

Functional methods and perturbation theory

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The framework of functional integrals in field theory is convenient for presenting a unified view of the perturbation expansion according to the number of loops. We review the calculation of the generating functional for irreducible Green functions and its renormalization properties. Calculations of the effective potential and Z -function are carried up to the order of two loops for the self-coupled scalar field. This is applied to compute the coefficients of the Callan-Symanzik equation which describes the short distance behavior of the theory. In conclusion, we present some speculations concerning the positivity of the coupling constant, and its relation to the on-shell two-particle scattering amplitude.

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I. INTRODUCTION

The present article is basically pedagogical in nature. It grew out of seminars and discussions that we had in the Theory Division at CERN during the autumn of 1973. The subject is quite broad and we had to choose only certain aspects of it.

In quantum field theory, the quantities with the greatest physical interest are the Green functions. It is in terms of the Green functions that the S -matrix is constructed and their analytic properties have been studied in detail. Furthermore the whole perturbation theory and the renormalization program are traditionally expressed in this language. Finally the powerful computational method of Feynman diagrams is designed specifically for the explicit calculation, order by order in perturbation theory, of the Green functions. A simple way to introduce formally these functions is by means of a generating functional. Let $\mathcal{L}(\varphi_i(x))$ be the Lagrangian density describing the system of n interacting fields $\varphi_i(x)$ $i = 1, \dots, n$, and $J_i(x)$ n c -number functions of space-time which transform with respect

to all the symmetries of \mathcal{L} in such a way that

$$\sum_{i=1}^n J_i(x) \varphi_i(x)$$

is an invariant. If we consider $\mathcal{L} + \sum J_i \varphi_i$ and calculate the vacuum-to-vacuum transition amplitude, we obtain a functional $S_{\text{dis}}[J_i]$ of J_i which, if functionally expanded in powers of J_i , gives the Green functions of the theory. On the other hand, by taking the functional Legendre transform of $\log S_{\text{dis}}[J_i]$ we obtain a functional $\Gamma[\varphi_{c_i}]$, where φ_{c_i} are the conjugate variables to J_i , which generates the one-particle-irreducible Green functions (Jona-Lasinio, 1964). The latter enable one to express the renormalization conditions.

However, there exist some problems, like, for example, the one associated with spontaneous symmetry breaking, for which a slightly different language is more convenient. It is obtained again by considering the functional $\Gamma[\varphi_{c_i}]$ but instead of expanding in powers of $\varphi_c(x)$, we expand around the point $\varphi_c(x) = \text{constant}$. We thus obtain a new infinite series of functions, each of which can also be computed order by order in perturbation, and which can be used to describe all properties of the theory (renormalization, symmetries etc. . .) as well as the ordinary Green functions.¹ Of course they are not the most convenient for the calculation of scattering amplitudes, since each one of them equals the sum of all Green functions taken at special points. Nevertheless they are very useful for other problems and here we attempt an introduction to their study. As we notice, the whole program of field theory can be carried through without ever mentioning the Green functions although, for reasons of physical transparency, we shall most often use the renormalization conditions defined in the traditional way. In order to simplify the notation we shall limit ourselves to the study of the simplest renormalizable theory in four dimensions, namely a massive, neutral scalar field interacting through a φ^4 coupling.

The paper is organized as follows. In Sec. II we review the renormalization conditions expressed in terms of the

¹ Coleman and Weinberg (1973) and references therein. A very nice review is given by Coleman in "Secret Symmetry," 1973 Ettore Majorana Summer School.

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Green functions, and introduce the generating functionals and the expansion around $\varphi_c(x) = \text{constant}$. Section III contains the loop expansion and the method of steepest descent which is the most convenient for calculations in this scheme. A paragraph on different regularization procedures is also added. In Sec. IV we give explicit examples of calculations. The first two functions in the expansion, namely the effective potential $V(\varphi_c)$ and the next function $Z(\varphi_c)$, are calculated up to and including two closed loops. In Sec. V we derive the renormalization group and the Callan-Symanzik equations directly for the generating functionals, and we use them in order to study the asymptotic properties of the theory for large values of φ_c . Finally in Sec. VI we use the results of the previous section in order to argue that the coupling constant in a φ^4 theory must be positive. Several technical details are gathered in Appendix A and B. Since the coupling constant can be viewed as the value of the four-meson scattering amplitude at the center of the Mandelstam triangle, we asked the question what is known for this quantity from first principles alone, without reference to any particular field theory model. Appendix C, written by A. Martin, contains some results pertaining to this question. This Appendix can in fact be read almost independently from the rest of the paper.

An interested reader not familiar with the formalism of generating functionals will presumably find it useful to read for instance the excellent presentation of Coleman and Weinberg (1973).¹ We could hardly reproduce its content here without repeating it step by step.

As we said earlier we hardly claim to give very many new results. As this work proceeded rather slowly several preprints appeared which cover some parts which we expected to be slightly more original. These articles will be quoted below. We thought nevertheless that functional methods becoming rather popular and bridging the gap between field theory and statistical mechanics are not yet too familiar. Hence this selection of topics might be useful. It was, however, hopeless to present a complete review nor to give a detailed bibliography. We apologize at once for our numerous omissions.

We thank our colleagues and friends at CERN and in Saclay for many suggestions and discussions.

II. GENERAL FORMALISM

In this section we review, mainly in order to establish the notations, the general formalism of renormalized perturbation theory.

A. Renormalization conditions

As stated above we shall study the simplest renormalizable field theory in four dimensions, namely a massive, neutral scalar field interacting through a φ^4 coupling. The Lagrangian density for such an interaction is given by:

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}\mu^2\varphi^2 - (\lambda/4!)\varphi^4 + \text{counter terms}, \quad (2.1)$$

where μ and λ are the renormalized mass and coupling constant respectively to be defined precisely below.

Let $\Gamma^{(2n)}(p_1, \dots, p_{2n})$ be the renormalized, $2n$ point, connected, one-particle-irreducible (1-PI) Green function which, for $n > 1$, depends on $8n - 10$ independent scalar products among the $2n - 1$ independent four vectors p . We shall always assume that a Wick rotation to Euclidean space (implying analytic continuation to purely imaginary times) has been performed. The $\Gamma^{(2n)}$'s are uniquely determined, order by order in perturbation theory, once a suitable set of renormalization conditions is specified. In φ^4 theory, renormalizability implies that two conditions on $\Gamma^{(2)}$ and one on $\Gamma^{(4)}$ are sufficient.

It is customary to take one of them to determine the value of the physical mass μ^2 , as the position of the pole of the complete propagator. We can therefore express a set of renormalization conditions by requiring that²:

$$\Gamma^{(2)} = -(p^2 + \mu^2) \left[1 - \frac{p^2 + M^2}{\mu^2} \sum(p^2, \mu^2, M^2, \lambda_M) \right] \quad (2.2)$$

$$\Gamma^{(4)}|_{\text{sym. point } M^2} = -\lambda_M. \quad (2.3)$$

The condition (2.2) means that the two-point function, which is the inverse of the complete propagator, vanishes at $p^2 = -\mu^2$. (We recall that in all our formulae p^2 has been rotated into Euclidean space). Furthermore this same function is normalized to the value $M^2 - \mu^2$ at an arbitrary point $p^2 = -M^2$. On the other hand, the condition (2.3) defines the renormalized coupling constant as the value of the Euclidean 4-point function at the symmetric point:

$$\begin{aligned} \text{sym. point } M^2: p_i^2 = -M^2, \quad (p_i + p_j)^2 = -\frac{4}{3}M^2 \\ i, j = 1, \dots, 4, \quad i \neq j. \end{aligned} \quad (2.4)$$

The function \sum in Eq. (2.2) is an $O(4)$ invariant dimensionless function of its arguments which is regular at $p^2 = -M^2$ and $p^2 = -\mu^2$. As a matter of fact, it can be shown that in the Euclidean space \sum is a real analytic function of p^2 , and this result holds true also for the $2n$ -point function.

Once (2.2) and (2.3) have been imposed, the $2n$ -point function $\Gamma^{(2n)}$ depends on the $8n - 10$ scalar products s_i , as well as on μ^2 , M^2 , and λ_M .

However, no physical quantity can depend on the arbitrary point M^2 . Indeed the renormalizability of the φ^4 theory implies that a change in the subtraction point M^2 , can be compensated by a change in the value of the coupling constant and a corresponding rescaling of the fields. We can therefore write:

$$\begin{aligned} \Gamma^{(2n)}(s_i, \mu^2, M_1^2, \lambda_{M_1}) \\ = Z_3^n(M_2^2, \mu^2, M_1^2, \lambda_{M_1}) \Gamma^{(2n)}(s_i, \mu^2, M_2^2, \lambda_{M_2}), \end{aligned} \quad (2.5)$$

where s_i stands for the $8n - 10$ scalar variables, and $Z_3^{1/2}$ rescales the fields. We shall have the opportunity to use these relations several times later.

² The renormalization conditions (2.2) and (2.3) are not the most general ones. One can avoid reference to the physical mass altogether and, furthermore, the points M^2 appearing in the definition of the 2 and 4-point functions need not be the same.

Among the choices implied by Eqs. (2.2) and (2.3) some are of particular interest. The "physical" Green functions, which are directly related to the S -matrix, are obtained by choosing $M^2 = \mu^2$. However, for practical calculations the choice $M^2 = 0$ is much more convenient and we shall adopt it in these notes (unless otherwise stated) dropping the subscript when referring to λ_0 . We shall therefore write

$$\Gamma^{(2)}(0) \equiv \Gamma^{(2)}(0, \mu^2, 0, \lambda) = -\mu^2, \quad (2.6)$$

$$\Gamma^{(2)}(-\mu^2) \equiv \Gamma^{(2)}(-\mu^2, \mu^2, 0, \lambda) = 0, \quad (2.7)$$

$$\Gamma^{(4)}(p_i = 0) \equiv \Gamma^{(4)}(p_i = 0, \mu^2, 0, \lambda) = -\lambda. \quad (2.8)$$

With such a choice the propagator $\Delta(p^2)$, which is the inverse of the 1-point 1-PI function:

$$\Delta(p^2) = -[1/\Gamma^{(2)}(p^2)] \quad (2.9)$$

has a pole at $p^2 = -\mu^2$ with residue given by

$$\begin{aligned} & \{ -(\partial/\partial p^2) \Gamma^{(2)}(p^2, \mu^2, 0, \lambda) |_{p^2 = -\mu^2} \}^{-1} \\ & = [1 + \sum (-\mu^2, \mu^2, 0, \lambda)]^{-1}, \end{aligned} \quad (2.10)$$

where \sum is defined in (2.2). Using (2.5) we can write (2.10) as

$$-(\partial/\partial p^2) \Gamma^{(2)}(p^2, \mu^2, 0, \lambda) |_{p^2 = -\mu^2} = Z_3(\mu^2, \mu^2, 0, \lambda). \quad (2.11)$$

We can also show that the value of the function Z_3 entering (2.11) has to be positive. Indeed the on-the-mass shell

normalized 2-point function satisfies a twice subtracted dispersion relation (Källén-Lehman representation):

$$\begin{aligned} -\Gamma^{(2)}(p^2, \mu^2, \mu^2, \lambda_{\mu^2}) &= (p^2 + \mu^2) + (p^2 + \mu^2)^2 \\ &\times \int_{4\mu^2}^{\infty} dq^2 \frac{\rho(q^2)}{(q^2 + p^2)(q^2 - \mu^2)^2} \quad \rho(q^2) \geq 0. \end{aligned} \quad (2.12)$$

Now, using (2.5), for $p^2 = 0$ we get

$$\Gamma^{(2)}(0, \mu^2, 0, \lambda) = Z_3(\mu^2, \mu^2, 0, \lambda) \Gamma^{(2)}(0, \mu^2, \mu^2, \lambda_{\mu^2})$$

or, equivalently, using (2.10) and (2.12)

$$Z_3^{-1}(\mu^2, \mu^2, 0, \lambda) = 1 + \mu^2 \int_{4\mu^2}^{\infty} dq^2 \frac{\rho(q^2)}{q^2(q^2 - \mu^2)^2} \geq 1. \quad (2.13)$$

B. Generating functionals and effective potential

A convenient way to study the properties of Green functions in perturbation theory is by introducing a generating functional. Let us assume that we add to the Lagrangian (2.1) a linear interaction with an external source $J(x)$ which is a c -number function of space-time, i.e., a term $J(x)\varphi(x)$. The generating functional of all Green functions, including the disconnected ones, is given by the vacuum-to-vacuum transition amplitude in the presence of the source J

$$S_{\text{disc}}[J] = \langle 0_{\text{out}} | 0_{\text{in}} \rangle_J.$$

Ignoring for the moment difficulties associated with renormalization, a formal expression of $S_{\text{disc}}[J]$ is given, in terms of path integrals, by

$$S_{\text{disc}}[J] = \frac{\int \exp\{-\int [\frac{1}{2}(\partial\psi)^2 + \frac{1}{2}\mu^2\psi^2 + (\lambda/4!)\psi^4 - J\psi] d^4x\} \mathfrak{D}[\psi]}{\int \exp\{-\int [\frac{1}{2}(\partial\psi)^2 + \frac{1}{2}\mu^2\psi^2 + (\lambda/4!)\psi^4] d^4x\} \mathfrak{D}[\psi]}, \quad (2.14)$$

where $\mathfrak{D}[\psi]$ is assumed to be some "positive measure" on the functions ψ and $(\partial\psi)^2$ is the square of the Euclidean gradient of ψ :

$$(\partial\psi)^2 = \sum_{\alpha=0}^3 \left(\frac{\partial\psi}{\partial x_{\alpha}} \right)^2.$$

By functionally expanding $S_{\text{disc}}[J]$ in powers of J , we obtain the Green functions $\tilde{G}_{\text{disc}}(x_1, \dots, x_{2n})$

$$S_{\text{disc}}[J] = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \int \prod_{i=0}^{2n} [d^4x_i J(x_i)] \tilde{G}_{\text{disc}}(x_1, \dots, x_{2n}), \quad (2.15)$$

where, because of the symmetry $\varphi \rightarrow -\varphi$ of (2.1), only the even order terms appear.

Similarly, one introduces the generating functional $S[J]$ of all connected Green functions $\tilde{G}(x_1, \dots, x_n)$:

$$S[J] = \log S_{\text{disc}}[J] \quad (2.16)$$

with an expansion analogous to (2.15), and finally the generating functional of 1-PI Green functions sometimes

called "vertex functions." It is given, in terms of $S[J]$ by a functional Legendre transformation

$$\Gamma[\varphi_c] = S[J] - \int d^4x J(x)\varphi_c(x), \quad (2.17)$$

where $\varphi_c(x)$, sometimes called the "classical field," is defined by

$$\varphi_c(x) \equiv \delta S[J] / \delta J(x). \quad (2.18)$$

Equation (2.17) should be understood as follows. One has to invert (2.18) thereby expressing J as a functional of φ_c and then replace it in (2.17) thus obtaining Γ as a functional of φ_c . In an expansion of Γ in powers of φ_c , one obtains the Γ functions introduced in the previous paragraph. In particular (2.17) gives:

$$J(x) = -\delta \Gamma[\varphi_c] / \delta \varphi_c(x) \quad (2.19)$$

$$\begin{aligned} \tilde{\Gamma}^{(2)}(x_1, x_2) &= \frac{\delta^2 \Gamma[\varphi_c]}{\delta \varphi_c(x_1) \delta \varphi_c(x_2)} = - \left(\frac{\delta^2 S[J]}{\delta J(x_1) \delta J(x_2)} \right)^{-1} \\ &= -[\tilde{\Delta}(x_1, x_2)]^{-1} \end{aligned} \quad (2.20)$$

evaluated at $\varphi_c = 0$.

It is useful to introduce an expansion of $\Gamma[\varphi_c]$ around the value $\varphi_c = \text{constant}$. More precisely, this is an expansion of the density of Γ in powers of the derivatives of φ_c . Taking into account translational invariance, we write

$$-\Gamma[\varphi_c] = \int d^4x [V(\varphi_c(x)) + \frac{1}{2}Z(\varphi_c(x))(\partial\varphi_c(x))^2 + \dots], \tag{2.21}$$

where V, Z, \dots are ordinary functions of $\varphi_c(x)$. It is easy to see that (2.21) corresponds to an expansion of the vertex functions around zero external momenta. In fact let us write:

$$\Gamma[\varphi_c] = \sum_{n=0}^{\infty} [(2n)!]^{-1} \int \prod_{i=1}^{2n} [d^4x_i \varphi_c(x_i)] \bar{\Gamma}^{(2n)}(x_1, \dots, x_n). \tag{2.22}$$

By translational invariance, $\bar{\Gamma}^{(2n)}$ is only a function of the $2n - 1$ variables $x_i - x_1$ and therefore, introducing the Fourier transforms written without a twiddle, we can write (2.22) as

$$\begin{aligned} \Gamma[\varphi_c] &= \sum_{n=0}^{\infty} [(2n)!]^{-1} \int \prod_{i=1}^{2n} [d^4x_i \varphi_c(x_i)] \\ &\times \prod_{j=1}^{2n} [d^4p_j / (2\pi)^4 \exp(ip_j x_j)] (2\pi)^4 \delta^4(\sum p_j) \Gamma^{(2n)} \\ &\times (p_1, \dots, p_n). \end{aligned} \tag{2.23}$$

Expanding $\Gamma^{(2n)}(p_1, \dots, p_n)$ around $p_i = 0$ and comparing with (2.21) we obtain

$$V(\varphi_c) = - \sum_{n=0}^{\infty} [(2n)!]^{-1} [\varphi_c]^{2n} \Gamma^{(2n)}(0, \dots, 0). \tag{2.24}$$

A similar expression, involving the derivatives of the vertex functions around $p_i = 0$, holds for $Z(\varphi_c)$ as well as for the higher order terms in (2.21). In order to obtain, as an example, the expression for $Z(\varphi_c)$ we consider the vertex function $\Gamma^{(2n)}(p_1, \dots, p_{2n})$, which is $\mathcal{O}(4)$ symmetric and invariant under permutations of the momenta. Define a function of $2n - 1$ independent momenta by

$$\begin{aligned} \hat{\Gamma}^{(2n)}(p_1, \dots, p_{2n-1}) \\ = \Gamma^{(2n)}(p_1 \dots p_{2n-1}, -p_1 - p_2 - \dots - p_{2n-1}). \end{aligned}$$

Clearly

$$\begin{aligned} \hat{\Gamma}^{(2n)}(p_1, \dots, p_{2n-1}) \\ = \hat{\Gamma}^{(2n)}(0, \dots, 0) + (p_1^2 + \dots + p_{2n-1}^2) \hat{\Gamma}_a^{(2n)} \\ + \left(\sum_{1 \leq i < j \leq 2n-1} p_i p_j \right) \hat{\Gamma}_b^{(2n)} + \dots \end{aligned}$$

By abuse of language, instead of writing,

$$\begin{aligned} \hat{\Gamma}_a^{(2n)} &= \frac{1}{2} [\partial^2 / (\partial p_1^0)^2] \hat{\Gamma}^{(2n)}(0, \dots, 0), \\ \hat{\Gamma}_b^{(2n)} &= (\partial^2 / \partial p_1^0 \partial p_2^0) \hat{\Gamma}^{(2n)}(0, \dots, 0), \end{aligned}$$

we write instead

$$\begin{aligned} \hat{\Gamma}_a^{(2n)} &= (\partial / \partial p_1^2) \hat{\Gamma}^{(2n)}(0, \dots, 0) \\ \hat{\Gamma}_b^{(2n)} &= [\partial / (\partial p_1 \cdot p_2)] \Gamma^{(2n)}(0, \dots, 0) \end{aligned}$$

even though for $n \geq 3$ the scalar products $p_i p_j$ are not independent variables. Z is then given by

$$\begin{aligned} Z(\phi_c) = \sum_{n=1}^{\infty} \frac{1}{(2n-2)!} [\phi_c]^{2n-2} \left\{ - \frac{2n-1}{n} \frac{\partial}{\partial p_1^2} \hat{\Gamma}^{(2n)}(0) \right. \\ \left. + \frac{n-1}{n} \frac{\partial}{\partial p_1 \cdot p_2} \hat{\Gamma}^{(2n)}(0) \right\}. \end{aligned} \tag{2.25}$$

Of course further terms in the expansion (2.21) could be introduced as generating functions for higher derivatives of Green functions. It is worth pointing out that, at least formally, $V(\varphi_c)$ can be given the physical meaning of the energy density of a state where the field $\varphi(x)$ takes throughout space the constant value φ_c .

The last step, before describing the algorithm of the perturbation series in terms of the generating functionals, is to express in this language the normalization conditions introduced in the previous paragraph. We thus obtain the equivalent of Eqs. (2.6) and (2.8):

$$(d^2/d\varphi_c^2) V |_{\varphi_c=0} = \mu^2 \quad (d^4/d\varphi_c^4) V |_{\varphi_c=0} = \lambda. \tag{2.26}$$

The other condition, namely $\Gamma^{(2)}(-\mu^2) = 0$, which ensures that μ is the physical mass and appears as the pole of the complete propagator, cannot be expressed in terms of V, Z etc... since they only involve the successive derivatives of the vertex functions around $p_i = 0$. We see that the traditional normalization scheme is not well adapted to this language. However we can abandon the idea of using one of the renormalization conditions in order to determine the value of the physical mass, and use instead of $\Gamma^{(2)}(-\mu^2) = 0$ a condition of the form

$$(d/dp^2) \Gamma^{(2)}(0) = -1 \quad \text{or} \quad Z(0) = 1. \tag{2.27}$$

From the point of view of the renormalization theory, the set of conditions (2.26) and (2.27) is certainly as good as any other, and in fact the Green functions calculated according to this prescription will be related to the physical ones by a finite renormalization of the general form of Eq. (2.5). For the practical purposes of this paper it will turn out that the two sets of normalization conditions (2.6) to (2.8) and (2.26)–(2.27) are equivalent, since the explicit calculations presented here are not performed at sufficiently high order to be affected by these finite renormalization effects.

III. THE LOOP EXPANSION AND THE METHOD OF STEEPEST DESCENT

In this section we show, through formal manipulations of Eq. (2.14), how we can reproduce the usual perturbation series. We emphasize again that this method is as heuristic as the ordinary canonical quantization procedure, and does not have more claims to rigor.

A. The loop expansion

We start by introducing a suitable book-keeping method to count the number of loops in perturbation theory³. For a connected diagram, if L is the number of loops, $I(E)$ the number of internal (external) lines, and V the number of vertices, we obtain

$$2I + E = 4V, \quad L = I - V + 1 = V + 1 - (E/2). \tag{3.1}$$

The first condition means that four lines meet at each

vertex, the internal ones connecting two vertices. The second condition counts the number of independent integration four-momenta L . The added +1 appears because of the factorization of the overall δ function of energy-momentum conservation. We see from (3.1) that the power V of the coupling constant does not determine the number of loops since a connected diagram of order λ^V can contain any number of loops $L \leq V + 1$.

The solution to this problem is well known. One introduces a new parameter \hbar^{-1} which multiplies the whole Lagrangian, not just the interaction part:

$$\exp \frac{S[J]}{\hbar} = \frac{\int \exp\{- (1/\hbar) \int [\frac{1}{2}(\partial\psi)^2 + \frac{1}{2}\mu^2\psi^2 + (\lambda/4!)\psi^4 - J\psi] d^4x\} \mathcal{D}[\psi]}{\int \exp\{- (1/\hbar) \int [\frac{1}{2}(\partial\psi)^2 + \frac{1}{2}\mu^2\psi^2 + (\lambda/4!)\psi^4] d^4x\} \mathcal{D}[\psi]} \tag{3.2}$$

In a connected diagram, each vertex will carry a factor $1/\hbar$, each propagator a factor \hbar and each external line a factor $1/\hbar$. Therefore each connected diagram has a power of $\hbar: (\hbar)^{L-1-E}$. In order to obtain a diagram in the expansion of $(1/\hbar)\Gamma[\varphi_c]$ we select a 1-PI diagram of $(1/\hbar)S[J]$ and multiply it by an inverse propagator for each external line. Therefore the terms in the expansion of $(1/\hbar)\Gamma[\varphi_c]$ will have factors

$$(\hbar)^{L-1} = (1/\hbar)(\hbar)^L.$$

Consequently in the series of $\Gamma[\varphi_c]$ the power of \hbar counts the number of loops. For $\hbar = 0$ we obtain the tree diagrams ("classical" approximation). We repeat that this is nothing more than a book keeping device and we do not have to assume that \hbar is "small".

B. Method of steepest descent

A standard way to handle formal expressions like (3.2) is to apply the method of steepest descent. This will be shown now to yield the desired \hbar expansion. It consists in expanding the exponent in the numerator of the rhs of (3.2) around the position $\psi_0[J]$ at which it is stationary. The denominator is merely a normalization factor designed to constrain $S[0] = 0$. Let ψ_0 be a solution of the classical (elliptic) equation:

$$(-\square + \mu^2)\psi_0 + \frac{\lambda}{3!}\psi_0^3 = J; \quad \square\psi_0 = \sum_{\alpha=0}^3 \partial_{\alpha^2}\psi_0. \tag{3.3}$$

Let us choose $J(x)$ to vanish at infinity. The solution of (3.3) is then unique if it is also required to vanish at infinity, hence $\psi_0[J] = 0$ if $J = 0$. It is clear from (3.3) that the relation between ψ_0 and J is \hbar independent.

We now expand the action $I[\psi]$

$$I[\psi] = \int [\frac{1}{2}(\partial\psi)^2 + \frac{1}{2}\mu^2\psi^2 + (\lambda/4!)\psi^4 - J\psi] d^4x \tag{3.4}$$

around the function ψ_0 by writing $\psi = \psi_0 + \tilde{\psi}$

$$I[\psi] = I[\psi_0] + \int [\frac{1}{2}(\partial\tilde{\psi})^2 + \frac{1}{2}(\mu^2 + (\lambda/2)\psi_0^2)\tilde{\psi}^2 + (\lambda/3!)\psi_0\tilde{\psi}^3 + (\lambda/4!)\tilde{\psi}^4] d^4x, \tag{3.5}$$

where we have used (3.3) in order to eliminate the linear terms. We now impose on the measure $\mathcal{D}[\psi]$ to be translationally invariant and we obtain

$$S[J] = S_0[J] + \hbar S_1[\psi_0] + \hbar^2 S_2[\psi_0] + \dots, \tag{3.6}$$

where

$$S_0[J] = - \int [\frac{1}{2}(\partial\psi_0)^2 + \frac{1}{2}\mu^2\psi_0^2 + (\lambda/4!)\psi_0^4 - J\psi_0] d^4x = -I(\psi_0) \tag{3.7}$$

and

$$\begin{aligned} \exp \frac{S[J] - S_0[J]}{\hbar} &= \frac{\int \exp\{- (1/\hbar) \int [\frac{1}{2}(\partial\psi)^2 + \frac{1}{2}(\mu^2 + \frac{1}{2}\lambda\psi_0^2)\psi^2 + (\lambda\psi_0/3!)\psi^3 + (\lambda/4!)\psi^4] d^4x\} \mathcal{D}[\psi]}{\int \exp\{- (1/\hbar) \int [\frac{1}{2}(\partial\psi)^2 + \frac{1}{2}\mu^2\psi^2 + (\lambda/4!)\psi^4] d^4x\} \mathcal{D}[\psi]} \\ &= \frac{\int \exp\{- \int [\frac{1}{2}(\partial\tilde{\psi})^2 + \frac{1}{2}(\mu^2 + \frac{1}{2}\lambda\psi_0^2)\tilde{\psi}^2 + \hbar^{1/2}(\lambda\psi_0/3!)\tilde{\psi}^3 + \hbar(\lambda/4!)\tilde{\psi}^4] d^4x\} \mathcal{D}[\tilde{\psi}]}{\int \exp\{- \int [\frac{1}{2}(\partial\tilde{\psi})^2 + \frac{1}{2}\mu^2\tilde{\psi}^2 + \hbar(\lambda/4!)\tilde{\psi}^4] d^4x\} \mathcal{D}[\tilde{\psi}]} \end{aligned} \tag{3.8}$$

To obtain the second expression we have rescaled the dummy integration field through $\psi \rightarrow \hbar^{1/2}\tilde{\psi}$. The first term $S_1[\psi_0]$ is found by integrating the exponential of a quadratic

form. The result is well known⁴, being apart from a common factor in both numerator and denominator the inverse

⁴In writing (3.9) we made use of the functional analogue of the identity

$$[(\det B)/(\det A)]^{1/2} = \int \prod_i dz_i \exp-(ZAZ)/\int \prod_i dz_i \exp-(ZBZ),$$

valid in principle for finite dimensional positive hermitian matrices A and B .

³The loop expansion and its relation to the steepest descent (or stationary phase method in the Minkowski region) were apparently introduced by Y. Nambu (1968). For a review, see Coleman and Weinberg (1973) and Coleman (1973).

square root of the determinant of this quadratic form:

$$\begin{aligned} S_1[\psi_0] &= -\frac{1}{2} \log \det K_{xy}(\psi_0)/K_{xy}(0) \\ &= -\frac{1}{2} \text{Tr} \log K_{xy}(\psi_0)/K_{xy}(0), \end{aligned} \quad (3.9)$$

where K_{xy} is the symmetric kernel

$$K_{xy}(\psi_0) = (\partial_x \partial_y + \mu^2 + \frac{1}{2} \lambda \psi_0^2) \delta^4(x - y). \quad (3.10)$$

We recall that in all the formulae (3.6) to (3.10), ψ_0 is understood to be a functional of J through the classical equation (3.3) which does not contain any \hbar i.e., ψ_0 is given, as a functional of J , only by tree diagrams of classical perturbation theory.

The higher order terms S_2, S_3 etc... in (3.6) can be read directly from (3.8). We see that the exponent in the integrand of the numerator of formula (3.8) represents an effective action of the form:

$$\begin{aligned} I_{3c}[\psi] &= \int [\frac{1}{2}(\partial\psi)^2 + \frac{1}{2}(\mu^2 + \frac{1}{2}\lambda\psi_0^2)\psi^2 \\ &+ (\hbar^{1/2}\lambda\psi_0/3!)\psi^3 + (\hbar\lambda/4!)\psi^4] d^4x. \end{aligned} \quad (3.11)$$

It has the following characteristics:

(a) There is no source term for ψ , in other words we must calculate only vacuum-to-vacuum diagrams.

(b) The "mass" in the propagators is $\mu^2 + \frac{1}{2}\lambda\psi_0^2(x)$. Note that in general $\psi_0(x)$ is x dependent. Hence the propagators are not merely diagonal in momentum space as is usually the case.

(c) There are trilinear as well as quadrilinear couplings with corresponding "coupling constants"

$$\hbar^{1/2}\lambda\psi_0(x)/3! \quad \text{and} \quad \hbar\lambda/4!.$$

If we call $V_3(V_4)$ the number of three (four) vertices, we obtain:

$$L = V_4 + \frac{1}{2}V_3 + 1. \quad (3.12)$$

It follows that for a given number of loops L , there is only a finite number of vacuum-to-vacuum connected diagrams which need to be calculated. Notice finally that the propagator is the inverse of the kernel (3.10) which appears in the expression for S_1 .

Having obtained the generating functional $S[J]$ by means of (3.6)–(3.11), we can perform the functional Legendre transformation (2.17) in order to calculate $\Gamma[\varphi_c]$.

A straightforward evaluation of this transformation requires first the evaluation of ψ_0 as a functional of J through (3.3), and then the inversion of (2.18) in order to obtain J as a functional of φ_c . This would give ψ_0 as a functional of φ_c and the Legendre transformation of (3.6) would determine $\Gamma[\varphi_c]$. Of course, it is not possible to perform this series of operations exactly, since they require an exact solution of the equations of motion. Therefore, in the same way as we used the loop expansion in order to evaluate $S[J]$ through (3.6), we shall try to determine $\Gamma[\varphi_c]$ in the form

$$\Gamma[\varphi_c] = \Gamma_0[\varphi_c] + \hbar\Gamma_1[\varphi_c] + \hbar^2\Gamma_2[\varphi_c] + \dots \quad (3.13)$$

We first observe that, to zeroth order in \hbar , φ_c equals ψ_0 which is the solution of the classical equation of motion (3.3). Indeed, using (2.18), (3.3), (3.6), and (3.7), we find:

$$\varphi_c \equiv \frac{\delta S[J]}{\delta J} = \frac{\delta S_0[J]}{\delta J} + 0(\hbar) = \psi_0 + 0(\hbar). \quad (3.14)$$

Therefore the required functional relation $\psi_0[\varphi_c]$ becomes trivial to zeroth order in \hbar , and the first term in (3.13) is clearly given by

$$\begin{aligned} \Gamma_0[\varphi_c] &= -\int [\frac{1}{2}(\partial\varphi_c)^2 + \frac{1}{2}\mu^2\varphi_c^2 + (\lambda/4!)\varphi_c^4] d^4x \\ &= -I[\varphi_c] - \int J\varphi_c d^4x. \end{aligned} \quad (3.15)$$

In order to calculate the next terms we write the corrections to (3.14) as $\varphi_c = \psi_0 + \hbar\tilde{\varphi}_c$. We then obtain

$$\begin{aligned} S[J] - \int J\varphi_c d^4x - \Gamma_0[\varphi_c] &= -I[\psi_0] - \Gamma_0[\varphi_c] + \hbar S_1[\psi_0] \\ &+ \hbar^2 S_2[\psi_0] + \dots - \int J\varphi_c d^4x \\ &= -I[\psi_0] + I[\varphi_c] + \hbar S_1[\psi_0] + \hbar^2 S_2[\psi_0] + \dots \\ &= \hbar^2 \int [\frac{1}{2}(\partial\tilde{\varphi}_c)^2 + \frac{1}{2}(\mu^2 + \frac{1}{2}\lambda\psi_0^2)\tilde{\varphi}_c^2] d^4x \\ &+ \hbar S_1[\varphi_c - \hbar\tilde{\varphi}_c] + \hbar^2 S_2[\varphi_c] + 0(\hbar^3), \end{aligned} \quad (3.16)$$

where we have used (3.5), (3.6), (3.7), and (3.15). Comparing with (3.13) we find:

$$\begin{aligned} \Gamma_1[\varphi_c] &= S_1[\varphi_c] = -\frac{1}{2} \text{Tr} \log [K_{xy}(\varphi_c)/K_{xy}(0)], \\ \Gamma_2[\varphi_c] &= S_2[\varphi_c] + \int [\frac{1}{2}(\partial\tilde{\varphi}_c)^2 + \frac{1}{2}(\mu^2 + \frac{1}{2}\lambda\psi_0^2)\tilde{\varphi}_c^2] d^4x \\ &- \tilde{\varphi}_c \frac{\delta S_1[\varphi_c]}{\delta \varphi_c} \end{aligned} \quad (3.17)$$

$$= S_2[\varphi_c] + \frac{1}{2}\tilde{\varphi}_c K(\varphi_c)\tilde{\varphi}_c - \tilde{\varphi}_c \frac{\delta S_1[\varphi_c]}{\delta \varphi_c}. \quad (3.18)$$

It is convenient to eliminate $\tilde{\varphi}_c$ from (3.18). To zeroth order in \hbar we write

$$\begin{aligned} \tilde{\varphi}_c &= (1/\hbar)[\varphi_c - \psi_0] = (1/\hbar)[\delta S/\delta J - \psi_0] \\ &= \frac{\delta S_1[\psi_0]}{\delta \psi_0} \frac{\delta \psi_0}{\delta J} \\ &= \frac{\delta S_1[\psi_0]}{\delta \psi_0} K^{-1}(\psi_0) = \frac{\delta S_1[\varphi_c]}{\delta \varphi_c} K^{-1}(\varphi_c). \end{aligned} \quad (3.19)$$

The final expression for $\Gamma_2[\varphi_c]$ is therefore the following:

$$\begin{aligned} \Gamma_2[\varphi_c] &= S_2[\varphi_c] - \frac{1}{2}(\delta S_1/\delta \varphi_c) K^{-1}(\varphi_c) (\delta S_1/\delta \varphi_c) \\ &= S_2[\varphi_c] - \frac{1}{2} \iint \frac{\delta S_1[\varphi_c]}{\delta \varphi_c(x)} \Delta_{xy}(\varphi_c) \frac{\delta S_1[\varphi_c]}{\delta \varphi_c(y)} d^4x d^4y. \end{aligned} \quad (3.20)$$

In (3.20), Δ is the propagator found before, which is equal to the inverse of the kernel K given by (3.10). We can compute the higher order terms of (3.13) in a similar way.

The meaning of $\Gamma[\varphi_c]$ is now transparent, Γ_0 obviously generates the trivial 1-PI tree diagrams. $S_1[\psi_0]$, with ψ_0 taken as a functional of J , generates all the one-loop, connected diagrams, thus $S_1[\varphi_c] = \Gamma_1(\varphi_c)$ gives the 1-PI, one-loop ones. Γ_2 contains two terms: $S_2[\varphi_c]$ generates the three vacuum-to-vacuum diagrams of Fig. 1 with the rules a), b) and c) explained above. The first two of these diagrams are 1-PI but the third is not. However it is precisely cancelled by the second term of (3.20). Therefore Γ_2 also generates the 2-loop 1-PI diagrams. It is straightforward but tedious to generalize this argument inductively to all orders⁵.

The lesson we have learned can be summarized as follows. To go from the expansion (3.6) of S to the expansion of Γ Eq. (3.13), we simply do the following:

- (i) drop the term $\int J\psi_0$
- (ii) replace ψ_0 by φ_c
- (iii) keep only the 1-PI diagrams [in terms of the propagator $\Delta(\varphi_c)$].

If we are only interested in the potential $V(\varphi_c)$ we must isolate an overall factor of the space-time volume $\int d^4x$, then set $\varphi_c = \text{constant}$ and change the sign as explained in (2.21).

Up to now we have implicitly assumed that the reader reacts instantly when shown a Lagrangian by figuring out the Feynman rules it generates. Nevertheless we conclude this section with some remarks on these rules and Wick's theorem derived in this formalism.

The Feynman rules are embodied in the following formula:

$$\begin{aligned} \langle \exp \int d^4x k(x)\psi(x) \rangle &= \frac{\int \exp\{-\int \frac{1}{2}\psi(x)\Delta_{xy}^{-1}(\varphi_c)\psi(y) d^4x d^4y + \int k(x)\psi(x) d^4x\} \mathcal{D}[\psi]}{\int \exp\{-\int \frac{1}{2}\psi(x)\Delta_{xy}^{-1}(\varphi_c)\psi(y) d^4x d^4y\} \mathcal{D}[\psi]} \\ &= \exp \frac{1}{2} \int k(x)\Delta_{xy}(\varphi_c)k(y) d^4x d^4y. \end{aligned} \tag{3.21}$$

Thus, expanding both sides and noticing that odd monomials in ψ have zero expectation value, we obtain:

$$\begin{aligned} [(2p)!]^{-1} \int d^4x_1 \dots d^4x_{2p} \langle \psi(x_1) \dots \psi(x_{2p}) \rangle k(x_1) \dots k(x_{2p}) \\ = (2^p p!)^{-1} \int d^4x_1 \dots d^4x_{2p} k(x_1) \dots k(x_{2p}) [(2p)!]^{-1} \\ \times \sum_{\text{perm.}} \Delta(x_{n_1}, x_{n_2}) \dots \Delta(x_{n_{2p-1}}, x_{n_{2p}}). \end{aligned} \tag{3.22}$$

We can divide the $(2p)!$ permutations of the rhs into classes having each $2^p p!$ members as follows: Starting from a given permutation we can obtain all the members of its class by interchanging the variables in each Δ separately and permuting the pairs of arguments among the Δ 's. Obviously all members of a given class yield the same result. By just taking a representative of each class we therefore obtain *Wick's theorem*:

$$\begin{aligned} \langle \psi(x_1) \dots \psi(x_{2p}) \rangle \\ = \sum_{\text{distinct terms}} \Delta(x_{\pi_1}, x_{\pi_2}) \dots \Delta(x_{\pi_{2p-1}}, x_{\pi_{2p}}). \end{aligned} \tag{3.23}$$

As we have already noticed, for the calculation of the effective potential we only need the expression of the kernel Δ for $\varphi_c = \text{constant}$. In this case it has the usual representation

$$\Delta_{xy}(\varphi_c) = \int \frac{d^4k}{(2\pi)^4} \frac{\exp[ik(x-y)]}{k^2 + \mu^2 + (\lambda/2)\varphi_c^2}. \tag{3.24}$$

However for φ_c varying with x one has in principle to solve the 4-dimensional equation:

$$[-\square + \mu^2 + \lambda\varphi_c^2(x)/2]\Delta_{xy} = \delta^4(x-y). \tag{3.25}$$

Fortunately we shall not be really faced with this uneasy task. However since we intend to compute not only $V(\varphi_c)$

⁵ We need not give here the complete argument since in the meantime it has been presented in a recent MIT preprint by R. Jackiw, "Functional evaluation of the effective potential."

but also $Z(\varphi_c)$ at least to low orders, we shall need a special case of (3.25) where $\varphi_c(x)^2$ is at most quadratic in x . Even though this is not exactly what is required we give below the solution for the case where $\varphi_c(x) = \varphi + ax$. The solution is obtained by noticing its relationship with the standard harmonic oscillator problem of quantum mechanics.

Let us recall that for a one dimensional oscillator with P, Q denoting the usual operators satisfying $[Q, P] = i$, one has:

$$\begin{aligned} \langle q_1 | \exp[-s(P^2 + \omega^2 Q^2)] | q_2 \rangle \\ = \left(\frac{\omega}{2\pi s \hbar 2s\omega}\right)^{1/2} \exp - \frac{\omega}{2s\hbar 2s\omega} \\ \times [(q_1^2 + q_2^2)ch2s\omega - 2q_1q_2]. \end{aligned} \tag{3.26}$$

Choosing the 0 direction along the vector a , and setting $\omega^2 = (\lambda a^2/2)$ it then follows easily that:

$$\begin{aligned} \Delta_{xy} &= \int \frac{d^3k_T}{(2\pi)^3} \int \frac{dk_0 dq_0}{2\pi} \int_0^\infty ds \\ &\times \exp[i\mathbf{k}_T(\mathbf{x}_T - \mathbf{y}_T) - s(k_T^2 + \mu^2)] \\ &\times \left[\frac{1}{2\pi\omega s \hbar 2\omega s} \right]^{1/2} \exp \left\{ i \left[k_0 \left(x_0 + \frac{\varphi}{|a|} \right) \right. \right. \\ &\left. \left. - q_0 \left(y_0 + \frac{\varphi}{|a|} \right) \right] \right\} \exp - \frac{1}{2\omega s \hbar 2\omega s} \\ &\times [(k_0^2 + q_0^2)ch2\omega s - 2k_0q_0] \\ &= \int \frac{d^3k_T}{(2\pi)^3} \int_0^\infty ds \exp[i\mathbf{k}_T(\mathbf{x}_T - \mathbf{y}_T) - s(\mathbf{k}_T^2 + \mu^2)] \\ &\times \left(\frac{\omega}{2\pi s \hbar 2\omega s}\right)^{1/2} \exp - \frac{\omega}{2a^2 s \hbar 2\omega s} \\ &\times [(\varphi_x^2 + \varphi_y^2)ch2\omega s + 2\varphi_x\varphi_y]. \end{aligned} \tag{3.27}$$

C. Regularization

The expressions derived so far are purely formal due to the well known divergences of perturbation theory. However, for renormalizable theories there exists a well-defined prescription which allows one to extract meaningful results in any given order of perturbation. (The question of the convergence of the whole series cannot yet be answered). The first step, usually called regularization, of any such renormalization program, is to replace all divergent expressions appearing in the theory by finite ones. It is only then that the next step, namely the enforcement of renormalization conditions such as (2.2)–(2.3), can be applied. There exist several ways to regularize a field theory and each one seems to be better adapted to certain uses, or certain theories, than others. A simple and elegant example of such a scheme is Zimmermann’s subtraction procedure which consists basically in subtracting the Feynman diagram integrands a sufficient number of times around the origin in momentum space until finite integrals are obtained. This method is very useful for giving rigorous proofs of the renormalizability of a theory, as well as for defining operators as monomials of the basic fields and their derivatives such as φ^{2n} , $\varphi \square \varphi$, etc... However it is expressed directly in terms of the Feynman diagrams and it is not known at present how to incorporate its prescriptions into the Lagrangian formalism we have been using so far. On the other hand, in order to perform explicit calculations and especially for theories with Gauge symmetries, the method of dimensional regularization is by far the most convenient.

There exist in addition many other regularization schemes, but most of the topics that will be discussed in these notes will be phrased in such a way as to avoid reference to any one in particular. However, we shall not hesitate for any given problem to appeal to the one that seems to us most suitable. For future use we shall present in this paragraph the classical Pauli–Villars regularization method through a dimensional cutoff Λ using the functional language. It presents some advantages for our case since: (i) it can be incorporated very nicely into the formalism, (ii) it is conceptually very simple since the cutoff Λ^2 can be viewed as the analog of the inverse lattice spacing in nonrelativistic statistical mechanics, and (iii) the derivation of the Callan–Symanzik equations is very simple if we use this method.

For this purpose, let us denote by μ_0 and λ_0 the bare mass and coupling constant of the theory. The method consists in replacing the bare propagator by a regularized one through

$$\frac{1}{k^2 + \mu_0^2} \rightarrow \int_0^\infty g_{\Lambda^2}(\alpha) \exp[-\alpha(k^2 + \mu_0^2)] d\alpha, \tag{3.28}$$

where $g_{\Lambda^2}(\alpha) \rightarrow 1$ for $\Lambda^2 \rightarrow \infty$ and for fixed Λ^2 vanishes sufficiently fast for $\alpha \rightarrow 0$ in order to make all integrals convergent. Formally this substitution, (which for a general $g_{\Lambda^2}(\alpha)$ leads to a nonlocal action) can still be cast into the form (2.14), i.e., there exists a positive measure $d\nu_\rho$ and a function C_ρ , with $C_\rho \neq 0$ in the support of $d\nu_\rho$, such that:

$$\exp(S_{\Lambda^2}[J]) = \frac{\int \mathcal{D}[\psi_\rho] \exp - \int d^4x \{ \int d\nu_\rho C_\rho \frac{1}{2} [(\partial\psi_\rho)^2 + \mu_\rho^2 \psi_\rho^2] + (\lambda_0/4!) [\int d\nu_\rho \psi_\rho]^4 - J \int d\nu_\rho \psi_\rho \}}{\text{(same with } J = 0)} \tag{3.29}$$

In (3.29) we can integrate over all uncoupled degrees of freedom, i.e., over all ψ_ρ ’s except the combination $\psi = \int d\nu_\rho \psi_\rho$. In order to do so we introduce a representation of the δ function, and we write (3.24) as follows

$$\begin{aligned} \exp(S_{\Lambda^2}[J]) &= (\int \mathcal{D}[\psi] \int \mathcal{D}[\alpha] \int \mathcal{D}[\psi_\rho] \exp \{ \int d^4x [i\alpha(x)\psi(x) - (\lambda_0/4!)\psi^4 + J\psi] \} \\ &\times \exp \{ - \int d^4x \int d\nu_\rho C_\rho \frac{1}{2} (\partial\psi_\rho)^2 + \frac{1}{2} \mu_\rho^2 \psi_\rho^2 + i(\alpha/C_\rho)\psi_\rho \}) \text{(same with } J = 0)^{-1}. \end{aligned} \tag{3.30}$$

The integrations on ψ_ρ and α can easily be done using (3.21) and yield

$$\exp(S_{\Lambda^2}[J]) = \frac{\int \mathcal{D}[\psi] \exp \{ -\frac{1}{2} \iint d^4x d^4y \psi(x) K_{\Lambda^2}(x, y) \psi(y) - \int d^4x [(\lambda_0/4!)\psi^4 - J\psi] \}}{\text{(same with } J = 0)}, \tag{3.31}$$

where $K_{\Lambda^2}(x, y)$ is the inverse of the propagator $\Delta_{\Lambda^2}(x, y)$

$$\Delta_{\Lambda^2}(x, y) = \int (d\nu_\rho/C_\rho) \Delta(x, y, \mu_\rho^2), \tag{3.32}$$

with $\Delta(x, y, \mu^2)$ the free propagator of mass μ^2 .

Hence we have the identification, comparing with (3.28)

$$\begin{aligned} &\int_0^\infty d\alpha g_{\Lambda^2}(\alpha) \exp[-\alpha(k^2 + \mu_0^2)] \\ &= \int (d\nu_\rho/C_\rho) (k^2 + \mu_\rho^2)^{-1} \\ &= \int_0^\infty d\alpha \exp[-\alpha(\mu_0^2 + k^2)] \\ &\times \int (d\nu_\rho/C_\rho) \exp[-\alpha(\mu_\rho^2 - \mu_0^2)] \end{aligned} \tag{3.33}$$

which gives

$$g_{\Lambda^2}(\alpha) = \int (d\nu_\rho/C_\rho) \exp[-\alpha(\mu_\rho^2 - \mu_0^2)]. \tag{3.34}$$

We see therefore that we have succeeded in formally incorporating the regularization (3.28) into our formalism. Now we have various choices each one characterized by different measures $d\nu_\rho/C_\rho$ and masses μ_ρ . Let us write $d\nu_\rho/C_\rho$ in the form $d\nu(\mu^2)/C(\mu^2)$. The most common choice is the standard Pauli–Villars regularization defined as

$$\begin{aligned} &d\nu(\mu^2)/C(\mu^2) \\ &= d\mu^2 [\delta(\mu^2 - \mu_0^2) + \sum_{K=1}^N (1/C_K) \delta(\mu^2 - \mu_0^2 - \Theta_K \Lambda^2)] \end{aligned} \tag{3.35}$$

with the sum running over a finite number of terms N , and C_K and Θ_K a set of numbers to be chosen so that $g_{\Lambda^2}(\alpha)$ has the required properties.

$$\begin{aligned}
 g_{\Lambda^2}(\alpha) &= \int d\mu^2 \{ \delta(\mu^2 - \mu_0^2) + \sum_{K=1}^N (1/C_K) \delta(\mu^2 - \mu_0^2 - \Theta_K \Lambda^2) \} \\
 &\times \exp[-\alpha(\mu^2 - \mu_0^2)] \\
 &= 1 + \sum_{K=1}^N (1/C_K) \exp[-\alpha \Theta_K \Lambda^2]. \tag{3.36}
 \end{aligned}$$

For $\Lambda^2 \rightarrow \infty$, $g_{\Lambda^2}(\alpha)$ will tend to one for any fixed $\alpha > 0$, while it is possible to choose C_K 's and Θ_K 's such that

$$\begin{aligned}
 1 + \sum_K (1/C_K) &= 0, \quad \sum_K (\Theta_K/C_K) = 0, \quad \dots, \\
 \sum_K (\Theta_K^n/C_K) &= 0, \tag{3.37}
 \end{aligned}$$

which ensure that

$$g_{\Lambda^2}(0) = 0, \quad g_{\Lambda^2}'(0) = 0, \quad \dots, \quad g_{\Lambda^2}^{(n)}(0) = 0. \tag{3.38}$$

It is clear that, given an integer $n > 1$ we can always choose N in (3.35) sufficiently large so that $g_{\Lambda^2}(\alpha)$ as well as its n first derivatives vanish for $\alpha = 0$. Notice also that for (3.37) to be true some of the C_K 's have to be chosen negative which means, in ordinary language, that the corresponding fields would have to be quantized in a space with negative metric.

A different choice for the measure $d\nu_p/C_p$ is one that gives the Gaussian form with

$$g_{\Lambda^2}(\alpha) = \Theta(\alpha - 1/\Lambda^2). \tag{3.39}$$

Clearly, due to the discontinuity of the Θ function, it is difficult to give an explicit representation. But any smooth choice, such as for instance $[th(\Lambda^2\alpha - 1) + th1]/[1 + th1]$ would allow it.

With the preceding examples, we wanted merely to illustrate the point that there exist several possible choices for the functions $g_{\Lambda^2}(\alpha)$ with the desired properties and they are well adapted for the functional formalism. In the rest of these notes we need not be more specific.

IV. EXAMPLES OF EXPLICIT CALCULATIONS: V AND Z UP TO ORDER \hbar^2

We interrupt here the formal discussions in order to illustrate, with some explicit calculations, the use of the equations derived in the previous section. We have chosen to calculate the quantities $V(\varphi_c)$ and $Z(\varphi_c)$, defined in Eqs. (2.24) and (2.25) respectively, up to two closed loops in perturbation theory.

A. Calculation of V

Corresponding to the expansion (3.13) we have

$$V(\varphi_c) = V_0(\varphi_c) + \hbar V_1(\varphi_c) + \hbar^2 V_2(\varphi_c) + \dots \tag{4.1}$$

Clearly, once an over-all factor $\int d^4x$ is isolated in $\Gamma[\varphi_c]$, we get $V(\varphi_c)$ by just setting $\varphi_c = \text{constant}$. We shall now compute the first three terms in (4.1). The first one is trivial. From (3.15) we get

$$V_0 = (\mu^2/2)\varphi_c^2 + (\lambda/4!)\varphi_c^4. \tag{4.2}$$

The normalization conditions (2.26) are fulfilled up to this order. Let us now look at V_1 . Here $\Gamma_1[\varphi_c]$ is given by (3.17). For constant φ_c the kernel K_{xy} becomes

$$\begin{aligned}
 K_{xy}(\varphi_c) &= (\partial_x \partial_y + \mu^2 + \lambda\varphi_c^2/2)\delta^4(x - y) \\
 &= \int [d^4k/(2\pi)^4] (k^2 + \mu^2 + \lambda\varphi_c^2/2) \exp[ik(x - y)]. \tag{4.3}
 \end{aligned}$$

Therefore we get:

$$\begin{aligned}
 \frac{1}{2} \text{tr} \log[K(\varphi_c)/K(0)] &= \frac{1}{2} \int d^4x \int [d^4k/(2\pi)^4] \\
 &\times \log[(k^2 + \mu^2 + \frac{1}{2}\lambda\varphi_c^2)/(k^2 + \mu^2)]. \tag{4.4}
 \end{aligned}$$

This gives us $V_1(\varphi_c)$ up to renormalization. In fact the integral in (4.4) is ultraviolet divergent, but the counter terms must be chosen such as to make the coefficients of φ_c^2 and φ_c^4 vanish in a power series expansion at the origin, since $V_0(\varphi_c)$ already satisfies the conditions (2.26). This renormalization is sufficient to give a finite result. We therefore write:

$$V_1(\varphi_c) = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \log \left[1 + \frac{\lambda\varphi_c^2/2}{k^2 + \mu^2} \right] + A\varphi_c^2 + B\varphi_c^4, \tag{4.5}$$

where the counterterms A and B will be determined by the conditions (2.26). We find:

$$\begin{aligned}
 V_1(\varphi_c) &= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[\log \left(1 + \frac{\lambda\varphi_c^2/2}{k^2 + \mu^2} \right) - \frac{\lambda\varphi_c^2/2}{k^2 + \mu^2} \right. \\
 &\left. + \frac{1}{2} \frac{(\lambda\varphi_c^2/2)^2}{(k^2 + \mu^2)^2} \right] = \mu^4 w(\lambda\varphi_c^2/2\mu^2), \tag{4.6}
 \end{aligned}$$

where the function $w(x)$ is given by:

$$\begin{aligned}
 w(x) &= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[\log \left(1 + \frac{x}{k^2 + 1} \right) - \frac{x}{k^2 + 1} \right. \\
 &\left. + \frac{1}{2} \frac{x^2}{(k^2 + 1)^2} \right] = [4(4\pi)^2]^{-1} [(x + 1)^2 \log(x + 1) \\
 &- (\frac{3}{2}x^2 + x)]. \tag{4.7}
 \end{aligned}$$

Let us make two simple remarks:

(i) In order to check whether we have made any mistake we can compare our result with the one of Coleman and Weinberg (1973). To do this we have to adapt our normalization conventions and instead of (2.26) we call λ the fourth derivative of the potential at a nonvanishing value M of the classical field φ_c :

$$d^4V(\varphi_c)/d\varphi_c^4 |_{\varphi_c=M} = \lambda \quad M \neq 0. \tag{4.8}$$

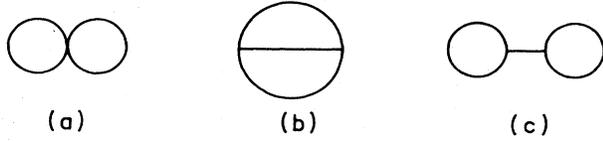


FIG. 1. The three connected diagrams to order \hbar^2 in the expansion of $S_2[\varphi_c]_\infty$. Only the first two are to be kept in order to find $\Gamma_2[\varphi_c]$. Heavy lines indicate a propagator $\Delta(\varphi_c)$.

Then we let the mass μ go to zero. The resulting $V(\varphi_c)$, using Eqs. (4.2) and (4.5), is

$$V(\varphi_c) = \frac{\lambda\varphi_c^4}{4!} + \frac{1}{(8\pi)^2} \left(\frac{\lambda\varphi_c^2}{2}\right)^2 \left[\log \frac{\varphi_c^2}{M^2} - \frac{25}{6} \right] + \dots \tag{4.9}$$

in agreement with Ref. 2).

(ii) The expression (4.9), considered as a function of λ , has no singularities for any finite value of the coupling constant. This is due to the zero mass limit we considered. For $\mu \neq 0$, the situation is different. Although $V_0 + V_1$, given by (4.2) and (4.6), is a very primitive approximation to the real potential, we see that singularities occur at $\varphi_c = \pm i(2\mu^2/\lambda)^{1/2}$. Since μ^2 is positive, these singularities are harmless for $\lambda > 0$. However for $\lambda < 0$ we would find singularities occurring for real φ_c and $V(\varphi_c)$ become complex.⁶ We shall later try to prove that $\lambda\varphi^4$ theory makes no sense for $\lambda < 0$. This perturbative argument should not be taken too seriously at this point. Notice finally that (4.9) cannot be obtained by simply taking the limit $\mu \rightarrow 0$ of

(4.2) and (4.6). This latter actually does not exist. We can instead calculate $V(\varphi_c)$ for $\mu \neq 0$ by using (4.8) as a renormalization condition; (4.9) will then be the limit of this expression for $\mu \rightarrow 0$.

Let us now turn to the evaluation of $V_2(\varphi_c)$. This is meant as a pedagogical exercise on renormalization. It also allows us to test explicitly the statements, to be made later, on the asymptotic behavior of perturbation theory.

We start from (3.8). Keeping only the \hbar^2 terms we get:

$$S_2 = \int \mathcal{D}[\psi] \exp - \int d^4x \left[\frac{1}{2}(\partial\psi)^2 + \frac{1}{2}(\mu^2 + \frac{1}{2}\lambda\varphi_c^2)\psi^2 \right] \times \{ -(\lambda/4!) \int d^4x \psi^4(x) + \frac{1}{2}(\lambda/3!)^2 \times \int \int d^4x d^4y \varphi_c(x)\psi^3(x)\varphi_c(y)\psi^3(y) \} \times (\int \mathcal{D}[\psi] \exp - \int d^4x \left[\frac{1}{2}(\partial\psi)^2 + \frac{1}{2}(\mu^2 + \frac{1}{2}\lambda\varphi_c^2)\psi^2 \right])^{-1}. \tag{4.10}$$

Using Wick's theorem Eq. (3.23) we see that to the first term corresponds the diagram (a) of Fig. 1. [Heavy lines indicate the propagator $\Delta(\varphi_c)$]. The second term gives rise to the two other diagrams, but the last one is to be disregarded in the calculation of Γ_2 according to Eq. (3.20), since it is not 1-PI. Consequently we obtain

$$\Gamma_2[\varphi_c] = -\frac{1}{8}\lambda \int d^4x \Delta(x, x | \varphi_c)^2 + 3(\lambda/3!)^2 \times \int \int d^4x d^4y \varphi_c(x)\varphi_c(y)\Delta(x, y | \varphi_c)^3. \tag{4.11}$$

Hence for $\varphi_c = \text{constant}$, using (3.24)

$$V_2(\varphi_c) = \frac{\lambda}{8} \iint \frac{d^4k_1 d^4k_2}{(2\pi)^8} \frac{1}{[k_1^2 + \mu^2 + \lambda\varphi_c^2/2][k_2^2 + \mu^2 + \lambda\varphi_c^2/2]} - \frac{\lambda^2\varphi_c^2}{12} \iiint \frac{d^4k_1 d^4k_2 d^4k_3}{(2\pi)^{12}} \frac{(2\pi)^4 \delta^4(k_1 + k_2 + k_3)}{[k_1^2 + \mu^2 + \lambda\varphi_c^2/2][k_2^2 + \mu^2 + \lambda\varphi_c^2/2][k_3^2 + \mu^2 + \lambda\varphi_c^2/2]} + \text{counterterms}. \tag{4.12}$$

As it stands, this expression is infinite and the counterterms are supposed to take care of this. However this time they are less trivial than in (4.5). We have to state how they are to be introduced. What we shall see however is that the prescription on V alone fixes them uniquely. This means that in the calculation of V to this order, no use is made of the condition (2.7) or, equivalently, of the conditions (2.27). These conditions will only come into play when mass insertions appear and this happens only to order \hbar^4 and higher. Therefore, as it was stated in Sec. II, up to

this order there is no difference between using (2.8) and (2.27). The counterterms are introduced in the usual way. We may imagine that three terms of order \hbar had been added already in the Lagrangian in order to account for the three infinite terms in φ_c^0 , φ_c^2 , and φ_c^4 occurring in the calculation of V_1 . They come now, to order \hbar^2 , when combined with the other terms in the Lagrangian. Furthermore there are the new counterterms of order \hbar^2 . The bookkeeping is simply summarized in the analog of Zimmerman's forest formula and yields explicitly

$$V_2(\varphi_c) = \frac{\lambda}{8} \left[\int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + \mu^2 + \lambda\varphi_c^2/2} - \frac{1}{k^2 + \mu^2} + \frac{\lambda\varphi_c^2/2}{(k^2 + \mu^2 + \lambda\varphi_c^2/2)^2} \right) \right]^2 - \frac{\lambda^2\varphi_c^2}{12} \int \frac{d^4k_1 d^4k_2 d^4k_3}{(2\pi)^{12}} \left(\frac{(2\pi)^4 \delta^4(k_1 + k_2 + k_3)}{(k_1^2 + \mu^2 + \lambda\varphi_c^2/2)(k_2^2 + \mu^2 + \lambda\varphi_c^2/2)(k_3^2 + \mu^2 + \lambda\varphi_c^2/2)} - \frac{3(2\pi)^4 \delta^4(k_1 + k_2)}{(k_1^2 + \mu^2)(k_2^2 + \mu^2)(k_3^2 + \mu^2 + \lambda\varphi_c^2/2)} \right) - \text{global counterterms}. \tag{4.13}$$

⁶The treatment of perturbation theory in the presence of external sources is discussed by J. Schwinger (1951) who in fact is also at the origin of many of the concepts used in the functional approach. In the context of electrodynamics, he also relates the occurrence of a complex effective potential with instability of the vacuum.

The remaining global counterterms are genuine \hbar^2 ones, and are there in order to ensure that the overall integral is of order φ_c^6 for small φ_c . The internal divergences have been cured by the subtractions inside the integrand. The reader will notice that for these terms the propagators have reas-

summed their standard form $[k^2 + \mu^2]^{-1}$ since subtractions are performed around $\varphi_c = 0$.

In order to simplify the notations we write

$$V_2(\varphi_c) = \mu^4 \left[\frac{\lambda}{8} \xi_1 \left(\frac{\lambda \varphi_c^2}{2\mu^2} \right) - \frac{\lambda^2 \varphi_c^2}{12 \mu^2} \xi_2 \left(\frac{\lambda \varphi_c^2}{2\mu^2} \right) \right], \quad (4.14)$$

and the result of an explicit calculation gives

$$\xi_1(x) = [1/(4\pi)^4] [(1+x) \log(1+x) - x]^2, \quad (4.15a)$$

$$\xi_2(x) = -[3/(4\pi)^4] \left[\frac{1}{2}(1+x) \log^2(1+x) - 2(1+x) \log(1+x) + 2x \right]. \quad (4.15b)$$

A brief description of the evaluation of the integrals is given in Appendix A.

Putting together (4.2), (4.6) and (4.14) we can write the effective potential up to order \hbar^2 in the form:

$$V(\varphi_c) = \frac{\mu^4}{\lambda} \mathcal{U} \left(\frac{\lambda \varphi_c^2}{2\mu^2}, \frac{\hbar \lambda}{(4\pi)^2} \right), \quad x = \frac{\lambda \varphi_c^2}{2\mu^2}, \alpha = \frac{\lambda \hbar}{(4\pi)^2}. \quad (4.16)$$

The \hbar expansion is the α expansion of \mathcal{U} , and we summarize our results for \mathcal{U} , up to two closed loops, in Table I. We conclude with the following observation. Apart from an over-all factor of μ^4 , μ and φ_c appear for simple dimensional reasons only through their ratio φ_c/μ . Therefore the behavior of V for large φ_c is related to that for small μ . We shall come back to this point later.

B. Calculation of Z

Parallel to the calculation of $V(\varphi_c)$ we can compute $Z(\varphi_c)$ defined in (2.21) and (2.25). One might at first think that instead of computing $\Gamma(\varphi_c)$ for a constant φ_c as was done to extract $V(\varphi_c)$ it is sufficient now to consider the case of a classical field φ_c varying linearly with x i.e., of the form $\text{const} + a \cdot x$. The pitfalls of such a method are slightly subtle and will be explained below. Thus we do not make such an assumption at this stage.

We shall use for Z the same loop expansion of the form

$$Z = Z_0 + \hbar Z_1 + \hbar^2 Z_2 + \dots \quad (4.17)$$

with obviously $Z_0 = 1$. Recalling (3.17) we have (Schwinger, 1951)

$$2\Gamma_1(\varphi_c) = \text{Tr}(\log \Delta(\varphi_c) - \log \Delta(0)). \quad (4.18)$$

Using four operators X_μ, P_μ satisfying $[X_\mu, P_\nu] = i\delta_{\mu\nu}$, we can write $\Delta(\varphi_c)$ as a matrix element

$$\Delta(x, y | \varphi_c) = \langle x | [P^2 + \mu^2 + \frac{1}{2}\lambda\varphi_c^2(X)]^{-1} | y \rangle \quad (4.19a)$$

of the operator

$$\Delta(\varphi_c) = [P^2 + \mu^2 + \frac{1}{2}\lambda\varphi_c^2(X)]^{-1}. \quad (4.19b)$$

Indeed, it is in terms of this operator that the log is introduced above, while trace requires to take the x, x matrix ele-

TABLE I. Contributions of the various diagrams to the value of the effective potential up to the order of two loops. The potential is written as

$$V(\varphi_c) = \frac{\mu^4}{\lambda} \mathcal{U} \left(\frac{\lambda \varphi_c^2}{2\mu^2}, \frac{\hbar \lambda}{(4\pi)^2} \right)$$

and x and α stand for $x = \lambda \varphi_c^2 / 2\mu^2, \alpha = \hbar \lambda / (4\pi)^2$.

Diagram	Contribution
	x
	$\frac{x^2}{6}$
	$(\alpha/4) ((1+x)^2 \log(1+x) - (x + \frac{1}{2}x^2))$
	$(\alpha^2/8) ((1+x) \log(1+x) - x)^2$
(a)	
	$\frac{1}{2}\alpha^2 x (\frac{1}{2}(1+x) \log^2(1+x) - 2(1+x) \times \log(1+x) + 2x)$
(b)	

ment and to sum over d^4x . Now let us use one of Schwinger's tricks which gives the following representation of the difference of logarithms of operators

$$\log A - \log B = \int_0^\infty (ds/s) [\exp(-Bs) - \exp(-As)]. \quad (4.20)$$

From this we obtain

$$\begin{aligned} & -2\Gamma_1(\varphi_c) + 2 \int d^4x V_1(\varphi_c(x)) \\ &= \int d^4x Z_1(\varphi_c(x)) (\partial\varphi_c(x))^2 + \dots \\ &= \text{Tr} \log(P^2 + \mu^2 + \frac{1}{2}\lambda\varphi_c^2(X)) \\ & - \int d^4x \langle x | \log(P^2 + \mu^2 + \frac{1}{2}\lambda\varphi_c^2(x)) | x \rangle \\ &= \int d^4x \int_0^\infty (ds/s) \{ \langle x | \exp - s[P^2 + \mu^2 \\ & + \frac{1}{2}\lambda\varphi_c^2(X)] | x \rangle - \langle x | \exp - s[P^2 + \mu^2 \\ & + \frac{1}{2}\lambda\varphi_c^2(x)] | x \rangle \}. \end{aligned} \quad (4.21)$$

Note in the last integral that the two terms differ in that in going from the first to the second the operator X is replaced by the c number x . Now if the functional $\Gamma(\varphi_c)$ were known as an expansion

$$- \int d^4x V(\varphi_c(x)) + \frac{1}{2} Z_1(\varphi_c(x)) (\partial\varphi_c(x))^2 + \dots$$

it would indeed be true that by substitution of $\varphi_c = \text{const} + a \cdot x$ the coefficient of a^2 would essentially yield $Z_1(\varphi_c)$ up

to known terms expressed in terms of V that are already computed. But recall that we are instead trying to isolate Z_1 that is we have to factor out an integration $\int d^4x$. This leads to ambiguities if we do not use test functions that vanish fast enough at infinity in order to allow for partial integrations. To illustrate these ambiguities let us give a simple example of a $\Gamma(\varphi_c)$ generated by a unique two point function itself taken as a first degree polynomial in p^2 . That is we consider the trivial functional

$$\begin{aligned} \mathcal{F}(\varphi_c) &= \int d^4x [\varphi_c^2(x) + \alpha(\partial\varphi_c(x))^2], \\ &= \iint d^4y d^4z \varphi_c(y) \varphi_c(z) \int \frac{d^4p}{(2\pi)^4} \\ &\times \exp[ip(y-z)](1 + \alpha p^2), \\ &= \iint d^4y d^4z \varphi_c(y) \varphi_c(z) (1 + \alpha \partial_y \cdot \partial_z) \delta^4(y-z). \end{aligned}$$

If we try to use a $\varphi_c(x)$ of the form $\varphi + a \cdot x$ we would write

$$\begin{aligned} \mathcal{F}(\varphi_c) &= \iint d^4y d^4z (\varphi + a \cdot y) (\varphi + a \cdot z) \\ &\times \left(1 - \alpha \frac{\partial^2}{\partial(y-z)^2} \right) \delta^4(y-z), \end{aligned}$$

and separate the variables as $y = x + u/2$, $z = x - u/2$ thus obtaining

$$\begin{aligned} \mathcal{F}(\varphi_c) &= \int d^4x \int d^4u [(\varphi + a \cdot x)^2 \\ &- \frac{1}{4}(a \cdot u)^2] [1 - \alpha(\partial^2/\partial u^2)] \delta^4(u) \\ &= \int d^4x (\varphi_c(x)^2 + \frac{1}{2}\alpha \partial\varphi_c(x)^2). \end{aligned}$$

Comparing with the original expression we see that the term in α occurs now multiplied with a factor $\frac{1}{2}$ instead of 1(!). The reason for this discrepancy is easily understood. As soon as we replaced $\varphi_c(x)$ by $\varphi + a \cdot x$ we dealt with infinite quantities and $\infty = \frac{1}{2}\infty$. This is why we have to be careful to keep $\varphi_c(x)$ depending on x in order to retain the possibility of performing an eventual integration by parts. Nevertheless we shall exploit the idea of linearization but in a slightly more sophisticated way. Returning to (4.21), the idea is to expand $\varphi_c^2(X)$ in the neighborhood of $\varphi_c^2(x)$. It turns to be sufficient, not to be bothered with indices, to assume a variation in one direction only (we take it to be the 0 direction) but one has to go up to quadratic terms. Thus we write

$$\begin{aligned} \varphi_c^2(X) &= \varphi_c^2(x) + a(x)\tilde{X}_0 + \frac{1}{2}b(x)\tilde{X}_0^2 + \dots \quad (4.22) \\ \tilde{X} &= X - x, \quad a^2(x) = 4\varphi_c^2(x)(\partial\varphi_c(x))^2 \\ b(x) &= \square\varphi_c^2(x) = 2(\partial\varphi_c(x))^2 + 2\varphi_c(x)\square\varphi_c(x). \end{aligned}$$

Using this expansion together with the formula (3.26) borrowed from quantum mechanics for the harmonic oscil-

lator we obtain

$$\begin{aligned} &\int d^4x Z_1(\varphi_c(x)) (\partial\varphi_c(x))^2 + \dots \\ &= - \int d^4x \int_0^\infty \frac{ds}{s} \int \frac{d^3k_T}{(2\pi)^3} \\ &\times \exp - s(k_T^2 + \mu^2 + \frac{1}{2}\lambda\varphi_c^2(x)) \\ &\times (x_0 | [\exp - s(P_0^2 + \frac{1}{2}\lambda a_x \tilde{X}_0 + \frac{1}{4}\lambda b_x \tilde{X}_0^2 + \dots) \\ &- \exp - sP_0^2] | x_0). \end{aligned}$$

Two terms have only been retained in the series expansion of $\varphi_c^2(X)$ since we are only interested in the coefficient of $\partial\varphi^2$. The method could of course be generalized. Now the integration over k_T can be performed and the matrix element evaluated with the result

$$\begin{aligned} &\int d^4x Z_1(\varphi_c(x)) (\partial\varphi_c(x))^2 + \dots \\ &= - \int d^4x \int_0^\infty \frac{ds}{s} \frac{1}{(4\pi)^2} \exp - s[\mu^2 + \frac{1}{2}\lambda\varphi_c^2(x)] \\ &\times \left[\left(\frac{2\omega_x s}{sh2\omega_x s} \right)^{1/2} \exp \left(\frac{\lambda a_x^2}{4b_x} s - \frac{\omega_x}{sh2\omega_x s} \frac{a_x^2}{b_x^2} \right. \right. \\ &\left. \left. \times (ch2\omega_x s - 1) \right) - 1 \right], \end{aligned}$$

with $\omega_x^2 = \lambda b_x^2/4$.

We expand the right hand side in powers of a_x and b_x , and keep terms up to order a_x^2 and b_x . The right hand side becomes

$$\begin{aligned} &- \frac{1}{(4\pi)^2} \int d^4x \int_0^\infty \frac{ds}{s^3} \exp[-s(\mu^2 + \frac{1}{2}\lambda\varphi_c^2(x))] \\ &\times \left(-\frac{\lambda b_x s^2}{12} + \frac{\lambda^2}{48} a_x^2 s^3 + \dots \right). \end{aligned}$$

We now recall the expressions given in (4.22) for a_x^2 and b_x :

$$\begin{aligned} &\int d^4x Z_1(\varphi_c(x)) (\partial\varphi_c(x))^2 + \dots \\ &= - \frac{1}{(4\pi)^2} \int d^4x \int_0^\infty ds \exp[-s(\mu^2 + \frac{1}{2}\lambda\varphi_c^2(x))] \\ &\times \left(\frac{1}{12} \lambda^2 \varphi_c^2(x) (\partial\varphi_c(x))^2 - \frac{1}{12} \lambda (1/s) \square\varphi_c^2(x) + \dots \right). \end{aligned}$$

Due to the appearance of a term $1/s$ in the last bracket, one might superficially think that a divergence occurs. This is not the case however since an integration by parts of the \square operator is allowed to factor out $(\partial\varphi_c(x))^2$ and this has the pleasant property of bringing down a factor s from the exponential. Everything settles down correctly, and we can now forget the x dependence and write

$$\begin{aligned} Z_1(\varphi_c) &= \frac{1}{12(4\pi)^2} \int_0^\infty ds \exp[-s(\mu^2 + \frac{1}{2}\lambda\varphi_c^2)] \lambda^2 \varphi_c^2, \\ Z_1(\varphi_c) &= \frac{1}{6} \frac{\lambda}{(4\pi)^2} \left(\frac{\lambda\varphi_c^2/2\mu^2}{1 + \lambda\varphi_c^2/2\mu^2} \right) = \frac{\alpha}{6} \frac{x}{1+x}. \quad (4.23) \end{aligned}$$

In the last expression we have reinstated our notations $\alpha = \lambda/(4\pi)^2$, $x = \lambda\varphi_c^2/2\mu$ (not to be confused of course with the four-dimensional configuration space variable). Note that $Z_1(0)$ vanishes (there is no wavefunction renormalization to the order of one loop in φ^4 theory).

We have gone through the one loop calculation in some detail to show that one could avoid summing series of (ordinary) diagrams with combinatorial factors and some complicated bookkeeping of derivatives. The price to be paid was that some care had to be exercised in choosing the correct x variation for the classical field φ_c .

We shall be much briefer for the two loop term. According to (4.11) the contribution to Z_2 can be split in two terms. One pertains to the first diagram of Fig. 1.

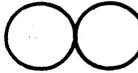
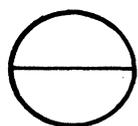
$$\int d^4x \frac{1}{2} Z_2^{(a)}(\varphi_c(x)) \partial\varphi_c(x)^2 = \text{factor of } \partial\varphi_c(x)^2 \text{ in } \int d^4x \frac{1}{2} \lambda \Delta(x, x | \varphi_c)^2. \tag{4.23}$$

The second contribution from the diagram (b) of Fig. 1 is:

$$\begin{aligned} \int d^4x \frac{1}{2} Z_2^{(b)}(\varphi_c(x)) \partial\varphi_c(x)^2 &= \text{factor of } \partial\varphi_c(x)^2 \text{ in} \\ &- (\lambda^2/12) (\iint d^4x d^4y \varphi_c(x) \varphi_c(y) \Delta^3(x, y | \varphi_c) \\ &- 3 \int d^4x \varphi_c(x)^2 \Delta(x, x | \varphi_c) \int d^4y \Delta^2(x, y | 0)) \\ &+ \text{over-all counterterm.} \end{aligned} \tag{4.24}$$

The value of the over-all counterterm is dependent on the normalization scheme adopted as explained in detail in Sec. II. We leave this choice free as appears in Table II

TABLE II. Contributions up to the order of two loops to the function Z . The variables x and α stand for $x = \lambda\varphi_c^2/2\mu^2$, $\alpha = \lambda\hbar/(4\pi)^2$.

Diagram	Contribution
Z_0 	1
Z_1 	$\frac{\alpha}{6} \frac{x}{1+x}$
	$\frac{\alpha^2}{12} x \left(\frac{\log(1+x)}{1+x} + \frac{x}{(1+x)^2} \right)$
Z_2 	$\begin{aligned} &\frac{\alpha^2}{6} \left[-\frac{1}{2} \log(1+x) + \frac{x}{1+x} + \frac{2x}{1+x} \right. \\ &\times (\log(1+x) + 4A/3 - 1) - \frac{x^2}{(1+x)^2} \\ &\times (\log(1+x) + \frac{2}{3}(A-1)) + \frac{4}{3} \frac{x^2}{(1+x)^2} \\ &\left. + \text{finite const.} \right] \end{aligned}$

The number A occurring in the last expression is

$$\begin{aligned} A &= - \int_0^1 du \frac{\log u}{1-u+u^2} = \frac{3}{2} \sum_0^\infty \left(\frac{1}{(1+3p)^2} - \frac{1}{(2+3p)^2} \right) \\ &= 1.1719536 \end{aligned}$$

where we have collected the results of the integrations. The actual calculations are summarized in Appendix A. For clarity of notation when referring to Z expressed in terms of x and α we shall denote it $z(\alpha, x)$ that is we set (with $\hbar = 1$)

$$\begin{aligned} Z(\varphi_c, \lambda) &= z(\alpha, x) \\ \alpha &= \lambda/(4\pi)^2 \quad x = \frac{1}{2} \lambda (\varphi_c^2/\mu^2). \end{aligned} \tag{4.25}$$

The numerical results of this section will be used in the sequel to discuss the asymptotic behavior of the φ^4 theory.

V. RENORMALIZATION GROUP AND CALLAN-SYMANZIK EQUATIONS

A. The renormalization group

In Sec. II we noticed that a simple relation exists between the Green functions calculated according to the different renormalization prescriptions. In particular, a change of the subtraction point $M_1^2 \rightarrow M_2^2$ is described by Eq. (2.5)

$$\begin{aligned} \Gamma^{(2n)}(s_i, \mu^2, M_1^2, \lambda_{M_1}) \\ = Z_3^n(M_2^2, \mu^2, M_1^2, \lambda_{M_1}) \Gamma^{(2n)}(s_i, \mu^2, M_2^2, \lambda_{M_2}). \end{aligned} \tag{2.5}$$

The factor Z_3 can be easily evaluated by applying (2.5) for $n = 1$. We thus obtain

$$Z_3(M_2^2, \mu^2, M_1^2, \lambda_{M_1}) = \frac{\Gamma^{(2)}(-M_2^2, \mu^2, M_1^2, \lambda_{M_1})}{M_2^2 - \mu^2}. \tag{5.1}$$

In the same way, Eq. (2.5) for $n = 2$ gives

$$\begin{aligned} \lambda_{M_2} &= - \frac{\Gamma^{(4)}(s_i, \mu^2, M_1^2, \lambda_{M_1})}{Z_3^2} \Big|_{\text{sym. point } M_2^2} \\ &\equiv R(M_2^2, \mu^2, M_1^2, \lambda_{M_1}), \end{aligned} \tag{5.2}$$

where the function R , defined by (5.2), satisfies the normalization condition

$$R(a, \mu^2, a, \lambda) = \lambda. \tag{5.3}$$

Using (5.2), (2.5) can be written as

$$\begin{aligned} \Gamma^{(2n)}(s_i, \mu^2, M_1^2, \lambda_{M_1}) \\ = Z_3^n \Gamma^{(2n)}(s_i, \mu^2, M_2^2, R(M_2^2, \mu^2, M_1^2, \lambda_{M_1})). \end{aligned} \tag{5.4}$$

If we set $M_2^2 = \tau M_1^2$, the realization (5.4) of the multiplicative group $\tau_1 * \tau_2 \rightarrow \tau_1 \cdot \tau_2$ of positive numbers is called the renormalization group (Stüeckelberg and Petermann, 1953; Gell-Mann and Low, 1954). Notice that the transformations of the group leave the physical mass, i.e., the pole of the complete propagator, unchanged. It is only in this case that simple relations, like Eq. (2.5), hold. Equation (5.4) is the functional equation of the group, and R is called the invariant charge, or invariant coupling constant.

Similar equations are satisfied by the generating functionals. For example, $\Gamma[\varphi_c]$, if we use (2.2) and (2.3) as normalization conditions, will depend on μ^2 , M^2 , and λ_{M^2} in addition to the functional dependence on $\varphi_c(x)$. Again all physical results are independent of M^2 in the sense that

if we change $M^2 = M_1^2 \rightarrow M_2^2 = \tau^2 M^2$ there exists a certain value of the coupling constant $\lambda_{M_2^2}$ which is a function of μ^2 , $\lambda_{M_1^2}$, M_1^2 , and M_2^2 , and a certain renormalization of the fields Z_3 given by (5.1) such that

$$\Gamma[\varphi_c, \mu^2, M_1^2, \lambda_{M_1}] = \Gamma[Z_3^{1/2}\varphi_c, \mu^2, M_2^2, \lambda_{M_2}], \quad (5.5)$$

or, using (5.2):

$$\begin{aligned} \Gamma[\varphi_c, \mu^2, M_1^2, \lambda_{M_1}] \\ = \Gamma[Z_3^{1/2}\varphi_c, \mu^2, M_2^2, R(M_2^2, \mu^2, M_1^2, \lambda_{M_1})]. \end{aligned} \quad (5.6)$$

This equation is the functional equation of the renormalization group, analogous to (5.4). In fact, (5.4) can be recovered from (5.6) if we functionally expand this last one in powers of $\varphi_c(x)$.

The functional equations (5.4) or (5.6) are very useful, as we shall show in this section. However one could object that, if we want to use entirely the formalism of the generating functionals and not talk about Green functions, these equations are not very convenient since the normalization conditions (2.2) and (2.3), upon which they are based, cannot be expressed in closed form in terms of $V(\varphi_c)$, $Z(\varphi_c)$ etc. The reason is that they involve values of the Green functions for external momenta different from zero and hence they need an infinite number of terms in the development (2.21). We shall continue to use Eq. (5.6), but, for the reader who wants to avoid reference to Green functions altogether, we recall that one can use a variety of normalization conditions expressible entirely in terms of $V(\varphi_c)$ and $Z(\varphi_c)$. In fact, in the same way that we could normalize the Green functions away from the origin in momentum space, we can normalize $\Gamma[\varphi_c]$ away from the origin in classical field space. We can, for example, use a set of normalization conditions of the form

$$d^2V(\varphi_c)/d\varphi_c^2 |_{\varphi_c=0} = m^2, \quad (5.7a)$$

$$d^4V(\varphi_c)/d\varphi_c^4 |_{\varphi_c=M} = \bar{\lambda}_M, \quad (5.7b)$$

$$Z(M) = 1, \quad (5.7c)$$

where M and m are positive masses. Notice that m is not the physical mass and $\bar{\lambda}_M$ is not the value of the four-point function at some symmetry point, but for the renormalization program to be carried through, the set of constants m^2 , $\bar{\lambda}_M$ is as good as μ^2 , λ_M which in turn is as good as any other set. From the three conditions (5.7) only the first one can be expressed in terms of Green functions in a closed form since it simply means that $\Gamma^{(2)}(p^2=0) = m^2$. The other two conditions involve the values of all Green functions at $p_i = 0$. Again all physical quantities are independent of the point M in the sense that if for a certain functional $\Gamma[\varphi_c]$ defined through a set of constants M_1 , m_1^2 , $\bar{\lambda}_{M_1}$ we change M_1 to M_2 , there exist new constants m_2^2 and $\bar{\lambda}_{M_2}$ such that all physical quantities remain unchanged. The change $M_1 \rightarrow M_2$ can be absorbed into new values of the constants m^2 and $\bar{\lambda}$ and a rescaling of the fields. We therefore write the analog of Eq. (5.5) as

$$\Gamma[\varphi_c, m_1^2, M_1, \bar{\lambda}_{M_1}] = \Gamma[\bar{Z}_3^{1/2}\varphi_c, m_2^2, M_2, \bar{\lambda}_{M_2}]. \quad (5.8)$$

Notice that, since m^2 is no more the value of the physical mass, it does not remain unchanged in the transformation.

The equation (5.8) is only useful if we can express \bar{Z}_3 , m_2^2 and $\bar{\lambda}_{M_2}$ in terms of M_2 , m_1^2 , M_1 and $\bar{\lambda}_{M_1}$ in the same way as we did for Z_3 and λ_{M_2} in Eqs. (5.1) and (5.2). In fact, writing (5.8) for $Z(\varphi_c)$, and using (5.7c) we obtain

$$\Gamma(\bar{Z}_3^{-1/2}M_2, m_1^2, M_1, \bar{\lambda}_{M_1}) = \bar{Z}_3. \quad (5.9)$$

Since $Z(\varphi_c, m^2, M, \bar{\lambda}_M)$ is known order by order in the loop expansion, (5.9) gives \bar{Z}_3 as a function of M_2 , m_1^2 , M_1 and $\bar{\lambda}_{M_1}$. In the same way, writing (5.8) for $V(\varphi_c)$ and using (5.7a) and (5.7b) we obtain

$$\begin{aligned} m_2^2 &= m_1^2 \bar{Z}_3^{-1}, \\ \bar{\lambda}_{M_2} &= \bar{Z}_3^{-2} (d^4V/d\varphi_c^4) |_{\varphi_c=\bar{Z}_3^{-1/2}M_2} \equiv \bar{R}(M_2, m_1^2, M_1, \bar{\lambda}_{M_1}). \end{aligned} \quad (5.11)$$

Therefore the functional equation of the renormalization group reads, in this case,

$$\begin{aligned} \Gamma[\varphi_c, m_1^2, M_1, \lambda_{M_1}] \\ = \Gamma[\bar{Z}_3^{1/2}\varphi_c, \bar{Z}_3^{-1}m_1^2, M_2, R(M_2, m_1^2, M_1, \lambda_{M_1})]. \end{aligned} \quad (5.12)$$

The equation (5.12) is strictly equivalent to (5.6) although some of the quantities appearing in the latter, like the physical mass μ , have more direct physical meaning. Either one can be used in order to study the asymptotic properties of the theory. Before doing that, we shall derive another functional relation, the Callan-Symanzik equation, which will be proved to be equivalent to the renormalization group equation in the asymptotic region.

B. The Callan-Symanzik equation (Callan, 1970; Symanzik, 1970)⁷

The renormalization group Eq. (5.6) or (5.12) derived in the previous paragraph describe the invariance of the theory under a change of the renormalization point. In this paragraph we shall derive a similar equation which will describe the response of the system under a change of scale. Having obtained (5.6) or (5.12) we know that there is no loss of information in working at a particular value of the normalization point, therefore we shall choose to define the theory using conditions (2.6) to (2.8) which correspond to $M = 0$.

Up to now we have worked exclusively in terms of renormalized quantities, but it turns out that the physical interpretation of the Callan-Symanzik equation is much more transparent when derived starting from the unrenormalized Green functions, since the properties of Feynman amplitudes are much simpler when a cutoff ensures convergence in terms of bare quantities. We emphasize that this change of language is dictated only by pedagogical reasons, and in fact an alternative derivation without reference to cutoff and bare objects can be found in the literature.⁷

Let Γ_{un} be the generating functional of the unrenormalized but regularized Green functions. It will depend on the cutoff Λ , the bare mass μ_0 and coupling constant λ_0 . If μ

⁷ C. G. Callan (1970); K. Symanzik (1970). For a more formal proof see, for instance, J. H. Lowenstein (1971).

and λ are the renormalized quantities defined by (2.6) to (2.8), the renormalizability of the theory tells us that:

$$\Gamma[\varphi_c(x), \mu^2, \lambda] = \lim_{\Lambda \rightarrow \infty} \Gamma_{un}[Z_3^{1/2}\varphi_c(x), \mu_0^2, \lambda_0, \Lambda]. \quad (5.13)$$

The bare quantities μ_0 and λ_0 , as well as the wavefunction renormalization Z_3 , all diverge when $\Lambda \rightarrow \infty$, but the left-hand-side is finite. The conditions (2.6) to (2.8) can be expressed in terms of the unrenormalized Green functions and can be regarded as giving μ_0 , λ_0 and Z_3 as functions of μ , λ , and Λ . For dimensional reasons we have

$$\begin{aligned} \lambda_0 &= \lambda_0(\mu/\Lambda, \lambda), & \mu_0 &= \mu g(\mu/\Lambda, \lambda), \\ Z_3^{1/2} &= Z_3^{1/2}(\mu/\Lambda, \lambda). \end{aligned} \quad (5.14)$$

Two ingredients will be used in the derivation.

(i) We first make the trivial remark that since the limit (5.13) exists, it is invariant when we scale $\Lambda \rightarrow \tau\Lambda$. We shall now define two new, τ -dependent, renormalized quantities.

A coupling constant λ_τ through

$$\lim_{\Lambda \rightarrow \infty} \frac{\lambda_0(\mu/\tau\Lambda, \lambda)}{\lambda_0(\mu/\Lambda, \lambda_\tau)} = 1, \quad (5.15a)$$

and a shift of wavefunction renormalization through

$$\rho^{1/2}(\tau, \lambda) = \lim_{\Lambda \rightarrow \infty} \frac{Z_3^{1/2}(\mu/\tau\Lambda, \lambda)}{Z_3^{1/2}(\mu/\Lambda, \lambda_\tau)}. \quad (5.15b)$$

The fact that ρ is independent of μ is trivial since it is dimensionless, what is not trivial however, is that the limits (5.15) exist at all. This can be shown by expressing them in terms of renormalized quantities.

(ii) The second ingredient is provided by ordinary dimensional analysis. Since Γ is dimensionless, it is unchanged when all dimensional quantities are scaled simultaneously. Specifically

$$\Gamma_{un}[\tau\varphi_c(\tau x), \tau\mu_0, \lambda_0, \tau\Lambda] = \Gamma_{un}[\varphi_c(x), \mu_0, \lambda_0, \Lambda].$$

Let us now apply (i) and (ii). We first write (5.13) for slightly different arguments:

$$\begin{aligned} &\Gamma\left[\frac{\rho^{1/2}(\tau, \lambda)}{\tau}\varphi_c(x/\tau), \mu^2, \lambda_\tau\right] \\ &= \lim_{\Lambda \rightarrow \infty} \Gamma_{un}\left[Z_3^{1/2}(\mu/\Lambda, \lambda_\tau)\frac{\rho^{1/2}(\tau, \lambda)}{\tau}\right. \\ &\quad \left.\times \varphi_c(x/\tau), \mu^2 g^2(\mu/\Lambda, \lambda_\tau), \lambda_0(\mu/\Lambda, \lambda_\tau), \Lambda\right] \\ &= \lim_{\Lambda \rightarrow \infty} \Gamma_{un}[Z_3^{1/2}(\mu/\tau\Lambda, \lambda)(1/\tau)\varphi_c(x/\tau), \mu^2 g^2 \\ &\quad \times (\mu/\Lambda, \lambda_\tau), \lambda_0(\mu/\tau\Lambda, \lambda), \Lambda], \end{aligned} \quad (5.16)$$

where we have used (5.15a) and (5.15b). We now change $\Lambda \rightarrow \Lambda/\tau$, and scale all dimensional quantities by τ . The

result is

$$\begin{aligned} &\Gamma\left(\frac{\rho^{1/2}(\tau, \lambda)}{\tau}\varphi_c(x/\tau), \mu^2, \lambda_\tau\right) \\ &= \lim_{\Lambda \rightarrow \infty} \Gamma_{un}(Z_3^{1/2}(\mu/\Lambda, \lambda)\varphi_c(x), \tau^2\mu^2 g^2 \\ &\quad \times (\tau\mu/\Lambda, \lambda_\tau), \lambda_0(\mu/\Lambda, \lambda), \Lambda). \end{aligned} \quad (5.17)$$

Subtracting (5.13) from (5.17), we finally obtain the Callan-Symanzik equation

$$\begin{aligned} &\Gamma\left(\frac{\rho^{1/2}(\tau, \lambda)}{\tau}\varphi_c(x/\tau), \mu^2, \lambda_\tau\right) - \Gamma[\varphi_c(x), \mu^2, \lambda] \\ &= \Delta[\varphi_c(x), \mu^2, \lambda, \tau], \end{aligned} \quad (5.18)$$

where Δ is given by

$$\begin{aligned} &\Delta[\varphi_c(x), \mu^2, \lambda, \tau] \\ &= \lim_{\Lambda \rightarrow \infty} \int_{\mu^2 g^2(\mu/\Lambda, \lambda)}^{\tau^2 \mu^2 g^2(\tau\mu/\Lambda, \lambda_\tau)} d\mu_0^2 \frac{\partial}{\partial \mu_0^2} \\ &\quad \times \Gamma_{un}[Z_3^{1/2}(\mu/\Lambda, \lambda)\varphi_c(x), \mu_0^2, \lambda_0(\mu/\Lambda, \lambda)] \end{aligned} \quad (5.19)$$

and is exhibited as an explicit integral over mass insertions (the $\partial/\partial\mu_0^2$ operation). It is this physical meaning of Δ which is more involved to show when one works directly in terms of renormalized quantities.

We shall argue later that (5.18) is in fact equivalent to (5.6) or (5.12) in the asymptotic region. This should not be too surprising since they both express the same content of renormalizability.

C. The differential equations

It is sometimes easier to extract the physical information contained in Eqs. (5.6) or (5.12) and (5.18) by transforming them into the equivalent differential equations.

Let us start with the renormalization group equations (5.6) or (5.12). By differentiating the former with respect to M_2^2 and then setting $M_1^2 = M_2^2 = M^2$, we find

$$\begin{aligned} &\left\{ M^2 \frac{\partial}{\partial M^2} + \beta' \left(\frac{M^2}{\mu^2}, \lambda_M \right) \frac{\partial}{\partial \lambda_M} - \frac{1}{2} \gamma' \left(\frac{M^2}{\mu^2}, \lambda_M \right) \varphi_c \frac{\delta}{\delta \varphi_c} \right\} \\ &\quad \times \Gamma[\varphi_c, \mu^2, M^2, \lambda_M] = 0, \end{aligned} \quad (5.20)$$

where

$$\beta' \left(\frac{M^2}{\mu^2}, \lambda_M \right) = M^2 \frac{\partial R(M_2^2, \mu^2, M^2, \lambda_M)}{\partial M_2^2} \Big|_{M_2^2=M^2}, \quad (5.21)$$

$$\gamma' \left(\frac{M^2}{\mu^2}, \lambda_M \right) = -M^2 \frac{\partial Z_3(M_2^2, \mu^2, M_1^2, \lambda_M)}{\partial M_2^2} \Big|_{M_2^2=M^2}. \quad (5.22)$$

Equation (5.20) is the functional analog of the Ovsianikov (1956) equations for the Green functions. The correspond-

ing equation, derived in exactly the same way from (5.12), reads:

$$\left\{ M^2 \frac{\partial}{\partial M^2} + \tilde{\beta}' \left(\frac{M^2}{m^2}, \tilde{\lambda}_M \right) \frac{\partial}{\partial \tilde{\lambda}_M} + \tilde{\gamma}' \left(\frac{M^2}{m^2}, \tilde{\lambda}_M \right) \right. \\ \left. \times \left[m^2 \frac{\partial}{\partial m^2} - \frac{1}{2} \varphi_c \frac{\delta}{\delta \varphi_c} \right] \right\} \Gamma[\varphi_c, m^2, M^2, \tilde{\lambda}_M] = 0, \quad (5.23)$$

with $\tilde{\beta}'$ and $\tilde{\gamma}'$ given by formulae similar to (5.21) and (5.22) in which we replace R, Z_3, λ_M and μ^2 by $\tilde{R}, \tilde{Z}_3, \tilde{\lambda}_M$ and m^2 .

Let us now use the same method in order to derive the differential form of the Callan–Symanzik equation. We differentiate (5.18) with respect to τ and then put $\tau = 1$. We then obtain

$$\left\{ \mu(\partial/\partial\mu) + \beta(\lambda)(\partial/\partial\lambda) + \frac{1}{2}\gamma(\lambda) \int dx \varphi_c(x) [\delta/\delta\varphi_c(x)] \right\} \\ \times \Gamma[\varphi_c(x), \mu^2, \lambda] = \delta[\varphi_c(x), \mu^2, \lambda], \quad (5.24)$$

where

$$\beta(\lambda) = \partial\lambda_\tau/\partial\tau |_{\tau=1} \quad \lambda_\tau |_{\tau=1} = \lambda, \quad (5.25)$$

$$\gamma(\lambda) = \partial\rho(\tau, \lambda)/\partial\tau |_{\tau=1} \quad \rho(\tau, \lambda) |_{\tau=1} = 1, \quad (5.26)$$

$$Z_\mu(\mu/\Lambda, \lambda) = (\partial/\partial\tau) \log[\tau^2 g^2(\tau\mu/\Lambda, \lambda_\tau)] |_{\tau=1}, \quad (5.27)$$

$$\delta[\varphi_c(x), \mu^2, \lambda] \\ = \lim_{\Lambda \rightarrow \infty} Z_\mu(\mu/\Lambda, \lambda) \mu_0^2 (\partial/\partial\mu_0^2) \\ \times \Gamma_{un}[Z_3^{1/2} \varphi_c(x), \mu_0^2, \lambda_0, \Lambda]. \quad (5.28)$$

Equation (5.28) shows that $\delta[\varphi_c(x), \mu^2, \lambda]$ is the generating functional of the 1PI Green functions containing an arbitrary number of external lines and one zero momentum mass insertion. When expanded in power series in φ_c , (5.24) gives the well-known form of the Callan–Symanzik equations for Green functions. In this way, we can express $\beta(\lambda)$ and $\gamma(\lambda)$ in terms of renormalized Green functions by using the normalization conditions for $\Gamma^{(2)}$ and $\Gamma^{(4)}$, Eqs. (2.6)–(2.8). The result is

$$\gamma(\lambda) = -2 - (1/\mu^2) \delta^{(2)}(0, \mu^2, \lambda), \quad (5.29)$$

$$\beta(\lambda) = -2\lambda\gamma(\lambda) - \delta^{(4)}(p_i = 0, \mu^2, \lambda). \quad (5.20)$$

The normalization of $\delta^{(2)}$ is chosen such that it satisfies

$$\delta^{(2)}(-\mu^2, \mu^2, \lambda) = 2\mu^2 \frac{\Gamma^{(2)}(p^2, \mu^2, \lambda)}{p^2 + \mu^2} \Big|_{p^2 = -\mu^2} \\ = -2\mu^2 [1 - \Sigma(-\mu^2, \mu^2, \lambda)].$$

The functional equation (5.24) implies corresponding equations for $V(\varphi_c)$, $Z(\varphi_c)$ etc... Indeed, let us expand $\delta[\varphi_c(x), \mu^2, \lambda]$ around $\varphi_c = \text{const.}$ as we did for $\Gamma[\varphi_c(x), \mu^2, \lambda]$ in Eq. (2.21)

$$\Gamma[\varphi_c(x)] = -\int d^4x V(\varphi_c) + \frac{1}{2} Z(\varphi_c) (\partial\varphi_c)^2 + \dots \quad (2.21)$$

$$\delta[\varphi_c(x)] = -\int d^4x \delta_V(\varphi_c) + \frac{1}{2} \delta_Z(\varphi_c) (\partial\varphi_c)^2 + \dots \quad (5.31)$$

Applying the operator of the lhs of (5.24) on (2.21) we obtain

$$[\mu(\partial/\partial\mu) + \beta(\lambda)(\partial/\partial\lambda) + \frac{1}{2}\gamma(\lambda)\varphi_c(\partial/\partial\varphi_c)] V(\varphi_c, \mu^2, \lambda) \\ = \delta_V(\varphi_c, \mu^2, \lambda), \quad (5.32)$$

$$\{\mu(\partial/\partial\mu) + \beta(\lambda)(\partial/\partial\lambda) + \frac{1}{2}\gamma(\lambda)[2 + \varphi_c(\partial/\partial\varphi_c)]\} \\ \times Z(\varphi_c, \mu^2, \lambda) = \delta_Z(\varphi_c, \mu^2, \lambda). \quad (5.33)$$

In the last section we had found convenient to use the functions $\mathcal{U}(\alpha, x)$ and $z(\alpha, x)$ defined in (4.16) and (4.25). We shall further define:

$$g(\alpha, x) = (\partial^2/\partial x^2) \mathcal{U}(\alpha, x), \quad (5.34)$$

$$\delta_V(\varphi_c, \mu^2, \lambda) \equiv (\mu^4/\lambda) \delta_V(\alpha, x), \quad \delta_Z(\varphi_c, \mu^2, \lambda) \equiv \delta_Z(\alpha, x), \quad (5.35)$$

and write (5.32) and (5.33) as

$$[x(\partial/\partial x) - \beta_\theta(\alpha)(\partial/\partial\alpha) - \gamma_\theta(\alpha)] g(\alpha, x) = \delta_\theta(\alpha, x), \quad (5.36)$$

$$[x(\partial/\partial x) - \beta_z(\alpha)(\partial/\partial\alpha) - \gamma_z(\alpha)] z(\alpha, x) = \delta_z(\alpha, x), \quad (5.37)$$

where we have set

$$\tau(\alpha) = 2 - [\beta(\lambda)/\lambda] - \gamma(\lambda) \quad (5.38)$$

$$\beta_\theta(\alpha) = \frac{1}{(4\pi)^2} \frac{\beta(\lambda)}{\tau(\alpha)}, \quad \gamma_\theta(\alpha) = \frac{2\gamma(\lambda) + [\beta(\lambda)/\lambda]}{\tau(\alpha)},$$

$$\delta_\theta(\alpha, x) = -\frac{1}{\tau(\alpha)} \frac{\partial^2}{\partial x^2} \delta_V(\alpha, x) \quad (5.39a)$$

$$\beta_z(\alpha) = \beta_\theta(\alpha), \quad \gamma_z(\alpha) = \frac{\gamma(\lambda)}{\tau(\alpha)},$$

$$\delta_z(\alpha, x) = -\frac{1}{\tau(\alpha)} \delta_Z(\alpha, x). \quad (5.39b)$$

Equations (5.20) or (5.23) and (5.24) or (5.36)–(5.37) determine the asymptotic behavior of the theory for large φ_c or large x . Indeed, order by order in \hbar , we can show, using Weinberg's theorem, that the rhs of (5.24) or (5.36)–(5.37) is negligible. In an analogous way, we can show that this limit is obtained by letting $\mu^2 \rightarrow 0$ in (5.20). In both cases we are left with a linear homogeneous first order partial differential equation, the solution of which is nothing other than the asymptotic form of the functional equation we started with, namely (5.4) and (5.18).

An explicit expression of the functions β 's and γ 's pertaining to g and z is given in Appendix B where we find:

$$\beta_\theta(\alpha) = \beta_z(\alpha) = \frac{1}{2}\alpha[3\alpha - \frac{7}{6}\alpha^2 + \dots], \\ \gamma_\theta(\alpha) = \frac{1}{2}[3\alpha - \frac{3}{2}\alpha^2 + \dots], \\ \gamma_z(\alpha) = -\frac{1}{2}(\alpha^2/6) + \dots \quad (5.39c)$$

D. The dilatation flow

The recognition of the existence of the flow $\lambda \rightarrow \lambda_\tau$ with $\lambda_1 = \lambda$ is a far reaching one, intrinsic to any renormalizable

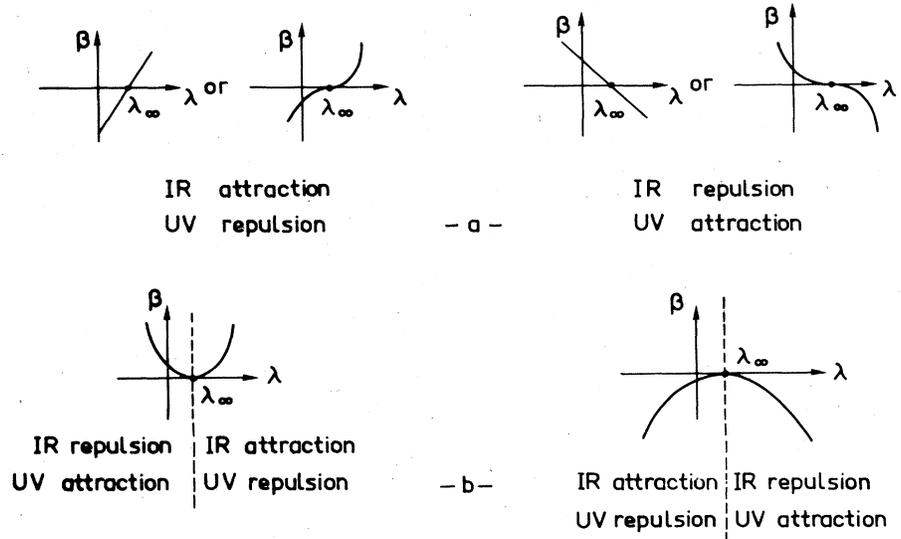


FIG. 2. Summary of the various stable or unstable situations in the vicinity of a zero of β in the ultraviolet (uv) or infrared (ir) region. We have distinguished the case of an odd (a) or even (b) zero. The even case can be thought of as the collapse of two nearby odd zeros.

field theory. Let us first study the underlying elementary group structure.⁸ From the definition (5.15a) we immediately deduce

$$\lambda(\tau_1\tau_2, \lambda) = \lambda(\tau_2, \lambda(\tau_1, \lambda)). \tag{5.40}$$

Hence, using as variable $t = \log \tau$, we obtain that for all t

$$\partial\lambda(t, \lambda)/\partial t = \beta(\lambda_t), \tag{5.41}$$

where $\beta(\lambda)$ is defined in (5.25). We see that $\beta(\lambda)$ is the generator of the flow and can be compared to the Hamiltonian of the Schrödinger equation.⁹ If $\beta(\lambda)$ is known, the solution can be written explicitly

$$t = \log\tau = \int_{\lambda}^{\lambda_t} \frac{d\mu}{\beta(\mu)}. \tag{5.42}$$

The critical points of the flow are given by the zeroes of β . If λ happens to be chosen at such a zero, then it is invariant in t . Furthermore β determines also the stability properties of the solution. Let us denote by λ_{∞} a zero of β and let us study the solution in the vicinity of λ_{∞} . Assume that

$$\beta(\lambda) \sim \omega(\lambda - \lambda_{\infty})^n + \dots \tag{5.43}$$

⁸ This approach was initiated by Kadanoff and Wilson, who investigated the consequences on the effective action of grouping the variables in domains of varying size. The parallel here is the mapping $\Gamma[\phi, \mu^2, \lambda] \rightarrow \Gamma[\rho^{1/2}(\tau, \lambda)\phi, \mu^2, \lambda_{\tau}] \equiv \Gamma_{\tau}$. The idea is that as $\tau \rightarrow \infty$, Γ_{τ} has a limit which is the fixed point of the transformation, and it is assumed to describe the asymptotic behavior of Γ . Notice finally that one could generalize this approach and study, instead of the scaling $\Lambda \rightarrow \tau\Lambda$, the general transformation $\Lambda \rightarrow f(\Lambda)$ where $f(\Lambda)$ has a fixed point at infinity. For a review of these ideas see Wilson and Kogut, "The renormalization group and the ϵ -expansion," to appear in Physics Reports in 1975. This general framework is also presented in a language very close to the one followed in this work in the lectures by J. Zinn-Justin, "Wilson's theory of critical phenomena and renormalized perturbation theory," Cargèse, 1973.

⁹ We can look at the Callan-Symanzik equation

$$[\partial/\partial t + \beta\lambda(\partial/\partial\lambda) + n\gamma(\lambda)] \Gamma_{\text{asympt.}}^{(2n)} = 0$$

as a Schrödinger equation with $H = H_0 + V$, $H_0 = \beta(\lambda)\lambda(\partial/\partial\lambda)$, and $V = n\gamma(\lambda)$. We can therefore write $\lambda_t = \exp ut \lambda_{t=0} \exp -ut$, etc.

We shall limit our discussion to a finite, integer n . In the vicinity of λ_{∞} , (5.43) gives

$$\omega t = \int_{\lambda}^{\lambda_t} \frac{d\mu}{(\mu - \lambda_{\infty})^n} = \begin{cases} \log \left| \frac{\lambda_t - \lambda_{\infty}}{\lambda - \lambda_{\infty}} \right| & \text{for } n = 1 \\ \frac{1}{n-1} \left[\left(\frac{1}{\lambda - \lambda_{\infty}} \right)^{n-1} - \left(\frac{1}{\lambda_t - \lambda_{\infty}} \right)^{n-1} \right] & \text{for } n \geq 1. \end{cases} \tag{5.44}$$

If $n = 1$ we shall be attracted by λ_{∞} if $\omega t \rightarrow -\infty$. Since one cannot cross the zero we find that $\lambda_t - \lambda_{\infty} \cong [\beta(\lambda)/\omega] \times e^{\omega t} \rightarrow 0$ if $\omega t \rightarrow -\infty$. Hence

if $\omega > 0$ we find an infrared ($t \rightarrow -\infty$) attractor,
if $\omega < 0$ we find an ultraviolet ($t \rightarrow +\infty$) attractor.

This is in fact the situation for any odd n since $-(\lambda_t - \lambda_{\infty})^{1-n}$ is always negative.

For even n , the situation changes. If $\lambda > \lambda_{\infty}$ then $\lambda_t > \lambda_{\infty}$, and again $\lambda_t \rightarrow \lambda_{\infty}$ as $\omega t \rightarrow -\infty$. If $\lambda < \lambda_{\infty}$, then $\lambda_t < \lambda_{\infty}$ and $\lambda_t \rightarrow \lambda_{\infty}$ as $\omega t \rightarrow \infty$. We illustrate all cases in Fig. 2.

The situation when β has several zeros is now clear: attractors and repulsors alternate on the λ axis. Notice that multiple zeros should be counted accordingly.

From the above discussion we conclude that a critical point is characterized by three parameters:

- (i) its location λ_{∞} ,
- (ii) its associated "frequency" ω which determines the rate of approach. (The approach is slower for smaller ω .)
- (iii) Its index n . For $n = 1$ we have exponential decay to stability $\exp(-|\omega t|)$, while for $n > 1$ we have only power approach $|\lambda_t - \lambda_{\infty}| \sim |\omega t|^{1-n}$. The larger the index n , the slower the approach.

We can carry a parallel discussion for the function $\gamma(\lambda)$. Using the definition (5.15b) we find:

$$\begin{aligned} (\partial/\partial t) \log g(t, \lambda) &= \gamma(\lambda(t, \lambda)), \\ \rho(t, \lambda) &= \exp \left[\int_0^t dt' \gamma(\lambda(t', \lambda)) \right] = \exp \left[\int_\lambda^{\lambda_t} d\mu \frac{\gamma(\mu)}{\beta(\mu)} \right], \end{aligned} \tag{5.45}$$

i.e., ρ is driven along the flow. If λ_∞ is a critical point, there is no *a priori* reason why it should be a special point of γ . For a simple uv attractor of the form

$$\beta(\mu) \simeq -|\omega|(\mu - \lambda_\infty),$$

we assume that a limited expansion of γ around the value λ_∞ is possible and we write $\gamma(\mu) = \gamma(\lambda_\infty) + 0(\mu - \lambda_\infty)$. We then find

$$\rho(\tau, \lambda) \sim \rho_{finite} \tau^{\gamma(\lambda_\infty)}. \tag{5.46}$$

Analogous formulae hold for higher order zeros. For example if $\beta \sim \omega(\lambda - \lambda_\infty)^2$ with $\omega > 0$, $\lambda < \lambda_\infty$, we write $\gamma(\mu) \sim \gamma(\lambda_\infty) + (\mu - \lambda_\infty)\gamma'(\lambda_\infty) + \dots$ and we get from (5.45)

$$\rho(\tau, \lambda) = \tau^{\gamma(\lambda_\infty)} (\log \tau)^{-\gamma'(\lambda_\infty)/\omega} (\dots) \tag{5.47}$$

which is a typical situation when nearby singularities collapse to a point.

We conclude this paragraph with a brief description of what can be learned about the functions $\beta(\lambda)$ and $\gamma(\lambda)$ of the φ^4 theory from perturbation calculations alone. Obviously no information can be obtained about the possible presence of critical points away from the origin, but necessarily $\beta(0) = \gamma(0) = 0$ for any renormalizable field theory, at least when computed in any finite order of perturbation. Moreover such a computation gives $\beta(\lambda)$ as an analytic function (in fact a polynomial). The best one can hope is that, although β and γ need not be analytic around $\lambda = 0$, perturbation theory correctly gives their successive derivatives near the origin so that the essential features of Fig. 2 do not change. Needless to say that any other method that would provide information about the analytic properties of β and γ around $\lambda \sim 0$ would be very interesting.¹⁰

With this assumption, since $\lambda = 0$ will always be a critical point, we can determine its nature from perturbation. From our previous discussion we see that it is sufficient to compute the lowest, non vanishing order of $\beta(\lambda)$. Referring back to our definition Eq. (5.25) we take the derivative with respect to τ at $\tau = 1$ of (5.15a) before the limit $\Lambda \rightarrow \infty$ is taken. We then find:

$$\begin{aligned} \Lambda(\partial/\partial \Lambda)\lambda|_{\mu, \Lambda} &= (\partial\lambda\tau/\partial\tau)|_{\tau=1; \Lambda, \mu} (\partial\lambda_0/\partial\lambda)|_{\Lambda, \mu} = -\mu(\partial/\partial\mu)\lambda_0|_{\Lambda, \Lambda}. \end{aligned} \tag{5.48}$$

¹⁰ S. Adler's idea of a "mode expansion" of the path integral could in fact provide a nonperturbative method of studying field theory and in particular its asymptotia. See for instance his article, Phys. Rev. D **8**, 2400 (1973).

Instead of looking at λ_0 as a function of μ and λ , let us invert the relation and look at λ as a function of μ and λ_0 , i.e., $\lambda = \lambda[(\mu/\Lambda), \lambda_0]$. We therefore obtain from (5.45), taking the limit $\Lambda \rightarrow \infty$

$$\beta(\lambda) = \lim_{\Lambda \rightarrow \infty} \mu(\partial/\partial\mu)\lambda(\mu/\Lambda, \lambda_0)|_{\lambda_0}. \tag{5.49}$$

A similar equation holds for $\gamma(\lambda)$ for which we find

$$\gamma(\lambda) = \lim_{\Lambda \rightarrow \infty} \mu(\partial/\partial\mu) \log Z_{\beta}^{-1}(\mu/\Lambda, \lambda_0)|_{\lambda_0}. \tag{5.50}$$

Equations (5.49) and (5.50) show that one way to calculate perturbatively β and γ (and in fact the easiest one) consists of the following

- (i) Express everything in terms of the cut-off Λ , the physical mass μ and the bare coupling constant λ_0 .
- (ii) Compute $\mu(\partial/\partial\mu)$ of λ and $\log Z_{\beta}^{-1}$ keeping λ_0 and Λ fixed.
- (iii) Re-express λ_0 in terms of physical quantities and take the limit $\Lambda \rightarrow \infty$.

For the reader who may feel uneasy with this interchange of limits, we simply notice that he can avoid reference to any cutoff altogether and use instead Eqs. (5.29) and (5.30) which express β and γ directly in terms of renormalized quantities. The result in perturbation theory is, of course, the same.

The actual calculation of β and γ is done by several authors.¹¹ A brief description, using our notation, is given in Appendix B. The result, up to order \hbar^2 , is

$$\beta(\lambda) = \lambda[3\alpha - \frac{1}{3}\alpha^2 + \dots], \tag{5.51}$$

$$\gamma(\lambda) = -\frac{1}{6}\alpha^2 + \dots, \tag{5.52}$$

where we recall that $\alpha = \lambda/(4\pi)^2$.

VI. POSITIVITY, BOUNDEDNESS AND SIGN OF THE COUPLING CONSTANT

There exists an old belief among theorists that the physical coupling constant in φ^4 theory must be positive otherwise all sorts of horrors may appear. There are several intuitive arguments which support this belief.

(i) Let us first be very naive and forget about divergences of perturbation theory and renormalization. In other words let us just stay at the level of tree diagrams. In this case $\lambda = \lambda_0$ and the path integral (2.14) makes sense only for $\lambda > 0$. Alternatively, one could argue that the potential $V(\varphi_c)$ in this approximation becomes unbounded below if $\lambda < 0$. The trouble, of course, with this argument is that λ_0 is ill defined, and the above "proof" does not seem to apply to λ if higher orders are taken into account. For example, looking at the expression for $V(\varphi_c)$ [Eq. (4.16)] there is no obvious way to guess the correct sign of λ without making arbitrary assumptions about the contribution of the higher orders.

(ii) Alternatively we can look at the first few terms in the expansion of V or Z given in Sec. IV. As we have noticed

¹¹ For a summary, see for instance Callan's lectures at Cargèse (1973), and references therein.

already, if $\lambda < 0$, singularities occur for real φ_c . In fact we saw that V becomes complex to lowest order if

$$1 + (\lambda\varphi_c^2/2\mu^2) = 0.$$

Since, for a massive theory, V must be real we conclude that for $\lambda < 0$ the theory makes no sense. Again this argument can be considered at best as heuristic, since it is based only on the first few terms in the \hbar expansion.

It is obvious that, no rigorous proof can be given as long as perturbation theory, in one form or another, remains our only line of approach. The purpose of this section is to clarify the assumptions involved and state how much can be said starting from general principles. We shall follow an argument due to Coleman¹²; it is based on two working assumptions and one conjecture which we shall explain later. The assumptions are:

(i) Although the renormalization group and the Callan-Symanzik equations have been obtained by perturbation theory considerations, they are assumed to have a much more general validity, and in fact to govern the asymptotic properties of the exact solution. Notice that this last statement contains a very questionable assumption, namely that terms which are asymptotically negligible order by order in perturbation theory by a solid power of \hbar^2 , do not sum up to give non-negligible terms.

(ii) The point $\lambda = 0$ is a critical point and its nature can be determined by perturbation theory.

These seem to us to be very mild assumptions and we think that perturbation theory would make no sense would they be violated.

We now write the Callan-Symanzik equation for V . Using (5.36), (5.39), (5.51), and (5.52) we find:

$$[x(\partial/\partial x) - (\frac{3}{2}\alpha^2 + \dots)(\partial/\partial\alpha) - \frac{3}{2}(\alpha - \frac{1}{2}\alpha^2 + \dots)] \times g_{as}(\alpha, x) = 0, \tag{6.1}$$

where g_{as} is the asymptotic form of $g(\alpha, x)$ for large x , and we have used Weinberg's theorem and assumption (i) in order to get rid of the right hand side. The general solution of (6.1) is given by the corresponding asymptotic form of Eq. (5.18) which, for $g(\alpha, x)$, reads:

$$g_{as}(\alpha, x) = g_{as}(\alpha_x, 1) \exp \left[- \int_1^x \gamma(\alpha_{x'}) dx' \right], \tag{6.2}$$

with α_x satisfying the equation

$$[x(\partial/\partial x) + \beta_g(\alpha)(\partial/\partial\alpha)]\alpha_x = 0, \quad \alpha_1 = \alpha. \tag{6.3}$$

According to the general discussion of Sec. V, and using assumption (ii), in the region of $\alpha < 0$ and sufficiently close to zero, the origin is uv attractive, i.e., the theory is asymptotically free. In this case the same assumption

¹² Although positivity of the interaction Lagrangian is very much at the root of many developments in the domain of asymptotic freedom, there does not seem to exist in the literature a full discussion of this point. We heard Coleman's argument during seminars he was giving during the summer of 1973.

allows us to use the low-order estimates of $\beta(\alpha)$ and $\gamma(\alpha)$, and (6.3) gives

$$\alpha_x \sim (1/\alpha - \frac{3}{2} \log x)^{-1}. \tag{6.4}$$

On the other hand, the lowest order estimate of $g_{as}(\alpha_x, 1)$ is a positive number ($\sim \frac{1}{3}$), and (6.2) tells us that

$$\lim_{x \rightarrow \infty} g_{as}(\alpha, x) = \frac{1}{3} (1 - \frac{3}{2} \alpha \log |x|)^{-1}. \tag{6.5}$$

This in turn means that the potential $V(\varphi_c)$ goes to $-\infty$ for large φ_c , which contradicts the intuitive requirement of the potential being bounded from below.

Let us recapitulate Coleman's argument. First one writes the asymptotic form of the Callan-Symanzik equation for the effective potential. One observes that, if $\lambda < 0$, the origin is uv attractive, which means that, for large values of the classical fields, the effective coupling constant tends to zero. In this case one assumes that reliable conclusions can be drawn from lowest order perturbation theory and then one finds that the potential is unbounded from below. On the contrary, if $\lambda > 0$, the origin is uv repulsive, i.e., the effective coupling constant is driven away towards higher values for large φ_c , and lowest order perturbation theory becomes unreliable.

As we see, the argument relies heavily on the fact that, the effective potential must be bounded from below being the energy density of a state with φ_c as the expectation value of the field. This is a special case of a general property formulated by Symanzik.¹³ Since this problem is more interesting than the particular application that is made here and seems to the authors of this paper not to be fully elucidated,¹⁴ we think it worthwhile to present briefly what it amounts to. We recall that among other applications the requirement that the interaction part of a bare Lagrangian (assumed to be non derivative) to define a positive functional on classical function space is very much at the root of the general discussion of asymptotic freedom of various field theories.

Let us be slightly more general and discuss some renormalizable field theory in Euclidean space, involving only Bose fields in finite number, distinguished by a discrete index that we omit for brevity. Dual to this field is a source J , and G_{dis} is a functional of J that is formally expressed as an infinite "Taylor" series in terms of the $2n$ Green functions, each of which in turn is at best a formal renormalized power series in the various coupling constants. Now the path integral representation suggests some global properties of the functional. However, we know that it is plagued by infinities. Nevertheless we could use some Pauli-Villars regularization curing almost all infinities except perhaps the vacuum ones. These are discretely concealed in the normalization by requiring that $G_{dis}(0) = 1$. If we think in terms of path integrals a careful study shows

¹³ See Symanzik (1970) where further references on this point can be found.

¹⁴ We thank Glaser for several discussions on this point. He seemed to be convinced that it was very unlikely that a proof of inequality (6.7) could be given on the basis of Wightman's axioms only. If true, these inequalities require most likely some information on the underlying Lagrangian structure.

that these last infinities have two sources: (i) infinite volume of space time, (ii) infinite number of degrees of freedom even in a finite volume for a continuous system. Nevertheless we shall assume that to a certain extent these are not fundamental difficulties and as long as the cutoff is large and the bare quantities expressed as functions of Λ and the "physical ones" some sensible statements can be made that remain valid in the limit $\Lambda \rightarrow \infty$. In other words we would like to extrapolate to the renormalized theory global statements that are true when various cutoffs are imposed. This is where more work would certainly have to be done.

Having thus presented our framework we may think of the renormalized functional $G_{\text{dis}}(J)$ in the Euclidean region as a functional average

$$G_{\text{dis}}(J) = \langle \exp(\int d^4x J(x)\varphi(x)) \rangle. \tag{6.6}$$

This summarizes the positivity of the measure on the function space φ and the fact that $G_{\text{disc}}(0) = 1$. This measure might not be given for the renormalized theory by the Feynman path integral formula. But the assumption that such a measure exists is a far reaching statement. A weaker form of (6.6) is obtained by expanding the exponential. It asserts that the renormalized Green functions define positive kernels. For any symmetric function of n arguments in a suitable test space one should have

$$\int d^4x_1 \dots d^4x_{2n} \psi(x_1, \dots, x_n) G_{\text{disc}}^{(2n)}(x_1, \dots, x_{2n}) \times \psi^*(x_{n+1}, \dots, x_{2n}) \geq 0 \quad \forall n \quad \forall \psi_n. \tag{6.7}$$

According to Symanzik¹³ a generalization of Bochner's theorem ensures that from (6.7) follows the existence of a measure in such a way that (6.6) be true. Clearly an inequality like (6.7) cannot be tested by using perturbation theory, and thus can at best be proved within a general framework of axioms for field theory. It is remarkable that the case $n = 2$ is always true as the reader will easily convince himself.¹⁴

We now show some consequences of (6.6). Recall the standard Hölder inequality. For two positive functions f and g and any measure $d\mu$ if the following integrals exist one has for every $0 \leq \alpha \leq 1$

$$\int d\mu f^\alpha g^{(1-\alpha)} \leq (\int d\mu f)^\alpha (\int d\mu g)^{1-\alpha}. \tag{6.8}$$

Extending this result to functionals and writing

$$G_{\text{disc}}(J) = \exp G(J), \tag{6.9}$$

$$G(\alpha J_1 + (1 - \alpha)J_2) < \alpha G(J_1) + (1 - \alpha)G(J_2). \tag{6.10}$$

This inequality means that $G(J)$ is a concave functional. In Fig. 3(a) we have described pictorially this inequality.

The Legendre transformation which allows one to go over to the one-particle irreducible generating functional $\Gamma(\varphi)$ can be also obtained by a minimization procedure. Namely one defines an intermediate quantity

$$\Gamma'(J, \varphi_c) = G(J) - \int d^4x J(x)\varphi_c(x) \tag{6.11}$$

and sets by virtue of (6.10)

$$\Gamma(\varphi_c) = \min_J \Gamma'(J, \varphi_c). \tag{6.12}$$

If we exclude for simplicity cases of broken symmetry, it is clear from the concavity (6.10) that the extremum is unique and is reached for a $J(x)$ satisfying

$$\varphi_c(x) = \delta G(J) / \delta J(x),$$

and $\Gamma(\varphi_c)$ coincides with the usual definition of the Legendre transform. Now the added term in (6.11) being linear, we also have

$$\Gamma'(\alpha J_1 + (1 - \alpha)J_2, \varphi_c) \leq \alpha \Gamma'(J_1, \varphi_c) + (1 - \alpha) \Gamma'(J_2, \varphi_c). \tag{6.13}$$

Consequently the minimum $\Gamma(\varphi_c)$ is convex

$$\begin{aligned} & \Gamma(\alpha \varphi_{c_1} + (1 - \alpha)\varphi_{c_2}) \\ &= \min_J \Gamma'(J, \alpha \varphi_{c_1} + (1 - \alpha)\varphi_{c_2}) \\ &= \min_J [\alpha(G(J) - \int d^4x J(x)\varphi_{c_1}(x)) \\ &+ (1 - \alpha)(G(J) - \int d^4x J(x)\varphi_{c_2}(x))] \\ &= \min_J (\alpha \Gamma'(J, \varphi_{c_1}) + (1 - \alpha) \Gamma'(J, \varphi_{c_2})) \\ &\geq \alpha \Gamma(\varphi_{c_1}) + (1 - \alpha) \Gamma(\varphi_{c_2}). \end{aligned} \tag{6.14}$$

As a by product of inequality (6.14) we have with the definition of the effective potential:

$$\begin{aligned} & \Gamma(\varphi_c) = -\int d^4x V(\varphi_c(x)) + \frac{1}{2} Z(\varphi_c(x)) (\partial \varphi_c(x))^2 + \dots \\ & V(\alpha \varphi_{c_1} + (1 - \alpha)\varphi_{c_2}) \leq \alpha V(\varphi_{c_1}) + (1 - \alpha) V(\varphi_{c_2}) \\ & \forall \varphi_{c_1}, \varphi_{c_2} \quad \text{and} \quad 0 \leq \alpha \leq 1. \end{aligned} \tag{6.15}$$

This means that $V(\varphi_c)$ is concave [Fig. 3(c)], and if it exists at all it is necessarily bounded from below. We learn furthermore that if it admits derivatives, its second derivative with respect to φ_c (in the case of a unique field) or the quadratic form of its second derivative $\partial^2 V / \partial \varphi_{c_a} \partial \varphi_{c_b}$ (in the case of several fields) have to be positive.

To summarize our brief description of Symanzik's positivity, we see that the crucial step is the inequality (6.7) from which follow the boundedness and the concavity of the renormalized effective potential. We refer to Symanzik (1970) for the discussion of broken symmetry. All these properties are clearly analogous to those of the partition function in statistical mechanics.

In Appendix C we shall describe interesting consequences of these assumptions using entirely different means which combine the above positivity in Euclidean space with the underlying Hilbert space norm of states in the Minkowskian region of the theory. Although we shall not be able to go as far as one would wish, this combination shows indeed how tight is field theory when one tries to combine all available information.

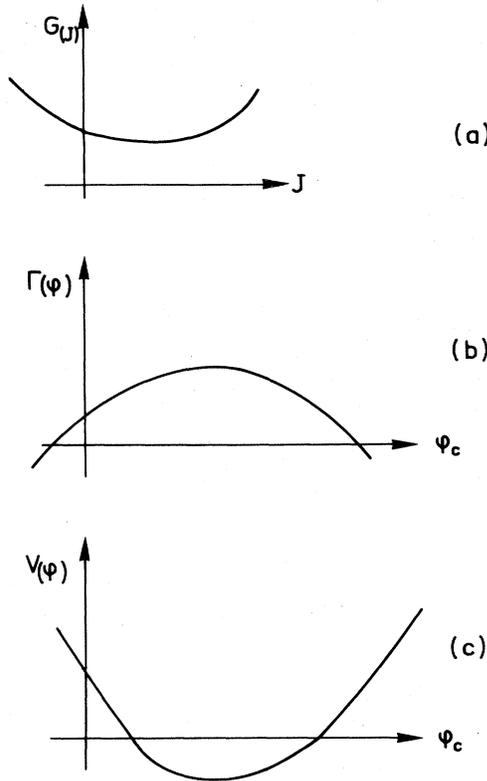


FIG. 3. The convexity properties of (a) the generating functional $G(J)$ of connected Green functions, (b) the generating functional $\Gamma(\varphi)$ of one particle irreducible Green functions, (c) of the effective potential $V(\varphi)$.

APPENDIX A. EVALUATION OF INTEGRALS

We first give here for completeness the explicit evaluation of the integrals occurring in (4.13) and (4.14) yielding the second-order contribution to the effective potential. We compute in turn ξ_1 and ξ_2 :

$$\xi_1(x) = u(x)^2$$

$$u(x) = \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{1}{k^2 + 1 + x} - \frac{1}{k^2 + 1} + \frac{x}{(k^2 + 1)^2} \right\}.$$

We use Schwinger's parametric form

$$\frac{1}{A} = \int_0^\infty d\alpha \exp(-\alpha A) \quad \text{for } A > 0.$$

Then

$$u(x) = \int_0^\infty d\alpha \int \frac{d^4k}{(2\pi)^4} [\exp\{-\alpha(k^2 + 1 + x)\} - \exp\{-\alpha(k^2 + 1)\} + \alpha x \exp\{-\alpha(k^2 + 1)\}],$$

$$= \frac{1}{(4\pi)^2} \int_0^\infty d\alpha \exp(-\alpha) \frac{[\exp(-\alpha x) - 1 + \alpha x]}{\alpha^2}.$$

We integrate this expression twice by parts

$$u(x) = [1/(4\pi)^2] \left[-x + (1+x) \int_0^\infty d\alpha \times \log \alpha \{ \exp(-\alpha) - (1+x) \exp[-(1+x)\alpha] \} \right],$$

$$u(x) = [1/(4\pi)^2] [(1+x) \log(1+x) - x].$$

Hence

$$\xi_1(x) = [1/(4\pi)^4] [(1+x) \log(1+x) - x]^2. \tag{A1}$$

The second function is $\xi_2(x)$ given by

$$\xi_2(x) = \frac{1}{(2\pi)^8} \int^{(3)} d^4k_1 d^4k_2 d^4k_3$$

$$\times \left\{ \frac{\delta(k_1 + k_2 + k_3)}{(k_1^2 + 1 + x)(k_2^2 + 1 + x)(k_3^2 + 1 + x)} - 3 \frac{\delta(k_1 + k_2)}{(k_1^2 + 1)^2(k_3^2 + 1 + x)} \right\}.$$

We use for the propagators the same parametrization as before except that to take into account the global counterterms at a later stage, we cut all α integrations at $1/\Lambda^2$. The integration over four-momenta is trivial and

$$(4\pi)^4 \xi_2(x) = \int_{1/\Lambda^2}^{\infty(3)} d\alpha_1 d\alpha_2 d\alpha_3$$

$$\times \left\{ [\exp(-\sum_1^3 \alpha_i(1+x)) / (\prod_{1 \leq i < j \leq 3} \alpha_i \alpha_j)^2] - 3[\exp(-\alpha_1(1+x) - (\alpha_2 + \alpha_3)) / \alpha_1^2(\alpha_2 + \alpha_3)^2] \right\}$$

+ global counterterm

$$= (1+x) \left[f_1\left(\frac{1+x}{\Lambda^2}\right) - 3f_2\left(\frac{1+x}{\Lambda^2}\right) \right]$$

+ global counterterm.

Since $\varphi^2 \xi_2$ enters the expression for V , the counterterm is a first degree polynomial in x designed to have $\xi_2 = 0(x^2)$ for small x . Now we evaluate $f_1(u)$ and $f_2(u)$ close to $u = 0$

$$f_1(u) = \int_u^{\infty(3)} d\alpha_1 d\alpha_2 d\alpha_3 \exp(-\sum_1^3 \alpha_i) / (\prod_{1 \leq i < j \leq 3} \alpha_i \alpha_j)^2.$$

Taking the derivative with respect to u yields

$$-f_1'(u) = 3 \exp(-u) \int_u^\infty \int_u^\infty \frac{d\alpha d\beta \exp[-(\alpha + \beta)]}{[\alpha\beta + u(\alpha + \beta)]^2}$$

$$= \frac{3 \exp(-u)}{u^2} \int_1^\infty \int_1^\infty d\alpha d\beta \frac{\exp[-u(\alpha + \beta)]}{(\alpha\beta + \alpha + \beta)^2}.$$

From the knowledge that internal divergences occur we add

and subtract in the integrand a term designed to compensate them

$$\begin{aligned}
 -f_1'(u) &= \frac{3 \exp(-u)}{u^2} \left\{ \int_1^\infty \int_1^\infty d\alpha d\beta \exp[-u(\alpha + \beta)] \right. \\
 &\times \left(\frac{1}{(\alpha\beta + \alpha + \beta)^2} - \frac{1}{(\alpha + 1)^2(\beta + 1)^2} \right) \\
 &\left. + \left[\int_1^\infty \frac{d\alpha}{(\alpha + 1)^2} \exp[-u\alpha] \right]^2 \right\}.
 \end{aligned}$$

In the double integral we may replace $\exp[-u(\alpha + \beta)]$ by $1 - u(\alpha + \beta)$ with an error $\sim u^2(\log u)^n$ which is negligible in $f_1(u)$ once subtraction will be performed and u set equal to zero. Now

$$\begin{aligned}
 \int_1^\infty d\alpha \int_1^\infty d\beta [1 - u(\alpha + \beta)] &\left(\frac{1}{(\alpha\beta + \alpha + \beta)^2} \right. \\
 &\left. - \frac{1}{(\alpha + 1)^2(\beta + 1)^2} \right) = (\log \frac{4}{3} - \frac{1}{4}) + u(\log \frac{4}{3} - \frac{1}{2}), \\
 \int_1^\infty d\alpha \frac{\exp(-u\alpha)}{(\alpha + 1)^2} &= \frac{1}{2} + u \log u - (\gamma + \frac{1}{2} - \log 2) + 0(u),
 \end{aligned}$$

where γ is the Euler constant (which will disappear in the final result)

$$\gamma = \Gamma'(1) = \int_0^\infty d\alpha \log \alpha \exp(-\alpha).$$

Putting all this together

$$\begin{aligned}
 -f'(u) &= \frac{3e^{-u}}{u^2} \left\{ (\log \frac{4}{3} - \frac{1}{4}) + u(\log \frac{4}{3} - \frac{1}{2}) \right. \\
 &+ \left[\frac{1}{2} + u \log u - (\gamma + \frac{1}{2} - \log 2) u \right]^2 + \dots \left. \right\}, \\
 &= 3\{u^{-2} \log \frac{4}{3} + u^{-1} \log u + u^{-1}(\log 2 - 1 - \gamma) + \dots\}, \\
 f_1(u) &= -3\{u^{-1} \log \frac{3}{4} + \frac{1}{2}(\log u)^2 \\
 &+ \log u(\log 2 - \gamma - 1) + \dots\}.
 \end{aligned}$$

The calculation of f_2 is simpler

$$f_2(u) = \left(\int_u^\infty d\alpha \frac{\exp(-\alpha)}{\alpha^2} \right) \int_{1/\Lambda^2}^\infty d\alpha \int_{1/\Lambda^2}^\infty d\beta \frac{\exp[-(\alpha + \beta)]}{(\alpha + \beta)^2}.$$

The first integral in this product is

$$\int_u^\infty d\alpha \frac{\exp(-\alpha)}{\alpha^2} = 1/u + \log u - \gamma - 1 + \dots.$$

The second is

$$\begin{aligned}
 \int_{1/\Lambda^2}^\infty d\alpha \int_{1/\Lambda^2}^\infty d\beta \frac{\exp[-(\alpha + \beta)]}{(\alpha + \beta)^2} &= \int_{2/\Lambda^2}^\infty d\alpha (\alpha - 2/\Lambda^2) [\exp(-\alpha)/\alpha^2] \\
 &= \log \Lambda^2 - \log 2 - 1 + \gamma + \dots.
 \end{aligned}$$

Hence

$$\begin{aligned}
 f_2(u, \Lambda^2) &= (\log \Lambda^2 - \log 2 - 1 + \gamma + \dots) \\
 &\times (1/u + \log u - \gamma - 1 + \dots).
 \end{aligned}$$

Returning to ξ_2

$$\begin{aligned}
 -\frac{1}{3}(4\pi)^4 \xi_2(x) &= (1+x) \left[\frac{1}{2}(\log \Lambda^2 - \log(1+x))^2 \right. \\
 &- (\log \Lambda^2 - \log(1+x))(\log 2 - \gamma - 1) \\
 &+ \left(\log \frac{\Lambda^2}{2} - 1 + \gamma \right) [\Lambda^2 - (1+x)(\log \Lambda^2 \\
 &- \log(1+x)) - \gamma - 1] + \text{first degree polynomial} \\
 &= \frac{1}{2}(1+x) \log^2(1+x) - (1+x) \log(1+x) \\
 &\times \log \Lambda^2 + (\log 2 - \gamma - 1)(1+x) \log(1+x) \\
 &+ (1+x) \log(1+x)(\log \Lambda^2 - \log 2 - 1 + \gamma) \\
 &+ \text{first degree polynomial}.
 \end{aligned}$$

As expected the virtue of the internal subtractions has been to cancel the nonpolynomial divergences, $(1+x) \log(1+x) \log \Lambda^2$. Taking now into account the constraints $\xi_2(0) = \xi_2'(0) = 0$, the remaining finite part is thus

$$\begin{aligned}
 \xi_2(x) &= -(3/(4\pi)^4) \left(\frac{1}{2}(1+x) \log^2(1+x) \right. \\
 &- 2(1+x) \log(1+x) + 2x \left. \right). \tag{A2}
 \end{aligned}$$

The results (A1) and (A2) reappear in the text as formula (4.15a) and (4.15b).

We turn now to the evaluation of the two second order contributions to Z . We have called them $Z_2^{(a)}$ and $Z_2^{(b)}$. For shortness let A stand for the rhs of (4.23)

$$\begin{aligned}
 A &= \frac{1}{8}\lambda(4\pi)^{-4} \int d^4x \left[\int_0^\infty ds(x) \exp(-s[P^2 + \mu^2 \right. \\
 &+ \frac{1}{2}\lambda\varphi_c^2(X)]) | x) - \text{counterterm} \left. \right]^2. \tag{A3}
 \end{aligned}$$

Proceeding as in Sec. IV we expand $\varphi_c^2(X)$ in the neighborhood of $\varphi_c(x)^2$ up to second order and find

$$\begin{aligned}
 A &= \frac{1}{8}\lambda(4\pi)^{-4} \int d^4x \left[\int_0^\infty \frac{ds}{s^2} \exp(-s(\mu^2 + \frac{1}{2}\lambda\varphi_c^2(x)) \right. \\
 &\times [1 - (\lambda s^2/12) \square \varphi_c(x)^2 + (\lambda^2 s^3/12)\varphi_c^2(x) \\
 &\times (\partial\varphi_c(x))^2 + \dots] - \exp(-s\mu^2)(1 - \frac{1}{2}\lambda s\varphi_c^2(x)) \left. \right]^2.
 \end{aligned}$$

Again a partial integration of $\square\varphi_c(x)^2 = \partial(2\varphi_c(x)\partial\varphi_c(x))$ is necessary. Identifying the factor of $\frac{1}{2}\partial\varphi_c(x)^2$ in the x integration we find

$$Z_2^{(\omega)}(x) = \frac{\alpha^2}{12} x \left(\frac{\log(1+x)}{1+x} + \frac{x}{(1+x)^2} \right). \quad (A4)$$

Similarly we call B the rhs of (4.24). Taking as variables $(x+y)/2$ and $x-y$, we express it as

$$\begin{aligned} B = & -\frac{\lambda^2}{12} \left\{ \int d^4x \int d^4u \varphi_c(x + \frac{1}{2}u) \varphi_c(x - \frac{1}{2}u) \right. \\ & \times \left[(x + \frac{1}{2}u \mid \int_0^\infty ds \exp(-s(P^2 + \mu^2 \right. \\ & \left. + \frac{1}{2}\lambda\varphi_c(X)^2) \mid x - \frac{1}{2}u) \right]^3 - 3 \int d^4x \varphi_c^2(x) (x \mid \\ & \times \int_0^\infty ds_1 \exp(-s_1(P^2 + \mu^2 + \frac{1}{2}\lambda\varphi_c^2(X) \mid x) \\ & \left. \times \int_0^\infty ds_2 \int_0^\infty ds_3 \int \exp(-(s_2 + s_3)(k^2 + \mu^2)) \frac{d^4k}{(2\pi)^4} \right\} \\ & + \text{overall counterterm.} \end{aligned} \quad (A5)$$

Again we expand $\varphi_c^2(X)$ around $\varphi_c^2(x)$, i.e., around the middle point with the result

$$\begin{aligned} B = & -\frac{\lambda^2}{12} \left[\int d^4x \int d^4u \varphi_c(x + \frac{1}{2}u) \varphi_c(x - \frac{1}{2}u) \right. \\ & \times \int_0^\infty \prod_1^3 ds_i \int \prod_1^3 \frac{d^3k_T^i}{(2\pi)^3} [(4\pi)^{3/2} (\prod_1^3 s_i)^{1/2}]^{-1} \\ & \times \exp\{-\sum_1^3 s_i [k_T^2 + \mu^2 + \frac{1}{2}\lambda\varphi_c^2(x)]\} \exp(iu_T \cdot \sum_1^3 k_i^T) \\ & \times \prod_1^3 \frac{2\omega_x s_i}{sh2\omega_x s_i} \exp\left(-\frac{\omega_x}{sh2\omega_x s_i} [(a_x^2/b_x^2)(ch2\omega_x s_i - 1) \right. \\ & \left. + \frac{1}{4}u_0^2(ch2\omega_x s_i + 1)] + \frac{\lambda a_x^2}{4b_x} s_i\right) - 3 \int d^4x \varphi_c^2(x) \\ & \times \int_0^\infty ds_1 \int \frac{d^3k_T}{(2\pi)^3} \exp\{-s_1[k_T^2 + \mu^2 + \frac{1}{2}\lambda\varphi_c^2(x)]\} \\ & \times \frac{1}{(4\pi)^2} \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{\exp[-(s_1 + s_2)\mu^2]}{(s_1 + s_2)^2} \\ & \times (4\pi s_1)^{-1/2} \left(\frac{2\omega_x s_1}{sh2\omega_x s_1} \right)^{1/2} \times \exp[-\omega_x th\omega_x s_1 (a_x^2/b_x^2) \\ & \left. + (\lambda a_x^2/4b_x) s_1 \right] + \text{overall counterterm} \end{aligned}$$

with a_x, b_x, ω_x having the same meaning as in Sec. IV. Expanding in powers of these quantities up to the required order as well as $\varphi_c(x + \frac{1}{2}u)\varphi_c(x - \frac{1}{2}u)$ and keeping only the coefficient of $\partial\varphi_c(x)^2$, we find

$$\begin{aligned} Z_2^{(b)}(\varphi_c) = & -\frac{\lambda^2}{6(4\pi)^4} \int_0^{\infty(3)} ds_1 ds_2 ds_3 \\ & \times \left\{ \exp(-\sum_1^3 s_i(\mu^2 + \frac{1}{2}\lambda\varphi_c^2)) / (s_1 s_2 + s_2 s_3 + s_3 s_1)^2 \right. \\ & \times \left[\frac{s_1 s_2 s_3}{(s_1 s_2 + s_2 s_3 + s_3 s_1)} \left(\sum_1^3 s_i \lambda \varphi_c^2 - 1 \right) \right. \\ & \left. + (1/12)\lambda^2 \varphi_c^4 \sum_1^3 s_i^3 + (\lambda/3)\varphi_c^2 \sum_1^3 s_i^2 - (\lambda^2/6) \right. \\ & \left. \times \varphi_c^4 \left(\sum_1^3 s_i \right) \left(\sum_1^3 s_i^2 \right) \right] - \frac{3 \exp[-s_1(\mu^2 + \frac{1}{2}\lambda\varphi_c^2)]}{s_1^2 (s_1 + s_2)^2} \\ & \left. \times [(\lambda^2/12)\varphi_c^4 s_1^3 + (\lambda/3)\varphi_c^2 s_1^2 - (\lambda^2/6)\varphi_c^4 s_1^3] \right\} \\ & + \text{overall } \varphi_c \text{ independent logarithmic counterterm.} \end{aligned}$$

We introduce our reduced variables x and α and organize this into the contribution of six terms

$$\begin{aligned} Z_2^{(b)} = & -\frac{\alpha^2}{6} \left[-u_0 + 2x(u_1 + u_2) - x^2 u_3 \right. \\ & \left. + \frac{2x^2}{1+x^2} (u_4 - u_5) \right] + \text{counterterm,} \end{aligned} \quad (A6)$$

with the following identification

$$\begin{aligned} u_0 = & \int_{\mu^2/\Lambda^2}^{\infty(3)} \frac{ds_1 ds_2 ds_3}{(s_1 s_2 + s_2 s_3 + s_3 s_1)^3} s_1 s_2 s_3 \\ & \times \exp[-(1+x)(s_1 + s_2 + s_3)] \\ = & -\frac{1}{2} \log \frac{\mu^2(1+x)}{\Lambda^2} + 0 \left(\frac{\mu^2}{\Lambda^2} \right) \end{aligned} \quad (A7)$$

$$u_1 = -\frac{\partial}{\partial x} u_0 = -\frac{1}{2} \frac{1}{(1+x)} \quad (A8)$$

$$\begin{aligned} u_n = & \int_0^{\infty(3)} ds_1 ds_2 ds_3 s_1^n \left[\frac{\exp[-(1+x)(s_1 + s_2 + s_3)]}{(s_1 s_2 + s_2 s_3 + s_3 s_1)^2} \right. \\ & \left. - \frac{\exp[-(1+x)s_1 - (s_2 + s_3)]}{s_1^2 (s_2 + s_3)^2} \right], \end{aligned}$$

$$n = 2, 3.$$

We set $s_2 + s_3 = \sigma, (s_2 - s_3)/2 = v$, integrate over v , and parametrize the s_1, σ space as $s_1 + (\sigma/2) = ach\theta, s_1 = ash\theta$. The result is after an other integration over a

$$\begin{aligned} u_n = & \int_0^\infty d\theta sh\theta^n \left(\frac{(1/sh\theta) + \log[(1+e^{-\theta})/(1-e^{-\theta})]}{(1+x)^{n-1} (sh\theta + 2e^{-\theta})^{n-1}} \right. \\ & \left. - \frac{1}{sh^2\theta e^{-\theta}} \frac{1}{[(1+x)sh\theta + 2e^{-\theta}]^{n-1}} \right) \\ n = & 2, 3. \end{aligned}$$

We now change variable to $z = \log(1 + e^{-\theta})/(1 - e^{-\theta})$. The result for u_2 is

$$u_2 = \frac{1}{1+x} \left[-\log(1+x) + 1 - \frac{4}{3} \log 2 - \int_0^\infty dz \frac{z}{(1+sh^2 \frac{1}{2}z)(1+4sh^2 \frac{1}{2}z)} \right].$$

The remaining integral is with $z = -\log x$

$$\begin{aligned} & \int_0^\infty dz \frac{z}{(1+sh^2 \frac{1}{2}z)(1+4sh^2 \frac{1}{2}z)} \\ &= \frac{4}{3} \int_0^1 dx \frac{\log x}{(1+x)^2} - \frac{4}{3} \int_0^1 dx \frac{\log x}{x^2 - x + 1} \\ &= -\frac{4}{3} \log 2 + \frac{4}{3} A. \end{aligned}$$

The integral giving the number A is a transcendental one

$$\begin{aligned} A &= - \int_0^1 dx \frac{\log x}{x^2 - x + 1} \\ &= \frac{\text{dilog exp}(i\pi/3) - \text{dilog exp}(-i\pi/3)}{\exp(i\pi/3) - \exp(-i\pi/3)} \\ &= \sum_1^\infty \frac{1}{n^2} \frac{\sin(\pi/3)}{\sin(\pi/3)} \\ &= \frac{3}{2} \sum_0^\infty \left(\frac{1}{(1+3p)^2} - \frac{1}{(2+3p)^2} \right) = 1.1719536. \quad (\text{A9}) \end{aligned}$$

Finally

$$u_2 = \frac{1}{1+x} \left[-\log(1+x) + 1 + \frac{4}{3} A \right]. \quad (\text{A10})$$

Similarly

$$u_3 = \frac{1}{(1+x)^2} \left[-\log(1+x) + \frac{4}{3} - \frac{8}{9} \log 2 - 2 \int_0^\infty dz \times \frac{z(1+2sh^2 \frac{1}{2}z)}{(1+sh^2 \frac{1}{2}z)(1+4sh^2 \frac{1}{2}z)^2} \right].$$

Now

$$\begin{aligned} & \int_0^\infty dz \frac{z(1+2sh^2 \frac{1}{2}z)}{(1+sh^2 \frac{1}{2}z)(1+4sh^2 \frac{1}{2}z)^2} \\ &= -\frac{4}{9} \int_0^1 dx \frac{\log x}{1-x+x^2} + \frac{4}{9} \int_0^1 dx \frac{\log x}{(1+x)^2} \\ &\quad - \frac{2}{3} \int_0^1 dx \frac{x \log x}{(1-x+x^2)^2}. \end{aligned}$$

We recognize in the first integral the number A , the second is proportional to $\log 2$ and for the third it is again proportional to A . Finally

$$\int_0^\infty dz \frac{z(1+2sh^2 \frac{1}{2}z)}{(1+sh^2 \frac{1}{2}z)(1+4sh^2 \frac{1}{2}z)^2} = \frac{2}{3} A - \frac{4}{9} \log 2.$$

The complete value of u_3 is therefore

$$u_3 = -[1/(1+x)^2][\log(1+x) + \frac{4}{3}(A-1)]. \quad (\text{A11})$$

The remaining integrals are

$$u_4 = \int_0^\infty d^3s_1 d^3s_2 d^3s_3 \frac{s_1 s_2 s_3 \exp[-(s_1 + s_2 + s_3)]}{(s_1 s_2 + s_2 s_3 + s_3 s_1)^2}$$

and

$$u_5 = \int_0^\infty d^3s_1 d^3s_2 d^3s_3 \frac{s_1 \exp[-(s_1 + s_2 + s_3)]}{(s_1 s_2 + s_2 s_3 + s_3 s_1)}.$$

We note that

$$\begin{aligned} (u_5 - u_4)/(1+x)^2 &= -(du_2/dx + u_3) = \frac{2}{3} [(1+x)^2]^{-1} \\ u_5 - u_4 &= \frac{2}{3}. \quad (\text{A12}) \end{aligned}$$

Inserting formulae (A7) (A8), (A10), (A11), and (A12) in (A6) we obtain the desired contribution after the subtraction of a constant

$$\begin{aligned} Z_2^{(b)} &= \frac{1}{6} \alpha^2 \left\{ -\frac{1}{2} \log(1+x) + x/(1+x) \right. \\ &\quad + [2x/(1+x)][\log(1+x) + \frac{4}{3} A - 1] \\ &\quad - [x^2/(1+x)^2][\log(1+x) + \frac{4}{3}(A-1)] \\ &\quad \left. + \frac{4}{3} [x^2/(1+x)^2] + \text{finite constant} \right\} \quad (\text{A13}) \end{aligned}$$

as quoted in Table II.

APPENDIX B. CALCULATION OF β AND γ TO ORDER \hbar^2

We shall follow the conventional procedure to compute the functions β and γ according to the discussion of Sec. D. We will then apply these results to study the asymptotic behavior of V and Z as discussed in Sec. V.C in order to check our calculations.

All that is required is thus the cutoff dependence of the coupling constant and wavefunction renormalization.

We recall that the steps are the following:

- (i) Introduce a cutoff Λ in a coherent fashion in order to give a meaning to the Feynman perturbative amplitudes expressed in terms of bare quantities.
- (ii) Use as parameters, the cutoff Λ , the physical mass μ , and the bare coupling constant λ_0 .
- (iii) Compute $\mu(\partial/\partial\mu)$ of λ and $\log Z_3^{-1}$, keeping λ_0 fixed.
- (iv) Re-express λ_0 in terms of physical quantities, and then let the cutoff Λ go to infinity

$$\beta = \mu(\partial/\partial\mu)\lambda = -\Lambda(\partial/\partial\Lambda)\lambda$$

$$\gamma = \mu(\partial/\partial\mu) \log Z_3^{-1} = -\Lambda(\partial/\partial\Lambda) \log Z_3^{-1} \quad (\lambda_0 \text{ fixed}). \quad (\text{B1})$$

It is required for consistency that after the above derivatives are taken and everything is expressed in terms of physical quantities all the $\log \Lambda$'s dependence cancels.

TABLE III. Calculation of the infinite parts of $\Gamma_{(2)}(p^2)$ and $\Gamma_{(4)}(0)$ up to \hbar^2 .

	\hbar^0	\hbar^1	\hbar^2
$\Gamma_{(2)}(p^2)$			+
	$-(\mu_0^2 + p^2)$	$-\frac{\lambda_0}{2} \int \frac{d^4k}{(2\pi)^4} \Delta_\Lambda(k^2, \mu_0^2)$	$\frac{\lambda_0^2}{3!} \int \frac{d^4k_1 d^4k_2 d^4k_3}{(2\pi)^8} \delta^{(4)}(k_1 + k_2 + k_3 - p) \Delta_\Lambda(k_1^2, \mu_0^2) \Delta_\Lambda(k_2^2, \mu_0^2) \Delta_\Lambda(k_3^2, \mu_0^2) + \text{irrelevant and } p \text{ independent}$
		$-\frac{1}{2} \frac{\lambda_0}{(4\pi)^2} \left[\Lambda^2 - \mu_0^2 \log \frac{\Lambda^2}{\mu_0^2} - \mu_0^2(1 + \gamma) \right]$	$-\frac{\lambda_0^2}{6(4\pi)^2} \left[3\Lambda^2 \log \frac{\Lambda^2}{\mu_0^2} - 3\mu_0^2 \left(\frac{1}{2} \left(\log \frac{\Lambda^2}{\mu_0^2} \right)^2 + (\gamma + 1 - \log 2) \log \frac{\Lambda^2}{\mu_0^2} - \frac{p^2}{2} \log \frac{\Lambda^2}{\mu_0^2} + \psi_{\Gamma(1/2)}(p^2) \mu_0^2 \right) \right] + \text{irrelevant}$
$\Gamma_{(4)}(0)$			+ +
	$-\lambda_0$	$\frac{3\lambda_0^2}{2} \int \frac{d^4k}{(2\pi)^4} [\Delta_\Lambda(k^2, \mu_0^2)]^2$	$-3\lambda_0^3 \int \frac{d^4k_1 d^4k_2 d^4k_3}{(2\pi)^8} \delta^{(4)}(k_1 + k_2 + k_3) [\Delta_\Lambda(k_1^2, \mu_0^2)]^2 \Delta_\Lambda(k_2^2, \mu_0^2) \Delta_\Lambda(k_3^2, \mu_0^2) - \frac{1}{2} \lambda_0^3 \left[\int \frac{d^4k}{(2\pi)^4} [\Delta_\Lambda(k^2, \mu_0^2)]^2 \right]^2 - \frac{1}{2} \lambda_0^3 \int \frac{d^4k_1}{(2\pi)^4} \Delta_\Lambda(k_1^2, \mu_0^2) \int \frac{d^4k_2}{(2\pi)^4} [\Delta_\Lambda(k_2^2, \mu_0^2)]^2$
		$\frac{3}{2} \frac{\lambda_0^2}{(4\pi)^2} \left[\log \frac{\Lambda^2}{\mu_0^2} + \gamma - 1 - \log 2 \right]$	$-\frac{3\lambda_0^3}{(4\pi)^4} \left[\frac{1}{2} \left(\log \frac{\Lambda^2}{\mu_0^2} \right)^2 + \log \frac{\Lambda^2}{\mu_0^2} (\gamma - \log 2) + cte \right] - \frac{3}{4} \frac{\lambda_0^3}{(4\pi)^4} \left[\left(\log \frac{\Lambda^2}{\mu_0^2} \right)^2 + 2 \log \frac{\Lambda^2}{\mu_0^2} (\gamma - 1 - \log 2) + (\gamma - 1 - \log 2)^2 \right] - \frac{3}{4} \frac{\lambda_0^3}{(4\pi)^4} \left[\frac{\Lambda^2}{\mu_0^2} - \log \frac{\Lambda^2}{\mu_0^2} - \gamma - 1 \right]$

We turn now to the explicit calculation of these functions up to order \hbar^2 . We introduce a cutoff Λ according to the generalized Pauli-Villars prescription. To be precise we make the choice

$$g_{\Lambda^2}(\alpha) = \theta(\alpha - 1/\Lambda^2).$$

The regularized propagator $\Delta_\Lambda(k^2, \mu_0^2)$ is thus

$$\Delta_\Lambda(k^2, \mu_0^2) = \frac{\exp[-(k^2 + \mu_0^2)/\Lambda^2]}{(k^2 + \mu_0^2)}. \tag{B2}$$

We now compute simply $\Gamma_{(2)}(p^2)$ and $\Gamma_{(4)}(0)$ in terms of $\lambda_0, \mu_0^2, \Lambda^2$ up to two loops. Only those contributions which are unbounded when $\Lambda \rightarrow \infty$ are of interest.

The diagrams, the corresponding integrals and the asymptotic expansion in Λ^2 are displayed in Table III.

It turns out, as it is in fact clear from the onset, that the seagull contributions could altogether be omitted (this corresponds in the operator formalism to work with a Wick-ordered Lagrangian). We have kept them here in order to check their cancellation and hence our algebra. Also we did not bother to compute to order \hbar^2 a p -independent term corresponding to the last diagram in Table III for $\Gamma_{(2)}(p^2)$ since for our calculation only the p -dependent part of $\Gamma_{(2)}$ enters at the order of 2 loops.

We are now ready to obtain β and γ to order \hbar^2 . With the help of Table III, we find:

$$Z_3^{-1} = 1 + \frac{\hbar^2}{12} \left(\left[\frac{\lambda_0}{(4\pi)^2} \right]^2 \log \frac{\Lambda^2}{\mu_0^2} + cte \right) + O(\hbar^3)$$

$cte \equiv$ independent of Λ ,

$$\mu_0^2 = \mu^2 - \frac{1}{2} \frac{\lambda_0}{(4\pi)^2} \hbar \left(\Lambda^2 - \mu^2 \log \frac{\Lambda^2}{\mu^2} - \mu^2(1 + \gamma) + O(\hbar^3) \right)$$

$$\lambda = Z_3^2 \left[\lambda_0 - \frac{3}{2} \frac{\lambda_0^2}{(4\pi)^2} \hbar \left(\log \frac{\Lambda^2}{\mu_0^2} + (\gamma - 1) - \log 2 \right), \right.$$

$$\left. + \frac{3}{4} \frac{\lambda_0^3}{(4\pi)^4} \hbar^2 \left(\left(\log \frac{\Lambda^2}{\mu_0^2} \right)^2 + 2 \log \frac{\Lambda^2}{\mu_0^2} (\gamma - 1 - \log 2) \right. \right.$$

$$\left. + (\gamma - 1 - \log 2)^2 \right) + \frac{3}{4} \frac{\lambda_0^3}{(4\pi)^4} \hbar^2 \left(\frac{\Lambda^2}{\mu_0^2} - \log \frac{\Lambda^2}{\mu_0^2} \right.$$

$$\left. - (1 + \gamma) \right) + \frac{3\lambda_0^3}{(4\pi)^4} \hbar^2 \left(\frac{1}{2} \left(\log \frac{\Lambda^2}{\mu_0^2} \right)^2 \right.$$

$$\left. + \log \frac{\Lambda^2}{\mu_0^2} (\gamma - \log 2) + cte \right) + O(\hbar^3).$$

The notation γ is for the Euler constant $\gamma = \Gamma'(1)$.

We now eliminate μ_0^2 in favor of μ^2 and observe indeed the cancellation of seagulls. We set:

$$\alpha = \lambda/(4\pi)^2 \quad \alpha_0 = \lambda_0/(4\pi)^2$$

and find

$$Z_3^{-1} = 1 + \hbar^2[(\alpha_0^2/12) \log(\Lambda^2/\mu^2) + cte] + 0(\hbar^3) \quad (\text{B4})$$

$$\begin{aligned} \alpha = \alpha_0 Z_3^{-2} & \left\{ 1 - \frac{3}{2} \hbar \alpha_0 \left(\log \frac{\Lambda^2}{\mu^2} + \gamma - 1 - \log 2 \right) \right. \\ & + \hbar^2 \alpha_0^2 \left(\frac{9}{4} \left(\log \frac{\Lambda^2}{\mu^2} \right)^2 + \log \frac{\Lambda^2}{\mu^2} \left[\frac{3}{2} (\gamma - 1 - \log 2) \right. \right. \\ & \left. \left. + 3(\gamma - \log 2) \right] + cte \right\} + 0(\hbar^3). \end{aligned} \quad (\text{B4})$$

Terms proportional to Λ^2 have disappeared, only $\log \Lambda$ terms survive.

We now apply the definitions of β and γ

$$\begin{aligned} \beta &= -\Lambda(\partial/\partial\Lambda)\lambda|_{\lambda_0, \mu}, \text{ hence} \\ & - [1/(4\pi)^2]^{1/2} \beta \\ &= \frac{\partial}{\partial \log \Lambda^2} \alpha = -2 \left(\frac{\partial}{\partial \log \Lambda^2} \log Z_3^{-1} \right) \alpha \\ &+ Z_3^2 \alpha_0 \left\{ -\frac{3}{2} \hbar \alpha_0 + \hbar^2 \alpha_0^2 \left[\frac{3}{2} \log(\Lambda^2/\mu^2) \right. \right. \\ & \left. \left. + \frac{3}{2} (\gamma - 1 - \log 2) + 3(\gamma - \log 2) \right] \right\}. \end{aligned}$$

To order \hbar^2 the Z_3^2 in front of the last bracket can be set equal to 1.

Finally we re-express everything in terms of α , that is:

$$\begin{aligned} & - [1/(4\pi)^2]^{1/2} \beta \\ &= -\frac{\alpha^3}{6} \hbar^2 + \left\{ -\frac{3}{2} \hbar \left(\alpha + \frac{3}{2} \hbar \alpha^2 \left[\log \frac{\Lambda^2}{\mu^2} \right. \right. \right. \\ & \left. \left. + \gamma - 1 - \log 2 \right] \right)^2 + \frac{9}{2} \hbar^2 \alpha^3 \log \frac{\Lambda^2}{\mu^2} \\ & \left. + \frac{3}{2} \hbar^2 \alpha^3 (3\gamma - 1 - 3 \log 2) \right\} + \dots \end{aligned}$$

The expected "miracle" occurs, namely all divergent terms disappear and we are left with

$$\beta(\lambda) = \lambda [3\hbar\alpha - (17/3)\hbar^2\alpha^2 + \dots] \quad \alpha = \lambda/(4\pi)^2. \quad (\text{B5})$$

Note that all γ 's, \log^2 's, etc... which were due to the particular kind of cutoff chosen have also disappeared.

The calculation of γ is even simpler. Recall that

$$\begin{aligned} \gamma &= -\Lambda(\partial/\partial\Lambda) \log Z_3^{-1} |_{\mu, \lambda_0} \\ \gamma &= -\Lambda(\partial/\partial\Lambda) \hbar^2 (\alpha_0^2/12) [\log(\Lambda^2/\mu^2) + cte] \\ &= -\hbar^2 (\alpha_0^2/6). \end{aligned}$$

Hence to order \hbar^2

$$\gamma = -\hbar^2 (\alpha^2/6). \quad (\text{B6})$$

Formulae (B5) and (B6) reappear in the text with $\hbar = 1$ as (5.51) and (5.52). We can now apply this calculation to the asymptotic behavior of V and Z as in Sec. V.C. According to (5.38) and (5.39)

$$\begin{aligned} \beta_g(\alpha) &= \frac{1}{2} \frac{\beta/(4\pi)^2}{1 - \frac{1}{2}(\beta/\lambda + \gamma)} = \frac{1}{2} \alpha \frac{[3\hbar\alpha - (17/3)\hbar^2\alpha^2 + \dots]}{1 - \frac{1}{2}[3\hbar\alpha] + \dots} \\ &= \frac{1}{2} \alpha [3\hbar\alpha - (7/6)\hbar^2\alpha^2 + \dots] \end{aligned} \quad (\text{B7})$$

$$\begin{aligned} \gamma_g(\alpha) &= \frac{1}{2} \frac{2\gamma + \beta/\lambda}{1 - \frac{1}{2}(\beta/\lambda + \gamma)} = \frac{1}{2} \frac{3\hbar\alpha - (17/3)\hbar^2\alpha^2 - \frac{1}{3}\hbar^2\alpha^2}{1 - \frac{3}{2}\hbar\alpha + \dots} \\ &= \frac{1}{2} [3\hbar\alpha - \frac{3}{2}\hbar^2\alpha^2 + \dots] \end{aligned} \quad (\text{B8})$$

$$\gamma_s = \frac{1}{2} \frac{\gamma}{1 - \frac{1}{2}(\beta/\lambda + \gamma)} = -\frac{1}{2} (\hbar^2\alpha^2/6) + \dots \quad (\text{B9})$$

That means we should now verify that

$$\begin{aligned} [x(\partial/\partial x) - \frac{3}{2}(\alpha^2 + \dots)(\partial/\partial\alpha) - \frac{3}{2}(\alpha - \alpha^2/2 + \dots)] \\ \times g_{\text{as}}(\alpha, x) = 0 \quad \text{mod}(\alpha^3), \end{aligned} \quad (\text{B10})$$

with g given by (5.34) and g_{as} extracted from Table I as the asymptotic behavior of g to order α^2 . Thus we have:

$$g_{\text{as}} = \frac{1}{3} + \frac{1}{2}\alpha \log x + (3\alpha^2/4)[(\log x)^2 - \frac{1}{3}\log x - 1] \quad (\text{B11})$$

We readily find combining (B7), (B8) and (B11) that (B10) is indeed satisfied. Equation (B10) reappears in the text as (6.11). Similarly we should verify that

$$[x(\partial/\partial x) - \frac{3}{2}(\alpha^2 + \dots)(\partial/\partial\alpha) + \alpha^2/12] z_{\text{as}}(\alpha, x) = 0 \quad \text{mod}(\alpha^3), \quad (\text{B12})$$

where according to Table II we have

$$z_{\text{as}}(\alpha, x) = 1 + \frac{1}{6}\alpha + \frac{1}{6}\alpha^2(\log x + cte). \quad (\text{B13})$$

Again, using (B7) and (B9) we find that (B12) is verified.

Of course we could have proceeded in the reverse way. Namely use our expressions for V and Z to derive β and γ . It is nice however to see the various approaches lead to the same consistent result. As far as practical experience teaches it seems that the calculations of the infinite parts in terms of a well defined cutoff procedure give a quicker and simpler way to obtain the result whenever they are possible.

APPENDIX C. POSITIVITY OF THE COUPLING CONSTANT AND ASYMPTOTIC BEHAVIOR OF THE SCATTERING AMPLITUDE

This Appendix can be read independently of the rest of this paper. We have seen in Sec. VI that in φ^4 theory it is

very likely that the coupling constant if small has to be positive. Furthermore we could as well use as renormalized quantity not the off-shell value of the four-point amplitude at zero momenta, but some other value evaluated for instance when external momenta are on shell. Relating this information to the usual unitarity will provide further constraints on the theory. In order to make contact with the usual language of S -matrix theory we shall speak of the on-shell four-point function as the $\pi^0\pi^0$ scattering amplitude in a fictitious world where we would neglect isospin, baryons, ... Also s, t, u will have their usual meaning as square center of mass energies in the three identical channels of $\pi^0\pi^0$ scattering.

Using standard conventions for on shell unitarity, the scattering amplitude is in fact the analytic continuation of $\Gamma^{(4)}(p_1, p_2, p_3, p_4)$ with $p_i^2 = -\mu^2$ in our Euclidean metric. To simplify further, we take μ as the unit of energy. Hence we see that we can interpret the renormalized coupling constant as proportional to the opposite of the value of the $\pi^0\pi^0$ scattering amplitude at the symmetry point, $F(s = \frac{4}{3}, \frac{4}{3}, \frac{4}{3})$. We shall see that the knowledge of the "sign" of the coupling constant will have some observable consequences on the $\pi\pi$ on-shell amplitude. If one has more information like the fact that not only $F(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}) < 0$, but also $\text{Re}F < 0$, in the physical sheet of the real s, t, u plane the amplitude will be even more constrained. The latter hypothesis is in fact no more unrealistic as the previous one, for there is no *a priori* reason to choose the normalization point at the center of the Mandelstam triangle. Here we want to make a nonexhaustive list of facts which follow from these hypotheses.

(1) From an S -matrix point of view, there is no visible objection to having $F(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}) > 0 (\lambda < 0)$. Indeed Atkinson (1968) has elaborated a procedure of construction of a $\pi\pi \rightarrow \pi\pi$ amplitude which:

- (i) satisfies a Mandelstam representation, and in particular unsubtracted dispersion relations for $|t| < 4$;
- (ii) is symmetric in s, t, u ;
- (iii) has elastic unitarity exactly satisfied for $4 < s < 16$;
- (iv) satisfies the partial wave amplitude inequalities required by unitarity for $s > 16$.

From positivity and unsubtracted dispersion relations, we get

$$F(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}) = 2 \frac{1}{\pi} \int_4^\infty \frac{A(x, \frac{4}{3})}{x - \frac{4}{3}} dx, \tag{C1}$$

where A is the absorptive part

$$A(x, \frac{4}{3}) = \sum (2l + 1) \text{Im}f_l(x) P_l[1 + 8/\frac{1}{4}(3x - 4)] > 0; \tag{C2}$$

hence $F(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}) > 0$. In (C2) $P_l(x)$ is the l -th Legendre polynomial.

Of course we cannot guarantee that the full content of the unitarity condition can be satisfied because one looks here only at the two-body sector. Unfortunately, for the time being, this is the only thing which can be done rigorously.

(2) If $F(s, t = 0) \rightarrow 0$ for $s \rightarrow \infty$ and also $F(s, t) \rightarrow 0$ for $s \rightarrow \infty$ and $-\epsilon < t \leq 0$ (ϵ as small as one wishes), then

$F(s, t, u)$ is positive inside the whole region $s < 4, t < 4, u < 4$ and, in particular,

$$F(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}) > 0$$

$$F(4, 0, 0) > 0 \quad (\text{positive scattering length}). \tag{C3}$$

Proof

Jin and Martin (1964) have shown that the (even) number of subtractions does not change from $-\epsilon < t \leq 0$ to $t = 4$. By assumption, there are no subtractions for $-\epsilon < t \leq 0$, so there are no subtractions for $0 < t < 4$. If there are no subtractions, positivity in $0 < t < 4, s$ outside the cuts, is evident. We get positivity inside the whole triangle by circular permutations of s, t and u .

$$(3) \text{ If } F(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}) < 0$$

$$\lim_{s \rightarrow \infty} (\sup_{-\epsilon < t \leq 0} |F(s, t)|) \neq 0. \tag{C4}$$

This is just the inverse of the previous statement.

This result would be of a great interest if one knew that the high energy amplitude is not purely real. Assume that $(\text{Im}F/\text{Re}F) > \alpha$, then:

$$\text{Im}F(s, t = 0) \simeq C s \sigma_t. \tag{C5}$$

Hence the first conclusion is:

$$\lim_{s \rightarrow \infty} s \sigma_t(s) \neq 0.$$

But then, if $\lim(s\sigma_t(s))$ is finite and nonzero, we can easily see by inserting into a dispersion relation that $\text{Re}F \rightarrow \infty$ and hence $s\sigma_t(s) \rightarrow \infty$. Hence in the asymptotic behavior of σ_t a *mass* must explicitly appear.

If the assumption $(\text{Im}F/\text{Re}F) > \alpha$ is not made, the conclusions are much weaker: we get that in $-\epsilon < t \leq 0$ there is a nonzero set where $\lim |F| \neq 0$.

Then, for s large

$$\sigma_{el} > \frac{1}{s^2} \int_{-\epsilon}^0 |F(s, t)|^2 dt$$

$$\sigma_{el} > \text{const} s^{-2} \tag{C6}$$

where the constant is strictly positive.

(4) The assumption $|F| \rightarrow C \neq 0$ for $s \rightarrow \infty$ uniformly at all physical angles, is not tenable.

This assumption would correspond to believing that the naïve lowest order perturbation with a finite, non-zero coupling constant holds.

Indeed if $|F| \rightarrow C$ at all angles,

$$\sigma_{el} \simeq C^2/s, \quad s\sigma_{el} \rightarrow C' \neq 0, \tag{C7}$$

then:

$$F(s, t = 0)$$

$$= F(2, t = 0) + \frac{(s - 2)^2}{\pi} \int_4^\infty \frac{2s' ds' s' \sigma_{total}}{(s' - 2)^2 (s'^2 - s^2)}. \tag{C8}$$

It is easy to see that if $s'\sigma_t > C$

$$F(s, t = 0) \rightarrow -\infty \quad (\text{C9})$$

for $s \rightarrow i\infty$. Therefore, by the Phragmen-Lindelöf theorem, we cannot have

$$|F(s, t = 0)| \rightarrow C \quad \text{for } s \rightarrow \pm\infty. \quad (\text{C10})$$

(5) Assume that $F(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}) < 0$ propagates to the whole physical sheet in the form $\text{Re}F < 0$. What are the consequences?

One consequence is that the total cross-section cannot rise to ∞ . More specifically there exists a sequence of energies $s_1, s_2, \dots, s_n, s_n \rightarrow \infty$ for which σ_{total} is bounded and $|F|/s$ is bounded.

This is a direct consequence of the Khuri-Kinoshita or Jin-MacDowell theorems. Let us give a direct proof. We use the variable $z = (s - 2)^2$. We look at the forward amplitude. $(F(z))^2 = G(z)$ is analytic in a cut plane and bounded by $z(\log z)^4$. So

$$(F(z))^2 = A + Bz + \frac{z^2}{\pi} \int \frac{2 \text{Im}F(z') \text{Re}F(z') dz'}{z'^2(z' - z)}. \quad (\text{C11})$$

We know that $\text{Im}F(z') > 0$; if $\text{Re}F(z') < 0$ we have for $z < 0$

$$0 < |F(z)|^2 < A + Bz. \quad (\text{C12})$$

Taking $z = -x$:

$$\frac{x^2}{\pi} \int \frac{2 \text{Im}F(z') |\text{Re}F(z')| dz'}{z'^2(z' + x)} < |A| + |B|x \quad (\text{C13})$$

for $x \rightarrow +\infty$.

From this we deduce easily that the bound $|F(z)|^2 < A' + B'|z|$ holds in any complex direction. Then, by

standard techniques it can be shown that an average of $(F(z))^2$ also satisfies such a bound on the real axis. Notice also that A and B are expressible in terms of $-F(s = 2, t = 0)$ and $(d^2/ds^2)F(s = 2, t = 0)$ for which there exist absolute bounds (Martin, 1965; Lukaszuk and Martin, 1967). So the knowledge of $\text{Re}F < 0$ gives us a numerical bound on the limit of the total cross section for $s \rightarrow \infty$, which is proportional to $(|B|)^{1/2}$.

Another consequence of $F < 0$ in the triangle $s < 4, t < 4, u < 4$ is that the Bonnier-Vinh Mau (1968) lower bound of $F(4, 0, 0)$ (the scattering length !) can be considerably improved, following for instance the ideas of Grassberger and Kühnelt (Grassberger and Kühnelt, 1973).

Is it possible to forbid completely this situation? So far we do not know.

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