cdot k-m)\gamma_\nu[i\gamma\cdot(k-p)-m]}{(2\pi)^4[k^2+m^2-i\epsilon][(k-p)^2+m^2-i\epsilon]}, \tag{3.21}$$

is seen, from power counting, to be quadratically divergent (Tr means trace). Infrared infinities do not occur. In order to regularize the operator $\Pi_{\mu\nu}(p)$, known as the polarization tensor (Schweber, 1962), we define $\Pi_{\mu\nu}$ over a space of $2\omega$ dimensions. The resulting expression

$$\Pi_{\mu\nu}(p,2\omega)=ie^2$$
$$\times{\rm Tr}\int\frac{d^{2\omega}k\,\gamma_\mu(i\gamma\cdot k-m)\gamma_\nu[i\gamma\cdot(k-p)-m]}{(2\pi)^{2\omega}[k^2+m^2-i\epsilon][(k-p)^2+m^2-i\epsilon]} \tag{3.22}$$

is then a well defined analytic function of the regulating parameter $\omega$. Combining propagators and shifting the integration variable from $k_\mu$ to $[k_\mu-(1-x)p_\mu]$ we find that

$$\Pi_{\mu\nu}(p,2\omega)=ie^2\,{\rm Tr}\int_0^1dx\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}(k^2+H)^{-2}$$
$$\times\{\gamma_\mu[i\gamma\cdot k+i\gamma\cdot p(1-x)-m]$$
$$\times\gamma_\nu[i\gamma\cdot(k-px)-m]\}, \tag{3.23}$$

where

$$H=x(1-x)p^2+m^2-i\epsilon.$$

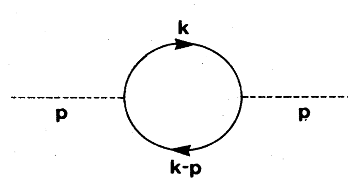



FIG. 5. Vacuum polarization in quantum electrodynamics to order $e^2$.

We observe again that the shift of origin mentioned above is justified, since Eq. (3.22) is known to converge. The shift is not permitted for the quadratically divergent four-dimensional integral (3.21). (In Minkowski space such a shift may only be applied to convergent and logarithmically divergent integrals.)

The numerator in Eq. (3.23) may be simplified with the help of the trace relations (Akyeampong and Delbourgo, 1973a,b)

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu) = 2^\omega\eta_{\mu\nu}, \tag{3.24a}$$

$$\mathrm{Tr}(\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta) = 2^\omega(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\beta}\eta_{\alpha\nu}). \tag{3.24b}$$

It is important to note that the dimensionality $\omega$ introduced through Eqs. (3.24) does not affect the Slavnov–Taylor identities, since the number of closed fermion loops in these identities is the same in every term. Applying formulas (3.8) and (3.9), together with [cf. Eq. (A4)]

$$\int \frac{d^{2\omega}k\, k^\alpha k^\beta}{(2\pi)^{2\omega}(k^2+H)^2} = \frac{i\pi^\omega\Gamma(1-\omega)}{(2\pi)^{2\omega}2H^{1-\omega}}\eta^{\alpha\beta}, \tag{3.25}$$

we obtain from Eq. (3.23)

$$\Pi_{\mu\nu}(p,2\omega) = \frac{-e^2}{(2\pi)^\omega}\int_0^1 dx\left\{\frac{\Gamma(1-\omega)}{2H^{1-\omega}}\right.$$
$$\times(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\beta}\eta_{\alpha\nu})\eta^{\alpha\beta}$$
$$+\frac{\Gamma(2-\omega)}{H^{2-\omega}}[x(1-x)(\eta_{\mu\sigma}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\sigma\alpha} + \eta_{\mu\alpha}\eta_{\sigma\nu})$$
$$\left.\times p^\sigma p^\alpha + m^2\eta_{\mu\nu}]\right\} \tag{3.26}$$

or finally

$$\Pi_{\mu\nu}(p,2\omega) = (p^2\eta_{\mu\nu} - p_\mu p_\nu)\Pi(p^2,2\omega), \tag{3.27a}$$

$$\Pi(p^2,2\omega) = \frac{2e^2\Gamma(2-\omega)}{(2\pi)^\omega}\int_0^1 \frac{dx\, x(1-x)}{[x(1-x)p^2+m^2]^{2-\omega}}. \tag{3.27b}$$

In deriving (3.27), we also employed the relation ('t Hooft and Veltman, 1973)

$$\eta^{\alpha\beta}\eta_{\alpha\beta} = 2\omega. \tag{3.28}$$

The remaining $x$ integral can be computed using the formula [see p. 287, no. 8, of Gradshteyn and Ryzhik (1965)]

$$\int_0^u dx\, x^{\nu-1}(u-x)^{\mu-1}(x+\beta)^\lambda$$
$$=\beta^\lambda u^{\mu+\nu-1}B(\mu,\nu)\,{}_2F_1(-\lambda,\,\nu;\,\mu+\nu;\,-u/\beta),$$
$$|\arg(u/\beta)| < \pi,\quad \mathrm{Re}\,\mu > 0,\quad \mathrm{Re}\,\nu > 0 \tag{3.29}$$

so that the system (3.27) reduces to

$$\Pi_{\mu\nu}(p^2,2\omega) = (p^2\eta_{\mu\nu} - p_\mu p_\nu)\Pi(p^2,2\omega), \tag{3.30a}$$

$$\Pi(p^2,2\omega) = \frac{e^2\Gamma(2-\omega)(m^2)^{\omega-2}}{3(2\pi)^\omega}\,{}_2F_1\!\left(2-\omega,\,2;\,\tfrac{5}{2};\,\frac{p^2}{4m^2}\right). \tag{3.30b}$$

It is evident from (3.30) that the ultraviolet infinities in the original expression (3.21) manifest themselves as poles of the $\gamma$ function in the limit $\omega \to 2^+$ [cf. Eq. (3.18) for the electron self-energy]. To eliminate these poles, we construct, in analogy with four-dimensional quantum electrodynamics, the expression

$$\Pi_f(p^2,2\omega) \equiv \Pi(p^2,2\omega) - \Pi(0,2\omega), \tag{3.31}$$

which gives the observable (physical) contribution of the photon self-energy (Schweber, 1962). In our case

$$\Pi_f(p^2,2\omega) = \frac{e^2\Gamma(2-\omega)(m^2)^{\omega-2}}{3(2\pi)^\omega}$$
$$\times\left[{}_2F_1\!\left(2-\omega,\,2;\,\tfrac{5}{2};\,-\frac{p^2}{4m^2}\right) - 1\right], \tag{3.32}$$

which is finite for *all* values of $\omega$, and leads to the regularized quantity

$$\lim_{\omega\to2^+}\Pi_f(p^2,2\omega) \equiv \Pi_{\mathrm{Reg}}(p^2,\,2\omega=4). \tag{3.33}$$

Let us turn now to the question of gauge invariance. It is well known (Jauch and Rohrlich, 1959) that the polarization tensor $\Pi_{\mu\nu}(p^2)$ in four-dimensional space-time is proportional to the gauge-invariant quantity $(p^2\eta_{\mu\nu} - p_\mu p_\nu)$. Not only does this quantity reappear in the generalized version of $\Pi_{\mu\nu}$, Eq. (3.30a), but it reappears there *independent* of the regulating parameter $\omega$. Consequently, the relation

$$p^\mu\Pi_{\mu\nu}(p^2,2\omega) = p^\mu(p^2\eta_{\mu\nu} - p_\mu p_\nu)\Pi_f(p^2,2\omega) = 0 \tag{3.34}$$

is identically satisfied for all values of $\omega$, particularly for $\omega = 2$. It is clear then that the continuous dimension method preserves the gauge-invariant character of the polarization tensor. For non-Abelian fields the problem of satisfying gauge invariance becomes much more acute due to the appearance of ghost particles which destroy the unitarity of the $S$ matrix. We return to this problem in Sec. VI.

## C. Limitation of dimensional regularization

### 1. Can the $\epsilon_{\alpha\beta\mu\nu}$ tensor be generalized to arbitrary dimensions?

The technique of dimensional regularization must be applied with care if the final Slavnov–Taylor identities contain factors of $\gamma^5$ or, equivalently, of the totally antisymmetric tensor $\epsilon_{\alpha\beta\mu\nu}$, where

$$\gamma^5 = (1/4!)\epsilon_{\alpha\beta\mu\nu}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu. \tag{3.35}$$

The basic reason for this caution is that the $\epsilon_{\alpha\beta\mu\nu}$ symbol is only defined in *four-dimensional* space, whereas the appropriate Slavnov–Taylor identities must be satisfied in any space, whether integral or complex. The solution to this dilemma hinges decisively on the possibility of generalizing Eq. (3.35) to dimensions other than four (Zumino, 1972). Akyeampong and Delbourgo (1973a,b) have adopted the view that one should work with $\gamma^5$ rather than with the $\epsilon_{\alpha\beta\mu\nu}$ tensor. The transition to arbitrary dimensions is then achieved by making the replacement

$$\gamma^5 \to \Gamma_{[K}\Gamma_L\Gamma_M\Gamma_{N]},$$

where the indices $K$, $L$, $M$, $N$ range from 0, 1, 2, ... up to

$2l-1$, and where the four Γ's are anticommuting. With this definition of the matrix $\gamma^5$ for *even*-dimensional spaces the four-dimensional term $i\psi^+\gamma^5\psi$ represents only *one* component of the chiral quantity $i\psi^+\Gamma_{[K}\Gamma_L\Gamma_M\Gamma_{N]}\psi$ in $2l$ dimensions. Whether a similar definition is also possible in odd dimensions remains to be seen.[12] 't Hooft and Veltman (1972a) employ, for example, a $\gamma^5$ which anticommutes with the first four $\gamma$ matrices, but which commutes with the remaining $\gamma$'s. The drawback of this particular generalization is that the Slavnov–Taylor identities, connected with the spinor line, may be violated (Bardeen, Gastmans, and Lautrup, 1972).

## 2. Anomalies

The difficulty with $\epsilon_{\alpha\beta\mu\nu}$, in physical terms, is that it gives rise to the well known Bell–Jackiw–Adler anomalies (Bell and Jackiw, 1969; Adler, 1969) such as the axial-vector current anomaly which arises in nucleon-nucleon scattering and leads to a breakdown of the Slavnov–Taylor identities as soon as $2\omega\neq 4$. Another example is the anomaly associated with the axial-vector triangle graph in spinor electrodynamics (Fig. 6), which is of the form (Adler, 1969)

$$-(k_1+k_2)_\lambda T_{\mu\nu}{}^\lambda = 2m_0T_{\mu\nu}+8\pi^2k_1{}^\alpha k_2{}^\beta\epsilon_{\alpha\beta\mu\nu}, \tag{3.36}$$

where $m_0$ is the bare electron mass; the structure of $T_{\mu\nu}$ is governed by parity conservation and Lorentz invariance.

Fortunately the appearance of anomalies of the type discussed above does not prevent us *per se* from using the continuous dimension method. For it is sometimes possible to make the anomalies cancel by a judicious redefinition of the associated spinor fields, as was demonstrated in the case of one-loop anomalies by Bouchiat, Iliopoulos, and Meyer (1972), Gross and Jackiw (1972), and Wess and Zumino (1971). This is true not only for Abelian, but also for non-Abelian models, where it is possible to remove the anomalies by doubling the number of fundamental fermions (Gross and Jackiw, 1972). The possibility of cancellations at higher orders has also been studied. According to Bardeen (1972), anomalies arising from higher-order diagrams will create no additional difficulties provided the anomalies can be made to cancel at the one-loop level.

Since the presence of anomalies (Bardeen, 1969) can be somewhat of a problem, both in Abelian and non-Abelian gauge theories, it is reasonable to ask which gauge models of the weak and electromagnetic interactions are likely to be *free of anomalies*. The answer to this question, as Georgi and Glashow (1972b) and others have shown, depends basically on the underlying group structure of the particular model (Gross and Jackiw, 1972; Wess and Zumino, 1971). To illustrate this for the non-Abelian case, consider the interaction between the gauge boson fields $W_\mu{}^a$ and the spinor fermion fields $\psi$. The Lagrangian for the interaction reads[13]

$$\mathcal{L}=-\tfrac{1}{4}G_{\mu\nu}{}^aG_a{}^{\mu\nu}+\bar\psi[i\not\partial-g\gamma^\mu W_\mu(1-\gamma_5)]\psi, \tag{3.37}$$

[12] It may turn out, in fact, that a generalization of $\gamma^5$ to odd-dimensional spaces is really not needed, since one is interested in returning eventually to physical four-space.

[13] We follow the work of Gross and Jackiw (1972).

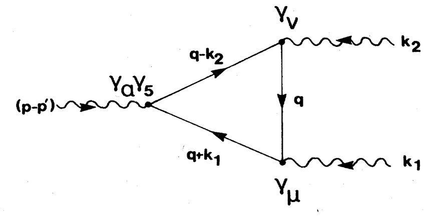



FIG. 6. Axial-vector triangle graph in spinor electrodynamics which leads to the anomaly in Eq. (3.36).

with

$$G_{\mu\nu}{}^a=\partial_\mu W_\nu{}^a-\partial_\nu W_\mu{}^a+gf_b{}^a{}_cW_\mu{}^bW_\nu{}^c, \tag{3.38}$$

$$W_\mu=\tfrac{1}{2}\lambda_aW_\mu{}^a, \tag{3.39}$$

$$[\lambda_a,\lambda_b]=2if_a{}^c{}_b\lambda_c, \tag{3.40}$$

where the Hermitian matrices $\lambda_a$ generate a representation of the Lie algebra, and $f_a{}^c{}_b$ are totally antisymmetric structure constants [cf. Eq. (1.8)]. Although the Lagrangian (3.37) is formally invariant under gauge transformations of the second kind, the same is not true for the corresponding perturbation series, where gauge invariance is broken by the appearance of axial-vector current anomalies. For the Lagrangian (3.37), the divergence of the current

$$J_\mu{}^a=\tfrac{1}{2}\bar\psi\gamma_\mu(1-\gamma_5)\lambda^a\psi \tag{3.41}$$

is given by (Gross and Jackiw, 1972)

$$\partial^\mu J_\mu{}^a=(\mathrm{const})g^3\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\,\{\lambda^a[2\,\partial_\mu W_\nu\,\partial_\alpha W_\beta-i\,\partial_\mu(W_\nu W_\alpha W_\beta)]\} \tag{3.42}$$

$$=(\mathrm{const})g^3\,d_{abc}\epsilon^{\mu\nu\alpha\beta}\,\partial_\mu[W_\nu{}^b(4\,\partial_\alpha W_\beta{}^c+f_{cde}W_\alpha{}^dW_\beta{}^e)], \tag{3.43}$$

where

$$d_{abc}=\tfrac{1}{4}\,\mathrm{Tr}\,[\lambda_a\{\lambda_b\lambda_c\}]. \tag{3.44}$$

If the group symbol $d_{abc}$ of the representation is zero, for all $a$, the non-Abelian anomaly on the right-hand side of (3.43) is absent. The proportionality of the anomaly to $d_{abc}$ implies quite generally that a gauge model will be anomaly-free if the corresponding group symbol $d_{abc}$ is zero, in which case the theory is said to be *safe* (Georgi and Glashow, 1972b). For example, a theory based on $SU(2)$ is safe, since $d_{abc}$ vanishes for all representations of $SU(2)$. The Lie algebra $SU(3)$, on the other hand, is not safe. Georgi and Glashow (1972b) have constructed a list of Lie algebras for which $d_{abc}$ is zero and which are accordingly safe for constructing models of the weak and electromagnetic interactions.

We conclude that if we are dealing with weak interactions or with some other interaction leading to an axial-vector current, the continuous dimension method must be employed judiciously. In particular, if the emerging anomalies cannot be made to cancel, dimensional regularization must be applied with care to prevent the appearance of ambiguities. The same conclusion holds for gauge theories involving chiral transformations, such as the Weinberg–Salam model (Weinberg, 1967; Salam, 1968).

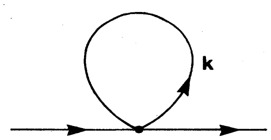



FIG. 7. Massless tadpole diagram.

## IV. EXTENSION OF DIMENSIONAL REGULARIZATION TO MASSLESS FIELDS

### A. General remarks

The technique of dimensional regularization, defined and illustrated in the preceding two sections, works admirably for massive fields provided no anomalies occur from partially conserved axial-vector currents. For massless fields, however, the technique requires modification in order to cope with genuine infrared divergences. The trick of inserting a finite mass into the integral and then allowing it to approach zero at the end of the calculation is, in general, not a satisfactory prescription. To see this, consider the massive integral

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}(k^2+m^2)^\alpha} = \frac{\Gamma(\alpha-\omega)}{(4\pi)^\omega (m^2)^{\alpha-\omega}\Gamma(\alpha)} \equiv K(m,\omega,\alpha), \quad m^2 \neq 0 \tag{4.1}$$

which converges for $\omega$ complex; the parameter $\alpha$ is arbitrary but fixed. All integrals in this and the subsequent section are defined over Euclidean space in order to facilitate comparison with the published literature. We note first of all that the limit

$$\lim_{m^2\to 0} K(m,\omega,\alpha) \tag{4.2}$$

may or may not exist, depending on the relative magnitudes of $\alpha$ and $\omega$. But even if it did exist, another problem could arise as we approach four-space (provided the original amplitude is infrared divergent to begin with), because in general

$$\lim_{\omega\to 2^+}[\lim_{m^2\to 0} K(m,\omega,\alpha)] \neq \lim_{m^2\to 0}[\lim_{\omega\to 2^+} K(m,\omega,\alpha)], \tag{4.3}$$

so that the massless integral

$$\lim_{\omega\to 2^+}\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}(k^2)^\alpha}$$

cannot be derived unambiguously from the massive integral (4.1).[14]

The insertion of a finite mass term into the propagator is unsatisfactory for yet another reason: it spoils the gauge symmetry in the original theory, provided such a symmetry existed in the first place. For these reasons it is probably fair to say that the transition from a massive to a massless theory creates at least as many problems as it solves (Van Dam and Veltman, 1970; Zakharov, 1970). Before describing a possible solution to this infrared dilemma, we illustrate by means of a concrete example why the technique described in Sec. II.B.1 may lead, for massless particles, to ambiguous results.

Let us evaluate the massless integral

$$I = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}k^2}, \tag{4.4}$$

associated with the tadpole shown in Fig. 7, in two distinct ways (Leibbrandt and Capper, 1974a). First, using Eqs. (2.3) and (2.4), we obtain

$$I = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \int_0^\infty \exp(-xk^2)dx = \int_0^\infty \frac{dx\, x^{-\omega}}{(4\pi)^\omega}, \tag{4.5}$$

which diverges as $\omega \to 2^+$, the infinity arising specifically from the lower limit of integration $x = 0$. In view of this divergence the interchange of the $x$ and $k$ integrations in (4.5) is strictly speaking not permissible.

In our second approach we express (4.4) as

$$I = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}k^2}\cdot\frac{(k-p)^2}{(k-p)^2} \tag{4.6a}$$

$$= p^2\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}k^2(k-p)^2} - 2\int \frac{d^{2\omega}k\; p\cdot k}{(2\pi)^{2\omega}k^2(k-p)^2} + \int \frac{d^{2\omega}k\; k^2}{(2\pi)^{2\omega}k^2(k-p)^2}, \quad p^2 \neq 0. \tag{4.6b}$$

Evaluating the three integrals in (4.6b) separately we find that

$$I = (4\pi)^{-\omega}\{(p^2)^{\omega-1}\Gamma(1-\omega)[(1-\omega)B(\omega-1,\omega-1) - 2(1-\omega)B(\omega-1,\omega) + (1-\omega)B(\omega-1,\omega+1)]\} + [\omega(4\pi)^{-\omega}(p^2)^{\omega-1}\Gamma(1-\omega)B(\omega,\omega)], \tag{4.7}$$

where the $\beta$ function is defined by

$$B(x,y) = \int_0^1 dt\, t^{x-1}(1-t)^{y-1}, \quad \mathrm{Re}\, x > 0, \quad \mathrm{Re}\, y > 0. \tag{4.8}$$

A naive reduction of Eq. (4.7) yields $I = 0$. To see that this need not be the case, we observe that each of the terms in the bracket $\{\ldots\}$ is analytic in the finite strip $\mathfrak{D}_1$: $1 < \mathrm{Re}\,\omega < 2$, whereas the last expression involving $\Gamma(1-\omega)B(\omega,\omega)$ is only defined in the domain $\mathfrak{D}_2$: $0 < \mathrm{Re}\,\omega < 1$. Since the domains of definition $\mathfrak{D}_1$ and $\mathfrak{D}_2$ do *not overlap* ($\mathfrak{D}_1 \cap \mathfrak{D}_2 = \phi$), it is not clear whether we are justified in making cancellations between the analytic continuations of the corresponding functions in (4.7). The difficulty with Eq. (4.7), therefore, is lack of uniqueness in the limit $\omega \to 2^+$.

We conclude that the application of the continuous dimension method to the massless integral (4.4) yields either infinity, zero, or a finite value, depending on the method of computation. Similar inconsistencies emerge in other massless integrals.

[14] Of course, for "healthy" theories, such as quantum electrodynamics, it does not really matter if the limits in Eq. (4.3) cannot be interchanged, provided the arithmetic is performed *consistently* throughout the entire calculation. It is also worthwhile recalling that off-mass-shell amplitudes are *not* infrared divergent and that there are some real infinities in closed loop diagrams which are cancelled by similar infinities in phase space integrals. The author is grateful to Professor M. Veltman for pointing these facts out to him.

### B. Redefinition of the generalized Gaussian integral

#### 1. The Gaussian integral

Since the continuous dimension method outlined in Sec. II.B is clearly inadequate to cope with the infrared divergences arising from massless particles, Leibbrandt and Capper (1974a) have proposed the following redefinition of the generalized Gaussian integral in $2\omega$-dimensional Euclidean space[15]:

*Definition:*

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \exp[-xk^2 + 2b\cdot k] \equiv (4\pi x)^{-\omega} \times \exp[b^2/x - xf(\omega)], \quad x > 0 \tag{4.9}$$

where the vector $b_\mu$ is also defined over $2\omega$-space and $x$ behaves like a $c$ number. The new function $f(\omega)$, which is *not* unique, is called the *continuity function* and satisfies the following four conditions:

(i) $f(\omega)$ is a nonzero analytic function of the complex variable $\omega = \sigma + i\tau$;

(ii) $f(\omega) = 0$ for $\omega = \pm \frac{1}{2}\lambda$, $\lambda = 0, 1, 2, \ldots$;

(iii) $f^{(l)}(\omega) = 0$ for $\omega = \pm \frac{1}{2}\lambda$, $\lambda = 0, 1, 2, \ldots$, and $l \leq l_0$, where $l_0$ is *finite*; $l$ denotes the number of ordinary derivatives with respect to $\omega$;

(iv) $\mathrm{Re}[f(\omega)] > 0$ for *any* $\mathrm{Re}\,\omega \neq \pm \frac{1}{2}\lambda$, $\lambda = 0, 1, 2, \ldots$ and for *some* $\mathrm{Im}\,\omega$.

These properties are discussed extensively in Leibbrandt and Capper (1974a). Here we merely observe that property (iv) guarantees that the $x$ integration is well defined, since the integral

$$\int_0^\infty x^{-\omega} \exp[-xf(\omega)]dx, \tag{4.10}$$

with $\mathrm{Re}[f(\omega)] > 0$, yields immediately Euler's $\gamma$ function; by comparison, the integral $\int_0^\infty x^{-\omega}dx$ does not exist [cf. Eq. (4.5)].

A continuity function $f(\omega)$ satisfying the above four properties is, for example, given by (Capper and Leibbrandt, 1974)

$$f(\omega) = 1 - \cos(2\pi \cos(2\pi(\cos(\ldots(\cos 2\pi\omega)\ldots)))), \tag{4.11}$$

which contains $n$ *nested* cosine functions, $n$ being a finite integer. The function (4.11) satisfies two additional criteria:

(v) $f^{(l)}(\omega) = 0$ for $\omega = \pm \frac{1}{2}\lambda$, $\lambda = 0, 1, 2, \ldots$, and where $2^n > l + 1$; $n$ has the same meaning as in (4.11), while $l$ is the same as in property (iii);

(vi) $f^{(l)}(\omega) \neq 0$ for $\omega = \pm\frac{1}{2}\lambda$, $\lambda = 0, 1, 2, \ldots$, and $2^n \leq l + 1$.

There exist other functions with the required properties (i)–(vi), but the representation (4.11) is particularly easy to handle. It is certainly superior to an earlier version of $f(\omega)$, especially with regard to the calculation of higher-order diagrams (Leibbrandt and Capper, 1974b). We finally emphasize that the introduction of $f(\omega)$ in definition (4.9) is *not* a gauge-invariant procedure for $\omega \neq 2$. Since it is possible, however, by property (v) to make an arbitrarily large number of derivatives of $f(\omega)$ vanish, the recipe (4.9) to (4.11) can be made *gauge-invariant to any finite order*.

#### 2. The lowest-order tadpole integral

The modified definition (4.9) of the complex dimensional Gaussian integral permits a consistent treatment of the massless tadpole integral (4.4). Application of Eqs. (2.3) and (4.9) yields forthwith

$$I = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}k^2} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \int_0^\infty \exp(-xk^2) \tag{4.12}$$

$$= (4\pi)^{-\omega} \int_0^\infty dx\, x^{-\omega} \exp[-xf(\omega)] \tag{4.13}$$

with $f(\omega)$ given by (4.11). Since $\mathrm{Re}[f(\omega)] > 0$, it is possible to integrate (4.13):

$$I = (4\pi)^{-\omega}\Gamma(1 - \omega)[f(\omega)]^{\omega-1}. \tag{4.14}$$

Expanding both $[f(\omega)]^{\omega-1}$ and $\Gamma(1 - \omega)$ in (4.14) about $\omega = 2$, where

$$\Gamma(1 - \omega) = -\{1/(2 - \omega) + \psi(2) + \tfrac{1}{2}(2 - \omega)[\tfrac{1}{3}\pi^2 + \psi^2(2) - \psi'(2)] + O((2 - \omega)^2)\}, \tag{4.15}$$

we obtain

$$I = -4i\pi^3(4\pi)^{-\omega}[(2 - \omega) + (2 - \omega)^2\psi(2) + O((2 - \omega)^3)]. \tag{4.16}$$

Clearly, as $\omega \to 2^+$, the massless integral (4.12) reduces to zero:

$$\lim_{\omega\to 2^+} \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}k^2} = 0. \tag{4.17}$$

This result must be interpreted with care: We are not claiming that the tadpole (4.4) is really zero, but rather that the continuous dimension method regularizes these highly divergent (four-dimensional) integrals formally to zero.

Definition (4.9) may be used to demonstrate that the result (4.14) is consistent with the integral (4.6a). Calculation of the three integrals in Eq. (4.6b) is straightforward and yields (Leibbrandt and Capper, 1974a)

$$I = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}k^2}\cdot\frac{(k - p)^2}{(k - p)^2} = (4\pi)^{-\omega}\Gamma(1 - \omega)[f(\omega)]^{\omega-1}, \tag{4.18}$$

in exact agreement with Eq. (4.14). Hence the modified Gaussian formula (4.9) leads to a consistent value for the tadpole integral (4.4).

[15] This approach to dimensional regularization differs, for *massless* particles, somewhat from that of 't Hooft and Veltman. The difference arises basically from step (ii) in Sec. II.B.1. The author is grateful to Dr. G. 't Hooft for his clarifying remarks about this matter.

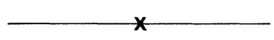


FIG. 8. Diagram giving rise to a$\delta^4(0)$ term.

### 3. $\delta^4(0)$ terms

$\delta^4(0)$ terms arise in theories which contain two or more derivatives in a nonlinear interaction Lagrangian (see Fig. 8). In the context of dimensional regularization, such terms can formally be replaced by the integral $\int d^{2\omega}k$; the latter may be evaluated by multiplying the integrand by $1 = k^2/k^2$. Applying the parametrization (2.3), we get

$$I_0 = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \cdot \frac{k^2}{k^2} = \int_0^\infty dx \int \frac{d^{2\omega}k\, k^2}{(2\pi)^{2\omega}} \exp(-xk^2), \quad (4.19)$$

which may be further reduced with the help of definition (4.9) and the formulas

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \exp(-xk^2) = (4\pi x)^{-\omega} \exp[-xf(\omega)], \quad (4.20)$$

$$\int \frac{d^{2\omega}k\, k^2}{(2\pi)^{2\omega}} \exp(-xk^2) = (4\pi)^{-\omega}[\omega x^{-(1+\omega)} + x^{-\omega} f(\omega)] \times \exp[-xf(\omega)]. \quad (4.21)$$

Hence

$$I_0 = \omega(4\pi)^{-\omega} \int_0^\infty dx\, x^{-1-\omega} \exp[-xf(\omega)] + f(\omega)(4\pi)^{-\omega} \int_0^\infty dx\, x^{-\omega} \exp[-xf(\omega)], \quad (4.22)$$

which on integration yields the result

$$I_0 = [f(\omega)]^{\omega}(4\pi)^{-\omega}[\omega\Gamma(-\omega) + \Gamma(1-\omega)]. \quad (4.23)$$

In the limit as $\omega \to 2^+$, the right-hand side of (4.23) approaches zero inside the bracket, so that $\delta^4(0)$ formally vanishes.

## C. Proof of a 't Hooft–Veltman conjecture

The continuous dimension method has the additional advantage of treating highly divergent Feynman integrals such as $\int d^4k(k^2)^n$, $n = 0, 1, 2, \ldots$, in a consistent manner, as was first pointed out by 't Hooft and Veltman (1972c) in the case of *massive* particles. They conjectured that no inconsistencies occur, for example, in the Slavnov–Taylor identities (Capper, Leibbrandt, and Ramón Medrano, 1973) if one assumes that

$$\int \frac{d^{2\omega}k(k^2)^{\beta-1}}{(2\pi)^{2\omega}} = 0, \quad \text{for} \quad \omega, \beta \text{ complex}. \quad (4.24a)$$

Equation (4.24a) implies, among other things, that integrals over a polynomial vanish identically within the context of dimensional regularization:

$$I \equiv \int \frac{d^{2\omega}k(k^2)^{\beta-1}}{(2\pi)^{2\omega}} = 0, \quad \beta = 0, 1, 2, \cdots. \quad (4.24b)$$

Within our definition, using the function $f(\omega)$, Eq. (4.24b) has recently been proven for $\beta = 1, 2, 3, \ldots$ and *any* $\omega$, and for $\beta = 0$ in the limit $\omega \to 2^+$. In actual fact the integral (4.24a) is zero for sufficiently large $\omega$ ($\omega > 1 - \beta$) for any given $\beta$ if one first introduces a *mass* $m$, and then takes the limit $m \to 0$. [This follows, for example, from Eq. (A1).] By analyticity in $\omega$ the integral (4.24a) is then zero everywhere. Consequently there exists a definition whereby Eq. (4.24a) holds for all $\omega$ and $\beta$.

The major steps in the proof of the 't Hooft–Veltman conjecture (4.24b) for *massless* particles—massless integrals over a polynomial occur for example in one-loop and two-loop graviton-graviton calculations—can be summarized as follows (Leibbrandt and Capper, 1974b). One first applies Eqs. (2.3) and (4.9) to the integral (4.4), which leads to (4.13). Differentiation of

$$\int d^{2\omega}k \exp(-xk^2) = \pi^{\omega} x^{-\omega} \exp[-xf(\omega)] \quad (4.25)$$

$\beta$ times with respect to $x$ then yields

$$\int d^{2\omega}k(k^2)^{\beta} \exp(-xk^2) = \pi^{\omega} \sum_{j=0}^{\beta} \frac{\Gamma(\beta+1)\Gamma(\omega+j)}{\Gamma(j)\Gamma(\beta-j+1)\Gamma(\omega)} \cdot [f(\omega)]^{\beta-j} x^{-\omega-j} \times \exp[-xf(\omega)], \quad \beta = 0, 1, 2, \cdots, \quad (4.26)$$

in which case the integral in (4.24b) becomes

$$I = (4\pi)^{-\omega}[f(\omega)]^{\omega+\beta-1}\Gamma(1-\omega) \times \sum_{j=0}^{\beta} \frac{(-1)^j\Gamma(\beta+1)}{\Gamma(j+1)\Gamma(\beta-j+1)}. \quad (4.27)$$

The last expression reduces, on account of the formula

$$\sum_{j=0}^{\beta} \frac{(-1)^j\Gamma(\beta+1)}{\Gamma(j+1)\Gamma(\beta-j+1)} = \begin{cases} 1 & \text{for} \quad \beta = 0, \quad (4.28) \\ 0 & \text{for} \quad \beta \geq 1, \quad (4.29) \end{cases}$$

to the desired result

$$\int \frac{d^{2\omega}k(k^2)^{\beta-1}}{(2\pi)^{2\omega}} = \begin{cases} (4\pi)^{-\omega}[f(\omega)]^{\omega+\beta-1}\Gamma(1-\omega) & \text{for} \quad \beta = 0, \quad (4.30) \\ 0 & \text{for} \quad \beta \geq 1. \quad (4.31) \end{cases}$$

For $\beta = 0$, we recover from Eq. (4.30) the tadpole result (4.14), while Eq. (4.31) with $\beta = 1$ verifies that $\delta^4(0)$ terms are formally zero in the framework of dimensional regularization. Finally, for $\beta > 1$ Eq. (4.31) shows that integrals over a polynomial do indeed vanish. This completes our proof of the 't Hooft–Veltman conjecture (4.24b) for *massless* particles.

# V. APPLICATION TO YANG–MILLS FIELDS

## A. Introduction

The continuous dimension method has recently been applied by several authors to gauge field theories of the Yang–Mills type (Yang and Mills, 1954; Shaw, 1954) which represent the simplest example of theories associated with a local non-Abelian gauge group. Bardeen, Gastmans, and Lautrup (1972), for example, have computed the muon anomaly at the one-loop level in the Weinberg $SU(2) \times U(1)$ model (Weinberg, 1967). The results of their calculation agree with those of Fujikawa, Lee, and Sanda (1972) working in the $R_\xi$ formalism. Jones (1974) has performed a two-loop calculation of the coefficient function $\beta(g)$ which appears in the Callan–Symanzik renormalization group equations (Callan, 1970; Symanzik, 1970a).

### B. The muon (g − 2) factor in the Georgi–Glashow model

In this section we use dimensional regularization to study the muon $(g-2)$ factor in the Georgi–Glashow model. Georgi and Glashow (1972a) have constructed a spontaneously broken gauge model of weak and electromagnetic interactions with $SO(3)$ symmetry which contains no additional neutral currents apart from the electromagnetic one.[16] This model, which contains two gauge fields $W^\pm$ along with the photon $A \equiv W_3$, consists of two muon ($\mu$) triplets and singlets and two electron ($e$) triplets and singlets. The muon triplets are of the form

$$\psi_{\mu R} = \tfrac{1}{2}(1-\gamma_5)\begin{pmatrix} Y^+ \\ Y^0 \\ \mu^- \end{pmatrix}, \qquad \psi_{\mu L} = \tfrac{1}{2}(1+\gamma_5)\begin{pmatrix} Y^+ \\ Y^0 \cos\beta + \nu_\mu \sin\beta \\ \mu^- \end{pmatrix}, \tag{5.1}$$

while the two left (L)- and right (R)-handed muon singlets read

$$s_{\mu L} = \tfrac{1}{2}(1+\gamma_5)(Y^0 \sin\beta - \nu_\mu \cos\beta), \qquad s_{\mu R} = \tfrac{1}{2}(1-\gamma_5)\nu_\mu, \tag{5.2}$$

where $Y^+$, $Y^0$ are heavy muonic leptons and $\beta$ is the mixing angle between the physical muon neutrino $\nu_{\mu L}$ and $Y_L{}^0 \equiv \frac{1}{2}(1-\gamma_5)Y^0$. A similar set of triplets and singlets, with heavy leptons $X^+$, $X^0$ and with the same mixing angle $\beta$, exists for the electron and its neutrino $\nu_e$. The total Lagrangian for the Georgi–Glashow model, which is invariant under a local non-Abelian gauge group, consists of five components:

$$\mathcal{L} = \mathcal{L}_{K.E.} + \mathcal{L}_{m^0} + \mathcal{L}_{W^-l} + \mathcal{L}_\phi + \mathcal{L}_W, \tag{5.3}$$

where $\mathcal{L}_{W^-l}$,

$$\mathcal{L}_{W^-l} = \sum_{l=e,\mu} e\mathbf{W}_\lambda \cdot (\bar{\psi}_l \times \gamma_\lambda \psi_l), \tag{5.4}$$

describes the coupling of the muon and electron to the gauge field $\mathbf{W}_\lambda$. We shall refrain from listing explicitly the remaining terms in Eq. (5.3)—they can be found in Primack and Quinn (1972)—since the purpose of the above summary is merely to familiarize the reader with the structure and nomenclature of this simple model.

Our discussion of the weak correction to the anomalous magnetic moment of the muon follows closely the work of Primack and Quinn (1972). To begin with we recall that the $\mu^-$ electromagnetic vertex is of the form (Abers and Lee, 1973)

$$V_\alpha = \bar{u}(p+\tfrac{1}{2}q)\left[\frac{p_\alpha}{m_\mu}F_1(q^2) + i\frac{\sigma_{\alpha\beta}q^\beta}{2m_\mu}(F_1(q^2)+F_2(q^2))\right] \times u(p-\tfrac{1}{2}q), \tag{5.5}$$

where $m_\mu$ is the mass of the muon and $(p-\frac{1}{2}q)$ and $(p+\frac{1}{2}q)$ its initial and final momentum, respectively. $F_1$ and $F_2$ denote, respectively, the electric and magnetic form factors as functions of the momentum transfer $q_\alpha$. The anomalous magnetic moment of the muon is defined as

$$(F_2(0))_\mu = [\tfrac{1}{2}(g-2)]_\mu \equiv a_\mu. \tag{5.6}$$

Whereas the one-photon exchange contribution to $a_\mu$ [see Fig. 9(a)] is the same and equal to $\alpha/2\pi$ in all models [see, for example, Weinberg (1967) and Georgi and Glashow (1972a)], the corrections to $a_\mu$ are definitely model-dependent. In the Georgi–Glashow model the corrections to second order in $e$ come from diagrams 9(b), (d), and (c) which yield, respectively (Primack and Quinn, 1972),

$$(a_\mu)^\nu = \frac{\alpha \sin^2\beta m_\mu{}^2}{8\pi M_W{}^2}\frac{10}{3}, \tag{5.7}$$

$$(a_\mu)^\phi = \frac{\alpha(m_{Y^+}-m_\mu)^2 m_\mu{}^2}{8\pi M_W{}^2 M_\phi{}^2}\int_0^1 \frac{dx(2x^2-x^3)}{x^2\rho - x + 1}, \tag{5.8}$$

$$(a_\mu)^{Y^0} = \frac{-\alpha m_\mu m_{Y^0}\cos\beta}{2\pi M_W{}^2}\left[\frac{1}{2} + 3\int_0^1 \frac{dx\, x^2}{x+(1-x)r}\right], \tag{5.9}$$

where

$$r \equiv m_{Y^0}{}^2/M_W{}^2, \quad \rho \equiv m_\mu{}^2/M_\phi{}^2$$

and $m_{Y^+}$, $m_{Y^0}$, $M_\phi$, and $M_W$ denote, respectively, the masses of the $Y^+$, $Y^0$, $\phi$, and $W$ particles.[17] Using dimensional regularization, we shall concentrate on the derivation in (5.9) of the factor $(\frac{1}{2})$ inside the square bracket. In the

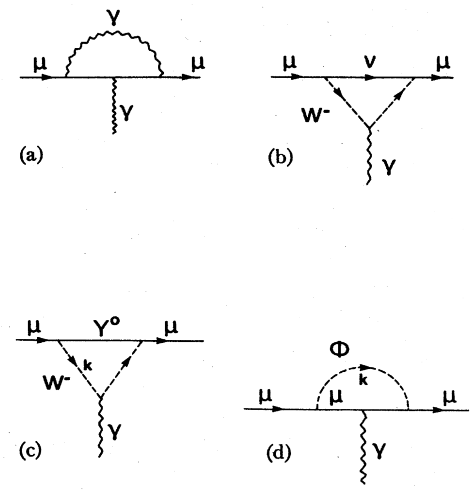



FIG. 9. These four diagrams contribute to the muon $(g-2)$ factor to order $e^2$ in the $SO(3)$ model. In the text we are chiefly concerned with diagram (c).

[16] The various aspects of spontaneous symmetry breaking are discussed in a recent article by Bernstein (1974).

[17] *After* spontaneous symmetry breaking, the originally massless vector bosons acquire a mass $M_W \sim (53 \sin\beta)$ Ge V, while the photon $A = W^0$ remains massless.

unitary gauge or $U$ formalism (Primack, 1972; Weinberg, 1973), where the $W$ propagator reads

$$\frac{g_{\alpha\beta} - k_\alpha k_\beta/M_W^2}{k^2 - M_W^2}, \tag{5.10}$$

the contribution from diagram 9(c) equals ($q = p' - p$)

$$\Gamma^\lambda(p',p) = ie^2\int \frac{d^4k}{(2\pi)^4}\bar{u}(p')\gamma_\sigma[m_{Y^0}\cos\beta + \tfrac{1}{2}(\not{p} - \not{k})(1 + \cos^2\beta - \gamma_5\sin^2\beta)]$$
$$\times \gamma_\tau u(p)\frac{(k+q)^\alpha(k+q)^\sigma/M_W^2 - g^{\alpha\sigma}}{(k+q)^2 - M_W^2}$$
$$\cdot\frac{k^\beta k^\tau/M_W^2 - g^{\beta\tau}}{k^2 - M_W^2}\,\frac{1}{(p-k)^2 - m_{Y^0}{}^2}[(k-q)^\alpha g^{\beta\lambda} + (2q+k)^\beta g^{\alpha\lambda} - (2k+q)^\lambda g^{\alpha\beta}]. \tag{5.11}$$

The computation of $\Gamma^\lambda$ is nontrivial on at least two counts. In the first place, the integrals appear to be highly divergent due to the special form (5.10) of the propagators, and secondly Eq. (5.11) contains the factor $\gamma_5$. The first problem is readily dealt with by using a powerful cutoff procedure such as dimensional regularization. But if the latter is employed, we are faced with another problem, that of generalizing the $\gamma_5$ matrix *unambiguously* to arbitrary dimensions which, as pointed out in Sec. III.C, is still not rigorously solved. The problem can be circumvented by calculating only the leading term in $(g - 2)$ which is of order $m_\mu m_{Y^0}/M_W^2$, so that in the first square bracket of (5.11) only the expression $m_{Y^0}\cos\beta$ survives. Generalizing Eq. (5.11) to $2\omega$ dimensions and observing that the numerator term proportional to $1/M_W^4$ vanishes at $q = 0$, we obtain

$$\Gamma^\lambda(2\omega,p',p) = -ie^2 m_{Y^0}\cos\beta(\bar{u}(p')\gamma_\sigma\gamma_\tau u(p))$$
$$\times\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\left\{(k+q)^\alpha(k+q)^\sigma\frac{g^{\beta\tau}}{M_W^2} + k^\beta k^\tau\frac{g^{\alpha\sigma}}{M_W^2} - g^{\alpha\sigma}g^{\beta\tau}\right\}$$
$$\times[(k-q)^\alpha g^{\beta\lambda} + (2q+k)^\beta g^{\alpha\lambda} - (2k+q)^\lambda g^{\alpha\beta}]$$
$$\times[((k+q)^2 - M_W^2)(k^2 - M_W^2)\times((p-k)^2 - m_{Y^0}{}^2)]^{-1}. \tag{5.12}$$

The computation of the term $g^{\alpha\sigma}g^{\beta\tau}$ in the curly bracket in (5.12) is straightforward and will not be discussed any further. For the evaluation of the divergent integrals associated with the first and second terms, *viz.*

$$[(k+q)^\alpha(k+q)^\sigma g^{\beta\tau} + k^\beta k^\tau g^{\alpha\sigma}]/M_W^2,$$

a gauge-invariant cutoff procedure is virtually mandatory.[18] Combining propagators and simplifying the resulting integrand, we can show that the relevant integrals reduce to the form

$$I(2\omega,p) = \int\frac{d^{2\omega}k(4k\cdot pk_\mu - k^2p_\mu)}{(2\pi)^{2\omega}(k^2+c)^3}. \tag{5.13}$$

[18] A naive computation of this expression in four dimensions yields zero. See the comment in the appendix of Primack and Quinn (1972).

Application of Eq. (2.8c) together with

$$\int\frac{d^{2\omega}k\,k_\mu k_\nu}{(2\pi)^{2\omega}(k^2+c)^3} = \frac{i\pi^\omega\Gamma(2-\omega)g_{\mu\nu}}{(2\pi)^{2\omega}2c^{2-\omega}\Gamma(3)} \tag{5.14}$$

yields

$$I(2\omega,p) = \frac{i\pi^\omega(2-\omega)\Gamma(2-\omega)}{(2\pi)^{2\omega}\Gamma(3)c^{2-\omega}}p_\mu. \tag{5.15}$$

As $\omega \to 2^+$, the last expression reduces to $(i/32\pi^2)p_\mu$ which leads precisely to the factor $(\frac{1}{2})$ in Eq. (5.9). This completes our discussion of the $(a_\mu)^{Y^0}$ contribution.

The weak corrections to the muon anomalous magnetic moment have also been computed by Jackiw and Weinberg (1972) and by Bars and Yoshimura (1972) in Weinberg's $SU(2)\times U(1)$ model, employing the $U$ formalism, and by Fujikawa, Lee, and Sanda (1972) in the $R_\xi$ gauge [see also Hagiwara (1974)]. To compute the muon anomaly, the latter three authors apply the $R_\xi$ formalism in both the Georgi–Glashow and the Weinberg model.[19] The important message emerging from these various model calculations of the muon $(g - 2)$ factor is simply this: For the sake of internal consistency and in order to minimize the appearance of ambiguities, a gauge-invariant regularization scheme should be employed whenever possible.[20]

## VI. APPLICATION TO QUANTUM GRAVITY

### A. Pure quantum gravity

#### 1. Introduction

The purpose of this section is to calculate the one-loop contributions to the graviton self-energy (Fig. 10) in the context of covariant quantization by employing the modified technique of dimensional regularization [cf. Eq. (4.9)]. Since quantum gravity belongs to the class of non-Abelian gauge theories, it is considerably harder to regularize than the two Abelian examples discussed in Sec. III. Calculations are further complicated by the fact that we are dealing here with a massless spin-two theory. Despite these rather severe obstacles, the continuous dimension method correctly preserves the gauge symmetry of the underlying Einstein–Hilbert Lagrangian, provided all contributions from fictitious particles are properly included.

Einstein's gravitational field is described by the Lagrangian density

$$\mathcal{L} = (2/\kappa^2)\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}, \tag{6.1}$$

where $g^{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ the Ricci tensor defined by

$$R_{\mu\nu} = \Gamma_{\mu\rho,\nu}{}^\rho - \Gamma_{\mu\nu,\rho}{}^\rho - \Gamma_{\mu\nu}{}^\sigma\Gamma_{\sigma\rho}{}^\rho + \Gamma_{\sigma\nu}{}^\rho\Gamma_{\mu\rho}{}^\sigma \tag{6.2}$$

with

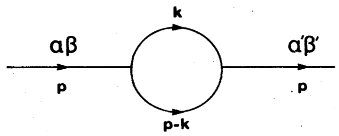
$$\Gamma_{\mu\nu}{}^\rho = \tfrac{1}{2}g^{\rho\sigma}(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}), \tag{6.3}$$

and $g \equiv \det g_{\mu\nu}$; $\kappa^2 = 32\pi G$ is the gravitational constant in natural units $\hbar = c = 1$ and $G$ is the Newtonian constant.

[19] For a review of this topic, see Primack (1972).

[20] A clear discussion of ambiguities arising, for example, in the $U$ gauge from the use of the Pauli–Villars or the "proper-time" regularization schemes can be found in Fujikawa, Lee, and Sanda (1972).

FIG. 10. Massless graviton loop.




We can simplify subsequent calculations appreciably by expressing the Lagrangian density (6.1) in terms of the tensor density $\tilde{g}^{\alpha\beta}$ of weight $+1$ (Capper, Leibbrandt, and Ramón Medrano, 1973):

$$\tilde{g}^{\alpha\beta} \equiv \sqrt{-g}\, g^{\alpha\beta}, \tag{6.4}$$

in which case

$$\mathcal{L} = \frac{1}{2\kappa^2}\left(\tilde{g}^{\alpha\sigma}\tilde{g}_{\lambda\mu}\tilde{g}_{\beta\nu} - \frac{1}{2(\omega-1)}\tilde{g}^{\alpha\sigma}\tilde{g}_{\mu\beta}\tilde{g}_{\lambda\nu} - 2\delta_{\beta}{}^{\sigma}\delta_{\lambda}{}^{\alpha}\tilde{g}_{\mu\nu}\right) \times \tilde{g}_{,\alpha}{}^{\mu\beta}\tilde{g}_{,\sigma}{}^{\lambda\nu}. \tag{6.5}$$

The new version of the Einstein–Hilbert Lagrangian exhibits a pole at $\omega = 1$ which is connected with the fact that the total Lagrangian $L = \int \mathcal{L}\, d^{2\omega}x$ reduces in two dimensions to a surface integral. It is also worth noting that the original Lagrangian (6.1) is independent of the dimension of the space, whereas (6.5) contains $\omega$ explicitly.

## 2. Fictitious particles

For gauge theories, the Green's functions and vertices are most easily derived from a generating functional $Z$ by the method of functional derivatives (Schwinger, 1951; Symanzik, 1954; Zumino, 1960). In the case of quantum gravity, $Z$ reads

$$Z[j_{\mu\nu}] = \int d[\tilde{g}^{\mu\nu}]\Delta[\tilde{g}^{\mu\nu}] \exp\left\{i\int dx\left[\mathcal{L} + \frac{1}{\kappa}\tilde{g}^{\mu\nu}j_{\mu\nu} - \frac{1}{\alpha\kappa^2}(\partial_\mu\tilde{g}^{\mu\nu})^2\right]\right\}, \tag{6.6}$$

where $j_{\mu\nu}$ is an external source function and $\mathcal{L}$ is given by (6.5). The factor $\Delta[\tilde{g}^{\mu\nu}]$ gives rise to fictitious particles, while $-(\alpha\kappa^2)^{-1}(\partial_\mu\tilde{g}^{\mu\nu})^2$ breaks the gauge symmetry. The last two terms play an essential role in the quantization of the gravitational field, especially in the derivation of the Feynman rules. The gauge term removes the degeneracy in the free part of the Lagrangian so that the propagators become *unique*. The expression $\Delta[\tilde{g}^{\mu\nu}]$, on the other hand, compensates for an *infinite volume* factor which arises from integrating over points in the function space of the field variables $\tilde{g}^{\mu\nu}$ (Faddeev and Popov, 1967; Popov and Faddeev, 1972). If the physical graviton field $\phi^{\mu\nu}$ is defined by

$$\tilde{g}^{\mu\nu} \equiv \delta^{\mu\nu} + \kappa\phi^{\mu\nu}, \tag{6.7}$$

then[21]

$$(\Delta[\tilde{g}^{\mu\nu}])^{-1} = \int d[\xi_\lambda]\, d[\eta_\nu] \times \exp\{i\int dx\, \eta_\nu[\delta_{\nu\lambda}\Box - \kappa(\phi_{\mu\nu,\lambda\mu} - \phi_{\mu\rho}\delta_{\nu\lambda}\,\partial_\mu\,\partial_\rho - \phi_{\mu\rho,\mu}\delta_{\nu\lambda}\,\partial_\rho + \phi_{\mu\nu,\mu}\,\partial_\lambda)]\xi_\lambda\}, \tag{6.8}$$

where $\xi_\lambda$ and $\eta_\nu$ represent *fictitious* particles, also known as Feynman–DeWitt–Faddeev–Popov ghosts. [See Feynman (1963), DeWitt (1967a,b), and also Mandelstam (1968).] In pure quantum gravity, these ghosts are real massless *vector* particles. Moreover, they are unphysical since they occur only in oriented closed loops, called fictitious "fermion" loops (Fig. 11). The name "fermion" comes from the rule which assigns—in analogy with Furry's theorem in quantum electrodynamics—a factor of $(-1)$ to each closed loop. The fictitious particles $\xi_\lambda$ and $\eta_\nu$ are needed to cancel the longitudinal, i.e., unphysical polarizations arising from closed loops. Their purpose is to restore both the unitarity of the scattering matrix $S$ and the transversality of the scattering amplitudes (Faddeev and Popov, 1967; Fradkin and Tyutin, 1970).

[21] In view of definition (6.7) it is no longer necessary to distinguish between upper and lower indices on $\phi_{\mu\nu}$.

## 3. Slavnov–Taylor identities

Having gained some insight into the structure of the generating functional $Z[j_{\mu\nu}]$, we are ready now to continue with Eq. (6.6). Employing the technique of 't Hooft (1971a) we obtain the following expression for the Slavnov–Taylor identity (Slavnov, 1972; Taylor, 1971):

$$(2/\alpha) < T\phi_{\mu\nu,\mu}(x)\phi_{\lambda\beta,\lambda}(y) > = -\delta_{\nu\beta}\delta(x-y), \quad \alpha \neq 0 \tag{6.9}$$

which is true to all orders in the gravitational constant $\kappa$ and holds for any gauge specified by the parameter $\alpha$. Since the Slavnov–Taylor identity (6.9) is a theoretical result which is clearly independent of the number of dimensions $2\omega$, the central question is: Does our space-dependent regularization scheme preserve the space-independent identity (6.9) in a practical calculation? Capper, Leibbrandt, and Ramón Medrano (1973) have demonstrated for the graviton self-energy that the continuous dimension method does indeed preserve (6.9) to second order in $\kappa$ [see also Brown (1973)]. Since their calculation is already of tremendous complexity (there are over ten thousand terms), it will be some time before the identity (6.9) is verified to fourth and higher order.

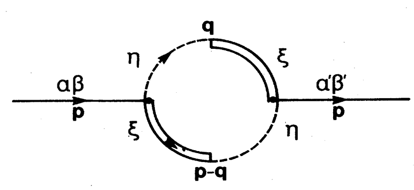
Symmetry and Lorentz covariance demand that the total contribution from the pure graviton loop and the fictitious particle loop be written as ($g$ means graviton)

$$\begin{aligned} T_{\alpha\beta\alpha'\beta'}{}^{(g)}(p) = {}& \kappa^2[p_\alpha p_\beta p_{\alpha'} p_{\beta'} T_1{}^{(g)}(p^2) \\ &+ \delta_{\alpha\beta}\delta_{\alpha'\beta'}T_2{}^{(g)}(p^2) + (\delta_{\alpha\alpha'}\delta_{\beta\beta'} + \delta_{\beta\alpha'}\delta_{\alpha\beta'})T_3{}^{(g)}(p^2) \\ &+ (\delta_{\alpha\beta}p_{\alpha'}p_{\beta'} + \delta_{\alpha'\beta'}p_\alpha p_\beta)T_4{}^{(g)}(p^2) \\ &+ (\delta_{\alpha\alpha'}p_\beta p_{\beta'} + \delta_{\beta\alpha'}p_\alpha p_{\beta'} + \delta_{\alpha\beta'}p_\beta p_{\alpha'} \\ &\quad + \delta_{\beta\beta'}p_\alpha p_{\alpha'})T_5{}^{(g)}(p^2)]. \end{aligned} \tag{6.10}$$

The invariant amplitudes $T_i{}^{(g)}(p^2)$, $i = 1, 2, \ldots, 5$, which are crucial for the discussion of the Slavnov–Taylor identities, possess the following form:

$$T_1{}^{(g)}(p^2) = [8(4\omega^2-1)]^{-1} \times (2\omega^4 - 5\omega^3 + 35\omega^2 + 16\omega)I_1, \tag{6.11a}$$

FIG. 11. Fictitious particle loop in pure quantum gravity. The solid line represents a graviton; $\eta$ and $\xi$ denote fictitious particles.



$$T_2^{(g)}(p^2) = [32(\omega-1)^2(4\omega^2-1)]^{-1} \times(-14\omega^4-7\omega^3+36\omega^2+9\omega)I_1, \tag{6.11b}$$

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

$$T_5^{(g)}(p^2) = [32(4\omega^2-1)]^{-1} \times(-16\omega^3-18\omega^2+15\omega+8)I_1, \tag{6.11e}$$

where

$$I_1 \equiv \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}k^2(k-p)^2} = (4\pi)^{-\omega}\Gamma(2-\omega) \times\int_0^1 d\xi[\xi(1-\xi)p^2+f(\omega)]^{\omega-2}, \tag{6.12}$$

and $f(\omega)$ is the continuity function defined in (4.11).

In order to verify (6.9) we must first construct from (6.10) the *connected* Green's function

$$Q_{\nu\sigma\mu\lambda}^{(g)}(p) = D_{\nu\sigma\alpha\beta}(p^2)T_{\alpha\beta\alpha'\beta'}^{(g)}(p)D_{\alpha'\beta'\mu\lambda}(p^2), \tag{6.13}$$

where $D_{\alpha\beta\mu\nu}(p^2)$ is the free massless spin-two propagator:

$$D_{\alpha\beta\mu\nu}(p^2) = (2p^2)^{-1}(\delta_{\alpha\mu}\delta_{\beta\nu}-\delta_{\alpha\beta}\delta_{\mu\nu}+\delta_{\alpha\nu}\delta_{\beta\mu}). \tag{6.14}$$

Substitution of (6.10) and (6.14) into the right-hand side of (6.13) yields

$$Q_{\nu\sigma\mu\lambda}^{(g)}(p) = (4p^2)^{-2}\kappa^2\{a_{1\nu\sigma\mu\lambda}T_1^{(g)}+(\omega-1)^2a_{2\nu\sigma\mu\lambda}T_2^{(g)} +[a_{3\nu\sigma\mu\lambda}+(\omega-2)a_{2\nu\sigma\mu\lambda}]T_3^{(g)}+(\omega-1)a_{4\nu\sigma\mu\lambda}T_4^{(g)} +a_{5\nu\sigma\mu\lambda}T_5^{(g)}\}, \tag{6.15}$$

where the kinematical coefficients $a_{1\nu\sigma\mu\lambda}, \ldots, a_{5\nu\sigma\mu\lambda}$ are not needed for the discussion here. For the Green's function (6.15), the Slavnov–Taylor identity (6.9) implies that

$$p_\nu p_\mu Q_{\nu\sigma\mu\lambda}^{(g)}(p) = 0, \tag{6.16}$$

or equivalently

$$T_3^{(g)}+p^2T_5^{(g)} = 0 \tag{6.17}$$

$$(p^2)^2T_1^{(g)}+4(\omega-1)^2T_2^{(g)} +4(\omega-1)(T_3^{(g)}-p^2T_4^{(g)}) = 0. \tag{6.18}$$

Equations (6.16)–(6.18) contain a number of remarkable features which we shall briefly summarize. (a) Identities (6.16) and (6.17) are both independent of the dimensionality of the space, whereas Eq. (6.18) holds for all values of $\omega$, except for $\omega = 1$ (we recall from earlier discussions that the Einstein–Hilbert Lagrangian is not defined in two dimensions). (b) It is amazing as well as gratifying to find that the *$\omega$-dependent* amplitudes $T_i^{(g)}$ in (6.11) satisfy the theoretical identities (6.17) and (6.18). In short, dimensional regularization respects the Slavnov–Taylor identity (6.9).

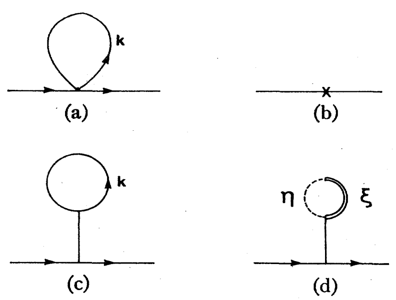



FIG. 12. Lowest-order contributions to the graviton self-energy. Diagrams (a), (c), and (d) are tadpoles with $\eta$, $\xi$ denoting fictitious particles again, while diagram (b) gives rise to $\delta^4(0)$ terms.

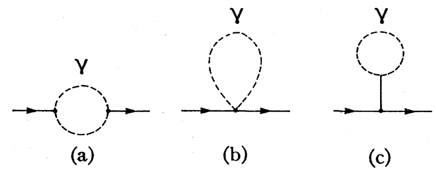



FIG. 13. Lowest-order photon contributions to the graviton self-energy. The solid lines represent gravitons, the dashed lines photons ($\gamma$).

(c) In view of the contributions from fictitious particles—these give rise to longitudinal polarizations—the connected Green's function $Q^{(g)}$ no longer satisfies the naive identity

$$p_\nu Q_{\nu\sigma\mu\lambda}^{(g)}(p) = 0. \tag{6.19}$$

We should point out that the total graviton amplitude (6.10) includes, in addition to Figs. 10 and 11, four other second-order graphs (Fig. 12). The tadpole diagram 12(a) and the $\delta^4(0)$ diagram 12(b) have already been shown to vanish in the context of dimensional regularization [Eqs. (4.17) and (4.23), respectively], whereas diagrams 12(c) and 12(d) contain zero-momentum propagators of mass zero and are consequently harder to evaluate. However, since both 12(c) and 12(d) satisfy the Slavnov–Taylor identities, we shall not examine these graphs any further here.

### 4. Structure of pole term and counter Lagrangian

Expanding the right-hand side of Eq. (6.15) about $\omega = 2$ and continuing factors such as $\ln(p^2)$ analytically from Euclidean to Minkowski space, we can express the total amplitude for the graviton self-energy conveniently as

$$Q_{\nu\sigma\mu\lambda}^{(g)} = \kappa^2(4p^4)^{-1}[Q_{\nu\sigma\mu\lambda}^{(g)\,\mathrm{pole}}+Q_{\nu\sigma\mu\lambda}^{(g)\,\mathrm{finite}}], \tag{6.20}$$

where

$$Q_{\nu\sigma\mu\lambda}^{(g)\mathrm{finite}} = Q_{\nu\sigma\mu\lambda}^{(g)\,\mathrm{Real}}+iQ_{\nu\sigma\mu\lambda}^{(g)\,\mathrm{Im}}. \tag{6.21}$$

The detailed structure of $Q^{(g)\,\mathrm{Real}}$ and $Q^{(g)\,\mathrm{Im}}$ is given in Eqs. (5.18) and (5.19) of Capper, Leibbrandt, and Ramón Medrano (1973),[22] while the pole term reads

$$Q_{\nu\sigma\mu\lambda}^{(g)\,\mathrm{pole}}(p) = [240(4\pi)^2(2-\omega)]^{-1}[328a_{1\nu\sigma\mu\lambda} -59a_{2\nu\sigma\mu\lambda}+81a_{3\nu\sigma\mu\lambda}+104a_{4\nu\sigma\mu\lambda}-81a_{5\nu\sigma\mu\lambda}] +O(\omega-2). \tag{6.22}$$

The pole term may be eliminated by means of appropriate counterterms in the Lagrangian. Moreover, the extraction of $Q^{(g)\,\mathrm{finite}}$ from $Q^{(g)}$ in (6.13) is consistent with gauge invariance, since each one of the expressions $Q^{(g)\,\mathrm{Real}}$, $Q^{(g)\,\mathrm{Im}}$, and $Q^{(g)\,\mathrm{pole}}$ satisfies the Slavnov–Taylor identities (6.17) to (6.18) separately. We postpone a detailed discussion on the possibility of renormalizing quantum gravity until Sec. VI.C.

## B. Corrections to the graviton propagator

### 1. Photon correction

Using dimensional regularization, Capper, Duff, and Halpern (1974) have evaluated the one-loop photon correction to the graviton self-energy (Fig. 13). We examine

[22] There is a correction in that paper: the over-all factor $(120\ p^4)^{-1}$ in (5.18) and (5.19) should be replaced by $(240(4\pi)^4)^{-1}$.

the highlights of their calculation, paying particular attention to the Slavnov–Taylor identities and to the extraction of the finite part of the photon contribution.

The interaction of gravitons with photons is described by the Lagrangian density ($\gamma$ means photon)

$$\mathcal{L}^{(\gamma)} = (2/\kappa^2)\sqrt{-g}\, g^{\mu\nu}R_{\mu\nu} - \tfrac{1}{4}\sqrt{-g}\, g^{\mu\nu}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}, \quad (6.23)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (6.24)$$

while the connected Green's function [Fig. 13(a)] reads

$$Q_{\mu\nu\rho\sigma}{}^{(\gamma)}(p) = D_{\mu\nu\alpha\beta}(p^2)T_{\alpha\beta\tau\lambda}{}^{(\gamma)}(p)D_{\tau\lambda\rho\sigma}(p^2), \quad (6.25)$$

where $T^{(\gamma)}$ is the total photon self-energy contribution and $D_{\mu\nu\alpha\rho}$ is given by (6.14). The integrals associated with the tadpole diagrams 13(b) and 13(c) yield zero again. Invariance of (6.23) under general coordinate transformations leads to the identity

$$p_\mu Q_{\mu\nu\rho\sigma}{}^{(\gamma)}(p) = 0, \quad (6.26)$$

or equivalently, to the three identities

$$T_1{}^{(\gamma)} - 2(\omega - 1)T_4{}^{(\gamma)} = 0, \quad (6.27a)$$

$$T_1{}^{(\gamma)} - 4(\omega - 1)^2T_2{}^{(\gamma)} - 4(\omega - 2)T_3{}^{(\gamma)} + 4T_5{}^{(\gamma)} = 0, \quad (6.27b)$$

$$T_3{}^{(\gamma)} + T_5{}^{(\gamma)} = 0, \quad (6.27c)$$

where the structure of the invariant photon amplitudes $T_i{}^{(\gamma)}$, $i = 1, 2, \ldots, 5$, is not needed here for the discussion. We see, just as in the case of the graviton amplitude, that the identities (6.27a) and (6.27b) involve $\omega$ explicitly. To regularize $Q_{\mu\nu\rho\sigma}{}^{(\gamma)}$, one first expands (6.25) about $\omega = 2$,

$$Q_{\mu\nu\rho\sigma}{}^{(\gamma)} = \frac{1}{2-\omega}Q_{\mu\nu\rho\sigma}{}^{(\gamma)\text{pole}} + Q_{\mu\nu\rho\sigma}{}^{(\gamma)\text{finite}}, \quad (6.28)$$

and then eliminates $Q_{\mu\nu\rho\sigma}{}^{(\gamma)\text{ pole}}$ by means of the *four-dimensional* counterterm

$$\Delta\mathcal{L}^{(\gamma)} = \frac{1}{2-\omega}\left[\frac{\sqrt{-g}}{60(4\pi)^2}(R^2 - 3R_{\mu\nu}R^{\mu\nu})\right]. \quad (6.29)$$

Since the extraction of $Q^{(\gamma)\text{ finite}}$ turns out to be consistent with the Slavnov–Taylor identities (6.27), we conclude that the continuous dimension method preserves the gauge-invariant character of the photon-graviton Lagrangian (6.23). It is also worth noting that the photon function $Q^{(\gamma)}$ is *transverse*, satisfying the naive identity (6.26), in contrast to $Q^{(g)}$, which satisfies the weaker identity (6.16). Consequently we find that the counterterms for $Q^{(\gamma)}$ are covariant, whereas the graviton counterterms are generally noncovariant.

### 2. Neutrino correction

For the one-loop neutrino contribution to the pure graviton propagator [Fig. 14(a)] the connected Green's function

$$Q_{\tau\sigma\mu\lambda}{}^{(\nu)}(p) = D_{\tau\sigma\alpha\beta}(p^2)T_{\alpha\beta\alpha'\beta'}{}^{(\nu)}(p)D_{\alpha'\beta'\mu\lambda}(p^2) \quad (6.30)$$

satisfies two distinct identities (Capper and Duff, 1974b): the Slavnov–Taylor identity ($\nu$ means neutrino)

$$p_\tau Q_{\tau\sigma\mu\lambda}{}^{(\nu)}(p) = 0, \quad (6.31)$$

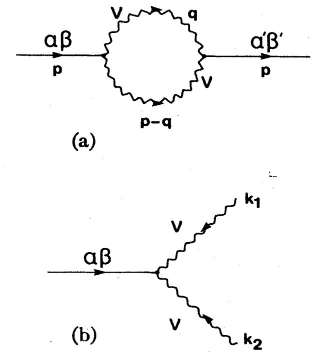



FIG. 14. (a) One loop-neutrino contribution to the graviton self-energy. The wavy lines represent neutrinos ($\nu$). (b) The graviton-neutrino vertex.

which expresses gravitational gauge invariance, and the identity

$$Q_{\sigma\sigma\mu\lambda}{}^{(\nu)}(p) = 0, \quad \text{for all } \omega \quad (6.32)$$

which arises from contracting the indices at one of the vertices in Fig. 14(a).

Since the neutrino amplitude satisfies the identity (6.31), one can remove the ultraviolet-divergent portion of $Q^{(\nu)}$ by augmenting the original graviton-neutrino Lagrangian by the counterterm

$$\Delta\mathcal{L}^{(\nu)} = \frac{1}{2-\omega}\frac{\sqrt{-g}}{120(4\pi)^2}(R^2 - 3R_{\mu\nu}R^{\mu\nu}), \quad (6.33)$$

which is the same, except for a factor of $+\frac{1}{2}$, as the photon term (6.29). The Lagrangian (6.33) meets the requirements of (6.31), but fails to satisfy the second identity (6.32). The reason is that $\Delta\mathcal{L}^{(\nu)}$ is strictly a four-dimensional quantity, whereas the trace identity (6.32) is seen to hold in any dimension. The problem can be resolved by replacing (6.33) by the $2\omega$-dimensional expression

$$\Delta\mathcal{L}^{(\nu)}(\omega) = \frac{1}{2-\omega}\frac{2^\omega\sqrt{-g}}{32(4\pi)^2(4\omega^2-1)} \times\left[\frac{\omega}{2}R^2 - (2\omega - 1)R_{\mu\nu}R^{\mu\nu}\right], \quad (6.34)$$

which is generally covariant as well as conformally invariant for all $\omega$. $\Delta\mathcal{L}^{(\nu)}(\omega)$ satisfies both identities, (6.31) and (6.32).

## C. A second look at quantum gravity

### 1. Trace anomalies

As pointed out in Sec. III, dimensional regularization must be applied with care if the Slavnov–Taylor identities contain the $\epsilon_{\alpha\beta\mu\nu}$ tensor, since the latter is only defined in four-space. There exists another type of anomaly which is connected with the fact that the trace operation depends explicitly on the dimensionality of the space, as seen from the relation $\delta_{\mu\mu} = 2\omega$.

To illustrate this second type of anomaly let us go back to the neutrino Green's function $Q^{(\nu)}$ satisfying the trace identity (6.32). The latter is equivalent to (Capper and

Duff, 1974a)

$$Q_1^{(\nu)} + 2\omega Q_4^{(\nu)} + 4Q_5^{(\nu)} = 0, \quad \text{for all } \omega \tag{6.35a}$$

$$2\omega Q_2^{(\nu)} + 2Q_3^{(\nu)} + Q_4^{(\nu)} = 0, \quad \text{for all } \omega \tag{6.35b}$$

where

$$Q_i^{(\nu)}(\omega,p^2) = q_i^{(\nu)}(\omega)(p^2)^{-2}I(\omega,p^2), \quad i = 1, \ldots, 5 \tag{6.36}$$

with

$$q_1^{(\nu)}(\omega) = (\omega - 1)2^{\omega}[32(4\omega^2 - 1)]^{-1};$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{6.37}$$

$$q_5^{(\nu)}(\omega) = -(2\omega - 1)2^{\omega}[64(4\omega^2 - 1)]^{-1}.$$

$I(\omega,p^2)$ is given by Eq. (6.12). Notice the $\omega$ dependence in (6.35). We find as usual that

$$Q_i^{(\nu)} = (1/(2-\omega))Q_i^{(\nu)\,\text{pole}} + Q_i^{(\nu)\,\text{finite}}, \quad i = 1, \ldots, 5 \tag{6.38}$$

where

$$Q_i^{(\nu)\text{pole}} = \frac{\kappa^2}{(4\pi)^2(p^2)^2} q_i(2) \tag{6.39}$$

and

$$Q_i^{(\nu)\text{finite}} = \frac{\kappa^2}{(4\pi)^2(p^2)^2}\left\{q_i(2)\left[\psi(1) + 2 - \ln\left|\frac{p^2}{4\pi\mu^2}\right|\right] - q_i'(2)\right\} + O(\omega - 2). \tag{6.40}$$

The prime in $q_i'$ denotes differentiation with respect to $\omega$, while the arbitrary parameter $\mu$, with the dimension of mass, arises from a redefinition of the gravitational coupling constant in $2\omega$ dimensions: $\kappa^2 \propto G/(\mu^2)^{\omega-2}$ (Capper, Duff, and Halpern, 1974).

It turns out that the pole term (6.39) respects the trace identity (6.35), but that the finite part (6.40) does *not*. This breakdown of conformal invariance can be explained as follows. Since $Q_i^{(\nu)}$ satisfies (6.35) the $q_i^{(\nu)}$ terms, being proportional to $Q_i^{(\nu)}$, will likewise satisfy (6.35) and so will the pole term $Q_i^{(\nu)\,\text{pole}}$ in (6.39). But what about $Q_i^{(\nu)\text{finite}}$ which involves, according to (6.40), derivatives of $q_i(\omega)$? To explore this situation, we differentiate (6.35) with respect to $\omega$, yielding

$$(Q_1^{(\nu)\prime} + 2\omega Q_4^{(\nu)\prime} + 4Q_5^{(\nu)\prime}) + 2Q_4^{(\nu)} = 0, \tag{6.41a}$$

$$(2\omega Q_2^{(\nu)\prime} + 2Q_3^{(\nu)\prime} + Q_4^{(\nu)\prime}) + 2Q_2^{(\nu)} = 0. \tag{6.41b}$$

Since $q_i^{(\nu)}$ is proportional to $Q_i^{(\nu)}$, it is clear that $q_i^{(\nu)\prime}$ cannot possibly satisfy the *differentiated* Slavnov–Taylor identities (6.41) and neither can $Q_i^{(\nu)\,\text{finite}}$. We conclude that the extraction of the finite part of $Q_{\sigma\sigma\mu\lambda}^{(\nu)}$ by the technique of Sec. II—this would, in particular, include the use of a four-dimensional counterterm—violates the trace identity (6.35). The situation can be rectified in this instance by replacing (6.33) with the $2\omega$-dimensional counterterm (6.34). It remains to be seen whether this treatment of trace anomalies can also be applied in higher orders of perturbation theory.

### 2. Is quantum gravity renormalizable?

We have seen that pure gravity can be regularized to lowest order in the gravitational coupling constant $\kappa$. However, since the counterterms needed to cancel the one-loop divergences are not of the same form as the terms in the original Lagrangian (6.1), it follows that pure gravity is unrenormalizable even at the one-loop level. We also observe that the counterterms associated with (6.1) are called "Slavnov–Taylor invariant" because they satisfy the Slavnov–Taylor identity (6.16), rather than the naive identity

$$p_\nu Q_{\nu\mu\sigma\lambda}(p) = 0.$$

The addition of matter fields to the pure Einstein–Hilbert Lagrangian complicates the analysis even further ('t Hooft and Veltman, 1974). 't Hooft (1973) has shown, for example, that the presence of a scalar field spoils the renormalizability of quantum gravity already at the one-loop level. A similar conclusion has been reached by Deser and his collaborators (Deser and van Nieuwenhuizen, 1974a,b; Deser, Tsao, and van Nieuwenhuizen, 1974). It is also discouraging to realize that the photon and neutrino counterterms in Eqs. (6.29) and (6.34), respectively, possess the same sign so that the divergences of these and other matter loops cannot be made to cancel one another. [This problem is further analyzed in the appendix of Capper and Duff (1974a).]

The renormalization problems discussed above are of course quite divorced from the immediate technical problem of regularizing quantum gravity in a gauge-invariant manner. And, for this task, dimensional regularization remains remarkably well suited.

## VII. HIGHER-ORDER DIAGRAMS

### A. Multiple-loop integrals

#### 1. General remarks

The main purpose of this section is to extend the continuous dimension method to multiple-loop Feynman integrals of the form (Eden *et al.*, 1966; Nakanishi, 1971)

$$F(p) = \int \frac{d^4k_1}{(2\pi)^4}\cdots\int\frac{d^4k_L}{(2\pi)^4} J(k,p)/\prod_{i=1}^{N}(q_i^2 + m_i^2), \tag{7.1}$$

where $L$ is the total number of closed loops, $N$ denotes the number of internal propagators ($N > L$), and $k_1, k_2, \ldots, k_L$ are linearly independent loop momenta. The momentum $q_i$ of the $i$th propagator of mass $m_i > 0$, $i = 1, 2, \ldots, N$, is a linear combination of the loop momenta $k_l$ and of the external momenta $p_j$. There is no need for an $i\epsilon$ term in (7.1), since each one of the integrals is defined over four-dimensional Euclidean space. The functional form of $J(k,p)$ depends on the structure of the particular diagram under consideration and will be left arbitrary for the time being. In $2\omega$-dimensional space, the integral (7.1) reads

$$F(2\omega,p) = \int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\cdots\int\frac{d^{2\omega}k_L}{(2\pi)^{2\omega}} \cdot J(k,p)/\prod_{i=1}^{N}(q_i^2 + m_i^2). \tag{7.2}$$

It would be premature at this stage of the theory to list ground rules for evaluating multiple-loop integrals such as (7.2), since only relatively few higher-order diagrams have been computed so far [see, for example, Jones (1974) and Lee and Sciaccaluga (1974)]. Nevertheless, there do exist several basic rules that can be quoted with a reasonable degree of confidence. One of these rules asserts that the *same* $\omega$ must be employed for each new loop integral, contrary to Ashmore's suggestion advocating a different $\omega_l$ for each $l$th loop. Perhaps the most convincing argument in support of the same $\omega$ comes from gauge invariance. To see this we first recall that the method of analytic regularization[23] (Speer, 1968, 1969) employs a different regulating parameter $\lambda_i$ for each new loop:

$$\int \frac{d^4k_1}{(2\pi)^4}\cdots\int\frac{d^4k_L}{(2\pi)^4}\cdot J(k,p)/\prod_{i=1}^{N}(q_i^2+m_i^2)^{\lambda_i}. \tag{7.3}$$

But analytic regularization is generally *not* gauge-invariant since, according to Veltman (1974), "Gauge invariance implies a tight interplay between the numerator of an integrand and its denominator. Changing either of the two will generally destroy gauge invariance." A similar situation exists in the case of dimensional regularization. If we were to assign in a multiple-loop diagram a different label to each integral, we would introduce by virtue of the special rules (2.8) to (2.10) a new function consisting of different $\omega$'s. As this would lead to the same problem already encountered in the technique of analytic regularization, it is mandatory that all integrals be labelled by the *same* $\omega$.

To facilitate comparison with the published literature, we briefly discuss the extension of multiple-loop diagrams in the 't Hooft–Veltman approach (Sec. II.C.1), then in the spirit of Sec. II.B. For a graph consisting of $L$ loops, the expression

$$\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\cdots\int\frac{d^{2\omega}k_L}{(2\pi)^{2\omega}}f(k,p) \text{ becomes}$$

$$\Rightarrow\int\frac{d^4\hat{k}_1}{(2\pi)^4}\cdots\int\frac{d^4\hat{k}_L}{(2\pi)^4}\cdot\int\frac{d^{2\omega-4}K_1}{(2\pi)^{2\omega-4}}\cdots\int\frac{d^{2\omega-4}K_L}{(2\pi)^{2\omega-4}} \times g(\hat{k}_l;K_l^2;p)\equiv I(2\omega,p), \tag{7.4}$$

$p_j$ being external momenta. According to Bardeen (1972), we can express $I(2\omega,p)$ succinctly as

$$I(2\omega,p)=\left(\prod_{l=1}^{L}\int\frac{d^4\hat{k}_l}{(2\pi)^4}\right)Q_{(2\omega)}g^{(2\omega)}(\hat{k}_i\cdot\hat{k}_j,\hat{k}_i\cdot\hat{p}_j), \tag{7.5}$$

where

$$Q_{(2\omega)}=(\det D)^{2-\omega},\quad D=(d_{ij}) \tag{7.6}$$

and

$$\begin{aligned} d_{ij} &= (2\pi)^{-1}\,\partial_{\hat{k}_i\cdot\hat{k}_j}, && \text{for } i\neq j\\ &=\pi^{-1}\,\partial_{\hat{k}_i^2}, && \text{for } i=j. \end{aligned} \tag{7.7}$$

The second way of attacking multiple-loop diagrams is to employ a specific propagator parametrization such as

$$1/\prod_{i=1}^{N}(q_i^2+m_i^2)=(N-1)!\int_0^1\cdots\int_0^1 d\alpha_1\cdots \times d\alpha_N\delta\left(1-\sum_{i=1}^{N}\alpha_i\right)\left[\sum_{j=1}^{N}\alpha_j(q_j^2+m_j^2)\right]^{-N}, \tag{7.8}$$

which reduces a typical Feynman integral to the form (Eden, 1962)

$$I(2\omega,p)=\int_0^1 d\alpha_1\cdots\int_0^1 d\alpha_N\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\cdots\int\frac{d^{2\omega}k_L}{(2\pi)^{2\omega}} \cdot J(k,p)\delta\left(1-\sum_{i=1}^{N}\alpha_i\right)/[b_1k_1^2+\cdots+b_Lk_L^2+c]^N, \tag{7.9}$$

where $N$ and $L$ denote, respectively, the total number of initial propagators and internal momenta, while $c$ and the coefficients $b_l$, $l=1,2,\ldots,L$, are explained in Eden (1962). $J(k,p)$ depends typically on various masses and scalar products between external and/or internal momenta, none of which alter significantly the analytical structure of the total integral. The $2\omega$-dimensional $k$ integrals in (7.9) may be attacked with the aid of formulas such as (3.8), (3.9), and (3.25).

## 2. Exponential parametrization

It is often more convenient to employ instead of (7.8) the *exponential* parametrization (2.3). For $N$ massive propagators we have

$$1/\prod_{i=1}^{N}(q_i^2+m_i^2)=\left(\prod_{i=1}^{N}\int_0^\infty d\alpha_i\right)\exp\left[-\sum_{i=1}^{N}\alpha_i(q_i^2+m_i^2)\right],$$

$$m_i^2\neq 0,\quad i=1,2,\ldots,N \tag{7.10}$$

so that the generalized Feynman integral (7.2) becomes

$$F(2\omega,p)=\left(\prod_{i=1}^{N}\int_0^\infty d\alpha_i\right)\left(\prod_{l=1}^{L}\int\frac{d^{2\omega}k_l}{(2\pi)^{2\omega}}\right)J(k,p) \times\exp\left[-\sum_{i=1}^{N}\alpha_i(q_i^2+m_i^2)\right], \tag{7.11}$$

each of the internal $k_l$ vectors being defined over a $2\omega$-dimensional space, while the *external* momenta $p_j$ are as usual only defined over the first four dimensions.

In order to gain some insight into the technical problems arising from the reduction of the $k$ integrals in (7.11), let us consider a two-loop Feynman diagram $G$ with $E$ external lines. The associated integral has the typical form [we follow the notation of Abers and Lee (1973)]

$$F_G(2\omega,p)=\left(\prod_{i=1}^{N}\int_0^\infty d\alpha_i\right)\int\frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\frac{d^{2\omega}k_2}{(2\pi)^{2\omega}}\left(\prod_{m,n}k_m\cdot k_n\right) \times\left(\prod_{s,r}k_s\cdot p_r\right)\exp\left[-\sum_{i=1}^{N}\alpha_i(q_i^2+m_i^2)\right], \tag{7.12}$$

where each of the subscripts $m$, $n$, and $s$ ranges over 1, 2; momentum conservation implies that there are only $(E-1)$ linearly independent external momentum vectors $p_r$, $r=1,2,\ldots,E-1$. The first step in the reduction of the $k$ integrals in (7.12) is to recall that the momentum $q_i$ of the $i$th propagator is in general a linear combination of the loop momenta $k_1$, $k_2$ and of the external momenta $p_r$, so that the argument of the exponential function in (7.12) can

[23] See the discussion in Sec. I.A.2.

be expressed as (Eden *et al.*, 1966)

$$\sum_{i=1}^{N} \alpha_i(q_i^2 + m_i^2) = k^T\cdot A(\alpha)k + k^T\cdot B(\alpha)p + p^T\cdot C(\alpha)p + \sum_{i=1}^{N} \alpha_i m_i^2. \tag{7.13}$$

The order of the matrices $A(\alpha)$, $B(\alpha)$, and $C(\alpha)$ is $2\times 2$, $2\times(E-1)$, and $(E-1)\times(E-1)$, respectively; $k$ and $p$ are column matrices,

$$k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \qquad p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{E-1} \end{bmatrix}, \tag{7.14}$$

and their transposes $k^T$ and $p^T$ row vectors. Substitution of (7.13) into (7.12) yields forthwith

$$F_G(2\omega,p) = \left(\prod_{i=1}^{N}\int_0^\infty d\alpha_i\right)\int \frac{d^{2\omega}k_1}{(2\pi)^{2\omega}}\frac{d^{2\omega}k_2}{(2\pi)^{2\omega}}(\prod_{m,n} k_m\cdot k_n) \times(\prod_{s,r} k_s\cdot p_r)\exp[-k^T\cdot A(\alpha)k - k^T\cdot B(\alpha)p - p^T\cdot C(\alpha)p - \sum_{i=1}^{N}\alpha_i m_i^2]. \tag{7.15}$$

Rather than integrate over $d^{2\omega}k_1$ and $d^{2\omega}k_2$ directly, as was done in earlier sections of this review, we follow the approach of 't Hooft and Veltman [cf. Eqs. (2.11) and (7.4)] who separate the $2\omega$-dimensional momentum space over $k$ into a physical four-dimensional space, characterized by the four-vector $\hat{k}$, plus a $(2\omega-4)$-dimensional space over the vector $K$. Thus

$$k = (\hat{k},K) \tag{7.16a}$$

and, for the external momenta,

$$p = (\hat{p},0). \tag{7.16b}$$

Since the $K$ space is orthogonal to the physical $\hat{k}$ space, the first two terms on the right-hand side of (7.13) reduce to

$$k^T\cdot A(\alpha)k + k^T\cdot B(\alpha)p = \hat{k}^T\cdot A(\alpha)\hat{k} + \hat{k}^T\cdot B(\alpha)\hat{p} + K^T\cdot A(\alpha)K; \tag{7.17}$$

in addition,

$$d^{2\omega}k_i = d^4\hat{k}_i\, d^{2\omega-4}K_i, \quad i = 1, 2 \tag{7.18a}$$

$$k\cdot p = \hat{k}\cdot\hat{p}, \tag{7.18b}$$

and

$$k\cdot k = \hat{k}^2 + K^2. \tag{7.18c}$$

The two-loop integral (7.15) is therefore composed of a *sum* of terms of the type [cf. Eq. (7.18c)]

$$F_G(2\omega,p) = \left(\prod_{i=1}^{N}\int_0^\infty d\alpha_i\right)\left(\int\frac{d^4\hat{k}_1}{(2\pi)^4}\frac{d^4\hat{k}_2}{(2\pi)^4}\right)(\prod_{m,n}\hat{k}_m\cdot\hat{k}_n) \times(\prod_{s,r}\hat{k}_s\cdot\hat{p}_r)\exp[-\hat{k}^T\cdot A(\alpha)\hat{k} - \hat{k}^T\cdot B(\alpha)\hat{p} - \hat{p}^T\cdot C(\alpha)\hat{p} - \sum_{i=1}^{N}\alpha_i m_i^2]\int\frac{d^{2\omega-4}K_1}{(2\pi)^{2\omega-4}}\frac{d^{2\omega-4}K_2}{(2\pi)^{2\omega-4}}(\prod_{m,n}K_m\cdot K_n) \times\exp[-K^T\cdot A(\alpha)K]. \tag{7.19}$$

In order to perform the $K$ integrations, we follow the standard procedure of first diagonalizing the $2\times 2$ matrix $A(\alpha)$, labelling its eigenvalues by $a_j(\alpha)$, $j = 1, 2$, and then applying the following generalized Gaussian integrals $(a > 0)$:

$$\int\frac{d^{2\omega-4}K}{(2\pi)^{2\omega-4}}\exp[-aK^2] = \frac{1}{(2\sqrt{\pi a})^{2\omega-4}}, \tag{7.20a}$$

$$\int\frac{d^{2\omega-4}K}{(2\pi)^{2\omega-4}}K_\mu\exp[-aK^2] = 0, \tag{7.20b}$$

$$\int\frac{d^{2\omega-4}K}{(2\pi)^{2\omega-4}}K_\mu K_\nu\exp[-aK^2] = \frac{\delta_{\mu\nu}}{2a(2\sqrt{\pi a})^{2\omega-4}}, \tag{7.20c}$$

$$\int\frac{d^{2\omega-4}K}{(2\pi)^{2\omega-4}}K^2\exp[-aK^2] = \frac{\omega-2}{a(2\sqrt{\pi a})^{2\omega-4}}. \tag{7.20d}$$

Other formulas follow from the basic expression (7.20a) by partial differentiation with respect to $a(\alpha)$. Reduction of the $K$ integrals also brings into play certain important identities, some of which contain explicitly the dimension $\omega$ of the space; for example,

$$\delta_{\mu\nu}\delta_{\mu\nu} = 2\omega-4, \quad \delta_{\mu\mu} = 2\omega-4, \tag{7.21}$$

$$\delta_{\mu\nu}K_\mu = K_\nu \tag{7.22}$$

and, if the structure of the integrand demands it, identities such as

$$\mathrm{Trace}(\gamma_\mu\gamma_\nu) = 2^{(\omega-2)}\delta_{\mu\nu}. \tag{7.23}$$

We summarily denote all $\omega$-dependent terms arising from such identities by the polynomial factor $g(\omega)$. We further deduce from Eqs. (7.20) that the eigenvalues $a_j$ behave like

$$[a_j(\alpha)]^{-(\omega-2+d_j)}, \quad j = 1, 2$$

where $d_j$ depends strictly on the number of $K_{j\mu_1}$, $K_{j\mu_2}$, $K_{j\mu_3}$, ... vectors in the integrand; $d_j$ can be either an integer or zero, $d_j \geq 0$. The integral (7.19) therefore assumes the form[24]

$$F_G(2\omega,p) = \left(\prod_{i=1}^{N}\int_0^\infty d\alpha_i\right)\frac{g(\omega)}{[a_1(\alpha)]^{\omega-2+d_1}[a_2(\alpha)]^{\omega-2+d_2}} \cdot\int\frac{d^4\hat{k}_1}{(2\pi)^4}\frac{d^4\hat{k}_2}{(2\pi)^4}(\prod_{m,n}\hat{k}_m\cdot\hat{k}_n)(\prod_{s,r}\hat{k}_s\cdot\hat{p}_r) \cdot\exp[-\hat{k}^T\cdot A(\alpha)\hat{k} - \hat{k}^T\cdot B(\alpha)\hat{p} - \hat{p}^T\cdot C(\alpha)\hat{p} - \sum_{i=1}^{N}\alpha_i m_i^2]. \tag{7.24}$$

We do not reduce the right-hand side of (7.24) any further. Suffice it to say that the two $\hat{k}$ integrals are convergent for complex values of $\omega$ and can therefore be evaluated by the usual techniques. There remain $N$ integrals over $\alpha_1, \alpha_2, \ldots, \alpha_N$. As might be expected, the computational aspects of these parameter integrals can be rather forbidding, just as in the case of ordinary four-dimensional integrals. Regarding the singularity structure of (7.24) we note that two-loop diagrams in general give rise to poles of

[24] Factors of $2\pi$, originating from the $K$ integrations, are omitted.

order up to two at the "physical" value $2\omega = 4$. The analysis of these integrals may be further complicated by the presence of overlapping divergences. For a discussion of these matters we refer the reader to 't Hooft and Veltman (1972a, 1973).

It is clear how the two-loop formula (7.24) can be extended to describe diagrams with $L$ loops. The corresponding integral reads

$$F(2\omega,p) = \left(\prod_{i=1}^{N}\int_0^\infty d\alpha_i\right) g(\omega)/\prod_{j=1}^{L}[a_j(\alpha)]^{\omega-2+d_j}$$
$$\times\left(\prod_{l=1}^{L}\int\frac{d^4\hat{k}_l}{(2\pi)^4}\right)\left(\prod_{m,n}\hat{k}_m\cdot\hat{k}_n\right)\left(\prod_{s,r}\hat{k}_s\cdot\hat{p}_r\right)$$
$$\times\exp\Big[-\hat{k}^T\cdot A(\alpha)\hat{k} - \hat{k}^T\cdot B(\alpha)\hat{p} - \hat{p}^T\cdot C(\alpha)\hat{p} - \sum_{i=1}^{N}\alpha_i m_i^2\Big], \qquad (7.25)$$

where the elements of the column matrices

$$\hat{k} = \begin{bmatrix}\hat{k}_1\\ \hat{k}_2\\ \vdots\\ \hat{k}_L\end{bmatrix},\quad \hat{p} = \begin{bmatrix}\hat{p}_1\\ \hat{p}_2\\ \vdots\\ \hat{p}_{E-1}\end{bmatrix} \qquad (7.26)$$

are still Lorentz four-vectors, but where $A(\alpha)$, $B(\alpha)$, and $C(\alpha)$ are now, respectively, $L\times L$, $L\times(E-1)$, and $(E-1)\times(E-1)$ matrices.

Let us summarize the main features of this section. We have shown, by using the exponential parametrization for momentum space propagators, how the technique of dimensional regularization can readily be extended from one-loop graphs to multiple-loop diagrams. The exponential parametrization is particularly convenient to apply here, since it leads to generalized Gaussian integrals that are easy to evaluate. We have also seen [recall formulas such as (7.21)–(7.23) and (2.7)–(2.10)] that the generalization to and subsequent computation of multiple-loop diagrams are both gauge-invariant procedures. This observation leads to the important conclusion that the continuous dimension method respects gauge invariance to all orders of perturbation theory, a property which is essential in proving the renormalizability of gauge theories in general. This completes our discussion of multiple-loop diagrams.

### B. Vertex diagram

The basic integral associated with the third-order vertex diagram Fig. 15,

$$I(m) = \int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\{(k^2+m_1^2)[(k-p_2)^2+m_2^2]$$
$$\times[(k+p_3)^2+m_3^2]\}^{-1}, \qquad (7.27)$$

is most easily evaluated by using the exponential parametrization (7.10), which leads to

$$I(m) = (4\pi)^{-\omega}\int_0^\infty d\alpha\, d\beta\, d\gamma\,(\alpha+\beta+\gamma)^{-\omega}$$
$$\times\exp[-(\alpha\beta\bar{p}_3^2+\alpha\gamma\bar{p}_2^2+\gamma\beta\bar{p}_1^2+\alpha^2m_1^2$$
$$+\beta^2m_3^2+\gamma^2m_2^2)/(\alpha+\beta+\gamma)], \qquad (7.28)$$

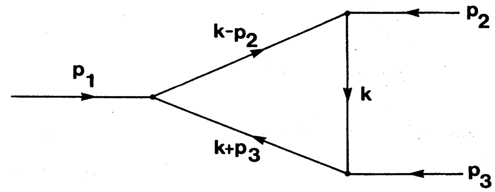



FIG. 15. Pure graviton triangle diagram.

where

$$\bar{p}_1^2 = p_1^2+m_2^2+m_3^2,\quad p_1^2 = (p_2+p_3)^2, \qquad (7.29a)$$
$$\bar{p}_2^2 = p_2^2+m_1^2+m_2^2,\quad \bar{p}_3^2 = p_3^2+m_1^2+m_3^2. \qquad (7.29b)$$

If we define new variables $\xi$, $\tau$, $\lambda$ by $\alpha = \xi\lambda$, $\beta = \tau\lambda$, $\gamma = \lambda(1-\xi-\tau)$ with Jacobian $|J| = \lambda^2$, the integral (7.28) reads

$$I(m) = (4\pi)^{-\omega}\int_0^1 d\xi\int_0^{1-\xi}d\tau\int_0^\infty d\lambda\,\lambda^{2-\omega}$$
$$\times\exp\{-\lambda[A(p)+B(m)]\}, \qquad (7.30)$$

where the structure of $A(p)$ and $B(m)$ need not be known explicitly here. The computation of the massive integral $I(m)$ is straightforward though tedious and will not be discussed any further. Instead we shall briefly examine the corresponding massless integral which, on account of the infrared problem, is substantially trickier to handle than (7.30). Setting $m_1 = m_2 = m_3 = 0$ in Eq. (7.27) and employing the modified Gaussian formula (4.9), we obtain in place of (7.30)

$$I(0) = (4\pi)^{-\omega}\int_0^1 d\xi\int_0^{1-\xi}d\tau\int_0^\infty d\lambda\,\lambda^{2-\omega}$$
$$\times\exp\{-\lambda[A(p)+f(\omega)]\}, \qquad (7.31)$$

where

$$A(p) \equiv \tau\xi p_3^2+\xi(1-\xi-\tau)p_2^2$$
$$+\tau(1-\xi-\tau)p_1^2 \qquad (7.32)$$

is the same as in $I(m)$. Hence the only *formal* difference between $I(m)$ and $I(0)$ lies in the continuity function $f(\omega)$ which replaces the massive term $B(m)$ in Eq. (7.30). Integration over $\lambda$ and $\tau$ yields

$$I(0) = \frac{(p_1^2)^{\omega-3}\Gamma(3-\omega)}{2(\omega-2)(4\pi)^\omega}\int_0^1 d\xi$$
$$\times R^{-1}[z_1^{\omega-2}{}_2F_1(\omega-2,\tfrac{1}{2};\omega-1;z_1/R^2)$$
$$-z_0^{\omega-2}{}_2F_1(\omega-2,\tfrac{1}{2};\omega-1;z_0/R^2)], \qquad (7.33)$$

where

$$R = (c_0+c_1\xi+c_2\xi^2)^{\frac{1}{2}},\quad c_0 = \tfrac{1}{4}(1+4fp_1^{-2}),$$
$$c_1 = -(p_2\cdot p_3)p_1^{-2},\quad c_2 = [(p_2\cdot p_3)^2-p_2^2p_3^2]p_1^{-4}, \qquad (7.34)$$
$$z_0 = [f+\xi(1-\xi)p_2^2]p_1^{-2},\quad z_1 = [f+\xi(1-\xi)p_3^2]p_1^{-2}.$$

As long as $\omega$ remains complex, the right-hand side of (7.33) is a well defined function of $\omega$. In particular, $f(\omega)$ guarantees that the $\xi$ integrand in (7.33) contains neither *real* poles nor end point singularities. Since the finiteness of $I(0)$ as $\omega\to 2^+$ is also readily established (Capper and Leibbrandt, 1974), we conclude once again that the continuous dimension method yields the same result as other more conventional regularization procedures.

## VIII. CONCLUDING REMARKS

In this review we have applied the technique of dimensional regularization to divergent Feynman integrals in the context of Abelian as well as non-Abelian gauge theories. The technique has, in our opinion, three distinct advantages. In the first place, it is simpler and more elegant than other, more conventional regularization methods such as the Pauli–Villars prescription. Secondly, dimensional regularization is powerful enough to handle efficiently and on the same footing both ultraviolet and genuine infrared divergences. Finally, and this is its most significant feature, the technique is eminently well suited for dealing with gauge theories, since it preserves the local gauge symmetry of the underlying Lagrangian. The preservation of this symmetry was demonstrated in Sec. III for the vacuum polarization tensor in quantum electrodynamics, and in Sec. VI by regularizing non-Abelian massless spin-two quantum gravity to lowest order in the gravitational coupling constant. It was specifically shown there that the sum of the graviton and fictitious-particle contributions to the graviton propagator satisfies Slavnov–Taylor identities [Eqs. (6.17) and (6.18)] and that the finite portion of this sum can be extracted in a manner which is consistent with these identities. In summary, dimensional regularization permits a consistent gauge-invariant treatment of divergent Feynman amplitudes to all orders in perturbation theory.

Although the concept of dimensional regularization is easy to grasp once the notion of analytic continuation is clearly understood, the method should not be applied indiscriminately to any model possessing gauge symmetry. Before embarking on an explicit calculation, it is best to ascertain first whether or not the underlying theory—be it Abelian or non-Abelian—(i) is massive, (ii) is massless, or (iii) contains, through Slavnov–Taylor identities or otherwise, factors of $\gamma^5$. Let us briefly examine these three possibilities.

For massive theories the prescription of 't Hooft and Veltman (1972a), Bollini and Giambiagi (1972), and Ashmore (1972, 1973) works remarkably well, as demonstrated in Secs. III.A and V, and ambiguities can easily be avoided. For massless theories the prescription given in Sec. II.B.1 requires modification (Leibbrandt and Capper, 1974a) due to the appearance of infrared divergences connected specifically with massless particles (see Sec. IV.B). The modification involves basically a redefinition of the original $2\omega$-dimensional Gaussian integral which permits a consistent treatment of massless tadpoles as well as $\delta^4(0)$ terms (Sec. IV.C) and preserves, moreover, the crucial Slavnov–Taylor identities associated with the graviton self-energy loop (Sec. VI).

Finally, extreme care must be exercised whenever the theory contains anomalies characterized by $\epsilon_{\alpha\beta\mu\nu}$ or $\gamma^5$. Since the latter has only been generalized successfully to even-dimensional spaces (Sec. III.C), the continuous dimension method could create ambiguity problems if these anomalies persist in the final Slavnov–Taylor identities. Although the anomalies can sometimes be made to cancel by a judicious redefinition of the fundamental fermion fields, it is nevertheless desirable to continue the search for a $\gamma^5$ matrix valid in arbitrary dimensions, so that the technique of dimensional regularization may be applied unambiguously to an even greater variety of physical models than has hitherto been the case.

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## APPENDIX. USEFUL INTEGRATION FORMULAS

The following list of $2\omega$-dimensional integrals is divided into two categories: the integrals in the first category hold for massive particles ($m \neq 0$), whereas those in category B are valid for integrals associated with massless fields ($m = 0$).

### A. Massive integrals

Formulas (A1)–(A6) below are taken from Appendix A of 't Hooft and Veltman (1972a). In transferring them we have, for the sake of consistency, replaced the complex variable $n$ by $2\omega$ and divided each integral by $(2\pi)^{2\omega}$. These integrals hold for $m^2 \neq 0$, $\omega$ arbitrary, and are particularly useful in connection with the discussion in Secs. III and V.

$$\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}(q^2+2k\cdot q+m^2)^\alpha} = \frac{i}{(4\pi)^\omega(m^2-k^2)^{\alpha-\omega}} \cdot \frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)}, \tag{A1}$$

$$\int \frac{d^{2\omega}q\, q_\mu}{(2\pi)^{2\omega}(q^2+2k\cdot q+m^2)^\alpha} = \frac{i}{(4\pi)^\omega(m^2-k^2)^{\alpha-\omega}} \cdot \frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)}(-k_\mu), \tag{A2}$$

$$\int \frac{d^{2\omega}q\, q^2}{(2\pi)^{2\omega}(q^2+2k\cdot q+m^2)^\alpha} = \frac{i}{(4\pi)^\omega(m^2-k^2)^{\alpha-\omega}} \cdot \frac{1}{\Gamma(\alpha)} \times\{\Gamma(\alpha-\omega)k^2+\Gamma(\alpha-1-\omega)\omega(m^2-k^2)\}, \tag{A3}$$

$$\int \frac{d^{2\omega}q\, q_\mu q_\nu}{(2\pi)^{2\omega}(q^2+2k\cdot q+m^2)^\alpha} = \frac{i}{(4\pi)^\omega(m^2-k^2)^{\alpha-\omega}} \cdot \frac{1}{\Gamma(\alpha)} \times\{\Gamma(\alpha-\omega)k_\mu k_\nu+\Gamma(\alpha-1-\omega)\tfrac{1}{2}\delta_{\mu\nu}(m^2-k^2)\}, \tag{A4}$$

$$\int \frac{d^{2\omega}q\, q_\mu q_\nu q_\lambda}{(2\pi)^{2\omega}(q^2+2k\cdot q+m^2)^\alpha} = \frac{i}{(4\pi)^\omega(m^2-k^2)^{\alpha-\omega}}\cdot\frac{1}{\Gamma(\alpha)} \times\{-\Gamma(\alpha-\omega)k_\mu k_\nu k_\lambda - \Gamma(\alpha-1-\omega) \times\tfrac{1}{2}(\delta_{\mu\nu}k_\lambda+\delta_{\mu\lambda}k_\nu+\delta_{\nu\lambda}k_\mu)(m^2-k^2)\}, \quad \text{(A5)}$$

$$\int \frac{d^{2\omega}q\, q^2 q_\mu}{(2\pi)^{2\omega}(q^2+2k\cdot q+m^2)^\alpha} = \frac{i}{(4\pi)^\omega(m^2-k^2)^{\alpha-\omega}}\cdot\frac{1}{\Gamma(\alpha)} \times(-k_\mu)\{\Gamma(\alpha-\omega)k^2+\Gamma(\alpha-1-\omega)(\omega+1) \times(m^2-k^2)\}. \quad \text{(A6)}$$

## B. Massless integrals

The following massless integrals (Capper, Leibbrandt, and Ramón Medrano, 1973) arise in the treatment of quantum gravity in Secs. VI and VII, for which a gauge-invariant cutoff procedure is absolutely essential. There are no factors of $i$ present, since all integrals are defined over Euclidean space.

$$\int \frac{d^{2\omega}q}{(2\pi)^{2\omega}q^2(q-p)^2} = \frac{1}{(4\pi)^\omega\Gamma(2\omega-2)} \times\Gamma(2-\omega)\Gamma(\omega-1)\Gamma(\omega-1)(p^2)^{\omega-2} \equiv I_1, \quad \text{(B1)}$$

$$\int \frac{d^{2\omega}q\, q_\mu}{(2\pi)^{2\omega}q^2(q-p)^2} = p_\mu I_2, \quad \text{(B2)}$$

$$\int \frac{d^{2\omega}q\, q_\mu q_\nu}{(2\pi)^{2\omega}q^2(q-p)^2} = \delta_{\mu\nu}I_3 + p_\mu p_\nu I_4, \quad \text{(B3)}$$

$$\int \frac{d^{2\omega}q\, q_\mu q_\nu q_\gamma}{(2\pi)^{2\omega}q^2(q-p)^2} = p_\mu p_\nu p_\gamma I_5 + E_{\mu\nu\gamma}I_6, \quad \text{(B4)}$$

$$\int \frac{d^{2\omega}q\, q_\mu q_\nu q_\gamma q_\sigma}{(2\pi)^{2\omega}q^2(q-p)^2} = p_\mu p_\nu p_\gamma p_\sigma I_7 + G_{\mu\nu\gamma\sigma}I_8 + H_{\mu\nu\gamma\sigma}I_9, \quad \text{(B5)}$$

where

$$E_{\mu\nu\gamma} \equiv \delta_{\mu\nu}p_\gamma + \delta_{\nu\gamma}p_\mu + \delta_{\gamma\mu}p_\nu, \quad \text{(B6)}$$

$$G_{\mu\nu\gamma\sigma} \equiv \delta_{\mu\nu}p_\gamma p_\sigma + \delta_{\nu\gamma}p_\mu p_\sigma + \delta_{\nu\sigma}p_\mu p_\gamma + \delta_{\mu\gamma}p_\nu p_\sigma + \delta_{\mu\sigma}p_\nu p_\gamma + \delta_{\gamma\sigma}p_\mu p_\nu, \quad \text{(B7)}$$

$$H_{\mu\nu\gamma\sigma} \equiv \delta_{\mu\nu}\delta_{\gamma\sigma} + \delta_{\mu\sigma}\delta_{\nu\gamma} + \delta_{\nu\sigma}\delta_{\mu\gamma}. \quad \text{(B8)}$$

The integrals $I_2, \ldots, I_9$ have the following simple structure in terms of the basic integral $I_1$ in Eq. (B1):

$$I_2 = \tfrac{1}{2}I_1, \quad \text{(B9)}$$

$$I_3 = \frac{-p^2}{4(2\omega-1)}I_1, \quad \text{(B10)}$$

$$I_4 = \frac{\omega}{2(2\omega-1)}I_1, \quad \text{(B11)}$$

$$I_5 = \frac{(\omega+1)}{4(2\omega-1)}I_1, \quad \text{(B12)}$$

$$I_6 = \frac{-p^2}{8(2\omega-1)}I_1, \quad \text{(B13)}$$

$$I_7 = \frac{(\omega+1)(\omega+2)}{4(4\omega^2-1)}I_1, \quad \text{(B14)}$$

$$I_8 = \frac{-(\omega+1)p^2}{8(4\omega^2-1)}I_1, \quad \text{(B15)}$$

$$I_9 = \frac{(p^2)^2}{16(4\omega^2-1)}I_1. \quad \text{(B16)}$$

Formulas (B9)–(B16) must of course be understood in the spirit of Sec. IV, where it was demonstrated that the analytic continuation of these integrals hinges decisively on the application of Eq. (4.9) rather than on Eq. (2.4). A detailed analysis of this mathematical problem can be found in Leibbrandt and Capper (1974a,b).

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