Graded Lie algebras in mathematics and physics (Bose–Fermi symmetry)

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Graded Lie algebras have recently become a topic of interest in physics in the context of "supersymmetries," relating particles of differing statistics. In mathematics, graded Lie algebras have been known for some time in the context of deformation theory. In this paper we discuss basic properties of graded Lie algebras and present various new constructs for producing examples of such algebras. In addition we present a short survey of the role played by graded Lie algebras in mathematics and review in some detail the recent applications of supersymmetry in the physics of particles and fields.

CONTENTS

INTRODUCTION

The application of invariance considerations to relativistic quantum field theory has recently undergone further advances with the introduction of so-called "supergauges."[1] J. Wess and B. Zumino (1974a), following a similar algorithm postulated in dual models (Neveu and Schwarz, 1971; Ramond, 1971; Aharonov, Casher, and Susskind, 1971; Gervais and Sakita, 1971), and simultaneously and independently Volkov and Akulov (1973), introduced transformations relating states of different quantum statistics types. Indeed, since 1965 there had been attempts (Bella, 1973; Joseph, 1972; see references in these articles for previous work) to describe the composition of the spectrum of hadron states through a symmetry whose supermultiplets would combine bosons and fermions. However, the symmetry generator connecting a boson state to a fermion state (an "odd" generator in our treatment) is itself a fermion, and its local density can thus only involve an odd number of fermion fields. The fields–canonical momenta relations (in this case anticommutators) therefore do not provide the necessary information required for Lie algebra commutators between two odd generators. There was thus no way of pursuing this approach while using Lie algebras, and this effort was discontinued.

The problem has now been resolved (Volkov and Akulov, 1973; Wess and Zumion, 1974a) through the introduction of a different algebraic construct. As we shall show in this article, the dual model "supergauge" and the new supersymmetry in Minkowski space all involve graded Lie algebras (GLA)'s, in which an anticommutator appears as the relevant Lie product between two odd generators.

Graded Lie algebras have appeared in the mathematical literature in another context, namely in deformation theory.

The first basic example of graded Lie algebras was provided by Nijenhuis (1955) and then by Frölicher and Nijenhuis (1957) in their paper. The basic role that this object plays in the theory of deformation of algebraic structures was discovered by Gerstenhaber (1963, 1964) in a fundamental series of papers, while Spencer and collaborators were developing applications to pseudogroup struc-

* NSF Grant GP 30673 and Sloan Foundation Fellow.

† Partially supported by the United States–Israel Binational Science Foundation.

‡ Partially supported by the National Science Foundation and the John Simon Guggenheim Foundation.

[1] More recently, the term "supersymmetry" has been adopted for the space–time case with no local dependence in the parameters.

tures on manifolds (Kodaira and Spencer, 1958, 1959, 1960, 1961; Spencer, 1962; Kodaira, Nirenberg, and Spencer, 1962). For two alternative views of this subject see Guillemin and Sternberg (1966) and Kumpera and Spencer (1973). The general role of GLA in deformation theory was presented by Nijenhuis and Richardson (1964).

From a rather different point of view (motivated primarily by problems of second quantization), the subject was introduced by Berezin and Kac (1972), who discuss[2] the analogs of the classical theorems relating Lie algebras to Lie groups.

In this paper we discuss some of the basic properties of graded Lie algebras and construct various examples. The plan of the paper is as follows: In Sec. I we give the basic definitions. In Sec. II, which is the heart of the paper, we develop several different types of examples. Particularly noteworthy is a construction which associates a graded Lie algebra structure to a filtered associative algebra satisfying certain conditions. Applied to the ring of differential operators on a manifold this construction yields the usual algebra of Poisson brackets. But it applies equally well to something like the Clifford algebra and is hence (via tensor product constructions) likely to be related to the role of anticommutators in quantum field theory. Also noteworthy is the construction (via spinors) of an algebra intimately connected with conformal algebra on Minkowski space. This is the GLA of supersymmetry, introduced by Volkov and Akulov and by Wess and Zumino.

In Sec. III we discuss some issues connected with the notion of simplicity of a graded Lie algebra. In Sec. IV we prove the analog of the Poincaré–Birkhoff–Witt theorem for graded Lie algebras. This asserts that the associated graded algebra of the universal enveloping algebra of a graded Lie algebra, U, is isomorphic to the tensor product of the symmetric algebra in the even elements and the exterior algebra in the odd elements. In Sec. V we give a quick sketch of the use of graded Lie algebras in deformation theory. In Sec. VI we discuss the methods recently applied by physicists in the construction of representations of supersymmetry. In Sec. VII we review the physical applications of supersymmetry and discuss the various model theories to which that GLA has been applied. We discuss the results in terms of improved renormalizability, interrelated fields, masses, interactions and coupling strengths. We also study the possible ways of explicit or spontaneous supersymmetry breakdown. The Appendix provides physicists with the essential definitions and results of Clifford algebras and exterior (Grassman) algebras.

I. DEFINITIONS AND ELEMENTARY PROPERTIES

Let $L = \oplus_c L_k$ be a graded vector space; in other words L is a vector space, and the most general element of L can be written uniquely as a finite sum of its components, each component lying in one of the vector spaces L_k.[3] The index k can range over some given Abelian group as indexing group; however we shall principally be interested in cases where the indexing set is either the group of integers, $\mathbf{Z}$, or the two element group, $\mathbf{Z}_2$. In what follows we shall adopt the convention that x will denote an element of L_k, that y will denote an element of L_l, and that z will denote an element of L_m. We say that L is a graded Lie algebra if we are given a bilinear map, denoted by $[\ ,\ ]$, of $L \times L \to L$ such that the following three conditions hold:

$$[L_k, L_l] \subset L_{k+l} \tag{1.1}$$

$$[x,y] = -(-1)^{kl}[y,x] \tag{1.2}$$

and

$$[x,[y,z]] = [[x,y],z] + (-1)^{kl}[y,[x,z]]. \tag{1.3}$$

Here the meaning of the factor $(-1)^n$ is clear for either $\mathbf{Z}$ or $\mathbf{Z}_2$ as indexing group. For more general indexing group we take it to denote some character of the group with values in the group $\{\pm 1\}$ which must be given as an additional piece of the structure. Condition (1.1) simply says that the bracket multiplication is consistent with the grading. Condition (1.2) is the graded version of anticommutativity. Notice that for odd elements it says that "multiplication" is commutative, i.e., the bracket then represents an anticommutator. These are thus our "supergauge" generators. We see that the even elements are Lie algebra generators (i.e., physically, they connect states of similar statistics).

Condition (1.3) is the graded version of Jacobi's identity. For even x it asserts that left multiplication by x is a derivation of the bracket multiplication, while for odd x it asserts that left multiplication by x is an antiderivation.

We shall call an operator D: $L \to L$ such that D: $L_n \to L_{n+k}$ and

$$D(yz) = (Dy)z + (-1)^{kl} y Dz \tag{1.4}$$

a (graded) derivation of degree k. This notion makes sense for any graded algebra A, that is for any graded vector space $A = A_n$ with a bilinear map $A \times A \to A$ such that $A_n \times A_m \to A_{m+n}$. (No associativity or commutativity condition on A is assumed.)

Again notice that L_0 is a Lie algebra in the old fashioned sense and so is the direct sum of all the even L_k's.

We can construct graded Lie algebras as follows: Let $V = \oplus V_n$ be a graded vector space. (For instance, to illustrate a physical application, we might let V be the relevant piece of the Hilbert space of states, graded according to fermion number.) We let $\mathrm{End}_k(V)$ consist of those linear maps, x, of V into itself such that $xV_n \subset V_{n+k}$. [Thus an $x \in \mathrm{End}_k(V)$ is given by a whole string of linear maps, one from each V_n to V_{n+k}; in general $\mathrm{End}_k(V)$ will thus be infinite dimensional even though the individual V_n's, and hence $\mathrm{Hom}(V_n, V_{n+k})$, are finite dimensional.) It is clear that if $x \in \mathrm{End}_k(V)$ and $y \in \mathrm{End}_l(V)$ then the composition xy lies in $\mathrm{End}_{k+l}(V)$. We define a graded Lie algebra structure on $\mathrm{End}(V) = \oplus \mathrm{End}_k(V)$ by setting

$$[x,y] = x \cdot y - (-1)^{kl} y \cdot x. \tag{1.5}$$

It is obvious that conditions (1.1) and (1.2) are satisfied and a straightforward verification shows that condition (1.3) is

[2] Formula 2.3 of Berezin and Kac (1970) defines a particular GLA, somewhat similar to the GLA introduced in #2 of Nijenhuis and Richardson (1964).

[3] Here the vector spaces are over any field of characteristic different from two and the L_k's will usually be finite dimensional. In much of what follows the L_k's can be modules over a commutative ring whose characteristic is different from two.

satisfied. Similarly, if $A = \oplus A_n$ is any graded associative algebra the above bracket gives a graded Lie algebra called the commutator algebra of A. If A is any graded algebra, associative or not, it is easy to check that the set of graded derivations of A is a Lie subalgebra, Der A, of End A.

It is clear how to define a homomorphism from one graded Lie algebra to another, where they are both indexed by the same group. We require the map to be gradation preserving as well as being a homomorphism of the bracket structure. By a *representation* of a graded Lie algebra, L, on a graded vector space V we shall mean a homomorphism of L into the graded Lie algebra End (V).

II. EXAMPLES

A. The algebra End (*V*) for *V* two dimensional—the Fermi–Dirac anticommutator

Suppose that $V = V_0 \oplus V_1$ where both V_0 and V_1 are one-dimensional vector spaces over the complex numbers. Then

$$\text{End } V = L_{-1} \oplus L_0 \oplus L_1.$$

In other words, the algebra $\mathbf{Z}$ graded with L_i one dimensional for $i = -1$, 1, two dimensional for $i = 0$, and all other L_i trivial. We can define it using real 2×2 matrices,

$$e = 1 \qquad x = \tfrac{1}{2}(\sigma_x + i\sigma_y)$$
$$h = \tfrac{1}{2}(\sigma_z + 1) \quad y = \tfrac{1}{2}(\sigma_x - i\sigma_y)$$

the indexing given by the eigenvalues of h:

$$h, \quad e \in L_0, \quad x \in L_1, \quad y \in L_{-1}$$

with the bracket relations

$$[h,x] = x, \quad [h,y] = -y, \quad [x,y] = e, \tag{2.1}$$

all other brackets vanishing. It is easy to check the graded version of the Jacobi identity, etc.

One well-known physical realization of this scheme is the Jordan–Wigner quantization (1928; see also Fock, 1932) scheme (which incorporates the Pauli principle) for Fermi–Dirac annihilation and creation operators (we restrict ourselves to one state)

$$\begin{aligned} &[b,b^*]_+ = 1, \quad [b,b]_+ = 0, \quad [b^*,b^*]_+ = 0, \\ &[N,b^*]_- = b^*, \quad [N,b]_- = -b; \end{aligned} \tag{2.2}$$

thus

$$N \sim h, \quad b^* \sim x, \quad b \sim y, \quad e = 1. \tag{2.3}$$

Note that in this example, the graded vector space $V = V_k$ is the Fermi Fock space, with V_0 for the vacuum, V_1 for the one fermion state. Indeed, the odd generators b, b^* connect a boson (the vacuum) with a fermion.

B. The Frölicher–Nijenhuis algebra of a vector space

Let W be a vector space and set $V = \Lambda W^*$, the exterior algebra (see Appendix) over W^*, so that $V_m = \Lambda^m W^*$. Notice that V is not only a graded vector space but is actually a graded algebra which is graded-commutative in the sense that if $\omega \in V_m$ and $\nu \in V_n$ then $\nu \wedge = \omega\ (-1)^{mn} \times \omega \wedge \nu$. We can consider those elements of End (V) which are graded derivations, i.e., which satisfy

$$x(\omega \wedge \nu) = (x\omega) \wedge \nu + (-1)^{km}\omega \wedge x\nu, \quad \omega \in V_m. \tag{2.4}$$

The set of x satisfying the above equations for various k define a graded Lie subalgebra of End (V) which we denote by Der (V). If W is finite dimensional then V is finite dimensional and its terms of positive degree are generated by $V_1 = W^*$. Thus every derivation is determined by its action on V_1 and we can identify $\text{Der}_k\ (V)$ with

$$\text{Hom } (W^*,\Lambda^{k+1}W^*) = \Lambda^{k+1}W^* \otimes W. \tag{2.5}$$

With this identification, the action of an element $\omega \otimes u$ on a form will be written as $(\omega \otimes u) \wedge^- \nu$, where the symbol $\wedge^-$ means the combination of first interior product by u (i.e., contraction of the antisymmetric covariant tensor ν with the contravariant vector u) followed by exterior multiplication by ω. It is easy to check that if $\omega \otimes u = x$ and $\gamma \otimes v = y$, then

$$[x,y] = (\omega \otimes u) \wedge^- \gamma \otimes v - (-1)^{kl}(\gamma \otimes v) \wedge^- \omega \otimes u. \tag{2.6}$$

This defines the Frölicher–Nijenhuis algebra for the case where W is finite dimensional. In general it is convenient to take L_k to consist of all alternating $(k+1)$ multilinear maps of $Wx\cdots xW \to W$. (For finite dimensional W this definition of L_k coincides with $\Lambda^{k+1}W^* \otimes W$.) We then define

$$\begin{aligned} x \wedge y(&w_0,\ldots,w_{k+l}) \\ &= \Sigma \text{sgn}(\epsilon) y(\epsilon(w_{\epsilon_0},\ldots,w_{\epsilon_k}),w_{\epsilon_{k+1}},\ldots,w_{\epsilon_{k+l}}), \end{aligned} \tag{2.7}$$

where the sum is taken over all permutations, ϵ, of $0, \ldots, k+l$ with the first $k+1$ and last l elements in increasing order. Then the Lie bracket is again defined as

$$[x,y] = x \cdot y - (-1)^{kl} y \cdot x. \tag{2.8}$$

The fact that this defines a Lie algebra multiplication now follows from the finite dimensional case since we need verify identities involving only finitely many vectors at a time. See Sec. V for a description of the role played by this example and the following in the theory of deformations of mathematical structures.

C. Cohomology algebras

Suppose we start with a graded Lie algebra, L, together with a differential operator, d, of degree n. Thus d maps L_k into L_{k+n} for each k and satisfies the identities

$$d[x,y] = [dx,y] + (-1)^{kn}[x,dy] \tag{2.9}$$

and

$$d^2 = 0. \tag{2.10}$$

We define $Z_k \subset L_k$ to consist of those x satisfying $dx = 0$ and observe that the first identity on d implies that the Z_k fit together form a subalgebra, Z, of L. We let B_k consist of those x of the form $x = dw$ and observe that the identities imply that $B_k \subset L_k$ and that the B's fit together to form an ideal, B, of Z. Thus the quotient spaces $H_k(L) = Z_k/B_k$ fit together to form a graded Lie algebra, $H(L)$, called the cohomology algebra of L (relative to the operator, d). For example, suppose we take L to be the Frölicher–Nijenhius algebra (example Sec. II.B) of a vector space, W, and let z be an element of L_1 [i.e., $\text{Der}_1(V)$] which satisfies the condition $[z,z]=0$. Then left multiplication by z defines a differential operator on L.

Since z is in L_1 it defines an antisymmetric bilinear map of $W \times W \rightarrow W$. The identity

$$[z,z]=0 \tag{2.11}$$

is exactly the classical Jacobi identity for this bilinear map, as can be seen by applying (2.7), so that

$$[z,z]=z \wedge z(w_0,w_1,w_2)-(-1)^{1\times 1}z \wedge z(w_0,w_1,w_2) \tag{2.12}$$

yielding

$$z(z(w_0,w_1),w_2)+z(z(w_1,w_2),w_0)+z(z(w_2,w_0),w_1)=0. \tag{2.13}$$

Thus z defines a standard Lie algebra structure on W. A computation shows that, up to a shift in degree, $H(L)$ coincides with the usual cohomology of W, considered as a Lie algebra under the multiplication defined by z (Chevalley and Eilenberg, 1948).

D. The (*f,d*) algebra of Gell–Mann, Michel, and Radicati

This is an algebra graded by $\mathbf{Z}_2$, where, as vector spaces, both L_0 and L_1 are taken to be the space of n by n matrices with trace zero. The bracket of an element of L_0 with an element of either L_0 or L_1 is given by the usual commutator of matrices while the bracket of two elements of L_1 is given by the "traceless anticommutator," i.e.,

$$[x,y]=xy+yx-(2/n)(\mathrm{Tr}\,xy)I \tag{2.14}$$

for x and y both in L_1, where I is the identity matrix. This construction is valid for matrices over any field. A slight modification makes it also work for the case where we take L_0 and L_1 both to be $su(n)$, the algebra of skew Hermitian matrices.[4] Here multiplication by an element of L_0 is commutation as before, while for two elements of L_1 we define

$$[x,y]=i(xy+yx-(2/n)(\mathrm{Tr}\,xy)I). \tag{2.15}$$

A routine check shows that the axioms for a graded Lie algebra are satisfied.

One can also notice that if $V=W+W$ is a $\mathbf{Z}_2$ graded vector space then the "diagonal matrices" of the form

$$\left(\begin{array}{c|c} a & 0 \\ \hline 0 & a \end{array}\right) \text{ in } \mathrm{End}_0\, V \text{ and } \left(\begin{array}{c|c} 0 & x \\ \hline x & 0 \end{array}\right) \text{ in } \mathrm{End}_1\,(V)$$

form a graded Lie subalgebra of End V. The scalar matrices

$$\left(\begin{array}{c|c} cI & 0 \\ \hline 0 & cI \end{array}\right)$$

form an ideal and the (f,d) algebra [for $sl(n)$] is the quotient algebra.

Michel and Radicati (see Michel, 1969) have used the (f,d) algebra extensively to study the orbits of $SU(3)$ in the octet space. Among the applications of this approach, one can reproduce the variational equation for a spontaneous breakdown of the symmetry by a Hamiltonian which obeys $SU(3)$, and is assumed to behave like an octet component.

Their variational equation is a quadratic equation which can be formulated as follows: The (f,d) algebra has an outer derivation of odd degree, l, where $l=0$ on L_0 and $l: L_1 \rightarrow L_0$ sends a matrix x into the same matrix, but now considered as an element of L_0. Notice that l is equivariant with respect to the action of $su(n)$ [or $sl(n)$] on both L_0 and L_1 so that the equation for a derivation is automatically satisfied for any expression that involves at least one element of L_0. If x and y are elements of L_1 we have

$$0=l[x,y]=[lx,y]-[x,ly] \tag{2.16}$$

in view of the definitions of the bracket relation. Cabibbo's (1968) original equation, rederived by Michel and Radicati in their algebra, can be written as (see also Brout, 1967)

$$[x,x]=-lx. \tag{2.17}$$

It is interesting to remark that the (f,d) algebra is simple, in the sense that it has no ideals, and yet it possesses an outer derivation, l. This is in contrast to the situation of classical Lie algebra theory. We shall return to this point in Sec. IV.

We shall see, in Sec. II.H, that the (f,d) algebra can be realized as a subalgebra of another graded Lie algebra closely associated with the geometry of certain bounded complex domains.

E. The di-spin algebra *sl*(2)

This algebra is $\mathbf{Z}$ graded with L_i one dimensional for $i=-2,-1,0,1,2$, and all the other L_i trivial. The even terms form the algebra $sl(2)$ while the odd terms form the two-dimensional representation of $sl(2)$. To describe the bracket relations explicitly we choose bases

$$e \in L_2, \quad x \in L_1, \quad h \in L_0, \quad y \in L_{-1}, \quad \text{and} \quad f \in L_{-2}.$$

We assume that e, h, f form a standard basis for $sl(2)$ so that they satisfy the relations

$$[h,e]=2e, \quad [h,f]=-2f, \quad [e,f]=h \quad [\text{brackets for } sl(2)]. \tag{2.18}$$

As we want the bracket between two elements of L_1 to be nontrivial, we can, by multiplying our original choice of basis by a scalar, if necessary, arrange that

$$[x,x]=e. \tag{2.19}$$

The axioms for a graded Lie algebra then require that $[h,x]=x$, so that x is the maximal weight vector for the two-dimensional representation of $sl(2)$. This determines a choice of y by setting $[f,x]=y$ and the fact that the odd terms form a representation for $sl(2)$ then yields the bracket relations

$$[h,x]=x, \quad [h,y]=-y, \quad [e,x]=0, \quad [f,x]=y, \quad [e,y]=x, \quad \text{and} \quad [f,y]=0. \tag{2.20}$$

A computation using the axioms then shows that we must have

$$[x,y]=-(1/2)h \quad \text{and} \quad [y,y]=-f. \tag{2.21}$$

[4] We follow the convention of using capital letters for the group [$SL(n)$, $SU(n)$, etc.] and lower case characters for the algebra [$sl(n)$, $su(n)$, etc.].

To summarize, the nonzero bracket relations are given by

$$\begin{array}{lll} [h,e] = 2e, & [h,x] = x, & [x,x] = e, \\ [h,f] = -2f, & [h,y] = -y, & [y,y] = -f, \\ [e,f] = h, & [f,x] = y, & [x,y] = -(1/2)h, \\ & [e,y] = x, & \end{array}$$

all other brackets among generators being zero. It is very easy to find the irreducible finite dimensional (graded) representations of this algebra. Indeed, any such representation is, in particular, a representation of $sl(2)$. Let $v = v_n$ be a highest weight vector for this representation. Then $hxv = [h,x]v + xhv = xv + xhv$ so that we must have $xv = 0$. Similarly, we see that if v is of weight n, then yv must be of weight $n - 1$ and cannot vanish if $n > 1$, since we must have $2y^2v = -fv$. Repeated arguments of this type show that the irreducible representation of $sl(2)$ spanned by v_n together with the irreducible $sl(2)$ representation spanned by yv_n is invariant under the whole algebra. This shows that any finite dimensional irreducible representation of the di-spin algebra consists of the direct sum of a spin $n/2$ and a spin $(n - 1)/2$ representation space for $sl(2)$. We let V_n be the one-dimensional subspace spanned by v_n, let V_{n-1} be the space spanned by yv_n, etc. so that V_{n-j} is spanned by y^jv_n. Then $V = V_k$ is the graded vector space of the representation. According to the preferred convention for choice of basis in each V_k one gets an explicit matrix representation.

With the basis written as $v_n, v_{n-1}, v_{n-2}, \ldots, v_{-n}$ the matrices are:

f

$$\left|\begin{array}{cccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & & \\
n & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & & \\
0 & n-1 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & & \\
0 & 0 & n-1 & 0 & 0 & 0 & 0 & \cdots & & & & & & \\
0 & 0 & 0 & n-2 & 0 & 0 & 0 & \cdots & & & & & & \\
0 & 0 & 0 & 0 & n-2 & 0 & 0 & \cdots & & & & & & \\
0 & 0 & 0 & 0 & 0 & n-3 & 0 & \cdots & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & \ddots & & & & & & & \\
 & & & & & \cdot & \cdot\, 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & & & & & \ddots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & & & \cdots & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & \cdots & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
 & & & & \cdots & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
 & & & & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right|$$

e

$$\left|\begin{array}{cccccccccccccc}
0 & 0 & 1 & 0 & 0 & 0 & \cdots & & & & & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & & & & & \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & \cdots & & & & & \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & \cdots & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 3 & 0 & 0 & \cdots & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 3 & 0 & 0 & \cdots & & \\
 & & & & & \cdots & 0 & \ddots & 0 & 0 & \cdots & & & \\
 & & & & & & \cdots & 0 & \ddots & 0 & 0 & \cdots & & \\
 & & & & & & & \cdots & 0 & n-2 & 0 & 0 & 0 & 0 \\
 & & & & & & & & \cdots & 0 & n-2 & 0 & 0 & 0 \\
 & & & & & & & & \cdots & 0 & 0 & n-1 & 0 & 0 \\
 & & & & & & & & \cdots & 0 & 0 & 0 & n-1 & 0 \\
 & & & & & & & & \cdots & 0 & 0 & 0 & 0 & n \\
 & & & & & & & & \cdots & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & \cdots & 0 & 0 & 0 & 0 & 0
\end{array}\right|$$

y

$$\left|\begin{array}{ccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & & & \\
-n/2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & & & \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & & & \\
0 & 0 & -\frac{(n-1)}{2} & 0 & 0 & 0 & 0 & \cdots & & & & & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & & & & & & & \\
0 & 0 & 0 & 0 & -\frac{(n-2)}{2} & 0 & 0 & \cdots & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & \ddots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \cdots & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 & & & & & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\
 & & & & & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right|$$

x

$$\left|\begin{array}{cccccccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & & & & & \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & & & & & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & & & & & \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 \cdots & & & & & \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 1 & \ddots & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & & \ddots & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \cdots & 0 & 0 & 0 & \frac{n-2}{2} & 0 & 0 & 0 & 0 \\
 & & & & & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & & \cdots & 0 & 0 & 0 & 0 & 0 & \frac{n-1}{2} & 0 & 0 \\
 & & & & & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n/2 \\
 & & & & & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right|$$

h

$$\left|\begin{array}{cccccccccc}
n & 0 & 0 & 0 & 0 & & & & & \\
0 & n-1 & 0 & 0 & 0 & & & & & \\
0 & 0 & n-2 & 0 & 0 & & & & & \\
0 & 0 & 0 & n-3 & 0 & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & 0 & 0 & -(n-3) & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & -(n-2) & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 & -(n-1) & 0 \\
 & & & & 0 & 0 & 0 & 0 & 0 & -n
\end{array}\right|$$

F. Dual models (strings)

We now turn to the "supergauge" transformations of the so-called dual models (Neveu and Schwarz, 1971; Ramond, 1971; Aharonov, Casher, and Susskind, 1971; Gervais and Sakita, 1971). Numerous reviews (Schwarz, 1973; Veneziano, 1974; Rebbi, 1974) will assist readers who happen to be unfamiliar with this relatively recent development of dispersion theory, following Veneziano's (1968) discovery of a crossing-symmetric relativistic strong-interaction amplitude satisfying in addition a bootstrap condition in the form of finite energy sum rules, FESR, (Dolen, Horn, and Schmidt, 1967; Igi and Matsuda, 1967; Logunov, Soloviev, and Tavkhelidze, 1967) and possessing appropriate Regge asymptotic behavior. The attempt to unitarize Veneziano's representation has yielded systems which now tend to be regarded as infinite-component field theories, rather than as on-mass shell amplitudes. Nambu, Nielsen, Susskind, and others have replaced the factorized Veneziano model by a quantized one-space dimensional relativistic string moving in Minkowski space–time. The quantum excitations of the string reproduce the Veneziano spectrum; the gauge conditions (including the "supergauge") are necessary for the removal of ghost states. The next development adjoined a continuous-spin structure to the linear string, yielding two related models:

(a) The Neveu–Schwarz (1971) model, yielding a spectrum of bosons.

(b) The Ramond model (1971), with a fermion spectrum.

Both are produced by the same algebraic relations, with "ordinary" Lie generators L_m ($m = 0, \pm1, \pm2, \ldots$) and what we now recognize as "odd" generators of a graded Lie algebra: G_r ($r = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$) in the Neveu–Schwarz model, and F_m in the Ramond model. The Neveu–Schwarz model is given by the algebra:

$$\begin{aligned}[][L_m,L_n] &= (m-n)L_{m+n} + \tfrac{1}{8}d(m^3-m)\delta_{m,-n},\\ [L_m,G_r] &= (\tfrac{1}{2}m - r)G_{m+r},\\ \{G_r,G_s\} &= 2L_{r+s} + \tfrac{1}{2}d(r^2-\tfrac{1}{4})_{r,-n}.\end{aligned} \tag{2.22}$$

d is the dimension of the space–time in which we embed the string. One hopes $d = 4$, though this result depends at present upon the introduction of internal degrees of freedom; the actual result is $d = 10$ for convergence to be ensured.

We see that the grading N_a of the Neveu–Schwarz $N = \oplus N_a$ algebra is a $\mathbf{Z}$ grading, with

$$L_m \in N_{2m}, \quad G_r \in N_{2r}.$$

The L_m are thus even, the G_r odd. All N_a are one dimensional, except for N_0 which contains L_0 and the identity operator. The index a is given by the commutator

$$[-2L_0,x_a] = ax_a. \tag{2.23}$$

The Ramond model has only $\mathbf{Z}_2$ grading. The algebra is similar, except that G_r is replaced by F_m, $m = 0, \pm1, \pm2, \ldots$, i.e., the same values as for the even generators. We are thus led to use the grading

$$R = \bigoplus_a R^a, \quad a = 0, 1$$

$$L_m \subset R^0$$

$$F_m \subset R^1.$$

G. The conformal algebra o(2,3), the symplectic and the Poisson algebras

Let us start with a specific construction for the conformal algebra $o(2,3)$. Later on we shall give a more general construction. For $o(2,3)$ we construct a graded Lie algebra whose even components, L_{-2}, L_0, L_2, fit together to form the algebra of infinitesimal conformal transformations on $\mathbf{R}^3$ endowed with the indefinite metric of signature $+ - -$. We consider $\mathbf{R}^3$ to be the space of real symmetric matrices

$$X = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.24}$$

so that $\det X = ac - b^2$ is a quadratic form of signature $(+ - -)$, with coordinates such as $\{(a+c)/2, (a-c)/2, b\}$. We let the group $GL(2,\mathbf{R})$ act on the space of symmetric matrices by letting the nonsingular matrix A take the symmetric matrix

$$X \rightsquigarrow AXA^t. \tag{2.25}$$

We let L_{-2} denote the space of such X with this action. Similarly, we let L_2 denote the space of symmetric matrices under the action

$$W \rightsquigarrow A^{t-1}WA^{-1}. \tag{2.26}$$

On L_{-2} the matrix A multiplies $\det X$ by $\det A^2$ while on L_2 the matrix A multiplies $\det W$ by $\det A^{-2}$. In particular, the group $SL(2,\mathbf{R})$ acts as isometries on both spaces. If we regard $o(2,3)$ as infinitestimal conformal transformations, then L_{-2} is the three-dimensional subalgebra of translations on $\mathbf{R}^3$ while L_2 consists of the infinitesimal proper conformal transformations.

We let $L_0 = gl(2,\mathbf{R})$ with $GL(2,\mathbf{R})$ acting via the adjoint representation, i.e., sending $B \in gl(2,\mathbf{R})$ into ABA^{-1}. We let L_{-1} be the space of column two vectors with the action $x \rightsquigarrow Ax$ and we let L_1 denote the space of row two vectors with the action $z \rightsquigarrow zA^{-1}$. Finally, we define bracket relations as follows (a more succinct but abstract description of these bracket relations will be given at the end of this subsection):

$$\begin{aligned}
&[L_{-2},L_{-2}] = 0, &&\\
&[L_{-2},L_{-1}] = 0, &&\\
&[x,y] = x\otimes y^t + y\otimes x^t, && \text{for } x,y\in L_{-1}\\
&[a,X] = aX + Xa^t, && a\in L_0,\ X\in L_{-2}\\
&[a,x] = ax, && a\in L_0,\ x\in L_{-1}\\
&[a,b] = ab - ba, && a,b\in L_0\\
&[a,z] = -za, && a\in L_0,\ z\in L_1\\
&[a,W] = -a^tW - Wa, && a\in L_0,\ W\in L_2\\
&[x,z] = x\otimes z, && x\in L_{-1},\ z\in L_1\\
&[X,z] = -Xz^t, && X\in L_{-2},\ z\in L_1\\
&[X,W] = -XW, && X\in L_{-2},\ W\in L_2\\
&[z,w] = z^t\otimes w + w^t\otimes z, && z,w\in L_1\\
&[x,W] = -(Wx)^t, && x\in L_{-1},\ W\in L_2\\
&[L_1,L_2] = 0, &&\\
&[L_2,L_2] = 0. &&
\end{aligned} \tag{2.27}$$

All remaining brackets are determined by the requirement of commutativity or anticommutativity, i.e., by condition (1.3). Notice that all brackets are equivariant with respect to the action of $GL(2,\mathbf{R})$ and that the bracket (on the left) by an element of L_0 is just the infinitesimal version of the action of $GL(2,\mathbf{R})$. This immediately implies that the Jacobi identity automatically holds for any expression involving at least one $a \in L_0$; Jacobi's identity is trivially verified for terms of all strictly negative or all strictly positive degree since all expressions vanish. Similarly for an expression involving two elements of L_{-2} and one element of L_1 or two elements of L_2 and one element of L_{-1}. We check the remaining cases. In what follows X, Y will denote elements of L_{-2}; x, y of L_{-1}; z, w of L_1, and Z, W of L_2. Then

$$0 = [W,[X,Y]]$$

and

$$[[W,X],Y] + [X,[W,Y]] = XWY + YWX \\ -[(YWX + XWY)] = 0.$$

A similar computation shows that

$$0 = [X,[W,Z]] = [[X,W],Z] + [W,[X,Z]].$$

We have

$$[X,[x,W]] = -[X,(Wx)]^t, \\ = XWx$$

while

$$[[X,x],W] = 0$$

and

$$[x,[X,W]] = XWx$$

so that

$$[X,[x,W]] = [[X,x],W] + [x,[X,W]].$$

A similar computation shows that

$$[W,[z,X]] = [[W,z],X] + [z,[W,X]].$$

The remaining cases are equally straightforward and are left to the reader.

As we mentioned, the Lie algebra $L_{-2} \oplus L_0 \oplus L_2$ can be regarded as the algebra of infinitesimal conformal transformations; the subspace L_{-2} corresponds to infinitesimal translations, the subspace L_0 to infinitesimal linear conformal transformations and the subspace L_2 corresponds to the infinitesimal "proper conformal transformations." Notice that under the action of $SL(2)$ we can identify L_{-2} with L_2 by sending the matrix X into its "cofactor matrix," X^f, defined by the equation $XX^f = (\det X)I$. However, under the scalar matrices (acting as "scale transformations") the spaces L_{-2} and L_2 behave as dual to one another. This algebra and the di-spin algebra of example E represent graded generalizations of conformal algebras in $(+ - -)$ and in one dimension, and will lead us to the Wess–Zumino (1974a) algebra $\mathcal{W}$, when we reach Minkowski space.

Notice that we can perform the same construction over any field and for any n. Of course we no longer have the interpretation of the algebra $L_{-2} \oplus L_0 \oplus L_2$ as the conformal algebra. However, it is clear that $L_{-2} \oplus L_0 \oplus L_2$ is a simple Lie algebra and that the diagonal matrices (in L_0) form a Cartan subalgebra. Thus the rank is n while the dimension is $n^2 + 2n(n+1)/2$ which determines the algebra (if over $\mathbf{C}$) as being of type B_l or C_l. If Λ is the diagnoal matrix with entries $(\lambda_1, \ldots, \lambda_n)$ then it is clear that the roots of the algebra, when evaluated on Λ, are exactly $\lambda_i + \lambda_j$, $-\lambda_i - \lambda_j$ (all, i,j), and $\lambda_i - \lambda_j$ $(i \neq j)$. This shows that the structure is of type C_l, i.e., we are dealing with the symplectic algebra. (Of course for $l = 2$ we have $B_2 = C_2$ and the conformal algebra in three dimensions is an orthogonal algebra in five dimensions which is isomorphic (if over $\mathbf{C}$) to the symplectic algebra in four dimensions.) We can actually find a representation of the entire graded algebra which clearly exhibits the even part as the symplectic algebra: Let V_1 be an n-dimensional vector space (which we may identify as the space of row vectors) and V_{-1} its dual space. We let $V_0 = \mathbf{R}$ (or, more generally the ground field). We then consider $W = V_{-1} \oplus V_1$ as a symplectic vector space and let $\mathrm{sp}(W)$ act on $V = V_{-1} \oplus V_0 \oplus V_1$ by acting trivially on V_0. Thus $\mathrm{sp}(W)$ is realized as all matrices of the form

$$\begin{array}{c} \\ n \\ 1 \\ n \end{array}\begin{array}{c} \begin{array}{ccc} n & 1 & n \end{array} \\ \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ c & 0 & -a^t \end{pmatrix} \end{array},$$

where $b = b^t$ and $c = c^t$ are symmetric matrices and a is an arbitrary $n \times n$ matrix. We then let L_1 and L_{-1} consist of matrices of the form

$$\begin{pmatrix} 0 & x & 0 \\ 0 & 0 & x^t \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ z^t & 0 & 0 \\ 0 & -z & 0 \end{pmatrix}.$$

If we are interested only in the $\mathbf{Z}_2$ gradation we can consider a similar construction for any symplectic vector space W (not necessarily decomposed as $W = V_{-1} \oplus V_1$). Indeed we can identify the Lie algebra $\mathrm{sp}(W)$ with the space of symmetric tensors $S^2(W)$, where the symmetric tensor $u[= \frac{1}{2}(u \otimes v + v \otimes u)]$ acts as

$$(uv)(w) = u\langle v,w\rangle + v\langle u,w\rangle \tag{2.28}$$

with $\langle\ ,\ \rangle$ denoting the symplectic form. If we let $L_1 = W$ and define the map of $L_1 \times L_1 \to L_0$ by $[u,v] = uv$ with $L_0 = \mathrm{sp}(W)$ acting in the usual fashion we obtain a $\mathbf{Z}_2$ graded Lie algebra. We can construct an analogous $\mathbf{Z}_2$ graded Lie algebra using multiplication and Poisson bracket for functions defined on any symplectic manifold. (Souriau, 1970; Sternberg, 1964). Indeed, let X be a symplectic manifold with symplectic form $\mathcal{W}$. Let $L_0 = F(x) = L_1$ be the space of C^∞ functions on X and set

$$[f,g] = f,g = \mathcal{W}(df,dg), \quad f, g \in L_0 \tag{2.29}$$

where $\mathcal{W}(df,dg)$ is the value of the fundamental two form $\mathcal{W}$ (considered as a contravariant form via the identification of TX with T^*X given by $\mathcal{W}$) on df, dg. Similarly we set

$$[f,\theta] = \{f,\theta\} = \mathcal{W}(df,d\theta) \in L_1, \quad f \in L_0, \quad \theta \in L_1 \tag{2.30}$$

while

$$[\theta,\varphi] = \{\theta,\varphi\} = f \in L_0, \quad \text{for } \theta \in L_1. \tag{2.31}$$

Jacobi's identity holds for any expression involving two or three elements of L_0, since it reduces to the Jacobi identity for Poisson brackets. The equation

$$[f,[\varphi,\theta]] = [[f,\varphi],\theta] + [\varphi,[f,\theta]]$$

just asserts that a Poisson bracket is a derivation for multiplication. Finally

$$[[\theta,\varphi],\lambda] = \theta w(d\varphi,d\lambda) + \varphi w(d\theta,d\lambda) \tag{2.32}$$

clearly vanishes under cyclic sum and so we have verified Jacobi's identity in all cases. Notice that if we take X to be a vector space we can consider the subalgebra of all quadratic functions in L_0 and all linear functions in L_1. This reduces to the algebra $W \oplus \text{sp}W$ considered above. If we are given a splitting $W = V_{-1} \oplus V_1$ we may split the linear functions up into p's and q's. We may then take L_{-2} to consist of functions quadratic in the p's, take L_{-1} to be the functions linear in the p's, take L_0 to consist of functions of the form $a_{ij}p_iq_j$, take L_1 to be functions linear in the q's and L_2 to consist of functions quadratic in the q's. In this way we recover the original example. For $n = 2$ we get the $o(2,3)$ algebra—and for $n = 1$ we get the di-spin algebra. Let us return to the case of the large Poisson bracket algebra but consider the situation where the symplectic manifold, X, is the cotangent bundle, T^*M, of a manifold M. In this case it makes invariant sense to say that a function is a polynomial in the cotangent direction, i.e., that the function is a polynomial in the p's when expressed in terms of q,p coordinates with q position coordinates coming from M. Let L_{2k} denote the space of functions which are homogeneous polynomials of degree $k + 1$ in the p's. Thus L_{-2} consists of functions independent of the p's, and L_0 consists of functions linear in the p's, etc. Similarly, let L_{-1} consist of functions independent of the p's, let L_1 consist of functions linear in the p's, and, generally, L_{2i-1} consist of functions which are homogeneous polynomials of degree i in the p's. As before, we define the bracket by an even degree element to be the usual Poisson bracket while the bracket of two odd elements is ordinary multiplication. It is now easy to check that $\oplus_{k=-2}^{\infty} L_k$ is a $\mathbf{Z}$ graded Lie algebra. In Sec. II.J we shall show how to associate a "Poisson" algebra with a filtered associative algebra satisfying additional conditions. These conditions are satisfied for the ring of differential operators on a manifold and the associated graded Lie algebra (having only even elements) is the usual Poisson bracket algebra. These conditions are also satisfied by the Clifford algebra. In Sec. II.K we show how to form the tensor product of two such algebras to obtain a third algebra of the same type. It turns out that the graded Poisson algebra described above is the "Poisson algebra" associated with the tensor product of the one-dimensional Clifford algebra with the algebra of differential operators.

H. The spin-conformal algebra $\mathcal{W}$ on Minkowski space as introduced by Wess and Zumino (1974a)

It is possible to modify the construction of the preceding example to the case of the Lorentz metric on Minkowski space. We let L_{-2} be the space of Hermitian 2×2 matrices. If

$$X = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \tag{2.33}$$

then

$$\det X = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{2.34}$$

is just the Minkowski length of X. If $A \in GL(2,\mathbf{C})$ then AXA^* is again Hermitian and

$$\det(AXA^*) = |\det A|^2 \det X. \tag{2.35}$$

Thus $GL(2,\mathbf{C})$ acts as conformal linear transformations on Minkowski space and $SL(2,\mathbf{C})$ acts as Lorentz transformations; indeed this provides the standard realization of $SL(2,\mathbf{C})$ as the double covering of the Lorentz group. Notice that the unitary scalar matrices of the form $e^{i\phi}I$ (ϕ real) act trivially, so that the representation of $GL(2,\mathbf{C})$ on the space of symmetric matrices is not faithful. We let L_{-2} denote the space of Hermitian matrices with the action as described above and let L_2 denote the space of Hermitian matrices with the action

$$W \rightsquigarrow A^{*-1}WA^*.$$

We define, for $X \in L_{-2}$ and $W \in L_2$

$$\begin{aligned} &[X,W] = XW \text{ (matrix multiplication)} \\ &[X_1,X_2] = 0 \quad X_i \in L_{-2} \\ &[W_1,W_2] = 0 \quad W_i \in L_2. \end{aligned} \tag{2.36}$$

All operations are equivariant under the action of $GL(2,\mathbf{C})$ and we define the bracket by $L_0 = gl(2,\mathbf{C})$ to be the infinitesimal version of this action. Then Jacobi's identity automatically holds and it is easy to see that $L_{-2} \oplus L_0 \oplus L_2$ is a sixteen-dimensional real Lie algebra which is the direct sum of the fifteen-dimensional conformal algebra [$su(2,2)$ or $o(2,4)$] plus a one-dimensional center consisting of the imaginary scalar matrices. We might expect to procede as before, letting L_{-1} consist of column vectors, x, with the action $x \rightsquigarrow Ax$, i.e., spinors, and define

$$[x_1,x_2] = x_1 \otimes x_2^* + x_2 \otimes x_1^*.$$

This does indeed define a symmetric bilinear map of $L_{-1} \times L_{-1} - L_{-2}$ which is equivariant with respect to the action of $GL(2,\mathbf{C})$. Similarly we would define L_1 to consist of row vectors w with the action $w \rightsquigarrow wA^{-1}$ and the bracket

$$[w_1,w_2] = w_1^* \otimes w_2 + w_2^* \otimes w_1.$$

The problem comes in defining $[x,w]$. The only possible nontrivial choice (up to a scalar factor) is to set $[x,w] = x \otimes w$. But then Jacobi's identity requires

$$\begin{aligned}[x,[w_1,w_2]] &= [[x,w_1],w_2] + [[x,w_2],w_1] \\ &= -(w_2x)w_1 - (w_1x)w_2\end{aligned}$$

and this last expression is not bilinear in x and

$$w_1^* \otimes w_2 + w_2^* \otimes w_1.$$

The way out of this problem is to modify the action of $gl(2,\mathbf{C})$ on L_{-1} and L_1 by changing the action of the purely imaginary scalar matrices, precisely the ones that gave no effect on L_{-2} and L_2. It is simplest to explain the construction in matrix form. We can write the most general element of the sixteen-dimensional algebra $L_{-2} \oplus L_0 \oplus L_2$ as

$$\begin{pmatrix} a & b \\ c & -a^* \end{pmatrix} a \in gl(2,\mathbf{C}),\ b = b^*,\ c = c^*.$$

[This identifies the algebra as $u(2,2)$, the linear transformations preserving the form $x_1\bar{x}_3 + x_2\bar{x}_4$.]

We now enlarge this matrix by expanding it to a 5×5 matrix

$$\begin{bmatrix} a & 0 & b \\ 0 & \lambda_a & 0 \\ c & 0 & -a^* \end{bmatrix},$$

where

$$\lambda_a = -2i(im\ \mathrm{Tr}a). \tag{2.37}$$

Thus the situation is quite similar to the symplectic case, except for the imaginary part of the trace of a, which contributes to the middle position. We add the matrices

$$\begin{pmatrix} 0 & x & 0 \\ 0 & 0 & x^* \\ 0 & 0 & 0 \end{pmatrix} \in L_{-1}$$

and

$$\begin{pmatrix} 0 & 0 & 0 \\ w & 0 & 0 \\ 0 & -w^* & 0 \end{pmatrix} \in L_1$$

then

$$\left[\begin{pmatrix} a & 0 & b \\ 0 & \lambda_a & 0 \\ c & 0 & -a^* \end{pmatrix}, \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & x^* \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & ax - \lambda_a x & 0 \\ -x^*c & 0 & \lambda_a x^* + x^*a^* \\ 0 & cx & 0 \end{pmatrix}$$

and the matrix on the right is of the desired form since λ_a is purely imaginary and $c = c^*$. Similarly

$$\left[\begin{pmatrix} a & 0 & b \\ 0 & \lambda_a & 0 \\ c & 0 & -a^* \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ +w & 0 & 0 \\ 0 & -w^* & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & bw^* & 0 \\ -w_a + \lambda_a & 0 & wb \\ 0 & a+{}^*w^* + \lambda_a w^* & 0 \end{pmatrix}.$$

Finally

$$\begin{pmatrix} 0 & x & 0 \\ w & 0 & x^* \\ 0 & -w^* & 0 \end{pmatrix}\begin{pmatrix} 0 & y & 0 \\ z & 0 & y^* \\ 0 & -z^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & y & 0 \\ z & 0 & y^* \\ 0 & -z^* & 0 \end{pmatrix}\begin{pmatrix} 0 & x & 0 \\ w & 0 & x^* \\ 0 & -w^* & 0 \end{pmatrix}$$
$$= \begin{pmatrix} x\otimes z + y\otimes w & 0 & x\otimes y^* + y\otimes x^* \\ 0 & \lambda & 0 \\ -(w^*\otimes z + z^*\otimes w) & 0 & -w^*\otimes y^* - z^*\otimes x^* \end{pmatrix},$$

where

$$\lambda = wy - x^*z^* + zx - y^*w^* = -2i[\mathrm{Im}\,\mathrm{Tr}(x \otimes z + y \otimes w)].$$

If we let $V = V_{-1} \oplus V_0 \oplus V_1$, similarly to the symplectic case, we have explicitly shown that $L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ is a graded Lie subalgebra of End V and hence Jacobi's identity automatically holds. Needless to say, the construction works for $u(n,n)$ for any value of n.

Getting back to the case $n = 2$, notice that the action of $iI \in gl(2,\mathbf{C})$ on L_{-1} sends $x \rightsquigarrow -3ix$ and on L_1 sends $w \rightsquigarrow 3iw$ in contrast to the expected action.

Notice also that the action of the full conformal algebra, $o(2,4)$, on $L_{-1} \oplus L_1$ is irreducible. Thus $L_{-1} \oplus L_1$ provides an irreducible eight-dimensional representation for the conformal algebra which is, therefore, the spin representation of the conformal algebra. The elements of this spin representation of $o(2,4)$ were introduced, in a geometric context, by Penrose (1967) who calls them "twistors." Roughly speaking, we can regard a Dirac spinor, x, as being the "square root" of the null vector $x \otimes x^*$. Similarly, a pair (x,w) (satisfying the additional condition $\mathrm{Re}\, x\bar{w} = 0$) can be regarded as the "square root" of a geometric object determining a null line. See Penrose (1967) for the details. Sternberg and Wolf (1975) have extended the preceding example by replacing the row and column vectors, x and w, by rectangular matrices of size n by k and k by n. Then an expression such as $x \otimes w^* + w \otimes x^*$ is replaced by the sum of matrix products, $xw^* + wx^*$. Otherwise everything remains the same except that the "center block" is now a k by k skew Hermitian matrix (which is the generalization to dimension k of a purely imaginary number) and the entire $(n + k)$ by $(n + k)$ matrix is subject to the constraint that its trace be zero. This suggests the possibility of using the middle $u(k)$ for the purpose of generating internal symmetries of fermions. This graded Lie algebra of $(n + k)$ by $(n + k)$ matrices is closely connected to the ordinary Lie algebra $u(n, n + k)$. Indeed, the odd part of the graded Lie algebra is a complex vector space, whose complex structure is invariant under the action of the even part, L_0. Furthermore, there is a Hermitian form, H, from $L_{\mathrm{odd}} \times L_{\mathrm{odd}} \to L_{\mathrm{even}} \otimes \mathbf{C}$, such that Re$H$ gives the graded Lie algebra bracket from $L_{\mathrm{odd}} \times L_{\mathrm{odd}} \to L_{\mathrm{even}}$ and the imaginary part makes $L_{\mathrm{even}} \oplus L_{\mathrm{odd}}$ into the ordinary Lie algebra $u(n, n + k)$. This phenomenon, as pointed out by Sternberg and Wolf, is quite general: Suppose one has a real Lie algebra, L_{even}, acting on a complex vector space, L_{odd}, and an equivariant Hermitian form, H, from $L_{\mathrm{odd}} \otimes L_{\mathrm{odd}} \to L_{\mathrm{even}}$. (Equivariant means with respect to the given action of L_{even} on L_{odd} and the complex extension of the adjoint action, to give an action of L_{even} on $L_{\mathrm{even}} \otimes \mathbf{C}$). Then Re$H$ defines a graded Lie algebra structure on $L_{\mathrm{even}} \oplus L_{\mathrm{odd}}$ if and only if ImH defines an ordinary Lie algebra structure on the same space. Sternberg and Wolf show that the Lie algebra of the group of automorphisms of any bounded homogeneous domain in several complex variables has such a structure. Thus a graded Lie algebra is associated with each of the bounded domains. The spin conformal algebra introduced above corresponds to the algebra of the group of automorphisms of the space of all 4×5 matrices, Z, satisfying $I - ZZ^* > 0$.

In order to show the connection between the spin conformal algebra and the pseudounitary algebras more clearly, Sternberg and Wolf (1975) have rewritten the spin conformal algebra in a slightly different form, which differs from the form written above by multiplying appropriate matrix entries by i or by $e^{2\pi i/4}$ and $e^{-2\pi i/4}$. We now briefly rewrite the algebra in the form presented by Sternberg and Wolf in order to exhibit an interesting relation with the (f,d) algebra.

On complex $m + k + m$ space, $\mathbf{C}^{m+k+m}$ they introduce the pseudo-Hermitian form given by the matrix

$$J = \begin{pmatrix} 0 & 0 & I_m \\ 0 & I_k & 0 \\ I_m & 0 & 0 \end{pmatrix},$$

where I_m is the $m \times m$ identity matrix, and I_k is the $k \times k$ identity matrix. The algebra $u(m, m + k)$ consists of all $(m + k + m) \times (m + k + m)$ matrices which infinitesimally preserve the form, J, i.e., which satisfy the equation

$$XJ + JX^* = 0. \tag{2.38}$$

The algebra $su(m, m + k)$ consists of those matrices satisfying (2.38) and, in addition, the condition tr $X = 0$. A direct computation shows that an X satisfying (2.38) must be of the form

$$X = \begin{bmatrix} A & E & B \\ -F^* & D^* & -E^* \\ C & F & -A^* \end{bmatrix},$$

where A is an arbitrary $m\times m$ complex matrix E and F are arbitrary complex matrices with m rows and k columns

B and C are skew Hermitian $m\times m$ matrices, and

D is a skew Hermitian $k \times k$ matrix.

We can write $u(m, m+k) = g = g_{-2} + g_{-1} + g_0 + g_1 + g_2$, where g_{-2} consists of matrices containing nonzero entries in the C position, g_{-1} consists only of F's, g_0 consists only of A's and D's, g_1 consists of E's and g_2 of B's. If we let $[\ ,\ ]_L$ denote the usual Lie bracket, i.e., the commutator, then

$$[g_i, g_j]_L \subset g_{i+j}.$$

We then define a graded Lie algebra structure on g by using the commutator bracket for any expression involving at least one even term, and, for odd terms, define the graded Lie algebra bracket of $g_{\text{odd}} \times g_{\text{odd}} \rightarrow g_{\text{even}}$ by setting

$$\left[\begin{matrix} 0 & E_1 & 0 \\ -F_1^* & 0 & -E_1^* \\ 0 & F_1 & 0 \end{matrix}, \begin{matrix} 0 & E_2 & 0 \\ -F_2^* & 0 & -E^* \\ 0 & F_2 & 0 \end{matrix}\right] = i\begin{bmatrix} E_1F_2^*+E_2F_1^* & 0 & E_1E_2^*+E_2E_1^* \\ 0 & E_1^*F_2+E_2^*F_1+F_1^*E_2+F_2^*E_1 & 0 \\ F_1F_2^*+F_2F_1^* & 0 & F_1E_2^*+F_2E_1^* \end{bmatrix}.$$

The fact that this multiplication makes g into a graded Lie algebra follows from the explicit realizations of both the Lie bracket and graded Lie bracket as imaginary and real parts of a Hermitian form. Of course the Jacobi identity can also be verified directly. It is also easy to check that this algebra is indeed isomorphic to the spin conformal algebra (once we factor out the trace). Notice that if we consider the case $k = m = n$, and take the subalgebra obtained by considering those elements of L_{even} of the form

$$\begin{pmatrix} A & 0 & A \\ 0 & 2A & 0 \\ A & 0 & A \end{pmatrix} \quad A \in u(n)$$

and those elements of L_{odd} of the form

$$\begin{pmatrix} 0 & B & 0 \\ B & 0 & B \\ 0 & B & 0 \end{pmatrix} \quad B \in u(n)$$

then we get a graded subalgebra isomorphic to the (f,d) algebra (again after factoring out the trace).

Returning to the spin conformal algebra, with $m = 2$, $k = 1$, notice that the matrices

$$\gamma(X) = \begin{pmatrix} 0 & 0 & X \\ 0 & 0 & 0 \\ -X^a & 0 & 0 \end{pmatrix}, \quad X \in su(2)$$

satisfy the Dirac conditions

$$[\gamma(X),[\gamma(X),u]] = \|X\|^2 u$$

for $u \in L_{\text{odd}}$, where

$$X = i\begin{pmatrix} x_0 + x_3 & x_1 + 2x_2 \\ x_1 - 2x_2 & x_0 - x_3 \end{pmatrix}$$

and

$$\|X\| = x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

If we add the terms of degree zero with $A \in sl(2,\mathbf{C})$ then we get a subalgebra isomorphic to the usual algebra built up from the Dirac matrices (and the Lorentz matrices as their commutators). It is interesting to observe that this algebra makes its appearance as an (orthogonal) subalgebra of the conformal algebra and to speculate on the possible physical significance of this fact.

We return to the applications of this graded Lie algebra $\mathcal{W}$ in Sec. VI.

I. Another algebra associated with $sl(n-1)$

Let $V = V_0 + V_1$ be a $\mathbf{Z}_2$ graded vector space. Then End V contains a one-dimensional ideal lying in $\text{End}_0\, V$, namely the multiples of the linear transformation J, where $J_{1V_0} = id$ and $J_{1V_1} = -id$. We can thus form the quotient algebra, whose dimensionality will be $n^2 - 1$.

Let V_0 be $(n-1)$ dimensional, and V_1 one dimensional. Thus L_0 will be $gl(n-1)$, L_1 will be $2n-2$ dimensional. The total dimensionality of the graded Lie algebra will be $n^2 - 1$. Note that the excluded generator is not the identity.

Gell-Mann and Ne'eman (1974) have taken $n = 3$. L_0 can then be identified with spin and fermion number, and the odd operators then appear as "square roots" of rotations. Complexifying, one can have a system

$$\{G_a, G_b\} = (\rho_{\mu\nu})_{ab} J^{\mu\nu}, \tag{2.39}$$

where the $(\rho_{\mu\nu})_{ab}$ can be given in terms of d-type coefficients and the identity. This algebra might serve as an alternative to the spin-conformal one and is connected with it. However, it should be written relativistically in order to be useful; this implies using Wigner rotations etc., to preserve the $J^{\mu\nu}$ in motion.

J. Graded Lie algebras associated with Clifford algebras

Let V be a vector space over a field, K, of characteristic unequal to two, and let Q be a quadratic form on V. We recall the definition of the Clifford algebra, $C_Q(V)$, determined by V and Q. Let $T(V) = T^0(V) \oplus T^1(V) \oplus T^2(V) \oplus \cdots = K \oplus V \oplus V \otimes V \oplus \cdots$ be the full tensor algebra over V and let $I(Q)$ be the two-sided ideal generated by the elements $v \otimes v - Q(v)1$ as v ranges over V. Then $C_Q(V)$ is defined to be $T(V)/I(Q)$. The composition of the maps $V \rightarrow T(V) \rightarrow C_Q(V)$ is an injection and allows us to identify

V as a subspace of $C_Q(V)$. If φ is a linear map of V into an associative K algebra A with unit such that $\varphi(x)^2 = Q(x)\cdot 1$, then φ extends to an algebra homomorphism of $C_Q(V) \to A$ which is uniquely determined and which we shall continue to denote by φ. Thus the algebra $C_Q(V)$ can be characterized as the universal algebra with respect to maps: $V \to A$ described above. For two elements, x and y, of V we have the relation

$$xy + yx = 2(x,y)1, \tag{2.40}$$

where (,) is the scalar product determined by the quadratic form, Q.

The filtration by degree on $T(V)$, where

$$F^qT(V) = \bigoplus_0^q T^i(V)$$

induces a filtration, F^qC on the algebra $C = C_Q(V)$ so that F^qC consists of those expressions which can be written as sums of products involving at most q factors of elements of V. The associated graded algebra is just the exterior algebra ΛV which is graded commutative. Actually, the "cancellation law" in the multiplication always drops degree by two so that C is a $\mathbf{Z}_2$ graded (but not graded commutative) algebra where C^0 consists of sums of even products of elements of V and C^1 consists of sums of odd products (Atiyah *et al.*, 1964). The elements of C^0 are filtered by even degrees and the elements of C^1 are filtered by odd degrees in the sense that if two elements x and y lie in $F^q(C^0)$ and $x - y \in F^{q-1}(C)$ (q even) then $x - y \in F^{q-2}(C)$, and similarly for x and y in $F^q(C^1)$ (q odd).

Let C be such a filtered graded algebra, so that C is $\mathbf{Z}_2$ graded and $\mathbf{Z}$ filtered in the above sense, with the associated $\mathbf{Z}$ graded algebra graded commutative. We claim that the graded algebra, with a shift of two in gradation degree, inherits the structure of a graded Lie algebra, which we shall call the Poisson algebra associated with C and denote by $P(C)$. More precisely, let us set $P_k(C) = F^{k+2}(C)/F^k(C)$. If $x \in P_k(C)$ and $y \in P_l(C)$ we can find $\mathbf{x} \in F^{k+2}(C)$ such that $\mathbf{x}/F^k(C) = x$ and $\mathbf{y} \in F^{l+2}(C)$ such that $\mathbf{y}/F^l(C) = y$, where $\mathbf{x}$ is in C^0 or C^1 according as k is even or odd and similarly for $\mathbf{y}$. Since the graded algebra associated with the filtration on C is graded commutative, we know that the expression $\mathbf{xy} - (-1)^{kl}\mathbf{yx}$ which *a priori* belongs to $F^{k+2+l+2}(C)$ actually has a vanishing highest order piece, and hence lies in filtration degree two less, i.e., in $F^{k+l+2}(C)$. We define

$$[x,y] = (\mathbf{xy} - (-1)^{kl}\mathbf{yx})/F^{k+l}(C). \tag{2.41}$$

Observe that this is independent of the choice of $\mathbf{x}$ or $\mathbf{y}$, Indeed, if for example we chose some other $\mathbf{x}$, then $\mathbf{x} - \mathbf{x}$ has the same parity as $\mathbf{x}$ and the filtration degree is two less so that using $\mathbf{x}$ or $\mathbf{x}$ gives the same answer modulo $F^{k+l}(C)$. Once we know that the bracket operation on $P(C) = \oplus P_k(C)$ is well defined, the Jacobi identity is automatic, because it holds for the commutator bracket of three elements $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ when we consider C as a $\mathbf{Z}_2$ graded algebra. Also notice that the Poisson bracket is a (graded) derivation of the graded commutative structure of gr$C$.

For the case of the Clifford algebra, a direct verification shows that $P_0[C_Q(V)] = \Lambda^2(V)$ can be identified with the Lie algebra $o_Q(V)$, the infinitesimal orthogonal transformations relative to the form Q provided that the form Q is nondegenerate, and the bracket of P_0 on the P_i is the induced action of o_Q on Λ^{i+2}. (More generally, the form Q gives a map of $V \to V^*$ and thus from $V \otimes V \to V \otimes V^*$ which we can consider as $\mathrm{Hom}(V,V)$ if V is finite dimensional and contained in $\mathrm{Hom}(V,V)$ otherwise. The image of $P_0 = \Lambda^2(V) \subset V \otimes V$ gives a map $P^0 \to \mathrm{Hom}(V,V)$ and induces an action of P_0 on the P_i which coincides with the bracket.) The subspace P_{-2} is always a central ideal of $P(C_Q(V))$, so that one can form the quotient algebra.

In case $\dim V = 4$, a direct computation shows that $[P_1,P_1] = 0$, so that one can make a graded Lie algebra out of $V \oplus \Lambda^2(V) \oplus \Lambda^3(V)$. In general $[\Lambda^i(V),\Lambda^j(V)] = 0$ for $i + j > n - 2$ if V is an n-dimensional vector space. Indeed, if we choose a basis of V, then any pair of expressions of the form $v_{k_1} \wedge \ldots \wedge v_{k_i}$ and $v_{l_1} \wedge \ldots \wedge v_{l_j}$ must have at least two v's in common, and thus their product drops by at least two in filtration degree, and the commutator drops by at least four. Thus $P_{-1} \oplus P_0 \oplus \ldots \oplus P_{n-3}$ forms a graded Lie algebra if we quotient out P_{-2}.

K. Tensor products with graded algebras

Let A and B be two graded associative algebras. We make their tensor product, $A \otimes B$, into a graded associative algebra by setting

$$(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j \tag{2.42}$$

and

$$(x_i \otimes y_j)(x_m \otimes y_n) = (-1)^{jm} x_i x_m \otimes y_j y_n. \tag{2.43}$$

It is easy to check that if, for example, both A and B are graded commutative, then so is $A \otimes B$. For instance, if $A = \Lambda(V)$ and $B = \Lambda(W)$ then $A \otimes B = \Lambda(V \oplus W)$. If $A = C_{Q_1}(V_1)$, the Clifford algebra of V_1 relative to Q_1 (which is not graded commutative) and $B = C_{Q_2}(V_2)$ then $A \otimes B = C_{Q_1 \oplus Q_2}(V_1 \oplus V_2)$. (Indeed, the map $x \oplus y \to x \otimes 1 + 1 \otimes y$ satisfies $(x \otimes 1 \oplus 1 \otimes y)^2 = Q_1(x) + Q_2(y)$, and hence, by the universal property of the Clifford algebra, gives a map of $C_{Q_1 \oplus Q_2}(V_1 \oplus V_2)$ into $A \otimes B$ which is easily seen to be an isomorphism.)

Let A be a graded commutative algebra and let L be a graded Lie algebra. We make $A \otimes L$ into a graded Lie algebra by setting

$$(A \otimes L)_n = \bigoplus_{i+j=n} A_i \otimes L_j \tag{2.44}$$

and

$$[a \otimes x, b \otimes y] = (-1)^{(\deg a)(\deg x)} ab \otimes [x,y]. \tag{2.45}$$

A straightforward verification shows that this multiplication is indeed graded anticommutative and that the graded version of Jacobi's identity does indeed hold. As a special case, if L is an old fashioned Lie algebra, we may consider it as a graded Lie algebra with $L = L_0$. Then if A is any graded commutative alegbra we can form the graded Lie algebra, $A \otimes L$. Thus, for example, $\Lambda(V) \otimes L$ becomes a $\mathbf{Z}$ graded Lie algebra for any auxiliary vector space, V, and $C_Q(V) \otimes L$ becomes a $\mathbf{Z}_2$ graded algebra. The technique of multiplying a graded Lie algebra, L, by a graded commutative algebra, A, and, particularly, by the exterior algebra has been extensively employed, first in the implementation of the "supergauge" program in dual models (see references to original papers and recent reviews in our discussion of example II.F), and more recently in supersymmetry (see for example Salam and Strathdee, 1974e). We discuss the latter in some detail in Sec. VI. Notice that the even terms in $A \otimes L$ which, as usual, form an ordinary Lie algebra, are combinations of products of even terms in A with even terms in L and odd terms in A with odd terms in L. Thus, for instance, if we take $A = \Lambda V$, where V is Minkowski space,

$$L_{\text{even}} \oplus (\Lambda^1(V) \otimes L_{\text{odd}}) \oplus (\Lambda^2(V) \otimes L_{\text{even}}) \oplus (\Lambda^3(V) \otimes L_{\text{odd}}) \oplus (\Lambda^4(V) \otimes L_{\text{even}})$$

is an ordinary Lie algebra. The Lie algebras constructed in this manner in the case of supersymmetry have substantial radicals and possess large dimensionalities as compared with the dimensions of L.

If A is a graded commutative associative algebra and B is a graded associative algebra, we can form a graded algebra $A \otimes B$. If L denotes the commutator algebra of B, then an immediate verification shows that the commutator algebra of $A \otimes B$ is $A \otimes L$.

Suppose that A is a graded commutative algebra which (with a shift of two in the grading) also carries a graded Lie algebra structure so that the Lie bracket acts as a graded derivation of the associative multiplication. Thus we assume that A possesses a graded commutative multiplication *and* a Lie multiplication such that

$$[A_k, A_l] \subset A_{k+l} \tag{2.46}$$

and that

$$[x, yz] = [x,y]z + (-1)^{kl} y[x,z]. \tag{2.47}$$

For instance, if C is $\mathbf{Z}_2$ graded and $\mathbf{Z}$ filtered, as in the case of the Clifford algebra with a graded commutative associated graded algebra $A = \text{Gr}C$, then A inherits such a structure as we indicated above. Suppose that B also carries the same structure. Then so does $A \otimes B$, where the associative structure is as we defined it above and where

$$[a_1 \otimes b_1, a_2 \otimes b_2] = (-1)^{b_1 a_2}([a_1,a_2] \otimes b_1 b_2 + a_1 a_2 \otimes [b_1,b_2]). \tag{2.48}$$

A straightforward, if rather tedious, computation with the signs shows that Jacobi's identity is satisfied and that the Lie bracket is a derivation of the associative multiplication. If we take A to be a graded commutative algebra and give it a trivial Lie bracket structure (so that all brackets are zero), and take $B = L$ to be a graded Lie algebra and give it a trivial commutative multiplication (so that all associative products are zero), then the above construction reduces to the previous construction of $A \otimes L$. Suppose that $A = \text{Gr}C$ and $B = \text{Gr}D$, where C and D are $\mathbf{Z}_2$ graded and $\mathbf{Z}$ filtered algebras with A and B carrying the inherited structures as indicated. Then $C \otimes D$ is a $\mathbf{Z}_2$ graded algebra and carries the obvious induced filtration whose associated graded algebra is $A \otimes B$. A direct verification shows that the induced Lie algebra structure on $A \otimes B$ is the one described above, i.e., that

$$\text{Gr}(A \otimes B) \sim \text{Gr}A \otimes \text{Gr}B = C \otimes D. \tag{2.49}$$

As an illustration of this construction, let C be the Clifford algebra of a one-dimensional vector space with nondegenerate quadratic form, so that C is generated by 1 and $\mathbf{e}$ with $1 \in C^0$ and $\mathbf{e} \in C^1$ and where $\mathbf{e}^2 = 1$. Its associated graded algebra is generated by 1 and e with the associative multiplication $1 \cdot e = e$ and $e^2 = 0$ and Lie multiplication $[1,1] = 0 = [1,e]$ and $[e,e] = 2$. Let D denote the ring of differential operators on a differentiable manifold, M. We shall think of D as a $\mathbf{Z}_2$ graded ring having only even elements and shall filter D by even degrees by letting $F^{2k}D$ consist of all differential operators of degree at most k. Then $B = \text{Gr}D$ consists of functions on T^*M which are polynomials in the cotangent variables. The functions which are homogeneous polynomials in the p's of degree k constitute B_{2k} and the inherited Lie bracket on B is just the Poisson bracket. Then $A \otimes B = \text{Gr}(C \otimes D)$ is just the Poisson algebra introduced in example G. Indeed, if we let $e' = (1/\sqrt{2})e$ and write the even elements in $A \otimes B$ as $1 \otimes f$ and the odd elements as $e' \otimes h = \phi$, we obtain exactly the bracket relations described in example G.

Notice that this suggests that an algebra of the form $C \otimes D$, where C is a Clifford algebra (or some other $\mathbf{Z}_2$ graded, $\mathbf{Z}$ filtered algebra), really lies behind the formalism of the commutation–anticommutation relations of field theory. In other words, in a certain sense we can regard the ring of differential operators as "quantum mechanics" whose "classical approximation" is given by the Poisson bracket algebra. In this sense it would appear that an algebra of the form $C \otimes D$ should enter into the "quantum mechanics" (Sternberg, 1964; Souriau, 1970) whose "classical approximation" is the algebra of graded commutators–anticommutators. Indeed the "quantum mechanics" would consist of using just the $\mathbf{Z}_2$ graded structure on $C \otimes D$ and let the ($\mathbf{Z}_2$) graded Lie algebra be the corresponding commutator algebra (with no notice taken of the filtration). It would be interesting to try to derive the relation between spin and statistics from this viewpoint.

L. Filling in the odd terms

Suppose we are given a graded Lie algebra, L^e, with only even terms $L^e = \oplus L_{2k}$. (Thus L^e is a Lie algebra in the classical sense.) We can always find a graded Lie algebra $L = \oplus L_k$ whose even terms coincide with the given terms and whose odd terms generate. Indeed, let us set

$$L_{2k+1} = L_{2k}, \quad k \neq -1 \tag{2.50}$$

and

$$L_{-1} = L_{-2} + \{u\}, \tag{2.51}$$

where we write an element of L_{2k+1} as θz, for $z \in L_{2k}$. We set

$$[L_{2k}, L_{2l+1}] = [L_{2k}, L_{2l}] \subset L_{2(k+l)+1}. \tag{2.52}$$

$$[z, \theta\omega] = \theta[z,\omega] \tag{2.53}$$

with the additional understanding that

$$[L_{2k}, u] = 0 = [u, L_{2l}] \tag{2.54}$$

if either k or $l = -1$. (In particular $[u,u] = 0$.)

If k, $d \neq -1$ we set

$$[L_{2k+1}, L_{2l+1}] = 0 \tag{2.55}$$

and similarly for $k = -1$ except that we set

$$[u,\theta z] = z. \tag{2.56}$$

This last equation shows that L is generated by its odd terms. Jacobi's identity is automatically verified for any expression involving at least two even terms. An expression involving at least two odd terms vanishes unless it contains exactly one u, and

$$\begin{aligned}[u,[w,\theta z]] &= [u,\theta[w,z]] = [w,z] \\ &= 0 + [w,z] \\ &= [[u,w],\theta z] + [w,[u,\theta z]]\end{aligned}$$

and

$$\begin{aligned}0 = [u,[\theta w,\theta z]] &= [w,\theta z] - [\theta w,z] \\ &= [[u,\theta w],z] - [\theta w,[u,\theta z]]\end{aligned}$$

so that Jacobi's identity holds. This algebra is, of course, very artificial.

III. A CRITERION FOR SIMPLICITY

We define the notion of subalgebra and ideal of a graded Lie algebra in the usual manner. Thus I is an ideal of L if $x \in I$ and $y \in L$ implies $[x,y] \in I$. We say that I is a graded ideal if the graded components of every element, x, of I belong to I, i.e., if $x = \oplus x_j$ and $x \in I$ then all the $x_j \in I$. We say that L is simple if it contains no nontrivial ideals and if its multiplication is not trivial. We say that it is graded simple if it does not have any graded ideals. We now present a criterion, which appears at first glance to be a bit complicated, but is conveniently verified in practice, which provides a sufficient condition for simplicity. We assert that if L verifies the following conditions then L is simple.

(a) L_0 contains an element, d, such that $[d,x] = kx$ for any $x \in L_k$, for all k.
(b) $[L_{-1},L_1] = L_0$.
(c) L contains no graded ideals lying entirely in $\oplus^{-2} L_j$.
(d) L_0 acts irreducibly on L_{-1}.
(e) If $k \geq 0$, then $[x_k,L_{-1}] = 0$ implies $x_k = 0$.

Notice that condition (a) implies that every ideal must be graded. Indeed, if $x = \oplus x_j$ lies in I so does $[d,x] = \oplus jx_j$ and so does $\oplus j^r x_j$ for any r. For r sufficiently large we can solve for the individual x_j in terms of the $j^s x_j$ $(s \leq r)$, which shows that $x_j \in I$. By condition (d) either $I_{-1} = L_{-1}$ or $I_{-1} = 0$. If $I_{-1} = L_{-1}$ then $L_0 = I_0$ by (b), but then, by (a), $d \in I$, which implies that $L_k = I_k$ for $k \neq 0$. Thus $L = I$. If $I_{-1} = 0$, then $[I_0,L_{-1}] = 0$, which, by (e), implies that $I_0 = 0$. Proceeding inductively, we see that $I_k = 0$ implies that $I_{k+1} = 0$ and hence that $I \supset \oplus^{-2} L_k$, which, by (c), implies that $I = 0$.

This immediately implies, for example, that the symplectic graded algebra, example G of Sec. II, is simple. It does not apply to the Gell-Mann (f,d) algebra which is directly seen to be simple. Also notice that the only role played by L_{-1} was that it was the first nonvanishing negative component. For instance, if L is a graded Lie algebra with only even components (and thus a classical Lie algebra), we need only replace L_{-1} by L_{-2} and L_1 by L_2 to obtain a useful criterion for simplicity of classical Lie algebras.

The spin-conformal algebra $\mathcal{W}$ of Wess and Zumino in Sec. II.H is simple. However, the subalgebra V suggested by Volkov and Akulov (1973) and Salam and Strathdee (1974a) (which is the algebra actually applied in physics to date), $L_0 + L_1 + L_2$, is not simple.

Notice that it is *not* true that a graded representation of a graded simple algebra is completely reducible. Indeed, we have already seen one example of this phenomenon in Sec. II.D. Here is another example: for any nondegenerate quadratic form, Q, on an n-dimensional vector space, v, the algebra

$$L = \overset{n-3}{\oplus} P_k[C_Q(V)]/P_{-2}(C_Q(V))$$

has no graded ideals. Indeed, P_0 is the orthogonal algebra which acts irreducibly on the spaces $P_i = \Lambda^{i+2}(V)$. If I were a nontrivial graded ideal then we must have $I_j = \Lambda^{j+2}(V)$ for some j. Since $[P_{-1},P_i] \neq 0$ for any $i > 0$ we conclude that $I_0 \neq 0$ and hence $I_0 = P_0$ and then that $I_k = P_k$ for all $k \leq n-3$.

On the other hand, we can construct the algebra $L' = L \oplus \{d\}$ where $d \in L_0'$ and $[d,x] = kx$ for $x \in L_k$. Then L is an ideal in L' and has no complementary ideal, i.e., L' is not completely reducible under L. This same example shows that if we define the analog of the Killing form, $f(a,b) = \mathrm{Tr}(\mathrm{ad}\, a)(\mathrm{ad}\, b)$ then the analog of Cartan's criterion does not hold. Indeed, it is clear that in any graded algebra we must have $f(L_i,L_j) = 0$ for $j \neq -i$. On the other hand, for odd elements we have $\mathrm{Tr}(\mathrm{ad}\, a)(\mathrm{ad}\, b) = 2\,\mathrm{Tr}\,\mathrm{ad}([a,b])$. For $a \in P_{-1}$ and $b \in P_1$ the element $[a,b] \in o(Q)$ is an infinitesimal orthogonal matrix and hence has zero trace. Thus, although L has no graded ideals, the Killing form f is degenerate.

It would be interesting to see which theorems from the classical theory of Lie carry over, and with what modifications, to graded Lie algebras.

IV. THE BIRKHOFF–WITT THEOREM FOR GRADED LIE ALGEBRAS

In the classical theory of Lie algebras, a central role is played by the universal enveloping algebra. Roughly speaking, if L is the Lie algebra of a Lie group G, then the ring of left invariant differential operators gives the enveloping algebra of L. (We give the more precise, and more algebraic definitions below.) Thus the various Casimir operators lie in the center of the enveloping algebra, and, in general, the universal enveloping algebra is important in representation theory. The first basic structural fact about enveloping algebras is the Birkhoff–Witt theorem. In this section we state and prove the corresponding theorem for graded Lie algebras.

Let L be a graded Lie algebra. In what follows, we shall regard L as $L_0 \oplus L_1$, where L_0 is the sum of the even graded pieces and L_1 is the sum of the odd graded pieces. This is not necessary, but it saves us some indices.

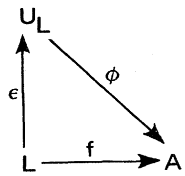
The universal enveloping algebra of L, called U_L, is an associative algebra with the following properties:

1. There is a canonical linear map $\epsilon: L \to U_L$ satisfying
$$\epsilon(x)\epsilon(y) - (-1)^{kl}\epsilon(y)\epsilon(x) = \epsilon([x,y]).$$

2. If $f: L \rightarrow A$ is any linear map from L to an associative algebra, A, satisfying

$$f(x)f(y) - (-1)^{kl}f(y)f(x) = f([x,y]),$$

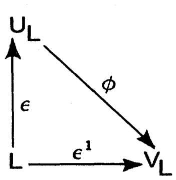
then there is a unique homomorphism $\phi: U_L \rightarrow A$ such that $f = \phi \circ \epsilon$, or such that



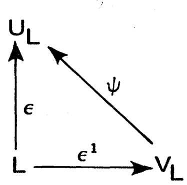
commutes.

Proposition: The universal enveloping algebra exists and is unique (up to isomorphism).

Proof: This follows standard arguments. To construct U_L, let T_L be the tensor algebra generated by L, and let $\mathscr{I}$ be the ideal generated by all expressions of the form $x \otimes y - (-1)^{kl}y \otimes x - [x,y]$. Then $U_L = T_1/\mathscr{I}$. The map ϵ takes x into the tensor x of degree 1. That U_L has the right property is easy to check: given f, the universal property of T_L says that there is a unique homomorphism $\psi: T_L \rightarrow A$ which extends f to T_L. But $\psi(x \otimes y - (-1)^{kl}y \otimes x - [x,y]) = f(x)f(y) - (-1)^{kl}f(y)f(x) - [x,y] = 0$, and so ψ takes $\mathscr{I}$ to 0. Thus ψ defines ϕ on U_L. For uniqueness, suppose that V_L and ϵ^1 also satisfy the definition. Then property 2 says that we have maps ϕ: $U_L \rightarrow V_L$ and $\psi = V_1 \rightarrow U_L$ such that

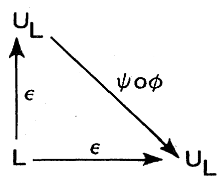


and



commute.

But then, combining these,



commutes. By the uniqueness of the extending homomorphism, $\psi \circ \phi =$ identity. Similarly, $\phi \circ \psi =$ identity, and so $U_L \cong V_L$.

The Birkhoff–Witt theorem gives more information about the structure of U_L. We need a modification of the Birkhoff–Witt theorem for ordinary Lie algebras, as given, for example, in Serre (1964). What follows (and most of what preceded) is an adaptation of Serre's treatment.

Let $\{x_i\}$ $(i \in I)$ be a basis of L; for convenience we assume that I is well-ordered and that the elements of L_0 precede those of L_1. A word of I will be an expression $M = (i_1, \cdots, i_m)$, with $i_1 \leq i_2 \leq \cdots \leq i_m$, and with $i_j < i_k$ if $x_{i_j} \in L_1$. (Of course, than $x_{i_k} \in L_1$, too). We then write $x_M = (x_{i_1}) \cdots \epsilon(x_{i_m})$. The length of M is $l(M) = m$.

Theorem. Assume L is free over k. Then the set of monomials x_M (where M is a word) is a basis for U(L).

Proof: (1) The monomials span U_L. To begin with, note that the monomials x_N (no ordering restriction on N) span U_L, because L generates U_L. If $i_1 > i_2$, then we can replace $\epsilon(x_{i_1})\epsilon(x_{i_2})$ by $\pm\epsilon(x_{i_2})\epsilon(x_{i_2}) + \epsilon([x_{i_1},x_{i_2}])$. Similarly, if $x_i \in L_1$, then $\epsilon(x_i)\epsilon(x_i) = \frac{1}{2}\epsilon([x_i,x_i])$. By repeating these operations, we are able to see that any x_N can be written as a linear combination of x_M's, where the M's are words. That proves half the theorem.

(2) The monomials are linearly independent. We first let V be the free k module with basis $\{z_M\}$, where M runs through all words. If $i \in I$, $M = (i,i_1,\ldots i_m)$, and $i < i_1$ (or $i = i_1$ and $x_{i_1} \in L_0$), we shall say $i \leq M$ and let $iM = (i,i_1,\ldots i_m)$.

We shall show that V can be made into an L-module with $x_i z_M = z_{iM}$ for $i \leq M$. Given this, the linear independence follows easily. For V is then a U_L module as well. Consider $z_\emptyset$ ($\emptyset =$ empty set). Then $x_M z_\emptyset = z_M$, as induction on $l(M)$ shows. [If $l(M) = 0$, then $x_M = 1$ and the result holds; otherwise, $M = iN$, $i \leq N$, and $x_M z_\emptyset = x_i x_N z_\emptyset = x_i z_N = z_{iN}$ by induction.] So if $\Sigma c_M x_M = 0$, then $0 = \Sigma c_M x_M z_\emptyset = \Sigma c_M z_M$, and hence $c_M = 0$ because the z_M are linearly independent.

Everything, therefore, depends on defining the module structure. We need to define $x_i z_M$ for all i, m. Let $\alpha_i = \deg x_i$. We assume inductively that if $l(N) \lesssim l(M)$, then $x_j z_N$ is defined for all $j \in I$ and $x_j z_N$ is a k-linear combination of z_i's with $l(L) \leq l(N) + 1$, and that if $j < i$, then $x_j z_M$ is defined. Then define (where $\alpha_i = \deg x_i$)

$$\begin{aligned} x_i z_M &= z_{iM} \text{ if } i \leq M; \\ &= (-1)^{\alpha_i\alpha_j}x_j(x_i x_N) + [x_i,x_j]z_N \text{ if } M = jN \\ &\quad \text{with } j < i; \\ &= \tfrac{1}{2}[x_i,x_i]z_N \text{ if } M = iN \text{ and } x_i \in L. \end{aligned}$$

These expressions on the right are well-defined and continue the inductive hypotheses. Thus $x_i z_M$ is now always defined.

We need to show that $x_i x_j z_N - (-1)^{\alpha_i\alpha_j}x_j x_i z_N = [x_i x_j]z_N$. There are a number of cases. We may always assume $i \geq j$ (and $i > j$ if $x_j \in L_1$), by symmetry. We shall not give all the cases here, but enough to make the argument plain.

(1) $j \leq N$ and $i > j$ (or $i = j$ and $x_i \in L_1$); then the result comes from the second line of the definition.

(2) $N = kL$ and $i > j > k$. We need to show that

$$(*) x_i x_j x_k z_L - (-1)^{\alpha_i\alpha_j}x_j x_i x_k z_L = [x_i,x_j]x_k z_L.$$

By induction, we know that (*) holds if we permute i, j, and k cyclically. Also, by induction on $l(N)$, we may assume that

$$x_k x_l z_L = (-1)^{\alpha_k\alpha_l}x_l x_k z_L + [x_k,x_l]z_L, \forall k,l \in I.$$

So (*) becomes

$$\begin{aligned} (*1) x_i x_j x_k z_L - (-1)^{\alpha_i\alpha_j}x_j x_i x_k z_L &= [[x_i,x_j],x_k]z_L \\ &+ (-1)^{\alpha_k(\alpha_i+\alpha_j)}x_k x_i x_j z_L - (-1)^{\alpha_i\alpha_j+\alpha_j\alpha_k+\alpha_k\alpha_i}x_k x_j x_i z_l. \end{aligned}$$

We know that the versions of (*1) with i, j, k permuted cyclically (*2 and *3, say) are true. To prove (*1), it suffices to show that some linear combination of (*1), (*2), and (*3) gives an identity. Of course, (*1) must enter nontrivially.

But $(-1)^{\alpha_i\alpha_k}(*1)+(-1)^{\alpha_j\alpha_i}(*2)+(-1)^{\alpha_k\alpha_j}(*3)$ is of the form

$$Q=Q+((-1)^{\alpha_i\alpha_k}[[x_i,x_j],x_k]+(-1)^{\alpha_j\alpha_k}[[x_k,x_i],x_j]+(-1)^{\alpha_i\alpha_j}[[x_j,x_k],x_i])z_i.$$

This last is true, by the Jacobi identity. So the result works for (2).

(3) There remain cases where some of i, j, k are equal and the basis element is in L_1. For instance, if $i=j=k$, then $[x_i,x_j]x_kz_L=2x_kx_kx_kz_L$ and the result is clear. The other cases are similar. This proves the theorem.

The Birkhoff–Witt theorem is usually given in a somewhat different form. We need a number of definitions first.

Let U_L^n be the subspace of U_L generated by all products of the form $\epsilon(x_1)\cdots\epsilon(x_m)$, where $m\le n$, let $\mathrm{Gr}_n(U_L)=U_L^n/U_L^{n-1}$, and let

$$\mathrm{Gr}(U_L)=\bigoplus_{n=0}^{\infty}\mathrm{Gr}_n(U_L).$$

In some sense, $\mathrm{Gr}_n(U_L)$ consists of expressions "purely" of length (n_1). Since $U_L{}^mU_L{}^n\subseteq U_L{}^{m+n}$, we can define (by passage to the quotient) a multiplication on $\mathrm{Gr}(U_L)$ with $\mathrm{Gr}_m(U_L)\subseteq\mathrm{Gr}_{m+n}(U_L)$. Clearly $\mathrm{Gr}(U_L)$ is generated by the image of $\epsilon(L)$ (in U_L). In fact, $\epsilon(L)=U_L{}^1$.

If x or y is in L_0, then $\epsilon(x)\epsilon(y)-\epsilon(y)\epsilon(x)=\epsilon([x,y])\in U_L{}^1$, and thus $\epsilon(x)$ and $\epsilon(y)$ commute modulo $U_L{}^1$. Thus if we let $\epsilon^{\sim}(x)$ be the image of $\epsilon(x)$ in $\mathrm{Gr}(U_L)$, $\epsilon^{\sim}(x)$ and $\epsilon^{\sim}(y)$ commute. Similarly, if x and y are in L_1, then $\epsilon^{\sim}(x)$ and $\epsilon^{\sim}(y)$ anticommute.

Now let $S(L_0)$ be the symmetric algebra over L, $\Lambda(L_1)$ the exterior algebra over L_1, and $\mathcal{V}$ their tensor product. There is a natural map ϕ from $\mathcal{V}$ to $\mathrm{Gr}(U_L)$. For, according to general nonsense, there is a natural map from $T(L)$ to $\mathrm{Gr}(U_L)$ [$T(L)$ is the tensor algebra over L], and the above commutation and anticommutation relations show that it factors through $\mathcal{V}$. Moreover, ϕ is onto, since $\phi(x_i)=\epsilon^{\sim}(x_i)$ and the elements $\epsilon^{\sim}(x_i)$ generate $\mathrm{Gr}(U_L)$.

Theorem (Birkhoff–Witt). ϕ is an isomorphism if L is k-free.

Proof. We need to show that ϕ is injective. As before, let $x_M=\epsilon(x_{i_1})\cdots\epsilon(x_{i_m})$ if $M=(i_1,\ldots,i_m)$ is a word; let $x_M^{\sim}$ be its image in $\mathrm{Gr}(U_L)$. To say that ϕ is injective is to say that the only relation of the form

$$\sum_{l(M)=n}C_Mx_M=0(\mathrm{mod}U_{n-1}L)$$

is the trivial one. But because the ordered monomials span $U(L)$, this last statement amounts to saying that the only relation of the form

$$\sum_{l(M)=n}C_Mx_M=\sum_{l(M)\le n}C_Mx_M$$

has the left-hand side 0. This is a consequence of the previous theorem.

For ordinary Lie algebras, the filtration on U gives rise to a commutative algebra structure on Gr U and hence also a Poisson bracket structure for which $L\sim\mathrm{Gr}_0U$. It would be nice to have an analogous theorem for graded algebras.

V. A VERY BRIEF SKETCH OF DEFORMATION THEORY

Here we only outline an illustrative example of the subject in very bare form. We refer the reader to the article of Nijenhuis and Richardson (1966) for a very readable exposition of this example and to their paper (1964) for the presentation of the theory in its general form. Suppose we start with a vector space, W, and ask for a description of all possible (classical) Lie algebra structures that W can carry. Thus we are looking for all possible maps $\mu: W\times W\to W$ which are antisymmetric and which satisfy Jacobi's identity. To say that μ is antisymmetric means that

$$\mu\in\mathrm{Hom}\,(W\wedge W,W)=\mathrm{Der}_1\,(\Lambda W).$$

The Jacobi condition says that $[\mu,\mu]=0$ in the F–N algebra of W. Thus the set of all Lie algebra structures consists of the algebraic variety C, of all solutions of the system of homogeneous quadratic equations $[\mu,\mu]=0$ in $\mathrm{Der}_1\,(\wedge W)$. Now we are really interested in classifying such structures up to isomorphism. This means the following: Let A be any nonsingular linear transformation of W. Then A acts on $\mathrm{Der}_1\,(\wedge W)$ by sending $\mu\rightsquigarrow\mu_A$, where $\mu_a(x,y)=A\mu(A^{-1}x,A^{-1}y)$. We wish to regard μ and μ_A as the same.

Thus we are interested in the structure of the "orbits" of C under the action of the general linear group, $G(V)$. Deformation theory studies this problem from an infinitesimal point of view: Suppose we are given an analytic curve

$$\mu_t=\mu+t\varphi_1+t^2\varphi_2+\cdots \tag{5.1}$$

in $\mathrm{Der}_1\Lambda W$. We wish to find conditions for this curve to lie on C. We also wish to regard as "trivial" a curve of the form

$$\mu_t=\mu_{A(t)}, \tag{5.2}$$

where $A(t)$ is a curve in $GL(V)$ with $A(0)=id$.

The condition for μ_t to lie in C is $[\mu_t,\mu_t]\equiv 0$. Expanding in terms of t we get

$$[\mu_t,\mu_t]\equiv 2t[\mu,\varphi_1]+t^2([\varphi_1,\varphi_1]+2[\mu,\varphi_2])+\cdots. \tag{5.3}$$

Since $[\mu,\mu]=0$, left multiplication defines a d operator on $\mathrm{Der}\,(\Lambda V)$ which we denote by D. The first condition on φ, asserts that

$$D\varphi_1=0, \tag{5.4}$$

the second, that

$$D\varphi_2=-\tfrac{1}{2}[\varphi_1,\varphi_1], \tag{5.5}$$

and, in particular, that $[\varphi_1,\varphi_1]$ is a coboundary.

If $A(t)=id+ta+\cdots$, where $a\in\mathrm{Hom}(N,W)=\mathrm{Der}_0\,\Lambda W$ then a direct computation shows that

$$\mu_{A(t)}=\mu+t[a,\mu]+\cdots.$$

Thus the first order triviality condition on μ_t is given as

$$\varphi_1=-D_\mu a. \tag{5.6}$$

Thus the possible "first-order" nontrivial deformations correspond to the cohomology space $H^1(\Lambda W,D_\mu)$. The second order condition, that $[\varphi_1,\Psi_1]$ be a coboundary, is independent of the choice of Ψ_1 in its cohomology class. If

$$H^2(\Lambda W,D_\mu)=\{0\} \tag{5.7}$$

then every "first-order" derivation can be extended to a "second-order derivation." Actually one can justify this whole formal theory by use of the implicit function theorem to relate the H^1 and H^2 to the study of the orbit through μ. We refer the reader to Nijenhuis and Richardson (1964, 1966).

VI. SUPERSYMMETRY

A. The GLA $\mathcal{W}$ and $\mathcal{V}$

The first introduction of a GLA as a supersymmetry of space–time (i.e., as a symmetry containing the Poincaré group) is due to Volkov and Akulov (1973). They adjoined to the Poincaré algebra $\mathcal{P}$ a set of four odd generators, behaving like a Majorana spinor. This is isomorphic to the $L_0\oplus L_1\oplus L_2$ subalgebra $\mathcal{V}$ of the spin-conformal algebra of example 2*H* which we shall denote by $\mathcal{W}$. Volkov and Akulov were exploring the hypothesis that the neutrino's masslessness might indicate that it is a Goldstone particle, necessary to the (nonlinear) realization of an exact symmetry of the physical world, a hypothesis suggested earlier by one of the present authors (YN), and which had failed because of the statistics issue (Joseph, 1972; Bella 1973). They therefore had to introduce conserved generators which do not destroy the vacuum, and behave like the physical neutrinos under the Poincaré group, i.e., spinors. To preserve Fermi statistics, the new generators were now also required to anticommute. The GLA $\mathcal{V}$ thus had fourteen generators (this came out of the requirement of algebraic closure), was not simple, and had to be realized nonlinearly. For ψ the neutrino field,

$$\delta\psi=\zeta+ia(\bar{\zeta}\gamma_\mu\psi)\partial_\mu\psi, \tag{6.1}$$

where a is a constant of dimension (-4), i.e., the fourth power of a length, and ζ is a four-spinor (Majorana) anticommuting parameter. Considering that the neutrino does not seem to fit a Majorana description, the actual physical content would appear rather speculative, but the algebraic innovation remains interesting. Apart from the nonlinear realization, the $\mathcal{V}$ algebra is indeed the one that has been extensively applied as a supersymmetry, after its reintroduction by Salam and Strathdee (1974a) as the physically interesting subalgebra of the spin-conformal $\mathcal{W}$ GLA of Sec. II.H, discovered by Wess and Zumino (1974a). It is interesting that Salam and Strathdee even returned to the question of a possible Goldstone role for the neutrino (Salam and Strathdee, 1974c).

We now turn to a more detailed study of $\mathcal{W}$.

We preserve the energy metric of Sec. II.H, $g^{00}=1$, $g^{11}=g^{22}=g^{33}=-1$, and $g^{\mu\nu}=0$ for $\mu\neq\nu$. For the convenience of the reader who is accustomed to the conventions common in the physical literature, we shall rewrite the bracket relations of the observables associated to the spin conformal algebra in terms of a standard basis of Hermitian operators. The physical algebraic system consists in the conformal algebra $SU(2,2)$ for the even gradation

$$\begin{aligned}
[J^{\mu\nu},J^{\rho\sigma}]&=+ig^{\mu\sigma}J^{\nu\rho}-ig^{\mu\rho}J^{\nu\sigma}+ig^{\nu\rho}J^{\mu\sigma}-ig^{\nu\sigma}J^{\mu\rho},\\
[J^{\mu\nu},P^\lambda]&=-ig^{\mu\lambda}P^\nu+ig^{\nu\lambda}P^\mu,\\
[P^\mu,P^\nu]&=0,\\
[J^{\mu\nu},K^\lambda]&=-ig^{\mu\lambda}K^\nu+ig^{\nu\lambda}K^\mu,\\
[K^\mu,K^\nu]&=0,\\
[K^\mu,P^\nu]&=-2ig^{\mu\nu}D-2iJ^{\mu\nu},\\
[D,P^\mu]&=iP^\mu,\\
[D,K^\mu]&=-iK^\mu,\\
[D,J^{\mu\nu}]&=0,
\end{aligned}\tag{6.2}$$

where $J^{\mu\nu}$ are the Lorentz group generators, P^μ the translations, K^μ the pure conformal transformations, and D the dilation operator. Note that the indexing for the gradation is provided by doubling the eigenvalue of D, so that the X of Sec. II.H correspond to K^μ, and the W to P^μ. In the context of adjoining $L_{-1}\oplus L_1$ we are led to add a sixteenth (scalar) operator to the even set. This will be E, corresponding as shown in Sec. II.H to the infinitesimal action of $e^{i\phi}\,1$ (ϕ real), acting trivially on the even generators,

$$[E,J_{\mu\nu}]=[E,P^\mu]=[E,K^\mu]=[E,D]=0. \tag{6.3}$$

To construct $L_{-1}\oplus L_1$ in terms of physical operators, we use the Dirac matrices

$$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu},\quad \sigma_{\mu\nu}=(i/2)[\gamma_\mu,\gamma_\nu] \tag{6.4}$$

and the special matrices γ_5, β, and C:

$$\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3=(1/4!)\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \tag{6.5}$$

$$\gamma_\mu{}^+=\beta\gamma_\mu\beta^{-1}, \tag{6.6}$$

$$\begin{aligned}
&-\gamma_\mu{}^\sim=C^{-1}\gamma_\mu C,\quad C^+=C^{-1},\quad C^\sim=-C,\quad C\gamma_5=\gamma_5{}^\sim C,\\
&C\gamma_5{}^\sim=\gamma_5 C.
\end{aligned}\tag{6.7}$$

We have to distinguish here between *, denoting complex conjugation, and +, which now stands for Hermitian conjugation; $\sim$ denotes transposition. To form the adjoint spinor we utilize β,

$$\bar{\psi}=\psi^+\beta. \tag{6.8}$$

We use a Majorana representation, i.e., for $\mu,\nu=1,2,3,0$ we have

$$\begin{aligned}
&\gamma^{\mu *}=-\gamma^\mu;\quad \sigma^{\mu\nu *}=-\sigma^{\mu\nu};\quad \gamma^{\mu +}=g^{\mu\mu}\gamma^\mu;\\
&\sigma^{\mu\nu +}=g^{\mu\mu}g^{\nu\nu}\sigma^{\mu\nu};\quad \gamma_5{}^+=\gamma_5{}^\sim=-\gamma_5;\\
&(\gamma_5\gamma^\mu)^+=g^{\mu\mu}\gamma_5\gamma^\mu;\quad (\gamma_5\sigma^{\mu\nu})^+=-g^{\mu\mu}g^{\nu\nu}\gamma_5\sigma^{\mu\nu}.
\end{aligned}\tag{6.9}$$

We find $\beta=b\gamma_0$, $C=c\gamma_0$. We choose $b=1$, $c=-1$. One possible representation is

$$\begin{aligned}
&\gamma_1=-i\rho_1\sigma_1,\quad \gamma_2=-i\rho_3,\quad \gamma_3=-i\rho_1\sigma_3,\\
&\gamma_0=-\rho_2,
\end{aligned}\tag{6.10}$$

which ensures $-\gamma_0 C=1$; $\gamma_5=i\rho_1\sigma_2$.

Note the following reality and symmetry properties:

$$\begin{aligned}
&\gamma_0{}^+=\gamma_0;\quad (\gamma_0\gamma_\mu)^+=\gamma_0\gamma_\mu;\quad (\gamma_0\sigma_{\mu\nu})^+=\gamma_0\sigma_{\mu\nu};\\
&(\gamma_0\gamma_5\sigma_{\mu\nu})^+=\gamma_0\gamma_5\sigma_{\mu\nu};\quad (\gamma_0\gamma_5\gamma_\mu)^+=-\gamma_0\gamma_5\gamma_\mu;\\
&(\gamma_0\gamma_5)^+=\gamma_0\gamma_5\\
&(\gamma_\mu C)^*=\gamma_\mu C;\quad (\sigma_{\mu\nu}C)^*=\sigma_{\mu\nu}C;\quad (\gamma_5\sigma_{\mu\nu}C)^*=\gamma_5\sigma_{\mu\nu}C;\\
&(\gamma_5\gamma_\mu C)^*=\gamma_5\gamma_\mu C;\quad C^*=-C;\quad (\gamma_5 C)^*=-\gamma_5 C\\
&(\gamma_\mu C)^\sim=\gamma_\mu C;\quad (\sigma_{\mu\nu}C)^\sim=\sigma_{\mu\nu}C;\quad (\gamma_5\sigma_{\mu\nu}C)^\sim=\gamma_5\sigma_{\mu\nu}C;
\end{aligned}$$

$$C^{\sim} = -C; \quad (\gamma_5 C)^{\sim} = -\gamma_5 C; \quad (\gamma_5\gamma_\mu C)^{\sim} = -\gamma_5\gamma_\mu C. \tag{6.11}$$

The charge-conjugate spinor is given by

$$\psi \to C\bar{\psi}^{\sim} = \psi^c, \tag{6.12}$$

and for our choice of phases $\psi^c = \psi^*$. We also note that a Majorana spinor is one which is equal to its charge conjugate,

$$\psi_M = \psi_M{}^c \text{ and with our phases, } \psi_M = \psi_M{}^*,$$
$$\bar{\psi}_M = \psi^{\sim}\gamma_0 \tag{6.13}$$

which reduces the number of its complex components to two or real components to four in this choice; we shall transform later to two complex ones, which can then be identified with the column and row vectors introduced in Sec. II above. From the symmetry properties in Eq. (6.11) we find, for ψ and χ anticommuting Majorana spinors:

$$\bar{\psi}\chi = \bar{\chi}\psi; \quad \bar{\psi}\gamma_\mu\chi = -\bar{\chi}\gamma_\mu\psi; \quad \bar{\psi}\sigma_{\mu\nu}\chi = -\bar{\chi}\sigma_{\mu\nu}\psi;$$
$$i\bar{\psi}\gamma_5\gamma_\mu\chi = i\bar{\chi}\gamma_\sigma\gamma_\mu\psi; \quad \bar{\psi}\gamma_\sigma\chi = \bar{\chi}\gamma_\sigma\psi. \tag{6.14}$$

The odd generators of $\mathcal{W}$ are Q_α and R_β ($\alpha,\beta = 1\ldots 4$ in Dirac spinor space) and are Majorana spinors, thus involving two complex or four real functions each. The even–odd Lie brackets are the commutators (Wess and Zumino, 1974a; Corwin, Ne'eman, and Sternberg, 1974; Dondi and Sohnius, 1974)

$$\begin{aligned}
[J^{\mu\nu},Q_\alpha] &= -\tfrac{1}{2}(\sigma^{\mu\nu})_{\alpha\beta}Q_\beta,\\
[J^{\mu\nu},R_\beta] &= -\tfrac{1}{2}(\sigma^{\mu\nu})_{\alpha\beta}R_\beta,\\
[P^\mu,Q_\alpha] &= [K^\mu,R_\alpha] = 0,\\
[K^\mu,Q_\alpha] &= -(\gamma_5\gamma^\mu)_{\alpha\beta}R_\beta,\\
[P^\mu,R_\alpha] &= -(\gamma_5\gamma^\mu)_{\alpha\beta}Q_\beta,\\
[D,Q_\alpha] &= (i/2)Q_\alpha,\\
[D,R_\alpha] &= -(i/2)R_\alpha,\\
[E,Q_\alpha] &= 3i(\gamma^5)_{\alpha\beta}Q_\beta,\\
[E,R_\alpha] &= -3i(\gamma^5)_{\alpha\beta}R_\beta,
\end{aligned} \tag{6.15}$$

and the odd–odd brackets consist of the anticommutators,

$$\begin{aligned}
\{Q_\alpha,R_\beta\} &= -(\gamma_5\sigma_{\mu\nu}C)_{\alpha\beta}J^{\mu\nu} - i(C)_{\alpha\beta}E + 2i(\gamma_5 C)_{\alpha\beta}D,\\
\{Q_\alpha,Q_\beta\} &= -2(\gamma_\mu C)_{\alpha\beta}P^\mu,\\
\{R_\alpha,R_\beta\} &= -2(\gamma_\mu C)_{\alpha\beta}K^\mu.
\end{aligned} \tag{6.16}$$

We can also introduce "adjoint" spinors $\bar{Q}_\alpha$ and $\bar{R}_\alpha$ as per Eq. (6.11). This is especially useful in view of further generalizations in which we shall introduce internal degrees of freedom. For $su(n)$, $n \geq 3$, the covariant and contravariant representations are not equivalent [**3** and **3*** in $su(3)$, etc.] and this will require distinguishing between Q_α and $\bar{Q}_\alpha$. The bracket relations are

$$\begin{aligned}
\{Q_\alpha,\bar{R}_\beta\} &= (\gamma_5\sigma_{\mu\nu})_{\alpha\beta}J^{\mu\nu} - 2i(\gamma_5)_{\alpha\beta}D + i\delta_{\alpha\beta}E,\\
\{Q_\alpha,\bar{Q}_\beta\} &= 2(\gamma_\mu)_{\alpha\beta}P^\mu,\\
\{R_\alpha,\bar{R}_\beta\} &= 2(\gamma_\mu)_{\alpha\beta}K^\mu.
\end{aligned} \tag{6.17}$$

B. Representations of $\mathcal{V}$

Working with the conformal group as a symmetry implies massless particles (provided the symmetry is not spontaneously broken by a Goldstone boson). We first construct some physically relevant representations of the subalgebra corresponding to mass zero.

The generators of the subalgebra $\mathcal{V}$ are:

$$\begin{aligned}
&J^{\mu\nu} \in L_0,\\
&Q_\alpha \in L_1,\\
&P^\mu \in L_2,
\end{aligned} \tag{6.18}$$

i.e., the even gradation corresponds to the Poincaré algebra. For massless particles, the helicity λ (taken here with the same sign as J^{12}) is the only remaining quantum number in the little group of the Poincaré group.

For $p^+ = p^0 + p^3 \neq 0$, $p^- = p^0 - p^3 = 0$, $p^1 = p^2 = 0$ on the states, the little group is generated by $[(J^{12}, J^{23} - J^{20}, J^{31} - J^{01})]$.

In the odd set of (6.16), only the Q_α are in the little group. Using for example the representation (6.10) we find that the only two nonvanishing Q_α are Q_1 and Q_4. These are not J^{12} (helicity) eigenvectors, and we recombine them into helicity $+\frac{1}{2}$ and $\frac{1}{2}$ operators.

$$\begin{cases}
(Q_1 - iQ_4)/\sqrt{2} \equiv x, \quad (Q_1 + iQ_4)\sqrt{2} \equiv y\\
\{x,y\} = e, \quad \text{for} \quad 2P^+ \equiv e\\
[h,x] = x, \quad [h,y] = -y \quad \text{for} \quad 2J^{12} \equiv h.
\end{cases} \tag{6.19}$$

This is just the GLA of our Sec. II.A. Its defining 2×2 representation acts on a vector space containing one fermion and one boson state (helicities $\frac{1}{2}$, 0 or any $(n + \frac{1}{2}$; $n/2)$.

To discuss the representations, we notice that $\alpha_1 = x + y$, $\alpha_2 = -i(x - y)$ define the Clifford algebra C_2, as can be computed from our defining brackets. It is a $2^2 = 4$ dimensional vector space with basis α_1, α_2, $\alpha_1\alpha_2$, e. Its only irreducible representation is the defining set of 2×2 matrices (see for example Boerner, 1963). However Q_1 and Q_4 are not parity eigenvectors, as can be seen by using Eq. (6.10) in the parity transformation,

$$Q_\alpha \to \eta_{\mathcal{P}}(\gamma_0)_{\alpha\beta}Q_\beta \tag{6.20}$$

which has (choosing $\eta_{\mathcal{P}} = 1$ here)

$$Q_1 \to -iQ_3, \quad Q_4 \to iQ_2.$$

To conserve parity, we therefore adjoin a 2-space representing states with $P^- \neq 0$, $P^+ = P^1 = P^2 = 0$. We find,

$$(Q_3 + iQ_2)/\sqrt{2} \equiv x', \quad (Q_3 - iQ_2)/\sqrt{2} \equiv y', \quad 2P^- \equiv e',$$
$$2J^{12} \equiv h'. \tag{6.21}$$

This time, we pick $n = -1$, getting eigenvalues $(0,-\frac{1}{2})$ or $[-n/2, -(n+1)/2]$ for the helicities λ.

The Fermi states, being helicity eigenstates, have to consist of combinations of the type $(\psi_1 \mp i\psi_4)$ of the real components of a Majorana "neutrino," just as we calculated for the Q_α in Eq. (6.19). Parity thus consists in complex-conjugation, leading to the conjugate space. The bosons thus also can be written as $(u \pm iv)$, u a scalar and v a pseudoscalar. We still return to this simplest of all representations when we construct appropriate "superfields," i.e., field representations of supersymmetry. Note however that the findings of Volkov and Soroka (1973) fit within this picture: the massless graviton, with $\lambda = \pm 2$, gets a companion with $\lambda = \pm\frac{3}{2}$.

Actually, there have been to date very few applications of the full algebra $\mathcal{W}$. Instead, following Salam and Strathdee (1974a), the nonsimple Volkov–Akulov subalgebra $\mathcal{V}$ was used in its linear realizations ("supersymmetry"). However, any Lagrangian which is invariant under that algebra, and which is in addition made invariant under the conformal group (by making all masses and all dimensional couplings vanish), will also be invariant under the $\mathcal{W}$ GLA.

Taking here the $M \geq 0$ case, we use the rest frame to find the "little" GLA. First we note that the Eqs. (6.16) and (6.17) reduce to

$$\begin{cases} \{Q_\alpha,Q_\beta\} = 2\delta_{\alpha\beta}M \\ \text{or} \\ \{Q_\alpha,\bar{Q}_\beta\} = 2(\gamma_0)_{\alpha\beta}M. \end{cases} \tag{6.22}$$

The first bracket defines C_4, the fourth-order Clifford algebra (dimensionality $2^4 = 16$); it is just the algebra of Dirac γ^μ matrices in a Euclidean metric. Its only representation is in four-dimensional matrices (Boerner, 1963). We thus know that all $M \neq 0$ representations of $\mathcal{V}$ will reduce into four-dimensional subspaces, just as the $M = 0$ ones worked in doubled two-spaces.

To get the "little" GLA in a more familiar form, we diagonalize γ_0. In our representation (6.10) $\gamma_0 = -\rho_2$, so that the appropriate unitary operator U will act in ρ space only, with

$$U(-\rho_2)U^{-1} = \rho_3. \tag{6.23}$$

This will rearrange our (Majorana spinor) odd operators

$$Q_\alpha \rightarrow UQ_\alpha$$

which has as components

$$\begin{aligned} Q_{\rm I} &= (Q_1 + iQ_3)/\sqrt{2} \\ Q_{\rm II} &= (Q_2 + iQ_4)/\sqrt{2} \end{aligned} \tag{6.24}$$

and their Hermitian (here just complex) conjugates $Q_{\rm I}^+$, $Q_{\rm II}^+$. Transforming the matrices $(-\frac{1}{2}\sigma^{ij}) \rightarrow U(-\frac{1}{2}\sigma^{ij})U^{-1}$ we get the appropriate Clebsch–Gordan coefficients for the angular momentum commutators in (6.15). We find that J^{23} is now diagonal,

$$\begin{aligned} &[J^{23},Q_{\rm I}] = \tfrac{1}{2}Q_{\rm I}, \quad [J^{23},Q_{\rm II}] = -\tfrac{1}{2}Q_{\rm II}, \\ &[J^{23},Q_{\rm I}^+] = -\tfrac{1}{2}Q_{\rm I}^+, \quad [J^{23},Q_{\rm II}^+] = \tfrac{1}{2}Q_{\rm II}^+, \\ &[J^{12},Q_{\rm I}] = -Q_{\rm II}, \quad [J^{12},Q_{\rm II}] = -Q_{\rm I}, \\ &[J^{12},Q_{\rm I}^+] = Q_{\rm II}^+, \quad [J^{12},Q_{\rm II}^+] = Q_{\rm I}^+, \\ &[J^{31},Q_{\rm I}^+] = iQ_{\rm II}, \quad [J^{31},Q_{\rm II}] = -iQ_{\rm I}, \\ &[J^{31},Q_{\rm I}^+] = iQ_{\rm II}^+, \quad [J^{31},Q_{\rm II}^+] = -iQ_{\rm I}^+. \end{aligned} \tag{6.25}$$

The $Q_{A={\rm I,II}}$ and Q_A^+ thus form independent 2-spinors. They fulfill

$$\{Q_A,Q_B^+\} = 2\delta_{AB}M, \quad \{Q_A,Q_B\} = 0, \quad \{Q_A^+,Q_B^+\} = 0. \tag{6.26}$$

The brackets (6.25), (6.26) together with the angular momentum commutation relations define the little GLA:

$$\begin{aligned} &J^{ij},M \in L_0 \ (d = 4), \\ &Q_A \in L_{-1} \ (d = 2), \\ &Q_A^+ \in L_1 \ (d = 2). \end{aligned}$$

Since γ_0 is diagonal in this representation, the spinors Q_A and Q_A^+ are parity eigenstates with opposite eigenvalues (note that $\eta_{\mathcal{P}}$ will have to be i or $-i$ here).

An irreducible representation of the little GLA is thus obtained [Salam and Strathdee (1974b)] operating with the Q_A and Q_A^+ on the $(2j + 1)$ dimensional carrier space of any representation (j,M) of the (Wigner) little group of the Poincaré group. Because of (6.26), and taking (j_3 corresponds here to the "new" J^{23} direction)

$$Q_A|j,j_3,\chi_p,M\rangle = 0 \tag{6.27}$$

in analogy to an annihilation operator, we have four possible actions of

$$\begin{aligned} Q_A^+: \quad &Q_{\rm I}^+|j,j_3,\chi_p,M\rangle, \quad Q_{\rm II}^+|j,j_3,\chi_p,M\rangle, \\ &Q_{\rm I}^+Q_{\rm II}^+|j,j_3,\chi_p,M\rangle \quad \text{and} \quad |j,j_3,\chi_p,M\rangle. \end{aligned}$$

The first two change the spin, j_3, parity, and statistics of the states according to (6.25) and the eigenvalues of γ_0. The $Q_{\rm I}^+Q_{\rm II}^+$ action preserves j_3 and j but inverts the parity. We thus have a $4(2j + 1)$ dimensional Fock space, with subspaces

$$\begin{aligned} &|j,j_3,-\chi_p,M\rangle, \quad |j,j_3,\chi_p,M\rangle, \quad |j + \tfrac{1}{2}, j_3, \chi_p\eta_p, M\rangle, \\ &|j - \tfrac{1}{2}, j_3, \chi_p\eta_p, M\rangle. \end{aligned}$$

Notice that fermions and bosons have the same mass.

These rest states are then boosted to any $\mathbf{p}$ by a Lorentz transformation $U(L_{\mathbf{p}})$. The action of Q_A and Q_A^+ on the boosted states can be derived from our knowledge of the spinor behavior of the Q_A and Q_A^+ under Lorentz transformations.

An additional result derived by Salam and Strathdee relates to the action of the Q_α on two-particle states. Since we have to preserve (6.16),

$$\{Q_\alpha^{(1)} + Q_\alpha^{(2)}, Q_\beta^{(1)} + Q_\beta^{(2)}\} = -2(\gamma_\mu C)_{\alpha\beta}(p_{(1)} + p_{(2)})^\mu,$$

the cross terms have to vanish,

$$\{Q_\alpha^{(1)},Q_\beta^{(2)}\} = 0. \tag{6.28}$$

Thus Q_α acts as an antiderivation [Eq. (1.4)]

$$\begin{aligned} Q_\alpha|v_1v_2\rangle = &|v_1'v_2\rangle\langle v_1'|Q_\alpha(p_1)|v_1\rangle \\ &+ (-1)^{kl}|v_1v_2'\rangle\langle v_2'|Q_\alpha(p_2)|v_2\rangle, \end{aligned} \tag{6.29}$$

where $k = 1$ is the grading of $Q_\alpha \in \mathrm{End}_1$, and $l = 0$ or 1 according to whether $v_1' \in V^l$ is a boson or a fermion.

We present here an additional diagonalization of the Q_α which will prove to be useful in the construction of field representations. It corresponds to diagonalizing $-i\gamma_5$ [$= \rho_1\sigma_2$ in our representation (6.10)]. The transformed Q_α' then reduce into two 2-spinors corresponding to (we denote the chiral projection operators by γ^R and γ^L)

$$\begin{aligned} Q^R = (1 - i\gamma_5)/2Q' = \gamma^R Q', \quad Q^L = (1 + i\gamma_5)/2Q' \\ = \gamma^L Q'. \end{aligned} \tag{6.30}$$

The Q' is no longer real, since it is given by $UQ = Q'$, $U^+ = U^{-1}$. The Majorana condition should then be redefined. We get

$$\begin{aligned} Q'^c &= C((UQ)^+\gamma_0)^\sim = C\gamma_0{}^\sim U^*Q^* = -\gamma_0 CU^*Q^* \\ &= -cU^*Q^* \end{aligned}$$

and in our choice of $c = -1$, the condition becomes

$$Q' = UQ = Q'^c = -cU^*Q^* = U^*Q^*.$$

With γ_5 real and anti-Hermitian, we thus find

$$(Q_a{}^R)^* = \gamma_{ab}{}^L Q_b{}'^* = \gamma_{ab}{}^L Q_b{}' = Q_a{}^L. \tag{6.31}$$

The two 2-spinors $Q_a{}^R$ and $Q_a{}^L$ are thus conjugate, and behave like the Q_A and $Q_A{}^+$ of (6.25), except that J^{31} is now the diagonal projection of spin. Thus

$$[J^{31},Q_1{}^R] = \tfrac{1}{2}Q_1{}^R;\quad [J^{31},Q_2{}^R] = -\tfrac{1}{2}Q_2{}^R;$$
$$[J^{31},Q_1{}^L] = -\tfrac{1}{2}Q_1{}^L;\quad [J^{31},Q_2{}^L] = \tfrac{1}{2}Q_2{}^L.$$

The graded Lie brackets are in general

$$\{Q_a{}^L,Q_b{}^L\} = 0,\quad \{Q_a{}^R,Q_b{}^R\} = 0,\quad \{Q_a{}^R,Q_b{}^L\} = 2(\mathbf{1}\cdot P^0 + \sigma_3 P^2 - \sigma_1 P^1 - \sigma_2 P^2)_{ab} \tag{6.32}$$

and for $M \neq 0$ and rest states

$$\{Q_a{}^R,Q_b{}^L\} = 2\delta_{ab}M,$$

while for the $M = 0$ case, we again see the reduction into two subspaces with $P^0 + P^2 \neq 0$, $P^0 - P^2 = P^1 = P^3 = 0$ for the first, and the parity-inverted states for the second subspace. The $Q_1{}^R$ and $Q_1{}^L$ are in one subspace, and the $Q_2{}^R$ and $Q_2{}^L$ in the other.

For $M \neq 0$ and rest, $Q_a{}^R$ and $Q_a{}^L = Q_a{}^{R*}$ can thus be treated as annihilation and creation operators in the construction of representation of states or fields.

C. Realization on a Grassmann algebra as a generalized (Berezin–Kac) Lie group; superfields

Berezin and Kac (1970), following a similar idea of Lazard (1955) and motivated by problems of second quantization, introduced the notion of Lie groups with commuting and anticommuting parameters. Their idea is the following. Let G be an (ordinary) Lie group. Then G is a differentiable manifold, and the group multiplication defines a differentiable map of $G \times G \to G$. The group axioms impose some conditions on this map. Let $F(G)$ denote the ring of smooth functions on G, and $F(G \times G)$ the ring of smooth functions on $G \times G$. The multiplication then gives a map from $F(G) \to F(G \times G)$, sending the function f of the single G variable into the function ϕf of two G variables defined by

$$(\phi f)(x,y) = f(xy). \tag{6.33}$$

The various group axioms, such as associativity, existence of identity, and existence of inverses, can all be formulated in terms of the map ϕ. Since we can assume that the group coordinates are chosen so that the multiplication is given by analytic functions, we can, without harm, replace the ring of smooth functions by the ring of formal power series, say F_G. Then $F_{G\times G}$ can be identified with $F_G \hat{\otimes} F_G$ (completed tensor product). This is simply the assertion that any polynomial in two variables can be written as a sum of products of polynomials in one variable, and hence any formal power series in two variables can be written as a (formal power series) limit of such sums. We can then write all the group axioms as a series of conditions on the map, ϕ: $F_G \to F_G \times F_G$. Now let us replace the ring F_G by an arbitrary ring F of formal power series in variables x_i, where each x_i has degree d_i, and the multiplication of the x's is graded commutative, i.e., $x_i x_j = (-1)^{d_i d_j} x_j x_i$. We impose the formal analogs of the group axioms, and obtain what Berezin and Kac call a "Lie group with commuting and anticommuting parameters." If $f \in F$, then it follows from the right unit axiom that

$$\phi_f(x,y) = f(y) + \Sigma X_f{}^i(y)x_i + \text{(higher order terms in } x_i). \tag{6.34}$$

The maps $f \rightsquigarrow X_f{}^i$ are linear, and can be thought of as the analogs of infinitesimal right translation. If we put the obvious gradation on F, then the map $f \rightsquigarrow X_f{}^i$ is a graded derivation of F, and the X^i form a graded Lie algebra. In this way one associates a graded Lie algebra with each such "formal Lie group." Conversely, starting with a graded Lie algebra, by use of the analog of the Campbell–Hausdorff formula, Berezin and Kac show how to construct a "formal Lie group in commuting and noncommuting variables" to each graded Lie algebra. The correspondence between the graded Lie algebras and the "formal Lie groups" is functorial in the usual sense.

In applying GLA's as supergauges in the dual models (see references in Sec. II.F), the Berezin–Kac method was used, with Grassmann algebra elements appearing as group parameters. Wess and Zumino (1974a) applied the same technique in order to construct field representations and invariant Lagrangians. Salam and Strathdee (1974a, e) systematized the approach, which was further developed by Ferrara, Wess, and Zumino (1974). However, as we shall show, although the method does yield very useful results, its foundations are unclear and lack consistency (Rühl and Yunn, 1974; Ne'eman, 1974). Rühl and Yunn (1974) and Goddard (1974) have recently tried to supply a better set of basic assumptions and have indeed removed some of the inconsistencies, except for difficulties with an indefinite metric and for the fact that the Minkowski space coordinates x^μ are still nilpotent elements obeying the requirement $(x^\mu)^{n+1} = 0$, (with $n = 4$ in the conventional solution). In general, it also seems doubtful whether indeed the generalized group can be used as a symmetry, except very close to the identity. We shall now describe the formalism, using the notation of the Appendix.

We use an N-dimensional vector space (over the complex field) $V(\equiv \Lambda^1 V)$, generating a 2^N dimensional Grassmann algebra

$$\Lambda V = \bigoplus_{r=0}^{N} \Lambda^r V.$$

The basis vectors of V are $v_1, v_2, \ldots, v_N$; since the Grassmann algebra is graded commutative (A2), the elements of V anticommute,

$$v_i \wedge v_j = (-1) v_j \wedge v_i.$$

We shall write this property as

$$\{v_i,v_j\} = 0 \quad \text{for any} \quad i,\ j,\ v_{i,j} \in V \tag{6.35}$$

with multiplication thus being defined by the $\wedge$ operation.

We shall also use extensively the elements of $\Lambda^2 V \equiv W$, resulting from $v_i v_j$ products. In this case, graded commutativity ensures that the elements $w_a \in W$ commute. The Minkowski space coordinates are identified with elements of W, $x^\mu \in W = \Lambda^2 V$. If we attach a reflection operation R to the $v_i \in V$, $v_i \to -v_i$, the entire ΛV splits into two parts,

$$\begin{cases} \Lambda V = \Lambda V^{(-)} + \Lambda V^{(+)} \\ \Lambda^r V \subset \Lambda V^{(-)} \quad \text{if } r \text{ is odd}, \\ \qquad \subset \Lambda V^{(+)} \quad \text{if } r \text{ is even}. \end{cases}$$

"Superfields" are "local" fields, in the variables $\theta_\alpha \in \Lambda' V$, $x^\mu \in \Lambda^2 V$. θ_α is a Majorana spinor,

$$\theta = \theta^c = C\gamma_0{}^\sim \theta^* \tag{6.36}$$

which amounts to a true reality condition

$$\theta_\alpha = \theta_\alpha{}^*$$

in the representation (6.10). As to the coordinate, it should be real in any case,

$$x^\mu = (x^\mu)^*.$$

Thus V is at least four dimensional. Indeed, a four-dimensional quasi-Minkowski coordinate [it is not a true Minkowski coordinate since $(x^\mu)^{N+1} = 0$] in $\Lambda^2 V$ can be constructed from two θ, $\theta' \in V$,

$$x^\mu = \bar{\theta}\gamma^\mu\theta' \tag{6.37}$$

which, by Eq. (6.9) is Hermitian and real. From Eq. (6.14) we observe that

$$\bar{\theta}\gamma^\mu\theta' = -\bar{\theta}'\gamma^\mu\theta$$

which can be rewritten, using (6.36), as

$$\theta_\alpha{}^\sim (\gamma^0\gamma^\mu)_{\alpha\beta}\theta_\beta' = -\theta_\alpha{}'^\sim (\gamma^0\gamma^\mu)_{\alpha\beta}\theta_\beta. \tag{6.38}$$

We observe in this expression the (generalized) matrix structure of the Λ operation between two Majorana-like elements of V. It is still antisymmetric, because $\gamma^0\gamma^\mu$ is symmetric; the antisymmetry is thus derived from (6.35),

$$\{\theta_\alpha,\theta_\beta'\} = 0 \tag{6.39}$$

and the $\gamma^0\gamma^\mu$ matrices preserve this feature while taking care of the spinor indices.

We now turn to the action of the Q_α on these elements. From

$$\{Q_\alpha,Q_\beta\} = -2(\gamma_\mu C)_{\alpha\beta}P^\mu \tag{6.16}$$

we know that the doubled action of the Q_α represents a translation in W. We can thus guess that Q_α represents such a translation in V, acting in analogy to

$$P_\mu \sim -i\frac{\partial}{\partial x^\mu}, \quad [P_\mu,x^\nu] = -i\delta_\mu{}^\nu. \tag{6.40}$$

As far as its action on V is concerned, $Q_\alpha \sim \Gamma_{\alpha\beta}(\partial/\partial\theta_\beta)$. Thus Q_α is in V^* or in $\Lambda_1 V$. Note that for $\partial/\partial\theta_\alpha$, an element in V^*, we are in the larger $\oplus\Lambda_s{}^r V$. Thus

$$\left\{\frac{\partial}{\partial v_i},\frac{\partial}{\partial v_j}\right\} = 0, \quad \left\{\frac{\partial}{\partial v_i},v_j\right\} = \delta_{ij}. \tag{6.41}$$

Q_α will thus bracket with θ_β as $\mathrm{End}_1\, V$,

$$\{Q_\alpha,Q_\beta\} = iC_{\alpha\beta}, \tag{6.42}$$

where $C \in V_0$ appears as the appropriate metric for Majorana spinors, so that $\Gamma_{\alpha\beta} = iC_{\alpha\beta}$. To obtain an infinitesimal translation by a "constant" parameter $\epsilon_\alpha \neq \epsilon_\alpha(\theta)$

$$\theta_\alpha \to \theta_\alpha + \epsilon_\alpha, \quad \{Q_\alpha,\epsilon_\beta\} = 0 \tag{6.43}$$

with $\epsilon_\alpha \in V$, we have to act with $\bar{\epsilon}_\alpha Q_\alpha$, where we use $\bar{\epsilon}_\alpha$ rather than ϵ_α in order to obtain the necessary tensor construction as in Eq. (6.40). Note that exponentiation by $\bar{\epsilon}_\alpha$ follows the Berezin–Kac (1970) method of generating a generalized Lie group. Integration is defined through

$$\int dv_i = 0; \quad \int v_i\, dv_i = 1; \quad \int v_i\, dv_j = \{dv_i,dv_j\} = 0. \tag{6.44}$$

Note that $\epsilon \neq \epsilon(\theta)$ and $\{Q,\epsilon\} = 0$, as against Eq. (6.41), require additional dimensions in V.

The resulting action is then a commutator bracket, as needed for infinitesimal group action,

$$i[\bar{\epsilon}_\alpha Q_\alpha,\theta_\beta] = \epsilon_\beta. \tag{6.45}$$

The action on $x^\mu = \bar{\theta}'\gamma^\mu\theta$ is thus bound to be

$$i[\bar{\epsilon}_\alpha Q_\alpha,x^\mu] = \bar{\epsilon}\gamma^\mu\theta. \tag{6.46}$$

Assuming now the existence of a "superfield" $\phi(x_\mu,\theta_\alpha)$, we can use a Taylor series to identify the structure of the infinitesimal operator $\bar{\epsilon}_\alpha Q_\alpha$,

$$\begin{aligned} U\phi(x_\mu,\theta_\alpha)U^{-1} &= \phi(x_\mu - \bar{\epsilon}\gamma_\mu\theta,\, \theta_\alpha - \epsilon_\alpha) \\ &= \phi(x_\mu,\theta_\alpha) - \bar{\epsilon}\gamma_\mu\theta\frac{\partial}{\partial x^\lambda}\phi(x^\mu,\theta_\alpha) \\ &\quad - \epsilon_\alpha\frac{\partial}{\partial\theta_\alpha}\phi(x_\mu,\theta_\alpha) + O(\epsilon^2), \end{aligned}$$

where the generalized group element is

$$\begin{cases} U = 1 - i\bar{\epsilon}_\alpha Q_\alpha \\ (\bar{\epsilon}_\alpha Q_\alpha)^+ = \bar{\epsilon}_\alpha Q_\alpha. \end{cases} \tag{6.47}$$

This yields the explicit structure

$$Q_\alpha(\Lambda V) = \left(iC_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} - i(\gamma_\mu)_{\alpha\beta}\theta_\beta\frac{\partial}{\partial x_\mu}\right)\Lambda V. \tag{6.48}$$

We now come to one of the difficulties or inconsistencies of this picture. If we regard $\bar{\epsilon}Q$ as a Lie group generator, we get, by Eq. (6.17),

$$[\bar{\epsilon}_\alpha Q_\alpha,\bar{\epsilon}_\beta Q\epsilon] = \bar{\epsilon}_\alpha\{Q_\alpha,\bar{Q}_\beta\eta\epsilon_\beta = 2\bar{\epsilon}_\alpha(\gamma_\mu)_{\alpha\beta}\epsilon_\beta P^\mu. \tag{6.49}$$

However, from Eq. (6.14) we know that this vanishes, since $\bar{\psi}\gamma_\mu\chi = -\bar{\chi}\gamma_\mu\psi$. Even if we do not sum over the α and β indices, we shall at least have vanishing expressions for $\mu = 0$, since $\gamma_0{}^2 = 1$. This covers in fact the entire little algebra for $M \neq 0$ [Eq. (6.22)]. We are thus faced with two choices: either the Lie algebra is Abelian, or, as we already noted from Eq. (6.43), ϵ_α and ϵ_β' have to lie in new subspaces of V, which differ from each other and also do not contain the θ_α. In these new subspaces, we may be able to ensure nonvanishing of the right-hand side. Indeed, the simplest solution is to add eight dimensions, so as to have different $\bar{\epsilon}_\alpha$ and ϵ_β' on the right-hand side. The P^μ are then multiplied by $16 - 4 = 12$ new dimensions in Λ^2V.

Note that all of this is necessary because the superfield $\phi(x_\mu,\theta_\alpha)$ is acted upon by a Lie group. However, if we allow for finite transformations, θ_α will have "crept" into the new ϵ_α, ϵ_α' subspaces, and our efforts will have been to no avail. Still, we dare not allow Eq. (6.49) to have a vanishing rhs since we would then lose the connection with our starting point, in which Q_α acted as the "square-root" of P^μ. We have by all means to recover Eq. (6.16) or (6.17), even though the information will now be supplied by a commutator.

It turns out that we can also add only two dimensions to V, so that $N \geq 6$, and disconnect the new dimensions from the spinor indices in ϵ_α. Goddard (1974) has constructed this system, using two new dimensions. We denote their basis elements as v_5 and v_6. This seems the most economical solution. It may have been hinted at by Salam and Strathdee (1973a), but in their solution the number of odd generators would be doubled: $(v_a Q_\alpha)$ with $a = 5, 6$. Rühl and Yunn (1974) have pursued this method and come up with 26 generators instead of 14 for $\mathcal{V}$. This results from six for $J^{\mu\nu}$, eight for $\bar{\epsilon}Q$ and $\bar{\epsilon}'Q$, 12 for $\epsilon\epsilon' P_\mu$. Goddard has found a way of avoiding the doubling. In the chiral picture (6.30)–(6.32) we see that only cross terms in γ^R and γ^L contribute. This is true beyond the rest frame used there. Thus Goddard introduces the matrix

$$g = \tfrac{1}{2}(1 - i\gamma_5)v_5 + \tfrac{1}{2}(1 + i\gamma_5)v_6 = \gamma^R v_5 + \gamma^L v_6 \qquad (6.50)$$

and writes

$$\epsilon_\alpha = g_{\alpha\beta}\xi_\beta, \qquad (6.51)$$

where ξ_β is a c-number Majorana spinor,

$$\xi = \xi^c = C\bar{\xi}^\sim \qquad (6.52)$$

and is real in our (6.10) representation. The γ^R and γ^L are Hermitian. Under complex conjugation, v_5 and v_6 are made to obey

$$v_5{}^* = v_6, \quad v_6{}^* = v_5 \qquad (6.53)$$

so that, since γ_0 anticommutes with the γ_5 in g,

$$\bar{\epsilon}_\alpha = (g\xi)^+\gamma_0 = \bar{\xi}_\beta g_{\beta\alpha} \qquad (6.54)$$

and

$$\bar{\epsilon}Q = \bar{\xi}gQ = \bar{\xi}_\alpha(v_5\gamma^R + v_6\gamma^L)_{\alpha\beta}Q_\beta = \bar{\xi}_\alpha S_\alpha \qquad (6.55)$$

and the S_α fulfill the role of generators of a Lie algebra,

$$\begin{aligned}[S_\alpha,S_\beta] &= -g_{\alpha\gamma}\{Q_\gamma,Q_\delta\}g_{\beta\delta} \\ &= 2(g\gamma_\mu Cg^\sim)_{\alpha\beta}P^\mu = 2i(\gamma_5\gamma_\mu C)_{\alpha\beta}v_5v_6P^\mu \\ &= 2i(\gamma_5\gamma_\mu C)_{\alpha\beta}T^\mu. \end{aligned} \qquad (6.56)$$

Note that the new (even) Lie generators S_α yield a new set of "translations" T^μ. Just as the $S_\alpha \subset L_1\Lambda V$, so is T^μ in $L_2\Lambda V\Lambda V$. We can thus "replace" the physical GLA $\mathcal{V}$ of Eq. (6.18) by a "generalized Lie algebra" including both T^μ and P^μ for the sake of covariance considerations,

$$\begin{aligned}\mathcal{L}:\ (J^{\mu\nu},P^\mu,S_\alpha,T^\mu):\ &[P^\mu,S_\alpha] = 0, \quad [T^\mu,S_\alpha] = 0, \\ &[P^\mu,T^\mu] = 0, \\ &[J^{\mu\nu},T^\lambda] = -ig^{\mu\lambda}T^\nu + ig^{\nu\lambda}T^\mu. \end{aligned} \qquad (6.57)$$

Only $J^{\mu\nu}$ and P^μ are "physical," in the sense that they do not involve nilpotent elements. We shall return to $\mathcal{L}$ when studying symmetry and unitarity aspects.

The two subspaces of $V = V_\theta + V_\epsilon$, where V_θ has v_{1-4} as basis, and V_ϵ has v_{5-6}, generate subspaces ΛV_θ ($d = 16$) and ΛV_ϵ ($d = v$) of $\Lambda V^{(+)}$ and $\Lambda V^{(-)}$. Goddard's method utilizes these subspaces for $(\theta_\alpha)^n$ and $(\epsilon_\alpha)^m$, thus allowing only infinitesimal transformations of θ_α in Eq. (6.43). The Lie group is thus physically applied only very close to the identity.

We now follow Salam and Strathdee (1974a). Due to the anticommuting properties of θ_α, any function $f(\theta)$ must be a polynomial. Since the monomials $\theta_{\alpha_1}\theta_{\alpha_2}\cdots\theta_{\alpha_n}$ have to be completely antisymmetric, expanding $\phi(x^\mu,\theta_\alpha)$ in powers of θ_α is a finite operation terminating at $n = 4$. The even monomials belong in the $\Lambda V_\theta{}^{(+)}$, the odd ones in $\Lambda V_\theta{}^{(-)}$. Altogether, $\phi(x^\mu,\theta)$ is 16 dimensional as long as we do not allow finite transformations in ϵ_α. Expanding in θ, we get [using Eq. (6.14)]

$$\begin{cases}\phi(x,\theta) = A(x) \\ \qquad + \bar{\theta}\psi(x) \\ \qquad + \tfrac{1}{4}\bar{\theta}\theta F(x) + \tfrac{1}{4}\bar{\theta}\gamma_5\theta G(x) + \tfrac{1}{4}(i\bar{\theta}\gamma_5\gamma_\nu\theta)A_\nu(x) \\ \qquad + \tfrac{1}{4}\bar{\theta}\theta\bar{\theta}\chi(x) \\ \qquad + \tfrac{1}{32}(\bar{\theta}\theta)^2 D(x). \end{cases} \qquad (6.58)$$

We have altogether (before any subsidiary conditions or equations of motion) eight spinor and eight boson components. Foregoing the difficulty about the nilpotence of x^μ, which does not involve (6.58), we find that $A(x)$, $F(x)$, and $D(x)$ are scalar fields, $G(x)$ is a pseudoscalar, and $A_\mu(x)$ an axial vector field. Besides these Bose fields, we have two (Dirac) spinor fields ψ and χ. We can impose a "Hermiticity" condition on the superfield,

$$\phi(x,\theta)^+ = \phi(x,\theta), \qquad (6.59)$$

where $+$ implies besides complex conjugation a reversal of the order of anticommuting factors. The Bose fields then make eight real components, and the spinors are Majorana spinors. Had we started with a pseudoscalar $\phi(x,\theta)$, all parities would be inverted. We can also define $\phi^\mu(x,\theta)$, a "vector" superfield, or $\phi_\alpha(x,\theta)$, a "spinor" superfield, according to the Poincaré transformation properties

$$\phi'(x',\theta') = \phi(x,\theta); \quad \phi_\alpha'(x',\theta') = a_\alpha{}^\beta(\Lambda)\phi_\beta(x,\theta) \text{ etc.} \qquad (6.60)$$

The variation of the fields in Eq. (6.59) under (6.43) or (6.47)–(6.48) can be found from the equations leading to (6.48). Identifying coefficients in (6.58) we find,

$$\begin{cases}\delta A = \bar{\epsilon}\psi, \\ \delta\psi = -\epsilon\eth A + \tfrac{1}{2}\epsilon F + \tfrac{1}{2}\epsilon\gamma_5 G + \tfrac{1}{2}\epsilon i\gamma_5\gamma_\nu A^\nu, \\ \delta F = \tfrac{1}{2}\bar{\epsilon}\chi - \bar{\epsilon}\eth\psi, \\ \delta G = -\tfrac{1}{2}\bar{\epsilon}\gamma_5\chi - \bar{\epsilon}\gamma_5\eth\psi, \\ \delta A_\nu = \tfrac{1}{2}\bar{\epsilon}i\gamma_5\gamma_\nu\chi - \bar{\epsilon}i\gamma_5\gamma_\nu\eth\psi, \\ \delta\chi = -\epsilon(\eth F + \eth\gamma_5 G) + \epsilon i\gamma_5\gamma_\nu\eth A^\nu - \tfrac{1}{2}\epsilon D, \\ \delta D = -2\bar{\epsilon}\eth\chi. \end{cases} \qquad (6.61)$$

Notice that the numbers of fermion and boson components are always equal, as required by our study of the "little" algebra.

In counting components, we did not consider subsidiary conditions, However, A_ν clearly has to obey a condition reducing it to three components (in the $M \neq 0$ case). If we impose

$$\partial^\mu A_\mu = 0$$

we can replace A_ν in Eq. (6.58) by $A_\nu + \partial_\nu B$, B a pseudoscalar field. χ_α can also be replaced by $\chi_\alpha + (i\gamma_\mu \partial^\mu \psi)_\alpha$ and D by $D - \frac{1}{4}\partial_\mu \partial^\mu A$. In that case, (6.61) will also have

$$\delta B = -\bar{\epsilon}\gamma_5 \psi - i(\partial_\lambda \partial^\lambda)^{-1} \bar{\epsilon}\gamma_5 \eth \chi$$

and an additional contribution to the ψ variation,

$$\delta\psi' = \tfrac{1}{2}\epsilon i \gamma_5 \gamma_\nu \partial_\nu B.$$

Also, the $\delta\chi$, δA_ν, δD can now be re-expressed in terms of contributions which involve them only,

$$\delta A_\nu = \tfrac{1}{2}\bar{\epsilon} i\gamma_5 \gamma_\nu \chi - (\partial^\mu \partial^\nu / \partial_\lambda \partial^\lambda) \tfrac{1}{2} \bar{\epsilon} i \gamma_5 \gamma_\nu \chi$$
$$\delta \chi = i\tfrac{1}{2}\epsilon \gamma_5 \sigma^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) - \tfrac{1}{2}\epsilon D.$$

Indeed, the superfield $\phi(x,\theta)$ is not irreducible. It can be made irreducible by applying a covariant and supersymmetric condition,

$$W_{\mu\nu}\phi = 0, \qquad (6.62)$$

where, using the representation (6.49) for Q_α, which we shall denote as $Q(\Lambda V)$,

$$W_{\mu\nu} = P_\mu W_\nu - P_\nu W_\mu; \quad W_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma} + \tfrac{1}{4}\bar{Q} i \gamma_5 \gamma_\mu Q. \qquad (6.63)$$

This condition cancels three fields, which now make four fermions and four boson components:

$$D = 0, \quad \chi = 0, \quad A_\nu = 0.$$

Such covariant and superinvariant conditions as (6.62) can be constructed from powers of $Q(\Lambda V)$, $\bar{Q}(\Lambda V)$ and their chiral projections. Ferrara, Wess, and Zumino (1974), Salam and Strathdee (1974e), O'Raifeartaigh (1974), and Nilsson and Tchrakian (1975) have developed such a calculus. It is based on the application of the 16 elements of the Clifford algebra (6.22) in its ΛV realization, using Eq. (6.48), thus yielding differential equations.

To construct supersymmetric couplings, one utilizes the above method of identifying coefficients of powers of θ. For instance, if

$$\Phi_3(x,\theta) = \Phi_1(x,\theta)\Phi_2(x,\theta)$$

we can identify

$$A_3(x) = A_1(x)A_2(x),$$
$$\bar{\psi}_3(x) = A_1\bar{\psi}_2 + \bar{\psi}_1 A_2,$$

etc.

Note that since the variation of D in (6.61) was only a divergence, the "D_3" component can be used as a Lagrangian density (Wess and Zumino, 1974a). For the case $\phi_1 = \phi_2$ and $W_{\mu\nu}\phi = 0$ one finds,

$$\text{“}D_3\text{''} = \epsilon^{\alpha\beta\gamma\delta}(-\tfrac{1}{4}A\partial^2 A + \tfrac{1}{2}i\bar{\psi}\chi\psi + \tfrac{1}{4}(\partial_\mu B)^2 + F^2 + G^2$$

or

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu A)^2 + \tfrac{1}{2}(\partial_\mu B)^2 + \bar{\psi} i \eth \psi + 2F^2 + 2G^2 - \tfrac{1}{2}\partial_\mu(A\partial_\mu A) \qquad (6.64)$$

which is indeed an example of a Lagrangian density. The fields F and G have no dynamics and satisfy equations of motion

$$F = 0, G = 0.$$

Note that the equations of motion for ψ, A, and B reduce the (massless) states to one fermion and one boson.

The $\Phi(x,\theta)$ are reducible. One can also work with chiral projections, by imposing conditions

$$\tfrac{1}{2}(1 \mp i\gamma_5)_{\alpha\beta} Q_\beta{}^v \phi = 0, \qquad (6.65)$$

where Q^v stands for the ΛV representation of Q as in (6.48). These superfields are now irreducible. The scalar (i.e., no spinor or vector index on ϕ itself) superfield ϕ_R is then composed of A_-, ψ_R, and F_-. They transform according to

$$\begin{cases} \delta A_\pm = \bar{\epsilon}\psi_{L,R}, \\ \delta\psi_{L,R} = \gamma_{L,R}(F_\pm - i\eth A_\pm)\epsilon, \\ \delta F_\pm = -\bar{\epsilon} i \eth \psi_{L,R}. \end{cases} \qquad (6.66)$$

We identify $\phi_- = \phi_R$, $\phi_+ = \phi_L$, i.e., $\phi_- = (\phi_+)^*$, though one could also have unconnected projections.

Examples of superfields and Lagrangians will appear in our review of the physical examples in which renormalization and other properties were studied. We refer the reader to the above mentioned articles (Ferrara, Wess, Zumino, 1974; Salam and Strathdee, 1974e) for other examples of superfields, both spinorial (ϕ_α, $\phi_\alpha{}^\mu$ etc.) and tensoral (ϕ^μ,$\phi^{\mu\nu}$ etc.). Furthermore, Capper (1974) has developed Feynman diagrams reproducting the superfield couplings; these are economical when studying the divergences of multiloop diagrams.

Considering the physical complications involved in the use of the Grassmann algebra substrate, it may be necessary at some stage to possess a formalism producing the field multiplets directly from the GLA. One can use the (6.32) set, just as the (6.26) subalgebra was used to construct irreducible representations. To construct nonunitary irreducible field multiplets (Salam and Strathdee, 1974e) one applies $Q_R{}^a$ and $Q_R{}^{a*}$ to a "lowest" representation $\mathfrak{D}(j_1,j_2)$ of the proper Lorentz group. Assuming

$$Q_R{}^a \phi_{j_1 j_2}(x) = 0$$

we get four submultiplets: two from the action of $Q_L{}^1$, $Q_L{}^2$, [in $(\frac{1}{2},0)$] and one from their joint action $\sim(0.0)$, plus the original

$$\phi_{j_1 j_2}(x).$$

The total dimensionality is thus $4(2j_1 + 1)(2j_2 + 1)$. One can also have a supermultiplet with inverted parities by starting with

$$Q_L{}^a \phi_j(x) = {}_{12j}0.$$

These representations are however generally reducible. One can extract pieces by contraction with powers of $\partial/\partial x^\mu$, i.e., graded analogs of subsidiary conditions.

In constructing irreducible representations, it is important to recall that considering, as in Sec. II.A, the boson and fermion states as forming a two-dimensional graded vector space V; the boson and fermion quantum fields $\phi(x)$ and

$\psi(x)$ themselves represent $\mathrm{End}_0\, V$ and $\mathrm{End}_1\, V$ operators, respectively. Their GLA brackets with the $Q_L{}^a$ and $Q_R{}^a$ are thus fixed by (1.2). Indeed, one may recover the entire (6.61) set, without the ϵ parameters, by bracketing the Q_α directly with the fields $\psi(x)$, $A(x)$, etc. Summing up, for G_k a GLA generator,

$$[G_k,\Phi_{j_1j_2}(x)] = -(-1)^{2(j_1+j_2)k}[\Phi_{j_1j_2}(x),G_k]. \tag{6.67}$$

D. Inclusion of internal symmetries

Let the indices i, $j = 1, \ldots, n$ denote an internal symmetry such as the $SU(2)$ of I-spin, or $SU(3)$. We then have, in addition to Eqs. (6.16) and (6.17), a set (Salam and Strathdee, 1974b)

$$\begin{cases}\{Q_{\alpha i},Q_{\beta j}\} = -2\delta_{ij}(\gamma_\mu C)_{\alpha\beta}P^\mu \\ [P_\mu,Q_{\alpha i}] = 0.\end{cases} \tag{6.68}$$

Restricting the system to rest states, we get a Clifford algebra, C_{4n}, whose dimensionality is 2^{4n} and whose matrix representation acts on a 2^{2n} vector space.

$$\{Q_{\alpha i},Q_{\beta j}\} = 2\delta_{ij}\delta_{\alpha\beta}M.$$

Thus, for isospin [$SU(2)$] and assuming that the $Q_{\alpha i}$ transform as an isospinor ($n = 2$), we find the symmetry realized over a 16-dimensional carrier space. (The Clifford algebra will have 256 base elements $Q_{\alpha i}$, $i[Q_{\alpha i},Q_{\beta j}]$, etc....) In fact, we can start with any (j,I) multiplet as the lowest state, and construct a representation with $16(2j+1)(2I+1)$ dimensions. The quantum numbers of the states in the case $j = 0$, $I = 0$ are given by the action of the $2n$ raising operators only; their graded products form a smaller Clifford algebra C_{2n}, whose dimensionality is indeed 2^{2n}(= 16 for I-spin), which will indeed create the 2^{2n} states of the carrier space. This enables us to get their quantum numbers directly; $(\frac{1}{2},\frac{1}{2})$, $\Lambda^2(\frac{1}{2},\frac{1}{2})$, $\Lambda^3(\frac{1}{2},\frac{1}{2})$, $\Lambda^4(\frac{1}{2},\frac{1}{2})$. In this case these are just the 16 matrices of the Dirac–Clifford algebra. They reduce to $(j,I)^p$ multiplets:

$$(0,0)^+ \oplus (\tfrac{1}{2},\tfrac{1}{2})^\eta \oplus (1,0)^- \oplus (0,1)^- \oplus (\tfrac{1}{2},\tfrac{1}{2})^{-\eta} \oplus (0,0)^+.$$

Going back to the C_{4n} of (6.69) we note that $\Lambda^2 Q_{\alpha i}$ will form the Lie algebra $SO(8) \supset SO(6) \sim SU(4)$, so that the 16 states can be grouped in $SU(4)$ (Wigner) supermultiplets $1 \oplus 4 \oplus 6 \oplus 4^* \oplus 1$.

Indeed, we can use a generalization of Eq. (6.32) instead of (6.68):

$$\{Q_{ai},Q_{bj}^*\} = 2\delta_{ab}\delta_{ij}M, \quad a, b = 1, 2 \tag{6.69}$$
$$\{Q_{ai},Q_{bj}\} = 0, \quad \{Q_{ai}^*,Q_{bj}^*\} = 0$$

for rest states. Here we have the same number of odd generators $4n$; the results are the same except that $\Lambda^2 Q$ now contains $i[Q_{ai},Q_{bj}^*] = S_{ai}{}^{bj}$ which is clearly the $su(n)$ algebra, the rest of $SO(8)$ being given by $[Q,Q]$ and $[Q^*,Q^*]$. Note that this "little" GLA now has $Q_{ai} \in L_{-1}$; $Q_{ai}^* \in L_1$; 1, $S_{ai}{}^{bj} \in L_0$.

The (6.69) bracket can be generalized for cases where the representation $\mathbf{n}$ differs from $\mathbf{n}^*$, such as the $SU(3)$ case:

$$\{Q_{\alpha i},Q_{\beta j}^*\} = 2\delta_{\alpha\beta}\delta_{ij}M, \quad \alpha = 1,\ldots,4. \tag{6.70}$$

The Clifford algebra is now C_{8n}, $d = 2^{8n}$, acting on a 2^{4n} dimensional carrier space. Salam and Strathdee (1974f) have constructed the $O(3)$ case (fitting 6.69) and discussed the totally antisymmetric features of the multiplets, due to the graded commutativity and filtered structure of the Clifford algebra. It seemed difficult to reconcile with the physical states in the quark model assignments. However, it was soon noted (Wess, 1974) that if one introduces $SU(3)_{\mathrm{color}} \otimes SU(3)_{GN}$, the totally antisymmetric representations will indeed contain the observed states whenever the color indices will contract or antisymmetrize to a singlet.

Salam and Strathdee (1974b), Dondi and Sohnius (1974), Lopuszański and Sohnius (1974), and Firth and Jenkins (1974) have further studied the isospin case and written down some of the Casimir operators of that GLA.

We shall leave the case of a local gauge symmetry and the problems relating to fermionic charge operators to our discussion of physical applications of supersymmetry.

VII. APPLICATIONS OF SUPERSYMMETRY

A. General symmetry considerations

All supersymmetric models upon $\mathcal{W}$ or its extension by internal degrees of freedom [as in Eqs. (6.68)–(6.70)] have in common two simplifying features:

$$[P^\mu,Q_{\alpha i}] = 0 \tag{7.1}$$

and

$$H = \sum_{\alpha,i} Q_{\alpha i}{}^2. \tag{7.2}$$

Conservation is thus guaranteed. In the case of the R_α of (6.15), which do not commute with H, conservation is ensured by

$$\frac{d}{dt}[K^\mu,Q_\alpha] = (-\gamma_5\gamma^\mu)_{\alpha\beta}\frac{d}{dt}R_\beta = 0. \tag{7.3}$$

These examples can be generalized in the following theorem: "A GLA $\mathcal{G}$ is conserved if its even subalgebra $\mathcal{L}$ (the Lie algebra) is conserved, and if its odd generators $\mathcal{O}$ transform irreducibly under $\mathcal{L}$ and contain at least one nonnilpotent generator $\mathcal{O}_a$."

Clearly, $[\mathcal{O}_a,\mathcal{O}_a] \subset \mathcal{L}$ and does not vanish, so that $(d/dt)\mathcal{O}_a = 0$, leading to $(d/dt)\mathcal{O}_i = 0$ for all i, through the action of $\mathcal{L}$.

We now give a preliminary discussion of the role of the Noether theorem (for recent advances see J. Schwinger, 1951; Orzalesi, 1970; Y. Dothan, 1972; J. Rosen, 1974) in the case of a GLA, and in particular for $\mathcal{W}$. From (6.64) as a Lagrangian, and using the variations in (6.61) and the condition (6.62) we find the conserved (spinor–vector) current,

$$j_\alpha{}^\mu(x) = ((\gamma^\lambda\partial_\lambda(A(x) - B(x)\gamma_5)\gamma^\mu\psi(x)))_\alpha$$
$$-2i(((F(x) + \gamma_5 G(x))\gamma^\mu\psi))_\alpha \tag{7.4}$$

and

$$Q_\alpha = \int d^3x j_\alpha{}^0(x). \tag{7.5}$$

If we use the superfield $\phi(\theta_\alpha,x^\mu)$, we can recover the conservation of Q_α and $j_\alpha{}^\mu$ from the solvable generalized Lie algebras of Rühl and Yunn or of Goddard. However, this implies a fictitious nilpotent x^μ and a Hilbert space over the Grassman elements. The correct answer thus consists in applying the GLA directly. At present, the reinterpretation for GLA of the Noether theorems is in accordance with the following scheme:

$$\left.\begin{array}{l}\text{Symmetry of Action } \mathcal{A}\\ \quad\text{of Lagrangian density } \mathcal{L}\\ \qquad\text{(up to } \partial^\mu\mathcal{L}'\text{)}\\ \quad\text{of } S\text{-matrix}\\ U\mathcal{A}U^{-1}=\mathcal{A},\quad U\mathcal{L}U^{-1}=\mathcal{L}\\ \quad+\partial^\mu\mathcal{L}',\quad USU^{-1}=S\end{array}\right\}\rightleftarrows\begin{cases}(d/dt)Q=0\\ \partial_\mu j^\mu=0\\ \\ Q=\int d^3x j^0(x)\\ U=\exp i\alpha Q.\end{cases}\tag{7.6}$$

We replace the lhs by a statement at the level of the algebras,

$$[Q,\mathcal{A}]=0,\quad [Q,\mathcal{L}]=\partial_\mu\mathcal{L}^l,\quad [Q,S]=0\}$$
$$\rightleftarrows\begin{cases}(d/d.)Q=0,\quad \partial_\mu j^\mu=0\\ Q=\int d^3\times j^0(x)\end{cases}\tag{7.7}$$

and we now generalize the brackets to include the GLA multiplication. The physical interpretation of this algebraic version is again equivalent to a symmetry: the discrete permutations of field (or superfield) components produced by the Q charges as generators of the symmetric group in n elements ($n=r+1$, $2r+1$, $2r$, $2r$, 7, for the Lie algebras A_r, B_r, C_r, D_r, G_2). This is in analogy to the realization of the discrete symmetry group of parity by the matrices γ_0 and $\mathbf{1}$ for spinors. For a GLA, we use the same counting, after first replacing it by the Lie algebra acting on the same bounded homogeneous domain (Sternberg and Wolf, 1975).

The inverse Noether theorem yields either a Lie algebra or a GLA, according to whether the conserved currents (or charges) all have integer spin, or contain a subset with half-integer spin. This results from the same considerations as in the discussions leading to (6.67).

We hope that the methods discussed in Secs. II.J and II.K and the relations between graded and ordinary Lie algebras, as discussed in Sternberg and Wolf (1975) will be used to discuss the Noether theorems from a more geometrical point of view.

The GLA $\mathcal{V}$ and its extensions (6.68)–(6.70) represent algebras which contain the Poincaré algebra $\mathcal{P}$, or $\mathcal{P}$ and $\mathcal{F}$ (the $SU(3)_{GN}\otimes SU(3)_{\text{color}}$) as subalgebras. As GLA, they do not come directly under the cases which have been studied and classified by L. O'Raifeartaigh (1965) or under the no-go theorem of S. Coleman and J. Mandula (1967). However, Goddard (1974) has used Eqs. (6.55)–(6.57) to construct the Lie algebra "equivalent" to $\mathcal{V}$, i.e., having the same vector space as carrier space for their representations. According to Levi's theorem, any Lie algebra E can be written uniquely as a semidirect sum

$$E=\Lambda+\Sigma,\tag{7.8}$$

where Λ is semisimple, and Σ solvable, i.e., for $\Sigma^{(1)}=\Sigma$, $\Sigma^{(n)}=[\Sigma^{(n-1)},\Sigma^{(n-1)}]$, a commutator bracket, $\Sigma^{(n)}=0$ for some n. O'Raifeartaigh then proves that there are four classes of inclusions of $\mathcal{P}\subset E$: ($\mathcal{P}=J^{\rho\sigma}+P^\mu$)

(1) $J^{\rho\sigma}\subset\Lambda;\quad P^\mu=\Sigma$

(2) $J^{\rho\sigma}\subset\Lambda;\quad P^\mu\subset\Sigma,\ \mathbf{\Sigma}-P^\mu\neq 0,\ [\Sigma^\nu,\Sigma^\lambda]=0$

[example: inhomogeneous isl(6,c) with 72 "translations"]

(3) $J^{\rho\sigma}\subset\Lambda;\quad P^\mu\subset\Sigma,\ \Sigma^{(n)}=0$

[this includes (6.57)]

(4) $\mathcal{P}\cap\Sigma=0$

[example: the conformal algebra $su(2,2)$].

As we can see, the case (6.57) studied by Goddard is in class (3). The O'Raifeartaigh theorem then forbids mass-splitting within a multiplet, if at least one state has a discrete m^2 eigenvalue for $P_\mu P^\mu|1\rangle$. However, we can deduce the same result directly from Eq. (7.1) for $\mathcal{V}$ and any extension by $\mathcal{F}$, provided Eq. (7.1) holds.

The Coleman–Mandula (1967) theorem has been extended by Haag, Lopuszanski, and Sohnius (1974) to GLA symmetries of the S matrix. However, it should be remarked that symmetry breaking according to the Goldstone scheme will tend to violate the requirement of additivity assumed by Haag *et al.* in their "no-go" theorem.

Goddard (1974) succeeds in defining a complex-valued inner product in a quadrupled Hilbert space (one each for v_5, v_6, $v_5\wedge v_6$, 1), but loses positive-definiteness. In either case, the Coleman–Mandula theorem doesn't apply.

B. Improved renormalizability in a Yukawa and ϕ^4 interaction

The first example of a supersymmetric interaction was provided by Wess and Zumino (1974b). They added to the free Lagrangian (6.64),

$$\mathcal{L}_{\text{free}}=\tfrac{1}{2}(\partial_\mu A)^2+\tfrac{1}{2}(\partial_\mu B)^2+i\bar\psi\partial\!\!\!/\psi+2F^2+2G^2,\tag{7.9}$$

a mass term

$$\mathcal{L}_m=2m(FA+GB-\tfrac{1}{2}\bar\psi\psi)\tag{7.10}$$

and an interaction

$$\mathcal{L}_g=g[F(A^2-B^2)+2GAB-\bar\psi(A-\gamma_5B)\psi].\tag{7.11}$$

The terms (7.9)–(7.11) all transform invariantly up to a 4-divergence, under (6.61) as amended through the introduction of the field B [see discussion after (6.61)]. One can also add a term [see δF in (6.66)]

$$\mathcal{L}_\lambda=\lambda F.\tag{7.12}$$

A and F are scalar fields, B and G are pseudoscalars, and ψ is a Majorana spinor. F and G are auxiliary and satisfy the equations of motion,

$$\begin{aligned}-F&=\tfrac{1}{4}g(A^2-B^2)+\tfrac{1}{2}mA+\lambda/4\\ -G&=\tfrac{1}{2}gAB+\tfrac{1}{2}mB.\end{aligned}\tag{7.13}$$

Eliminating F and G from the Lagrangian, we find

$$\begin{aligned}\mathcal{L}=&\tfrac{1}{2}(\partial_\mu A)^2+\tfrac{1}{2}(\partial_\mu B)^2+\bar\psi(i\partial\!\!\!/-m)\psi-\tfrac{1}{2}m^2(A^2+B^2)\\ &-\tfrac{1}{2}gmA(A^2+B^2)-\tfrac{1}{8}g^2(A^2+B^2)^2-g\bar\psi(A-\gamma_5B)\psi\\ &-\tfrac{1}{2}\lambda[\tfrac{1}{4}\lambda+mA+\tfrac{1}{2}g(A^2-B^2)],\end{aligned}\tag{7.14}$$

which represents a nonlinear realization of supersymmetry, corresponding to the elimination of F and G in the linear (6.61), (6.66). We can regroup the part of the "potential" which involves the A and B fields only,

$$V = -\mathcal{L}(A,B) = \tfrac{1}{2}m^2[A + (\lambda/2m)]^2 + \tfrac{1}{2}m^2B^2 + \tfrac{1}{4}g\lambda(A^2 - B^2) + \tfrac{1}{2}mgA(A^2 + B^2) + \tfrac{1}{8}g^2(A^2 + B^2)^2. \tag{7.15}$$

$-\mathcal{L}(A,B)$ is the "potential" V whose extrema we shall later study in our search for Goldstone-like solutions. Note that the λF term can be eliminated by a shift in A. Salam and Strathdee (1974e) have shown how to derive (7.14) using the superfield calculus. It results from writing

$$\mathcal{L} = \tfrac{1}{8}(\bar{Q}Q)^2(\phi_+\phi_-) - \tfrac{1}{2}\bar{Q}Q(P(\phi_+) + P(\phi_-)), \tag{7.16}$$

with $\phi_- = \phi_+{}^*$ as in the discussion of (6.66). $Q = Q(\Lambda V)$ is given by (6.48), P is a polynomial of order three. It is apparent that the four-volume integral vanishes, trivially, so that

$$\delta \int d^4 \times \mathcal{L} = \int d^4 \times \bar{\epsilon}Q(\Lambda V)\mathcal{L} = \bar{\epsilon}(\partial/\partial\bar{\theta})d^4 \times \mathcal{L} + \text{surface term} = 0.$$

The relevant terms in $\mathcal{L}$ are obtained by setting $\theta = 0$, yielding (7.14).

Before we study the effects of renormalization (and disregarding the $\mathcal{L}_\lambda$ term at this stage), we already observe in (7.14) the expected result of a symmetry: A, B, and ψ have related bare masses. The three interactions (ϕ^3, ϕ^4, and the Yukawa term) have related couplings $\frac{1}{2}gm$, $\frac{1}{8}g^2$, g. Supersymmetry thus does indeed play the role of a symmetry [which we can interpret as a discrete symmetry as explained in Eq. (7.7)]. After elimination of F and G, the conserved current is

$$j_\alpha{}^\mu = (\eth(A - \gamma_5B)\gamma^\mu\psi + im(A + \gamma_5B)\gamma^\mu\psi + \tfrac{1}{2}ig(A + \gamma_5B)^2\gamma^\mu\psi)_\alpha; \quad \partial_\mu j_\alpha{}^\mu = 0. \tag{7.17}$$

The conservation equation can be checked directly, using the equations of motion and the identity

$$\psi(\bar{\psi}\psi) \equiv \gamma_5\psi(\bar{\psi}\gamma_5\psi). \tag{7.18}$$

Wess and Zumino (1974b) showed that the theory of Eq. (7.14) is less divergent than if the masses and couplings were independent. For instance, in the one-loop approximation, the quadratic divergence of the mass renormalization for A and B cancels out. The logarithmic divergence of the vertex correction to the Yukawa interaction also cancels between the A and B terms, leaving a finite vertex correction.

In its original form, before elimination of F and G, the theory can be regularized (by the method of Pauli and Villars, for instance) without spoiling supersymmetry. Thus, the Ward identities following from Eq. (7.17) in perturbation theory are expected to be satisfied. If one uses $\mathcal{L}_{\text{free}} + \mathcal{L}_m$ as the unperturbed Lagrangian, one finds as propagators

$$\langle AA\rangle = \langle BB\rangle \sim \Delta_c,$$
$$\langle FF\rangle = \langle GG\rangle \sim \Box\Delta_c,$$
$$\langle AF\rangle = \langle BG\rangle \sim -m\Delta_c.$$

In the one-loop approximation, there is only one renormalization needed, a logarithmically divergent wave function renormalization constant Z, common to A, B, ψ, F, and G,

$$\begin{cases} Z = 1 - 4g^2I \\ I = -i\displaystyle\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+m^2)^2} = \frac{1}{16\pi^2}\int^\infty \frac{d\chi}{\chi}. \end{cases} \tag{7.19}$$

No diagonal mass is generated for either A or B.

The quadratic divergence of the self-energy cancels out and the remaining logarithmically divergent contribution is proportional to $-p^2$. Similarly, the ψ self-energy is proportional to $i\gamma^\mu p_\mu$, and the corrections to the off-diagonal mass terms mFA and mGB cancel. Thus the only mass renormalization is that due to the wave function renormalizations,

$$m_r = mZ. \tag{7.20}$$

Corrections to gFA^2, $-gFB^2$, $2gGAB$ cancel, and the finite corrections to the Yukawa terms vanish for zero external momenta. One finds

$$g_r = gZ^{\frac{3}{2}}. \tag{7.21}$$

No divergent trilinear or quadrilinear interactions are generated. Iliopoulos and Zumino (1974) and Tsao (1974) have investigated this model in higher orders. For two-loop diagrams they calculated explicitly the various contributions and again found no mass and vertex corrections. They proved to all orders that the theory is renormalized with one single renormalization constant, Z, the wave function renormalization, so that Eqs. (7.20) and (7.21) hold. Note that theories like (7.10)–(7.12) are renormalizable even without supersymmetry (i.e., with arbitrary m_i and g_{ijk}), but supersymmetry has resulted in highly improved renormalizability. There is thus a possibility that some *a priori* nonrenormalizable model might become renormalizable when supersymmetry is imposed.

The full set of Ward identities corresponding to $\mathcal{V}$ supersymmetry has been derived by Iliopoulos and Zumino. They have also adapted a regularization scheme based upon the insertion of higher derivative terms in $\mathcal{L}$, in particular in the kinetic energy term $\mathcal{L}_{\text{free}}$. They use the insertion

$$\mathcal{L}_\xi = \xi[\tfrac{1}{2}(\partial_\mu\Box A)^2 + \tfrac{1}{2}(\partial_\mu\Box B)^2 + i\Box\bar{\psi}\eth\Box\psi + 2(\Box F)^2 + 2(\Box G)^2. \tag{7.22}$$

$\mathcal{L}_\xi$ transforms like $\mathcal{L}_{\text{free}}$ under the Q_α. It is sufficient to make all diagrams finite, including tadpoles.

Explicit symmetry breaking (in contradistinction to "spontaneous" breaking) is tried by the above authors in the form of a term

$$\mathcal{L}_{\text{SB}} = cA \tag{7.23}$$

(rather than $\mathcal{L}_\lambda$, which was invariant under $\mathcal{V}$). $\mathcal{L}_{\text{SB}}$ is not invariant under $\mathcal{V}$, and breaks current conservation,

$$\partial_\mu j^\mu = c\psi. \tag{7.24}$$

However, the entire renormalization program is unaffected, with only finite corrections appearing due to $\mathcal{L}_{\text{SB}}$. The masses are now only related by the equation

$$m_A{}^2 + m_B{}^2 = 2m_\psi{}^2 \tag{7.25}$$

derived in the tree approximation. In higher order the equation gets finite corrections.

The $\mathcal{L}_{SB}$ term can be eliminated by a simultaneous shift of A and F, $A \to A + a$, $F \to F + f$, with the equations

$$\begin{cases} 4f + 2ma + ga^2 = 0 \\ 2mf + 2gaf + c = 0 \end{cases} \tag{7.26}$$

which ensure vanishing of linear terms in A or F. Eliminating f we get a cubic equation for a,

$$a(2m + ga)\{(m + ga) + \tfrac{1}{2}c\} = 0. \tag{7.27}$$

Taking the limit $c \to 0$, this has three solutions,

$$a_1 = 0, \quad a_2 = -2m/g, \quad a_3 = -m/g$$

(a_3 is the "central" value). Taking in Eq. (7.14) $B = \psi = 0$ and $A \to a$ we have a "potential" $-\mathcal{L}(a) = V(a)$

$$\begin{aligned} V(a) &= \tfrac{1}{2}m^2a^2 + \tfrac{1}{2}gma^3 + \tfrac{1}{8}g^2a^4 + ca \\ &= \tfrac{1}{2}a^2(m + \tfrac{1}{2}ga)^2 + ca. \end{aligned} \tag{7.28}$$

Our solutions a_i correspond to the stationarity points of $V(a)$. We see that $V(a_1) = 0$, $V(a_2) = -2mc/g \to 0$, $V(a_3) = \frac{1}{8}(m^4/g^2) - c(m/g) \to \frac{1}{8}(m^4/g^2)$ so that a_1 and a_2 produce minima, and a_3 is a maximum. This is unstable, with no possible stabilization through a sign change. From Eq. (7.14) we see that (for $c \to 0$, i.e., vanishing of explicit symmetry breaking)

$$m_\psi = m + ga \to 0 \quad \text{for} \quad a_3$$

so that this is a "Goldstone spinor" solution, which is, however, unstable. Notice that Eq. (7.25) then requires one of the two bosons to be a tachyon, if the other one is massive. Indeed, from Eq. (7.14) we have to first order in g

$$\begin{aligned} m_A{}^2 &= -m^2 - 3gma - \tfrac{3}{2}g^2a^2; \quad \text{for} \quad a_3, m_A{}^2 = \tfrac{1}{2}m^2 \\ m_B{}^2 &= -m^2 - 3gma - \tfrac{1}{2}g^2a^2; \quad \text{for} \quad a_3, m_B{}^2 = -\tfrac{1}{2}m^2. \end{aligned}$$

Salam and Strathdee (1974c) have investigated directly the idea of a Goldstone spinor in that same Lagrangian, with similar results.

To include the effects of an internal symmetry, Salam and Strathdee (1974e) have rewritten Eq. (7.16) with ϕ_+ and $\phi_- = \phi_+{}^*$ as 3×3 matrices of superfields, behaving like the real representation (3,3) of $SU(2)_L \otimes SU(2)_R$ (we use the dimensionalities in this notation, i.e., $I_L = 1$, $I_R = 1$). The result is

$$\begin{aligned} \mathcal{L} = {} & \tfrac{1}{8}(\bar{Q}Q)^2 \operatorname{Tr}(\phi_-{}^{\sim}\phi_+) - \tfrac{1}{4}(\bar{Q}Q) \operatorname{Tr}(\phi_+{}^{\sim}\phi_+ + \phi_-{}^{\sim}\phi_-) \\ & + g\bar{Q}Q(\det\phi_+ + \det\phi_-). \end{aligned} \tag{7.29}$$

In terms of component fields this is

$$\begin{aligned} \mathcal{L} = {} & \partial_\mu A_-{}^{ia}\partial_\mu A_+{}^{ia} + F_-{}^{ia}F_+{}^{ia} + \tfrac{1}{2}i\bar{\psi}^{ia}\eth\psi^{ia} \\ & + M(A_+{}^{ia}F_+{}^{ia} + A_-{}^{ia}F_-{}^{ia} - \tfrac{1}{2}\bar{\psi}^{ia}\psi^{ia}) \\ & + g\epsilon^{ijk}\epsilon^{abc}(A_+{}^{ia}A_+{}^{jb}F_+{}^{kc} - \tfrac{1}{2}A_+{}^{ia}\bar{\psi}^{jb}(1 + i\gamma_5)\psi^{kc} \\ & \quad + A_-{}^{ia}A_-{}^{jb}F_-{}^{kc} - \tfrac{1}{2}A_-{}^{ia}\bar{\psi}^{jb}(1 - i\gamma_5)\psi^{kc}), \end{aligned} \tag{7.30}$$

where

$$A_\pm = (1/\sqrt{2})(A \pm iB), \quad F_\pm = (1/\sqrt{2})(F \pm iG), \tag{7.31}$$

and ψ^{ia} is again a Majorana spinor. Note that "spontaneous" breaking of supersymmetry occurs when a massless ψ^{ia} mixes with the vacuum when transforming under the Q_α. Looking at Eq. (6.61) or (6.66) or at a chiral-summed form of the latter,

$$\begin{cases} \Delta A = -C\psi, \\ \Delta B = -C\gamma_5\psi, \\ \Delta\psi = (F + G\gamma_5) - \tfrac{1}{2}i\eth(A + B\gamma_5), \\ \Delta F = \tfrac{1}{2}iC\eth\psi, \\ \Delta G = \tfrac{1}{2}iC\gamma_5\eth\psi, \end{cases} \tag{7.32}$$

we see that F is the scalar field which is connected to ψ under the transformation

$$\delta_{\alpha\beta}\Delta\psi = \{Q_\alpha, \psi_\beta\}. \tag{7.33}$$

Note that for didactic reasons we have written (7.32) as Δ, a discrete transformation ($\epsilon = 1$) involving only the GLA, without going through the Grassmann elements ΛV. Taking Eq. (7.33) between vacuum states we should get vanishing contributions, except if the vacuum is not superinvariant. In the latter case,

$$\langle\Delta\psi\rangle = \langle F\rangle. \tag{7.34}$$

Indeed, Salam and Strathdee (1974c) showed that the a_3 solution (Iliopoulos and Zumino, 1974) of Eq. (7.27) corresponds (for $c = 0$) to

$$\langle F\rangle = m^2/4g - \tfrac{1}{4}\lambda \tag{7.35}$$

so that the vacuum does break supersymmetry "spontaneously" as in Eq. (7.34), with the massless Majorana ψ as Goldstone fermion. For (7.30) they got the equations

$$\begin{aligned} & M\langle A_\pm{}^{ia}\rangle + g\epsilon^{ijk}\epsilon^{abc}\langle A_\pm{}^{jb}\rangle\langle A_\pm{}^{kc}\rangle = -\langle F_\mp{}^{ia}\rangle \\ & M\langle F_\mp{}^{ia}\rangle + 2g\epsilon^{ijk}\epsilon^{abc}\langle A_\mp{}^{jb}\rangle\langle F_\mp{}^{kc}\rangle = 0. \end{aligned}$$

Diagonalizing the $\langle A_+{}^{ia}\rangle$ and choosing an $SU(2)$ symmetric solution $\langle A_+{}^{ia}\rangle = \lambda\cdot\mathbf{1}$, the equations reduce to the cubic

$$\lambda(M + 2g\lambda)(M + 4g\lambda^*) = 0. \tag{7.36}$$

There are thus three solutions λ_i, all conserving parity:

$$\lambda_1 = 0, \quad \lambda_2 = -M/2g, \quad \lambda_3 = -M/4g.$$

λ_1 corresponds to unbroken supersymmetry. For λ_3 one finds

$$\langle A_\pm{}^{ia}\rangle = -(M/4g)\delta^{ia}, \quad \langle F_\pm{}^{ia}\rangle = -(M^2/8g)\delta^{ia},$$

i.e., both supersymmetry ($\langle F\rangle \neq 0$) and $SU(2)_L \times SU(2)_R$ (since $\langle A\rangle \neq 0$) are spontaneously broken. This is an unstable solution, with some mesons acquiring imaginary masses. λ_2 is stable; it has $\langle F_\pm\rangle = 0$, so that there is no spontaneous supersymmetry breaking. It turns out that the internal symmetry is spontaneously broken instead, with the entire $I = 1$ supermultiplet staying at $M_1 = 0$, while the $I = 0, 2$ supermultiplets have $M_0 = M$, $M_2 = 2M$. Changing to a partly local gauge $SU(2)_{\text{local}} \otimes SU(2)_{\text{global}}$, Salam and Strathdee (1974e) showed that some of the fields in that $I = 1$ superfield (which now contains a Yang–Mills field) acquire masses. The Higgs mechanism is working and provides masses for the spinor and vector fields.

Summing up the situation with respect to spontaneous supersymmetry breaking, we do not yet have here a stable example where this really occurs ($\langle F\rangle \neq 0$). Iliopoulos and Zumino (1974, see the Appendix A) even conjectured that it might be forbidden, but we shall see in the next example (supersymmetry with an Abelian local gauge) that it can be done.

Ferrara, Iliopoulos, and Zumino (1974) have investigated the Gell-Mann–Low ("renormalization group") eigenvalue equations for the $\mathcal{V}$ supersymmetric model in (7.14). They find that as implied by Eqs. (7.20) and (7.21), there is no eigenvalue solution other than $g = 0$, a theory of free fields. Thus the effective coupling increases indefinitely with k^2. The result is the same for $m \to 0$, a $\mathcal{W}$ supersymmetric theory.

The improved renormalizability of Lagrangian theories due to supersymmetry has led to the expectation that some otherwise unrenormalizable Lagrangians might become renormalizable when supersymmetry is imposed. There is as yet no example where this has happened. Lang and Wess (1974) and Woo (1974) tried Lagrangians with A^6, B^6, $\bar\psi A^2\psi$, $\bar\psi AB\gamma_5\psi$ $\mathcal{V}$ supersymmetric terms. Although there were numerous divergence cancellations, the theory remains unrenormalizable.

C. Supersymmetry and Abelian gauges; existence of a Goldstone–Higgs case

Wess and Zumino (1974c) have constructed a model theory which appears to involve a minimal set of fields sufficient for the inclusion of an interaction resembling electrodynamics, i.e., a coupling to a conserved charge resulting from an Abelian local gauge. This is the final Lagrangian, after the elimination of several auxiliary fields:

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\sum_{i=1,2}\left[(\partial_\mu A_i)^2 + (\partial_\mu B_i)^2 + i\bar\psi_i\partial\!\!\!/\psi_i\right] \\ &\quad - \tfrac{1}{2}\sum_{i=1,2}\left[m^2(A_i^2 + B_i^2) + im\bar\psi_i\psi_i\right] \\ &\quad - \tfrac{1}{4}V_{\mu\nu}V^{\mu\nu} + \frac{i}{2}\bar\chi\partial\!\!\!/\chi \\ &\quad - e\left[V^\mu(A_1\overleftrightarrow{\partial_\mu}A_2 + B_1\overleftrightarrow{\partial_\mu}B_2 - i\bar\psi_1\gamma_\mu\psi_2)\right. \\ &\quad \left. + i\bar\chi\{(A_1 + \gamma_5 B_1)\psi_2 - (A_2 + \gamma_5 B_2)\psi_1\}\right] \\ &\quad + \frac{e^2}{2}\left[V_\mu V^\mu\Big(\sum_{i=1,2}(A_i^2 + B_i^2)\Big) + (A_1B_2 - A_2B_1)^2\right]. \end{aligned} \tag{7.37}$$

Here the charged fields A, B, and ψ are given in terms of their real components [and $A\overleftrightarrow{\partial_\mu}B \equiv A\partial_\mu B - (\partial_\mu A)B$]

$$\{A = (1/\sqrt{2})(A_1 + iA_2),\quad B = (1/\sqrt{2})(B_1 + iB_2),\quad \psi = (1/\sqrt{2})(\psi_1 + i\psi_2). \tag{7.38}$$

We observe that beyond the electromagnetic interactions of the massive charged fields $A(0^+)$, $B(0^-)$, and $\psi(\frac{1}{2})$, we have a Yukawa-like coupling of the same strength, of χ, ψ, and A or B (as if the electromagnetic field were replaced by the spinor χ). The masses of A, B, and ψ (they embrace two real $J_{\max} = \frac{1}{2}$ representations) are equal, and the couplings are only e and $e^2/2$. This model is similar to the analogous minimal extension of gravitation (Volkov and Soroka, 1973) which involves $J = \frac{3}{2}, 1, \frac{1}{2}$ together with the $J = 2$ graviton. Our fields V^μ and χ belong to a single $J_{\max} = 1$ multiplet and are both massless. Obviously, this theory is both superinvariant and electric-gauge invariant.

The construction of this model involves some complication, with the original Lagrangian appearing as an infinite power series in e and highly nonrenormalizable. However, superinvariance, gauge invariance, and an additional symmetry corresponding to the commutator of Q_α with the "electric charge" local gauge generator provide a choice of the latter hybrid gauge such that $\mathcal{L}$ is greatly simplified. In the one-loop approximation, supersymmetry causes various cancellations between divergent contributions and the model is renormalizable. The masses remain equal within the multiplets, and

$$e_{\text{renorm.}} = e(1 + ie^2 I),\quad I = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^4}. \tag{7.39}$$

In both vacuum polarization and light-by-light scattering there is no need for any special treatment (regularization or other) of the diagrams to ensure gauge invariance of the results.

Fayet and Iliopoulos (1974) have added to this Lagrangian a parity-breaking supersymmetric and gauge-invariant term. It amounts to the appearance of off-diagonal mass terms between A_iB_j,

$$-\xi e(A_1B_2 - A_2B_1). \tag{7.40}$$

Diagonalizing, they obtain the fields $\tilde A_i$, $\tilde B_i$:

$$\tilde A_1 = (A_1 + B_2)/\sqrt{2},\quad \tilde A_2 = (A_2 + B_1)/\sqrt{2},\quad \tilde B_1 = (B_1 - A_2)/\sqrt{2},\quad \tilde B_2 = (B_2 - A_1)/\sqrt{2} \tag{7.41}$$

with mass terms

$$-\tfrac{1}{2}(m^2 + \xi e)(\tilde A_1^2 + \tilde B_1^2) - \tfrac{1}{2}(m^2 - \xi e)(\tilde A_2^2 + \tilde B_2^2) \tag{7.42}$$

and a quartic self-coupling,

$$-\tfrac{1}{8}e^2(\tilde A_1^2 - \tilde A_2^2 + \tilde B_1^2 - \tilde B_2^2)^2. \tag{7.43}$$

We note that the masses of the A, B components are thus no longer equal to the mass of ψ, i.e., supersymmetry appears broken. The role of the Goldstone particle is played by the massless χ spinor.

There are now two cases. Taking $\xi e > 0$, we have

case a: $m^2 - \xi e > 0$
case b: $m^2 - \xi e < 0$.

In case a, the origin is an absolute minimum of the A, B fields. Ordinary ("electric") gauge invariance in unbroken, V^μ is massless and the "electromagnetic" interaction is given by

$$eV^\mu\Big[-\sum_{i=1,2}(A_i\overleftrightarrow{\partial_\mu}\tilde B_i) + i\bar\psi_1\gamma_\mu\psi_2\Big] + \frac{e^2}{2}V_\mu V^\mu\Big[\sum_{i=1,2}(\tilde A_i^2 + \tilde B_i^2)\Big]. \tag{7.44}$$

In case b, ordinary gauge symmetry is also spontaneously broken, and we shall see that we have a Higgs–Kibble mechanism at work. Observing that the "potential" is invariant under rotations in the $(\tilde A_2\tilde B_2)$ plane, we settle on the direction of $\tilde{A}_2$ and translate that field,

$$\tilde{A}_2 \rightarrow \tilde{A}_2 + a, \quad a^2 = 2(\xi e - m^2)/e^2 > 0 \tag{7.45}$$

which yields modified mass terms (due to the quartic interaction),

$$m^2(\tilde{A}_1) = 2m^2; \quad m^2(\tilde{B}_1) = 2m^2; \quad m^2(\tilde{A}_2) = 2(\xi e - m^2) > 0; \quad m^2(\tilde{B}_2) = 0; \quad m^2(V_\mu) = \xi e - m^2 > 0. \tag{7.46}$$

As for the spinors, the mass term in the Lagrangian becomes

$$-\tfrac{1}{2}im(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2) - ie\bar{\chi}a(\gamma_5\psi_2 - \psi_1). \tag{7.47}$$

This will be diagonalized into the new spinor fields:

$$\eta_1 = \tfrac{1}{2}[(1+\cos\beta)\psi_1 - (1-\cos\beta)\gamma_5\psi_2 - \sqrt{2}\sin\beta\chi],$$
$$\eta_2 = \tfrac{1}{2}[(1-\cos\beta)\gamma_5\psi_1 + (1+\cos\beta)\psi_2 + \sqrt{2}\sin\beta\gamma_5\chi],$$
$$\zeta = (\sqrt{2})^{-1}\sin\beta(\psi_1 + \gamma_5\psi_2) + \cos\beta\chi,$$
$$\beta = \arctan(ga/m),$$

and the masses will then be

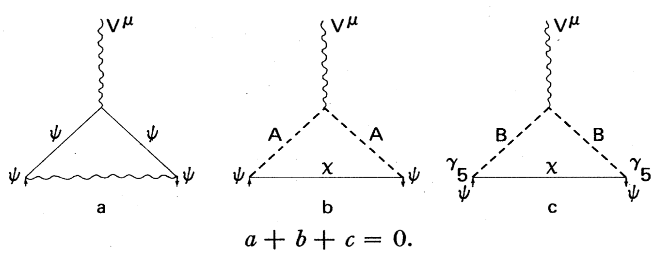
$$m(\eta_1) = m(\eta_2) = (m^2 + e^2a^2)^{\frac{1}{2}} = (2\xi e - m^2), \quad m(\zeta) = 0. \tag{7.48}$$

We observe that spontaneous breaking of supersymmetry is still due to a massless Goldstone spinor, the ζ. On the other hand, spontaneous breakdown of the Abelian gauge invariance has occurred, and as a result of the Higgs–Kibble mechanism, the gauge field V^μ has acquired a mass. The $\tilde{B}_2$ field is the Goldstone boson which was supposed to help break gauge invariance spontaneously and which is now removable, due to the emergence of a longitudinal component in V^μ. Thus the Higgs–Kibble mechanism has acted for gauge invariance, while supersymmetry is still in a pure Goldstone situation. Fayet and Iliopoulos have checked on the role of the ζ by studying the vacuum expectation values $\langle A_2\rangle$, $\langle B_1\rangle$, $\langle F_2\rangle$, $\langle G_1\rangle$, etc.

The renormalizability of this Abelian gauge supersymmetric model has not been checked beyond the one-loop approximation. However, some insight into the cancellations due to supersymmetry was provided by Ferrara and Remiddi (1974). They noted that a Pauli term in the electromagnetic interaction

$$(g-2)\frac{1}{4m}\bar{\psi}\sigma_{\mu\nu}\psi V^{\mu\nu}$$

would break supersymmetry, which thereby forbids the existence of an anomalous magnetic moment. When checking how this occurs in terms of Feynman diagrams, they found that the Yukawa $\chi A\psi$ and $\chi\gamma_5 B\psi$ interactions provide the necessary cancellations:



$$a + b + c = 0.$$

D. The Yang–Mills field; fermion number gauges

Salam and Strathdee (1974d) and Ferrara and Zumino (1974) have constructed the supersymmetric version of the Yang–Mills field, i.e., a supersymmetric Lagrangian which is also invariant under the action of a local non-Abelian gauge group (a gauge "of the second kind"). We refer the reader to the original papers and to the review by Salam and Strathdee (1974f) for the details of the construction. The end result involves n "matter fields" which are realized by a $J_{\max} = \frac{1}{2}$ supermultiplet each ("scalar" superfields, in the notation of these authors) and one $J_{\max} = 1$ supermultiplet (the Yang–Mills superfield), all lying in the adjoint representation of the gauge algebra. In principle, the matter fields are not restricted to the adjoint representation: also, they can be massive, whereas the $J_{\max} = 1$ superfield is massless because of gauge invariance. However, there is then no way of introducing a fermion number group. The spinor component of the $J_{\max} = 1$ gauge field is a Majorana spinor, which cannot carry a charge since it is its own charge conjugate by definition. This can be set right through a mixing with another Majorana spinor, belonging to the matter fields. For this purpose, the matter fields become massless (like the gauge field) and are restricted to the adjoint representation of the gauge group. Masses may then be due to spontaneous symmetry breaking, but this appears impossible in the linear version and the hope would thus be that it can be due to higher order corrections (S. Coleman and E. Weinberg, 1973).

The final Lagrangian becomes

$$\mathcal{L} = \mathrm{Tr}\{-\tfrac{1}{4}V_{\mu\nu}V^{\mu\nu} + \tfrac{1}{2}(D_\mu A)^2 + \tfrac{1}{2}(D_\mu B)^2 + \tfrac{1}{2}i\bar{\Phi}\not{D}_\mu\Phi - ig\bar{\Phi}[A + \gamma_5 B, \Phi] + \tfrac{1}{2}g^2(i[A,B]^2)\}. \tag{7.49}$$

All fields are written as $N \times N$ matrices, over which the trace is taken for $\mathcal{L}$.

$$D_\mu C^k = \partial_\mu C^k + gf^{klm}V_\mu{}^l C^m, \tag{7.50}$$

$$\Phi = \frac{1}{\sqrt{2}}(\psi_V + i\psi_A), \tag{7.51}$$

where ψ_V is the Majorana field in the V^μ multiplet, and ψ_A belongs to a matter multiplet with A and B. The baryonic gauge is given by

$$\Phi \rightarrow e^{i\alpha}\Phi, \quad \Phi^* \rightarrow e^{-i\alpha}\Phi^*.$$

The conserved supercurrent is given by

$$j^\mu = \mathrm{Tr}\{-\tfrac{1}{4}V_{\nu\rho}[\gamma^\nu,\gamma^\rho]\gamma^\mu\Phi + ig[A,B]\gamma_5\gamma^\mu\Phi - i\not{D}(A - \gamma_5 B)\gamma^\mu\Phi\}. \tag{7.52}$$

All coupling constants in Eq. (7.49) are given by g, the gauge coupling. The theory will be asymptotically free provided the Callan–Symanzik function β in the Gell-Mann–Low eigenvalue problem remains negative (and if the theory remains renormalizable). For the case of n massive matter supermultiplets, and $SU(N)$ symmetry (Ferrara and Zumino, 1974)

$$\beta = -(g^3/16\pi^2)(3-n)N. \tag{7.53}$$

Thus $n < 3$ preserves asymptotic freedom, and $n = 3$ yields a finite renormalization constant ($\beta = 0$).

Suzuki (1974) has studied the possibility of spontaneous symmetry breakdown and the emergence of masses for the various fields. It appears that the supersymmetric limit is not realized as a local minimum in every possible direction in the parameter space of independent couplings. Nevertheless, asymptotic freedom will not be ruined by the inclusion of "soft" explicit or spontaneous breaking. "Soft" implies canonical dimensionality less than four. The β functions are unaffected, but new and superrenormalizable couplings enter into the renormalization group equation. A search for asymptotic freedom with a massive gauge supermultiplet failed to produce such an example.

A model in which the gauge is chiral and thus doubles the gauge supermultiplet provides an alternative way of generating a complex spinor gauge field capable of carrying a baryonic charge (R. Delbourgo, A. Salam, and J. Strathdee, 1974). The theory is symmetric between $V - A$ and $V + A$. Explicit masses are still forbidden, however, in the matter fields.

In Sec. VI [see Eqs. (6.68)–(6.70)] we discussed a "nontrivial" inclusion of internal symmetry (the Yang–Mills and other cases we discussed here being "trivial" in the sense that they do not involve a supersymmetry GLA other than $\mathcal{V}$ or $\mathcal{W}$). As described in Sec. VI, the main physical result appears to consist of a reproduction of the quark model states when the internal group is taken to be $SU(3)_{\text{color}}\times SU(3)_{GN}$. The local gauge result in (7.49) might on the other hand be taken to represent an important physical requirement imposed on phenomenological hadron fields in their $m\rightarrow 0$ limit, which would explain saturation at the three-quark level through the requirement that physical states belong to the adjoint representation of the internal algebra. However, this would point to a special role for baryons in $\underset{\sim}{8}$ as against $\underset{\sim}{10}$.

VIII. RESULTS AND PROSPECTS

The actual physical results to date can thus be summed up in the following list:

(1) Like every other symmetry, quantum statistics (Bose–Fermi) independence (supersymmetry) implies very strong constraints on couplings and masses. Although all models studied to date are only formal models which do not correspond to reality, perhaps a modification of the method might lead to an explanation of some of the observed regularities in the mass spectrum of the hadrons. These observed regularities, which indicate relations involving small integers between masses of fermions and bosons, have been connected to various aspects of the quark model in a heuristic fashion. Perhaps these relations might emerge from a supergauge symmetry for a more sophisticated Lagrangian. Some better understanding of a relativistic quark model may already be provided by the GLA approach [see Eqs. (6.68)–(6.70)].

(2) Renormalizability is sometimes improved. In the examples cited the supersymmetry greatly reduces the number of renormalization constants. Previously unrelated types of interactions become connected via a supergauge invariant Lagrangian, which may thus help in unification schemes.

(3) We remind the reader that an independent pathway leading to supersymmetry was evolved in the search for a Goldstone role for fermions. Furthermore, other attempts were made to use fermions in spontaneous symmetry breakdown of the supersymmetric models in strong and weak interactions. It seems that the classical solution for a stable symmetry breakdown cannot be used in the more sophisticated models, but radiative corrections may improve the situation.

(4) The straightforward generalization of the Yang–Mills gauge requires the fermion field to behave like the Yang–Mills vector field under the internal symmetry. If we impose the graded Lie symmetry upon the phenomenological fields, this would explain the appearance of the baryons in an octet, and thus the nonexistence of quarks. This seems very intriguing. On the other hand, one may include the internal degrees of freedom nontrivially in the context of a larger GLA imposed on the fundamental fields. To reproduce the observed multiplets, the internal symmetry has to contain the color variable as well. We thus are led to the most interesting challenge: is there a formalism which will fix uniquely the structure of the fundamental system and its interactions: quarks, leptons, Yang–Mills "color" gauge fields, Higgs–Kibble fields to provide masses, the Weinberg–Salam intermediate bosons, etc.

ACKNOWLEDGMENT

One of the authors (Y.N.) would like to thank Professor M. Gell-Mann for numerous discussions and collaborative efforts in attempting to understand the origins of supersymmetry and some of its applications.

APPENDIX

In this appendix we quickly review some of the basic facts concerning Clifford algebras and exterior algebras. Let V be a vector space over a field K. (In the cases of interest to us K will either be the field of real numbers or the field of complex numbers.) We recall that the tensor algebra, $T(V)$, is the graded, associative algebra

$$T(V) = T^0(V) + T^1(V) + T^2(V) + \cdots = K + V + V\otimes V + \cdots,$$

where $T^0(V) = K$, $T^1(V) = V$, and $T^k(V)$ is the space of contravariant tensors of degree k, i.e., $T^k(V)$ is the k-fold tensor product of V with itself. We regard V as the subspace, $T^1(V)$, of $T(V)$. The algebra $T(V)$ has the following "universal" property: let l be any linear map of V into some associative algebra, A with unit. Then there exists a unique homomorphism, φ, of $T(V)$ into A such that φ coincides with l on elements of V.

Let Q be some quadratic form on V, and let $I(Q)$ be the two sided ideal in $T(V)$ generated by the elements $v\otimes v - Q(v)1$ as v ranges over V. The quotient algebra, $T(V)/I(Q)$ is called the Clifford algebra of Q, and will be denoted by $C_Q(V)$. No elements of V lie in $I(Q)$, and so the map $V\rightarrow T(V)/I(Q) = C_Q(V)$ is injective, and we can regard V as a subspace of $C_Q(V)$. If l is any map of V into an associative algebra A with unit satisfying the identity

$$l(v)^2 = Q(v)1,$$

then the homomorphism, φ, from $T(V)$ to A must vanish on the ideal $I(Q)$, and hence defines a homomorphism, which we continue to denote by φ, from $C_Q(V)$ to A. Thus $C_Q(V)$ can be regarded as the "universal algebra" among algebras satisfying the above identity.

The generators of the ideal $I(Q)$ are not homogeneous, and therefore, in general, the gradation of $T(V)$ is lost when we pass to the quotient — the product $v \cdot v = Q(v)$ has degree zero instead of degree two, and is not zero if $Q(v) \neq 0$. However, the generators all have even degrees, so that $C_Q(V)$ is $\mathbf{Z}_2$ graded: it makes sense to talk of even and odd elements, and they behave properly under product. There is one case where the generators of $I(Q)$ are homogeneous, and that is when Q is identically zero. In this case the algebra $C_Q(V)$ is called the exterior algebra or the Grassmann algebra of V and denoted by $\Lambda(V)$. The multiplication between two elements, μ and ν of $\Lambda(V)$, is denoted by $\mu \wedge \nu$. The algebra $\Lambda(V)$ is a graded algebra and is graded commutative, in the sense given to this term in the text. It is the universal algebra for maps satisfying $l(v)^2 = 0$.

The Clifford algebra $C_Q(V)$ is generated by 1 and the elements of V. Any graded derivation of $C_Q(V)$ must vanish on 1, and is thus determined by its action on elements of V, which can be arbitrary. In particular, any v^* in the dual space of V induces a derivation which is determined by sending v to $v^*(v)\cdot 1$. This graded derivation is known as the interior product by v^* and is denoted by i_{v^*}. Let us denote left multiplication by an element v of V by e_v, so that $e_v w = vw$. Then

$$i_{v^*}e_v w = i_{v^*}(vw) = v^*(v)w - v(i_{v^*}w) = v^*(v)w - e_v i_{v^*}w,$$

so that

$$i_{v^*}e_v + e_v i_{v^*} = v^*(v)id. \tag{A1}$$

This is valid for any Clifford algebra over V, in particular for the exterior algebra.

Suppose that V carries a nondegenerate symmetric bilinear form whose associated quadratic form is Q. Then we can identify V with V^* and write i_u for the interior product by an element of V, where $i_u v = (u,v)1$ and (,) is the given scalar product. Applied to the exterior algebra, ΛV, the operator i_u is recognized as the annihilation operator for fermions, and the operator e_v is recognized as the creation operator. In this case the equation (A1) becomes the familiar anticommutation relation

$$i_u e_v + e_v i_u = (u,v)id \tag{A2}$$

for fermions. If we set

$$r_v = e_v + i_v \quad \text{and} \quad s_v = e_v - i_v,$$

then

$$r_u r_v + r_v r_u = 2(u,v), \quad s_u s_v + s_v s_u = -2(u,v),$$

and

$$r_u s_v + s_v r_u = 0.$$

Thus r gives a representation of the algebra $C_Q(V)$ on $\Lambda(V)$, i.e., a homomorphism of $C_Q(\Lambda)$ into End $(\Lambda(V))$, while s gives a representation of the algebra $C_{-Q}(V)$ on $\Lambda(V)$. The elements r_u and s_u, as u ranges over V, generate the algebra End (ΛV), since from r_u and s_u we can recover e_u and i_u, and these generate since we can clearly move from any one element of ΛV to any other element by a succession of annihilations and creations.

Suppose that we are over the complex numbers and that V is even dimensional. Let $e_1, \ldots, e_{2n}$ be an orthonormal basis of V, and let W be a vector space of dimension n, i.e., half the dimension of V. Let $f_1, \ldots, f_n$ be an orthonormal basis of W and let us map V into End (ΛW) by sending e_j to r_{f_j} for $j \leq n$ and sending e_j to is_j for $j > n$. Then this gives a representation of $C_Q(V)$ as an irreducible algebra on ΛW, and so $C_Q(V)$ is a simple algebra. The representation of $C_Q(V)$ on ΛW is known as the spin representation.

For a more detailed discussion of various properties of Clifford algebras we refer the reader to Atiyah, Bott, and Shapiro (1964) and to Kastler (1961).

REFERENCES

Aharonov, Y., A. Casher, and L. Susskind, 1971, Phys. Lett. B **35**, 512.
Atiyah, M., R. Bott, and A. Shapiro, 1964, Topology **3**, 3.
Bella, G., 1973, Nuovo Cimento A **16**, 143.
Berezin, F. A., and G. I. Kac, 1970, Mat. Sb. (USSR) **82**, 124 (English translation 1970, **11**, 311).
Boerner, H., 1963, *Representations of Groups* (North-Holland, Amsterdam), Chap. VIII.
Brout, R., 1967, Nuovo Cimento A **47**, 932.
Cabibbo, N., 1968, in *Hadrons and their Interactions*, edited by A. Zichichi (Academic, New York).
Capper, D. M., 1974, Trieste report, IC/74/66.
Chevalley, C., and S. Eilenberg, 1948, Trans. Am. Math. Soc. **63**, 85.
Coleman, S., and J. Mandula, 1967, Phys. Rev. **159**, 1251.
Coleman, S., and E. Weinberg, 1973, Phys. Rev. D **7**, 1888.
Corwin, L., Y. Ne'eman, and S. Sternberg, 1974, Tel Aviv Univ. Report TAUP 448-74.
Delbourgo, R., A. Salam, and J. Strathdee, 1974, Phys. Lett. B **51**, 475.
Dolen, R., D. Horn, and C. Schmidt, 1967, Phys. Rev. Lett. **19**, 402.
Dondi, P. H., and M. Sohnius, 1974, Nucl. Phys. B **81**, 317.
Dothan, Y., 1972, Nuovo Cimento A **11**, 499.
Fayet, P., and J. Iliopoulos, 1974, Phys. Lett. B **51**, 461.
Ferrara, S., and E. Remiddi, 1974, CERN report TH 1935; (to be published).
Ferrara, S., J. Wess, and B. Zumino, 1974, Phys. Lett. B **51**, 239.
Ferrara, S., and B. Zumino, 1974, Nucl. Phys. B **79**, 413.
Firth, R. J., and J. D. Jenkins, 1974, Durham University report, to be published in Nucl. Phys.
Fock, V., 1932, Z. Phys. **75**, 622.
Frölicher, A., and Z. Nijenhuis, 1957, Proc. Natl. Acad. Sci. USA **43**, 239.
Gell-Mann, M., and Y. Ne'eman, 1964, *The Eightfold Way* (Benjamin, N. Y.).
Gell-Mann, M., and Y. Ne'eman, 1974 (unpublished).
Gerstenhaber, M., 1963, Ann. Math. **78**, 267.
Gerstenhaber, M., 1964, Ann. Math. **79**, 59.
Gervais, J. L., and B. Sakita, 1971, Nucl. Phys. B **34**, 633.
Goddard, P., 1974, Berkeley report LBL-3347; to be published in Nucl. Phys. B.
Guillemin, Y., and S. Sternberg, *Deformation theory of pseudogroup structures*, Mem. Am. Math. Soc. **64**, 1966.
Haag, R., J. T. Lopuszanski, and M. Sohnius, 1974, Karlsruhe University Preprint.
Iliopoulos, J., and B. Zumino, 1974, Nucl. Phys. B **76**, 310.
Igi, K., and S. Matsuda, 1967, Phys. Rev. Lett. **18**, 625.
Jordan, P., and E. Wigner, 1928, Z. Phys. **47**, 631.
Joseph, A., 1972, Nuovo Cimento A **8**, 217.
Kastler, D., 1961, *Introduction à électrodynamique quantique* (Dunod, Paris).
Kodaira, K., L. Nirenberg, and D. C. Spencer, 1962, Ann. Math. **75**, 536.
Kodaira, K., and D. C. Spencer, 1958, Ann. Math. **67**, 328–466; 1959, **70**, 145–166; 1960, **71**, 43–76; 1961, **74**, 52–100.

Kumpera, A., and D. C. Spencer, 1963, *Lie Equations* (Princeton University, Princeton).
Lang, W., and J. Wess, 1974, Nucl. Phys. B **81**, 249.
Logunov, A., L. D. Soloviev, and A. N. Tavkhelidze, 1967, Phys. Lett. B **24**, 181.
Lopuszanski, J. T., and M. Sohnius, 1974, Karlsruhe Univ. Report; to be published.
Michel, L., 1970, in *Group Representations in Mathematics and Physics*, Batelle Seattle 1969 Rencontres (Springer, Berlin-Heidelberg), p. 136.
Ne'eman, Y., 1974 (unpublished).
Neveu A., and J. H. Schwarz, 1971, Nucl. Phys. B **31**, 86.
Nijenhuis, A., 1955, Proc. K. Ned. Adad. Wet. A **58**, 3.
Nijenhuis, A., and R. W. Richardson, Jr., 1964, Bull. Am. Math. Soc. **70**, 406.
Nijenhuis, A., and R. W. Richardson, Jr., 1966, Bull. Am. Math. Soc. **72**, 1.
Nilsson, J. S., and D. H. Tehrakian, 1975, to be published in Nucl. Phys. B.
O'Raifeartaigh, L., 1965, Phys. Rev. **139**, B1053.
O'Raifeartaigh, L., 1974, "Weight-Diagrams for Superfields," DIAS report.
Orzalesi, C. A., 1970, Rev. Mod. Phys. **42**, 381.
Penrose, R., 1967, J. Math. Phys. **8**, 345.
Ramond, P., 1971, Phys. Rev. D **3**, 2415.
Rebbi, C., 1974, Phys. Rep. **12C**, 1.
Rosen, J., 1974, Ann. Phys. (N.Y.) **82**, 54, and **82**, 70.
Rühl, W., and B. C. Yunn, 1974, Trier-Kaiserlautern-University Report (to be published).
Salam, A., and J. Strathdee, 1974a, Nucl. Phys. B **76**, 477.
Salam, A., and J. Strathdee, 1974b, Nucl. Phys. B **80**, 499.
Salam, A., and J. Strathdee, 1974c, Phys. Lett. B **49**, 465.
Salam, A., and J. Strathdee, 1974d, Trieste Report IC/74/36, to be published in Phys. Lett.
Salam, A., and J. Strathdee, 1974e, Trieste Report IC/74/42 (to be published).
Salam, A., and J. Strathdee, 1974f, Trieste Report IC/74/80 (to be published).
Schwarz, J. H., 1973, Phys. Rep. **8C**, 269.
Schwinger, J., 1951, Phys. Rev. **82**, 914.
Serre, J. P., 1964, *Lie Algebras and Lie Groups* (Benjamin, New York), Chap. LA3.
Souriau J. M., 1970, *Structure des Systems Dynamiques* (Dunod, Paris).
Spencer, D. C., 1962, Ann. Math. **76**, 306–445.
Sternberg, S., 1964, *Lectures on Differential Geometry* (Prentice-Hall, Englewood Cliffs).
Sternberg, S., and J. A. Wolf, 1975, "Graded Lie Algebras and Bounded Homogeneous Domains," to be published.
Suzuki, M., 1974, Berkeley Report LBL 3308, to be published.
Tasao, H.-S., 1974, Brandeis University report.
Veneziano, G., 1968, Nuovo Cimento A **57**, 190.
Veneziano, G., 1974, Phys. Rep. **9C**, 199.
Volkov, D. V., and V. P. Akulov, 1973, Phys. Lett. B **46**, 109.
Volkov, D. V., and V. A. Soroka, 1973, JETP Lett. **18**, 529 (English translation **18**, 312).
Wess, J., 1974, Lectures at Karlsruhe Summer School.
Wess, J., and B. Zumino, 1974a, Nucl. Phys. B **70**, 39.
Wess, J., and B. Zumino, 1974b, Phys. Lett. **49B**, 52.
Wess, J., and B. Zumino, 1974c, CERN report TH 1857; to be published in Nucl. Phys. B.
Woo, G., 1974, Cambridge University report DAMTP 74/12.
Zumino, B., 1974, in *Proceedings of the XVII International Conference on High Energy Physics*, London, 1974, to be published. Available as CERN report TH.1901.