An introduction to the theory of dual models and strings

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The theory of a free string is first presented classically and quantum mechanically both in a covariant and a nonexplicitly covariant treatment. Then the concepts and techniques obtained from the free string are used to build the operator formalism of dual models. Both the conventional Veneziano model and the Neveu–Schwarz–Ramond model are presented. The self-consistency of these models at the tree and loop levels is investigated, and the interpretation of the results obtained is given in terms of the string picture.

CONTENTS

INTRODUCTION

Interactions among particles are classified into gravitational, weak, electromagnetic, and strong. Of these the first three are rather well understood. The electromagnetic interaction is of course the best understood of all because not only do we know the classical theory of electromagnetism but also, thanks to quantum electrodynamics, we can compute the quantum corrections to the classical theory. The agreement between the mathematical apparatus and experience in quantum electrodynamics is quite remarkable. For gravity, the classical theory is well understood, and although quantum gravity is still a rather academic subject, progress has been made towards computing corrections to the classical theory. In weak interactions there is now interest in the so-called unified theories of weak and electromagnetic interactions which have the advantage of being renormalizable so that finite corrections can be computed.

What these three fields have in common is the following: they are all described by quantum field theory, the coupling constants are small so that expansions in perturbation theory are relevant, they describe a small number of particles (for fields), they are all theories which have a gauge group leaving the Lagrangian invariant.

When we consider now the field of strong interactions we find that no theory yet exists which can claim to encompass all available data. The essential reasons for the theorists' failure to find such a theory are the following:

(1) The large number of particles or resonance states one has to take into account.

(2) The fact that the strong interactions coupling constants are not small.

By themselves, these facts do not necessarily dismiss the possibility that local field theories may describe strong interactions in terms of a few elementary fields. Indeed, a lot is learned about strong interactions by postulating that hadrons are made of spin 1/2 "quarks" (Gell-Mann (1964), Zweig (1964)). Further the interaction between quarks can be thought of as mediated by spin 1 gluons. From the theorists point of view, Yang–Mills theories of quarks and gluons are by far the most interesting [Fritzsch, Gell–Mann, Leutwyler (1973)], and hence one may speculate that strong, weak, and electromagnetic interactions could be unified within the framework of a gauge theory where the symmetry of the Lagrangian would solely be broken by the vacuum.

These theories however have to face the challenges that quarks are not seen experimentally so that one has to find a containment mechanism for them within the framework of field theory. Secondly even if such a mechanism exists, to go from the simple quark–gluon world to the actual hadronic world is not easy since it involves summing the whole perturbation expansion.

The S-matrix approach grew out of this need to bypass the problems associated with field theory and to deal only with physical quantities. The axioms of the S-matrix are Lorentz invariance, unitarity, T, C, and P invariance, analyticity in the complex plane of the Mandelstam variables, maximal analyticity in the complex plane of the angular momentum (we will refer nontechnically to this axiom as

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"Regge behavior"), factorizability of the residues of particle poles with real coupling constants, (particles coupled through imaginary coupling constants are usually referred to as "ghosts" and are according to our last axiom, highly undesirable in any theory).

The S-matrix can be built in a perturbative expansion in terms of a dimensionless coupling constant g. At the lowest order in g, the "Born term" of this expansion should satisfy Lorentz invariance; T, C, P invariance; analyticity with only poles; crossing symmetry; Regge behavior (without fixed poles); factorization of the pole residues with real coupling constants.

With the exception of Regge behavior without fixed poles, these postulates are satisfied by any Born term of a decent field theory. The distinction between field theories and dual models comes of the fact that Regge behavior without fixed poles and analyticity imply the existence of an infinite number of resonances even at the level of the Born approximation, and this clearly takes us beyond the realm of conventional field theory.

In addition to these natural postulates, one generally adds the requirement that all particle poles (except those which have vacuum quantum numbers) lie on linear Regge trajectories of universal slope $\alpha' \simeq 0.95$ GeV^{-2}. This has both a theoretical reason (in the sense that it is difficult to concoct satisfactory dual models with nonlinear trajectories) and an experimental reason, since both mesons and baryons seem to fall on such linear trajectories, as one can best verify for non-strange baryon states from $J = 3/2$ to $11/2$. One also adds the natural requirement that the spectrum in the Born approximation be as realistic as possible so that loop corrections may be small. Practically this amounts to requiring that the initial spectrum has the same features as one may expect from a quark–gluon scheme.

The presumed smallness of loop corrections in such an "ideal" model is linked to the observation that compared to the typical scale of strong interactions ($1/\alpha'^{1/2} \simeq 1$ GeV) resonances are narrow, the widths of the well-established resonances being at most 0.35. Further, properties which should be broken by loop diagram, like exchange degeneracy are well established experimentally and hence at least in certain energy regions, the loop contributions should be small.

Given their guesswork origin, dual models went much further along the way than one might have expected. First, generalizations of the Veneziano formula to N-point functions describing the scattering for N scalars were found [Bardakci and Ruegg (1968), Virasoro (1969), Chan (1968), Goebel and Sakita (1969), Bardakci and Ruegg (1969)]. Then it was shown that these N-point functions did satisfy the nontrivial property of factorizability of all particle poles [Fubini and Veneziano (1969), Bardakci and Mandelstam (1969) and an operator formalism was developed (Fubini, Gordon, Veneziano (1969), Susskind (1970)].

The spectrum which emerged from the factorization of the Veneziano model (often called "conventional model") revealed a degeneracy at each mass level, asymptotically increasing exponentially with the mass, a result which coincided with the prediction of the statistical bootstrap model of Hagedorn (1968) and Frautschi (1971). In addition, the study of inclusive reactions within the context of dual models revealed a sharp cutoff in the momentum transfer, a result also predicted by the statistical bootstrap model [Virasoro (1971), Gordon and Veneziano (1971), De Tar, Kang, Chung-I Tan and Weis (1971)].

A remaining very important problem was that this spectrum seemed to contain both positive and negative norm states (ghosts), a natural consequence of the Lorentz covariance of the factorization procedure. Virasoro (1970) discovered however that an infinite set of gauge identities were satisfied in the model, provided that the intercept of the leading trajecory $\alpha(0)$ was set equal to 1. This was clearly a step away from reality (where $\alpha(0) \simeq 1/2$ would be much preferred), but it also led to a more satisfactory situation from the theoretical point of view since it permitted to prove that in this case all negative norm states were indeed decoupled [Brower (1972), Goddard and Thorn (1972)]. An additional property of the model emerged namely that the no-ghost theorem held only if the dimension of spacetime was smaller or equal to a critical dimension (26 in the conventional model). Another dual model, closely related to the Veneziano model, but restricted to meson states of vacuum quantum numbers was discovered by Virasoro (1969) and generalized by Shapiro (1970), and it was also shown that if the intercept $\alpha(0)$ of the leading trajectory was 2, and D (dimension of space–time) less or equal to 26, its spectrum was also ghost-free.

For $D = 26$, it was shown that the model was unitarizable [Lovelace (1971)] and that a spectrum of bound states appeared even at the one-loop level, which was factorizable [Cremmer and Scherk (1972), Clavelli and Shapiro (1973)] and identical with the ghost-free physical states of the Virasoro-Shapiro model [Olive and Scherk (1973a)]. The Regge behavior produced by this new set of states was typical of what one could expect from a bare pomeron [Alessandrini, Amati and Morel (1972)] with slope 1/2 of the reggeon slope. Hence it was shown that there was room in the theory for a pomeron even though its intercept was most unrealistic (2 instead of 1).

Of the same lines as the Veneziano model, a modification of the conventional model based on the work of Bardakci and Halpern (1971) was proposed by Clavelli and Shapiro (1973) and Schwarz (1973). It has all the features we have previously described except that 26 is replaced by 26-N where N is an integer associated with an internal $SU(N)$ symmetry group. Although this model is not realistic it teaches us that the "critical" dimension of space time in these models has something to do with the symmetries of the model.

An even more sophisticated model could be built based on the work of Ramond (1971) and Neveu and Schwarz (1971a,b). In addition to the features we have mentioned before, it contains both fermions (half integral spins) and mesons (integral spins), and a kind of G parity classifies mesonic states in two categories. The positive definiteness of the spectrum holds if the intercept in the fermion sector is 1/2, in the meson sector 1, and if the dimension of spacetime is less or equal to 10. [Goddard and Thorn (1972), Schwarz (1972), Brower and Friedman (1973), Schwarz (1973)]. In addition, if $D = 10$, ghost-free fermion–fermion amplitudes can satisfactorily be constructed, and have duality [Olive and Scherk (1973b), Schwarz and Wu (1973), Corrigan (1974), Corrigan, Goddard, Olive and Smith (1973)]. All these results are dependent on a larger algebra of gauges working in this model than in the conventional model. The tachyon problem which plagues the conventional model with $\alpha(0) = 1$ is also improved since only one tachyon appears in this model which is not on the $\alpha(0) = 1$, but on the $\alpha(0) = 1/2$ trajectory (so-called "pion"). Hence it is clear that although still unphysical, this model is much more realistic than the conventional model.

It is hoped that there exists at least one, and maybe unique, model working for $D = 4$ and having the good features of the previous one. It has been shown that the inclusion of an SU(3) of color [Schwarz (1973)] or parastatistics [Hopkinson and Tucker (1974)] does bring down D from 10 to 4, but unfortunately the fermion–fermion scattering amplitudes in this model have wrong duality properties; so the search for a hypothetical "right model" still goes on.

It is quite likely however, that if such a "right model" was found, it may still have unphysical intercepts (massless vector mesons, spin 2 "graviton," massless spin 1/2 and may be associated tachyons). Hence it is quite possible that mechanisms of spontaneous symmetry breaking are needed in dual models as they are in the massless Yang–Mills theory. At least one such example has been found [Cremmer and Scherk, (1974)] and one can show that the renormalized mass of the singlet vector meson at the first level is non zero, in spite of the gauge identities. More general mechanisms may exist giving a mass to all vector mesons initially massless [Bardakci (1974)].

Another most important development in the history of the subject was the (almost) complete elucidation of its properties in terms of an underlying Lagrangian formalism describing one-dimensional structures (instead of pointlike), called "strings." This Lagrangian formalism exists at the classical level, at the first quantized level and it is almost certain that it also exists at the second quantized level. Originally proposed by Nambu (1969), Nielsen (1970) and Susskind (1970), the string picture became very clear after the work of Goddard, Goldstone, Rebbi and Thorn (1973) (G.G.R.T.) who were able to derive everything found previously about the free spectrum of the conventional model from a first quantized Lagrangian describing a free string. They found that the Lagrangian had a big invariance group which generated the Virasoro gauges, and that the conditions on D and $\alpha(0)$ came only after first quantization. Also it was possible to quantize the theory in two different ways, one where explicit covariance is maintained but where the absence of ghosts is nontrivial; the other one, where only spacelike oscillators appear, but where Lorentz invariance has to be proved. The first of these ways leads us directly into the operator formalism. The other way leads us into the functional integral formalism set up previously by Gervais and Sakita (1971, 1973) where scattering of strings is described by the simple picture of strings breaking and joining at the end [Mandelstam (1973)]. Hence from the string picture one easily obtains the "twig" diagrams introduced by Zweig (1964), and generalized to dual diagrams by Harari (1969) and Rosner (1969) provided that one localizes the "quarks" at the ends of the string and the string itself be identified with the neutral "glue" binding the quarks. Finally according to recent works of Ramond (1974) dealing with the free string and Kaku and Kikkawa (1974) the formalism of Mandelstam can be derived from a second quantized Lagrangian formalism where fields are quantized on null planes. These fields themselves depend on a space-time path rather than on a point.

Hence dual models, originally very close to the S-matrix approach have gone closer and closer towards field theory. The identification of dual models with conventional (local) field theories is still a fascinating subject. It was shown that if one lets the slope of the dual models α' go to zero and keeps some masses fixed, one can obtain from dual models various field theories: $\lambda\phi^3$ if the mass of the ground state scalar is held fixed [Scherk (1971), Nakanishi (1972)], massless Yang–Mills if the mass of the first vector meson is held at zero [Neveu and Scherk (1972a)], massive Yang–Mills theory with spontaneous symmetry breaking if the mass of the first vector meson is held nonzero [Gervais and Neveu (1972)]. It is also speculated that the inverse direction could also be followed. The essential distinction between dual models and field theories as we said previously, is that the first have Regge behavior at the tree level while the second do not. However, if one sums the perturbation expansion in field theories, the theory may eventually Reggeize, and it was shown that necessary criteria of Reggeization are met by the Yang–Mills-type theories [Cornwall, Levin, and Tiktopoulos (1973), Grisaru, Schnitzer, and Tsao (1973), Schnitzer (1973)]. 't Hooft (1973) was able to show that in a weak coupling limit of a Yang–Mills theory of quarks and gluons with an internal $SU(N)$ where N is very large, the dominant diagrams have the planar structure typical of duality diagrams and hence one could conjecture that the Born term in a dual expansion may well be an infinite resummation of field theoretic diagrams as suggested initially by Sakita and Virasoro (1970) and Nielsen and Susskind (1970). If this could fully be proved, it would bring a unification of dual models with gauge theories which have already much in common. Another line of approach followed by Nielsen and Olesen (1973) is to show that stringlike solutions exist even in local field theory in certain limits as exemplified by the existence of vortex lines in superconductivity.

As we shall see, in the models that we shall present, the ends of the strings ("quarks"?) are massless and the reader may wonder if an introduction of masses at the end of the string may not be the solution to the unphysical intercepts and could not also take care of the breaking of $SU(3)$. The classical theory of such a string exists but due to the non linearity of the problem it is impossible to solve the equations of motion classically and to set up a Hamiltonian formalism (Chodos and Thorn (1974) and unpublished works of Gell-Mann, Rebbi, Dashen, Schwarz). This is why spontaneous symmetry breaking seems a better solution to this problem.

We have deliberately presented many lines of thought so that the reader may realize the avenues still open in the studies of dual models. In the approach we shall follow we use the string picture classically and in the first quantization to set up all the concepts and machinery which will later be used in the operator formalism. In our opinion the treatment of interacting strings via operator formalism is not superior to functional integral techniques. We shall use the operator formalism mainly because of personal taste and also because of the general consensus that whatever has been proved via functional integrals needs rechecking at the level of the combinations of diagrams ['t Hooft and Veltman (1973)], so that the existence of an operator formalism is needed anyhow. Secondly, functional integrals and operator formalism correspond to the quantization of the same Lagrangian in two different gauges and hence it is precious to show that both formalisms lead to the same results (so far, this has always been the case). Since for lack of space it is impossible to treat both, we shall refer to the review article of Rebbi (1974) for a complete treatment of the string.

The guideline in our approach will be the unity between string pictures and operator formalism, and gauge invariance will be our Ariadne's thread throughout the paper. Because gauge invariance is so strongly emphasized, subjects like duality are treated rather sketchily and the reader is invited to read existing reviews on the subject, especially the review of Alessandrini, Amati, le Bellac, Olive (1971) where the duality properties of tree graphs are treated in great detail. We also recommend strongly the review of Schwarz (1973) which treats the no-ghost theorem in a different way and presents the 26-N model. Also recommended is the review of Veneziano (1974), more oriented towards the S-matrix. In Sec. I we shall start with Nambu's Lagrangian for the free string. We shall solve the equation of motion both in a covariant and in a transverse gauge, and the gauge identities of Virasoro shall appear at that stage already. In Sec. II we quantize the free string and prove the Lorentz covariance of the transverse gauge and the no-ghost theorem in the covariant approach; the conditions $D = 26, \alpha(0) = 1$ appear at that state. In Sec. III we follow the covariant treatment and introduce interactions which respect the gauge identities necessary for the decoupling of ghosts and obtain the dual amplitudes. Sec. IV is a short review of the results obtained from the loop diagrams and show how the Pomeron is obtained from the nonplanar one-loop diagram and can be identified with the closed string. Section V covers the spinning string (Neveu–Schwarz–Ramond model) beginning with the elegant equations of Wess and Zumino (1973), then solving them and quantizing them. Fermions and mesons are obtained from two opposite boundary conditions of the classical equations. The same line of construction is followed as in the Veneziano model and hence proofs are shortened. This section includes the most recent results on fermion–fermion amplitudes.

I. THE RELATIVISTIC STRING[1]

1. Classical theory of a relativistic string

We shall use the metric $g^{00} = -g^{ii} = +1\; g^{ij} = 0$, if $i \neq j$ and the units $\hbar = c = 1$.

The path followed by a point particle is classically parametrized by one parameter, τ, which can be the proper time. Here $x^\mu = x^\mu(\tau)$ describes the position of the particle for each value of the parameter τ. The relativistic action for a point particle is

$$S = -m \int_a^b [(\partial x^\mu/\partial\tau)^2]^{1/2}\, d\tau, \qquad (\mathrm{I}.1)$$

[1] The material covered in this section and the next one closely follows the work of G.G.R.T.

where the integrand is the length of an infinitesimal element of world line.

The simplest step beyond a pointlike object is a one-dimensional object, i.e., a string. It is parametrized by two internal coordinates σ and τ. The first can be thought of as labeling the points along the string, while the second plays the role of the proper time. During its evolution, the string spans a two-dimensional surface in space–time given by $x^\mu = x^\mu(\sigma, \tau)$.

If we identify τ with the time: $x^0 = t = \tau$, at a given time t, the string is a curve in the three-dimensional space, parameterized by the functions $x^i = x^i(\sigma, t)$.

We note that if we do not want the string to propagate faster than light at any of its points, its motion must obey

$$(\partial x/\partial t)^2 = 1 - (\partial x^i/\partial t)^2 \geq 0; \qquad (\mathrm{I}.2)$$

on the other hand, one also has

$$(\partial x/\partial\sigma)^2 = -(\partial x^i/\partial\sigma)^2 \leq 0. \qquad (\mathrm{I}.3)$$

If we make a Lorentz transformation $x_\mu' = \Lambda_{\mu\nu}x_\nu$; hence in general x_0 will depend on σ and τ. If we give the two-dimensional surface $x^\mu = x^\mu(\sigma, \tau)$, the evolution of the string in any Lorentz frame is obtained by slicing the surface with the family of planes $\eta^\mu x_\mu = \tau$.

In the particular frame we have previously discussed, at each point of the surface, there is a spacelike and a timelike tangent vector. This has to be true in any frame and the necessary and sufficient condition for this is expressed by

$$\left(\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x_\mu}{\partial\tau}\right)^2 - \left(\frac{\partial x^\mu}{\partial\tau}\right)^2\left(\frac{\partial x^\nu}{\partial\sigma}\right)^2 \geq 0. \qquad (\mathrm{I}.4)$$

This is deduced by requiring that

$$[\partial x^\mu/\partial\tau + \lambda(\partial x^\mu/\partial\sigma)]^2 \qquad (\mathrm{I}.5)$$

takes both positive and negative values when λ is varied. To simplify notations we shall use

$$\dot{x} = \partial x^\mu/\partial\tau, \qquad x' = \partial x^\mu/\partial\sigma, \qquad (\mathrm{I}.6)$$

and we shall often suppress Lorentz indices when not explicitly needed.

2. Action principle for the string

In analogy with the relativistic action of a point particle, Nambu (1970) suggested that the relativistic action for a free string be proportional to the area of the surface spanned in spacetime by the evolution of the string.

For a two-dimensional surface embedded in space–time, the area element is

$$d^2A = \left\{\left(\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x_\mu}{\partial\tau}\right)^2 - \left(\frac{\partial x^\mu}{\partial\sigma}\right)^2\left(\frac{\partial x^\nu}{\partial\tau}\right)^2\right\}^{1/2} d\sigma\, d\tau. \qquad (\mathrm{I}.7)$$

The quantity under the square root is positive if the surface is of the kind previously discussed.

We can also introduce a more symmetric notation for the space of the two parameters σ and τ

$$(\zeta^0, \zeta^1) = (\tau, \sigma). \tag{I.8}$$

The metric tensor of this two-dimensional space is given by

$$-g_{\alpha\beta}(\zeta) = \partial_\alpha x^\mu \partial_\beta x_\mu, \tag{I.9}$$

where the indices α and β take the values 0 and 1, and one checks that

$$\{-\det g(\zeta)\}^{1/2}\, d\sigma\, d\tau = d^2A. \tag{I.10}$$

The geometrical meaning of the surface element is reflected in its invariance under any change of parametrization. If we set

$$\tilde{\zeta}_i = \tilde{\zeta}_i(\zeta_0, \zeta_1), \tag{I.11}$$

then we have

$$g_{\alpha\beta}(\zeta) = \frac{\partial x^\mu}{\partial \zeta_\alpha}\frac{\partial x_\mu}{\partial \zeta_\beta} = \sum_{i,j} \frac{\partial x^\mu}{\partial \tilde{\zeta}_i}\frac{\partial \tilde{\zeta}_i}{\partial \zeta_\alpha}\frac{\partial x_\mu}{\partial \tilde{\zeta}_j}\frac{\partial \tilde{\zeta}_j}{\partial \zeta_\beta}, \tag{I.12}$$

which we can write as:

$$g_{\alpha\beta}(\zeta) = M_{\alpha i}{}^T g_{ij}(\tilde{\zeta}) M_{j\beta}, \tag{I.13}$$

but since: $M_{ab} = \partial\tilde{\zeta}_a/\partial\zeta_b$ $\quad T$: transposed, we see that:

$$-\det g = -\det\tilde{g}(\det M)^2 \tag{I.14}$$

and since

$$d^2\zeta = d\zeta_1\, d\zeta_2 = d\tilde{\zeta}_1\, d\tilde{\zeta}_2/|\det M| \tag{I.15}$$

the infinitesimal d^2A is invariant. So we postulate the following action for the string[2]:

$$S = -\frac{1}{2\pi\alpha'}\int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \{-\det g\}^{1/2}$$
$$= -\frac{1}{2\pi\alpha'}\int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \{(\dot{x}\cdot x')^2 - x'^2\dot{x}^2\}^{1/2}. \tag{I.16}$$

We have considered that the initial and final positions of the string are given by $x_\mu(\sigma, \tau_1)$, $x_\mu(\sigma, \tau_2)$ and that $0 \leq \sigma \leq \pi$. Here σ, τ are dimensionless parameters. Since $[S] = 0$ [in mass units] and $[d^2A] = -2$ $[\alpha'] = -2$, i.e., α' has the dimension of the inverse of a mass square, or the square of a length. We shall see that α' measures the slope of Regge trajectories.

3. Equations of motion for the string

Let us perform an infinitesimal variation of the path traced by the string during its evolution in order to find the equations of motion from the principle of least action:

$$x_\mu \to x_\mu + \delta x_\mu(\sigma, \tau),$$

$$\delta S = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left(\frac{\partial L}{\partial \dot{x}_\mu}\frac{d}{d\tau}\delta x^\mu + \frac{\partial L}{\partial x_\mu'}\frac{d}{d\sigma}\delta x^\mu\right),$$
$$= \int_0^\pi d\sigma \frac{\partial L}{\partial \dot{x}_\mu}\delta x^\mu\Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} d\tau \frac{\partial L}{\partial x_\mu'}\delta x^\mu\Big|_{\sigma=0}^{\sigma=\pi},$$
$$-\int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left(\frac{\partial}{\partial\tau}\frac{\partial L}{\partial \dot{x}_\mu} + \frac{\partial}{\partial\sigma}\frac{\partial L}{\partial x'^\mu}\right)\delta x^\mu. \tag{I.17}$$

To determine the trajectory of the string, we vary by keeping the initial and final positions of the string fixed

$$\delta x^\mu(\tau = \tau_1)$$
$$= 0 = \delta x^\mu(\tau = \tau_2), \qquad \delta x^\mu(\sigma = 0, \pi) \text{ is arbitrary.}$$

This gives us:

(1) the edge condition: $\dfrac{\partial L}{\partial x_\mu'} = 0, \qquad \sigma = 0, \pi,$ (I.18)

(2) the equations of motion: $\dfrac{\partial}{\partial\tau}\dfrac{\partial L}{\partial \dot{x}_\mu} + \dfrac{\partial}{\partial\sigma}\dfrac{\partial L}{\partial x_\mu'} = 0.$ (I.19)

We can also perform the variation in a different way: the initial and final positions of the string are not kept fixed, but only actual motions of the string are allowed, i.e., which satisfy the previous equations. This second type of variation is used to compute the momentum and the angular momentum of the string.

Let us perform a translation for instance

$$\delta S = \left\{\int_0^\pi d\sigma \frac{\partial L}{\partial \dot{x}_\mu}\Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} d\tau \frac{\partial L}{\partial x_\mu'}\Big|_{\sigma=\pi;\sigma=0}\right\}\delta x^\mu. \tag{I.20}$$

We can get the variation of δS for a small surface $\sigma_1 \leq \sigma \leq \sigma_2$, $\tau_1 \leq \tau \leq \tau_2$

$$\delta S = \int_{(c)} [d\sigma(\partial L/\partial \dot{x}_\mu) + d\tau(\partial L/\partial x_\mu')]\delta x^\mu. \tag{I.21}$$

From this we deduce the flow of energy momentum along *any* curve (c) on the surface spanned by the string. Let us define the energy momentum current on the surface σ, τ:

$$P_\tau{}^\mu = -\partial L/\partial \dot{x}_\mu, \qquad P_\sigma{}^\mu = -\partial L/\partial x_\mu', \tag{I.22}$$

$$P^\mu = \int_{(c)} (d\sigma P_\tau{}^\mu + d\tau P_\sigma{}^\mu)$$

if (c) is a closed curve, $P^\mu = 0$ because

$$\int_{(c)} d\sigma P_\tau{}^\mu + d\tau P_\sigma{}^\mu$$
$$= \text{flux of} \begin{pmatrix} P_\sigma{}^\mu \\ P_\tau{}^\mu \end{pmatrix} \text{through the line} \begin{pmatrix} d\sigma \\ d\tau \end{pmatrix}$$
$$= \int d\sigma\, d\tau \left(\frac{\partial}{\partial\tau}P_\tau{}^\mu + \frac{\partial}{\partial\sigma}P_\sigma{}^\mu\right) = 0. \tag{I.23}$$

[2] This action was first written by Nambu (1970). See also Chang and Mansouri (1972), Hara (1971), Goto (1971) and Minami (1972).

Since the second equation of motion states that the current

$$\begin{pmatrix} P_\sigma{}^\mu \\ P_\tau{}^\mu \end{pmatrix}$$

is conserved on the surface

$$(\partial/\partial\sigma)P_\sigma{}^\mu + (\partial/\partial\tau)P_\tau{}^\mu = 0. \tag{I.24}$$

The total momentum of the string, P^μ, is given by

$$P^\mu = \int_{(c)} d\sigma P_\tau{}^\mu + d\tau P_\sigma{}^\mu = \int_0^\pi d\sigma P_\tau{}^\mu, \tag{I.25}$$

where c is any curve going from one boundary to the other. It is *conserved* since

$$\begin{aligned} \partial P^\mu/\partial\tau &= \int_0^\pi d\sigma(\partial/\partial\tau)P_\tau{}^\mu \\ &= -\int_0^\pi d\sigma(\partial/\partial\sigma)P_\sigma{}^\mu \\ &= P_\sigma{}^\mu(\sigma=0) - P_\sigma{}^\mu(\sigma=\pi) = 0. \end{aligned} \tag{I.26}$$

Let us similarly perform an *infinitesimal* Lorentz transformation on the string:

$$x_\mu' = x_\mu + \delta\Lambda_{\mu\nu}x_\nu, \tag{I.27}$$

$$x'^2 \simeq x^2 + 2x_\mu x_\nu \delta\Lambda_{\mu\nu} = x^2 \text{ if: } \delta\Lambda_{\mu\nu} = -\delta\Lambda_{\nu\mu}, \tag{I.28}$$

$$\delta S = \int d\sigma P_\tau{}^\mu \delta\Lambda_{\mu\nu}x_\nu + \int d\tau P_\sigma{}^\mu \delta\Lambda_{\mu\nu}x_\nu, \tag{I.29}$$

$$\delta S = \tfrac{1}{2}\delta\Lambda_{\mu\nu} \times \left[\int d\sigma(P_\tau{}^\mu x_\nu - P_\tau{}^\nu x_\mu) + \int d\tau(P_\sigma{}^\mu x_\nu - P_\sigma{}^\nu x_\mu)\right]. \tag{I.30}$$

So we can define an angular momentum current on the surface

$$(M_\tau{}^{\mu\nu}, M_\sigma{}^{\mu\nu}) \; M_i{}^{\mu\nu} = P_i{}^\mu x^\nu - P_i{}^\nu x^\mu. \tag{I.31}$$

Because L is Lorentz invariant we know that $\delta S = 0$ which means that for a closed curve

$$\int_{(c)} d\sigma M_\tau{}^{\mu\nu} + d\tau M_\sigma{}^{\mu\nu} = 0, \tag{I.32}$$

this implies that the angular momentum current is conserved

$$\frac{\partial}{\partial\tau}M_\tau{}^{\mu\nu} + \frac{\partial}{\partial\sigma}M_\sigma{}^{\mu\nu} = 0. \tag{I.33}$$

This equation can also be checked directly by using the equations of motion.

Let us now write explicitly the equations of motion

$$P_\tau{}^\mu = -\frac{\partial L}{\partial \dot{x}_\mu} = \frac{1}{2\pi\alpha'}\frac{(\dot{x}\cdot x')x_\mu' - x'^2\dot{x}_\mu}{((\dot{x}\cdot x')^2 - x'^2\dot{x}^2)^{1/2}}, \tag{I.34}$$

$$P_\sigma{}^\mu = -\frac{\partial L}{\partial x_\mu'} = \frac{1}{2\pi\alpha'}\frac{(\dot{x}\cdot x')\dot{x}_\mu - \dot{x}^2 x'^\mu}{((\dot{x}\cdot x')^2 - x'^2\dot{x}^2)^{1/2}}. \tag{I.35}$$

So the equations of motion read:

$$(1)\ \text{edge condition: } \frac{(\dot{x}\cdot x')\dot{x}_\mu - \dot{x}^2 x_\mu'}{((\dot{x}\cdot x')^2 - x'^2\dot{x}^2)^{1/2}} = 0 \quad \text{for} \quad \sigma = 0, \pi, \tag{I.36}$$

$$(2)\ \frac{\partial}{\partial\tau}\left\{\frac{(\dot{x}\cdot x')x_\mu' - x'^2\dot{x}_\mu}{((\dot{x}\cdot x')^2 - x'^2\dot{x}^2)^{1/2}}\right\} + \frac{\partial}{\partial\sigma}\left\{\frac{(\dot{x}\cdot x')\dot{x}^\mu - \dot{x}^2 x'^\mu}{((\dot{x}\cdot x')^2 - x'^2\dot{x}^2)^{1/2}}\right\} = 0. \tag{I.37}$$

Under this form they are obviously unsolvable. We note the following identities however

$$\begin{aligned} &P_\tau{}^\mu\cdot x_\mu' = 0, \qquad P_\tau{}^2 + x'^2/4\pi^2\alpha'^2 = 0, \\ &P_\sigma{}^\mu\cdot \dot{x}_\mu = 0, \qquad P_\sigma{}^2 + \dot{x}^2/4\pi^2\alpha'^2 = 0. \end{aligned} \tag{I.38}$$

Since for $\sigma = 0, \pi$ we have $P_\sigma{}^\mu = 0$, we deduce that $\dot{x}^2(\sigma = 0, \pi) = 0$, i.e., the end points of the string *move at the speed of light.*

4. Covariant solution of the equations of motion

Let us now find general solutions to the equations of motion. As we have seen, the action integrand, and hence the equations of motion, are invariant under reparametrization $\tilde{\sigma} = \tilde{\sigma}(\sigma, \tau)$, $\tilde{\tau} = \tilde{\tau}(\sigma, \tau)$. Hence we can choose a parametrization which will give a simple form to the equations of motion. The simplest choice is an orthonormal system of coordinates on the surface, i.e.:

$$x'\cdot\dot{x} = 0, \qquad x'^2 + \dot{x}^2 = 0. \tag{I.39}$$

The + sign in the second equation is due to the fact that x'^2 is spacelike, while $\dot{x}$ is timelike. In this parametrization:

$$P_\tau{}^\mu = (1/2\pi\alpha')\dot{x}_\mu, \tag{I.40}$$

$$P_\sigma{}^\mu = (-1/2\pi\alpha')x'^\mu. \tag{I.41}$$

The edge condition is

$$x'^\mu = 0 \qquad \text{for} \quad \sigma = 0, \pi. \tag{I.42}$$

The equations of motion

$$\ddot{x}_\mu - x_\mu'' = 0. \tag{I.43}$$

The general solution of the equations of motion is

$$x_\mu = x_\mu^{(1)}(\sigma - \tau) + x_\mu^{(2)}(\sigma + \tau). \tag{I.44}$$

So we can solve in general the equations of motion. A general solution satisfying $x_\mu{}' = 0 \; \sigma = 0, \pi$ is

$$x_\mu = \sum_{n=0}^{+\infty} x_n{}^\mu \cos n\sigma, \tag{I.45}$$

and the equation of motion becomes

$$\ddot{x}_n{}^\mu + n^2 x_n{}^\mu = 0, \qquad n = 0, 1, 2, \cdots \tag{I.46}$$

Let us introduce

$$a_n{}^\mu = [1/2(2\alpha' n)^{1/2}](\dot{x}_n{}^\mu - inx_n{}^\mu), \tag{I.47}$$

from the equations of motion it follows that

$$a_n{}^\mu = a_n{}^\mu(0) \exp(-in\tau), \tag{I.48}$$

$$a_n{}^{*\mu} = a_n{}^{*\mu}(0) \exp(+in\tau). \tag{I.49}$$

This works for $n \neq 0$. If $n = 0$

$$\ddot{x}_n{}^\mu = 0 \tag{I.50}$$

has for solution

$$x_0{}^\mu = q_0{}^\mu + c\tau, \tag{I.51}$$

$$P_{\tau,0}{}^\mu = c/2\pi\alpha' = (1/\pi) p_0{}^\mu, \qquad x_0{}^\mu = q_0{}^\mu + 2\alpha' p_0{}^\mu \tau. \tag{I.52}$$

Solving for $x_n{}^\mu$

$$2(2\alpha' n)^{1/2} a_n{}^\mu = \dot{x}_n{}^\mu - inx_n{}^\mu, \tag{I.53}$$

$$x_n{}^\mu = i[(2\alpha')^{1/2}/n^{1/2}](a_n{}^\mu - a_n{}^{*\mu}), \tag{I.54}$$

so that

$$x_\mu(\sigma, \tau) = q_0{}^\mu + 2\alpha' p_0{}^\mu \tau - i(2\alpha')^{1/2} \times \sum_{n=1}^{\infty} \frac{[a_n{}^{*\mu}(0) \exp(in\tau) - a_n{}^\mu(0) \exp(-in\tau)]}{n^{1/2}} \cos n\sigma . \tag{I.55}$$

5. Constraints

The above formula solves the equations of motion. However, the constraints $\dot{x}\cdot x' = 0$, $\dot{x}^2 + x'^2 = 0$ have not yet been taken into account. We note that if we extend analytically $x_\mu(\sigma, \tau)$ from $0 \leq \sigma \leq \pi$ to $-\pi \leq \sigma \leq \pi$ we have

$$\dot{x}(-\sigma) = \dot{x}(\sigma), \tag{I.56}$$

$$x'(-\sigma) = -x'(\sigma). \tag{I.57}$$

We can unify the constraints by requiring that

$$(x' + \dot{x})^2 = 0 \qquad \text{for} \quad -\pi \leq \sigma \leq \pi, \tag{I.58}$$

$$x' + \dot{x} = 2\alpha' p_0{}^\mu + (2\alpha')^{1/2} \sum_{n=1}^{\infty} \{(n)^{1/2} a_n{}^* \exp[+in(\tau + \sigma)] + (n)^{1/2} a_n \exp[-in(\tau + \sigma)]\}. \tag{I.59}$$

Let us introduce some new variables which allow a certain simplification of the notation

$$\alpha_0 = 2\alpha' p_0{}^\mu \qquad \alpha_n = (2\alpha')^{1/2}(n)^{1/2} a_n \qquad \alpha_{-n} = \alpha_n{}^*. \tag{I.60}$$

In terms of these new variables, one now has

$$x' + \dot{x} = \sum_{-\infty}^{+\infty} \alpha_n{}^\mu \exp[-in(\tau + \sigma)], \tag{I.61}$$

$$(x' + \dot{x})^2 = -\sum_{-\infty}^{+\infty} \exp[-in(\tau + \sigma)] 2L_n 2\alpha', \tag{I.62}$$

$$L_n = (-1/2) \sum_{-\infty}^{+\infty} \alpha_{n-m}\alpha_m (1/2\alpha'). \tag{I.63}$$

So the constraint equations are now expressed as an infinite set of initial conditions (independent of time)

$$L_N = (-1/4\alpha') \sum_{-\infty}^{+\infty} \alpha_{N-n}\alpha_n = 0. \tag{I.64}$$

Note in particular the constraint $L_0 = 0$ which gives us the mass shell condition

$$M^2 = p^2 = -(1/2\alpha') \sum_{-\infty}^{+\infty} n a_n{}^{*\mu} a_n{}^\mu. \tag{I.65}$$

We have in this way solved the equations of motion and expressed the constraints in a completely covariant manner.

After quantization, the L_N conditions will become the gauge conditions discovered by Virasoro (1970). Chang and Mansouri (1972) were the first to notice that these conditions could be derived from the string Lagrangian and did not need to be imposed as additional conditions to the equations of motion. Because of the existence of the constraints, clearly the variables $a_n{}^\mu$, $a_n{}^{*\mu}$ are not dynamically independent. For quantization it is useful to give up momentarily Lorentz covariance and find a smaller set of dynamical variables which will be unconstrained.

6. Noncovariant solution of the equations of motion

The conditions $\dot{x}\cdot x' = 0$, $\dot{x}^2 + x'^2 = 0$ do not specify completely the choice of coordinates along the string since there exists an infinity of orthonormal systems on a surface. We are now going to choose a unique system.

$$\dot{x}\cdot x' = 0 \qquad \dot{x}^2 + x'^2 = 0. \tag{I.66}$$

Let us make a change of variables $\tilde{\sigma} = \tilde{\sigma}(\sigma, \tau)$, $\tilde{\tau} = \tilde{\tau}(\sigma, \tau)$ and impose that the orthonormality conditions are preserved. Since

$$\frac{\partial x}{\partial \tau} = \frac{\partial x}{\partial \tilde{\tau}}\frac{\partial \tilde{\tau}}{\partial \tau} + \frac{\partial x}{\partial \tilde{\sigma}}\frac{\partial \tilde{\sigma}}{\partial \tau} \qquad \frac{\partial x}{\partial \sigma} = \frac{\partial x}{\partial \tilde{\sigma}}\frac{\partial \tilde{\sigma}}{\partial \sigma} + \frac{\partial x}{\partial \tilde{\tau}}\frac{\partial \tilde{\tau}}{\partial \sigma}, \tag{I.67}$$

we obtain

$$\frac{\partial \tilde{\sigma}}{\partial \tau} = \frac{\partial \tilde{\tau}}{\partial \sigma} \qquad \frac{\partial \tilde{\sigma}}{\partial \sigma} = \frac{\partial \tilde{\tau}}{\partial \tau}. \tag{I.68}$$

This implies that

$$\frac{\partial^2}{\partial\tau^2}\tilde{\tau}-\frac{\partial^2}{\partial\sigma^2}\tilde{\tau}=0,\qquad \frac{\partial^2}{\partial\tau^2}\tilde{\sigma}-\frac{\partial^2}{\partial\sigma^2}\tilde{\sigma}=0 \tag{I.69}$$

i.e., the new variables satisfy the d'Alembert equation with respect to the old ones.

Let us choose an arbitrary timelike vector $n^\mu(n^2\geq 0)$. We can choose as our new variable

$$n_\mu x^\mu(\sigma,\tau)=\lambda\tau. \tag{I.70}$$

This is compatible with the d'Alembert equation since x^μ satisfies the d'Alembert equation. So we assume the constraints

$$\dot{x}\cdot x'=0,\qquad x'^2+\dot{x}^2=0,\qquad n_\mu x^\mu=\lambda\tau \tag{I.71}$$

since one has

$$P_\tau{}^\mu=(1/2\pi\alpha')\dot{x}_\mu \tag{I.72}$$

in orthonormal coordinates. Hence

$$n_\mu P_\tau{}^\mu=n_\mu\dot{x}_\mu/2\pi\alpha'=\lambda/2\pi\alpha' \tag{I.73}$$

and the total momentum is obtained by integrating σ from 0 to π. Then

$$n_\mu P^\mu=n_\mu\dot{x}_\mu/2\alpha'=\lambda/2\alpha',\qquad \lambda=2\alpha' n_\mu P^\mu, \tag{I.74}$$

and the parametrization is fixed by

$$n_\mu x^\mu=2\alpha'(n_\mu P^\mu)\tau. \tag{I.75}$$

The meaning of this equation is simple: we intersect the surface $x^\mu(\sigma,\tau)$ with the plane $n_\mu x^\mu=2\alpha'(n_\mu P^\mu)\tau$. In itself the function $x^\mu(\sigma,\tau)$ does not specify which parametrization is used, but once the surface is intersected by the plane $n_\mu x^\mu=2\alpha'(n\cdot P)\tau$, the τ variable is uniquely defined, as well as the lines of constant τ or the surface. Note how restricted this parametrization is since the lines of constant τ are in general not contained in a hyperplane.

Now that the lines of constant τ are defined, the orthogonal family of the lines of constant σ is also well defined and in principle we can deduce the σ parametrization from our choice of τ and the orthonormality condition. It is, however, easier to choose the σ parametrization in a specific way and show that together with the equations of motion it implies the orthonormality condition.

Define σ as follows

$$\int_0^\sigma d\sigma' n^\mu P_\mu{}^\tau(\sigma',\tau)=(\sigma/\pi)n^\mu P_\mu. \tag{I.76}$$

$P_\mu=$ total momentum

$$n^\mu P_\mu=\pi n^\mu P_\mu{}^\tau(\sigma,\tau) \tag{I.77}$$

and σ is proportional to the projection of the total energy of the string on P^μ or the energy–momentum density projection, i.e., $n^\mu P_{\mu,\tau}$ is constant along σ. So

$$n\cdot P_\tau=(n\cdot P)/\pi, \tag{I.78}$$

$$n\cdot x=2(n\cdot P)\tau \tag{I.79}$$

are the equations defining our choice of gauge. In addition we have also the equations of motion

$$(\partial/\partial\tau)P_\tau+(\partial/\partial\sigma)P_\sigma=0, \tag{I.80}$$

$$P_\sigma=0,\qquad \sigma=0,\pi. \tag{I.81}$$

Projecting the first of these on the vector n, and using (I.78) we deduce that

$$(\partial/\partial\sigma)n\cdot P_\sigma=0\quad\text{and from (I.81)}\quad n\cdot P_\sigma=0. \tag{I.82}$$

It is now easy to show that this last equation, together with the explicit expression of P_σ

$$P_\sigma=\frac{1}{2\pi\alpha'}\frac{(\dot{x}\cdot x')\dot{x}_\mu-\dot{x}^2x_\mu'}{[(\dot{x}\cdot x')^2-x'^2\dot{x}^2]^{1/2}} \tag{I.83}$$

implies that:

$$\dot{x}\cdot x'=0. \tag{I.84}$$

Similarly, using explicitly P_τ one gets that

$$x'^2+\dot{x}^2=0, \tag{I.85}$$

and hence our assertion that our gauge is an orthonormal gauge is verified.

We now define the transverse gauge as the gauge where n is a light-like vector $n=(1,-1,0,0)$. It is convenient to introduce light cone coordinates:

$$u_\pm=(1/2^{1/2})(u_0\pm u_1), \tag{I.86}$$

$$uv=u_+v_-+u_-v_+-u_iv_i, \tag{I.87}$$

u_i, v_i: transverse variables.

The equations defining the gauge are now

$$x^+=2\alpha' P^+\tau,$$
$$P_\tau{}^+=(1/\pi)P^+. \tag{I.88}$$

Let us see immediately that the only dynamical variables are the transverse ones. In a general gauge (even non orthonormal)

$$P_\tau{}^\mu\cdot x_\mu'=0,\qquad P_\tau{}^2+\frac{x'^2}{4\pi^2\alpha'^2}=0. \tag{I.89}$$

This reads

$$\underbrace{P_\tau{}^+}_{(1/\pi)P^+}x_-'+\underbrace{P_\tau{}^-x_+'}_{0}=P^ix_i' \tag{I.90}$$

and

$$x_-' = (\pi/P_+)P_\tau{}^i x_i', \tag{I.91}$$

$$(2/\pi)P_+P_\tau{}^- - P_\tau{}^iP_{\tau,i} = x_i'^2/4\pi^2\alpha'^2, \tag{I.92}$$

$$P_\tau{}^- = (1/2\pi p^+)[\pi^2(P_\tau{}^i)^2 + (x_i')^2/4\alpha'^2]. \tag{I.93}$$

As before, the equations of motion are

$$\ddot{x}_i - x_i'' = 0 \tag{I.94}$$

for the transverse variables. When the $x_i(\sigma, \tau)$ are known, x_+, x_- are known, up to an integration constant q_-. Hence the independent variables are $x_i(\sigma, \tau)$, q_-, p_+.

We can expand as before

$$x_i(\sigma,\tau) = q_i + 2\alpha' p_i\tau - i(2\alpha')^{1/2}\sum_{n=1}^{\infty}[a_n{}^{+i}\exp(in\tau) - a_n{}^i\exp(-in\tau)] \times \frac{\cos n\sigma}{n^{1/2}}. \tag{I.95}$$

Using the notations

$$\alpha_0{}^i = 2\alpha' p^i, \qquad \alpha_n{}^i = (2\alpha')^{1/2}a_n(n)^{1/2}, \qquad \alpha_{-n}{}^i = \alpha_n{}^{*i}, \tag{I.96}$$

we have

$$x_\mu(\sigma,\tau) = q_\mu + \alpha_{0\mu}\tau - i\sum_{n=1}^{\infty}[\alpha_{-n}{}^\mu \exp(in\tau) - \alpha_n{}^\mu\exp(-in\tau)]\frac{\cos n\sigma}{n}. \tag{I.97}$$

We can obviously express through the constraints $\alpha_n{}^-$ in terms of $\alpha_n{}^i$, $\alpha_0{}^+$. One finds:

$$\alpha_n{}^- = (1/2\alpha' p^+)L_n{}^\perp, \tag{I.98}$$

$$L_n{}^\perp = -\frac{1}{4\alpha'}\sum_{-\infty}^{+\infty}\alpha_{n-m}{}^i\alpha_m{}^i. \tag{I.99}$$

So a discrete set of independent variables is the

$$\{\alpha_n{}^i, q_-, p_+\}.$$

7. Mass and angular momentum of the string

$$M^2 = 2P_+P_- - P_iP^i \tag{I.100}$$

and according to the constraint giving us

$$\alpha_0{}^- = 2\alpha' P_- = (1/2\alpha' P_+)L_0{}^\perp \tag{I.101}$$

we get:

$$M^2 = 2P_+P_- - P_iP^i = (1/\alpha')\sum_{n=1}^{\infty} n a_n{}^{*i}a_n{}^i. \tag{I.102}$$

Hence we check that $M^2 \geq 0$, a fact not obvious in the covariant gauge. The total angular momentum is defined by

$$M^{\mu\nu} = \int_0^\pi d\sigma(x^\mu P_\tau{}^\nu - x^\nu P_\tau{}^\mu) = (1/2\pi\alpha')\int_0^\pi d\sigma(x^\mu\dot{x}^\nu - x^\nu\dot{x}^\mu). \tag{I.103}$$

Substituting the expression for x^μ in terms of oscillators we get

$$M^{\mu\nu} = q_\mu p_\nu - q_\nu p_\mu - i\sum_{n=1}^{\infty}[a_n{}^{*\mu}a_n{}^\nu - a_n{}^{*\nu}a_n{}^\mu]. \tag{I.104}$$

In this expression not only transverse oscillators enter. We define

$$M^{\mu\nu} = q_\mu p_\nu - q_\nu p_\mu + S_{\mu\nu}. \tag{I.105}$$

The classical spin of the string can be computed from the equation:

$$J^2 = \tfrac{1}{2}[S_{\mu\nu}S_{\mu\nu} - (2/M^2)P_\nu S_{\nu\rho}P_\sigma S_{\sigma\rho}]. \tag{I.106}$$

An interesting inequality between the spin and the mass can be obtained by considering an orthonormal parametrization of the string such that x^0 is identified with τ, and choosing the frame of reference to be the rest frame of the string. Then due to the fact that $a_n{}^0 = 0$, the second term in (I.106) vanishes. Comparing J^2 with α'^2M^4 one is led to prove the inequality (n and m are not summed upon)

$$(a_n{}^*\cdot a_m)(a_n\cdot a_m{}^*) - (a_n{}^*\cdot a_m{}^*)(a_n\cdot a_m) \leq nm(a_n{}^*\cdot a_n)(a_m{}^*\cdot a_m) \tag{I.107}$$

which follows directly from Schwarz's inequality. In this expression the scalar product is taken over all spacial components of $a_n{}^\mu$. Hence for all motions of the string

$$J \leq \alpha' M^2$$

and the equality is reached only for a rigid rotating string where only the first mode of oscillation is excited. The string picture accounts thus naturally for the nonexistence of particles of high spin and low mass (so-called "ancestors" in Regge literature).

II. HAMILTONIAN FORMALISM AND QUANTIZATION

1. Hamiltonian formalism for singular Lagrangians

The parameter τ has heretofore been considered as an evolution parameter. It is therefore normal to consider

$$P_\tau{}^\mu = -(\partial L/\partial\dot{x}^\mu) \tag{II.1}$$

as the momentum conjugate to $x^\mu(\sigma,\tau)$. σ then denotes the collection of points along the string. However if we try to express L as function of $x^\mu(\sigma,\tau)$, $(\partial/\partial\sigma)x^\mu(\sigma,\tau)$, $P_\tau{}^\mu$, it turns out to be impossible because of the relations

$$P_\tau{}^\mu x_\mu' = 0, \qquad P_\tau{}^2 + (x')^2/4\pi^2\alpha'^2 = 0. \tag{II.2}$$

This situation is similar to the case of a relativistic point particle where

$$L = -m[(\partial x^\mu/\partial\tau)^2]^{1/2} \qquad P^\mu = m\frac{\partial x^\mu/\partial\tau}{[(\partial x^\mu/\partial\tau)^2]^{1/2}} \quad \text{(II.3)}$$

and we have the constraint

$$P^2 - m^2 = 0 \quad \text{(II.4)}$$

In this case it is possible in order to establish a canonical formalism either:

(a) to disregard the constraints and compute all Poisson brackets and then afterwards to impose the constraints on the dynamical system. This is the Dirac (1950, 1958) method of quantization and it is covariant, but leads, when quantized, to an indefinite metric space.

(b) to reduce the number of degrees of freedom and eliminate all redundant variables. Then we have only to assume canonical Poisson brackets for the independent variables. This leads when quantized to a positive metric space, but the procedure is not explicitly covariant.

2. Covariant Hamiltonian formalism

We shall follow Dirac (1950, 1958) and Fadeev (1969) in the treatment of Lagrangians which imply primary constraints. Let us recall briefly the results of Dirac: Let us consider a Lagrangian

$$L(q_i, \dot{q}_i)$$

such that there is a set of algebraic constraints between the q_i and p_j denoted by the equations

$$\phi_\alpha(q_i, p_j) = 0, \qquad \alpha = 1, 2, \cdots M. \quad \text{(II.5)}$$

We assume that these equations are independent and irreducible in the sense that the surface M defined by these equations, called links or primary constraints, is such that any function vanishing on M is a superposition of constraints. We assume further that the constraints ϕ_α and the canonical Hamiltonian

$$H_0 = \sum_i p_i\dot{q}_i - L$$

form a closed algebra through Poisson brackets

$$\{\phi_\alpha, \phi_\beta\} = \sum_\gamma c_{\alpha\beta\gamma}\phi_\gamma, \quad \text{(II.6a)}$$

$$\{H_0, \phi_\alpha\} = \sum_\beta c_{\alpha\beta}\phi_\beta. \quad \text{(II.6b)}$$

Often (II.6b) is realized by the vanishing of H_0 itself due to algebraic relations. In these equations we are reminded that:

$$\{\zeta, \eta\} = \sum_i \frac{\partial\zeta}{\partial q_i}\frac{\partial\eta}{\partial p_i} - \frac{\partial\zeta}{\partial p_i}\frac{\partial\eta}{\partial q_i}$$

is the definition of the Poisson bracket.

Dirac showed then that the Hamiltonian of the system is given by

$$H = H_0 + \sum_\alpha v_\alpha\phi_\alpha(p_i, q_j), \quad \text{(II.7)}$$

where the v_α are constant in the p_i, q_j. Choosing them is equivalent in usual language to a choice of gauge. Note that H has not necessarily the dimensions of an energy! The "time" evolution of an arbitrary function $f(p, q)$ is given by

$$\dot{f}(p, q) = \{f, H\} + \partial f/\partial\tau. \quad \text{(II.8)}$$

The observables of the system are functions such that at a given time

$$\{f, \phi_\alpha\} = \sum_\beta d_{\alpha\beta}\phi_\beta. \quad \text{(II.9)}$$

The closed algebra of the constraints implies that

(1) if the constraints are applied at $\tau = 0$ they are valid at any later time.

(2) if a function is an observable at a given time, its time evolution is independent of the choice of the $v_\alpha(p_i, q_i)$.

Let us apply this first to a point particle whose Lagrangian is given by equation (II.3), so that we have the constraint expressed by Eq. (II.4). Then

$$H_0 = -P^\mu(\partial x_\mu/\partial\tau) - L = 0$$

identically so that (II.6b) is trivially satisfied. Equation (II.6a) is also trivially satisfied, since we have only one constraint. Hence

$$H \equiv v(p^2 - m^2).$$

The equation of motion is

$$\dot{x}_\mu = \{x_\mu, H\} = -2vp_\mu$$

expresses the motion on a straight line, but also specifies τ in terms of v. When the system is quantized the constraint $H = 0$ is now imposed on the state vector and we get the Klein–Gordon equation, while the τ evolution of the system is disregarded as irrelevant because τ and p^2 cannot be specified independently. So one is left with the equation

$$(p^2 - m^2)\,|\psi\rangle = 0.$$

Let us now apply this to the string where

$$q_i \to x^\mu(\sigma, \tau), \quad \text{(II.10)}$$

$$p_i \to P_\tau{}^\mu(\sigma, \tau) = -(\partial L/\partial\dot{x}^\mu). \quad \text{(II.11)}$$

We assume the canonical Poisson brackets

$$\{x^\mu(\sigma, \tau), x^\nu(\sigma', \tau)\} = 0, \qquad \{P_\tau{}^\mu(\sigma, \tau), P_\tau{}^\nu(\sigma', \tau)\} = 0, \quad \text{(II.12)}$$

$$\{x^\mu(\sigma, \tau), P_\tau{}^\mu(\sigma', \tau)\} = -g^{\mu\nu}\delta(\sigma - \sigma').$$

And we have the following links or primary constraints:

$$P_\tau \cdot x' = 0, \qquad P_\tau^2 + x'^2/4\pi^2\alpha'^2 = 0. \tag{II.13}$$

We extend the definition of σ from $[0, \pi]$ to $[-\pi, \pi]$ by

$$P_\tau^\mu(-\sigma) = P_\tau^\mu(+\sigma), \qquad x_\mu'(-\sigma) = -x_\mu'(\sigma),$$
$$-\pi \leq \sigma \leq +\pi. \tag{II.14}$$

Then the constraints can be expressed as

$$L_n = -\frac{1}{4}\int_{-\pi}^{+\pi} d\sigma \exp(in\sigma)[\pi(2\alpha')^{1/2}P_\tau + x'/(2\alpha')^{1/2}]^2$$
$$= 0 \tag{II.15}$$

for all n. They are indeed independent and irreducible. We check then that

$$H_0 = \int_0^\pi d\sigma\{-\dot{x}P_\tau - L\} = 0$$

identically since $L = -\dot{x}P_\tau$,

so that the second equation is trivially true.

Further the Poisson bracket algebra of the L_n closes as it should

$$\{L_n, L_m\} = i(m-n)L_{n+m}. \tag{II.16}$$

So the τ evolution of the system is solely generated by the constraints and

$$H = \sum_{-\infty}^{+\infty} v_n L_n. \tag{II.17}$$

The choice of the v's is arbitrary, and choosing a v is equivalent to choosing a gauge. A very convenient gauge is the one where:

$$H = L_0. \tag{II.18}$$

We can now compute the τ evolution of $x_\mu(\sigma, \tau)$

$$\dot{x}_\mu = \{x_\mu, H\} = \int_0^\pi d\sigma'\{(\pi\alpha' P_\tau^2 + x'^2/4\alpha'^2\pi), x_\mu(\sigma, \tau)\},$$

$$\dot{x}_\mu = 2\pi\alpha' P_{\tau\mu}(\sigma, \tau). \tag{II.19}$$

Similarly one gets

$$\dot{P}_\tau = \{P_\tau, H\} = (1/2\alpha'\pi)x_\mu''. \tag{II.20}$$

So we recover the equations of motion

$$\ddot{x}_\mu - x_\mu'' = 0, \tag{II.21}$$

and the constraints are

$$\dot{x}x' = 0, \qquad \dot{x}^2 + x'^2 = 0. \tag{II.22}$$

So the choice $H = L_0$ corresponds to the orthonormal parametrization seen before.

Expanding as before:

$$x_\mu = q_\mu + \alpha_{0\mu}\tau + i\sum_{n\neq 0}\alpha_n{}^\mu[(\cos n\sigma/)n], \tag{II.23}$$

$$P_{\tau\mu} = (1/2\pi\alpha')\{\alpha_{0\mu} + \sum_{n\neq 0}\alpha_{n\mu}\cos n\sigma\}. \tag{II.24}$$

Using the equation

$$\sum_{-\infty}^{+\infty}\cos n\sigma \cos n\sigma' = \pi[\delta(\sigma+\sigma') + \delta(\sigma-\sigma')], \tag{II.25}$$

we check that

$$\{\alpha_n{}^\mu, \alpha_m{}^\nu\} = 2in\alpha' g^{\mu\nu}\delta_{n,-m}, \tag{II.26}$$

$$\{q^\mu, \alpha_0{}^\mu\} = -2\alpha' g^{\mu\nu}. \tag{II.27}$$

Expressing these Poisson brackets in terms of the $a_n{}^\mu$ oscillators, we get

$$\{a_n{}^\mu, a_m{}^{\nu+}\} = ig^{\mu\nu}\delta_{n,m}, \tag{II.28}$$

$$\{q^\mu, p^\nu\} = -g^{\mu\nu}, \tag{II.29}$$

and

$$H = -\alpha' p^2 \sum_{n=1}^{\infty} n a_n{}^{\mu *} a_{n,\mu} = L_0 \tag{II.30}$$

then

$$\dot{a}_n{}^\mu = -\{H, a_n{}^\mu\} = -ina_n{}^\mu.$$

Hence

$$a_n{}^\mu(\tau) = a_n{}^\mu(0)\exp(-in\tau). \tag{II.31}$$

At $\tau = 0$ the constraints $L_n = 0$ are imposed and are then valid at any later time.

3. Quantization in the covariant formalism

We now regard the dynamical variables $a_n{}^\mu$, q^μ, p^μ as operators whose commutators are given by the correspondence principle

$$i\{P.B.\} \to [\text{commutator}].$$

So we postulate:

$$[a_n{}^\mu, a_m{}^{+\nu}] = -g^{\mu\nu}\delta_{n,m}, \tag{II.32}$$

$$[q^\mu, p^\nu] = -ig^{\mu\nu}, \tag{II.33}$$

and

$$[x^\mu(\sigma,\tau), P_\tau{}^\nu(\sigma',\tau)] = -g^{\mu\nu}\delta(\sigma-\sigma'). \tag{II.34}$$

We have now to define the Hilbert space on which the oscillators operate. So we introduce a $|o, p\rangle$ vector such that

$$a_n^\mu |o, p\rangle = 0, \qquad p^\mu |o, p\rangle = p^\mu |o, p\rangle, \tag{II.35}$$

where $|o, p\rangle$ is an eigenstate of the momentum operator.

The excitation levels of the string are now defined as any vector of the type

$$(\prod_n a_{n,\mu_n}{}^{+\lambda_n})\,|0\rangle. \tag{II.36}$$

This space has an indefinite metrix because of the $g^{\mu\nu}$ in Eqs. (II.32, 33). It is often customary in dual models to choose the unit of mass such that $2\alpha' = 1$ since it simplifies the equations a lot. In these units the Fubini–Veneziano (1970) fields are defined as follows:

$$Q_\mu(z) = q^\mu - ip^\mu \ln z - i\sum_{n=1}^{\infty}[a_n{}^+ z^n - a_n z^{-n}]/(n)^{1/2}, \tag{II.37}$$

$$P_\mu(z) = iz(dQ_\mu/dz) = p^\mu + \sum_{n=1}^{\infty}(n)^{1/2}[a_n{}^+ z^n + a_n z^{-n}]. \tag{II.38}$$

We note the connection with the fields introduced previously

$$Q_\mu(e^{i\tau}) = x_\mu(\tau, \sigma = 0)$$

and

$$P_\mu(z) = (\dot{x} + x')(\sigma, \tau) \qquad \text{for} \quad z = \exp[i(\tau + \sigma)]. \tag{II.39}$$

P^μ should not be confused with $P_\tau{}^\mu$.

The L_n constraints are now operators defined as

$$L_n = (-1/4\pi)\int_{-\pi}^{+\pi} d\sigma \exp(in\sigma):P^2\{\exp(i[\tau+\sigma)]\}:, \tag{II.40}$$

where the : : indicate normal ordering. It is readily checked that L_n is τ independent, and that L_n can be expressed as

$$L_n = -\tfrac{1}{2}\oint (dy/2i\pi y)y^n:P(y)^2:, \tag{II.41}$$

where integration is performed around the origin. More generally we can define "generalized Virasoro gauges" through the formula

$$L_f = \tfrac{1}{2}\oint (dy/2i\pi y)f(y):P(y)^2:. \tag{II.42}$$

To work out the algebra of the L_f operators we note that[3]

$$P_\mu(x)P_\nu(y) = :P_\mu(x)P_\nu(y): - g_{\mu\nu}[xy/(x-y)^2] \qquad \text{if} \quad |x| > |y|. \tag{II.43}$$

[3] The method which follows can be generalized for commuting any bilinear forms in the operator $P(x)$ [Brink, Olive, Scherk (1973)].

For $|x| < |y|$ the left hand side is not well defined (since the normal ordered product is non singular at $x = y$) and we must be careful when writing $P_\mu(x)P_\nu(y)$ that the condition $|x| > |y|$ is satisfied.

In this equation the quantity $-g_{\mu\nu}[xy/(x-y)^2]$ plays the role of the "contraction" used in Wick's theorem to express the product of operators in terms of their normal ordered product. Computing the commutator of two L operators we are led to

$$[L_f, L_g] = \frac{1}{4}\oint_{\Gamma_1}\frac{dx}{2i\pi x}\frac{dy}{2i\pi y}f(x)g(y):P^2(x)::P^2(y): - \frac{1}{4}\oint_{\Gamma_2}\frac{dx}{2i\pi x}\frac{dy}{2i\pi y}f(x)g(y):P^2(y)::P^2(x):, \tag{II.44}$$

where

$$\Gamma_1 = |x| > |y|, \qquad \Gamma_2 = |x| < |y|$$

have been chosen such that the products of operators inside the integrands are well defined. To evaluate $:P^2(x)::P^2(y):$ we apply Wick's theorem remembering that contractions within a normal ordered product do not occur. We get four terms with one contraction and two terms with two contractions of the type

$$:P(x)P(x)::P(y)P(y):$$

$$= g_{\mu\nu}\frac{xy}{(x-y)^2}g^{\mu\nu}\frac{xy}{(x-y)^2} = D\frac{x^2y^2}{(x-y)^4}, \tag{II.45}$$

where D is the dimension of spacetime. Hence we see clearly that the commutation will contain a c number depending on D, and that this is an inescapable consequence of the quantization procedure. This provides the reason why the theory depends so crucially on D. So

$$:P^2(x)::P^2(y): = :P^2(x)P^2(y): - [4xy/(x-y)^2]:P(x)P(y): + 2D[x^2y^2/(x-y)^4].$$

The rest of the computation is easy: We interchange x, y and notice that inside the normal ordered product we can interchange $P(x)$ and $P(y)$ freely, one obtains

$$[L_f, L_g] = \oint_\Gamma \frac{dx}{2i\pi x}\frac{dy}{2i\pi y}f(x)g(y)\{-[xy/(x-y)^2] \times :P(x)P(y): + D[x^2y^2/2(x-y)^4]\}, \tag{II.46}$$

where

$$\Gamma = |x| > |y| - |y| > |x|.$$

The x integration is evaluated simply by taking the residue at $x = y$ and one obtains

$$[L_f, L_g] = L_h + c, \tag{II.47}$$

where

$$h(y) = y(g'f - gf'),$$
$$c = -\frac{D}{24}\oint \frac{dy}{2i\pi}[(yf(y))''(yg(y))' - (yf(y))'(yg(y))''].\qquad\text{(II.48)}$$

The difference between the operator algebra and the Poisson bracket algebra is thus essentially the c-number term which depends on the dimension of spacetime and plays an essential role in dual models: For $f(x) = -x^n$, $g(y) = -y^m$ we obtain the Virasoro (1970) algebra

$$[L_n, L_m] = (n - m)L_{n+m} + (D/12)n(n^2 - 1)\delta_{m,-n}.\qquad\text{(II.49)}$$

So the quantized constraints form again a closed algebra up to c-number terms. Normal ordering problems arise when going from the classical form of L_0 to its quantum expression. Hence the classical constraints: $L_n = 0$ for all n are replaced by

$$\langle\psi_1 | (L_n - \alpha(0)\delta_{n,0}) | \psi_2\rangle = 0 \qquad \text{for all } n,\qquad\text{(II.50)}$$

where $\alpha(0)$ is an arbitrary c number. This amounts to imposing constraints in the weak sense as is done for instance in the Gupta–Bleuler formalism in quantum electrodynamics.

Noticing that

$$L_n = -\frac{1}{4\alpha'}\sum_{-\infty}^{+\infty} :\alpha_{n-m}\alpha_m: = -pa_n(n)^{1/2} + \cdots \qquad \text{if } n > 0,\qquad\text{(II.51)}$$

$$L_{-n} = -pa_n^{+}(n)^{1/2} + \cdots \qquad \text{if } n > 0,\qquad\text{(II.52)}$$

it is natural to assume that the weak constraints are fulfilled by imposing the subsidiary conditions

$$(L_n - \alpha(0)\delta_{n,0}) | \psi_2\rangle = 0 \qquad \text{for } n \geq 0 \text{ only},\qquad\text{(II.53a)}$$

$$\langle\psi_1 | (L_{-n} - \alpha(0)\delta_{n,0}) = 0 \qquad \text{for } n \geq 0 \text{ only}.\qquad\text{(II.53b)}$$

Because of the indefinite metric we are not sure that the solutions of these equations contain no negative norm states. The problem will be solved in a subsequent section and it will be shown that it is true provided that

$$\alpha_0 = 1 \quad \text{if } D \leq 26 \quad \text{or} \quad \alpha_0 \leq 1 \quad \text{if } D \leq 25.$$

The time evolution of x_μ is given by the equation

$$i\dot{x}_\mu = [x_\mu, L_0].\qquad\text{(II.54)}$$

If we examine the mass spectrum and the spin spectrum in the covariant quantized version of the theory, we find that the leading trajectory is now given by $J = \alpha(0) + \alpha' M^2$ so that $\alpha(0)$ is the intercept of the leading trajectory. The covariance of this quantization procedure follows from the fact that the momentum operator P^μ and the angular momentum operator

$$M^{\mu\nu} = \tfrac{1}{2}\int_0^\pi d\sigma(x^\mu P_\tau{}^\nu + P_\tau{}^\nu x^\mu - x^\nu P_\tau{}^\mu - P_\tau{}^\mu x^\nu)\qquad\text{(II.56)}$$

obey the algebra

$$[P^\mu, P^\nu] = 0,\qquad\text{(II.57a)}$$

$$[P^\mu, M^{\alpha\beta}] = i(g^{\mu\alpha}P^\beta - g^{\mu\beta}P^\alpha),\qquad\text{(II.57b)}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} + g^{\mu\sigma}M^{\nu\rho} - g^{\nu\sigma}M^{\mu\rho}),\qquad\text{(II.57c)}$$

as follows from the canonical commutators of Eqs. (II.32) and (II.33). These operators commute with the gauge operator L_n and hence define Lorentz transformations and translation of the states satisfying the equations (II.53a–b).

4. Hamiltonian formalism in the transverse gauge and quantization

We have obtained previously the following equations:

$$x^+ = 2\alpha' P^+\tau, \qquad P_\tau{}^+ = P^+/\pi,\qquad\text{(II.58)}$$

which define the transverse gauge.

$$x_-' = (\pi/P^+)P_\tau{}^i x_i',$$
$$P_\tau{}^- = (1/2\pi P^+)\{\pi^2(P_\tau{}^i)^2 + [(x_i')^2/4\alpha'^2]\},\qquad\text{(II.59)}$$

which express the primary constraints.

$$\ddot{x}_i - x_i'' = 0, \qquad x_i' = 0, \qquad \text{for } \sigma = 0, \pi,$$
$$P_\tau{}^i = (1/2\pi\alpha')\dot{x}^i,\qquad\text{(II.60)}$$
$$\dot{q}^- = 2\alpha' P_-,$$

which are the equations of evolution of the independent variables q^-, P_+, $P_\tau{}^i$, x^i. The last equation can be solved so that

$$q^- = q_0{}^- + 2\alpha' P_-\tau.$$

$q_0{}^-$ is now an independent dynamical variable and we postulate the following Poisson brackets among these variables

$$\{x^i, x^j\} = \{P^i, P^j\} = 0,$$
$$\{x^i(\sigma,\tau), P^j(\sigma',\tau)\} = \delta^{ij}\delta(\sigma - \sigma'),$$
$$\{q_0{}^-, P^+\} = -1,$$
$$\{P_+, x^i\} = \{P_+, P_i\} = \{q_-, x_i\} = \{q_-, P_i\} = 0.\qquad\text{(II.61)}$$

It is easy to find the Hamiltonian of this system. It is

$$H = 2\alpha' P_+P_- = \pi\alpha' \int_0^\pi d\sigma[(P_\tau{}^i)^2 + (x'^i)^2/(2\alpha'\pi)^2]. \qquad \text{(II.62)}$$

To check this, one verifies that the equations of motion follow from the canonical Poisson brackets and the Hamiltonian formalism

$$\dot{f} = (\partial f/\partial\tau) + \{f, H\}. \qquad \text{(II.63)}$$

Expanding in normal coordinates we have

$$H = \alpha' P_i{}^2 + \sum_{n=1}^{\infty} n a_n{}^{*i} a_n{}^i = 2\alpha' P_+P_- \qquad \text{(II.64)}$$

so that

$$M^2 - \frac{1}{\alpha'}\sum_{n=1}^{\infty} n a_n{}^{*i} a_n{}^i = 0. \qquad \text{(II.65)}$$

It is now trivial to quantize the system: we assume canonical commutators for the operators $a_n{}^i$, $a_n{}^{+i}$, $q_0{}^-$, P^+, P^i, q_i

$$[a_n{}^i, a_m{}^{+j}] = \delta^{ij}\delta_{mn} \text{ etc}, \cdots. \qquad \text{(II.66)}$$

Now because of the disappearance of the time component the space of the vectors

$$\prod_{n=1}^{\infty} (a_n{}^+)^{\lambda_n} \mid 0\rangle$$

has positive definite metric. The states of the string are hence described by purely transverse oscillators.

The nontransverse oscillators are expressed in terms of the transverse one through the equations

$$\alpha_n{}^- = (1/P^+)[L_n - \delta_{n0}\alpha(0)], \qquad \text{(II.67)}$$

where the c number $\alpha(0)$ appears due to normal ordering ambiguities in L_0. Here L_n is the transverse one

$$L_n = \tfrac{1}{2}\sum \alpha_{n-m}{}^i\alpha_m{}^i.$$

5. Covariance of the transverse gauge

It is possible to implement Lorentz transformations and translations in the transverse gauge provided that one can define generators of the Poincare group satisfying the following commutation relations of Eqs. (II.57a–c). The natural choices for P^μ and $M^{\mu\nu}$ are of course, as in the relativistic quantization, the energy momentum and angular momentum of the string. However, difficulties arise because of the peculiar algebra of the $\alpha_n{}^-$ operators

$$[\alpha_n{}^-, \alpha_m{}^-] = (1/P^+)(n-m)\alpha_{n+m}{}^- + \delta_{n,-m}\{-(2n/P^+) \times \alpha(0) + [(D-2)/12(P^+)^2]n(n^2-1)\}, \qquad \text{(II.68)}$$

while

$$[\alpha_n{}^i, \alpha_m{}^j] = n\delta^{ij}\delta_{n,-m}. \qquad \text{(II.69)}$$

Let us forget for a second about quantization and just consider Poisson brackets instead of commutators. Then in Eq. (II.68) only the first term on the right hand side survives and one can check that $M^{\mu\nu}$ and P^ν verify the algebra of Eqs. II.5 through Poisson brackets.

Coming back to the quantum case we have to verify the Lorentz algebra for the commutators of $M^{\mu\nu}$. It turns out that all the commutators give the expected result, except for: $[M^{i-}, M^{j-}]$ which should be zero. This commutator is the hardest one to evaluate explicitly because

$$M_{i-} = \tfrac{1}{2}(q^ip^- + p^-q^i) - q^-p^i - i\sum_{n=1} n^{-1}(\alpha_{-n}{}^i\alpha_n{}^- - \alpha_{-n}{}^-\alpha_n{}^i). \qquad \text{(II.70)}$$

This computation is very lengthy and we shall not describe it here. However we see that some anomalous terms proportional to $D-2$ and α_0 will arise because of the commutator [II.68]. Further when one commutes for instance

$$[\alpha_{-n}{}^i\alpha_n{}^-, \alpha_{-m}{}^j\alpha_m{}^-] = \alpha_{-n}{}^i[\alpha_n{}^-, \alpha_{-m}{}^j]\alpha_m{}^- + [\alpha_{-n}{}^i, \alpha_{-m}{}^j]\alpha_m{}^-\alpha_n{}^- + \alpha_{-n}{}^i\alpha_{-m}{}^{-j}[\alpha_n{}^-, \alpha_m{}^-] + \alpha_{-m}{}^j[\alpha_{-n}{}^i, \alpha_m{}^-]\alpha_n{}^-$$

and evaluate all these commutators, one would find cancellations if the order of terms was disregarded and anomalous c number ignored. However to obtain the desired cancellations some operators have to be commuted again which makes an essential difference between Poisson bracket computation which involves only the first stage and commutation which involves two stages. Finally, one obtains

$$[M^{i-}, M^{j-}] = \frac{2}{P^{+2}}\sum_{m=1}\left[m\left(1 - \frac{1}{24}(D-2)\right) + \frac{1}{m}\left(\frac{1}{24}(D-2) - \alpha(0)\right)\right](\alpha_{-m}{}^i\alpha_m{}^j - \alpha_{-m}{}^j\alpha_m{}^i). \qquad \text{(II.71)}$$

Therefore for arbitrary values of D and $\alpha(0)$, the theory is non-covariant. If we require covariance we have to set $D = 26$, $\alpha(0) = 1$. $\alpha(0) = 1$ can be understood easily because transversality for a vector meson requires it to be massless. However $D = 26$ has no such physical interpretation. The only mathematical reason for $D - 2 = 24$ is that in computing the commutator of the L_n a fourth order pole at $x = y$ occurs which introduces a 1/4! In a refinement of the Veneziano model where N quantum numbers are introduced, 26 is replaced by $26 - N$. [Clavelli and Shapiro (1973), Schwarz (1973), Bardakci and Halpern (1971)].

If we still think that the quantized string makes any sense, the fact that it can be quantized only in some peculiar dimension of space–time gives to the problem a kind of bootstrap aspect: Covariance and positiveness of the metric for an extended system determine the dimension of space-time. Problem: find out which extended system can be quantized for $D = 4$!

The choice $\alpha(0) = 1$ is more unfortunate because it implies the existence of a tachyon at $\alpha' M^2 = -1$. As we shall see by studying the Neveu–Schwarz model, tachyons are not an unsolvable problem in dual models. Several possibilities are open: First one can try to find a model with no tachyon. Second, tachyons may well disappear when the interaction between strings is included, as well as massless particles.

We have seen indeed that striking differences occur when the classical string is first quantized. It is reasonable to think that second quantization (interaction among strings) may again change things. It turns out that things seem to change for the better rather than for the worse and that not only is the second quantized theory consistent for $D = 26$, $\alpha_0 = 1$ but that it may cure some of its own problems (namely $\alpha(0) = 1$). The main justification for continuing our study of dual models rather than leave it there is that we shall see features emerging which seem independent from D, α_0 and have direct comparison with experiment.

6. No-ghost theorem in the covariant quantization

We now wish to show that the covariant quantization of the string is also ghost-free. The Fock space of the states with which we are working is not positive definite and is spanned by the states:

$$| r \rangle = \prod_{n=1}^{\infty} \prod_{\mu_n=0}^{d-1} (a_{n,\mu_n}{}^{+})^{\lambda_{n,\mu_n}} | 0, p \rangle.$$

The number operator is

$$R = -\sum_{n=1}^{\infty} n a_{n,\mu_n}{}^{+} a_n{}^{\mu_n} \qquad \text{and} \qquad R | r \rangle = M | r \rangle,$$

where

$$M = \sum_{n=1}^{\infty} \sum_{\mu_n} n \lambda_{n,\mu_n} \tag{II.72}$$

is the *level number* $M = 0, 1, 2, \cdots$.

We shall call R^M the space of states at the level M. Obviously this space contains both positive and negative norm states.

The *physical* states in the covariant quantization are these states which satisfy

$$L_n | \phi \rangle = 0, \qquad n \geq 1 \tag{II.73}$$

in addition to the *mass shell* condition

$$(L_0 - 1) | \phi \rangle = 0. \tag{II.74}$$

We set $\alpha(0) = 1$ for the moment and shall see later if it can or cannot be relaxed. As we have seen these constraints come from the orthonormality of the gauge which we have chosen. An on-shell state belonging to R^M has its mass given by

$$p^2 = 2(M - 1). \tag{II.75}$$

The physical subspace of R^M may or may not have positive definite norm. We shall prove that it does provided that $D \leq 26$.

Spurious states are those states belonging to R^M of the form

$$| S \rangle = \sum_{n=1}^{M} c_n L_{-n} | S_n \rangle. \tag{II.76}$$

They are called spurious because they are orthogonal to any physical state: if $| \phi \rangle$ is physical, and $| S \rangle$ is spurious then $\langle S | \phi \rangle = 0$.

We shall first build a subspace of R^M called *transverse*[4] subspace which has obviously a positive definite norm. This will be our first step towards proving the no-ghost theorem. The transverse states are defined as follows:

The momentum p of the states we shall talk about is constrained by $p^2 = 2(N - 1)$. We choose a frame where

$$p = [N/(2)^{1/2}, 0, \cdots, 0, (2)^{1/2}(1 - \tfrac{1}{2}N)].$$

We then pick the lightlike vector

$$k = (1/(2)^{1/2}, 0, \cdots, 0, -1/(2)^{1/2})$$

such that $k \cdot p = 1$. We shall call

$$K_n = k \cdot \alpha_n = k(n)^{1/2} a_n, \qquad K_n{}^{+} = K_{-n}.$$

We now define transverse states through the conditions

$$L_n | t \rangle = K_n | t \rangle = 0, \qquad n \geq 1. \tag{II.77}$$

It is clear that the transverse states have positive semidefinite norm. Indeed, in light cone coordinates

$$K_n = (n)^{1/2} a_{n,+} = \alpha_{n,+}.$$

And we have the commutators

$$[\alpha_{n,+}, \alpha_{m,+}] = [\alpha_{n,-}, \alpha_{m,-}] = 0,$$
$$[\alpha_{n,+}, \alpha_{m,-}] = -n \delta_{n,-m}. \tag{II.78}$$

So in light cone coordinates the last commutator creates negative norm states while the two first create zero-norm states.

Using $K_n | t \rangle = 0$ we deduce that $| t \rangle$ does not contain any $a_{n-}{}^{+}$ creation operator, and hence its norm is positive or zero. It is more difficult to prove that because of the L_n conditions, the transverse states have strictly positive norm and we need two lemmas to establish this.

Lemma 1.[5] Let us consider a transverse state $| t \rangle$ belonging to R^M, such that $\langle t | t \rangle \neq 0$. For such a $| t \rangle$ the states

$$| \{\lambda, \mu\}, f \rangle = L_{-1}{}^{\lambda_1} L_{-2}{}^{\lambda_2} \cdots L_{-n}{}^{\lambda_n} K_{-1}{}^{\mu_1} \cdots K_{-m}{}^{\mu_m} | t \rangle, \tag{II.79}$$

[4] Transverse states were originally introduced by Del Giudice, Fubini and Di Vecchia (1972). In our treatment we shall not need their explicit form.

[5] This lemma and the next one were derived by Goddard and Thorn (1972).

where

$$\sum r\lambda_r + \sum s\mu_s = N - M > 0$$

form a linear independent span of some subspace of R^N. Further, this subspace contains no solutions of

$$L_n \mid \phi\rangle = K_n \mid \phi\rangle = 0.$$

Proof:

(a) Suppose that we have a linear relation between the vectors $\mid \{\lambda, \mu\}, t\rangle$

$$[\sum_{\{\lambda,\mu\}} c[\lambda, \mu] L_{-1}^{\lambda_1} \cdots L_{-n}^{\lambda_n} K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m}] \mid t\rangle = 0 \tag{II.80}$$

$\mid t\rangle$ being obtained by applying creation operators on the vacuum. If we expand $L_{-1}^{\lambda_i}$ in terms of creation and annihilation operators the part made with creation operators must cancel with each other necessarily. If we pick the a_1^+ oscillators,

$L_{-1}^{\lambda_1} \cdots L_{-n}^{\lambda_n} K_{-1}^{\mu_1}$ contributes a term of the form

$$(p \cdot a_1^+)^{\lambda_1} (a_1^+ \cdot a_1^+)^{\lambda_2} (a_1^+ \cdot a_2^+)^{\lambda_3} \cdots (a_1^+ \cdot a_{n-1}^+)^{\lambda_n} (k \cdot a_1^+)^{\mu_1}$$

and these terms have to cancel with each other. Consider the terms in the sum which maximize $\lambda_1 + 2\lambda_2 + \lambda_3 + \cdots + \lambda_n + \mu_1$. When expanded, each will yield a term which will maximize this number, plus additional terms. These maximal terms have to cancel with each other. But obviously they are independent unless all the terms which maximize the above quantity have the same $\lambda_1, \lambda_2, \cdots \lambda_n, \mu_1$. But if this is the case there is no cancellation possible, and we therefore conclude that $\lambda_1 = \cdots = \lambda_n = \mu_1 = 0$. It is then obvious that all $C\{\lambda, \mu\}$ are zero.

(b) Let us now prove that this subspace contains no solution of $L_n \mid \phi\rangle = K_n \mid \phi\rangle = 0$. $n > 0$.

$$[L_m, K_n] = -nK_{m+n}, \tag{II.81a}$$

$$[K_m, K_n] = 0, \tag{II.81b}$$

$$[L_n, L_m] = (n-m)L_{n+m} + (D/12)n(n^2-1)\delta_{n,-m}. \tag{II.81c}$$

Let us apply first K_n on a state $\mid \{\lambda, \mu\}, t\rangle$. So we use $K_n L_{-m} = L_{-m} K_n + nK_{n-m}$.

Commuting K_n through the $L_{-m}^{\lambda_m}$ we produce some further K_r where $r < m$. We move them to the right until those K_r which have $r > 0$ will annihilate on $\mid t\rangle$. Some K_r with $r < 0$ will have been created in the process: The L_{-n} may not be any more in the order of Eq. (II.79) but we can bring them back in this order by using the commutation relations of Eq. (II.81c). And so we see that

$$K_n \mid \{\lambda, \mu\}, t\rangle = \sum_{\lambda',\mu'} C\{\lambda', \mu'\} \mid \{\lambda', \mu'\}, t\rangle,$$

where $\mu' = \sum s\mu_s' \geq \mu = \sum s\mu_s$. If $n = 1$ the only commutator which does not produce K_r with $r < 0$ is $[K_1, L_{-1}] = 1$. For this commutation μ is not increased.

Let us now consider a superposition of $\mid \{\lambda, \mu\}, t\rangle$ vectors (with $\mid t\rangle$ fixed) and ask that this superposition be annihilated by all K_n. The vectors in the superposition with maximal μ will be mapped into vectors with greater value of μ, which then have to cancel by themselves. This is impossible because of the independence theorem already proved and hence we conclude that $\lambda_2 = \lambda_3 = \cdots = \lambda_n = 0$. Applying again K_1 on a term which contains only $L_{-1}^{\lambda_1}$ and K_{-n} operators, we see again that $\lambda_1 = 0$ is the only solution. So we conclude that

$$\mid \phi\rangle = \sum_{\{\mu\}} c\{\mu\} K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m} \mid t\rangle. \tag{II.82}$$

Applying again now L_n of $\mid \phi\rangle$ we see that because

$$[L_n, K_{-m}] = mK_{n-m} \qquad \text{and} \qquad L_n \mid t\rangle = 0$$

we shall obtain a superposition of $\mid \{0, \mu\} t\rangle$ vectors where μ' will be increased. Then the only solution is $\mu_1 = \cdots = \mu_m = 0$ which is forbidden, since in our definition of $\mid \{\lambda, \mu\}, t\rangle$ states we have set $\sum r\lambda_r + \sum s\mu_s > 0$.

In this first lemma, what we have done is the following: Because of the commutation relations $[R, L_{-n}] = nL_{-n}$ and $[R, K_{-n}] = nK_{-n}$ L_{-n} and K_{-n} raise the eigenvalue of R by n in units. So starting from a transverse state belonging to R^M we have constructed an independent set of vectors belonging to $R^N: N = \sum r\lambda_r + \sum s\mu_s + M > M$; these vectors are obviously orthogonal to all transverse vectors of R^N. Further, the subspace of these vectors does not contain any transverse vector itself.

In order to obtain a complete basis for R^N we still have to vary $\mid t\rangle$ and this is done in the following lemma.

Lemma 2.

If $\mid t, M, \nu\rangle$ is an orthonormal basis for T^M, the states $\mid \{\lambda, \mu\}, t, M, \nu\rangle$ defined in Eq. (II.79) give a basis for the states with $R = \sum r\lambda_r + \sum s\mu_s + M = N$ and as N varies for the whole Fock space. Further T^M is positive definite.

Proof:

$\mid t, N, \nu\rangle$ is a transverse state belonging to R^N. The subspace of the transverse states is called T^N. Let us call G^N the set of all $\mid \{\lambda, \mu\}, t, M, \nu\rangle$ belonging to R^N such that $\sum r\lambda_r + \sum s\mu_s > 0$. We construct T^N and G^N as follows:

We start from the vacuum, i.e., $N = 0$. It satisfies $L_n \mid 0\rangle = K_n \mid 0\rangle = 0$ for $n > 1$. Hence $T^0 = \{\mid 0\rangle\}$ $G^0 = \{0\}$. From them we construct $G^1 = \{L_{-1} \mid 0\rangle, K_{-1} \mid 0\rangle\}$ and build its orthogonal complement. It is readily seen to be made of states annihilated by L_1, K_1 and is found to be

$$T^1 = \{a_{1,i}^+ \mid 0\rangle; \qquad i = 1, 2, \cdots, D-2\}.$$

From T^0, T^1 we build now $G^2 = \{L_{-1} a_{1,i}^+ \mid 0\rangle, L_{-2} \mid 0\rangle, K_{-1} a_{1,i}^+ \mid 0\rangle, K_{-2} \mid 0\rangle\}$ and construct its orthogonal complement T^2 and so on: Let us make sure that this construction is indeed possible and that at each stage T^N is made up of transverse states. We do it by recursion and assume that we have been able to construct $T^0, T^1, \cdots T^{N-1}$ and $G^0, G^1, \cdots$

G^{N-1} which is true for $N = 1, 2$. We now attempt to construct T^N, G^N.

G^N is constructed by raising all the states of $T^0, \cdots, T^{N-1}$ by L_{-n} and K_{-n} operators as done before. All the states thus obtained are $| \{\lambda, \mu\}, t, M, \nu\rangle$ states. They are all independent. We have shown this in the above Lemma for fixed $| t, M, \nu\rangle$. However, states with different $| t, M, \nu\rangle$ can never be dependent because it is readily seen[6] that

$$\langle\{\lambda', \mu'\}t', M', \nu' \mid \{\lambda, \mu\}, t, M, \nu\rangle = \delta_{MM'}\delta_{\nu\nu'}f(\lambda, \lambda', \mu, \mu', M, \nu).$$

This is due to the fact that the L_{-n}, K_{-n} can be commuted to the left and K_m, L_m to the right until they annihilate and the only surviving contributions will come from c number or L_0 operators which are diagonal in $| M, \nu\rangle$ [in M it is obvious, in ν because we have selected an orthonormal basis for $T^0, \cdots T^{N-1}$ by hypothesis]. So the $| \{\lambda, \mu\}, t, M, \nu\rangle$ states form a linearly independent basis of G^N. We have used here the recursion hypothesis that T^M is made of transverse states.

We now construct T^N as the orthogonal complement of G^N and show that it is made of transverse states. Since $| t, N, \nu\rangle$ belongs to T^N it has to be orthogonal to all vectors $| \{\lambda, \mu\}, t, M, \nu\rangle$ because of their independence just proved. Hence:

$$\begin{aligned}\langle t, N, \nu \mid & L_{-1}{}^{\lambda_1}L_{-2}{}^{\lambda_2}\cdots L_{-n}{}^{\lambda_n}K_{-1}{}^{\mu_1}\cdots K_{-m}{}^{\mu_m} \mid t, M, \nu'\rangle \\ &= \langle t, N, \nu \mid L_{-1}[L_{-1}{}^{\lambda_1-1}L_{-2}{}^{\lambda_2}\cdots L_{-n}{}^{\lambda_n} \\ &\quad \times K_{-1}{}^{\mu_1}\cdots K_{-m}{}^{\mu_m} \mid t, M, \nu'\rangle = 0 \qquad \text{(II.83)}\end{aligned}$$

for all $\lambda_1, \cdots \lambda_n, \mu_1, \cdots \mu_m, M, \nu$. The vectors

$$L_{-1}{}^{\lambda_1-1}L_{-2}{}^{\lambda_2}\cdots K_{-m}{}^{\mu_m} \mid t, M, \nu'\rangle$$

form a complete basis for R^{N-1} by hypothesis, and hence $\langle t, N, \nu \mid L_{-1} = 0$ identically. To prove now that

$$\langle t, N, \nu \mid L_{-2} = 0$$

we use the fact that by commuting L_{-n} with each other another independent basis of G^N is obtained by ordering the L_{-n} in the order $L_{-2}{}^{\lambda_2}L_{-1}{}^{\lambda_1}\cdots L_{-n}{}^{\lambda_n}$, (in fact, any order will do) and because we have an independent basis for R^{N-2} we conclude that $\langle t, N, \nu \mid L_{-2} = 0$. Hence

$$\langle t, N, \nu \mid L_{-n} = 0$$

by repeated commutations of L_{-1}, L_{-2}, and then we show easily that $\langle t, N, \nu \mid K_{-1}{}^{\mu_1}\cdots K_{-m}{}^{\mu_m} \mid t, M, \nu'\rangle = 0$ implies $\langle t, N, \nu \mid K_{-m} = 0$.

Hence the orthogonal complement of G^N, T^N is made up of transverse states. We still have to ensure a point which has been used in the recursion hypothesis, namely we can construct an orthonormal basis for T^N. This will be ensured if we can show that no zero norm vector occurs in T^N. We know already that T^N is positive semidefinite. In such a case it is easy to show that if $| t\rangle$ is a *zero norm* state belonging to a semi positive Hilbert space, then it has to be orthogonal to all the other states of this space, i.e., it has to be a *null state*. If there was such a state in T^N, it would also belong to its orthonormal complement G^N, i.e., in G^N there would be a transverse state. But by Lemma 1 we know that this is impossible and hence T^N contains no zero norm states and hence we can apply the orthonormalization procedure to it, which closes the recursion hypothesis. Hence the Lemma 2. In the wording of the Lemma 2 the $| \{\lambda, \mu\}, t, M, \nu\rangle$ states include the states where $\lambda_i = \mu_i = 0$, i.e., transverse states.

Hence we have been able to construct for each eigenvalue of R, a positive definite subspace T^N. Also we have constructed a complete basis for $R^N = T^N \oplus G^N$. This is so far independent of D and on the mass shell condition as well.

We now go back to the problem of the norm of *physical* states which at each level, span a subspace of R^N, which we shall call $P^N \supseteq T^N$. To compute their norm, it is useful to devise an operator which will project any state of R^N into T^N. This projection operator[7] is useful for many purposes and it is the reason why we construct it now rather than directly proving the no ghost theorem.[8] We shall first derive its form in a heuristic fashion and then check that it has all the required properties.

Physical states are such that $\langle\phi \mid (L_n - \delta_{n,0}) \mid \phi'\rangle = 0$ for all n. In terms of the operator $P(z)$ of which the L_n are Laurent coefficients, it means that $\langle\phi \mid (:P^2(z): + 2) \mid \phi'\rangle = 0$. If $:P^2(z): + 2 = 0$ was imposed as an operator statement it would mean that we are working with the transverse gauge and that: $2:P^+P^-: - :P_iP^i: = -2$. In this gauge P^+ as we have seen is completely determined in terms of P^i, P^- by the relation

$$P^+ = (1/2P^-)[-2 + :P_iP^i:] = P_{+,T}, \qquad \text{(II.84)}$$

where the T index refers to the transverse gauge.

Introducing the Laurent expansion of an arbitrary operator $x(z)$

$$x(z) = \sum_{-\infty}^{+\infty} \langle x(z)\rangle_n z^{-n}, \qquad \text{(II.85)}$$

$$\langle x(z)\rangle_n = \oint (dz/2i\pi z) z^n x(z). \qquad \text{(II.86)}$$

We see that in the transverse gauge the total momentum of the string in the $+$ direction, P_+ equals

$$P_{+,T} = \langle P_+(z)\rangle_0 = (1/2P^-)\langle[-2 + :P_iP^i:]\rangle_0. \qquad \text{(II.87)}$$

In the covariant quantization, the total momentum operator of the string along the $+$ direction is given simply by

$$P_+ = \langle P_+(z)\rangle_0 = \langle 2P_+(z)P_-(z)/2P_-(z)\rangle_0. \qquad \text{(II.88)}$$

[6] For the detailed reasoning, see Brower and Thorn (1971).

[7] It was derived by Brink and Olive (1973).

[8] The no-ghost theorem could now easily be proved using the method of Goddard and Thorn (1972). The original proof of the no ghost theorem, due to Brower (1972) follows a different line of reasoning.

Obviously the two operators $P_{+,T}$ and P_+ are different. However, since the two ways of quantizing the string should give the same physical result there should be subspaces of states in each R^N such that the matrix elements of $P_{+,T} - P_+$ vanish in these subspaces. Hence we form the following operator[9]

$$P_{+,T} - P_+ = \langle[1/2P_-(z)][-2 - 2{:}P_+(z)P_-(z){:} + {:}P_i(z)P^i(z){:}]\rangle_0$$
$$= \langle[1/P_-(z)][-1 - \tfrac{1}{2}{:}P^2(z){:}]\rangle_0. \quad \text{(II.89)}$$

But

$$-\tfrac{1}{2}{:}P^2(z){:} = \sum_{-\infty}^{+\infty} L_n z^{-n}, \quad \text{(II.90)}$$

and

$$1/P^-(z) = \sum_{-\infty}^{+\infty} D_n z^{-n}, \quad \text{(II.91)}$$

where

$$D_n = \oint (dz/2i\pi z) z^n [1/P^-(z)]. \quad \text{(II.92)}$$

Hence

$$P_{+,T} - P_+ = -D_0 + \sum_{-\infty}^{+\infty} D_{-n} L_n. \quad \text{(II.93)}$$

We have been very careless about normal ordering. Let us first check that the D_n coefficients are well defined. We see that

$$P^-(z) = k\cdot P(z) = 1 + \sum_{n=1}^{\infty} (K_{-n} z^n + K_n z^{-n})$$

because $k\cdot P = 1$. Hence we can set $P(z) = 1 + \epsilon$, and expand in powers of ϵ. The integration on z will pick up a finite number of terms for each power of ϵ and hence D_n is well-defined. Further, if we consider D_n with $n > 0$, we see that D_n is given by a series of terms each of which contains at least one K_m with $m > 0$. Hence, since $[K_i, K_j] = 0$, $D_n | 0\rangle = 0$, if $n > 0$. Similarly $(D_0 - 1) | 0\rangle = 0$. So if we require that $P_{+,T} - P_+$ vanishes on $| 0\rangle$, even off-the-mass shell, we are led to rewrite Eq. (II.93) in the following order

$$E = P_{+,T} - P_+ = (D_0 - 1)(L_0 - 1) + \sum_{n=1}^{\infty} (D_{-n} L_n + L_{-n} D_n). \quad \text{(II.94)}$$

where we have added a term $-(L_0 - 1)$ which vanishes on the mass shell. From now on, we shall work with this operator E and forget about the heuristic argument which led to it.

The algebra of the operators L_n, D_m together with the K_n closes

$$[L_n, L_m] = (n - m) L_{n+m} + (D/12) n(n^2 - 1) \delta_{n,-m}, \quad \text{(II.95a)}$$

$$[L_n, D_m] = -(2n + m) D_{n+m}, \quad \text{(II.95b)}$$

$$[D_n, D_m] = 0, \quad \text{(II.95c)}$$

$$[L_n, K_m] = -m K_{m+n}, \quad \text{(II.95d)}$$

$$[K_m, K_n] = 0, \quad \text{(II.95e)}$$

$$[K_m, D_n] = 0. \quad \text{(II.95f)}$$

Evaluating now the commutators of L_n, K_n, D_n with E yields

$$[L_n, E] = -nL_n + [(D - 26)/12] n^2 (n - 1) D_n, \quad \text{(II.96a)}$$

$$[K_n, E] = -nK_n, \quad \text{(II.96b)}$$

$$[D_n, E] = -nD_n. \quad \text{(II.96c)}$$

On a transverse state belonging to T^N we see readily that E vanishes

$$E | t, N, \nu\rangle = 0. \quad \text{(II.97)}$$

This is a consequence of the fact that each $D_0 - 1$, D_n contains K_i operators with $i > 0$ and hence annihilate $| t, N, \nu\rangle$, and that $L_n | t, N, \nu\rangle = 0$. Hence we have indeed found a subspace T^N on which $E = 0$. In order to find the other eigenvectors of, E we apply it on a basis of R^N

$$E L_{-1}^{\lambda_1} \cdots L_{-n}^{\lambda_n} K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m} | t, M, \nu\rangle.$$

Because of the D_n term in Eq. (II.96a) this cannot be computed unless $D = 26$. If we assume $D = 26$, using $(L_{-n} - n)E = EL_{-n}$ and $(K_{-n} - n)E = EK_{-n}$ we obtain

$$-(\sum_r r\lambda_r + \sum s\mu_s) L_{-1}^{\lambda_1} \cdots L_{-n}^{\lambda_n} K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m} | t, M, \nu\rangle.$$

So in the case $D = 26$ and in this case only we can assert that.

1. The eigenvectors of E span the whole Fock space.
2. The eigenvalues of E are negative integers.
3. The eigenvalue zero corresponds to, and only to, the transverse subspace.

It is now very easy to construct a projection operator onto the transverse subspace. Define

$$\mathfrak{I} = \oint (dy/2\pi i) y^{E-1}. \quad \text{(II.98)}$$

It is clear that (II.98) is a Hermitian projection operator unto the transverse subspace at each level N. We see easily that for each quantity $X_n = L_n$, K_n, D_n one has [for $D = 26$] $X_n E = (E - n) X_n$ and from this relation, one deduces

$$X_n \mathfrak{I} = \oint (dy/2i\pi y) y^{E-n} X_n. \quad \text{(II.99)}$$

but $E \leq 0$ and hence for $n \geq 1$ the integral on the right

[9] This derivation follows the work of Ramond (1973).

hand side vanishes

$$X_n\mathfrak{J} = 0 = \mathfrak{J}X_{-n} \qquad \text{if } n > 0. \tag{II.100}$$

Also, expanding

$$1 = \langle P^-(z)/P^-(z)\rangle_0 = \sum_{-\infty}^{+\infty} D_{-n}K_n$$

$$= \sum_1^\infty D_{-n}K_n + K_{-n}D_n + D_0, \tag{II.101}$$

where we have again used $K_0 = k\cdot p = 1$, we get

$$D_0 - 1 = \sum_1 D_{-n}K_n + K_{-n}D_n \tag{II.102}$$

and

$$(D_0 - 1)\mathfrak{J} = 0. \tag{II.103}$$

And hence $E\mathfrak{J} = 0$ so that

$$\mathfrak{J}^2 = \mathfrak{J}. \tag{II.104}$$

So this proves that $\mathfrak{J}$ is a projection operator, correctly normed, on the subspace $\mathfrak{J}^N$.

We now prove the no-ghost theorem for $D = 26$. Let us consider an on-shell physical state $|\,\phi\rangle$, belonging to P^N. Let us now compute $\langle\phi\,|\,\mathfrak{J}\,|\,\phi\rangle$. Because $\mathfrak{J}$ projects onto T^N, $\langle\phi\,|\,\mathfrak{J}\,|\,\phi\rangle \geq 0$. On the other hand $\langle\phi\,|\,\mathfrak{J}\,|\,\phi\rangle$ can be evaluated in the following way: Let us compute

$$\langle\phi\,|\,(y^E - 1)\,|\,\phi\rangle = \langle\phi\,|\,E\int_1^y dzz^{E-1}\,|\,\phi\rangle$$

$$= \langle\phi\,|\sum_{n=1}^\infty D_{-n}L_n\int_1^y dzz^{E-1}\,|\,\phi\rangle$$

$$= \langle\phi\,|\int_1^y dzz^{E-1}\sum_{n=1}^\infty D_{-n}L_n\,|\,\phi\rangle = 0. \tag{II.105}$$

Integrating over y this result we get

$$\langle\phi\,|\,\phi\rangle = \langle\phi\,|\,\mathfrak{J}\,|\,\phi\rangle \geq 0. \tag{II.106}$$

And this proves the absence of negative metric states among physical states. Note that we had to make use of the mass shell condition and the L_n conditions on both sides. For two different physical states we have also

$$\langle\phi_1\,|\,(\mathfrak{J} - 1)\,|\,\phi_2\rangle = 0. \tag{II.107}$$

And this expresses the completeness of transverse states with R^N. Transverse and physical states are not quite identical however. If $|\,\phi\rangle \in P^N$ we can write

$$|\,\phi\rangle = \mathfrak{J}\,|\,\phi\rangle + (1 - \mathfrak{J})\,|\,\phi\rangle. \tag{II.108}$$

Here $\mathfrak{J}\,|\,\phi\rangle$ belongs to T^N while $(1 - \mathfrak{J})\,|\,\phi\rangle$ is still a physical state, however it clearly is orthogonal to T^N and belongs to G^N; in fact, it is a zero norm state which is orthogonal to all physical states of P^N

$$\langle\phi'\,|\,(1 - \mathfrak{J})\,|\,\phi\rangle = 0.$$

Such states have the form

$$L_{-1}\,|\,t, N-1, \nu\rangle \qquad \text{or} \qquad (L_{-2} + \tfrac{3}{2}L_{-1}^2)\,|\,t, N-2, \nu\rangle$$

as one can check explicitly [Brink, Olive, Scherk (1973)].

Let us see if we can now enlarge the result of the theorem by relaxing some of the assumptions. First if $D < 26$, i.e., $D = 26 - N$ we can add N additional oscillators to the Fock space to enlarge it again. Then the no-ghost theorem for $D = 26$ ensures us that the physical states of the Fock space which do not have the N additional excitations have positive norm. Hence the no-ghost theorem holds for $D \leq 26$. However, the completeness of transverse states between physical states is no longer true: we have also additional "longitudinal" states.

$D > 26$ leads to negative norm states as can be seen by checking that

$$|\,\phi\rangle = (L_2 + \tfrac{3}{2}L_1^2)\,|\,\phi\rangle$$

is a physical state and its norm is

$$\langle\phi\,|\,\phi\rangle = 13 - (D/2).$$

So $D = 26$ is the upper bound above which the no-ghost theorem fails. For $D < 26$ we can also relax the assumption $\alpha(0) = 1$ by considering those states (let us fix $D = 4$ for instance) for which there are no excitations of $a_{n,i}{}^+$, $i = 5, 6,\cdots 26$, however, there is a conserved non zero 5th momentum p_4 which can be thought of as a conserved quantum number. Then the mass shell condition on such states $|\,\phi\rangle$ becomes

$$(L_0 - 1)\,|\,\phi\rangle = (R - \hat{p}^2/2 - 1)\,|\phi\rangle$$
$$= (R - p^2/2 - 1 + p_4{}^2/2)\,|\,\phi\rangle = 0,$$

where

$$\hat{p} = (p_0,\cdots, p_4, 0,\cdots),$$
$$p = (p_0,\cdots, p_3, 0,\cdots).$$

And the intercept of the leading trajectory is $\alpha(0) = 1 - p_4{}^2/2$ which can be chosen arbitrarily. Since we can add only a spatial component $p_4{}^2 \geq 0$ we have $\alpha(0) \leq 1$.

It is clear however that these modifications are somehow artificial and in fact we shall see that in spite of this excursion into a more realistic world we are led back in the following to our $D = 26$, $\alpha(0) = 1$ world rather quickly.

7. The closed string

In the preceding sections we have dealt with the theory of open strings. However we could have also considered closed strings, i.e., rings. The only difference in their mathematical treatment is the absence of the boundary condition of Eq. (I.18). Integrating σ from $[0, 2\pi]$, the new boundary condition, which expresses that the string is closed is now

$$x_\mu(\tau, \sigma + 2\pi) = x_\mu(\tau, \sigma),$$

and the action reads

$$S = (-1/4\pi\alpha') \int_{\tau_1}^{\tau_2} d\tau \int_0^{2\pi} d\sigma \{(\dot{x}\cdot x')^2 - x'^2\dot{x}^2\}^{1/2}.$$

The expansion of $x_\mu(\sigma, \tau)$ now contains both sines and cosines

$$x_\mu(\sigma, \tau) = q_\mu + p_\mu \tau + i \sum_{n\neq 0} [\exp(-in\tau)/n](\alpha_{n,\mu} \cos n\sigma + \bar{\alpha}_{n,\mu} \sin n\sigma).$$

We have two sets of gauge conditions

$$L_n = \bar{L}_n = 0,$$

where in the definition of L_n, $\bar{L}_n$ one uses

$$\alpha_0{}^\mu = \bar{\alpha}_0{}^\mu = (p^\mu/2)(2\alpha').$$

If α' is the same quantity as the one appearing in Eq. (I.13), one shows easily that for a closed classical string $J \leq (\alpha'/2)M^2$. One can see that easily by taking two rigidly rotating strings, sticking them together at the ends: the spin of the system doubles, the $(\text{mass})^2$ is multiplied by 4 and we have obtained a closed string.

When the theory is quantized one has two sets of mutually commuting annihilation and creation operators, $\alpha_n{}^\mu$, $\bar{\alpha}_m{}^\mu$. The gauge conditions now read

$$L_n \mid \psi\rangle = \bar{L}_n \mid \psi\rangle = 0, \qquad n \geq 1,$$
$$(L_0 - L_0) \mid \psi\rangle = 0,$$

and the mass shell condition

$$[L_0 + \bar{L}_0 - \alpha(0)] \mid \psi\rangle = 0.$$

The resulting model is the Virasoro–Shapiro model [Virasoro (1969), Shapiro (1970)]. It is ghost-free if $D = 26$, but now $\alpha(0) = 2$. Hence the theory contains a massless "graviton." A projection operator on the transverse states of this model can also be constructed. (Olive and Scherk (1973)).

Because one also cannot attach quantum numbers to a closed string it will play a particular role in reactions where vacuum quantum numbers can be exchanged in one channel. It may then well have some relation with the Pomeron singularity appearing in these processes. Experimentally, the Pomeron trajectory has slope $\alpha_p'(0) \approx .4$ $(\text{GeV})^{-2}$ below 100 GeV. At higher energies it decreases but we cannot expect the tree approximation to be valid there anyhow. So it agrees qualitatively with the $\frac{1}{2}\alpha'$ slope we have obtained. However of course $\alpha_p(0) \approx 1$ while we have obtained $\alpha_p(0) = 2$ in this model. Obviously this is the same type of difficulty we have met before and it is still unsolved.

Closed strings play an important role even in the open string theory. As we shall see the interaction for open strings is obtained by allowing them to break and join at the extremities. Then obviously a closed string can open itself into an open string and vice-versa. It turns out that when radiative corrections are computed they exhibit poles corresponding to this kind of transition. Hence open strings cannot be considered by themselves: even if we define the tree diagrams of the theory in terms of them, closed strings will appear in loop diagrams. [Olive, Scherk (1973a)].

III. THE INTERACTING STRING

1. Rules for dual diagrams

Ultimately we shall be interested in a dual theory working for $D = 4$ dimensions. So the normalization coefficients which follow have been derived for that case only. We choose the following normalization of the fully connected part of the S-matrix [Bogoliubov and Shirkov (1959)]

$$(p_1, \cdots, p_j \mid S \mid p_{j+1}, \cdots, p_N) = i(2\pi)^4 \delta^4\Big(\sum_{i=1}^{j} p_i - \sum_{k=j+1}^{N} p_k\Big) \prod_{i=1}^{N} ((2\pi)^3 2p_i{}^0)^{-1/2} \times T_N(p_1, \cdots, p_N). \tag{III.1}$$

To compute the scattering amplitudes T_N, we shall define a propagator for the string

$$D(p^2) = -\frac{i\alpha'}{(2\pi)^4} \frac{1}{L_0 - 1 - i\epsilon}. \tag{III.2}$$

This propagator is analogous to the Feynman propagator

$$\frac{-i}{(2\pi)^4} \frac{1}{m^2 - p^2 - i\epsilon},$$

since

$$L_0 - 1 = \alpha'[M^2 - p^2],$$

where

$$M^2 = (1/\alpha') \sum_{n=1}^{\infty} n a_n{}^+ a_n - (1/\alpha')$$

is the $(\text{mass})^2$ operator. So whenever p^2 equals one of the eigenvalues of the $(\text{mass})^2$ operator, the propagator blows up. The $i\epsilon$ prescription has been introduced in accordance with the Feynman rules. The propagator can be represented in two different graphical ways: In Fig. 1a) we remember that this is a string propagator and we represent the two boundaries of that string. The string itself spans through

FIG. 1. Dual propagator represented as: (a) a string propagator, (b) a solid line.

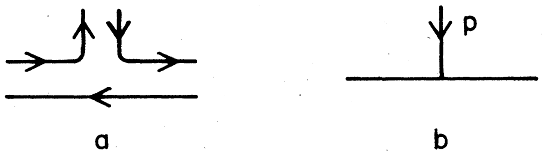


FIG. 2. Dual vertex represented: (a) in terms of quark lines, (b) as the absorbtion of a scalar.

its evolution the blank space in between. Diagrams built in accordance with this rule are usually called dual diagrams.[10] The orientation of the lines is added in order to be able to compute the factors arising from an internal $SU(N)$ symmetry as we shall see below.

In Fig. 1(b) the string picture has been forgotten and we simply represent the propagator by a line. One obtains then a set of diagrams called Feynman like diagrams (FLD's). Now we need to introduce a vertex for a 3 string interaction. It is difficult to do so directly and historically the vertex to emit the scalar ground state from a string line was first introduced (usually one speaks of "Reggeon" lines instead of string lines for historical reasons)

$$\mathcal{V}(p_i) = i[g/(\alpha')^{1/2}](2\pi)^4 V(p_i). \quad \text{(III.3)}$$

$V(p_i)$ shall be a dimensionless operator constructed from the a_n, a_n^+ operators and the zero-mode operator, and depending on the momentum p_i of the incoming scalar. Two possible representations of the vertex are shown in Fig. 2 (a) in terms of quark lines, or as (b) where only the absorption of a scalar is indicated on the F.L.D. In the following we shall work only with the dimensionless quantities $1/(L_0 - 1)$ and $V(p)$ and leave all normalization factors aside, and set $\alpha' = 1/2$ for convenience. We shall first consider tree-graphs, i.e., those which do not contain any loops, and more particularly "multiperipheral" trees

$$B_N = g^{N-2}\langle 0, p_1 \mid V(p_2)[1/(L_0 - 1)] \times V(p_3)\cdots[1/(L_0 - 1)]V(p_{N-1}) \mid 0, p_N\rangle$$

$\mid 0, p_N\rangle$ is an incoming ground state scalar on the right.
$\langle 0, p_1 \mid$ is an incoming ground state scalar on the left.

So that the graph is represented by Fig. 3(a) or by Fig. 3(b). Such an expression satisfies obviously the requirements of factorizability of the particle poles. The essential ingredient in the determination of the vertex is that it should satisfy Ward identities guaranteeing the absence of ghosts. The distinction between ghosts and non-ghost resonances holds only on-the-mass shell: so in order to see if the above amplitude contains ghost states in a given channel, say the channel where particles $(1, 2, \cdots i) \to (i + 1, \cdots N)$ we go to a particular pole in that channel, and require that $\pi_i = (p_1 + p_2 + \cdots + p_i)$ be constrained by $\pi_i^2 = 2(N - 1)$, where N is an integer. The residue of the pole in that channel is then given by the expression

$$R_N = \langle \psi_1 \mid \mathfrak{M}_N \mid \psi_2\rangle, \quad \text{(III.4)}$$

where

$$\mathfrak{M}_N = \oint (dx/2i\pi x) x^{L_0 - 1} \quad \text{(III.5)}$$

is the projector onto the Nth level, and

$$\mid \psi_2\rangle = V(p_{i+1})\cdots V(p_{N-1}) \mid 0, p_N\rangle, \quad \text{(III.6)}$$

$$\langle \psi_1 \mid = \langle 0, p_1 \mid V(p_2)\cdots V(p_i). \quad \text{(III.7)}$$

Defining

$$\mid \psi_1'\rangle = \mathfrak{M}_N \mid \psi_1\rangle, \quad \mid \psi_2'\rangle = \mathfrak{M}_N \mid \psi_2\rangle \quad \text{(III.8)}$$

R_N can be rewritten as

$$R_N = \langle \psi_1' \mid \psi_2'\rangle. \quad \text{(III.9)}$$

Since $\mathfrak{M}_N^2 = \mathfrak{M}_N$. The absence of ghosts will follow if we can prove that $\mid \psi_1'\rangle, \mid \psi_2'\rangle$ are physical states, i.e., that

$$L_n \mid \psi_i'\rangle = 0, \quad n \geq 1, \quad (L_0 - 1) \mid \psi_i'\rangle = 0, \quad i = 1, 2. \quad \text{(III.10)}$$

The second identity is obvious since $(L_0 - 1)\mathfrak{M}_N = 0$. So we have to require that

$$L_n \mathfrak{M}_N \mid \psi\rangle = 0. \quad \text{(III.11)}$$

Since

$$L_n \mathfrak{M}_N = \mathfrak{M}_N{}^n L_n,$$

where

$$\mathfrak{M}_N{}^n = \oint (dx/2i\pi x) x^{L_0 + n - 1}$$

and

$$\mathfrak{M}_N{}^n (L_0 + n - 1) = 0,$$

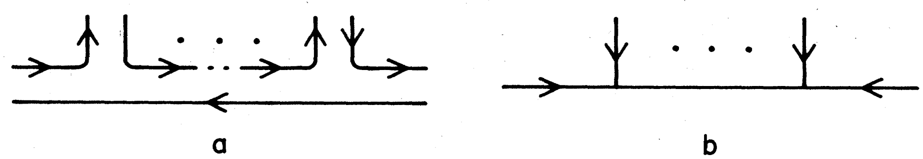
we are led to require that the $\mid \psi\rangle$ vectors satisfy the Ward identities

$$L_n \mid \psi\rangle = (L_0 + n - 1) \mid \psi\rangle, \quad n \geq 1. \quad \text{(III.12)}$$

Introducing

$$W_n = L_0 - L_n + n - 1$$

FIG. 3. Multiperipheral diagram represented: (a) in terms of quark lines, (b) as a Feynman-like diagram.



[10] Dual diagrams were originally introduced by Harari (1969) and Rosner (1969) and are a generalization of Zweig's "twig" diagram describing mesonic decays). (Zweig (1964)).

we must now look for a vertex operator $V(p)$ such that it satisfies nice commutation relations with the gauge operators W_n. The following vertex satisfies these requirements

$$V(p_j) = :\exp - ip_j\cdot Q(1): \tag{III.13}$$

where $Q(z)$ is the position operator defined in Eq. (II.37).

Commuting the gauge W_n through $V(p_j)$ yields

$$[W_n, V(p_j)] = (n/2)p_j^2. \tag{III.14}$$

And commuting again W_n through the adjacent propagation yields

$$[W_n, V(p_j)[1/(L_0-1)]] = V(p_j)[1/(L_0-1)]n(1+p_j^2/2). \tag{III.15}$$

Hence the gauge operators can be commuted past a vertex and a propagator if $p_j^2 = -2$, i.e., if all absorbed scalars are ground state tachyons on the mass shell. If this is satisfied,

$$W_n\,|\,\psi\rangle = 0, \qquad n > 0 \tag{III.16}$$

provided that

$$W_n\,|\,0, p_N\rangle = 0$$

which again implies the mass shell condition. Notice that we can generalize Eq. (III.16) by replacing $|\,0, p_N\rangle$ by any physical on-shell state of the string $|\,A\,\rangle$ without losing the gauge conditions.

It is now very easy to evaluate the amplitudes B_N. One uses coherent state techniques.

$$|f\rangle = \exp(fa^+)\,|\,0\rangle$$

is a coherent state, and satisfies

$$(1)\ \ a\,|f\rangle = f\,|f\rangle, \tag{III.17}$$

$$(2)\ \ \exp(ga^+)\,|f\rangle = |f+g\rangle, \tag{III.18}$$

$$(3)\ \ \langle f\,|\,g\rangle = \exp(f^*g), \tag{III.19}$$

$$(4)\ \ x^{a^+a}\,|f\rangle = |\,xf\rangle. \tag{III.20}$$

Each of the $N-3$ propagators is now written as

$$\int_0^1 \frac{dx_i}{x_i}\, x_i^{L_0-1} = \int_0^1 dx_i x_i^{-\alpha(s_i)-1} x_i^{-\Sigma n a_n^+ a_n}, \tag{III.21}$$

where

$$s_i = (p_1 + p_2 + \cdots + p_{i+1})^2, \qquad i = 1, 2, \cdots, N-3.$$

Using the properties of coherent states and taking the vacuum expectation of the zero mode one obtains

$$\langle 0, p_1\,|\,V(p_2)x_1^R V(p_3)\cdots x_{N-3}^R V(p_{N-1})\,|\,0, p_N\rangle = \prod_{1\le i<j\le N} (1-x_{ij})^{-p_i p_j} \tag{III.22}$$

where $x_{ij} = x_{i-1}x_i\cdots x_{j-2}$ is the product of the variables contained between i and j. So one obtains the following expression of the N-point function

$$B_N = g^{N-2}\int_0^1 \prod_{i=1}^{N-3} dx_i x_i^{-\alpha(s_i)-1} \prod_{1\le i<j\le N} (1-x_{ij})^{-p_i\cdot p_j}. \tag{III.23}$$

For $N = 4$ we obtain the Veneziano formula

$$\begin{aligned} B_4 &= g^2\int_0^1 dx x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1} \\ &= g^2 B(-\alpha(s); -\alpha(t)) \\ &= g^2[\Gamma(-\alpha(s))\Gamma(-\alpha(t))/\Gamma(-\alpha(s)-\alpha(t))], \end{aligned} \tag{III.24}$$

where

$$s = (p_1+p_2)^2, \qquad t = (p_2+p_3)^2.$$

2. Duality

The four-point function we have obtained is symmetric under the exchange of s and t. This was not imposed *a priori* but it is a consequence of our choice of vertices. Our FLD's hence have properties which are very much unlike those of ordinary Feynman graphs (see Fig. 4). In field theory both diagrams have to be added and contribute to the S-matrix. In dual models, only one of them must be included because it contains these two diagrams in a "hidden" way. The properties of tree diagrams with more than four legs are even more amazing. One can show that B_N is invariant under a *cyclic* interchange of the momenta $p_1, \cdots, p_N$. Further, B_N has all the singularities that one may deduce by using the rule of the last figure for internal lines, keeping the order of the external less fixed. For an example, see Fig. 5. The set of these properties is what one usually means by *duality*. Dual amplitudes were originally derived with $\alpha(0) \neq 1$. The modification is to include in the integrand of Eq. (III.23) a factor

$$\prod_i (1-x_i)^{\alpha(0)-1}$$

and to remember that $p_i^2 = -[\alpha(0)/\alpha']$. The proofs of the properties of duality were carried directly on the integral representations of the N point function.

Because of the property of duality, the counting of Feynman graphs is very different from the counting of FLD's: since each Feynman graph type of singularity can be obtained from an FLD drawn in multiperipheral form,

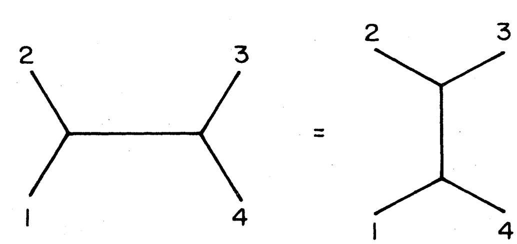


FIG. 4. The basic duality equation in graphical form.

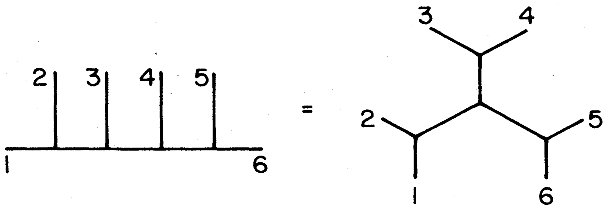


FIG. 5. Example of a duality transformation.

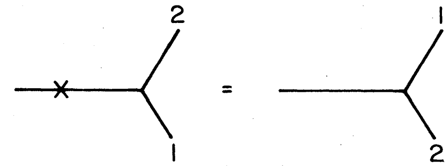


FIG. 7. Graphical representation of the action of the twist operator.

it is enough to obtain the whole amplitude (in the tree approximation) to sum over noncyclic permutations of the external legs of multiperipherally drawn FLD's

$$T_N = \sum_{\text{Non-cyclic}} B_N(\mathcal{P}p_1, \mathcal{P}p_2, \cdots \mathcal{P}p_N)$$

To a given FLD correspond many different FG's. Each FLD can be decomposed in a sum of

$$(2n-4)!/(n-1)!(n-2)!$$

subgraphs which have each the singularity structure of one, and only one F.G. [Nakanishi (1971)].

Duality is an obvious consequence of the underlying string picture: if we trust our set (a) of graphical representation of the Feynman rule, then simply by enlarging or shortening the space between the lines we obtain all the properties of duality. (That is why strings were also called "rubber bands" by Susskind (1970). Duality is a property which has a direct meaning in terms of graphs, and does not necessarily depend on the underlying operator formalism. So the properties of the compatibility of duality with factorization and absence of ghosts have to be checked at each state. The test is fully successful for tree graphs: one can define 3-Reggeon vertices operators [Sciuto (1969), Caneschi, Schwimmer and Veneziano (1969)] such that joining them with propagators one obtains N-Reggeon vertices [see the review of Alessandrini *et al.* (1971) for their definition] such that when the matrix elements of these N-Reggeon vertices are taken between ground states, one obtains the N point functions B_N. So equalities like those of Fig. (5) can be given a precise meaning in terms of the operators. The matrix elements of the N Reggeon vertices between *physical* states removes the last distinction between internal and external (scalar) lines: the external particules become any physical excited state of the string and this is a kind of *bootstrap* property where the scattering of a set of particles reproduce these same particles as bound states in all resonant channels. (Again, it is not surprising if we think in terms of the string.) The compatibility between the duality and absence of ghosts requires $\alpha(0) = 1$ and can be checked in all channels. Since the tree amplitudes are not explicitly D dependent we can assume $D \leq 26$, but in loop diagrams we shall be forced back to $D = 26$.

We have introduced the word "twist" which we shall have to use again in our discussion for loop diagrams. We shall not give its technical definition in terms of operators which can be found in Alessandrini *et al.* (1971), Schwarz (1973). Let us just say that in terms of the string it does represent a twist of the "rubber band" (see Fig. 6) and in operator language it is an operator which reverses the order of lines on its right (see Fig. 7).

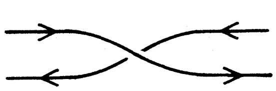


FIG. 6. Graphical representation of the twist in terms of quark lines.

An additional very pretty feature of tree diagrams is their behavior at high energy, where they exhibit Regge behavior rather than fixed pole behavior. This property does not depend on D or $\alpha(0)$, and distinguishes sharply dual models from field theories where Regge behavior can be obtained only by infinite sums of graphs.

By studying the high-energy behavior of N point functions one can show that it is completely compatible with Regge theory and that further one may get insights as to how a Reggeon calculus can be constructed. One can also study the inclusive cross section by computing discontinuities of the six point function and one exhibits a gaussian cutoff $\exp - 4\alpha' p^2$ in the transverse momentum, which qualitatively agrees with experiment at intermediate energies but is too strong at ISR and NAL energies: perhaps the loops are responsible for correcting this.

3. Quantum numbers

In the above dissucusion we have neglected to introduce quantum numbers. Duality takes it full predictive power when coupled with the requirement of the absence of exotic resonances; to be sure, this statement is a phenomenological one: it may have to be revised someday. However, as indicated above, we accept this requirement that no resonances occur in exotic channels. This distinguishes very strongly exotic and non-exotic channels. In an exotic channel the imaginary part of the amplitudes is zero in the tree approximation and of order g^2 when loops are included: exotic resonances may appear in a dual model when all loops are added, but they do not appear at any finite order of perturbation theory.

We now look for a mathematical formulation which can be consistent with the above FLD's rules and embodies the no-exotics requirement. We have assumed previously that all mesons have a $q\bar{q}$ content, i.e., belong to the 1 or 8 representation of $SU(3)$. Hence to each of the external mesons we associate a well-defined combination of the $\lambda_i[i = 0, 1, \cdots 8]$ matrices of $SU(3)$. We include the λ_0 matrix which is proportional to the unit matrix.

For each vertex we multiply the operator contribution by $(\lambda_i)_{\alpha\beta}$ and to each quark line one associates the contraction $\delta_{\alpha\beta}$. From this rule it results that the expression of Eq. (III.23) is multiplied by the factor: (Chan and Paton (1969)).

$$\tfrac{1}{2}\,\mathrm{Tr}[\lambda_1\lambda_2 \cdots \lambda_N] \tag{III.25}$$

(the factor $\frac{1}{2}$ is introduced here for normalization purposes). This choice has the following advantages:

1. Because a product of λ matrices is a λ matrix and that:

$$\mathrm{Tr}[\lambda_i\lambda_j] = 2\delta_{ij} \quad \text{(III.26)}$$

one can show that

$$\tfrac{1}{2}\,\mathrm{Tr}[\lambda_{a_1}\cdots\lambda_{a_n}] = \sum_{k=0}^{8} \tfrac{1}{2}\,\mathrm{Tr}[\lambda_{a_1}\cdots\lambda_{a_\mu}\lambda_k] \times \tfrac{1}{2}\,\mathrm{Tr}[\lambda_k\lambda_{a_{\mu+1}}\cdots\lambda_{a_N}]. \quad \text{(III.27)}$$

This, together with the factorization property of B_N shows that:

$$F_N = \tfrac{1}{2}\,\mathrm{Tr}[\lambda_1\cdots\lambda_N]B_N$$

is also factorizable and that the intermediate states also belong to the representation 1 and 8 of $SU(3)$. Hence no exotic resonances appear in the tree approximation. If all particles are identical, we retain crossing symmetry also.

This treatment of the symmetry properties can be in fact applied to a general $SU(N)$ group. Because in Eq. (III.36) k runs from 0 to 8 at each mass level we find both a singlet and an octet. An immediate consequence is the existence of degenerate singlet/octet trajectories which is well observed in nature (for instance $\rho - f$ degeneracy). This is what is called exchange degeneracy. In dual models exchange degeneracy is exact at the tree level, but breaks down at the loop levels as one would expect. So the symmetry group in dual models should, strictly speaking, be called $U(N)$ rather than $SU(N)$.

In the special case of $SU(2)$ because

$$\tau_a\tau_b = \delta_{ab} + i\epsilon_{abc}\tau_c$$

one can restrict the spectrum in the following way: let all external states be isospin 1 ρ mesons. Then we see that in even waves we shall obtain isospin 0 and in odd waves isospin 1 when we sum over noncyclic permutations of the external lines. This spectrum looks more like the experimental one, but it leads to trouble in loop diagrams: one has to introduce non orientable diagrams where mesons have a qq rather than a $q\bar{q}$ structure. Hence in what follows we shall keep the spectrum following from $U(N)$ and this will lead to the disappearance of these troublesome graphs. (Shapiro (1971)).

The assumption of duality and absence of exotics leads to many predictions in good agreement with experiments. These are described in the article by Jacob (1970) which we invite the reader to consult.

IV. LOOP DIAGRAMS

1. Loop diagrams and projection factors

In complete analogy with the Yang–Mills theory, or the quantized theory of gravitation it is not possible to define the loops in the dual model simply by using the propagators and vertices which we have used so far. The reason is that the Feynman rules have been derived in a particular gauge and that the disappearance of unphysical states from tree diagrams which is due to gauge invariance does not guarantee that these states do not appear in the loops if we use the same set of Feynman rules. In fact to ensure unitarity one has to modify the Feynman rules by introducing the so called Faddeev-Popov ghosts. (1967). An alternative derivation of the rules found by Faddeev and Popov is offered by Feynman's tree theorem. [Feynman (1972)]. Let us sketch very briefly Feynman's argument for a single-loop diagram containing scalar particles. Each propagator Δ_F can be expressed in terms of the retarded Δ_R and the Δ_+ Green's function so that:

$$\Delta_R = \Delta_F + \Delta_+$$

where

$$\Delta_R = 1/(p^2 - m^2 - i\epsilon p^0), \qquad \Delta_F = 1/(p^2 - m^2 + i\epsilon),$$
$$\Delta_+ = 2\pi i\theta(p^0)\delta(p^2 - m^2). \quad \text{(IV.1)}$$

If we consider a loop where all propagators are retarded, its value is obviously zero. Expressing then each propagator in terms of $\Delta_F + \Delta_+$ we obtain an identity which relates the loops containing all Δ_F propagators to a sum of "loops" containing at least one Δ_+. These "loops" are in fact, as is easily seen, integrals over tree diagrams, since the "cut" lines (those containing Δ_+) are on-the-mass shell. Feynman's proposal is to use this identity to *define* loops in the Yang–Mills or in quantized gravity. Now the entering and leaving particles on the "cut" lines are not only specified by their momenta and quantum numbers but also by their polarization tensors. Since we want to include in the unitarity sum only those states which appear in the tree diagrams, we have to include only the transverse states. The result is identical to the one obtained by Faddeev and Popov.

Similarly in dual models we can project each cut line onto the subspace of the transverse states using the Brink-Olive projection operator. This will restrict us to work in 26 dimensions rather than four. Although it is also possible to define loops in less than 26 dimensions, they exhibit bad properties which will be discussed later. The cut dual Δ_+ propagator is equal to:

$$\Delta_+ = \sum_{n=0}^{\infty} - 4\pi i\theta(p^0)\delta(p^2 - 2(n-1)) \oint (dx/2i\pi x)x^{L_0-1}. \quad \text{(IV.2)}$$

Each Δ_F propagator is represented in a parametric form as

$$1/(L_0 - 1) = \int_0^1 (dx_i/x_i)x_i^{L_0-1}. \quad \text{(IV.3)}$$

The summation over all states leaving and entering a cut loop is a trace over all oscillator modes. So we are led to consider for instance the following expression where the line 1 has been cut, 2, 3, $\cdots N$ are uncut

$$\oint \frac{dy}{2\pi iy} \oint \frac{dx_1}{2i\pi x_1} \prod_{i=2}^{N} \int_0^1 \frac{dx_i}{x_i} \mathrm{Tr}[y^E x_1^{L_0-1}V_1 x_2^{L_0-1}V_2\cdots V_N]. \quad \text{(IV.4)}$$

Using only the expression of E, the commutations of L_n and D_m operators, $L_n y^E = y^{E-n} L_n$, and the commutation of L_n with the vertices, Brink and Olive (1973) were able to show that the above expression is equal to:

$$\oint \frac{dx_1}{2i\pi x_1} \prod_{i=2}^{N} \int_0^1 \frac{dx_i}{x_i} f^{-2}(w) \ \mathrm{Tr}[x_1^{L_0-1} V_1 x_2^{L_0-1} V_2 \cdots V_N], \tag{IV.5}$$

where $w = x_1 \cdots x_N$ and $f(w)$ is the partition function

$$f(w) = \prod_{n=1}^{\infty} (1-w^n)^{-1}. \tag{IV.6}$$

A nontrivial aspect of this result is that although the definition of $\mathfrak{I}$ is not covariant the result obtained is covariant. We see that the Feynman rule of the graph has been modified by the projection operator and this is analogous to the Faddeev and Popov construction of loops in the Yang-Mills theory. Collecting together the various terms obtained from Feynman's tree theorem, one obtains the correct unitary expression for the loop:

$$F_N = \int i[dk/(2\pi)^{26}] \prod_{i=1}^{N} \int_0^1 (dx_i/x_i) f(w)^{-2} \times \mathrm{Tr}[x_1^{L_0-1} V_1 \cdots x_N^{L_0-1} V_N]. \tag{IV.7}$$

It differs by the $f(w)^{-2}$ factor from what one would have guessed if we had applied the ordinary Feynman rules. The integration dk is performed over the 26 dimensional space.

2. Classification of loop diagrams

We shall restrict our discussion to one-loop graphs although classification theorems are well known also for N-loop diagrams. [Kikkawa, Sakita, Virasoro (1969), Kikkawa, Klein, Sakita, Virasoro (1970), Gross, Neveu, Scherk and Schwarz (1970a)]. It is particularly simple to work with the alternate string picture of our Feynman rules. A one loop diagram is then represented by a surface with one hole, and the external momenta enter at the boundary of the surface. Nonorientable diagrams corresponding to a Mobius strip disappear in a theory with $U(N)$ invariance since cutting the strip gives us a qq content rather than a $q\bar{q}$ content. Hence we need only to consider an annulus, with particles entering and leaving at each side of the annulus. Two cases can be distinguished:

(a) no momenta enters the inside annulus,
(b) momenta enter the inside annulus.

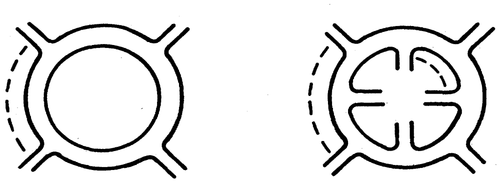


FIG. 8. (a) Planar and (b) nonplanar one-loop diagrams drawn in terms of quark lines.

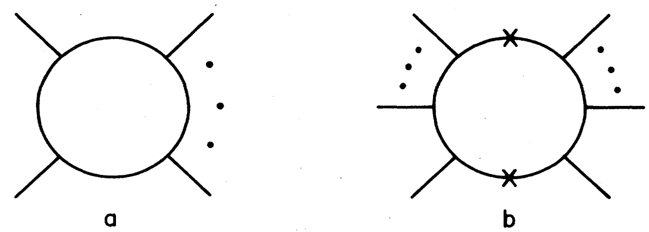


FIG. 9. (a) Planar, (b) nonplanar one-loop diagrams drawn as Feynman-like diagrams.

The first case is the planar, while the second is the non planar orientable loop. With our conventions (a) they are drawn as Fig. 8.

If we wish to draw this diagram in terms of our set (b) of Feynman rules we can do it in the second case by twisting the inside lines to the outside and obtain Fig. 9.

These diagrams are apt to be extremely misleading because of duality properties. For instance using the duality property several times one obtains the equalities displayed on Fig. 10. So in fact these graphs contain many Feynman graphs. If we define that two FLD's are equivalent if they are related by a duality transformation one can show for one loop, and more generally for n-loop diagrams that the correct unitary counting is to count each inequivalent FLD with weight one, a particularly simple result. (Gross, Neveu, Scherk and Schwarz (1970b), Frampton, Goddard and Wray (1971)). All the duality relations of Fig. 10 are easily checked from the expressions of the loops when the traces are computed. However the equivalence between the (a) and (b) representations is much more subtle and is discussed below. We shall concentrate first on the case of the nonplanar loop because the planar loop is a particular case of the non planar loop when no momenta are present on the inside boundary.

3. Evaluation of traces and integration

The integration over the momentum k is a Gaussian and hence can be performed in a region where we have no singularities and by performing a Wick rotation

$$\int d^D k x^{-\alpha' k^2} = i[\pi/-\alpha' \ln x]^{D/2} \tag{IV.8}$$

is the basis formula required for performing this integration.

The traces are computed by inserting inside the trace the unit operator written in terms of coherent states [Amati, Bouchiat, Gervais (1969)].

$$1 = \prod_{n=1}^{\infty} \int \frac{d^2 z_n}{\pi} \exp(-\sum_n |z_n|^2) \, |z_n\rangle\langle z_n|$$

and using the formula for the integral of a bilinear form (against a Gaussian!):

$$\int \frac{d^2 z}{\pi} \exp - [(z^* | (C-1) | z) + (z^* | A) + (B | z)] = \frac{1}{\det(1-C)} \exp - \left(B \left| \frac{1}{1-C} \right| A\right), \tag{IV.9}$$

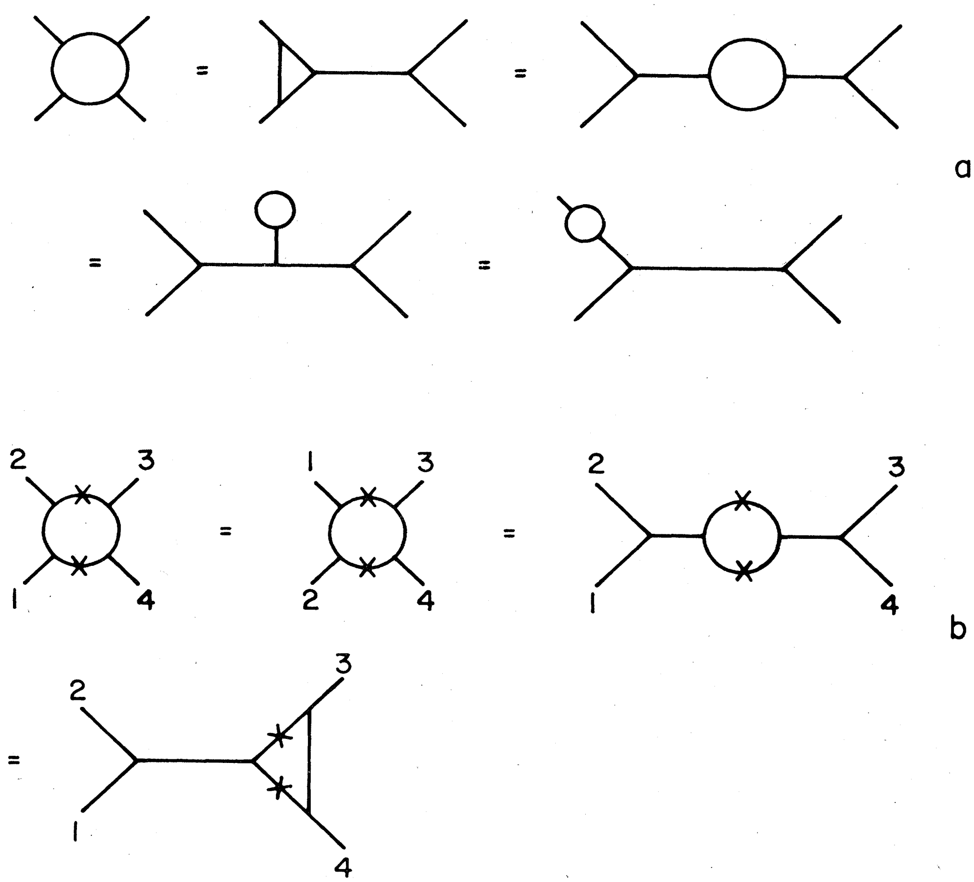


FIG. 10. Various duality identities verified by the following Feynman-like diagrams: (a) the planar loop, (b) the nonplanar loop.

where C is an infinite matrix, A, B are infinite vectors, and the integral is performed over all Z_n. For one-loop graphs, C is either diagonal or can be brought in a diagonal form. These two types of integrations combine with each other in a nice way so that one obtains the following expression for the loops: [Gross, Neveu, Scherk and Schwarz (1970b), Gross and Schwarz (1970)].

(a) *planar loop:*

$$F_N = (2\pi)^{13} g^N \int \prod_{i=1}^{N} dx_i w^{-2} \frac{f(w)^{24}}{(-\ln w)^{13}} \prod_{i\leq i<<j\leq N} \psi(C_{ji})^{-p_i\cdot p_j}, \tag{IV.10}$$

where

$$f(x) = \prod_{n=1}^{\infty} (1-x^n)^{-1}.$$

(b) *nonplanar loops:*

$$F_{N,M} = (2\pi)^{13} g^{N+M} \int \prod_{i=1}^{N} dx_i\, dy_j w^{-2} \frac{f(w)^{24}}{(-\ln w)^{13}} \times \prod_{i\leq i<j\leq N} [\psi(C_{ji}) \quad \text{or} \quad \psi_T(C_{ji})]^{-p_i\cdot p_j}. \tag{IV.11}$$

In the second case, we have N particles on the external boundary, M on the inside. The final expression, for the planar loop is a particular case of the second one for $M = 0$ C_{ji} is the product of the x and y variables between the legs i and j. The integration variable for the non planar loop has to be defined carefully.

Here $f(w)$ obtains raised at the power 24 rather than 26 due to the projection over physical states. The function ψ has to be used if p_i, p_j belong to the same boundary, ψ_T if they belong to different boundaries. They are expressed in terms of Jacobi θ functions [See Bateman (1953)]

$$\psi(x) = -2i\pi \exp \frac{\ln^2 x}{2\ln w}\, \theta_1\left(\frac{\ln x}{2\pi i}\middle|\frac{\ln w}{2\pi i}\right) \Big/ \theta_1'\left(0\middle|\frac{\ln w}{2\pi i}\right), \tag{IV.12a}$$

$$\psi_T(x) = 2\pi \exp \frac{\ln^2 x}{2\ln w}\, \theta_1\left(\frac{\ln x}{2\pi i}+\frac{1}{2}\middle|\frac{\ln w}{2\pi i}\right) \Big/ \theta_1'\left(0\middle|\frac{\ln w}{2\pi i}\right). \tag{IV.12b}$$

Under this form we can expand the parts of the integrand which do not contain logs in power series. One obtains then a series of cuts and one can verify perturbative unitarity. One can also check the duality equations of Fig. 10. The above loops do contain infrared divergences, and thresholds at negative (mass)^2 because of the tachyon, but we rather look for ultraviolet divergences. Clearly those must come from the $w = 1$ corner of the integration region because there the partition function blows up exponentially.

4. The Jacobi transformation

The Jacobi θ functions $\theta(\nu/\tau)$ have remarkable properties under the interchange of τ into $-1/\tau$. For instance

$$\theta_1(\nu \mid \tau) = i(1/\tau)^{1/2} \exp - i\pi(\nu^2/\tau)\theta_1 \left(\frac{\nu}{\tau}\bigg| -\frac{1}{\tau}\right).$$

This change of variable is equivalent to introducing the variable $q = \exp(2\pi^2/\ln w)$ which goes to zero when $w \to 1$ and angles $\theta_i = 2\pi \ln x_i/\ln w$. Using these properties of the θ functions, and transforming the partition function as well by use of the Hardy–Ramanujan formula

$$(-2\pi/\ln x)^{1/2} f(x) = x^{1/24} q^{-1/12} f(q^2)$$

and making these changes of variables one obtains a complementary expression for the loops: [Gross Neveu, Scherk and Schwarz (1970b), Cremmer and Scherk (1972)]

(a′) *planar loop:*

$$F_N = g^N \int_0^1 dq q^{-1-\alpha_p(0)} f(q^2)^{24} \int \prod_2^N d\theta_i \prod_{i<j} \bar{\psi}(\theta_i - \theta_j)^{-p_i \cdot p_j}, \tag{IV.13a}$$

(b′) *nonplanar loop:*

$$F_{N,M} = g^{N+M}(\pi/2)^{p^2} \int_0^1 dq q^{-1-\alpha_p(p^2)} f(q^2)^{24} \int_0^\pi d\theta$$

$$\times \int \prod_2^N d\theta_i \prod_2^M d\phi_i \prod_{i<j} (\bar{\psi} \text{ or } \bar{\psi}_T)^{-p_i \cdot p_j}, \tag{IV.13b}$$

where if p_i, p_j belong to the same boundary $\bar{\psi} = \bar{\psi}(\theta_i - \theta_j)$ or $\bar{\psi}(\phi_i - \phi_j)$; and if p_i, p_j belong to different boundaries, one uses $\bar{\psi}_T = \bar{\psi}_T(\theta + \phi_i + \theta_j)$; p is the sum of the momenta entering the outer boundary. Obviously $p = 0$ for the planar loop and again the planar loop is a particular case of the nonplanar loop. $\alpha_p(p^2)$ us defined as $\alpha_p(p^2) = 2 + \frac{1}{4}p^2$ (note that $\alpha' = \frac{1}{2}$). The functions $\bar{\psi}$, $\bar{\psi}_T$ are defined as

$$\bar{\psi}(\theta) = \sin\theta \prod_{n=1}^{\infty} \frac{(1 - 2q^{2n}\cos 2\theta + q^{4n})}{(1 - q^{2n})^2}, \tag{IV.14a}$$

$$\bar{\psi}_T(\theta) = \prod_{n=1}^{\infty} \frac{(1 - 2q^{2n-1}\cos 2\theta + q^{4n-2})}{(1 - q^{2n})^2}. \tag{IV.14b}$$

5. The Pomeron singularity and the closed string

Let us concentrate on the non planar loop first. In its new form it is very close now to the string picture we have drawn before: we can picture two concentric circles of radii q and 1, with the external momenta on the outside boundary characterized by the angles $\theta_1 = 0$, $\theta_2, \cdots \theta_N$; the internal momenta on the internal boundary of position given by the angles θ, $\theta + \phi_2$, $\theta + \phi_3, \cdots \theta + \phi_M$. The logs of $\bar{\psi}$ and $\bar{\psi}_T$ are precisely the Green's function of the two dimensional electrostatic problem of this annulus. Through functional formalism one obtains indeed the expression (IV.13) readily without Jacobi transformation. (Hsue, Sakita and Virasoro (1970)].

Let us see what happens when $q \to 0$. If $p^2 \leq -8$ the integrand converges. We can analytically continue in p^2 and find a series of *poles* in p^2 at $\alpha_p(p^2) = 2N$ $N = 0,1,\cdots$. Hence the nonplanar loop is *not ultraviolent divergent* at all! It is finite, and the ultraviolet divergences have been replaced by a series of poles. These poles are traditionally associated with the Pomeron in spite of the fact that the intercept is wrong $\alpha_p(0) = 2$, but after all so is the ρ intercept we started from. The reason is that the Chan Paton factors associated with the nonplanar loop are the product of the traces of the λ matrices of the outer boundary with the trace of the λ matrices on the inside boundary as one can see from Fig. 8: no quark goes from the inside to the outside of the graph: Hence the channel where this singularity occurs, has vacuum quantum numbers. Further, the channels dual to this channel have no resonances, and contain only pure background, as we see easily on Fig. 11.

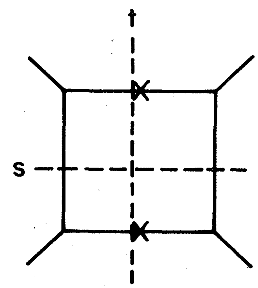


FIG. 11. s- and t-channel contents of the nonplanar box; the s channel exhibits reggeon and pomeron poles, the t-channel exhibits two-body unitarity cuts.

The s-channel of this graph contains the Pomeron poles and the t channel contains cuts and no resonances. This is in conformity with the Harari–Freund Ansatz. Note that it is almost a mathematical miracle that the Pomeron singularities turn out to be poles rather than logarithmic branch points. Lovelace (1971) was the first to notice that for this to happen, it was necessary to work in 26 dimensions, and assumed that the partition function appeared raised at the power 24 rather than 26, at a time where the Brink-Olive projection operator had not yet been found.

It remains now to have an interpretation of these new singularities. Obviously they are *bound states*. One must also show that these bound states are *factorizable, ghost-free* poles. Factorizability can be proved (Cremmer and Scherk (1972), Clavelli and Shapiro (1973)) by finding a proper propagator for the "Pomeron" state, $D_p(p^2)$ and a vertex for the transition between Reggeons and Pomerons such that the nonplanar loop now reads

$$F_{N,M} = \langle\psi_1 \mid \langle 0_{bc} \mid V(a^+, b, \bar{b}) \mid 0_a\rangle D_p(p^2)$$

$$\times \langle 0_a \mid V(a, b^+, \bar{b}^+) \mid 0_{bc}\rangle \mid \psi_2\rangle. \tag{IV.15}$$

The vertex V induces transition between the Reggeon space (characterized by the oscillators a_n, and the Pomeron space (characterized by the b_n, $\bar{b}_n$ oscillators). We shall not describe its rather complicated form.

$$D_p(p^2) = \int_0^1 (dq/q)\, f(q^2)^{-2} q^{L_0 + \bar{L}_0 - 2} \tag{IV.16}$$

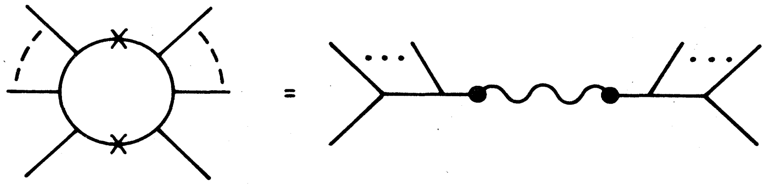


FIG. 12. Identity between the nonplanar loop and reggeon–pomeron–reggeon transitions.

is a strange propagator since it contains a partition function in it. The states $|\psi_1\rangle$ and $|\psi_2\rangle$ describe the external Reggeon trees ending with a propagator $1/L_0 - 1$.

The above equation has the graphical representation displayed on Fig. 12, where the wavy line is the Pomeron propagator, and the dot is the Reggeon–Pomeron vertex. It remains to be shown that this Pomeron sector is ghost free. Hence one has to find the Ward identities satisfied by the vertex V. Calling

$$\langle\phi_1| = \langle\psi_1|\,\langle 0_{bc}|\,V(a^+, b, \bar{b})\,|0_a\rangle,$$

one can show that $\langle\phi_1|$ satisfies the Ward identities

$$\langle\phi_1|\,L_{-n} = \langle\phi_1|\,\bar{L}_{+n}$$

for all n. This structure of Ward identities is very different from any we have worked with. It involves a pattern whereby an incident gauge L_{-n} of the Pomeron sector is partially transmitted through the vertex as a linear combination of L_{-m} gauges of the Reggeon sector, which annihilate on $\langle\psi_1|$, and is partially reflected as a L_n gauge to the right. This pattern is very similar to the one observed at the interface of two optical media of different indices.

It seems that the Pomeron sector is very similar to the Virasoro–Shapiro model (closed string) except that the gauges and the propagator are different. One can show that this difference is just a change of gauge: Let us exchange a closed string between the states $\langle\phi_1|$, $|\phi_2\rangle$; more precisely let us compute

$$R_M = \langle\phi_1|\,\mathfrak{I}_{vs} \oint (dq/2\pi i q)\, q^{L_0+\bar{L}_0-2}\,|\phi_2\rangle. \qquad \text{(IV.17)}$$

This is the residue at a certain pole: $p^2 = 8(M-1)$ of the exchange between $\langle\phi_1|$, and $|\phi_2\rangle$ of the *transverse* states of the Virasoro–Shapiro model, hence it is ghost free. Using the gauge identities satisfied by $\langle\phi_1|$, $|\phi_2\rangle$, one shows that: [Olive and Scherk (1973a)]

$$R_M = \langle\phi_1| \oint (dq/2\pi i q)\, f(q^2)^{-2} q^{L_0+\bar{L}_0-2}\,|\phi_2\rangle. \qquad \text{(IV.18)}$$

But this is precisely the residue of the Mth pole in the Pomeron sector of the amplitude (IV.15) and hence these residues are ghost free and saturated by the transverse states of the Shapiro–Virasoro model. So the "Pomeron" we have obtained is nothing else than the closed string and the vertex "Reggeon–Pomeron" is the vertex for opening a closed string into an open one and vice-versa. As amazing as it sounds it is not surprising if we have a blind faith in our set (a) of graphical rules. In Fig. 8 we see open strings arriving at the inside boundary, coalescing into a closed string which propagates to the outer boundary and breaks up finally into open strings. It is also possible to see in this figure an open string whose ends are attached on each of the boundaries propagating along the circle and closing upon itself. Hence the set (a) and (b) of graphical rules are really complementary pictures of each other. If we look at the last figure we see a loop diagram being equated with a tree diagram containing bound states. If we had worked with $D < 26$, none of this would have been true because the Pomeron singularity would have been made of cuts rather than poles and perturbative unitarity would have been violated. Hence the covariant quantization of the string which in principle allows us to work with $D \leq 26$ leads to inconsistency with unitarity at the one loop level if $D < 26$. On the other hand, if $D = 26$, perturbative unitarity is preserved, bound states appear where one may have expected divergences, and these bound states are just the excitation states of the closed string. So we are led to the equation

Bound states of Reggeons = Pomeron = Closed String.

A final point which is worth mentioning is that gauges also operate from the Reggeon to the Pomeron sector with a similar pattern of reflection and of transmission of an incident gauge. A consequence of these gauges is that the Reggeon states coupled to the Pomeron states are also transverse on the mass shell; This was not quite obvious *a priori* because we constructed the loop in such a way that only transverse Reggeons couple in the imaginary part. By duality transformation we obtain Fig. 12 on which it is not obvious that we have transverse Reggeons, but this is the case. So transversality and duality are still compatible with each other at the loop level. (Brink, Olive and Scherk (1973)).

6. The Pomeron and spontaneous symmetry breaking

Since bound states are created even at the one-loop level, there is a possibility that spontaneous symmetry breaking may occur because of the presence of bound state particles of zero mass. The analysis of the Pomeron sector at zero mass reveals the existence of a spin two particle ("graviton"), an antisymmetric tensor, and a scalar ("dilaton"). An interesting phenomenon occurs because of the presence of the antisymmetric tensor: it mixes by direct transitions with the singlet "photon" of the Reggeon sector to yield a massive antisymmetric tensor. If we were working with four dimensions, the antisymmetric tensor would be reducible to a scalar ("Goldstone boson") which, coupling directly to the massless "photon," would yield a massive vector boson. This mechanism of spontaneous symmetry breaking is a dynamical one since it occurs through bound states appearing from the unitarization of the theory. The $(\text{mass})^2$ acquired by the singlet photon is positive, of order g^2. If g^2 is large enough, one thus gets rid also of the singlet tachyon which is on the same trajectory as the isosinglet "photon." So we see that the nonplanar loop has reserved us a lot of interesting surprises. (Cremmer and Scherk (1973)). Similar conclusions were reached by Kalb and Ramond (1973), starting from the string formalism.

7. The planar loop

As one easily sees from Eq. (IV.10) the planar loop, contrary to the nonplanar, diverges exponentially when $w \to 1$. This would seem a hopeless disease if we did not know that under the alternate form of Eq. (IV.13a) the divergence is only quadratic $[\int_0 (dq/q^3)]$. Further, we know that the planar loop is just a particular case of the nonplanar one when no momentum enters the inside boundary,[11] and since the nonplanar loop is well defined, we should be able to make some sense out of the planar loop. Following the treatment of the nonplanar loop one can show that the planar one can be written as (Cremmer, Scherk (1972))

$$F_N = \langle \psi \mid \langle 0_{bc} \mid V(a^+, b, \bar{b}) \mid 0_a \rangle D_p(p^2 = 0) \times \frac{V}{g} (a = 0, b^+, \bar{b}^+) \mid 0_{bc} \rangle, \tag{IV.19}$$

where $\langle \psi \mid$ is a Reggeon tree. In this equation we see Reggeons being converted into a Pomeron line (closed loop), evaluated at zero momentum, which then is coupled through a particular vertex to the vacuum, i.e., it stops to propagate. Hence an alternate graphical representation of the planar loop is shown on Fig. 13. The $\times$ sign designates a coupling to the vacuum; since V is of order g, V/g is of order 1, and hence this is a large coupling. Again there is nothing surprising in this equality if we trust our string representation a) of the Feynman rules because we see there a closed string disappearing at the inside boundary. (See Fig. 8).

This equality shows that the loop is merely divergent because the propagator of the Pomeron is evaluated at $p^2 = 0$. So, performing an integration by part on q yields

$$F_N = \tfrac{1}{2} g^N \int_0^1 dq q^{-2} (\partial/\partial q) \{ f(q^2)^{24} \prod_{i<j} \bar{\psi}(\theta_i - \theta_j)^{-p_i \cdot p_j} \}. \tag{IV.20}$$

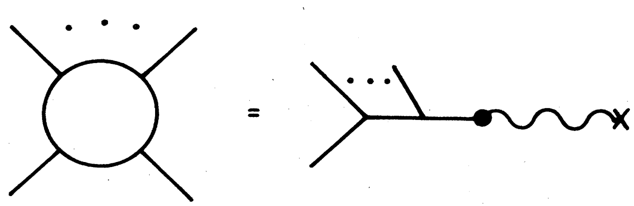


FIG. 13. Identity between the planar loop and reggeon–pomeron–vacuum transitions.

[11] The first attempt to treat the divergence of the planar loop by relating it to the nonplanar was due to Goddard (1971).

The term we have thrown away near $q = 0$ can be shown to be interpretable as a pure renormalization of the coupling constant g, and a wavefunction renormalization at the same time. (Neveu and Scherk (1972b)). To throw away such a term is the usual renormalization procedure which was originally proposed by Neveu and Scherk (1970) and Frye and Susskind (1970) when D was equal to 4; the result was finite and still unitary, crossing symmetric, dual and factorizable. It is possible to show that it is also compatible with the gauge identities.

However in the $D = 26$ case, we still have a logarithmic divergence present. This reflects the fact that at $p^2 = 0$ we have a scalar dilaton in the Pomeron sector, and this leads to a divergent propagator: $(1/p^2)\,|_{p^2=0}$. So we could stop there and make the following statement: The divergence of the planar loop is entirely due to the existence of massless particles in the model. Hence if we started with a realistic model having $\alpha_p(0) < 1$, $\alpha_p(0) < 2$, no such particles would appear and the theory would be finite.[12]

This impression is reinforced by the fact that in the Neveu–Schwarz–Ramond model cancellations occur between mesonic and fermionic loops such that only the logarithmic divergence remains [Green (1973)]. However there is also the tantalizing prospect that the argument could be followed the other way around and that a correct treatment of the divergence would lead to a model without massless particles (and maybe without tachyons). It is encouraging to see that such a mechanism exists in the Pomeron sector. The investigation into that direction is still preliminary, and is plagued by the nonexistence of a regularization scheme which makes the loop finite and keeps the gauge identities working as they should.

V. THE SPINNING STRING

1. Classical theory of a spinning string

We shall write directly the equations satisfied by a spinning string, rather than proceed from a Lagrangian formalism. In order to have the equations written in an elegant form, we work with the variables (ζ^0, ζ^1) rather than (τ, σ). Also we identify the position of a point along the string, x^μ, with a function of $\xi: x^\mu = \phi^\mu(\xi)$ which is meant to be a scalar with respect to coordinate transformations on the surface. We introduce the two-dimensional metric tensor

$$\eta^{ij} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the two-dimensional γ matrices satisfying

$$[\gamma^i, \gamma^j]_+ = 2\eta^{ij},$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

[12] Cremmer and Scherk, unpublished; this part also benefits from many stimulating discussions with D. Amati.

In addition to the field ϕ we introduce a spinor distribution on the surface

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

We shall take ψ to be Hermitian. Here ψ also has a Lorentz index, which shall not be exhibited in the following equations. ψ is an element of a Grassmann algebra such that it totally anticommutes with itself: $[\psi(\zeta), \psi(\zeta')]_+ = 0$, and commutes with ϕ. The equations of the free spinning string are: [Casher and Susskind (1971), Zumino (1973), Wess and Zumino (unpublished)]

$$\Box \phi = 0, \tag{V.1}$$

$$\gamma^i \partial_i \psi = 0, \tag{V.2}$$

$$\partial_i \phi \cdot \partial_j \phi + (i/4)\bar{\psi} \cdot (\gamma_i \partial_j + \gamma_j \partial_i)\psi = \lambda(\zeta)\eta_{ij}, \tag{V.3}$$

$$\gamma^i \gamma^j \partial_i \phi \cdot \psi = 0, \tag{V.4}$$

$$\partial_1 \phi = 0 \quad \text{for} \quad \zeta_1 = 0, \pi, \tag{V.5}$$

$$\psi_1(\zeta^0, 0) = \psi_2(\zeta^0, 0), \tag{V.6}$$

$$\psi_1(\zeta^0, \pi) = \epsilon\psi_2(\zeta^0, \pi), \qquad \epsilon = \pm 1. \tag{V.7}$$

If we set $\psi = 0$ in these equations, we see that (V.1) is identical to (I.43) (equation of motion of the scalar string in an orthonormal gauge), (V.3) is (I.39) (orthonormality condition), (V.5) is the usual edge condition (I.42), so that we recover all the equations of the previous model.

The new equations are the Dirac equation (V.2), the supergauge condition (V.5), the boundary conditions for the ψ field (V.6), and (V.7). In these boundary conditions if we set $\epsilon = +1$ we shall obtain the Ramond model [Ramond, (1971)] describing fermions of half-integer spin (in space–time), and if $\epsilon = -1$, we obtain the Neveu–Schwarz model describing integer spin mesons. [Neveu and Schwarz (1971a)].

Let us perform an infinitesimal conformal transformation $\delta\zeta^i = u^i$ where the infinitesimal parameter u_i satisfies

$$\partial_i u_j + \partial_j u_i - \eta_{ji}\, \partial^l u_l = 0. \tag{V.8}$$

The last equation is equivalent to Eqs. (I.68, I.69) which state that this transformation preserves the orthonormality condition, i.e., is a conformal change of coordinates. Under the conformal transformation if ϕ and ψ transform respectively as scalar/spinor, one can show that the set of the equations (V.1)–(V.4) are invariant. However there is a more general invariance of this set of equations than just the conformal transformations. The so-called supergauge transformations[13] also leave this set of equations invariant.

[13] Super gauges were originally introduced by Ramond (1971) in the fermionic model, and by Neveu, Schwarz and Thorn (1971) in the mesonic model. Supergauge transformations were introduced by Gervais and Sakita (1971).

They are defined as follows:

$$\begin{aligned} \delta\phi &= i\bar{\alpha}\psi, \\ \delta\psi &= \partial_i\phi\gamma^i\alpha + F\alpha, \\ \delta F &= i\bar{\alpha}\gamma^i\, \partial_i\psi, \end{aligned} \tag{V.9}$$

Here $\alpha(\zeta)$ is a two-dimensional spinor, Hermitian, totally anticommuting with itself and with ψ, commuting with ϕ and F. In the strict sense we cannot consider the above equations as classical, because they contain anticommuting quantities, but we shall nevertheless talk about a classical description, because commutators and anticommutators are zero. F is an auxiliary scalar field introduced to preserve a group structure; its equation of motion is simply $F = 0$. The parameter α of an infinitesimal supergauge transformation must satisfy the equation:

$$(\gamma_i\partial_j + \gamma_j\partial_i - \eta_{ij}\gamma^l\partial_l)\alpha = 0. \tag{V.10}$$

One can verify that the commutator of two supergauge transformations is a conformal transformation of parameter $u^i = 2i\bar{\alpha}_1\gamma^i\alpha_2$. The commutator of a supergauge transformation with a conformal transformation is a supergauge transformation. Hence supergauge transformations and conformal transformations form a Lie algebra whose parameters are anticommuting variables.

It is now clear why it is rather complicated to obtain a Lagrangian from which one can derive all the equations (V.1)–(V.7). It has to be invariant under generalized coordinate transformations and generalized supergauge transformations. This Lagrangian formalism does however exist [Wess and Zumino, in preparation, Iwasaki and Kikkawa (1973)]. As far as dual models are concerned one is more interested in the equations themselves although it is likely that it is the existence of an underlying Lagrangian structure which gives dual models their internal consistency.

Let us now solve the equations of motion and express the constraints in a manageable form. As before, equations (V.1) and (V.5) are solved by

$$\phi_\mu = q_\mu + \alpha_\mu{}^0\zeta^0 + i\sum_{-\infty}^{+\infty} \alpha_{n,\mu} \exp(-in\zeta^0)[(\cos n\zeta^1)/n] \tag{V.11}$$

and for the ψ field equations (V.2) and (V.7) are solved by setting

$$\psi_1 = \sum c_n \exp[-in(\zeta^0 + \zeta^1)], \tag{V.12}$$

$$\psi_2 = \sum c_n \exp[-in(\zeta^0 - \zeta^1)], \tag{V.13}$$

$$c_{-n} = c_n{}^* \tag{V.14}$$

if $\epsilon = +1$ n is summed over all integers, positive and negative, and if $\epsilon = -1$, n is summed over half-integers, positive and negative. In order to express the constraints in a manageable form we extend the definition of ζ^1 from $[0, \pi]$ to $[-\pi, +\pi]$ by defining the one-component field Ψ by:

$$\Psi(\zeta^0, \zeta^1) = \psi_1(\zeta^0, \zeta^1) \qquad \text{if} \quad \zeta^1[0, \pi]; \tag{V.16}$$

$$\Psi(\zeta^0, \zeta^1) = \psi_2(\zeta^0, -\zeta^1) \qquad \text{if} \quad \zeta^1[-\pi, 0]. \tag{V.17}$$

As one sees easily because of the Eqs. (V.12–13), $\Psi(\zeta^0, \zeta^1)$ is analytic at $\zeta^1 = 0$, and Ψ is defined on $[-\pi, +\pi]$ by

$$\Psi = \sum_n c_n \exp[-in(\zeta^0 + \zeta^1)]. \qquad \text{(V.18)}$$

In terms of the field Ψ which does not any longer contain a spinor index, the supergauge constraint (V.4) takes the form of a single equation

$$[(\partial_0 + \partial_1)\Phi]\cdot\Psi = 0 \qquad \text{(V.19)}$$

extended to the range $[-\pi, +\pi]$. This is nothing other than the equation

$$P_\mu \Psi^\mu = 0, \qquad \text{(V.20)}$$

where P is defined as before

$$P_\mu = \sum_n \alpha_{n,\mu} \exp[in(\zeta^0 + \zeta^1)]. \qquad \text{(V.21)}$$

Since both P_μ and Ψ^μ depend only on $\zeta^0 + \zeta^1$, this is a constraint on the time independent variables $\alpha_{n,\mu}$ and $c_{n,\mu}$.

The condition (V.3) turns out to be also expressible in terms of only P_μ and ψ, and one obtains

$$P^2 + i\Psi(\partial_0 + \partial_1)\Psi = 0. \qquad \text{(V.22)}$$

This again gives us a set of time-independent constraints. So the basic equations of this model are the definition of ϕ (V.11), of Ψ (V.18), and of the gauge conditions (V.20), (V.22).

In order to quantize this model one may either build a Hamiltonian formalism in the covariant formulation, or using the freedom given by the existence of the supergauge and conformal transformations solve explicitly the constraints (V.20) and (V.22) and set up a canonical formalism based on independent variables. We shall only describe the covariant way of proceeding but one can also use a non-covariant transverse gauge. [Iwasaki and Kikkawa (1973)].

2. Covariant quantization of the spinning string

From the work done in Sec. I.2 it is almost obvious how to proceed in order to set up an Hamiltonian formalism in the presence of the constraints (V.20) and (V.22). Rather than go through all the steps we present directly the quantized formalism. The quantized field ϕ and Ψ obey the equal ζ^0 commutation relation

$$[\phi^\mu(\zeta^0, \zeta^1), \partial_0\phi^\nu(\zeta^0, \zeta^{1\prime})]_- = -\pi g^{\mu\nu}\delta(\zeta^1 - \zeta^{1\prime}), \qquad \text{(V.23)}$$

$$[\Psi^\mu(\zeta^0, \zeta^1), \Psi^\nu(\zeta^0, \zeta^{1\prime})]_+ = -\pi g^{\mu\nu}\delta(\zeta^1 - \zeta^{1\prime}). \qquad \text{(V.24)}$$

The quantization of the ϕ proceeds as before. The Ψ field is expressed by Eq. (V.18) where the c_n are operators obeying the anticommutation relation

$$[c_n{}^\mu, c_m{}^\nu]_+ = -\tfrac{1}{2}g^{\mu\nu}\delta_{m+n,0}. \qquad \text{(V.25)}$$

If $\epsilon = +1$, n is integer. The c_n with $n > 0$ are regarded as annihilation operators, the c_n with $n < 0$ as creation operators. According to Eq. (V.25) the zero mode c_0 must satisfy $[c_0{}^\mu, c_0{}^\nu]_+ = -\tfrac{1}{2}g^{\mu\nu}$ and hence is proportional to the matrix γ^μ which satisfy the algebra

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}. \qquad \text{(V.26)}$$

The matrices γ^μ have of course nothing to do with the γ^i matrices of the previous section.

A realization of this algebra exists for any space–time of even dimension D, and the rank of the γ matrices is $2^{D/2}$. So we set $c_0{}^\mu = (i/2)\gamma^\mu$. Since $c_0{}^\mu$ is a matrix and the operators $c_n{}^\mu$ must anticommute with it, we define

$$c_n{}^\mu = [\gamma_5/(2)^{1/2}]d_n{}^\mu \qquad c_{-n}{}^\mu = [\gamma_5/(2)^{1/2}]d_n{}^{\mu+}, \qquad n > 0, \qquad \text{(V.27)}$$

where

$$[d_n{}^\mu, d_m{}^{\nu+}]_+ = -g^{\mu\nu}\delta_{m,n}. \qquad \text{(V.28)}$$

Here γ_5 is a matrix such that $\gamma_5{}^2 = +1$ and $[\gamma_5, \gamma^\mu]_+ = 0$. If $D/2$ is even, as in four dimensions

$$\gamma_5 = i\prod_{r=0}^{D-1}\gamma^\mu,$$

and if $D/2$ is odd

$$\gamma_5 = \prod_{r=0}^{D-1}\gamma^\mu.$$

Strictly speaking we should talk about γ_{D+1}, but the above notation is more familiar. Setting $z = \exp i(\zeta^0 + \zeta^1)$ we obtain the following expression for the field Ψ

$$\Psi^\mu = (i/2)\Gamma^\mu(z), \qquad \text{(V.29)}$$

where

$$\Gamma^\mu(z) = \gamma^\mu + i(2)^{1/2}\gamma_5\sum_{n=1}^{\infty}[d_n{}^{\mu+}z^n + d_n{}^\mu z^{-n}] \qquad \text{(V.30)}$$

is the field originally introduced by Ramond as a generalization of the γ matrices. This equation defines $\Gamma(z)$ for all complex numbers z.

If $\epsilon = -1$, n is half integer. We define the creation and annihilation operators $b_n{}^\mu$, $b_n{}^{+\nu}$ by

$$c_n{}^\mu = 2^{-1/2}b_n{}^\mu \qquad c_{-n}{}^\mu = 2^{-1/2}b_n{}^{\mu+}, \qquad n = 0, \qquad \text{(V.31)}$$

where

$$[b_n{}^\mu, b_m{}^{+\nu}]_+ = -g^{\mu\nu}\delta_{m,n}, \qquad \text{(V.32)}$$

and

$$\Psi^\mu(z) = 2^{-1/2}H^\mu(z), \qquad \text{(V.33)}$$

where

$$H^\mu(z) = \sum_{n=1/2}^{\infty}[b_n{}^{\mu+}z^n + b_n{}^\mu z^{-n}]. \qquad \text{(V.34)}$$

H^μ is the field introduced originally by Neveu and Schwarz. Because n is half-integrally moded, Ψ has a branch point at $z = 0$ of the square-root-type. Hence the two models have the same basic algebra, but the Ramond model will describe half integer spins while the Neveu–Schwarz model will describe integer spins.

To define the model one needs to take the constraints (V.20) and (V.22) into account. So we define first the supergauge operators:

$$\mathfrak{L}_f = \oint_{(\Gamma)} (dz/2i\pi z)\, f(z)\,(2)^{1/2}\Psi(z)\cdot P(z). \tag{V.35}$$

Whether Ψ is given by Eq. (V.29) or Eq. (V.33), $\mathfrak{L}_f$ is said to be an F_f gauge or a G_f gauge. The contour (Γ) depends upon the test function f. If $f = z^n$, we obtain the F_n and G_n gauges. For the F gauges, n is integer, and for the G gauges n is half integer, otherwise Eq. (V.35) is senseless. In these cases (Γ) simply encircles the origin.

Equation (V.22) can be rewritten as

$$P^2 - 2\Psi(z)z(d/dz)\Psi(z) = 0. \tag{V.36}$$

So one defines the L_f gauges as:

$$L_f = -\tfrac{1}{2}\oint (dz/2i\pi z)\, f(z)[:P^2(z): \\ - 2:\Psi(z)z(d/dz)\Psi(z):], \tag{V.37}$$

where again Ψ is given by Eq. (V.29) or Eq. (V.33), depending on the model. If $f(z) = z^n$ one obtains the L_n gauges.

In order to find the algebra of the gauges and supergauges, one can use Wick's theorem for normal products of anticommuting fields. For bilinear quantities in the Γ field one is led because of the zero mode to define the normal ordered product of two gamma matrices as:

$$:\gamma^\mu\gamma^\nu: = \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \gamma^\mu\gamma^\nu - g^{\mu\nu}.$$

This definition can be extended to a product of more than two matrices and satisfies the definitions which enables us to use Wick's theorem. The basic relation between normal and un-normal ordered products is

$$H^\mu(x)H^\nu(y) = :H^\mu(x)H^\nu(y): \\ - g_{\mu\nu}[(xy)^{1/2}/(x-y)] \quad \text{if } |x| > |y|, \tag{V.38}$$

$$-\tfrac{1}{2}\Gamma^\mu(x)\Gamma^\nu(y) = -\tfrac{1}{2}:\Gamma^\mu(x)\Gamma^\nu(y): \\ -\tfrac{1}{2}g_{\mu\nu}[(x+y)/(x-y)] \quad \text{if } |x| > |y|. \tag{V.39}$$

Because of the difference between the contractions away from $x = y$ the algebras of the gauges differ by c numbers in the two models. One obtains the following algebra of gauges and supergauges:

$\epsilon = 1$ (Ramond model):

$$[F_m, F_n]_+ = 2L_{m+n} + (D/2)m^2\delta_{m,-n}, \tag{V.40}$$

$$[L_m, F_n]_- = (m/2 - n)F_{m+n}, \tag{V.41}$$

$$[L_m, L_n]_- = (m-n)L_{m+n} + (D/8)m^3\delta_{m,-n}; \tag{V.42}$$

$\epsilon = -1$ (Neveu–Schwarz model:

$$[G_m, G_n]_+ = 2L_{m+n} + (D/2)(m^2 - \tfrac{1}{4})\delta_{m,-n}, \tag{V.43}$$

$$[L_m, G_n]_- = (m/2 - n)G_{m+n}, \tag{V.44}$$

$$[L_m, L_n]_- = (m-n)L_{m+n} + (D/8)m(m^2-1)\delta_{m,-n}. \tag{V.45}$$

The difference between the two algebras can be removed if one redefines L_0 in the Ramond model by:

$$L_0^{\text{new}} = \bar{L}_0 = L_0 + D/16. \tag{V.46}$$

Then the two algebras are identical.

Let us now define the spectrum of these models.

<u>$\epsilon = 1$</u> We shall impose as weak conditions on the state vectors of the model the F and L gauges by requiring:

$$F_n\,|\,\psi\rangle = 0 \tag{V.47}$$

$$L_n\,|\,\psi\rangle = 0 \qquad \text{for all} \quad n \geq 0. \tag{V.48}$$

There is a certain amount of ambiguity for the mass shell condition. One can either say that since there is no ambiguity due to normal ordering in the definition of the quantized F_0, one can impose $F_0\,|\,\psi\rangle = 0$ which leads to a massless ground state. Or one can argue that since there is an ambiguity in the quantum value of L_0, and F_0 is related to L_0 by $F_0^2 = L_0$, we should rather require the condition

$$[F_0 - im/(2)^{1/2}]\,|\,\psi\rangle = 0 \tag{V.49}$$

where m is the arbitrary mass of the ground state. Both procedures are in fact equivalent since the model turns out to be consistent only for $m = 0$, $D = 10$. Let us examine the spectrum one obtains in the second case (keeping in mind that we shall later put $m = 0$). The ground state is

$$|\,\psi\rangle = |\,0, p\rangle u(p),$$

where $u(p)$ is a spinor satisfying the Dirac equation $(\gamma p - m)u(p) = 0$. As one can easily see, the leading trajectory except for the ground state is doubly degenerate, since we can act on the vacuum either with $(a_1{}^\mu)^+$ n times or $n-1$ times and act again with $(d_1{}^\mu)^+$: One obtains states of same spin and same (mass)^2 since

$$L_0 + m^2/2)\,|\,\psi\rangle = (-p^2/2 + m^2/2 - \sum n a_n{}^+ a_n \\ - \sum n d_n{}^+ d_n)\,|\,\psi\rangle = 0. \tag{V.50}$$

This is maybe a bad aspect of the model since experimentally the Δ and N trajectories are nondegenerate. It is possible however that unitarity corrections may split the leading trajectory into two nondegenerate trajectories.

<u>$\epsilon = +1$</u> We can impose the G gauges to work without ambiguity, as well as the L_n gauges for $n > 0$. So we require

$$L_n\,|\,\psi\rangle = 0 \qquad \text{for} \quad n > 0, \tag{V.51}$$

$$G_r\,|\,\psi\rangle = 0 \qquad r = \tfrac{1}{2}, \tfrac{3}{2}, \cdots. \tag{V.52}$$

Since there is an ambiguity for L_0 in going from the classical to the quantum expression, the mass shell condition is defined by

$$[L_0 - \alpha_\pi(0)] \mid \psi\rangle = 0 \tag{V.53}$$

where $\alpha_\pi(0)$ is an arbitrary constant. As it turns out $\alpha_\pi(0)$ will be equal to 1/2. The mass shell condition is now

$$[-p^2/2 - \alpha_\pi(0) - \sum_{n=1}^{\infty} n a_n^{+}\cdot a_n - \sum_{m=1/2}^{\infty} m b_m^{+}\cdot b_m] \mid \psi\rangle = 0. \tag{V.54}$$

In this model there is a G parity which one can define as

$$G = (-1)^{\Sigma b_m^{+} b_m - 1} \tag{V.55}$$

and this G-parity will be preserved by the interaction.

The ground state is a scalar (or pseudo scalar since we have not yet defined the parity of our states), of (mass)^2 — $[\alpha_\pi(0)/\alpha']$. It will turn then to be a tachyon. There is a linear trajectory associated with this "pion." Above it there is another trajectory since using $b_{1/2}^{+}$ we raise the spin by one unit and the (mass)^2 by a half unit. The first state on that trajectory has mass zero [if $\alpha_\pi(0) = 1/2$], spin 1, G parity + 1 and hence is called the "ρ." It is remarkable that although $\alpha_\rho(0) = 1$, there is no tachyon on the ρ trajectory. The splitting $\alpha_\rho(0) - \alpha_\pi(0) = \frac{1}{2}$ is also the correct one.

3. No-ghost theorem for the covariant quantization

We shall give only the main outlines of the proofs of the no-ghost theorems for the fermion and meson models since they follow a similar line of argument as the one used in the conventional model. Because of the symmetry between $H^\mu(z)$ and $[i/(2)^{1/2}]\Gamma^\mu(z)$ we shall present the notations side by side for the Neveu–Schwarz ($\epsilon = -1$) and Ramond ($\epsilon = +1$) model.

The number operator R takes half-integral values if $\epsilon = -1$ and integral values for $\epsilon = +1$. At the Mth level, the space of states R^M contains both positive and negative norm states. The *physical* states are defined as the states which satisfy the gauge conditions (V.47) and (V.48) if $\epsilon = +1$, or (V.51) and (V.52) if $\epsilon = -1$, and are said to be on shell if in addition the mass shell condition (V.49) ($\epsilon = +1$) or (V.53) ($\epsilon = -1$) is satisfied. One wishes to show that the on-shell physical states have a positive norm. One defines first a smaller subspace of states, the *transverse* states, which in addition to the previous gauge conditions have to satisfy the supplementary conditions (k is the same light like vector which we had picked before)

$$K_n \mid t\rangle = 0, \tag{V.56}$$

$$H_n \mid t\rangle = 0, \qquad n > 0, \tag{V.57}$$

where

$$H_n = i[\gamma_5/(2)^{1/2}](k\cdot d_n) \qquad (\epsilon = +1)$$

or

$$H_n = k\cdot b_n \qquad (\epsilon = -1).$$

From a state $\mid t\rangle$ belonging to R^M one forms a set of states belonging to R^N defined as

$$F_{-1}^{\epsilon_1}\cdots F_{-a}^{\epsilon_a} H_{-1}^{\delta_1}\cdots H_{-b}^{\delta_b} L_{-1}^{\lambda_1}\cdots L_{-n}^{\lambda_n} \times K_{-1}^{\mu_1}\cdots K_{-m}^{\mu_m} \mid t\rangle \qquad (\epsilon = +1), \tag{V.58}$$

$$G_{-1/2}^{\epsilon_1}\cdots G_{1/2-a}^{\epsilon_a} H_{-1/2}^{\delta_1}\cdots H_{1/2-b}^{\delta_b} L_{-1}^{\lambda_1}\cdots L_{-n}^{\lambda_n} \times K_{-1}^{\mu_1}\cdots K_{-m}^{\mu_m} \mid t\rangle \qquad (\epsilon = -1), \tag{V.59}$$

where ϵ_i, δ_i = 0 or 1.

One can show that these states form a linearly independent basis of some subspace of R^N which does not contain any transverse states. Then by an iterative construction, one can show that together with the transverse states, the states defined by Eqs. (V.58–59) form a basis for the whole space. Further, the transverse states have positive definite norm. [Goddard and Thorn (1972), Corrigan and Goddard (1974)]. One then looks for a projection operator unto the space of transverse states. To do this one picks up the same lightlike vector k, and defines P_+, P_-, ψ_+, ψ_-. Then using the gauge conditions, written classically as

$$P\cdot\Psi = 0 \quad\text{and}\quad P^2 - 2z\Psi\Psi' = 0, \qquad \Psi' = (d/dz)\Psi(z),$$

we solve them for $P_{+,T}(z)$ as a function of P_-, $\vec{P}$, Ψ_-, $\vec{\Psi}$, and define $P_{+,T} = \langle P_{+,T}(z)\rangle_0$ where $P_{+,T}(z)$ is expressed in terms of only transverse and minus components. One then forms the difference between $P_{+,T}$ and $P_+ = \langle P_+(z)\rangle_0$ and one obtains [Ramond (1973)]:

$$P_{+,T} - P_+ = \left\langle (1/P_-)\left(1 - 2z\frac{\Psi_-\Psi_-'}{P_-}\right)(-\tfrac{1}{2}P^2 + z\Psi\cdot\Psi') + \frac{2z}{(P_-)^{3/2}}\frac{d}{dz}\left\{\frac{\Psi^-}{(P_-)^{1/2}}\right\} P\cdot\Psi\right\rangle_0. \tag{V.60}$$

Defining now the operators

$$D_n = \left\langle (1/P_-)\left(1 - 2z\frac{\Psi_-\Psi_-'}{P_-^{\,2}}\right)\right\rangle_n, \qquad n \text{ integer}, \tag{V.61}$$

$$B_n = \left\langle \frac{2^{1/2}}{(P_-)^{3/2}}\frac{d}{dz}\left\{\frac{\Psi_-}{(P_-)^{1/2}}\right\}\right\rangle_n, \quad n \text{ integer } (\epsilon = +1),\ \text{half-integer } (\epsilon = -1). \tag{V.62}$$

After taking care of the normal ordering ambiguities, one obtains the following expressions for $E = P_{+,T} - P_+$

$$\epsilon = +1 \qquad E = (\bar{L}_0 - \lambda)(D_0 - 1) + F_0 B_0 + \sum_{n=1}^{\infty} F_{-n}B_n - B_{-n}F_n + L_{-n}D_n + D_{-n}L_n, \tag{V.63}$$

$$\epsilon = -1 \qquad E = (L_0 - \alpha_\pi(0))(D_0 - 1) + \sum_{r=+1/2}^{\infty} (B_{-r}G_r + G_{-r}B_r) + \sum_{n=1}^{\infty} L_{-n}D_n + D_{-n}L_n, \tag{V.64}$$

where $\bar{L}_0$ has been defined in Eq. (V.46).

The algebra of the new operators B_n, D_m, is the following ($\epsilon = 1$)

$$[F_n, D_m]_- = 2B_{n+m}, \qquad [F_n, B_m]_+ = -\tfrac{1}{2}(3n+m)D_{n+m},$$
$$[L_n, D_m]_- = -(2n+m)D_{n+m},$$
$$[L_n, B_m]_- = -(\tfrac{3}{2}n+m)B_{n+m},$$
$$[D_n, D_m]_- = [B_n, B_m]_+ = [D_n, B_m]_- = 0. \qquad \text{(V.65)}$$

The same algebra obtains for $\epsilon = -1$ replacing F_n by G_n. By using this algebra one can show that for the fermions ($\epsilon = +1$)

$$[L_n, E]_- = -nL_n + D_n\{n^3(D/8 - 5/4) + n(3\lambda - D/8)\}, \qquad \text{(V.66)}$$

and hence one requires $D = 10$, $\lambda = D/16$. The last condition implies $\bar{L}_0 - \lambda = L_0$ and so the ground state fermion will be massless.

A similar computation in the meson model reveals $D = 10$, $\alpha_\pi(0) = \frac{1}{2}$, so that the "$\pi$" meson is a tachyon, the "ρ" meson is massless. In this case one can show that for both models

$$[X_n, E] = -nX_n \quad \text{for} \quad X_n = L_n, K_n, D_n, F_n/G_n, B_n, H_n.$$

Using the basis of states previously derived and these commutation relations, one shows that E has negative or zero eigenvalues (integer if $\epsilon = +1$), (half integer and integer if $\epsilon = -1$) and that the eigenvalue zero is reached only in the transverse subspace. The projectors unto the transverse subspace are then [Brink and Olive (1973), Corrigan and Goddard (1974a)]

$$\mathfrak{Z} = \oint (dy/2i\pi y)\, y^E \qquad \text{if} \quad \epsilon = +1, \qquad \text{(V.67)}$$

$$\mathfrak{Z} = \oint (dy/2i\pi y)\, y^{2E} \qquad \text{if} \quad \epsilon = -1. \qquad \text{(V.68)}$$

Using then the structure of the E operator, the commutation relations and the gauge conditions satisfied by on shell physical states one shows easily that if $\langle \phi_1 |$, $| \phi_2 \rangle$ are two on-shell physical states, then

$$\langle \phi_1 | \mathfrak{Z} | \phi_2 \rangle = \langle \phi_1 | \phi_2 \rangle \qquad \text{(V.69)}$$

and this proves the no-ghost theorem, originally proved by different methods by Goddard, Rebbi and Thorn (1972); Schwarz (1972): Brower and Friedman (1973). The same conditions on D, $\alpha_\pi(0)$, m, are also obtained if one quantizes the theory in a transverse gauge and then requires Lorentz covariance. [Iwasaki and Kikkawa (1973)].

As we have just seen, in the free case mesons and fermions are treated in a very symmetric way. As we are going to see, in the operator formalism based on the covariant orthonormal gauge, this is no longer true for the interacting theory, and it has not yet been possible (maybe for purely technical reasons) to give a unified covariant treatment of scattering amplitudes involving mesons and fermions. In the functional integration formalism based on the transverse gauge, such a symmetric treatment exists, and so this may be a disadvantage of the covariant gauge with respect to the transverse gauge. However the complicated apparatus of the covariant operator formalism finally gives the same set of amplitudes as the one obtained in the noncovariant approach, which also has its own degree of sophistication and therefore it is useful to have two independent derivations of the same result: it shows that the dual interacting string can really be treated in two different gauges, a nontrivial test of the consistency of the dual theory. We shall refer the reader for the transverse gauge treatment of the spinning string to: Mandelstam (1973a), (1973b), and (1974).

4. Meson-meson scattering

A mesonic string propagator in the covariant approach is simply represented by the dotted line to which corresponds the mathematical expression $1/(L_0 - 1/2)$. However, remembering that it is supposed to represent a spinning string propagator, it can also be represented as Fig. 14 where

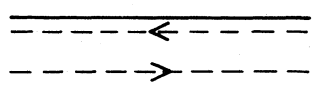


FIG. 14. A meson propagator drawn in the string picture.

the arrows going in opposite directions at the top and the bottom reflect the opposite sign in the boundary conditions at $\sigma = 0, \pi$. These arrows have nothing to do with those carrying $SU(N)$ quantum numbers and should not be confused with them. Similarly a three "point" vertex can be drawn either Fig. 15(a) or Fig. 15(b).

We shall work with the first kind of representation and show that because of the good choice of the vertex in the operator formalism, the amplitudes have all the good properties of duality that one may expect from the string diagrams. So one looks for a vertex $V(p)$ describing the absorption of a ground state "pion" by a mesonic propagator. In order to ensure the absence of ghosts $V(p)$ must have "good" anticommutation relations with the G_m gauges. Neveu and Schwarz introduced the vertex $V(p) = p\cdot H(1)V_0$ [Neveu and Schwarz (1971a)]

$$V(p) = p\cdot H(1)V_0 \qquad \text{(V.70)}$$

where

$$V_0 = :\exp - ip\cdot Q(1):$$

is the vertex of the conventional model. To show that V has the desired properties, one notes that

$$[G_m, V_0]_- = p\cdot H(1)V_0 \qquad \text{(V.71)}$$

for all m, and hence

$$[G_m, V]_+ = [G_m[G_m, V_0]_-]_+ = [G_m{}^2\, V_0]_- = [L_{2m}, V_0]_-$$
$$= (L_0 - mp^2 - \tfrac{1}{2})V_0 - V_0(L_0 - \tfrac{1}{2}). \qquad \text{(V.72)}$$

Let us consider an on-shell spurious state of the form: $\langle \psi | G_m = \langle \psi' |$. Since it is on shell, one has

$$\langle \psi | (L_0 + m - \tfrac{1}{2}) = 0. \qquad \text{(V.73)}$$

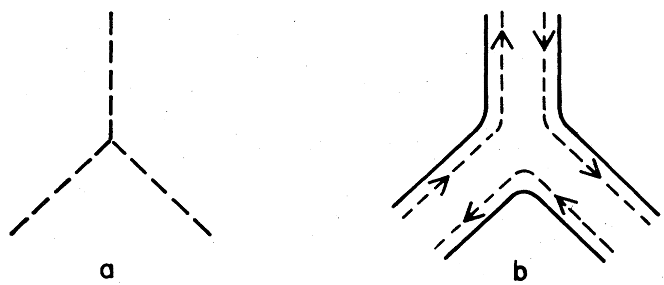


FIG. 15. A three-meson vertex drawn; (a) as a three-"point" coupling, (b) in the string picture.

If we want to prove that such a state is decoupled from any tree containing pions, we must show that:

$$\langle\psi \mid G_m V(p_2)\frac{1}{L_0-1/2}V(p_3)\cdots V(p_{N-1})\mid 0\rangle = 0. \tag{V.74}$$

Hence the equality (V.72) will be useful only if $p_2^2=-1$ which means that the absorbed scalar meson has the same mass as the "pion." In that case the first term on the right-hand side of (V.72) does not contribute. The second term cancels the propagator $1/(L_0-1/2)$. If one evaluates an expression containing a cancelled propagator, one obtains an expression of the type

$$\lim_{x\to 1}(1-x)^{-p_2\cdot p_3}.$$

Since the operator formalism is defined only in the region below the poles of the amplitude in any channel (otherwise it leads to divergent expressions), $p_2\cdot p_3$ is negative, and the above limit is zero. So expressions containing cancelled propagators give zero in the region where the integral representations are convergent, and there is no problem in continuing analytically this zero result outside of the region of convergence. (Neveu, Schwarz and Thorn (1971)).

Continuing the process of anticommuting G_m with vertices until it reaches the vacuum where it annihilates, one proves the desired identity (V.74) (provided that all the emitted pions are on shell). A similar result holds also for L_n. So the "tree states"

$$\mid\phi\rangle = V(p_2)[1/(L_0-1/2)]V(p_3)\cdots V(p_{N-1})\mid 0\rangle. \tag{V.75}$$

Satisfy the Ward identities

$$L_n\mid\phi\rangle = (L_0+n-\tfrac{1}{2})\mid\phi\rangle, \tag{V.76}$$

$$G_r\mid\phi\rangle = (L_0+r-\tfrac{1}{2})\mid\phi_0\rangle, \tag{V.77}$$

where $\mid\phi_0\rangle$ differs from $\mid\phi\rangle$ be replacing the first vertex $V(p_2)$ by V_0. For two such tree states $\mid\phi\rangle$ and $\mid\psi\rangle$ one can show that for $D=10$

$$\langle\psi\mid\mathfrak{M}_N\mathfrak{I}\mid\phi\rangle = \langle\psi\mid\mathfrak{M}_N\mid\phi\rangle \tag{V.78}$$

where

$$\mathfrak{M}_N = \oint (dx/2\pi i x)x^{2(L_0-1/2)}$$

is the projector unto the Nth level. This proves that transverse states saturate the residues of a mesonic tree, and, as a consequence, that no shost appears. The last part of the statement is still valid if $D\leq 10$.

Let us now evaluate the $\pi\pi\to\pi\pi$ amplitude. Up to normalization factors which include the coupling constant, it is given by

$$A_4 = \langle 0,k_1\mid k_2\cdot H(1)V_0(k_2)\int_0^1(dx/x)x^{L_0-1/2} \times V_0(k_3)k_3\cdot H(1)\mid 0,k_4\rangle. \tag{V.79}$$

The algebra of the a and b modes separate completely. The "a" part of the algebra is worked as before using coherent states, while the "b" part can be worked out using the relation

$$x^{L_0{}^b}H(z)x^{-L_0{}^b} = H(zx) \tag{V.80}$$

and also the contraction between two H's given by Eq. (V.38). One obtains

$$A_4 = -k_2\cdot k_3\int_0^1(dx/x)x^{-\alpha_\rho(s)+1}(1-x)^{-\alpha_\rho(t)-1}, \tag{V.81}$$

$$A_4 = -k_2\cdot k_3\frac{\Gamma(1-\alpha_\rho(s))\Gamma(-\alpha_\rho(t))}{\Gamma(1-\alpha_\rho(s)-\alpha_\rho(t))}, \tag{V.82}$$

where

$$\alpha_\rho(s) = \alpha_\pi(s)+\tfrac{1}{2} = 1+\tfrac{1}{2}s.$$

This expression is not obviously crossing symmetry, however using the fact that $k_2\cdot k_3=\alpha_\rho(t)$, one obtains

$$A_4 = \frac{\Gamma(1-\alpha_\rho(s))\Gamma(1-\alpha_\rho(t))}{\Gamma(1-\alpha_\rho(s)-\alpha_\rho(t))}. \tag{V.83}$$

This is the formula originally proposed by Lovelace (1968) and generalized by Shapiro (1969). It is crossing symmetric, does not have a ρ tachyon pole at $\alpha_\rho=0$, and would have the Adler zeroes if the intercept of the pion was zero rather than 1/2. It is obvious in this formalism that only amplitudes with an even number of pions can be constructed since all H's have to be paired to give a nonzero result. Hence the definition of the "G parity" operator given above.

In this formulation of the dual diagram rules, gauge invariance as we have seen is simple to check, but not cyclic symmetry, because the pions at the end of the chain do not have any factor of the type $k\cdot H$ associated with them, while for each pion emission inside the chain such a factor occurs. In order to prove the cyclic symmetry of the N point function one can go to an equivalent set of rules (the so called $\mathfrak{F}_1$ formalism) where cyclic symmetry is easy to prove, but where gauge invariance is less transparent. The formalism described previously is called the $\mathfrak{F}_2$ formalism, and historically the $\mathfrak{F}_1$ formalism was devised earlier. Let us consider a physical state, on the mass shell, in the previous $\mathfrak{F}_2$ formalism. Since

$$G_{1/2} \mid \phi_2 \rangle = 0 \quad \text{and} \quad (L_0 - \tfrac{1}{2}) \mid \phi_2 \rangle = 0 \quad \text{one has}$$
$$\mid \phi_2 \rangle = (2L_0 - G_{-1/2}G_{1/2}) \mid \phi_2 \rangle = G_{1/2}G_{-1/2} \mid \phi_2 \rangle. \tag{V.84}$$

An amplitude with two physical states at the end of the chain, $\mid \phi_2 \rangle$, $\mid \phi_2' \rangle$ is described by

$$A_N = \langle \phi_2' \mid V(k_2)(L_0 - 1/2)^{-1} \cdots (L_0 - 1/2)^{-1} \times V(k_{N-1}) \mid \phi_2 \rangle = \langle \phi_2' \mid V(k_2) \times (L_0 - 1/2)^{-1} \cdots (L_0 - 1/2)^{-1} V(k_{N-1}) G_{1/2}G_{-1/2} \mid \phi_2 \rangle. \tag{V.85}$$

Using the commutation relations (V.72) and (V.44) and cancelled propagator arguments, $G_{1/2}$ can be commuted to the left so that one obtains

$$A_N = \langle \phi_2' \mid G_{1/2} V(k_2)(L_0 - 1)^{-1} \cdots (L_0 - 1)^{-1} \times V(k_{N-1}) G_{-1/2} \mid \phi_2 \rangle. \tag{V.86}$$

In this new formalism, the propagator is now $(L_0 - 1)^{-1}$ so that it seems that a tachyon on the ρ trajectory appears: this is not the case as we have seen already. The vertices are unchanged, but the states at the end of the chain are described by the state vectors: $\mid \phi_1 \rangle = G_{-1/2} \mid \phi_2 \rangle$ and conversely $\mid \phi_2 \rangle = G_{-1/2} \mid \phi_1 \rangle$. These equations give the connection between $\mathfrak{F}_2$ and $\mathfrak{F}_1$ formalism. In the $\mathfrak{F}_1$ formalism, a pion at the end of a chain is now represented by $k \cdot b_{-1/2} \mid 0, k \rangle$ while in the $\mathfrak{F}_2$ it is represented by $\mid 0, k \rangle$. Because of this, to each pion whether at the end of the chain or not correspond a k factor and the proof of the cyclic symmetry is made much easier. The $\mathfrak{F}_1$ formalism appears less fundamental than the $\mathfrak{F}_2$, however as an intermediate technical instrument it can be very useful. The connection between these two formalisms was originally elucidated by Neveu, Schwarz and Thorn (1971). A very nice compact formula for the N-point function has been found by Fairlie and Martin (1973) involving integration over anticommuting numbers.

5. Fermion-meson scattering

A fermion propagator in the covariant quantization is represented by a solid, oriented line as in Fig. 16(a) or as a string propagator, as in Fig. 16(b) where arrows going in the same direction at the top and the bottom reflect the identical boundary conditions at $\sigma = 0, \pi$. An antifermion is represented in both pictures by changing the direction of the arrows. Similarly a fermion-fermion-meson vertex can be drawn either as Figs. 17(a) or (b). If we take the second picture literally, the emitted mesonic string should be the mesonic string introduced before, and indeed this will be the case.

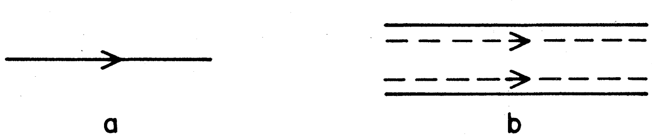


FIG. 16. Fermion propagator drawn; (a) as an ordinary Fermion propagator, (b) in the string picture.

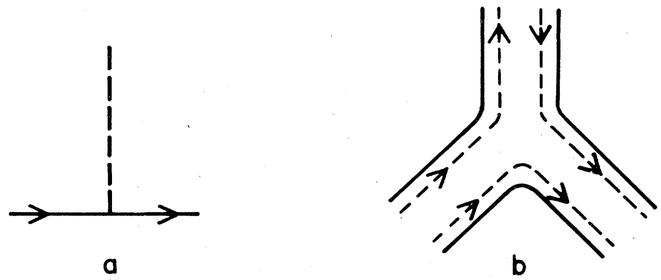


FIG. 17. Fermion–antifermion–meson vertex: (a) represented as a three-"point" coupling, (b) in the string picture.

Let us introduce a vertex for the emission of a ground state meson from a fermion line. Until now we have not assigned a given parity to the ground state of the Neveu–Schwarz model, and one can either define the ground state to be a scalar or a pseudoscalar. The states which have odd G parity have opposite parity in the two models defined by the choice of the parity of the ground state, while the even G-parity states keep same parity in both cases. It is customary to speak of the fermions as "quarks," and the version of the mesonic model where the ground state is a pseudoscalar is said to be a "gluonic" model while the mesons of the models where the ground state is a scalar are said to be $q\bar{q}$ mesons, a terminology which will be justified later. One can of course choose the opposite convention. (Schwarz (1971)).

In order to describe the emission of the ground state gluonic "pion," one introduces the generalized γ_5 matrix [Neveu and Schwarz (1971b)]

$$\Gamma_5 = \gamma_5(-1)^{\sum_{n=1}^{\infty} d_n{}^+ d_n} \tag{V.87}$$

which anticommutes with the generalized $\Gamma_\mu(z)$ matrix. Let us define $V_2 = \Gamma_5 V_0$ where V_0 is given by Eq. (III.13). Since

$$V_1 = [F_n, V_2]_+ = ip \cdot \Gamma \Gamma_5 V_0 \tag{V.88}$$

is independent on n, the commutator of V_1 with F_n is easily computed

$$[F_n, V_1]_- = [F_n{}^2, V_2]_- = [L_n, V_2]_- = (L_0 - np^2) V_2 - V_2 L_0. \tag{V.89}$$

For this gauge identity to be useful one needs $p^2 = -1$ as before, so that the ground state pseudoscalar gluon is on its mass shell. Then one defines a tree in the Ramond model as

$$\mid R1 \rangle = (1/L_0) V_1 (1/L_0) \cdots V_1 \mid 0 \rangle u(p), \tag{V.90}$$

where $u(p)$ is a spinor satisfying the massless Dirac equation.

Using the gauge property of V_1 just derived and using cancelled propagator arguments, one shows that

$$\langle \psi \mid F_n \mid R1 \rangle = 0 \qquad \text{for} \quad n = 0, 1, 2, \cdots \tag{V.91}$$

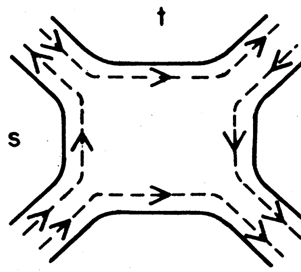


FIG. 18. Fermion-meson scattering displaying fermions in the s channel, and mesons in the t channel.

for any $\langle\psi|$ satisfying $\langle\psi|(L_0+n)=0$. This formalism is the R_1 formalism, where the gauge identities are simple to prove, but where the fermion has an unconventional Klein-Gordon propagator. This can be remedied easily by using Eq. (V.89) for $n=0$, $F_0^2=L_0$, and cancelled propagator arguments to prove that

$$|R1\rangle = |R2\rangle, \tag{V.92}$$

where

$$|R2\rangle = (1/F_0)V_2(1/F_0)V_2\cdots V_2|0\rangle u(p).$$

In the case of the $q\bar{q}$ model of mesons where the ground state is scalar, the vertex $V_2=V_0$ and $V_1=[F_n, V_2]_-$. Then

$$[F_n, V_1]_+ = (L_0-np^2)V_2 - V_2L_0 \tag{V.93}$$

and the same expressions and gauge conditions for $|R1\rangle$, $|R2\rangle$ are obtained. In order for the gauge condition to work for $n=0$ in this model it is crucial that the mass m of the ground-state fermion be set at zero: If not, the anticommutator (V.93) for $n=0$ means that $(F_0-im/2^{1/2})$ becomes converted into $[-F_0-im/(2)^{1/2}]$, up to cancelled propagators and the latter expression does not annihilate the ground-state fermion $|0\rangle u(p)$.

We shall now talk only about the "gluonic" model and in the end obtain the "$q\bar{q}$" model as bound states in the fermion-antifermion channel. A fermionic line emitting pseudoscalar "pions" is then described by:

$$F = \bar{u}(p_0)\langle 0|V_2(p_1)(1/F_0)V_2(p_3)\cdots(1/F_0) \times V_2(p_{N-1})|0\rangle u(p_N). \tag{V.94}$$

The no-ghost theorem for fermions can then be proved by inserting a factor $\mathfrak{I}-1$ on an internal line and by showing that this insertion gives zero on the mass shell. To do this, one converts first to the R_1 formalism, which is possible even in the presence of $\mathfrak{I}$, since F_0 commutes with $\mathfrak{I}$. There remains a last V_2 vertex (for instance the last on the left) which is converted to a V_1 by defining $\bar{u}'(p_0)$ such that $\bar{u}'(p_0)\langle 0|F_0=\bar{u}(p_0)\langle 0|$. Then the gauge identities are used to show that $\mathfrak{I}-1$ reduces to zero; the F_0 gauge is used working to the right since it does not annihilate on $\bar{u}'(p_0)\langle 0|$. So for $D=10$, the residues at each fermion pole are saturated by transverse states. [Corrigan and Goddard (1974)].

It now remains to find out the duality properties of a tree described by (V.94). In the R_1 formalism, one has

$$F = \bar{u}(p_0)\langle 0|V_2(p_1)(1/L_0)V_1(p_3) \times (1/L_0)\cdots V_1(p_{N-1})|0\rangle u(p_N), \tag{V.95}$$

and in this form the algebra of the a and d modes separate. Up to the spin factors, the fermion–meson scattering amplitude is proportional to $B(\frac{1}{2}-\alpha_F(s);-\alpha_\rho(t))$ where $\alpha_F(s)=\frac{1}{2}+s/2$. So we see that the fermions in the s channel are dual to gluons in the t-channel. This is indeed expected since if we draw the corresponding string graph, one obtains that result, as one sees easily on Fig. 18. However, to prove this duality between gluons and fermions in a factorized way one must find a vertex operator satisfying the graphical identity of Fig. 19.

We shall assume for simplicity that the emitted fermion state is the ground state. In order for the above equality to be possible one needs to find a vertex operator for the emission of a ground state fermion such that a meason emission vertex from the fermion line on the left can be slid to the right of the vertex and become a meson emission vertex for a mesonic line as shown on Fig. 20.

Since the space of the a modes is common to both fermions and mesons, one needs an operator which acts on the fermionic space on the left (d_n oscillators) and on the mesonic space on the right (b_n oscillators). If we express the fermionic tree on the left in the $R1$ formalism, we can write each propagator in terms of its integral representation, then use the equality $x^{L_0}\Gamma(z)x^{-L_0}=\Gamma(zx)$ to replace all propagators in terms of $\Gamma(z)$ matrices: similarly on the right of the vertex, all mesonic propagators can be absorbed into $H(z)$ fields, so that in order to prove the meson–fermion duality the vertex should convert Γ fields into H fields, and vice versa.

It would be too lengthy to give the details of the construction of this vertex which we shall denote by $V_F(p,z)$. $V_F(p,z)$ is defined by the following equations (Schwarz (1971), Thorn (1971), Corrigan and Olive (1972).

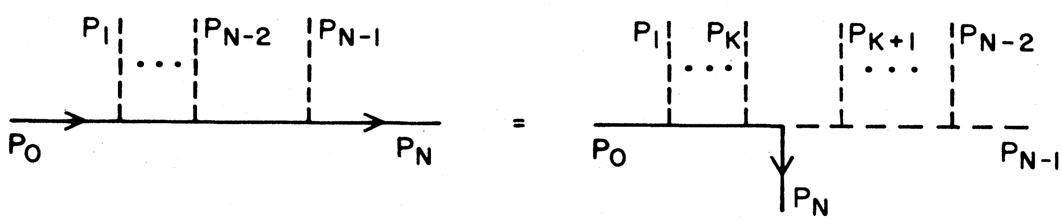


FIG. 19. Duality equation revealing that the emission of N "pions" from a fermion line contains the Neveu–Schwarz interaction in the cross channel.

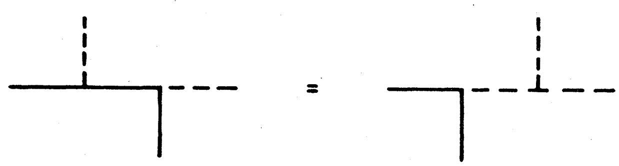


FIG. 20. Basic requirement for the fermion emission vertex.

$$V_F(p,z) = W_F(z)V_0(p,z)z^{p^2/2}, \tag{V.96}$$

$$W_F(z) = \exp zL_{-1}{}^{(d)}\tilde{W}_F(z), \tag{V.97}$$

where

$$\tilde{W}_F(z)\mid 0_b\rangle = \mid 0_d\rangle, \tag{V.98}$$

$$\tilde{W}_F(z)\gamma_5\frac{H^\mu(y)}{y^{1/2}} = -\frac{i}{2^{1/2}}\frac{\Gamma^\mu(y-z)}{(y-z)^{1/2}}\tilde{W}_F(z). \tag{V.99}$$

Equation (V.98) is an obvious boundary condition; Eq. (V.99) is the statement that the vertex converts H's into Γ's. Equations (V.96) and (V.97) are there to insure the correct duality properties of the vertex. The solution to equations (V.98) and (V.99) is given by

$$\tilde{W}_F(z) = \langle 0_b \mid \exp B(z) \mid 0_d\rangle \exp A(z), \tag{V.100}$$

where

$$A(z) = \tfrac{1}{2}\sum_{r,s>0} b_r A_{rs} b_s, \tag{V.101}$$

$$B(z) = -i\sum_{n=0}^{\infty}\sum_{r=1/2}^{\infty}\Gamma_{-n}B_{nr}(z)b_r\gamma_5, \tag{V.102}$$

and

$$\Gamma_0{}^\mu = \gamma^\mu; \qquad \Gamma_{-n}{}^\mu = 2^{1/2}\gamma_5 d_n{}^+. \tag{V.103}$$

Hence we see that the vertex $\tilde{W}_F(z)$ is a rather complicated object since it contains the exponential of bilinear form in the Fermi b_r operators. The infinite matrices A and B are given by

$$A_{rs}(z) = \tfrac{1}{2}(z)^{-r-s}[(s-r)/(s+r)] \times \binom{-1/2}{r-1/2}\binom{-1/2}{s-1/2}(-1)^{r+s-1}, \tag{V.104}$$

$$B_{nr} = (1/2^{1/2})(z)^{n-r}\binom{n-1/2}{r-1/2}(-1)^{n-r+1/2}, \tag{V.105}$$

In the $\mathfrak{F}_2$ formalism the amplitude for the process is given by

$$F = \langle R1 \mid V_F(1)\gamma_5 u(p)(L_0 - 1/2)^{-1}\mid \phi_2\rangle, \tag{V.106}$$

where $\mid\phi_2\rangle$ is a tree of the $\mathfrak{F}_2$ formalism $\langle R1\mid$, a fermionic tree in the R_1 formalism.

The gauge properties of the vertex $V_F(1)$ are very important, since they should guarantee that transverse states saturate the residues of fermionic and mesonic poles in meson–fermion scattering. One cannot however simply expect that an F_n gauge will be converted into an G_n gauge since n is an integer in the first case and a half integer in the second case. So the simplest possibility is for $V_F(1)$ to convert linear combinations of F gauges into linear combinations of G gauges. One can show indeed that it does so and the most general gauge identity satisfied by $V_F(1)$ is [Corrigan and Goddard (1973); Brink, Olive, Rebbi and Scherk (1973)]

$$F_\phi V_F(1) + V_F(1)[\gamma_5 G_\phi - [(ip\cdot\gamma)/2^{1/2}]f(1)] = 0, \tag{V.107}$$

where $\phi(x) = f(x)(1-1/x)^{1/2}$ and $f(x)$ is a function analytic in a domain including 0 and 1 with the exception of possible poles at 0. In the Eq. (V.35) which defines F_ϕ and G_ϕ, the contour is different: for G_ϕ, (Γ) is any circle for which $|z| < 1$, while for F_ϕ it is any circle for which $|z| > 1$. It is easy to see that with this contour prescription both G_ϕ and F_ϕ exist for the same function $\phi(x)$. The term proportional to $p\cdot\gamma$ is not harmful to the gauge condition since the emission vertex has to be multiplied on the right by $\gamma_5 u(p)$ which is annihilated by $p\cdot\gamma$. If one chooses $f(x) = x^{-n}$ $n = 0\ 1\ 2, \cdots$ one obtains an infinite sum of F_{-m} gauges which work to the left, a finite number of incident G_{-m} gauges working to the left and an infinite number of G_s gauges working to the right. This reminds one of the behavior of an incident plane wave at the interface of two media having different optical indices.

Defining

$$\langle F\bar{F}\mid = \langle R1 \mid V_0(1)\gamma_5 u(p) \tag{V.108}$$

and $\mid\phi_2\rangle$ as given by Eq. (V.75) to be a meson tree in the F_2 formalism, the gauge identities satisfied by $\mid\phi_2\rangle$, $\mid F\bar{F}\rangle$ can be written as (Olive and Scherk 1973b)

$$L_n\mathfrak{M}\mid\phi_2\rangle = 0, \tag{V.109}$$

$$G_r\mathfrak{M}\mid\phi_2\rangle = 0, \tag{V.110}$$

$$\langle F\bar{F}\mid L_{-n} = \langle F\bar{F}\mid (L_0 + \tfrac{5}{8}(n-1)), \tag{V.111}$$

$$\langle F\bar{F}\mid G_{-r} = \langle F\bar{F}\mid \sum_{s=1/2}^{\infty}\alpha_{rs}G_s, \tag{V.112}$$

where

$$\mathfrak{M} = \oint (dx/2i\pi x)x^{2(L_0-1/2)}$$

is the on shell projection operator, and

$$\alpha_{rs} = (-1)^{r+s+1}\frac{r}{r+s}\binom{-1/2}{r-1/2}\binom{-1/2}{s-1/2}. \tag{V.113}$$

Note that α_{rs} and A_{rs} are connected by

$$A_{rs}(1) = \tfrac{1}{2}(\alpha_{rs} - \alpha_{rs}).$$

The saturation of the meson pole residues by transverse states follows from the equality [Olive and Scherk (1973b)]

$$\langle F\bar{F} \mid \mathcal{J}\mathcal{M} \mid \phi_2\rangle = \langle F\bar{F} \mid \mathcal{M} \mid \phi_2\rangle. \qquad \text{(V.114)}$$

To prove this equality one needs not only the gauge identities (V.109–V.112) and the structure of the operator $\mathcal{J}$, but also to prove a mathematical identity

$$\sum_{r+s=n} \alpha_{rs}\left(s+\frac{n}{2}\right) = \frac{3n+1}{8}. \qquad \text{(V.115)}$$

Without this identity the left hand side of (V.114) would not even be covariant, and the Eq. (V.115) is one of the typical "miracles" exhibited by the operator formalism. The identity of the mesonic sector obtained by factorization of the poles dual to a fermion line with the transverse state of the Neveu–Schwarz model shows a non trivial interplay of the properties of duality and of gauge invariance, even if it could be expected on the basis of the string diagrams. Strictly speaking, it would have been possible to define only the fermion–fermion–meson vertex V_2 and to deduce the existence of the three-meson interaction and its form from duality, while the converse would not have been possible.

6. Fermion-fermion scattering

By fermion–fermion scattering we shall mean a process such as the one described in Fig. 21.

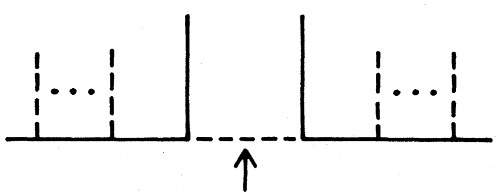


FIG. 21. Fermion–fermion scattering represented in the conventional way.

We shall assume by an economy principle, that we know all the vertices appearing in that figure already. However, it is by no means obvious that the propagator indicated by an arrow is simply $(L_0 - \frac{1}{2})^{-1}$. The reason for this is that the $\langle F\bar{F}_1 \mid$ and $\mid F\bar{F}_2\rangle$ on its left and right satisfy different Ward identities from those used to define the F_2 space, so that we may expect the propagator also to differ. Equation (V.112) looks as if we were still working in an orthonormal gauge, since in the average each G_r is zero between $\langle F\bar{F}_1 \mid$ and $\mid F\bar{F}_2\rangle$ states, but this vanishing in average is insured in a different way as in Eq. (V.52).

What one really needs however, is the residues at any pole of this propagator to be saturated by the transverse states of the Neveu–Schwarz model and this will define, together with the Regge behavior condition, which propagator one should use. So we require the residues at a given pole of the above amplitude in the intermediate meson channel to be

$$R = \langle F\bar{F}_1 \mid \mathcal{J}\mathcal{M} \mid F\bar{F}_2\rangle, \qquad \text{(V.116)}$$

where we write $\mathcal{M}$ as

$$\mathcal{M} = \int_{(c)} \frac{dx}{4\pi i x}\, x^{L_0-1/2}$$ (c) being a contour encircling the origin twice.

The mathematical challenge in this definition of R is that since is defined in a given Lorentz frame R is not necessarily Lorentz invariant. We can of course use the gauge conditions (V.111) and (V.112) and the structure of $\mathcal{J}$ to get rid of $\mathcal{J}$. Now the G gauges bounce to and fro along the internal and this gives rise to a covariant correction factor $1/\Delta(x)$ provided that a necessary and sufficient condition for Lorentz covariance be satisfied by the matrix α_{rs}

$$\sum_{u+v=N} \alpha_{tu}(u+N/2)\alpha_{vs} = (N/2-t)\alpha_{t+N,s} - \alpha_{t,s+N}(s+3N/2) + (s+t)\alpha_{t,s}. \qquad \text{(V.117)}$$

This is indeed true and is the second example of mathematical "miracle" happening in this computation. One now obtains the transverse state saturated, and hence ghost-free, covariant expression for the residue R

$$R = \langle F\bar{F}_1 \mid \int_{(c)} \frac{dx}{4\pi i x}\,\frac{x^{L_0-1/2}}{\Delta(x)} \mid F\bar{F}_2\rangle, \qquad \text{(V.118)}$$

where

$$\Delta(x) = \det(1 - M^2(x)), \qquad \text{(V.119)}$$

$$M_{mn} = -(-\lambda)^{m+n+1}\frac{m+1/2}{m+n+1}\binom{-1/2}{m}\binom{-1/2}{n}, \quad m,n = 1,2,\cdots, \qquad \text{(V.120)}$$

and $\lambda = x^{1/2}$.

The amplitude for fermion–fermion scattering is now defined as [Olive and Scherk (1973b)]

$$F = \left\langle F\bar{F}_1 \mid \int_0^1 \frac{dx}{x}\,\frac{x^{L_0-1/2}}{\Delta(x)} \mid F\bar{F}_2\right\rangle \qquad \text{(V.121)}$$

since the residues of this expression at any given pole in the intermediate meson channels are given by (V.118) and the amplitude will turn out to be Regge-behaved.

It now remains to evaluate expression (V.121) explicitly: this looks very complicated since one has to evaluate vacuum expectation values of exponentials of quadratic forms in the Fermi b operators. However, the theory of the normal form of a product of exponentials of linear and quadratic forms of Bose or Fermi operators is well known, so that no new mathematics is needed at that stage. An example of such mathematics is shown in the following formula: [Berezin (1966); Schwarz and Wu (1973)]

$$\exp\tfrac{1}{2}(b \mid X \mid b)\exp\tfrac{1}{2}(b^+ \mid Y \mid b^+) \mid 0\rangle = \det(1+XY)^{D/2} \times \exp\tfrac{1}{2}(b^+ \mid Y(1+XY)^{-1} \mid b^+) \mid 0\rangle, \qquad \text{(V.122)}$$

where X, Y are arbitrary matrices.

For the simple fermion–fermion scattering, without meson emission one obtains the expression for F_4 [Schwarz and Wu (1973a), Corrigan (1974)]

$$F_4 = \int_0^1 dx x^{-\alpha(s)-1}(1-x)^{-k_2\cdot k_3}\frac{\delta(x)^{D/2}}{\Delta(x)} \times \sum_{k=0}^{D}(1/k!)S_k\cdot\bar{S}_k[v^T(1-A^2)^{-1}v]: \qquad \text{(V.123)}$$

where

$$\delta(x) = \det(1-A^2(x)), \qquad \text{(V.124)}$$

$$A_{mn} = \tfrac{1}{2}(M_{mn}-M_{nm}), \qquad \text{(V.125)}$$

$$v_n = [1/(2)^{1/2}](-)^n\lambda^{n+1/2}\binom{-1/2}{n}. \qquad \text{(V.126)}$$

The S_k, $\bar{S}_k$ are antisymmetric tensors constructed from products of k, γ matrices sandwiched between the spinors $\bar{u}_1$, u_2 and $\bar{u}_3$, u_4, respectively. We see that the t channel contains singularities: it is however impossible to study these singularities without explicitly evaluating the infinite determinants occurring in the integrand and the quantity $v^T(1/1-A^2)v$. One can show that [Corrigan (1974)]

$$\Delta/\delta = 1 - v^T(1-A^2)^{-1}v \qquad \text{(V.127)}$$

so that one needs only to evaluate the infinite determinants Δ, δ. [Schwarz and Wu (1973)] guessed that in spite of their formidable expressions, $\delta(x)$ and $\Delta(x)$ are simple functions given by

$$\Delta(x) = (1-x)^{1/4} \qquad \text{(V.128)}$$

and

$$\delta(x) = \tfrac{1}{2}(1-x)^{-1/4}(1+(1-x)^{1/2}) \qquad \text{(V.129)}$$

and verified this by computer. Later, an analytic proof of these equations was found by setting up a system of differential equations for

$$\zeta_n = \sum_p (1-M)_{np}^{-1}v_p$$

and solving this system [Corrigan, Goddard, Olive, Smith (1973)].

Finally, the fermion–fermion amplitude takes the simple form

$$F_4 = 2^{-D/2}\int_0^1 dx x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1} \times \sum_{k=0}^{D}(1/k!)S_k\cdot\bar{S}_k x^{k/2}(1+(1-x)^{1/2}). \qquad \text{(V.130)}$$

This is precisely the form obtained by S. Mandelstam using functional techniques. Here F_4 is a sum of beta functions and hence is obviously Regge-behaved. It has poles in the s and t channels; the location of the poles depends on the dimensions of space–time D. For $D = 10$ $\alpha(t)$ is precisely equal to the pion trajectory: $\alpha(t) = \frac{1}{2} + t/2$. So the poles in the s and t channels occur at the same masses.

In order to find the duality properties of F_4, one needs to express (V.130) in terms of the quantities occurring in the t channel. Let us define the quantities T_k, $k = 0, 1, \cdots D$ by

$$T_k \doteq \sum_{A_k}\bar{u}_1\Gamma^{A_k}u_2\bar{u}_3\Gamma^{A_k}u_4, \qquad \text{(V.131)}$$

where the sum runs over all products Γ^{A_k} of distinct γ matrices. The analogous T_k' occurring in the t channel are defined as

$$T_k' = \sum_{A_k}\bar{u}_1\Gamma^{A_k}u_4\bar{u}_3\Gamma^{A_k}u_2, \qquad \text{(V.132)}$$

and the T_k' are related to the T_k by the general Fierz transformation in arbitrary space–time dimension D [Case (1956)]

$$T_k' = \sum_l T_l F_{lk},$$

where

$$F_{lk} = (-)^{lk}\frac{1}{2\pi i}\oint\frac{dz}{z}z^{-k}\left(\frac{1+z}{2^{1/2}}\right)^{D-l}\left(\frac{1-z}{2^{1/2}}\right)^{l}. \qquad \text{(V.133)}$$

Defining $c_{lk} = (-1)^{lk}F_{lk}$ a very nice mathematical identity is

$$x^{1/2l}(1+(1-x)^{1/2})^{1/2D-l} = \sum_k c_{lk}y^{1/2k} \times (1+(1-y)^{1/2})^{1/2D-l} \qquad \text{(V.134)}$$

where $y = 1-x$. Hence not only does the Fierz transformation allow us to go from the T_k to the T_k' but it also does the change of variable $x \to 1-x$ for us!

The scalar products $U_k = (1/k!)S_k\cdot\bar{S}_k$ are not however completely identical with the T_k because of the presence of a γ_5 matrix in the S_k which have an even index. This γ_5 comes from the definitions of the fermion vertex and is seen in Eq. (V.102). The relation between T_k and U_k is

$$U_k = T_k, \quad k \text{ odd}, \qquad U_k = T_{D-k}, \quad k \text{ even}.$$

Using this relation and the Fierz transformation one obtains the duality relations for the integrand

$$\sum_k U_k f_k(x) = \sum_l U_l' f_l(1-x), \qquad \text{(V.135)}$$

where

$$U_l' = (-1)^l T_l', \quad l \text{ odd or even}.$$

Hence the quantities with even label have changed parity under crossing: the ground state is a pseudoscalar in the s channel, but a scalar in the t channel. In general, states which have odd G parity have their parity changed under the crossing transformation. Hence the "$q\bar{q}$ mesons" can be defined and factorized as indicated at the beginning of the previous section. Note that apart from the change of parity

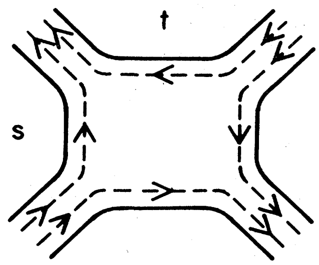


FIG. 22. Fermion–fermion scattering in the string picture revealing the similarity between s- and t-channel mesons.

this result could have been guessed from a glance at Fig. 22 by inspection of the s and t channel content.

More complicated diagrams such as those involving two fermion lines emitting mesons have also been computed and the results obtained indicate that the corresponding amplitudes are well-behaved [Schwarz and Wu (1974)]. Amplitudes involving more than two fermion lines have not been investigated yet, but it seems reasonable to think that, according to the string picture, they will reserve no surprises.

It is also possible to compute one- and N-loop diagrams [Goddard and Waltz (1971), Montonen (1973)] in the mesonic model, and one obtains similar results as in the conventional model. Although the theory of fermionic loops is still incomplete, the introduction of the planar fermion loop has been shown [Green (1973)] to make the theory more convergent. So it seems very likely that the quantized interacting spinning string theory has the same degree of internal consistency as the conventional spinless model.

CONCLUSION

We hope to have achieved our aim to convince the reader that the covariant treatment of dual models in the operator formalism, and the transverse string picture are two complementary faces of a single mathematical structure, having a high and maybe perfect degree of self-consistency. Whether or not these mathematical structures have anything to do with the real world is still unclear, and one has to wait to see whether more realistic models can be built or not. At the worst, it seems that the existing mathematical structures can be to hadron physics what the two dimensional Ising model is to the theory of ferromagnetism.

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