Recoil effects in allowed beta decay: The elementary particle approach*

Barry R. Holstein

Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01002

Allowed beta decay is studied using the elementary particle approach including all second forbidden terms. Model independent forms are given for various spectrum and correlation effects with parent polarization and/or orientation and for delayed particle emission correlations. The restrictions due to various symmetry assumptions are analyzed, as well as the problem of Coulomb corrections. We comment on past and future experiments in this realm.

CONTENTS

I. INTRODUCTION

Recent experiments have begun to probe the effects of so-called recoil terms in allowed ($\Delta J = 0, \pm 1$; "no") nuclear and hyperon beta decays (Garcia, 1971; Lee, Mo, and Wu, 1963; Lindquist *et al.*; 1971, Tribble and Garvey, 1974). Measurement of such terms, e.g., the induced tensor, can provide important information conerning nuclear structure (Preston, 1962), the correctness of the conserved vector current hypothesis (Bernstein and Lewis, 1958, Bouchiat, 1959, Gell-Mann, 1958), the existence of second class currents (Beg and Bernstein, 1972; Holstein and Treiman, 1971; Weinberg, 1958; Wilkinson, 1970b), etc. There are two distinct types of recoil terms which must be dealt with. The first, of which weak magnetism is an example, is of the order of momentum transfer q divided by the *nucleon* mass m. Since q is typically several MeV, while the nucleon mass is about one GeV, changes in decay spectra due to weak magnetism are generally a percent or so of the dominant Fermi and Gamow–Teller contributions. Nevertheless, these are measurable and can be appreciable if the leading contribution is suppressed for some reason. A second type of recoil term depends upon the nuclear radius R. For parity reasons it contributes quadratically in order $q^2R^2 \sim [6(q/m)A^{1/3}]^2$. Again such corrections can be of the order of several percent. Both types of effects must be included in analysis of precise allowed β-spectra measurements.

There have, of course, been numerous papers—even books—written about nuclear beta decay (Blin-Stoyle, 1973; Konopinski, 1966; Morita, 1973; Schopper, 1966; Wu, 1966), but many either omit recoil effects or include them in a model-dependent framework, generally based on the nuclear impulse approximation. The "elementary particle" viewpoint (Kim and Primakoff, 1965a; Kim and Primakoff, 1965b) attempts to reformulate beta decay theory so as to specify those expressions which are generally valid and those containing approximations. We attempt to separate the purely kinematic structure, the electromagnetic interaction between electron and nucleus, and the model-dependent analysis of nuclear matrix elements. Nuclei are, as far as calculations of the decay spectra are concerned, designated only by their intrinsic spin and parity and by their external four-momentum. In the next section we define notation and decompose the matrix element connecting initial and final nuclei via the weak hadronic current in terms of known kinematic terms multiplied by corresponding structure functions or form factors. In the absence of recoil there are, of course, only two such form factors—the Fermi and Gamow–Teller matrix elements. With inclusion of recoil to first order in q/m, q^2R^2 there are in general ten such structure functions, five associated with the polar vector and five with the axial vector current. For transitions involving low spins the number is, however, reduced. All spectra can be calculated in terms of the form factors without any assumptions about nuclear wave functions or the form of the hadronic current. Nuclear model dependence and effects due to the strong interactions are contained in the structure functions, which play the role of reduced matrix elements in the conventional analysis.

Section III analyzes the restrictions placed on these form factors by various assumptions of symmetries possessed by the hadronic current. Time reversal invariance requires that each structure function be real. The conserved vector current assumption of Feynman and Gell-Mann reduces the number of independent polar vector form factors from five to three and relates these three to measurable electromagnetic phenomena. The assumption that second class currents are not present in the weak current reduces the number of form factors to three polar vector and three axial vector terms if an analog decay is under discussion. In the case of nonanalog beta decay there is no reduction in the number of form factors, but a relation between the form factors of corresponding mirror transitions emerges.

Having produced the structural framework in which to analyze the decay process, the calculational machinery is activated, and the decay spectra are given in Sec. IV for (i) decay of polarized (and oriented) nuclei, (ii) β-particle correlation effects for a situation in which the daughter nucleus (from the β-decay of an unpolarized parent) itself undergoes a decay into a granddaughter nucleus and an additional alpha particle or (possibly polarized) photon. Section V quotes predictions for the form factors in terms

* Supported in part by the National Science Foundation.

 Copyright © 1974 American Physical Society

of the conventional impulse approximation, while Sec. VI discusses modifications of the previous formalism produced by the introduction of final state electromagnetic interactions between outgoing lepton and hadrons. Finally, in Sec. VII we look at the current experimental situation and suggest some possibilities for future experimental work.

Similar calculations can, of course, be performed for forbidden β-processes (Armstrong and Kim, 1972b). Nevertheless, we shall discuss here only the allowed case as it forms perhaps the most important class of β-decay phenomena and includes analog transitions, which, as we shall demonstrate, constitute an important reservoir of information about symmetries of the hadronic current.

II. DEFINITIONS

For definiteness we consider the case of electron decay. Modifications appropriate to positron decay appear in final formulas. We consider the reaction

$$\alpha \rightarrow \beta + e^- + \bar{\nu}_e.$$

Let p_1, p_2, p, k denote the respective four-momenta of parent nucleus, daughter nucleus, electron, and neutrino. The parent and daughter masses are M_1 and M_2. We also define

$$P = p_1 + p_2 \qquad q = p_1 - p_2 = p + k$$

$$M = \tfrac{1}{2}(M_1 + M_2) \qquad \Delta = M_1 - M_2.$$

For low energy β-decay processes the form of the interaction is thought to be known. It consists of the product of a leptonic current with a nuclear current. In the case that the electromagnetic interaction between the nuclei and the electron can be neglected the transition matrix element assumes a particularly simple form. For convenience, in this section we shall neglect such electromagnetic effects. The weak ($\Delta S = 0$) decay amplitude is then given by

$$T = (G_v/\sqrt{2}) \cos\theta_c \langle \beta \mid V_\mu(0) + A_\mu(0) \mid \alpha \rangle l^\mu, \tag{1}$$

where G_v is the usual weak-coupling constant ($G_v m_p^2 \simeq 10^{-5}$), $\theta_c = 15°$ is the Cabibbo angle, and l^μ is the matrix element of the lepton current[1]

$$l^\mu = \bar{u}(p)\gamma^\mu(1 + \gamma_5)v(k). \tag{2}$$

For strangeness changing decays we, of course, replace $\cos\theta_c$ by $\sin\theta_c$.

In the rest frame of the parent nucleus, let $E(\mathbf{p})$ denote the energy (three-momentum) of the electron and let $\hat{k}$ be a unit vector in the direction of the neutrino momentum. The maximum electron energy allowed by kinematics is E_0,

$$E_0 = \Delta\left(1 + \frac{m_e^2}{2M\Delta}\right) \bigg/ \left(1 + \frac{\Delta}{2M}\right), \tag{3}$$

[1] We utilize here the metric and conventions of J. D. Bjorken and S. D. Drell (1964) except for a change in sign for γ_5.

where m_e is the electron mass. Then, to first order in E/M the decay spectrum is given by

$$d^5\Gamma = \frac{|T|^2}{(2\pi)^5}\left[1 + \frac{3E - E_0 - 3\mathbf{p}\cdot\hat{k}}{M}\right] \times (E_0 - E)^2 pE\, dE\, d\Omega_e\, d\Omega_\nu. \tag{4}$$

For the familiar case of hyperon β-decay ($j = j' = \frac{1}{2}$), one can write for the matrix element of the weak current

$$\begin{aligned}\langle \beta \mid V_\mu + A_\mu \mid \alpha \rangle &= \bar{u}(p_2)[g_V\gamma_\mu + (1/2M)g_S q_\mu - i(g_M - g_V) \\ &\quad \times (1/2M)\sigma_{\mu\nu}q^\nu + g_A\gamma_\mu\gamma_5 + (1/2M)g_P q_\mu\gamma_5 \\ &\quad - ig_{II}(1/2M)\sigma_{\mu\nu}q^\nu\gamma_5]u(p_1).\end{aligned} \tag{5}$$

Here $g_V(g_A)$ is the conventional vector (axial-vector) coupling constant, $g_S(g_P)$ is the induced scalar (pseudoscalar), and $g_M(g_{II})$ is the weak magnetism (induced tensor) term. All aspects of the decay spectra can be analyzed in terms of these six form factors. Our procedure is to reduce the expression in Eq. (5) to a form involving only two-component (Pauli) spinors and then to generalize this version to an arbitrary allowed nuclear β decay.

Suppose parent (daughter) nucleus has spin $J(J')$ and spin component $M(M')$ along some axis of quantization. A general covariant decomposition of the hadron matrix element has been given by Stech and Schülke (1964). It is strictly valid in the Breit system but for nuclear beta decay the Breit and laboratory frames are equivalent for practical purposes. The nuclear current depends only on the momentum transfer $\mathbf{q}$. From rotational invariance and parity considerations we have then

$$\begin{aligned}\langle \beta \mid V_0 \mid \alpha \rangle &= \sum_{j\ \text{even}} \sum_{m=-j}^{j} C_{J'j;J}^{M'm;M}[4\pi/(2l+1)]^{1/2} \\ &\quad \times Y_j^m(\hat{q})F_j^V(q^2)(|\mathbf{q}|/2M)^j, \\ \langle \beta \mid \mathbf{V} \mid \alpha \rangle &= \sum_{l\ \text{odd}} \sum_{j=l-1}^{l+1} \sum_{m=-j}^{j} C_{J'j;J}^{M'm;M}[4\pi/(2l+1)]^{1/2} \\ &\quad \times \mathbf{T}_{jl}^m(\hat{q})F_{jl}^V(q^2)(|\mathbf{q}|/2M)^l, \\ \langle \beta \mid A_0 \mid \alpha \rangle &= \sum_{j\ \text{odd}} \sum_{m=-j}^{j} C_{J'j;J}^{M'm;M}[4\pi/(2l+1)]^{1/2} \\ &\quad \times Y_j^m(\hat{q})F_j^A(q^2)(|\mathbf{q}|/2M)^j, \\ \langle \beta \mid \mathbf{A} \mid \alpha \rangle &= \sum_{l\ \text{even}} \sum_{j=l-1}^{l+1} \sum_{m=-1}^{j} C_{J'j;J}^{M'm;M}[4\pi/(2l+1)]^{1/2} \\ &\quad \times \mathbf{T}_{jl}^m(\hat{q})F_{jl}^A(q^2)(|\mathbf{q}|/2M)^l,\end{aligned} \tag{6}$$

where the spherical harmonics and vector spherical harmonics are as defined by Rose (Rose, 1957). Arbitrary decay spectra can be completely analyzed in terms of form factors $F_j^V(q^2)$, $F_{jl}^V(q^2)$, $F_j^A(q^2)$, $F_{jl}^A(q^2)$. We, however, choose to deal with allowed ($\Delta J = 0, \pm 1$; no) decay only and we restrict consideration to terms of first order in q/m or q^2R^2, as discussed in the Introduction. Also, in order to reveal clearly the inherent symmetries, rather than the above parameterization, we prefer to utilize the definitions[2]

$$l^\mu\langle\beta \mid V_\mu \mid \alpha\rangle$$
$$\equiv \left(a(q^2)\frac{P\cdot l}{2M} + e(q^2)\frac{q\cdot l}{2M}\right)\delta_{JJ'}\delta_{MM'}$$
$$+ i\frac{b(q^2)}{2M} C_{J'1;J}{}^{M'k;M}(\mathbf{q}\times\mathbf{l})_k$$
$$+ C_{J'2;J}{}^{M'k;M}\left[\frac{f(q^2)}{2M} C_{11;2}{}^{nn';k} l_n q_{n'}\right.$$
$$\left.+ \frac{g(q^2)}{(2M)^3} P\cdot l(4\pi/5)^{1/2} Y_2{}^k(\hat{q})\mathbf{q}^2\right] + \cdots \tag{7a}$$

$$l^\mu\langle\beta \mid A_\mu \mid \alpha\rangle$$
$$\equiv C_{J'1;J}{}^{M'k;M}\epsilon_{ijk}\epsilon_{ij\lambda\eta}\frac{1}{4M}\left[c(q^2)l^\lambda P^\eta - d(q^2)l^\lambda q^\eta\right.$$
$$\left.+ \frac{1}{(2M)^2} h(q^2) q^\lambda P^\eta q\cdot l\right]$$
$$+ C_{J'2;J}{}^{M'k;M} C_{12;2}{}^{nn';k} l_n (4\pi/5)^{1/2} Y_2{}^{n'}(\hat{q})\frac{\mathbf{q}^2}{(2M)^2} j_2(q^2)$$
$$+ C_{J'3;J}{}^{M'k;M} C_{12;3}{}^{nn';k} l_n (4\pi/5)^{1/2}$$
$$\times Y_2{}^{n'}(\hat{q})\frac{\mathbf{q}^2}{(2M)^2} j_3(q^2) + \cdots. \tag{7b}$$

This decomposition is completely equivalent to the Stech–Schülke expression in Eq. (6) as may be confirmed by substitution of

$$a(q^2) = (1+\Delta/2\mathrm{M})^{-1}\left[F_0{}^V(q^2) + \frac{\Delta}{2M}(\tfrac{1}{3})^{1/2}F_{01}{}^V(q^2)\right],$$
$$b(q^2) = -(\tfrac{1}{2})^{1/2}F_{11}{}^V(q^2),$$
$$c(q^2) = -(1+\Delta/2M)^{-1}\left[F_{10}{}^A(q^2) + \frac{\Delta}{2M}F_1{}^A(q^2)\right.$$
$$\left. - \frac{1}{(10)^{1/2}}F_{12}{}^A(q^2)\frac{1}{(2M)^2}(2\Delta^2 + q^2)\right],$$
$$d(q^2) = (1+\Delta/2M)^{-1}\left[F_{10}{}^A(q^2) + F_1{}^A(q^2)\right.$$
$$\left. - \frac{6}{(10)^{1/2}}F_{12}{}^A(q^2)\left(M\Delta + \tfrac{1}{6}(\Delta^2 - q^2)\right)\frac{1}{(2M)^2}\right],$$
$$e(q^2) = (1+\Delta/2M)^{-1}[F_0{}^V(q^2) - (\tfrac{1}{3})^{1/2}F_{01}{}^V(q^2)],$$
$$f(q^2) = -F_{21}{}^V(q^2),$$
$$g(q^2) = F_2{}^V(q^2),$$
$$h(q^2) = -(1+\Delta/2M)^{-1}\frac{3}{(10)^{1/2}}F_{12}{}^A(q^2),$$
$$j_2(q^2) = F_{22}{}^A(q^2),$$
$$j_3(q^2) = -\mathrm{F}_{32}{}^A(q^2). \tag{8}$$

The expansion given in Eq. (7) can easily be generalized into a Cartesian and manifestly covariant form utilized by many practitioners of the elementary particle method (Armstrong and Kim, 1972b; Kim and Primakoff, 1965a; Kim and Primakoff, 1965b; Kudobera and Kim, 1973; Primakoff, 1967; Primakoff, 1970), as discussed in Appendix A. It is in this form that symmetry restrictions on invariant form factors $a(q^2), \cdots, j_3(q^2)$ are most apparent. For example, considering only the $\Delta J = 0$ terms in the analysis of the vector currents, our expansion reads

$$\delta_{JJ'}\delta_{MM'}(aP_\lambda + eq_\lambda)(l^\lambda/2M)$$
$$= \delta_{JJ'}\delta_{MM'}(1/2M)[(aP^0 + eq^0)l^0 - (a\mathbf{P} + e\mathbf{q})\cdot\mathbf{l}]$$
$$\cong \delta_{JJ'}\delta_{MM'}\left[\left(a + e\frac{\Delta}{2M}\right)l^0 + (a-e)\frac{\mathbf{q}\cdot\mathbf{l}}{2M}\right] \tag{9}$$

while corresponding terms in the Stech-Schülke decomposition are

$$\delta_{JJ'}\delta_{MM'}\left[F_0{}^V(q^2)l^0 + (\tfrac{1}{3})^{1/2}F_{01}{}^V(q^2)\frac{\mathbf{q}\cdot\mathbf{l}}{2M}\right]. \tag{10}$$

The relation between the two forms is easily found by equating

$$a + e(\Delta/2M) = F_0{}^V$$
$$a - e = (\tfrac{1}{3})^{1/2}F_{01}{}^V$$

which yields the results quoted in Eq. (8). However, the terms in Eq. (9) are manifestly covariant, and as we shall demonstrate in the following section, it is $a(q^2)$ and $e(q^2)$—*not* $F_0{}^V(q^2)$, $F_{01}{}^V(q^2)$—which have simple behavior with respect to symmetry properties of the hadronic current such as the conserved vector current hypothesis or the absence of second class currents. On the other hand, the Stech-Schülke representation in Eq. (6) is calculationally easier to handle due to the orthogonality properties of the spherical tensors employed. Our "hybrid" notation, with aspects both of the manifestly covariant Cartesian form discussed in Appendix A and the spherical tensor approach utilized by Stech and Schülke attempts to maintain the

---

[2] In terms of the hyperon decay form factors defined in Eq. (5) we have

$a = g_V \quad b = \sqrt{3}\, g_M \quad c = \sqrt{3}\, g_A \quad d = \sqrt{3}\, g_{II}$
$e = g_S \quad h = \sqrt{3}\, g_P$

Also, we note that we deviate here from the convention used in previous papers, wherein we changed the sign of the induced tensor for electron and positron decays.

calculational convenience afforded by spherical tensors *and* the straightforward identification of symmetry restrictions obvious in the Cartesian version.

Altough Eq. (7) utilizes ten invariant form factors—five polar vector and five axial vector—in the general decomposition of the current matrix elements, for low spin transitions, the triangle inequality satisfied by Clebsch-Gordan cofficients requires the vanishing of one or more terms. Thus we find

$$\begin{aligned}
&J = J' = 0 && b = c = d = f = g = h = j_2 = j_3 = 0\\
&\left.\begin{array}{l} J = 1,\ J' = 0\\ J = 0,\ J' = 1\end{array}\right\} && a = e = f = g = j_2 = j_3 = 0\\
&J = J' = 1 && j_3 = 0\\
&J = J' = \tfrac{1}{2} && f = g = j_2 = j_3 = 0\\
&\left.\begin{array}{l} J = \tfrac{1}{2},\ J' = \tfrac{3}{2}\\ J = \tfrac{3}{2},\ J' = \tfrac{1}{2}\end{array}\right\} && a = e = j_3 = 0.
\end{aligned} \tag{12}$$

All information about the structure of the nucleus and of the hadronic current is contained in the invariant form factors. No assumptions concerning nuclear models has gone into the decomposition given in Eq. (7). The connection with models for the transition current (e.g., the impulse approximation) and with nuclear wave functions need not be made at this stage. Decay spectra should be analyzed to reveal information about the *model-independent* structure functions.

III. SYMMETRIES

Thus far we have not assumed any particular restrictions on the hadronic currents V_μ, A_μ. There are, however, various theoretical ideas about their structure which permit (nuclear-) model independent predictions to be made concerning the values of some of the weak form factors. We shall explore several of these ideas in this section.

(i) T-invariance (Henley 1969; Holstein 1972; Kim and Primakoff 1969): The assumption of time reversal invariance reads[3]

$$T(V_\mu{}^\pm + A_\mu{}^\pm)T^{-1} = V^{\mp\mu} + A^{\mp\mu}. \tag{13}$$

Labelling states by their momentum and spin we have

$$T \mid \mathbf{p}, J, M\rangle = \langle -\mathbf{p}, J, -M \mid i^{2(J+M)}. \tag{14}$$

Then the T-invariance condition is

$$\begin{aligned}
&\langle \mathbf{p}'J'M' \mid V_\mu + A_\mu \mid \mathbf{p}JM\rangle\\
&\quad= \langle \mathbf{p}'J'M' \mid T^{-1}T(V_\mu + A_\mu)T^{-1}T \mid \mathbf{p}JM\rangle\\
&\quad= \langle -\mathbf{p}'J' - M' \mid V^\mu + A^\mu \mid -\mathbf{p}J - M\rangle^*(-)^{J+M-J'-M'}
\end{aligned} \tag{15}$$

which requires that all invariant form factors be real. Thus, detection of a phase difference between any two form factors signifies T-violation, apart from apparent phase differences induced by final state electromagnetic interactions (Brodine, 1970a; Brodine, 1970b; Callen and Treiman, 1967; Chen, 1969a, Chen, 1969b; Holstein, 1972).

(ii) Conserved vector current (Feynman and Gell-Mann, 1958; Gerschtein and Zeldovich, 1955): The conserved vector current (CVC) hypothesis of Feynman and Gell-Mann identifies the charge raising (lowering) vector current with that obtained from the electromagnetic current by means of isotopic spin rotation

$$V_\mu{}^\pm = \mp[I^\pm, V_\mu{}^{e\mathfrak{M}}] \qquad I^\pm = I_1 \pm iI_2. \tag{16}$$

Suppose that α, β are members of a common isotopic multiplet

$$\mid\alpha\rangle = \mid I, I_z\rangle \qquad \mid\beta\rangle = \mid I, I_z \pm 1\rangle.$$

Then

$$\begin{aligned}
\langle\beta \mid V_\mu{}^\pm \mid \alpha\rangle = \pm[(I \mp I_z)(I \pm I_z + 1)]^{1/2}\\
\times (\langle\beta \mid V_\mu{}^{e\mathfrak{M}} \mid \beta\rangle - \langle\alpha \mid V_\mu{}^{e\mathfrak{M}} \mid \alpha\rangle).
\end{aligned} \tag{17}$$

To second order in recoil the interaction of a nucleus with the electromagnetic field can be described in terms of three invariant form factors, whose values at $q^2 = 0$ correspond to its charge, magnetic dipole moment, and electric quadrupole moment. The multipole expansion of the electromagnetic interaction is conventionally written (Preston, 1962)

$$H_{e\mathfrak{M}} = Ze\phi_0 - \mathbf{M}\cdot\mathbf{B} - \tfrac{1}{6}\sum_{ij} Q_{ij}(\partial E_j/dx_i) + \cdots. \tag{18}$$

Here

$$\begin{aligned}
\mathbf{M} &= (e\mu/2m_p)(\mathbf{J}/J)\\
Q_{ij} &= [eQ/J(2J-1)][\tfrac{3}{2}(J_iJ_j + J_jJ_i) - \delta_{ij}J^2],
\end{aligned} \tag{19}$$

where μ is the magnetic dipole moment (measured in proton magnetons) and Q is the electric quadrupole moment of the nucleus with charge Ze. If we write for a nucleus of spin J

$$\begin{aligned}
&\langle\alpha \mid V_\mu{}^{e\mathfrak{M}} \mid \alpha\rangle\\
&\quad= \delta_{MM'}(P_\mu/2M)F_1(q^2)\\
&\qquad - iC_{J1;J}{}^{M'k;M}\epsilon_{ijk}g_{\mu j}(q^i/2M)F_2(q^2)\\
&\qquad + C_{J2;J}{}^{M'k;M}P_\mu/(2M)^3(4\pi/5)^{1/2}\\
&\qquad \times Y_2{}^k(\hat{q})\mathbf{q}^2F_3(q^2) + \cdots
\end{aligned} \tag{20}$$

then comparison with Eq. (18) yields

$$\begin{aligned}
F_1(0) &= Ze,\\
F_2(0) &= \left(\frac{J+1}{J}\right)^{1/2}\mu A,\\
F_3(0) &= -\left[\frac{(J+1)(2J+3)}{J(2J-1)}\right]^{1/2}\frac{2M^2}{3}Q,
\end{aligned} \tag{21}$$

[3] Here $V_\mu{}^-$ $(V_\mu{}^+)$ refers to the charge lowering (raising) current which are Hermitian conjugates of one another.

and the CVC condition gives

$$\begin{aligned} a(q^2) &= \pm M_F[F_1{}^\beta(q^2) - F_1{}^\alpha(q^2)], \\ b(q^2) &= \pm M_F[F_2{}^\beta(q^2) - F_2{}^\alpha(q^2)], \\ g(q^2) &= \pm M_F[F_3{}^\beta(q^2) - F_3{}^\alpha(q^2)], \\ M_F &= [(I \mp I_z)(I \pm I_z + 1)]^{1/2}, \\ e(q^2) &= f(q^2) = 0, \end{aligned} \tag{22}$$

which relates weak form factors to ones measurable in elastic electron scattering experiments from the initial and final nuclei.

If α, β are not members of a common isotopic multiplet the CVC condition relates the weak form factors to form factors measurable via inelastic electron scattering or by analysis of radiative decay widths. If

$$| \alpha \rangle = | I, I_z \rangle \qquad | \beta \rangle = | I', I_z \pm 1 \rangle,$$

we find

$$\begin{aligned} &\langle \beta \mid V_\mu{}^\pm \mid \alpha \rangle \\ &\quad = \mp[(I' \pm I_z + 1)(I' \mp I_z)]^{1/2} \langle I' I_z \mid V_\mu{}^{\mathrm{e}\mathfrak{M}} \mid I I_z \rangle \\ &\qquad \pm [(I \mp I_z)(I \pm I_z + 1)]^{1/2} \\ &\qquad \times \langle I' I_z \pm 1 \mid V_\mu{}^{\mathrm{e}\mathfrak{M}} \mid I I_z \pm 1 \rangle. \end{aligned} \tag{23}$$

Also since $\partial^\mu V_\mu{}^\pm = 0$ (in the absence of electromagnetic effects) we have

$$\begin{aligned} 0 = {} & (E_\alpha - E_\beta) \langle \beta \mid V_0(0) \mid \alpha \rangle \\ & - (\mathbf{P}_\alpha - \mathbf{P}_\beta) \cdot \langle \beta \mid \mathbf{V}(0) \mid \alpha \rangle. \end{aligned} \tag{24}$$

But

$$\begin{aligned} &\int dx^3 \langle \beta \mid V_0{}^\pm(\mathbf{x}, 0) \mid \alpha \rangle \\ &\quad = (2\pi)^3 \delta^3(\mathbf{P}_\alpha - \mathbf{P}_\beta) \langle \beta \mid V_0{}^\pm(0) \mid \alpha \rangle \\ &\quad = [\text{from Eq. (24)}]\ 0 \text{ unless } E_\alpha = E_\beta. \end{aligned} \tag{25}$$

Thus in the absence of isospin mixing we must have

$$a(q^2 = 0) = 0 \tag{26}$$

unless α, β are isotopic analogs.

Another way of seeing this is to note that

$$\begin{aligned} &\langle \beta \mid \partial^\mu V_\mu{}^\pm \mid \alpha \rangle \\ &\quad = i\delta_{JJ'}\delta_{MM'}[a(q^2)\Delta + e(q^2)(q^2/2M)] \\ &\qquad + C_{J'2;J}{}^{M'k;M} \left[\left(\frac{2}{3}\right)^{1/2} \frac{f(q^2)}{2M} + g(q^2) \frac{\Delta}{(2M)^2} \right] \\ &\qquad \times \left(\frac{4\pi}{5}\right)^{1/2} \mathbf{q}^2 Y_2{}^k(\hat{q}) + \cdots = 0 \end{aligned} \tag{27}$$

requires[4]

$$\begin{aligned} a(q^2) &= -(2M\Delta)^{-1} q^2 e(q^2) \\ g(0) &= -(2/3)^{1/2}(2M/\Delta) f(0) \end{aligned} \tag{28}$$

which, since $e(q^2)$ should not have a pole at $q^2 = 0$, reproduces Eq. (26). In addition we see that also in the case of nonanalog decay there exist only three independent form factors, since e, f are related to a, g via Eq. (28).

Although we may probe all three structure functions via inelastic electron scattering, radiative decay width measurements are only sensitive to the presence of the $M1$ amplitude (b) and the $E2$ amplitude (f). Defining

$$\begin{aligned} &\langle \beta \mid V_\mu{}^{\mathrm{e}\mathfrak{M}} \mid \alpha \rangle \\ &\quad \equiv \delta_{JJ'}\delta_{MM'}[F_1(q^2)P_\mu + F_4(q^2)q_\mu](2M)^{-1} \\ &\qquad - i[F_2(q^2)/2M]\, C_{J'1;J}{}^{M'k;M} \epsilon_{ijk} q^i g_{\mu j} \\ &\qquad + C_{J'2;J}{}^{M'k;M} \left[-\frac{F_5(q^2)}{2M} C_{11;2}{}^{nn';k} g_{\mu n} q_{n'} \right. \\ &\qquad \left. + \frac{F_3(q^2)}{(2M)^3} P_\mu (4\pi/5)^{1/2} Y_2{}^k(\hat{q}) \mathbf{q}^2 \right] + \cdots \end{aligned} \tag{29}$$

the rate for the transition $\alpha \to \beta\gamma$ is

$$\Gamma_{\alpha\to\beta\gamma} = (e^2\Delta^3/8\pi M^2)[\tfrac{2}{3} \mid F_2(0) \mid^2 + \tfrac{1}{5} \mid F_5(0) \mid^2]. \tag{30}$$

Often the transition is from an isotopic triplet state to an isotopic singlet. Then only the isovector component of the electromagnetic current can contribute and provided the $M1/E2$ ratio is known a measurement of the radiative width is sufficient to determine $F_2(0)$ and $F_5(0)$, which can be related to the weak decay parameters b, f via

$$b(q^2 = 0) = \pm\sqrt{2} F_2(0) \qquad f(q^2 = 0) = \pm\sqrt{2} F_5(0). \tag{31}$$

On the other hand, if the transition in question involves an istotopic spin sequence wherein both the iso-vector *and* iso-scalar pieces of the electromagnetic current contribute, then experiments must also be done on a mirror electromagnetic transition in order to isolate the isovector $M1$ and $E2$ coeficients via Eq. (23).

(iii) Second class currents (Beg and Bernstein, 1972; Cabibbo, 1964; Delorme and Rho, 1971a; Delorme and Rho, 1971b; Delorme and Rho, 1972; Holstein and Treiman, 1971; Kim, 1971; Kim and Fulton, 1971; Kubodera, Delorme, and Rho, 1973; Maiani, 1968; Weinberg, 1958; Wilkinson, 1970; Wilkinson and Alburger, 1970a; Wilkinson and Alburger, 1970b; Wilkinson and Alburger, 1971)—Weinberg has noted the utility of classifying currents as to their symmetry under the combined poerations of

[4] The relation between g, f is quoted only at $q^2 = 0$ since for nonzero momentum transfer the CVC condition involves an additional higher order form factor.

charge conjugation and charge symmetry. First class currents are defined by

$$GV_\mu^{\mathrm{I}}G^{-1} = V_\mu^{\mathrm{I}}$$
$$GA_\mu^{\mathrm{I}}G^{-1} = -A_\mu^{\mathrm{I}} \tag{32}$$

while the so-called second class currents satisfy

$$GV_\mu^{\mathrm{I}}G^{-1} = -V_\mu^{\mathrm{II}}$$
$$GA_\mu^{\mathrm{II}}G^{-1} = A_\mu^{\mathrm{II}}, \tag{33}$$

where $G = C\exp(-i\pi I_2)$. Assuming CPT invariance we have then

$$\exp(-i\pi I_2)J_\mu^{x\pm}\exp(i\pi I_2) = \epsilon_x J_\mu^{x\mp}, \tag{34}$$

with

$$\epsilon_{\mathrm{I}} = \mp 1 \qquad \text{if} \quad TJ_\mu^{\mathrm{I}}T^{-1} = \pm J^{\mathrm{I}\mu+}$$
$$\epsilon_{\mathrm{II}} = \pm 1 \qquad \text{if} \quad TJ_\mu^{\mathrm{II}}T^{-1} = \pm J^{\mathrm{II}\mu+} \tag{35}$$

For transitions within a common isotopic multiplet we have

$$\langle I, I_z \pm 1; \mathbf{p}', J, M' \mid J_\mu^x \mid I, I_z; \mathbf{p}, J, M\rangle$$
$$= -\epsilon_x\langle I, -I_z; \mathbf{p}, J, M \mid J_\mu^x \mid I, -I_z \pm 1; \mathbf{p}', J, M'\rangle^*. \tag{36}$$

Decomposing the current as a sum of iso-tensor operators of rank $R \geq 1$

$$J_\mu^x = \sum_{R=1} J_\mu^{xR} \tag{37}$$

we may utilize the Wigner–Eckart theorem. If $F = a, b, g, c, h, j_3$ we find

$$F^{xR}(q^2) = \epsilon_x(-)^R F^{xR*}(q^2) \tag{38}$$

while if $F = e, f, d, j_2$

$$F^{xR}(q^2) = -\epsilon_x(-)^R F^{xR*}(q^2), \tag{39}$$

where $F^{xR}(q^2)$ is the contribution to form factor $F(q^2)$ from the current J_μ^{xR}. Thus if $R = 1, 3, 5, \cdots$ (Holstein and Treiman, 1971; Kim and Primakoff, 1969; Rosen, 1972)

$$0 = a^{\mathrm{II}R}(q^2) = b^{\mathrm{II}R}(q^2) = c^{\mathrm{II}R}(q^2) = g^{\mathrm{II}R}(q^2)$$
$$= h^{\mathrm{II}R}(q^2) = j_3^{\mathrm{II}R}(q^2)$$
$$0 = d^{\mathrm{I}R}(q^2) = e^{\mathrm{I}R}(q^2) = f^{\mathrm{I}R}(q^2) = j_2^{\mathrm{I}R}(q^2) \tag{40}$$

while if $R = 2, 4, 6, \cdots$

$$0 = a^{\mathrm{I}R}(q^2) = b^{\mathrm{I}R}(q^2) = c^{\mathrm{I}R}(q^2) = g^{\mathrm{I}R}(q^2)$$
$$= h^{\mathrm{I}R}(q^2) = j_3^{\mathrm{I}R}(q^2)$$
$$0 = d^{\mathrm{II}R}(q^2) = e^{\mathrm{II}R}(q^2) = f^{\mathrm{II}R}(q^2) = j_2^{\mathrm{II}R}(q^2). \tag{41}$$

Under the usual assumption that the weak current is an isotopic vector we find that a, b, c, h, j_3 can receive contributions from first class currents only while j_2, e, f, d can receive contributions from second class currents only. In any case if a nonzero value for any of the form factors e, d, f, j_2 is confirmed for an analog transition this is evidence for either (i) the existence of a second class current or (ii) the existence of a first class current with R even, either of which would be anomalous according to the conventional picture of the hadronic current.

For transitions which are *not* within a common isotopic multiplet the conditions are not as restrictive, and we find

$$\langle I', I_z \pm 1; \mathbf{p}', J', M' \mid J_\mu^{x\pm} \mid I, I_z; \mathbf{p}, J, M\rangle$$
$$= (-)^{I-I'+1}\epsilon_x$$
$$\times \langle I', -I_z \mp 1; \mathbf{p}', J', M' \mid J_\mu^{x\mp} \mid I, -I_z; \mathbf{p}, J, M\rangle$$

first class

$$F_{\mathrm{I}}(q^2; I_z \to I_z \pm 1) = (-)^{I-I'}F_{\mathrm{I}}^*(q^2; -I_z \to -I_z \mp 1)$$

second class

$$F_{\mathrm{II}}(q^2; I_z \to I_z \pm 1)$$
$$= -(-)^{I-I'}F_{\mathrm{II}}^*(q^2; -I_z \to -I_z \mp 1). \tag{42}$$

Thus we relate the form factors of one transition to those of its mirror transition. There can be first *and* second class current contributions to each and every form factor. However, in relating the mirror decays we see that if the first class (second class) contributions to the form factors change sign, then the second class (first class) contributions will not. Thus the ft values for the two transitions must be identical if there are no second class currents.[5]

(iv) PCAC (Nambu, 1960; Gell-Mann and Lévy, 1960)—The partially conserved axial current (PCAC) hypothesis of Gell-Mann and Lévy relates the divergence of the axial current to the pion field with a factor of proportionality measured in pion beta decay

$$\partial_\mu A_\mu^{\pm} = \sqrt{2}F_\pi m_\pi^2 \phi_{\pi\pm}, \tag{43}$$

where $F_\pi = 94$ MeV is the pion decay constant. PCAC is, in principle, of no value unless the matrix element of the pion current is known. If we define

$$\langle\beta \mid j_\pi(0) \mid \alpha\rangle \equiv C_{J'1;J}{}^{M'k;M}\epsilon_{ijk}\epsilon_{ij\mu\nu}q^\nu P^\mu f_{\pi\beta\alpha}(q^2)$$
$$+ C_{J'3;J}{}^{M'k;M}(4\pi/7)^{1/2}Y_3{}^k(\hat{q}) \mid \mathbf{q} \mid^3 g_{\pi\beta\alpha}(q^2) + \cdots, \tag{44}$$

where $f_{\pi\beta\alpha}(q^2)$, $g_{\pi\beta\alpha}(q^2)$ is the p wave (f wave) emission amplitude then the PCAC condition for p wave pion

[5] Of course, the antisymmetric component of the lepton tensor changes sign for electron and positron transitions, but V, A interference is forbidden for the total decay rate.

emission is

$$c(q^2) + h(q^2)(q^2/4M^2) = [\sqrt{2}F_\pi m_\pi^2/(m_\pi^2 - q^2)]2Mf_{\pi\beta\alpha}(q^2). \quad (45)$$

Evaluation at $q^2 = 0$ gives

$$c(0) = 2\sqrt{2}MF_\pi f_{\pi\beta\alpha}(0) \quad (46)$$

which is the generalized Goldberger–Treiman equation (Goldberger and Treiman, 1958; Frazier and Kim, 1968) relating the off shell p-wave pion coupling constant to the axial vector form factor at zero momentum transfer. Taking the first derivative of Eq. (45) we learn

$$h(0) = \frac{4M^2}{m_\pi^2}c(0)\left[1 + m_\pi^2\left(\frac{f_{\pi\beta\alpha}'(0)}{f_{\pi\beta\alpha}(0)} - \frac{c'(0)}{c(0)}\right)\right]. \quad (47)$$

If we assume that $c'(0)$ and $f_{\pi\beta\alpha}'(0)$ are known we can predict the value of the induced pseudoscalar form factor h at $q^2 = 0$ (Armstrong and Kim, 1972b; Kim and Mintz, 1971).

Similar results obtain if the f wave coupling constant is known. No restriction is placed upon $j_2(q^2)$, but $j_3(q^2)$ and a form factor $n_3(q^2)C_{J'3;J}{}^{M'k;M}C_{14;3}{}^{nn';k}l_n(4\pi/9)^{1/2}Y_4{}^{n'}(\hat{q})\times[\mathbf{q}^4/(2M)^4]$ are related to $g_{\pi\beta\alpha}(q^2)$.

IV. SPECTRA

We have decomposed the nuclear matrix element into a number of kinematic structures, each accompanied by a corresponding invariant form factor, and we have seen the restrictions placed on these form factors by various assumed symmetries. It is now time to relate these form factors to quantities which can be measured experimentally. We compute the decay spectra (with and without detection of nuclear recoil) for the case of a polarized parent and for the case of delayed particle emission from the daughter nucleus of an unpolarized parent.

A. Polarized parent spectra—electron and neutrino momenta known

We suppose the parent nuclei to form an incoherent ensemble with respect to the spin projection M along an axis of quantization described by unit vector $\hat{n}$. The polarization (orientation) state of the parent is described by[6]

$$\Lambda^{(1)} = \langle M\rangle/J,$$
$$\Lambda^{(2)} = 1 - 3[\langle M^2\rangle/J(J+1)],$$
$$\Lambda^{(3)} = (\langle M\rangle/J) - 5[\langle M^3\rangle/J(3J^2+3J-1)],$$
$$\Lambda^{(4)} = 1 - \tfrac{5}{3}\langle M^2\rangle\frac{6J^2+6J-5}{(J-1)J(J+1)(J+2)} + 35/3\langle M^4\rangle\frac{1}{(J-1)J(J+1)(J+1)}, \quad (48)$$

which are proportional to the statistical population tensors of rank one through four, respectively.[7]

We construct, from the unit vectors $\hat{n}$, traceless symmetric tensors of rank J.

$$T_{i_1,i_2,\cdots i_J}{}^{(J)}(\hat{n}) = \hat{n}_{i_1}\hat{n}_{i_2}\cdots\hat{n}_{i_J} - (2J-1)^{-1}\sum_{\mathcal{P}}\delta_{i_1i_2}\hat{n}_{i_3}\cdots\hat{n}_{i_J} + [(2J-3)(2J-1)]^{-1}\sum_{\mathcal{P}}\delta_{i_1i_2}\delta_{i_3i_4}\hat{n}_{i_5}\cdots\hat{n}_{i_J} + \cdots. \quad (49)$$

The general spectrum then consists of the contraction of these tensors ($J = 0, 1, 2, 3, 4$) with $\mathbf{p}/E$ and/or $\hat{k}$ from the lepton tensor plus factors of $\mathbf{p}$ and/or $\mathbf{k}$ from nuclear recoil multiplied by the corresponding statistical population tensor. Using for notational convenience[8]

$$T^{(J)}(\hat{n}):[a, b, \cdots c] \equiv T_{i_1i_2\cdots i_J}{}^{(J)}(\hat{n})a_{i_1}b_{i_2}\cdots c_{i_J} \quad (50)$$

we define the spectral functions $f_i(E)$ $i = 1, 2, \cdots, 27$ via

$$d^5\Gamma = F_\mp(Z,E)[G_v^2\cos\theta_c)/(2\pi)^5] \times (E_0-E)^2pE\,dE\,d\Omega_e\,d\Omega_\nu\Bigg(f_1(E) + f_2(E)\frac{\mathbf{p}}{E}\cdot\hat{k} + f_3(E)\left(\left(\frac{\mathbf{p}}{E}\cdot\hat{k}\right)^2 - \frac{1}{3}\frac{p^2}{E^2}\right) + \Lambda^{(1)}\hat{n}\cdot\left[f_4(E)(\mathbf{p}/E) + f_5(E)\frac{\mathbf{p}}{E}\frac{\mathbf{p}}{E}\cdot\hat{k} + f_6(E)\hat{k}\right.$$

---

[6] Here $\langle\ \rangle$ designates the average value.

[7] In terms of the conventional statistical population tensors

$$R_K = \sum_M a(M)(-)^{J-M}C_{JJ;K}{}^{M-M;0},$$

$a(M)$ being the population of the Mth nuclear level

$$\Lambda^{(1)} = R_1[(J+1)(2J+1)/3J]^{1/2},$$
$$\Lambda^{(2)} = -R_2[(2J+3)(2J+1)(2J-1)/5J(J+1)]^{1/2},$$
$$\Lambda^{(3)} = -\frac{R_3}{3J^2+3J-1}\times\left[\frac{(2J+3)(2J+1)(2J-1)(J+2)(J+1)(J-1)}{7J}\right]^{1/2}$$
$$\Lambda^{(4)} = R_4\frac{2}{3}\left[\frac{(2J+5)(2J+3)(2J+1)(2J-1)(2J-3)}{9(J-1)J(J+1)(J+2)}\right]^{1/2}.$$

[8] Thus, for example,

$$T^{(1)}(\hat{n}):[a] = \hat{n}\cdot a$$
$$T^{(2)}(\hat{n}):[a,b] = \hat{n}\cdot a\,\hat{n}\cdot b - \tfrac{1}{3}a\cdot b$$
$$T^{(3)}(\hat{n}):[a,b,c] = \hat{n}\cdot a\,\hat{n}\cdot b\,\hat{n}\cdot c - \tfrac{1}{5}(b\cdot c\,\hat{n}\cdot a + a\cdot c\,\hat{n}\cdot b + b\cdot a\,\hat{n}\cdot c)$$
$$T^{(4)}(\hat{n}):[a,b,c,d] = \hat{n}\cdot a\,\hat{n}\cdot b\,\hat{n}\cdot c\,\hat{n}\cdot d - \tfrac{1}{7}(a\cdot b\,\hat{n}\cdot c\,\hat{n}\cdot d + a\cdot c\,\hat{n}\cdot b\,\hat{n}\cdot d + a\cdot d\,\hat{n}\cdot b\,\hat{n}\cdot c + b\cdot c\,\hat{n}\cdot a\,\hat{n}\cdot d + b\cdot d\,\hat{n}\cdot a\,\hat{n}\cdot c + c\cdot d\,\hat{n}\cdot a\,\hat{n}\cdot b) + 1/35(a\cdot b\,c\cdot d + a\cdot c\,b\cdot d + a\cdot d\,b\cdot c).$$

$$+f_7(E)\hat{k}\,\frac{\mathbf{p}}{E}\cdot\hat{k}+f_8(E)\,\frac{\mathbf{p}}{E}\times\hat{k}$$

$$+f_9(E)\,\frac{\mathbf{p}}{E}\times\hat{k}\,\frac{\mathbf{p}}{E}\cdot\hat{k}\Big]$$

$$+\Lambda^{(2)}T^{(2)}(\hat{n}):\Big\{f_{10}(E)[\mathbf{p}/E,\mathbf{p}/E]$$

$$+f_{11}(E)[\mathbf{p}/E,\mathbf{p}/E][(\mathbf{p}/E)\cdot\hat{k}]+f_{12}(E)[\mathbf{p}/E,\hat{k}]$$

$$+f_{13}(E)[\mathbf{p}/E,\hat{k}][(\mathbf{p}/E)\cdot\hat{k}]+f_{14}(E)[\hat{k},\hat{k}]$$

$$+f_{1\ }(E)[\hat{k},\hat{k}][(\mathbf{p}/E)\cdot\hat{k}]+f_{16}(E)\left[\frac{\mathbf{p}}{E},\frac{\mathbf{p}}{E}\times\hat{k}\right]$$

$$+f_{17}(E)\left[\hat{k},\frac{\mathbf{p}}{E}\times\hat{k}\right]\Big\}$$

$$+\Lambda^{(3)}T^{(3)}(\hat{n}):\Big\{f_{18}(E)[\mathbf{p}/E,\mathbf{p}/E,\mathbf{p}/E]$$

$$+f_{19}(E)[\mathbf{p}/E,\mathbf{p}/E,\hat{k}]+f_{20}(E)[\mathbf{p}/E,\hat{k},\hat{k}]$$

$$+f_{21}(E)[\hat{k},\hat{k},\hat{k}]+f_{22}(E)\left[\frac{\mathbf{p}}{E},\frac{\mathbf{p}}{E},\frac{\mathbf{p}}{E}\times\hat{k}\right]$$

$$+f_{23}(E)\left[\frac{\mathbf{p}}{E},\hat{k},\frac{\mathbf{p}}{E}\times\hat{k}\right]+f_{24}(E)\left[\hat{k},\hat{k},\frac{\mathbf{p}}{E}\times\hat{k}\right]\Big\}$$

$$+\Lambda^{(4)}T^{(4)}(\hat{n}):\Big\{f_{25}(E)[\mathbf{p}/E,\mathbf{p}/E,\mathbf{p}/E,\hat{k}]$$

$$+f_{26}(E)[\mathbf{p}/E,\mathbf{p}/E,\hat{k},\hat{k}]$$

$$+f_{27}(E)[\mathbf{p}/E,\hat{k},\hat{k},\hat{k}]\Big\}\Big). \qquad (51)$$

It is now a straightforward although tedious task to evaluate the spectral functions in terms of the invariant form factors. The results as well as a sketch of the techniques utilized are quoted in Appendix B. Note that we have here assumed that dominant Coulomb effects are contained in the energy-dependent Fermi function $F_{\mp}(Z,E)$. We discuss the correction to this approximation in Sec. VI.

B. Polarized parent spectra—neutrino unobserved

Of course, knowledge of the neutrino momentum involves the difficult job of detecting nuclear recoil, so that most experiments involve an average over neutrino momenta. In that case the spectrum becomes

$$d^3\Gamma=2F_{\mp}(Z,E)\,\frac{G_v^2\cos^2\theta_c}{(2\pi)^4}\,(E_0-E)^2pE\,dE\,d\Omega_e$$

$$\times\Big\{F_0(E)+F_1(E)\hat{n}\cdot\frac{\mathbf{p}}{E}$$

$$+F_2(E)\left(\hat{n}\cdot\frac{\mathbf{p}}{E}\hat{n}\cdot\frac{\mathbf{p}}{E}-\frac{1}{3}\frac{p^2}{E^2}\right)$$

$$+F_3(E)\left[\left(\hat{r}\cdot\frac{\mathbf{p}}{E}\right)^3-\frac{3}{5}\frac{\mathbf{p}^2}{E^2}\hat{n}\cdot\frac{\mathbf{p}}{E}\right]\Big\}. \qquad (52)$$

The spectral functions $F_i(E)$ are easily found in terms of the $f_i(E)$ and the results are listed in Appendix B.

C. Delayed α emission—electron and neutrino momentum known

Next, we analyze a situation in which the daughter nucleus produced in the β-decay process undergoes a subsequent transition to a final nucleus of spin J'' with emission of an accompanying α particle or (possibly polarized) photon. The latter is characterized by a unit vector $\hat{n}$ along its direction of motion in the laboratory frame (rest frame of the parent nucleus). The spectrum then contains certain kinematic-shift terms associated with the transformation to the lab frame from the rest frame of the β-decay daughter nucleus, wherein the subsequent transition is most simply characterized. In the following it is the spectral functions δ_i which express these purely kinematic effects. We find for the delayed α emission[9]

$$d^7\Gamma=F_{\mp}(Z,E)\,\frac{G_v^2\cos^2\theta_c}{2(2\pi)^6}\,(E_0-E)^2pE\,dE\,d\Omega_e\,d\Omega_\nu\,d\Omega_n$$

$$\times\Bigg(g_1(E)+g_2(E)\,\frac{\mathbf{p}}{E}\cdot\hat{k}+g_3(E)\left[\left(\frac{\mathbf{p}}{E}\cdot\hat{k}\right)^2-\frac{1}{3}\frac{\mathbf{p}^2}{E^2}\right]$$

$$+\delta_1(E,v^*,\tau_{J',J''}(L))\,\frac{\hat{n}\cdot\mathbf{p}}{E}$$

$$+\delta_2(E,v^*,\tau_{J',J''}(L))\hat{n}\cdot\frac{\mathbf{p}}{E}\frac{\mathbf{p}}{E}\cdot\hat{k}$$

$$+\delta_3(E,v^*,\tau_{J',J''}(L))\hat{n}\cdot\hat{k}$$

$$+\delta_4(E,v^*,\tau_{J',J''}(L))\hat{n}\cdot\hat{k}\,\frac{\mathbf{p}}{E}\cdot\hat{k}$$

$$+\tfrac{1}{10}\tau_{J',J''}(L)\,T^{(2)}(\hat{n}):\Big\{g_{10}(E)[\mathbf{p}/E,\mathbf{p}/E]$$

$$+g_{11}(E)[\mathbf{p}/E,\mathbf{p}/E]\,\frac{\mathbf{p}}{E}\cdot\hat{k}+g_{11}(E)[\mathbf{p}/E,\hat{k}]$$

$$+g_{12}(E)[\mathbf{p}/E,\hat{k}]\,\frac{\mathbf{p}}{E}\cdot\hat{k}+g_{13}(E)[\hat{k},\hat{k}]$$

$$+g_{15}(E)[\hat{k},\hat{k}]\,\frac{\mathbf{p}}{\mathbf{E}}\cdot\hat{k}+g_{16}(E)\left[\frac{\mathbf{p}}{E},\frac{\mathbf{p}}{E}\times\hat{k}\right]$$

$$+g_{17}(E)\left[\hat{k},\frac{\mathbf{p}}{E}\times\hat{k}\right]\Big\}$$

$$+\delta_8(E,v^*,\tau_{J'J''}(L))T^{(3)}(\hat{n}):[\mathbf{p}/E,\mathbf{p}/E,\hat{k}]$$

$$+\delta_9(E,v^*,\tau_{J'J''}(L))T^{(3)}(\hat{n}):[\mathbf{p}/E,\hat{k},\hat{k}]$$

$$+\tfrac{1}{10}\omega_{J'J''}(L)\,T^{(4)}(\hat{n}):\{g_{25}(E)[\mathbf{p}/E,\mathbf{p}/E,\mathbf{p}/E,\hat{k}]$$

$$+g_{26}(E)[\mathbf{p}/E,\mathbf{p}/E,\hat{k},\hat{k}]$$

$$+g_{27}(E)[\mathbf{p}/E,\hat{k},\hat{k},\hat{k}]\}\Bigg). \qquad (53)$$

[9] Identical results obtain for delayed proton or neutron emission if L is the angular momentum of the proton or neutron with respect to the daughter nucleus and the nucleon polarization is not detected.

The spectral functions $g_i(E)$ and $\delta_i(E, v^*, \tau_{J'J''}(L))$ are given in Appendix B. It is interesting to note that $g_i(E)$ and $f_i(E)$ (see Appendix B) are identical except for certain sign changes and alteration of the spin sequence dependence. Thus the same information is available in principle either by polarizing (orienting) the initial nucleus or by utilizing delayed particle emission in order to assess the polarization state of the daughter nucleus. Also we emphasize again that the $\delta_i(E, v^*, \tau_{J'J''}(L))$ are purely kinematic in origin and thus can be used as a check on the experimental procedure. Here v^* is the velocity of the α particle in the center of the mass frame of the β-decay daughter, while $\tau_{J'J''}(L)$, $\omega_{J'J''}(L)$ are coefficients which depend on the nuclear spin sequence and on the angular momentum L of the α particle with respect to the daughter nucleus. They are given in general by

$$\tau_{J',J''}(L) = 10\left[\frac{L(L+1)(2L+1)}{(2L-1)(2L+3)}\right]^{1/2} \times\left[\frac{(2J'-1)(2J'+1)(2J'+3)}{J'(J'+1)}\right]^{1/2} \times W(2J'LJ'';J'L)$$

$$\omega_{J',J''}(L) = 10\left[\frac{(L-1)L(L+1)(L+2)(2L+1)}{(2L-3)(2L-1)(2L+3)(2L+5)}\right]^{1/2} \times\left[\frac{(2J'-3)(2J'-1)(2J'+1)(2J'+3)(2J'+5)}{(J'-1)J'(J'+1)(J'+2)}\right]^{1/2} \times W(4J'LJ'';J'L). \quad (54)$$

For P-wave ($L = 1$) or D-wave ($L = 2$) emission they are given by

$$\tau_{J',J''}(L=1) = \frac{2}{J'(J'+1)}\begin{cases}(2J'+3)(J'+1) & J'=J''+1\\ -(2J'+3)(2J'-1) & J'=J''\\ (2J'-1)J' & J'=J''-1\end{cases}$$

$$\tau_{J',J''}(L=2) = \frac{10}{7J'(J'+1)}\begin{cases}2(2J'+3)(J'+1) & J'=J''+2\\ -(2J'+3)(J'-5) & J'=J''+1\\ -(2J'+5)(2J'-3) & J'=J''\\ -(2J'-1)(J'+6) & J'=J''-1\\ 2(2J'-1)J' & J'=J''-2\end{cases} \quad (55a)$$

$$\omega_{J',J''}(L=1) = 0$$

$$\omega_{J',J''}(L=2) = \frac{5\cdot 4}{9\cdot 7}\begin{cases}(2J'+5)(2J'+3)/J'(J'-1) & J'=J''+2\\ -2(2J'+5)(2J'+3)(2J'-3)/J'(J'+1)(J'-1) & J'=J''+1\\ 6(2J'+5)(2J'-3)/J'(J'+1) & J'=J''\\ -2(2J'-3)(2J'-1)(2J'+5)/J'(J'+1)(J'+2) & J'=J''-1\\ (2J'-3)(2J'-1)/(J'+1)(J'+2) & J'=J''-2\end{cases} \quad (55b)$$

D. Delayed α emission—neutrino unobserved

If the neutriono is unobserved, the spectrum simplifies to

$$d^5\Gamma = F_{\mp}(Z,E)\frac{G_v^2\cos^2\theta_c}{(2\pi)^5}(E_0-E)^2pE\,dE\,d\Omega_e\,d\Omega_{\hat{n}} \times\left\{G_0(E)+\Delta_1(E,v^*,\tau_{J'J''}(L))\hat{n}\cdot\frac{\mathbf{p}}{E} + \tfrac{1}{10}\tau_{J'J''}(L)G_2(E)\left[\left(\frac{\hat{n}\cdot\mathbf{p}}{E}\right)^2-\frac{1}{3}\frac{\mathbf{p}^2}{E^2}\right]\right\}, \quad (56)$$

where $G_i(E)$, $\Delta_1(E, v^*, \tau_{J'J''}(L))$ are easily found in terms of $g_i(E)$, $\delta_i(E, V^*, \tau_{J'J''}(L))$ and are given in Appendix B.

E. Delayed γ emission—electron and neutrino momenta known

For delayed emission of circularly polarized photons of multipolarity $E(l)$ or $M(l)$

$$d^7\Gamma = F_\mp(Z,E)\frac{G_v^2\cos^2\theta_c}{2(2\pi)^6}(E_0-E)^2pE\,dE\,d\Omega_e\,d\Omega_v\,d\Omega_{\hat{k}}$$

$$\times\Bigg(g_1(E)+g_2(E)\frac{\mathbf{p}}{E}\cdot\hat{k}+g_3(E)\left[\left(\frac{\mathbf{p}}{E}\cdot\hat{k}\right)^2-\frac{1}{3}\frac{p^2}{E^2}\right]$$

$$+\sigma\Upsilon_{J',J''}(l)(J'+1)\hat{n}\cdot\Bigg[\Bigg(g_4(E)$$

$$+\frac{1}{\sigma\Upsilon_{J'J''}(l)(J'+1)}\delta_1[E,1,\Gamma_{J'J''}(l)]\Bigg)\frac{\mathbf{p}}{E}$$

$$+\Bigg(g_5(E)+\frac{1}{\sigma\Upsilon_{J'J''}(l)(J'+1)}$$

$$\times\delta_2[E,1,\Gamma_{J'J''}(l)]\Bigg)\frac{\mathbf{p}}{E}\frac{\mathbf{p}}{E}\cdot\hat{k}$$

$$+\Bigg(g_6(E)+\frac{1}{\sigma\Upsilon_{J'J''}(l)(J'+1)}\delta_3[E,1,\Gamma_{J'J''}(l)]\Bigg)\hat{k}$$

$$+\Bigg(g_7(E)+\frac{1}{\sigma\Upsilon_{J'J''}(l)(J'+1)}\delta_4[E,1,\Gamma_{J'J''}(l)]\Bigg)\hat{k}$$

$$\times\frac{\mathbf{p}}{E}\cdot\hat{k}+g_8(E)\frac{\mathbf{p}}{E}\times\hat{k}+g_9(E)\frac{\mathbf{p}}{E}\times\hat{k}\frac{\mathbf{p}}{E}\cdot\hat{k}\Bigg]$$

$$+\tfrac{1}{10}\Gamma_{J'J''}(l)T^{(2)}(\hat{n}):\Bigg[\Bigg(g_{10}(E)$$

$$+\frac{10\sigma}{\Gamma_{J'J''}(l)}\delta_5[E,1,\Upsilon_{J'J''}(l)]\Bigg)[\mathbf{p}/E,\mathbf{p}/E]$$

$$+g_{11}(E)[\mathbf{p}/E,\mathbf{p}/E]\frac{\mathbf{p}}{E}\cdot\hat{k}$$

$$+\Bigg(g_{12}(E)+\frac{10\sigma}{\Gamma_{J'J''}(l)}\delta_6[E,1,\Upsilon_{J'J''}(l)]\Bigg)[\mathbf{p}/E,\hat{k}]$$

$$+g_{13}(E)[\mathbf{p}/E,\hat{k}]\frac{\mathbf{p}}{E}\cdot\hat{k}$$

$$+\Bigg(g_{14}(E)+\frac{10\sigma}{\Gamma_{J'J''}(l)}\delta_7[E,1,\Upsilon_{J'J''}(l)]\Bigg)[\hat{k},\hat{k}]$$

$$+g_{15}(E)[\hat{k},\hat{k}]\frac{\mathbf{p}}{E}\cdot\hat{k}+g_{16}(E)\left[\frac{\mathbf{p}}{E},\frac{\mathbf{p}}{E}\times\hat{k}\right]$$

$$+g_{17}(E)\left[\hat{k},\frac{\mathbf{p}}{E}\times\hat{k}\right]\Bigg\}$$

$$+\sigma\Sigma_{J'J''}(l)(J'+1)T^{(3)}(\hat{n})$$

$$:\Bigg\{g_{18}(E)[\mathbf{p}/E,\mathbf{p}/E,\mathbf{p}/E]$$

$$+\Bigg(g_{19}(E)+\frac{1}{\sigma\Sigma_{J'J''}(l)(J'+1)}$$

$$\times\delta_8[E,1,\Gamma_{J'J''}(l)]\Bigg)[\mathbf{p}/E,\mathbf{p}/E,\hat{k}]$$

$$+\Bigg(g_{20}(E)+\frac{1}{\sigma\Sigma_{J'J''}(l)(J'+1)}$$

$$\times\delta_9[E,1,\Gamma_{J'J''}(l)]\Bigg)[\mathbf{p}/E,\hat{k},\hat{k}]$$

$$+g_{21}(E)[\hat{k},\hat{k},\hat{k}]+g_{22}(E)\left[\frac{\mathbf{p}}{E}\times\hat{k},\frac{\mathbf{p}}{E},\frac{\mathbf{p}}{E}\right]$$

$$+g_{23}(E)\left[\frac{\mathbf{p}}{E}\times\hat{k},\frac{\mathbf{p}}{E},\hat{k}\right]+g_{24}(E)\left[\frac{\mathbf{p}}{E}\times\hat{k},\hat{k},\hat{k}\right]\Bigg\}$$

$$+\tfrac{1}{10}K_{J'J''}(l)T^{(4)}(\hat{n}):\Bigg\{g_{25}(E)[\mathbf{p}/E,\mathbf{p}/E,\mathbf{p}/E,\hat{k}]$$

$$+g_{26}(E)[\mathbf{p}/E,\mathbf{p}/E,\hat{k},\hat{k}]$$

$$+g_{27}(E)[\mathbf{p}/E,\hat{k},\hat{k},\hat{k}]\Bigg\}\Bigg),\qquad(57)$$

where $\sigma = +1$ for right-hand circular polarization

$= -1$ for left-hand circular polarization

and spectral functions $g_i(E)$, $\delta_i(E, 1, \Gamma_{J'J''}(l))$ are defined in Appendix B. The correlation coefficients $\Gamma_{J',J''}(l)$, $K_{J',J''}(l)$ are closely related to $\tau_{J',J''}(L)$, $\omega_{J',J''}(L)$ for delayed α emission.

$$\Gamma_{J'J''}(l) = (1-3/l(l+1))\tau_{J'J''}(l)$$
$$K_{J'J''}(l) = (1-10/l(l+1))\omega_{J'J''}(l)\qquad(58)$$

whereas $\Upsilon_{J'J''}(l)$, $\Sigma_{J'J''}(l)$ are new coefficients, defined in general by

$$\Upsilon_{J',J''}(l)=\left[\frac{2l+1}{l(l+1)}\right]^{1/2}\left[\frac{2J'+1}{J'(J'+1)}\right]^{1/2}W(1J'lJ'';J'l)$$

$$\Sigma_{J',J''}(l)$$
$$=\frac{3}{3J'^2+3J'-1}\left\{\frac{(2l+1)(l-1)(l+2)}{(2l-1)(2l+3)l(l+1)}\right]^{1/2}$$
$$\times\left[\frac{(2J'-1)(2J'+1)(2J'+3)(J'-1)(J'+2)}{J'(J'+1)}\right]^{1/2}$$
$$\times W(3J'lJ'';J'l)\ .\qquad(59)$$

For dipole ($l = 1$) or quadrupole ($l = 2$) emission we find

$$\Upsilon_{J',J''}(l=1) = [2J'(J'+1)]^{-1} \begin{cases} J'+1 & J' = J''+1 \\ 1 & J' = J'' \\ -J' & J' = J''-1 \end{cases}$$

$$\Upsilon_{J',J''}(l=2) = [6J'(J'+1)]^{-1} \begin{cases} 2(J'+1) & J' = J''+2 \\ (J'+3) & J' = J''+1 \\ 3 & J' = J' \\ -(J'-2) & J' = J''-1 \\ -2J' & J' = J''-2 \end{cases} \tag{60a}$$

$$\Sigma_{J',J''}(l=1) = 0$$

$$\Sigma_{J',J''}(l=2) = [7J'(J'+1)(3J'^2+3J'-1)]^{-1} \begin{cases} (J'+1)(J'+2)(2J'+3) & J' = J''+2 \\ -2(J'-2)(J'+2)(2J'+3) & J' = J''+1 \\ -12(J'-1)(J'+2) & J' = J'' \\ 2(J'-1)(J'+3)(2J'-1) & J' = J''-1 \\ -(J'-1)J'(2J'-1) & J' = J''-2 \end{cases} \tag{60b}$$

F. Delayed γ emission—neutrino unobserved

Finally, if the neutrino is unobserved the spectrum simplifies to

$$\begin{aligned} d^5\Gamma = {} & F_{\mp}(Z,E)[(G_v^2\cos^2\theta_c/2(2\pi)^5] \\ & \times (E_0-E)^2 pE\, dE\, d\Omega_e\, d\Omega_{\hat{n}} \\ & \times \bigg\{ G_0(E) + [\sigma \Upsilon_{J',J''}(l)(J'+1)G_1(E) \\ & + \Delta_1(E,1,\Gamma_{J',J''}(l))]\hat{n}\cdot\frac{\mathbf{p}}{E} \\ & + [\tfrac{1}{10}\Gamma_{J'J''}(l)G_2(E) + \sigma\Delta_2(E,1,\Upsilon_{J',J''}(l))] \\ & \times \left[\left(\hat{n}\cdot\frac{\mathbf{p}}{E}\right)^2 - \frac{1}{3}\frac{p^2}{E^2}\right] + \sigma\Sigma_{J',J''}(l)(J'+1)G_3(E) \\ & \times \left[\left(\hat{n}\cdot\frac{\mathbf{p}}{E}\right)^3 - \frac{3}{5}\frac{p^2}{E^2}\hat{n}\cdot\frac{\mathbf{p}}{E}\right]\bigg\}, \end{aligned} \tag{61}$$

where $G_i(E)$, $\Delta_i((E, 1, \Gamma_{J',J''}(l))$ are easily found in terms of $g_i(E)$, $\delta_i(E, 1, \Gamma_{J',J''}(l))$ and are quoted in Appendix B.

V. IMPULSE APPROXIMATION

The assumption of basic symmetries possessed by the weak hadronic current allows us to predict the values of some structure functions (e.g., a, b) in a model independent fashion. However, not all form factors can be calculated in this manner and it is necessary to have a procedure—even though model dependent—which allows estimates to be made concerning their magnitude.

The conventional nuclear physics calculation utilizes the so-called impulse approximation, which is based on the assumption that the nucleus consists of physical nucleons interacting with leptons in the same fashion as do free nucleons. That is, the hadronic current is taken as a simple sum of single nucleon currents. It is assumed that the interaction of nucleons within the nuclear volume does not appreciably affect their decay properties. Meson exchange and other many-body effects are neglected.

The first step in this procedure involves writing an interaction responsible for neutron β-decay. We take this to be

$$\begin{aligned} \mathcal{H}_{\rm int} = {} & \frac{G_V}{\sqrt{2}}\cos\theta_c \int d^4x \bar{\psi}_p(x)\{g_V(q^2)\gamma_\mu l^\mu(x) \\ & - i[g_S(q^2)/2m]\,\partial_\mu l^\mu(x) - (g_M(q^2) - g_V(q^2))/2m \\ & \times \sigma_{\mu\nu}\,\partial^\nu l^\mu(x) + g_A(q^2)\gamma_\mu\gamma_5 l^\mu(x) \\ & - i(2m)^{-1}g_P(q^2)\gamma_5\,\partial_\mu l^\mu(x) \\ & - g_{II}(q^2)(2m)^{-1}\sigma_{\mu\nu}\gamma_5\,\partial^\nu l^\mu(x)\} + \text{h.c.}, \end{aligned} \tag{62}$$

with $l^\mu(x) = \bar{u}(p)\gamma^\mu(1+\gamma_5)v(k)e^{iq\cdot x}$. Here $g_V(q^2)$, $g_A(q^2)$

are the usual vector, axial form factors—$g_A(0)/g_V(0) \cong 1.23$ (Christenson *et al.*, 1967, Christenson *et al.*, 1969) while $g_M(q^2)$, $g_P(q^2)$ are the weak magnetism, induced pseudoscalar form factors. According to CVC we must have

$$g_M(0) = \mu_p - \mu_n \cong 4.70 \qquad (63)$$

while nucleon PCA gives (Kim and Mintz, 1971)

$$g_P(q^2) \cong (2m)^2 g_A(q^2)/(q^2 - m_{\pi^2}), \qquad (64)$$

Finally $g_S(q^2)$, $g_{II}(q^2)$ are the induced scalar, induced tensor form factors. If we assume the absence of second class currents these form factors must vanish, as discussed in Sec. III since n, p constitute an isotopic doublet.

One now performs a Foldy–Wouthuysen transformation in order to find the nonrelativistic form of the interaction and generalizes the resultant expression to a system of A free nucleons (Huffaker and Greuling, 1963; Rose and Osborne, 1954). We find

$$\langle\beta \mid V_\mu^\pm + A_\mu^\pm \mid \alpha\rangle$$
$$= \int d^3r_1 \cdots d^3r_A \phi_\beta^*(r_1 \cdots r_A) \sum_{i=1}^{A} \delta^3(\mathbf{R})$$
$$\times (\hat{V}_\mu^\pm(r_i) + \hat{A}_\mu^\pm(r_i))\phi_\alpha(r_1 \cdots r_A), \qquad (65)$$

where $\mathbf{R}$ is the center-of-mass coordinate, and

$$V_0^\pm(\mathbf{r}_i) = \tau_i^\pm \exp(-i\mathbf{q}\cdot\mathbf{r}_i)[g_V(q^2) \pm (q_0/2m)g_S(q^2)],$$
$$\hat{V}^\pm(\mathbf{r}_i) = -i\tau_i^\pm[g_V(q^2)(2m)^{-1}\{\exp(-i\mathbf{q}\cdot\mathbf{r}_i), \boldsymbol{\nabla}_i\}$$
$$+ \exp(-i\mathbf{q}\cdot\mathbf{r}_i)((2\mathrm{m})^{-1}g_M(q^2)\boldsymbol{\sigma}_i \times \mathbf{q}$$
$$\pm g_S(q^2)(2m)^{-1}\boldsymbol{\sigma}_i\cdot\mathbf{q})],$$
$$\hat{A}_0^\pm(\mathbf{r}_i) = i\frac{g_A(q_2)}{2m}\tau_i^\pm\{\exp(-i\mathbf{q}\cdot\mathbf{r}_i), \boldsymbol{\sigma}_i\cdot\boldsymbol{\nabla}_i\}$$
$$+ \tau_i^\pm\left(\pm\frac{g_{II}(q^2)}{2m} - \frac{q_0}{(2m)^2}g_P(q^2)\right)$$
$$\times \exp(-i\mathbf{q}\cdot\mathbf{r}_i)\boldsymbol{\sigma}_i\cdot\mathbf{q},$$
$$\hat{\mathrm{A}}^\pm(\mathbf{r}_i) = -\tau_i^\pm\boldsymbol{\sigma}_i \exp(-i\mathbf{q}\cdot\mathbf{r}_i)$$
$$\times (g_A(q^2) \mp (q_0/2m)g_{II}(q^2))$$
$$- \tau_i^\pm\boldsymbol{\sigma}_i\cdot\mathbf{q}\exp(-i\mathbf{q}\cdot\mathbf{r}_i)[\mathbf{q}/(2m)^2]g_P(q^2). \qquad (66)$$

Here the upper (lower) signs designate electron (positron) decay.

Finally, we expand the exponential factors, take matrix elements, and compare with the decomposition given in Eq. (7). We find as the impulse approximation predictions[10]

$$a(q^2) \cong (1 + \Delta/2M)^{-1}g_V(q^2)$$
$$\times [\mathfrak{M}_F + \tfrac{1}{6}(q^2 - \Delta^2)\mathfrak{M}_{r^2} + \tfrac{1}{3}\Delta\mathfrak{M}_{r\cdot p}],$$
$$b(q^2) \cong A[g_M(q^2)\mathfrak{M}_{GT} + g_V(q^2)\mathfrak{M}_L],$$
$$c(q^2) \cong (1 + \Delta/2M)^{-1}g_A(q^2)[\mathfrak{M}_{GT} + \tfrac{1}{6}(q^2 - \Delta^2)\mathfrak{M}_{\sigma r^2}$$
$$+ [1/6(10)^{1/2}]\mathfrak{M}_{1y}(2\Delta^2 + q^2)$$
$$+ A(\Delta/2M)\mathfrak{M}_{\sigma L} + \tfrac{1}{2}\Delta\mathfrak{M}_{\sigma r p}\},$$
$$d(q^2) \cong (1 + \Delta/2M)^{-1}g_A(q^2)[-\mathfrak{M}_{GT} - \tfrac{1}{6}(q^2 - \Delta^2)\mathfrak{M}_{\sigma r^2}$$
$$+ (10)^{-1/2}\mathfrak{M}_{1y}(M\Delta + \tfrac{1}{6}(\Delta^2 - q^2))$$
$$+ A\mathfrak{M}_{\sigma L} + M\mathfrak{M}_{\sigma r p}] \pm Ag_{II}(q^2)\mathfrak{M}_{GT},$$
$$e(q^2) \cong (1 + \Delta/2M)^{-1}g_V(q^2)[\mathfrak{M}_F + \tfrac{1}{6}(q^2 - \Delta^2)\mathfrak{M}_{r^2}$$
$$- (2M/3)\mathfrak{M}_{r\cdot p} \pm Ag_S(q^2)],$$
$$f(q^2) \cong g_V(q^2)2M\mathfrak{M}_{\{r,p\}},$$
$$g(q^2) \cong -g_V(q^2)(4M^2/3)\mathfrak{M}_Q,$$
$$h(q^2) \cong -(1 + \Delta/2M)^{-1}$$
$$\times [g_A(q^2)2M^2(10)^{-1/2}\mathfrak{M}_{1y} + g_P(q^2)A^2\mathfrak{M}_{GT}],$$
$$j_i(q^2) \cong -(2M^2/3)g_A(q^2)\mathfrak{M}_{iy} \qquad i = 2, 3, \qquad (67)$$

where the script M's represent reduced matrix elements.

$$\mathfrak{M}_F = \langle\beta \,||\, \sum_i \tau_i^\pm \,||\, \alpha\rangle,$$
$$\mathfrak{M}_{GT} = \langle\beta \,||\, \sum_i \tau_i^\pm\boldsymbol{\sigma}_i \,||\, \alpha\rangle,$$
$$\mathfrak{M}_{r^2} = \langle\beta \,||\, \sum_i \tau_i^\pm r_i^2 \,||\, \alpha\rangle,$$
$$\mathfrak{M}_{\sigma r^2} = \langle\beta \,||\, \sum_i \tau_i^\pm r_i^2 \hat{r}_i \,||\, \alpha\rangle,$$
$$\mathfrak{M}_Q = (4\pi/5)^{1/2}\langle\beta \,||\, \sum_i \tau_i^\pm r_i^2 Y_2^k(\hat{r}_i) \,||\, \alpha\rangle,$$
$$\mathfrak{M}_{\{r,p\}} = (i/2m)\langle\beta \,||\, \sum_i \tau_i^\pm C_{11;2}^{nn';k}$$
$$\times r_{in}p_{in'} + p_{in}r_{in'}) \,||\, \alpha\rangle,$$
$$\mathfrak{M}_{r\cdot p} = (i/2m)\langle\beta \,||\, \sum_i \tau_i^\pm(\mathbf{r}_i\cdot\mathbf{p}_i + \mathbf{p}_i\cdot\mathbf{r}_i) \,||\, \alpha\rangle,$$
$$\mathfrak{M}_L = \langle\beta \,||\, \sum_i \tau_i^\pm(\mathbf{r}_i \times \mathbf{p}_i) \,||\, \alpha\rangle,$$
$$\mathfrak{M}_{\sigma L} = i\langle\beta \,||\, \sum \tau_i^\pm\boldsymbol{\sigma}_i \times (\mathbf{r}_i \times \mathbf{p}_i) \,||\, \alpha\rangle,$$
$$\mathfrak{M}_{Ky} = (16\pi/5)^{1/2}\langle\beta \,||\, \sum_i \tau_i^\pm r_i^2 C_{12;K}^{nn';k}$$
$$\times \sigma_{in}Y_2^{n'}(\hat{r}_i) \,||\, \alpha\rangle, \qquad K = 1, 2, 3$$
$$\mathfrak{M}_{\sigma r p} = (i/2M)\langle\beta \,||\, \sum_i \tau_i^\pm[\{\boldsymbol{\sigma}_i\cdot\mathbf{r}_i, \mathbf{p}_i\}$$
$$+ \{\boldsymbol{\sigma}_i\cdot\mathbf{p}_i, \mathbf{r}_i\}] \,||\, \alpha\rangle. \qquad (68)$$

It is interesting to compare these impulse approximation predictions with theoretical ideas discussed previously.

[10] The impulse approximation does not make a unique prediction for the second class contribution to d as has been pointed out by E. Henley and L. Wolfenstein (1971). Our value $d = \pm g_{II}A\mathfrak{M}_{GT}$ is based on the assumption of a divergenceless second class axial current, as discussed by J. Delorme and M. Rho (1971a).

T invariance is automatic if the n–p form factors are T-invariant. Since n and p constitute an isotopic doublet, only the isovector component of the weak current can contribute. Thus we should expect that for transitions within a common isotopic multiplet a, b, c, g, h, j_3 receive contributions *only* from first class currents—i.e., terms involving g_V, g_A, g_M, g_P whereas d, e, f, j_2 are purely second class—i.e., involving g_S, g_{II}.

This is perceived to be valid for the first class from factors a, b, c, g, h, j_3. However, d, e, f, j_2 obviously contain first class contributions. In the case of d the dominant terms for large A are

$$d \approx A[g_A(q^2)\mathfrak{M}_{\sigma L} \pm g_{II}(q^2)\mathfrak{M}_{GT}] \tag{69}$$

and $\mathfrak{M}_{\sigma L}$ can be shown to vanish when taken between states of a common isotopic multiplet.[11] For f, j_2 we may use similar arguments, or for f we note that if the nuclear force is assumed to be independent of momentum we can write

$$\mathfrak{M}\{_{r,p}\} \cong \tfrac{1}{2}\langle\beta\,||\,[H_{st}, \sum \tau_i^{\pm} C_{11;2}{}^{nn';k} r_{in} r_{in'}]\,||\,\alpha\rangle$$
$$= (\tfrac{2}{3})^{1/2}(\Delta/2)\mathfrak{M}_Q \tag{70}$$

which vanishes for analog transitions in the limit of isotopic spin invariance. For e however, a significant first class current contribution is apparently predicted by the naive impulse approximation.

The presence of a large e coefficient for analog decays is in contradiction to the CVC hypothesis, of course, since the electromagnetic current is purely first class. It is clear, however, that we should *expect* the impulse approximation to violate CVC since even if $g_S(q^2) = 0$ we have

$$\partial^\mu V_\mu{}^{\pm} = g_V(q^2) \sum_i \tau_i{}^{\pm}[iq_0 \exp(-i\mathbf{q}\cdot\mathbf{r}_i) - (2m)^{-1}\{\exp(-i\mathbf{q}\cdot\mathbf{r}_i), \mathbf{q}\cdot\nabla_i\}] \neq 0. \tag{71}$$

[11] We note that if

$$|\alpha\rangle = |I, I_z\rangle$$

and

$$|\beta\rangle = |I, I_z \pm 1\rangle$$

we have

$$\langle\beta\,|\,i\sum_i \tau_i{}^{\pm}\boldsymbol{\sigma}_i \times \mathbf{L}_i\,|\,\alpha\rangle$$
$$= \langle\beta\,|\,U^{-1}U\sum_i \tau_i{}^{\pm}\boldsymbol{\sigma}_i \times \mathbf{L}_i U^{-1}U\,|\,\alpha\rangle$$
$$= -\langle I, -I_z\,|\,i\sum_i \tau_i{}^{\pm}\boldsymbol{\sigma}_i \times \mathbf{L}\,|\,I, -I_i \pm 1\rangle^* \qquad U = \exp(-i\pi I_2).$$

However, by the Wigner–Eckart theorem and T invariance

$$\frac{\langle I, I_z \pm 1\,|\,i\sum_i \tau_i{}^{\pm}\boldsymbol{\sigma}_i \times \mathbf{L}_i\,|\,I, I_z\rangle}{\langle I, -I_z\,|\,i\sum_i \tau_i{}^{\pm}\boldsymbol{\sigma}_i \times \mathbf{L}_i\,|\,I, -I_z \mp 1\rangle^*} = \frac{C_{I,1;I}{}^{I_z\pm1,\mp1;I_z}}{C_{I,1;I}{}^{-I_z,\mp1;-I_z\mp1}} = 1\,.$$

Hence the matrix element vanishes.

Thus, even in the absence of electromagnetism, the vector current is not divergenceless in the impulse approximation (Armstrong and Kim, 1972b). This must be regarded as a shortcoming of the impulse approximation at recoil level and points up a reason that experiments should be analyzed in a model independent fashion. The validity of CVC or of the impulse approximation can then be decided by experiment. This suggests, in addition, possible limitations of the impulse approximation in recoil level for the axial current, to which we now turn.

A test of PCAC requires knowledge of the pion–nucleus coupling. If we assume a single particle model for the pion coupling we find

$$f_{\pi\beta\alpha}(q^2) = \sqrt{2}g_r(q^2)[A/(2M)^2]\mathfrak{M}_{GT}. \tag{72}$$

Then writing

$$c = (1 + \Delta/2M)^{-1}g_A(q^2)\mathfrak{M}_{GT} + \delta c$$
$$h = -(1 + \Delta/2M)^{-1}g_P(q^2)2MA^2\mathfrak{M}_{GT} + \delta h \tag{73}$$

we have

$$c(q^2) + h(q^2)[q^2/(2M)^2]$$
$$= g_A(q^2)[m_\pi{}^2/(m_\pi{}^2 - q^2)]\mathfrak{M}_{GT} + \delta$$
$$= \sqrt{2}F_\pi m_\pi{}^2(m_\pi{}^2 - q^2)^{-1}2Mf_{\pi\beta\alpha}(q^2) + \delta, \tag{74}$$

with $\delta = \delta c + [q^2/(2m)^2]\delta h \neq 0$. Here we have used the nucleon PCAC result Eq. (59) together with the Goldberger–Treiman relation

$$g_A = F_\pi g_r/m. \tag{75}$$

Even if PCAC is exact for the single nucleon transition, it is violated in the nuclear case by the terms which we have designated by δ. That this should be expected can be seen via the operator relation

$$\partial^\mu A_\mu{}^{\pm} = \sqrt{2}F_\pi m_\pi{}^2\phi_\pi{}^{\pm} - (q_0/2m)g_A(q^2)\sum_i \tau_i{}^{\pm} \times \{\exp(-i\mathbf{q}\cdot\mathbf{r}_i), \boldsymbol{\sigma}_i\cdot\nabla_i\}, \tag{76}$$

where we have used the single-particle value for $\phi_{\pi\pm}$

$$\phi_\pi{}^{\pm} = i\sqrt{2}g_r(q^2)(m_\pi{}^2 - q^2)^{-1}\sum_i \tau_i{}^{\pm} \times \exp(-i\mathbf{q}\cdot\mathbf{r}_i)[(\boldsymbol{\sigma}_i\cdot\mathbf{q})/2m]. \tag{77}$$

The term

$$(q_0/2m)g_A(q^2)\sum_i \tau_i{}^{\pm}\{\exp(-i\mathbf{q}\cdot\mathbf{r}_i), \boldsymbol{\sigma}_i\cdot\nabla_i\}$$

violates the PCAC condition in the impulse approximation, and again emphasizes the possible inadequacies of the impulse approximation in recoil order (Armstrong and Kim, 1972b; Holstein, 1974b; Krmpotic and Tadic, 1969).

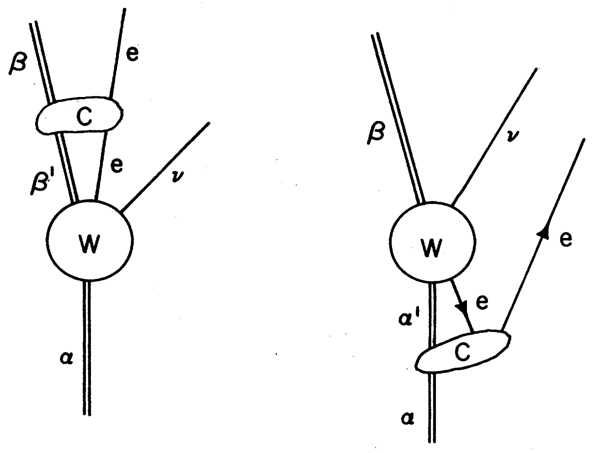

FIG. 1. Diagrams representing the Coulomb interaction between parent/daughter nuclei and electron.

At least some of these problems may be resolved by the presence of exchange terms, which produce an effective one body potential, as demonstrated by Delorme and Rho (Delorme and Rho, 1971a; Ohtsubo, Fujita, and Takeda, 1970).

VI. ELECTROMAGNETIC CORRECTIONS

Thus far except for the "knee-jerk" response of including the Fermi function in our decay spectra we have effectively pretended that the Hamiltonian for the world consists only of the strong and weak interactions. If we now include the electromagnetic interaction we find several changes in our previous results. The necessary modifications of our procedure are indicated by the Feynman diagrams in Fig. 1. The main contribution to the electromagnetic interaction is the Coulomb field between nucleus and outgoing lepton. One can take therefore to a good approximation $\alpha = \alpha'$ and $\beta = \beta'$. Assuming the nuclear Coulomb potentials produced by initial and final nuclei are identical, we then find (Armstrong and Kim, 1972a; Blin-Stoyle and Rosina, 1965; Huffaker and Laird, 1967; Schülke, 1964; Wilkinson 1970a)

$$\begin{aligned} T \cong\ & (G_V/\sqrt{2}) \cos\theta_c \int d^3r \int [d^3l/(2\pi)^3] \bar{\psi}_e(\mathbf{r}, \mathbf{p}) \\ & \times \exp(i\mathbf{l}\cdot\mathbf{r}) \gamma^\mu (1+\gamma_5) v(k) \\ & \times \langle \beta_{\mathbf{p}-\mathbf{l}-\mathbf{q}} \mid V_\mu + A_\mu \mid \alpha_0 \rangle, \end{aligned} \tag{78}$$

where $\psi_e(\mathbf{r}, \mathbf{p})$ is the solution to the Dirac equation in the presence of the nuclear Coulomb potential which reduces as $Z \to 0$ to $u(p) \exp(i\mathbf{p}\cdot\mathbf{r})$. In standard notation (Konopinski, 1966)

$$\begin{aligned} \psi_e(\mathbf{p}, \mathbf{r}) =\ & \left(\frac{(2\pi)^3}{m \mid \mathbf{p} \mid}\right)^{1/2} \sum_{\kappa,\mu} i^l C_{l,1/2;j}{}^{\mu-\rho,\rho;\mu} Y_l{}^{\mu-\rho *}(\hat{p}) \\ & \times \exp(-i\sigma_\kappa) \psi_{\kappa\mu}(r), \end{aligned} \tag{79}$$

where

$$\begin{aligned} \sigma_\kappa &= (\pi/2)(l(\kappa) + 1 - \gamma) + \eta_\kappa - \arg\Gamma(\gamma + i\nu) \\ \psi_{\kappa\mu}(r) &= \begin{pmatrix} g_\kappa(r) \chi_{\kappa,\mu}(r) \\ i f_\kappa(r) \chi_{-\kappa,\mu}(r) \end{pmatrix} \quad \begin{matrix} \gamma = (\kappa^2 - \alpha^2 Z^2)^{1/2} \\ \nu = Z_\alpha E / \mid \mathbf{p} \mid \end{matrix} \end{aligned} \tag{80}$$

$$\exp(2i\eta_\kappa) = \frac{-\kappa + i Z_\alpha (m_e / \mid \mathbf{p} \mid)}{\gamma + i\nu} \quad \begin{matrix} \kappa > 0 & 0 < \eta_\kappa \le \pi/2 \\ \kappa < 0 & 0 \ge \eta_\kappa > \pi/2 \end{matrix}$$

In the case of a point nucleus—$V(r) = -Z\alpha/r$

$$\begin{pmatrix} g_\kappa(r) \\ f_\kappa(r) \end{pmatrix} = \left(\frac{\mid \mathbf{p} \mid (E+m)}{\pi}\right)^{1/2} \begin{pmatrix} \mathrm{Re} \\ \mathrm{Im} \end{pmatrix} Q_\kappa(p, r), \tag{81}$$

where

$$\begin{aligned} Q_\kappa(p, r) =\ & 2 \exp(\pi\nu/2) [\mid \Gamma(\gamma + i\nu) \mid / \Gamma(2\gamma + 1] \\ & \times (\gamma + i\nu)(2pr)^{\gamma - 1} \exp(-ipr + i\eta_\kappa) \\ & \times F(\gamma + 1 + i\nu, 2\gamma + 1; 2ipr). \end{aligned}$$

Since recoil terms—b, d, f, etc.— are already suppressed in their contribution to the decay spectra, we disregard their presence in the subsequent discussion, although this cannot always be done with impunity (Bottino, Ciocchotti, and Kim, 1974). In this approximation—keeping only the leading terms in a and c—the electron must be in a state with total angular momentum $j = \frac{1}{2}$ and, following Armstrong and Kim (1972a) we may project out this component

$$\begin{aligned} & P_{j=1/2} \psi_e(p, r) \\ & \quad = N^* [w + x\gamma^0 + y \boldsymbol{\gamma}\cdot\hat{r} + z \boldsymbol{\gamma}\cdot\hat{r}\gamma^0] u(p), \end{aligned} \tag{82}$$

where

$$\begin{aligned} w &= \frac{1}{2}\left(\frac{2m}{E+m}\right)^{1/2} \left[g_{-1}(r) + \frac{E+m}{p} \exp(-i\delta) f_1(r) \right], \\ x &= \frac{1}{2}\left(\frac{2m}{E+m}\right)^{1/2} \left[g_{-1}(r) - \frac{E+m}{p} \exp(-i\delta) f_1(r) \right], \\ y &= \frac{i}{2}\left(\frac{2m}{E+m}\right)^{1/2} \left[f_{-1}(r) + \frac{E+m}{p} \exp(-i\delta) g_1(r) \right], \\ z &= \frac{i}{2}\left(\frac{2m}{E+m}\right)^{1/2} \left[f_{-1}(r) - \frac{E+m}{p} \exp(-i\delta) g_1(r) \right], \\ N &= \frac{1}{4\pi}\left(\frac{(2\pi)^3}{mp}\right)^{1/2} \\ & \qquad \times \exp i[\tfrac{1}{2}\pi(1-\gamma) + \eta_{-1} - \arg\Gamma(\gamma + i\nu)], \end{aligned} \tag{83}$$

and $\delta = (\pi/2) + \eta_1 - \eta_{-1}$. Assuming for simplicity that

$$a(q^2)/a(0) = c(q^2)/c(0) \equiv F(q^2) \tag{84}$$

we define the "weak charge" density $\rho(r)$

$$\rho(r) = \int [d^4q/(2\pi)^4 \exp(-i\mathbf{q}\cdot\mathbf{r}) 2\pi\delta(q^0) F(q^2). \tag{85}$$

The decay amplitude then becomes

$$\bar{u}(p)(t + \gamma_0 s)\gamma^\lambda(1+\gamma_5)v(k)M_\lambda(p_1, p_2), \tag{86}$$

where

$$\begin{aligned} M_\lambda(p_1, p_2) &= \delta_{JJ'}\delta_{MM'}a(0)(P_\lambda/2M) \\ &\quad + C_{J'1;J}{}^{M'k;M}\epsilon_{ijk}\epsilon_{ij\lambda\eta}(P^\eta/2M)c(0) \\ s &= (A, -D\mathbf{k}) \qquad t = (B, -C\mathbf{k}) \end{aligned} \tag{87}$$

and with

$$\begin{aligned} A &= \int d^3rN\rho(r)w^*(r)\exp(-i\mathbf{k}\cdot\mathbf{r}), \\ B &= \int d^3rN\rho(r)x^*(r)\exp(-i\mathbf{k}\cdot\mathbf{r}), \\ C &= \int d^3rN\rho(r)y^*(r)\hat{k}\cdot\hat{r}\exp(-i\mathbf{k}\cdot\mathbf{r}), \\ D &= \int d^3rN\rho(r)z^*(r)\hat{k}\cdot\hat{r}\exp(-i\mathbf{k}\cdot\mathbf{r}). \end{aligned} \tag{88}$$

Given a model for $\rho(r)$ we can calculate the decay spectra. Up to corrections of $\mathcal{O}(Z\alpha qR(q/m))$, $\mathcal{O}(Z\alpha(qR)^3)$ we find (cf. App. B)

$$\begin{aligned} &F_\mp(Z, E)F_i(E, u, v, s) \\ &\quad \to F_\mp(Z, E)[F_i(E, u, v, s) + \Delta F_i(E, u, v, s)], \end{aligned} \tag{89}$$

where $\Delta F_i(E, u, v, s)$ are given in Appendix C. Necessary changes for the case of positron emission are also there discussed.

Of course, these calculable modifications arising from replacement of the lepton plane wave function by a Coulomb wave function are unfortunately not the only radiative corrections. First, there are additional purely hadronic effects, wherein the initial (final) nuclei exchange virtual photons among themselves. These cannot alter the decay spectra, however, and can only affect the invariant form factors themselves—altering them slightly from the values which they would have in the absence of electromagnetism. Since their values in the absence of electromagnetism are generally unknown, this is not a serious problem. Secondly, there are inelastic terms wherein the exchange of a virtual photon with the outgoing electron or positron causes transitions to other nuclear states. These *can* become important in case the transition—though allowed—is hindered for some reason. However, we shall not discuss these further.

The form of the impulse approximation is also altered by the presence of the electromagnetic interaction. In order to maintain gauge invariance we must modify the interaction Hamiltonian [Eq. (62)] by the minimal substitution $\partial^\nu \to \partial^\nu \pm ieA^\nu \approx \partial^\nu \pm ig^{\nu 0}(Z\alpha/r)$ which amounts in Eqs. (66)–(67) to the change $q_0 \to q_0 \pm (Z\alpha/R)$.

VII. EXPERIMENTS IN RECOIL ORDER

In this, the concluding section, we examine briefly the experimental situation, both as to experiments which have been done or which can be done in recoil order which may assist in illumination of nuclear structure and/or of properties of the weak current. In this regard we note that our analysis has included second order forbidden terms—f, g, j_2, j_3—which have no counter part in hyperon decay and which are generally neglected in analysis of allowed decay. Detection of, e.g., the $[\hat{n}\cdot(\mathbf{p}/E)]^3\beta - \gamma$ circular polarization correlation coefficient $G_3(E)/G_0(E)$ [cf. Eq. (56)] would verify their presence[12] and allow evaluation of the size of such form factors, especially if mirror transitions such as

$$^{20}\text{F}(^{20}\text{Na}) \to {}^{20}\text{Ne}^*(1.63 \text{ MeV}) + e^-(e^+) + \bar{\nu}_e(\nu_e)$$
$$\searrow {}^{20}\text{Ne} + \gamma$$

could be utilized in order to isolate vector-axial interference terms.

Measurement of these "new" types of form factors would constitute a new probe of nuclear structure. In general, however, in order to simplify analysis of the spectral structure, it is useful to study transitions $J = J' = \frac{1}{2}$ (e.g., $^{19}\text{Ne} \to {}^{19}\text{F}$) or $J = 1$, $J' = 0$ (e.g., $^{12}\text{B}(^{12}\text{N}) \to {}^{12}\text{C}$) which eliminate consideration of at least some of the ten independent form factors. Yet there may be reasons—large energy release, analog transition, etc.—which suggest the use of alternate spin sequences.

A classic recoil order experiment was suggested by Gell-Mann (1958) and performed by Wu *et al.* (Lee, Mo, and Wu 1963; Wu, 1964) as a verification of the conserved vector current hypothesis. This involves the transitions

$$^{12}\text{B} \to {}^{12}\text{C} + e^- + \bar{\nu}_e$$

$$^{12}\text{N} \to {}^{12}\text{C} + e^+ + \nu_e$$

wherein the energy release is substantial ($\gtrsim 10$ MeV). As seen in Fig. 2 ^{12}B, ^{12}N are members of an isotopic triplet with the 15.11 MeV $J^p = 1^+$ excited state of ^{12}C. The experiment involves a careful measurement of the electron (positron) energy dependence of the spectral function $f_1(E)$. Writing

$$f_1(E) = x + yE \tag{90}$$

we predict for the slope

$$(y/x)^{\text{theo}} \cong \pm(4/3M)(b/c). \tag{91}$$

According to CVC, however, b can be predicted from the

[12] Of course, detection of any of the spectral functions $f_i(E)$ or $g_i(E)$ $i \geq 18$ [cf. Eqs. (51), (53), (57)] would signify the presence of these new form factors, but measurement of such terms involves the measurement of nuclear recoil.

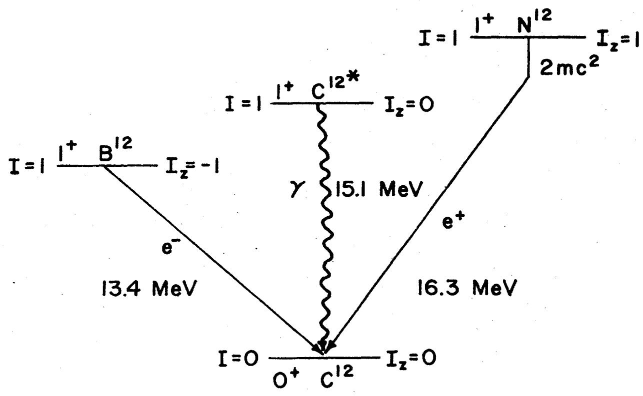

FIG. 2. Energy level diagram for the mass 12 triad used for CVC verification studies.

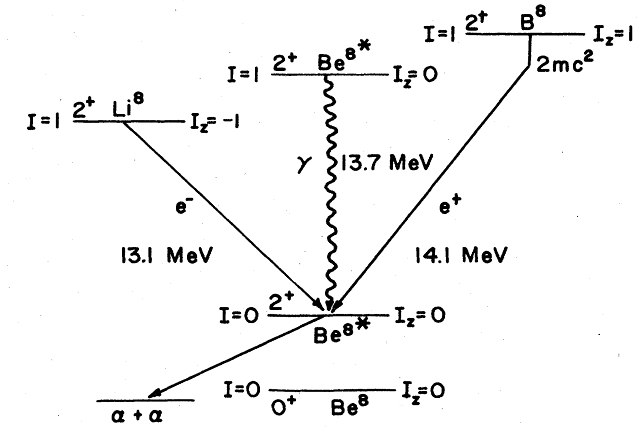

FIG. 3. Energy level diagram for the mass 8 triad. The ^{8}Li electron decay and the ^{8}B positron decay are followed by the alpha particle breakup of the 2.90 MeV ^{8}Be* level.

known width of the analog level while c is known from the ^{12}N(^{12}B) ft value. The predicted value of the slope (Gell-Mann, 1958) $\pm 6.4 \pm 0.5/m$ agrees nicely with the experimental number (Lee, Mo and Wu, 1963)

$$(y/x)^{\exp} = \begin{array}{c} 5.5 \pm 1.0/m \\ \\ -5.2 \pm 0.6/m \end{array} \tag{92}$$

thus confirming the CVC hypothesis in this transition.

An alternate test of CVC involves the β–α correlation in the mass 8 system (cf. Fig. 3) (Eichner *et al.*, 1966 Nordberg, Morinigo, and Barnes, 1962; Tribble and Garvey, 1974)

$$\begin{array}{l} {}^8\mathrm{Li} \rightarrow {}^8\mathrm{Be}^* + e^- + \bar{\nu}_e \\ \qquad\qquad \searrow \\ \qquad\qquad\quad \alpha + \alpha \\ \qquad\qquad \nearrow \\ {}^8\mathrm{B} \;\rightarrow {}^8\mathrm{Be}^* + e^+ + \nu_e . \end{array}$$

Here again the measurement is favored by a large energy release $\sim$15 MeV. We predict[13]

$$\begin{aligned} \delta &= [G_2(E)/G_0(E)]_{\mathrm{Li}} - [G_2(E)/G_0(E)]_{\mathrm{B}} \\ &\cong (E/M)[(b - d_{\mathrm{II}})/c], \end{aligned} \tag{93}$$

where d_{II} is the component of the induced tensor which is produced by a second class current. If the decay width and the $M1/E2$ ratio of the 2^+ analog state of ^{8}Be to the 2.90 MeV 2^+ state were known, then there would exist a model independent prediction for b. As it stands we utilize the impulse approximation which gives (Holstein 1971a; Tribble and Garvey, 1974)

$$5.6 \leq b/Ac \leq 7.1. \tag{94}$$

The experimental result (Tribble and Garvey 1974)

$$[(b - d_{\mathrm{II}})/Ac]^{\exp} = 6.8 \pm 0.4 \tag{95}$$

is quite consistent with the absence of second class currents and the validity of the CVC hypothesis. However, an experimental measurement of the radiative width in ^{8}Be would be very useful in removing the model dependence of this conclusion. We note from Eq. (91) that the Wu experiment is completely insensitive to a second class induced tensor.

In this same (mass 8) system, Wilkinson and Alburger have measured the electron (positron) endpoint energy dependence of $f_1(E_0/2)$, which is feasible due to the large width of the 2.90 Mev 2^+ level in ^{8}Be.

We predict, assuming the absence of second class vector currents and neglecting quadratic terms in energy,

$$\delta = \frac{f_1(E_0/2)_{\mathrm{Li}} - f_1(E_0/2)_{\mathrm{B}}}{f_1(E_0/2)_{\mathrm{Li}} + f_1(E_0/2)_{\mathrm{B}}} \cong 2\,\frac{c_{\mathrm{II}}}{c_{\mathrm{I}}} - \frac{d_{\mathrm{II}}}{3c_{\mathrm{I}}}\,\frac{E_0^+ + E_0^-}{M}, \tag{96}$$

where c_{II}, d_{II} represent the second class current contributions to the Gamow–Teller and induced tensor form factors and c_{I} is the usual first class Gamow–Teller term. The Wilkinson–Alburger experiment (Wilkinson and Alburger, 1971) is relevant to the energy dependent piece and indicates that

$$A^{-1} \mid d_{\mathrm{II}}/c_{\mathrm{I}} \mid < 1.2. \tag{97}$$

The interpretation of this result in terms of its implications with respect to second class currents is open to question. In terms of the naive impulse approximation (see Henley and Wolfenstein, 1971, however) we find $\mid g_{\mathrm{II}}(0) \mid < 1.5$, consistent with $g_{\mathrm{II}} = 0$ but not a very restrictive limit. Also Kubodera, Delorme, and Rho (1973) assess the meson exchange contribution to d_{II} and find (a) that it can be just as large as the impulse approximation prediction and (b) that large second class contributions in other systems are not inconsistent with a null effect in mass eight.

In any case the mass eight experiments indicate a large energy independent piece which must be explained either

[13] The terms in f/c and g/c are small compared to the weak magnetism contribution.

through electromagnetic asymmetries or via authentic second class contributions to c.

Some interesting measurements which are now underway (Calaprice, 1974; Morita, 1973) concern the energy dependence of the polarization correlation coefficient $A = F_1(E)/F_0(E)$ for analog transitions. Writing $A = u + vE$, we predict

$$\left(\frac{v}{u}\right)^{\text{theo}} \cong \pm\frac{1}{3M}\left[\frac{\delta_{JJ'}[J/(J+1)]^{1/2}2a(b+d) \mp [\gamma_{J,J'}/(J+1)]c(5b-d)}{\delta_{JJ'}[J/(J+1)]^{1/2}2ac \mp [\gamma_{J,J'}/(J+1)]c^2} - \frac{4bc}{a^2+c^2}\right]. \tag{98}$$

Experiments are currently underway on ^{19}Ne and on ^{12}B, ^{12}N, but results are as yet preliminary. In the case of ^{19}Ne the magnetic moments of ^{19}Ne (^{19}F) are known while in the mass 12 case the radiative width is known as mentioned previously. CVC makes a precise prediction for b in each case, while a, c are known from CVC and from measurements of the ft value. Thus $(v/u)^{\text{theo}}$ is known if no second class currents are present (i.e., $d = 0$ in the case of analog display) and the results of these experiments can be used to check this idea.

In another area of interest, Calaprice *et al.* (1969) have measured f_8/f_1 as a test of T invariance and find for $^{19}\text{Ne} \to {}^{19}\text{F} + e^+ + \nu_e$

$$\frac{f_8(E_0/2)}{f_1(E_0/2)} \approx \frac{(\frac{1}{3})^{1/2}\,\text{Im}\, a^*[2c + (E_0/M)d]}{|a|^2 + |c|^2} = 0.002 \pm 0.014. \tag{99}$$

However, Kim and Primakoff (1969) note that even this very precise result does not rule out a large T violating second class axial current since ^{19}Ne, ^{19}F form an isodoublet, whereby such a current can only contribute to the form factor d and is kinematically suppressed. They suggest examination of hindered allowed decays such as

$$^{32}\text{P} \to {}^{32}\text{S} + e^- + \bar{\nu}_e$$

wherein recoil terms may be enhanced. As seen from Eq. (B-6) a careful measurement of the energy dependence of the correlation should be considered in order to pinpoint the nature of any effects found.

Recoil effects have in general been omitted in the measurement of Coulomb mixing via circular polarization experiments (Bloom, Mann, and Miskel, 1962). Here the presence of recoil terms can mock the presence of bona fide mixing and thus inclusion of their presence in the analysis could be vital, as we have previously emphasized (Holstein, Shanahan, and Treiman, 1972). Measurement of energy dependence could be used in order to isolate the recoil contributions from real mixing terms, however.

Finally, an interesting possibility is a combined measurement of energy dependence for the electron neutrino correlation—$f_2(E)/f_1(E)$—*and* for the polarization asymmetry factor—$F_1(E)/F_0(E)$—in an analog decay, such as $^{19}\text{Ne} \to {}^{19}\text{F}$ or $n \to p$ wherein the corresponding magnetic moments are known. Then measurement of both b *and* d separately is feasible.[14] The mere presence of a nonzero d coefficient for analog decay is sufficient to prove the existence of second class currents—without model dependence, while the value of b would yield a test of CVC, so this would provide a doubly interesting measurement.

We thus see that a variety of experiments in recoil order have been completed, but an exciting range of additional possibilities remains, and one can anticipate a significant increase in the number of experiments involving recoil term sensitivity in the not too distant future.

ACKNOWLEDGMENT

We wish to express our gratitude to Professor S. Treiman for his encouragement and for many very useful conversations.

APPENDIX A

Most practitioners of the so-called elementary particle approach (Armstrong and Kim, 1972b; Kim and Primakoff, 1965a; Kim and Primakoff, 1965b; Kubodera and Kim, 1973; Primakoff, 1967; Primakoff, 1970) utilize a Cartesian formalism, wherein, for a particle of integral spin J one represents the spin state by a traceless symmetric tensor of rank J—the polarization tensor. For spin 1, this is just the polarization vector ξ_μ where μ ranges from 0 to 3. For a particle of spin J and 4-momentum p, which in its rest frame has spin projection M along the z axis we define

$$T_{\mu_1\cdots\mu_J}{}^J = \sum_{m_1\cdots m_J} \delta^{M,\Sigma_i m_i} \left[\frac{2^{J-\Sigma_i|m_i|}(J+M)!(J-M)!}{(2J)!}\right]^{1/2} \times \xi_{\mu_1}(m_1)\xi_{\mu_2}(m_2)\cdots\xi_{\mu_J}(m_J), \tag{A1}$$

where in the rest frame we have

$$\begin{aligned}\xi_\mu(+1) &= (0, -(\tfrac{1}{2})^{1/2}, -i(\tfrac{1}{2})^{1/2}, 0)\\ \xi_\mu(0) &= (0,0,0,1)\\ \xi_\mu(-1) &= (0, (\tfrac{1}{2})^{1/2}, -i(\tfrac{1}{2})^{1/2}, 0).\end{aligned} \tag{A2}$$

We now can write down manifestly covariant expressions for arbitrary integer spin transitions. We shall for simplicity

[14] Of course, b and d can be evaluated in principle by careful energy dependence measurements on *any* two of the correlation functions $f_i(E)$, $g_i(E)$ $1 \le i \le 17$ or $F_i(E)$, $G_i(E)$ $0 \le i \le 2$. However, those cited in Sec. VII are perhaps the easiest accessible experimentally.

treat only the $J \to J$ case, but it is easy to generalize for $J \to J \pm 1$.[15] Also, it is straightforward to treat arbitrary half-integral spin transitions in a similar fashion. For a spin $J \to$ spin J transition we define

$$\begin{aligned}
&\langle \beta \mid V_\lambda(0) \mid \alpha \rangle \\
&\equiv T_{\mu_1\cdots\mu_J}{}^{J*}(\beta) T_{\nu_1\cdots\nu_J}{}^{J}(\alpha)(-)^{J-1} g^{\mu_1,\nu_1}\cdots g^{\mu_{J-1},\nu_{J-1}} \\
&\times \Big\{ -g^{\mu_J,\nu_J}(aP_\lambda + eq_\lambda)(2M)^{-1} \\
&+ \frac{b}{2M}\left(\frac{J}{J+1}\right)^{1/2} (q^{\mu_J} g_\lambda{}^{\nu_J} - q^{\nu_J} g_\lambda{}^{\mu_J}) \\
&- \left[\frac{3(2J-1)J}{2(2J+3)(J+1)}\right]^{1/2} \frac{f}{2M}\Big[q^{\mu_J} g_\lambda{}^{\nu_J} + q^{\nu_J} g_\lambda{}^{\mu_J} \\
&- \tfrac{2}{3} g^{\mu_J,\nu_J}\left(q\cdot l - \frac{P\cdot q P\cdot l}{4M^2}\right)\Big] \\
&- 3\left[\frac{(2J-1)J}{(2J+3)(J+1)}\right]^{1/2} \frac{g}{4M^2}\frac{P_\lambda}{2M}\Big[q^{\mu_J} q^{\nu_J} \\
&- \tfrac{1}{3} g^{\mu_J,\nu_J}\left(q^2 - \frac{P\cdot q P\cdot q}{4M^2}\right)\Big]\Big\},
\end{aligned} \tag{A3a}$$

$$\begin{aligned}
&\langle \beta \mid A_\lambda(0) \mid \alpha \rangle \\
&= T_{\mu_1\cdots\mu_J}{}^{J*}(\beta) T_{\nu_1\cdots\nu_J}{}^{J}(\alpha)(-)^{J-2} g^{\mu_1,\nu_1}\cdots g^{\mu_{J-2},\nu_{J-2}} \\
&\times \Big\{ i\frac{c}{2M}\left(\frac{J}{J+1}\right)^{1/2} g^{\mu_{J-1},\nu_{J-1}} \epsilon_{\lambda\eta}{}^{\mu_J\nu_J} P^\eta \\
&- i\frac{d}{2M}\left(\frac{J}{J+1}\right)^{1/2} g^{\mu_{J-1},\nu_{J-1}} \epsilon^{\mu_J\nu_J}{}_{\lambda\eta} q^\eta \\
&+ i\frac{h}{(2M)^3}\left(\frac{J}{J+1}\right)^{1/2} q_\lambda g^{\mu_{J-1},\nu_{J-1}} \epsilon_{\lambda\eta}{}^{\mu_J\nu_J} q^\lambda P^\eta \\
&- i\frac{j_2}{(2M)^3}\left[\frac{3(2J-1)J}{2(2J+3)(J+1)}\right]^{1/2} g^{\mu_{J-1},\nu_{J-1}} \\
&\times [q^{\mu_J}\epsilon_{\sigma\lambda\eta}{}^{\nu_J} + q^{\nu_J}\epsilon_{\sigma\lambda\eta}{}^{\mu_J}] q^\sigma P^\eta \\
&- i\frac{10}{3}\frac{j_3}{(2M)^3}\left[\frac{3(J-1)J(2J-1)}{5(J+2)(J+1)(2J+3)}\right]^{1/2} \\
&\times \epsilon^{\mu_{J-1}\nu_{J-1}\sigma\eta} P_\eta \{ q^{\mu_J} q^{\nu_J} g_{\sigma\lambda} + q_\sigma q^{\nu_J} g_\lambda{}^{\mu_J} + q_\sigma q^{\mu_J} g_\lambda{}^{\nu_J} \\
&- \tfrac{2}{5}[q_\lambda - P\cdot q P_\lambda/(2M)^2] \\
&\times [q_\sigma g^{\mu_J,\nu_J} + g_\sigma{}^{\mu_J} q^{\nu_J} + g_\sigma{}^{\nu_J} q^{\mu_J}] \\
&- \tfrac{1}{5}[q^2 - P\cdot q P\cdot q/(2M)^2] \\
&\times [g_{\sigma\lambda} g^{\mu_J,\nu_J} + g_\lambda{}^{\mu_J} g_\sigma{}^{\nu_J} + g_\lambda{}^{\nu_J} g_\sigma{}^{\mu_J}]\}.
\end{aligned} \tag{A3b}$$

[15] A useful catalog of Cartesian tensor techniques can be found in P. L. Csonka, M. J. Moravcsik and M. D. Scadron (1966).

The form factors $a, b \cdots j_3$ defined above are identical to those used in the spherical formulation in Eq. (7), up to terms of $\mathcal{O}(q^2/M^2)$.

APPENDIX B

Before listing the spectral functions we include a brief sketch which indicates the procedure by which they are calculated. Keeping for simplicity only $\Delta J = 0, 1$ terms we write

$$\begin{aligned}
&l^\mu \langle \beta \mid V_\mu + A_\mu \mid \alpha \rangle \\
&= ((a/2M) P\cdot l + (e/2M) q\cdot l)\delta_{JJ'}\delta_{MM'} \\
&+ (i/4M) C_{J'1;J}{}^{M'k;M} \epsilon_{ijk} \\
&\times [2bq_i l_j - i\epsilon_{ij\mu\eta}(cP^\eta - dq^\eta) l^\mu]
\end{aligned} \tag{B1}$$

and note that this matrix element splits into two components according to transformation properties under spatial rotation—a term $P\cdot l$, $q\cdot l$ transforming as a scalar and a second piece, consisting of the term in brackets, transforming as a vector. Then, for example, in calculating β–γ correlations, we successively decompose parent and intermediate nuclear states into products of nuclear states and spherical harmonics relating to the weak current and the photon polarization. The work is simplified considerably by choosing the axis of quantization to be along the direction of photon momentum and specifying definite photon helicity. Thus for the vector terms and $E1$ radiation we write

$$| JM \rangle = C_{J'1;J}{}^{M'k;M} Y_1{}^k(S) \, | J'M' \rangle \tag{B2a}$$

with

$$| J'M' \rangle = C_{J''1;J'}{}^{M''r;M'} Y_1{}^r(\epsilon^*) \, | J''M'' \rangle, \tag{B2b}$$

where ϵ represents the photon polarization and

$$\mathbf{S}_k = (i/4M)\epsilon_{ijk}[2bq_i l_j - i\epsilon_{ij\mu\eta} l^\mu (cP^\eta - dq^\eta)]. \tag{B3}$$

A similar analysis is carried through for the scalar component of the current.

On then squares, expresses the products of spherical harmonics in rotationally invariant form, and multiplies by phase space as in Eq. (4), yielding

$$\begin{aligned}
&d\Gamma\alpha \text{ phase space} \\
&\quad \times [AS^*\cdot S + B(\hat{Q}\cdot S\hat{Q}\cdot S^* - \tfrac{1}{3}S^*\cdot S) + iC\hat{Q}\cdot S^* \times S \\
&\quad + 2\,\mathrm{Re}D\hat{Q}\cdot S(1/2M)(a^*P\cdot l^* + e^*q\cdot l^*) \\
&\quad + \mathrm{Re}E(1/(2M)^2)(aP\cdot l + eq\cdot l)(a^*P\cdot l^* + e^*q\cdot l^*)].
\end{aligned} \tag{B4}$$

Here $\hat{Q}$ is a unit vector describing the photon direction in the rest frame of the β-decay daughter nucleus, and A,

$B, \cdots E$ are combinations of Clebsch–Gordan coefficients. Finally we substitute

$$l_\mu^* l_\nu = p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu} p\cdot k \pm i\epsilon_{\alpha\mu\beta\nu} p^\alpha k^\beta \qquad \text{(B5)}$$

and normalize so that integration of $\hat{Q}$ over the sphere yields the standard allowed β-spectrum.

In order to write the spectral functions most conveniently we introduce the momentum transfer dependence of the Fermi and Gamow–Teller terms a and c. Defining

$$a(q^2) \equiv a_1 + a_2(q^2/M^2) + \cdots$$

$$c(q^2) \equiv c_1 + c_2(q^2)M^2) + \cdots$$

we note that for analog and nonanalog transitions, the CVC hypothesis gives $a_1 = M_F$ and $a_1 = 0$, respectively. Then we can define the spectral functions $F_i(E, u, v, s)$. Here u, v denote spin values and can take on integral or half-integral values, while s is an integer having either the value 0 or 1. The upper (lower) sign refers to electron (positron) decay.

$F_1(E, u, v, s)$

$$= |a_1|^2 + 2\,\mathrm{Re}a_1^*a_2(3M^2)^{-1}(m_e^2 + 4EE_0 + 2(m_e^2/E)E_0 - 4E^2) + |c_1|^2 + 2\,\mathrm{Re}c_1^*c_2(9M^2)^{-1} \times (11m_e^2 + 20EE_0 - 2(m_e^2/E)E_0 - 20E^2) - 2(E_0/3M)\,\mathrm{Re}c_1^*(c_1 + d \pm b) + (2/3M)E[3\,|a_1|^2 + \mathrm{Re}c_1^*(5c_1 \pm 2b)] - \frac{1}{3}\frac{m_e^2}{ME}\left[2\,|c_1|^2 + \mathrm{Re}c_1^*(d \pm 2b) - 3\,\mathrm{Re}a_1^*e - \mathrm{Re}c_1^*h\,\frac{E_0 - E}{2M}\right]$$

$F_2(E, u, v, s)$

$$= |a_1|^2 + 2\,\mathrm{Re}a_1^*a_2(m_e^2/M^2) - \tfrac{1}{3}|c_1|^2 - \tfrac{2}{3}\,\mathrm{Re}c_1^*c_2(1/M^2)(m_e^2 + 8EE_0 - 8E^2) + \tfrac{2}{3}(E_0/M)\,\mathrm{Re}c_1^*(c_1 + d \pm b) - \tfrac{4}{3}(E/M)\,\mathrm{Re}c_1^*(3c_1 \pm b)$$

$F_3(E, u, v, s)$

$$= -\frac{E}{M}\left[3\,|a_1|^2 + 4\,\mathrm{Re}a_1^*a_2\frac{E_0 - E}{M} - |c_1|^2 - \tfrac{4}{3}\,\mathrm{Re}c_1^*c_2\frac{E_0 - E}{M}\right]$$

$F_4(E, u, v, s)$

$$= \delta_{u,v}\left(\frac{u}{u+1}\right)^{1/2}\{\mathrm{Re}a_1^*[2c_1 - (E_0/M)(c_1 + d \pm b) + (E/M)(7c_1 + d \pm b) + (m_e^2/2M^2)h] + 2\,\mathrm{Re}(a_1^*c_2 + c_1^*a_2)(1/M^2)(m_e^2 + 2EE_0 - 2E^2)\}$$
$$\mp (-)^s\frac{\gamma_{u,v}}{u+1}\,\mathrm{Re}c_1^* \times [c_1 + 2c_2(1/M^2)(m_e^2 + 2EE_0 - 2E^2) - (E_0/2M)(c_1 + d \pm b) + (E/2M)(5c_1 - d \pm 3b)]$$
$$+ \frac{\lambda_{u,v}}{u+1}\,\mathrm{Re}c_1^*\left[-f\frac{E_0 + 9E}{2M} + (\tfrac{3}{2})^{1/2}g\,\frac{2E^2 - 5EE_0 + E_0^2 + 2m_e^2}{2M^2} \pm 3j_2\,\frac{4E^2 - EE_0 - E_0^2 - 2m_e^2}{4M^2}\right]$$

$F_5(E, u, v, s)$

$$= -\delta_{u,v}\left(\frac{u}{u+1}\right)^{1/2}\frac{E}{M}\,\mathrm{Re}\left\{a_1^*[5c_1 - d \pm b] + 4(a_1^*c_2 + c_1^*a_2)\frac{E_0 - E}{M}\right\}$$
$$\pm (-)^s\frac{\gamma_{u,v}}{u+1}\frac{E}{2M}\,\mathrm{Re}c_1^*\left[5c_1 + 8c_2\frac{E_0 - E}{M} - d \pm b\right]$$
$$+ \frac{\lambda_{u,v}}{u+1}\frac{E}{2M}\,\mathrm{Re}c_1^* \times \left[3f - (\tfrac{3}{2})^{1/2}g\,\frac{E_0 + 2E}{M} \pm 3j_2\,\frac{E_0 - 4E}{2M}\right]$$

$F_6(E, u, v, s)$

$$= \delta_{u,v}\left(\frac{u}{u+1}\right)^{1/2}\left\{\mathrm{Re}a_1^*\left[2c_1 + (E/M)(5c_1 - d \pm b) - (m_e^2/ME)\left(c_1 \pm b \mp e - h\frac{E_0 - E}{2M}\right)\right] + 2\,\mathrm{Re}(a_1^*c_2 + c_1^*a_2)(1/M^2)(m_e^2 + 2EE_0 - 2E^2)\right\}$$
$$\pm (-)^s[\gamma_{u,v}/(u+1)]\,\mathrm{Re}c_1^*[c_1 + 2c_2(1/M^2) \times (m_e^2 + 2EE_0 - 2E^2) - (E_0/M)(c_1 + d \pm b) + (E/2M)(7c_1 + d \pm 3b) - (m_e^2/2ME)(c_1 + d \pm b)]$$
$$+ \frac{\lambda_{u,v}}{u+1}\,\mathrm{Re}c_1^*\left[-f\,\frac{10E_0 - 9E - (m_e^2/E)}{2M} + (\tfrac{3}{2})^{1/2}g\,\frac{2E^2 + EE_0 - 2E_0^2 - 4m_e^2 + 3E_0(m_e^2/E)}{2M^2} \pm 3j_2\,\frac{-4E^2 + 7EE_0 - 2E_0^2 + 2m_e^2 - 3E_0(m_e^2/E)}{4M^2}\right]$$

$F_7(E, u, v, s)$

$$= \delta_{u,v}\left(\frac{u}{u+1}\right)^{1/2}\Big(\mathrm{Re}a_1^*[(E_0/M)(c_1 + d \pm b) + (E/M)(-7c_1 - d \mp b)] - 4\,\mathrm{Re}(a_1^*c_2 + c_1^*a_2)\frac{E(E_0-E)}{M^2}\Big)$$

$$\pm (-)^s \frac{\gamma_{u,v}}{u+1}\mathrm{Re}c_1^*\Big[-4c_2\frac{E(E_0-E)}{M^2} + (E_0/2M)(c_1 + d \pm b) - (E/2M)(7c_1 + d \pm b)\Big]$$

$$+ \frac{\lambda_{u,v}}{u+1}\mathrm{Re}c_1^*\Big[3f\frac{E_0-E}{2M} + (\tfrac{3}{2})^{1/2}g \times \frac{-E_0^2 + 2E_0E - E^2}{M^2} \pm 3j_2\frac{3E_0^2 - 7E_0E + 4E^2}{4M^2}\Big]$$

$F_8(E, u, v, s)$

$$= \mp\delta_{u,v}\left(\frac{u}{u+1}\right)^{1/2}\mathrm{Im}a_1^* \times [2c_1 - (E_0/M)(c_1 + d \pm b) + 2(E/M)(3c_1 \pm b)]$$

$$\mp (-)^s\frac{\lambda_{u,v}}{u+1}\mathrm{Im}c_1^*[(E_0/2M)(b \pm d) \mp (E/M)d \mp (m_e^2/4M^2)(1 + \Delta/2M)h]$$

$$+ \frac{\lambda_{u,v}}{u+1}\mathrm{Im}c_1^*\Big[\pm 5f(E_0/2M) \pm (\tfrac{3}{2})^{1/2}g \times \frac{E_0^2 - 2EE_0 - m_e^2 + 2E^2}{2M^2} + 3j_2\frac{E_0^2 - 2EE_0 + m_e^2}{4M^2}\Big]$$

$F_9(E, u, v, s)$

$$= \pm\delta_{u,v}\left(\frac{u}{u+1}\right)^{1/2} 6\,\mathrm{Im}a_1^*c_1(E/M) \pm \frac{\lambda_{u,v}}{u+1}\mathrm{Im}c_1^*g(\tfrac{3}{2})^{1/2}\frac{EE_0 - E^2}{M^2}$$

$F_{10}(E, u, v, s)$

$$= -\theta_{u,v}\frac{E}{2M}\mathrm{Re}c_1^*(c_1 + d \mp b) - \delta_{u,v}\left[\frac{u(u+1)}{(2u-1)(2u+3)}\right]^{1/2}\frac{E}{M}\mathrm{Re}a_1^* \times \Big[(\tfrac{3}{2})^{1/2}f + 3g(E/4M) \mp (\tfrac{3}{2})^{1/2}j_2\frac{E_0 - E}{2M}\Big]$$

$$+ \kappa_{u,v}(-)^s\frac{E}{2M}\mathrm{Re}c_1^* \times \Big[\pm 3f \pm (\tfrac{3}{2})^{1/2}g\frac{E_0 - E}{M} + 3j_2\frac{E_0 - 2E}{2M}\Big] + \epsilon_{u,v}3\,\mathrm{Re}c_1^*j_3\frac{2EE_0 + 5E^2}{8M^2}$$

$F_{11}(E, u, v, s)$

$$= -\delta_{u,v}\left[\frac{u(u+1)}{(2u-1)(2u+3)}\right]\mathrm{Re}a_1^* \times (\tfrac{3}{2}g \pm (\tfrac{3}{2})^{1/2}j_2)(E^2/2M^2) + \kappa_{u,v}(-)^s\mathrm{Re}c_1^*(\pm(\tfrac{3}{2})^{1/2}g + j_2)(E^2/2M^2) - \epsilon_{u,v}\mathrm{Re}c_1^*j_3(5E^2/8M^2)$$

$F_{12}(E, u, v, s)$

$$= -\theta_{u,v}\mathrm{Re}c_1^*[c_1 + 2c_2(1/M^2)(m_e^2 + 2EE_0 - 2E^2) - (E_0/2M)(c_1 + d \pm b) + (E/M)(3c_1 \pm b) - (m_e^2/2M^2)(1 + \Delta/2M)h]$$

$$- \delta_{u,v}\left[\frac{u(u+1)}{(2u-1)(2u+3)}\right]^{1/2}\mathrm{Re}a_1^*\Big[(\tfrac{3}{2})^{1/2}f(E_0/M) + 3g\frac{EE_0 - E^2}{2M^2} \pm (\tfrac{3}{2})^{1/2}j_2\frac{E_0^2 - 2EE_0 + m_e^2}{2M^2}\Big]$$

$$+ \kappa_{u,v}(-)^s\mathrm{Re}c_1^*\Big[\pm 3f\frac{E_0 - 2E}{2M} \pm (\tfrac{3}{2})^{1/2}g\frac{E_0^2 - 2EE_0 + m_e^2}{2M^2} - j_2\frac{-E_0^2 + 8EE_0 - 8E^2 + m_e^2}{4M^2}\Big]$$

$$+ \epsilon_{u,v}\mathrm{Re}c_1^*j_3\frac{E_0^2 + 9EE_0 - m_e^2 - 9E^2}{4M^2}$$

$F_{13}(E, u, v, s)$

$$= \theta_{u,v}(E/M)\,\mathrm{Re}c_1^*\Big(3c_1 + 4c_2\frac{E_0 - E}{M}\Big) - \delta_{u,v}\left[\frac{u(u+1)}{(2u-1)(2u+3)}\right]^{1/2} 3\,\mathrm{Re}a_1^*g\frac{EE_0 - E^2}{2M^2} - \epsilon_{u,v}9\,\mathrm{Re}c_1^*j_3\frac{EE_0 - E^2}{4M^2}$$

$F_{14}(E, u, v, s)$

$$= -\theta_{u,v}\frac{E_0 - E}{2M}\mathrm{Re}c_1^*$$

$$\times [c_1 + d \pm b + (m_e^2/2ME)(1+\Delta/2M)h]$$

$$- \delta_{u,v}\left[\frac{u(u+1)}{(2u-1)(2u+3)}\right]^{1/2} \mathrm{Re}a_1^*\left[(\tfrac{3}{2})^{1/2} f\frac{E_0-E}{M}\right.$$

$$+ 3g\frac{E_0^2 - 2EE_0 + E^2}{4M^2}$$

$$\left.\pm (\tfrac{3}{2})^{1/2} j_2 \frac{E^2 - EE_0 - m_e^2 + E_0(m_e^2/E)}{2M^2}\right]$$

$$+ \kappa_{u,v}(-)^s \mathrm{Re}c_1^*\left[\pm 3f\frac{E-E_0}{2M}\right.$$

$$\pm (\tfrac{3}{2})^{1/2} g\frac{E^2 - EE_0 - m_e^2 + E_0(m_e^2/E)}{2M^2}$$

$$\left.- 3j_2\frac{2E^2 - 3EE_0 - m_e^2 + E_0^2 + E_0(m_e^2/E)}{4M^2}\right]$$

$$+ \epsilon_{u,v} 3\,\mathrm{Re}c_1^* j_3$$

$$\times \frac{5E^2 - 12EE_0 + 2m_e^2 + 7E_0^2 - 2E_0(m_e^2/E)}{8M^2}$$

$F_{15}(E, u, v, s)$

$$= -\delta_{u,v}\left[\frac{u(u+1)}{(2u-1)(2u+3)}\right]^{1/2}$$

$$\times \mathrm{Re}a_1^*[\tfrac{3}{2}g \mp (\tfrac{3}{2})^{1/2} j_2]\frac{E_0^2 - 2EE_0 + E^2}{2M^2}$$

$$+ \kappa_{u,v}(-)^s \mathrm{Re}c_1^*[\mp (\tfrac{3}{2})^{1/2} g + j_2]\frac{E_0^2 - 2EE_0 + E^2}{2M^2}$$

$$- \epsilon_{u,v}\tfrac{5}{4}\mathrm{Re}c_1^* j_3\frac{E_0^2 - 2EE_0 + E^2}{2M^2}$$

$F_{16}(E, u, v, s)$

$$= -\theta_{u,v}(E/2M)\,\mathrm{Im}c_1^*(\pm d - b)$$

$$+ \delta_{u,v}\left[\frac{u(u+1)}{(2u-1)(2u+3)}\right]^{1/2}(E/M)$$

$$\times \mathrm{Im}a_1^*\left[\pm (\tfrac{3}{2})^{1/2} f + (\tfrac{3}{2})^{1/2} j_2\frac{E_0 - 2E}{2M}\right]$$

$$+ \kappa_{u,v}(-)^s (E/2M)\,\mathrm{Im}c_1^*\left[-f + (\tfrac{3}{2})^{1/2} g\frac{E_0-2E}{M}\right.$$

$$\left.\mp j_2(E_0/2M)\right] \mp \epsilon_{u,v} 7\,\mathrm{Im}c_1^* j_3(EE_0/4M^2)$$

$F_{17}(E, u, v, s)$

$$= -\theta_{u,v}\frac{E_0-E}{2M}\mathrm{Im}c_1^*(b \pm d)$$

$$+ \delta_{u,v}\left[\frac{u(u+1)}{(2u-1)(2u+3)}\right]^{1/2}\frac{E_0-E}{M}$$

$$\times \mathrm{Im}a_1^*\left[\pm (\tfrac{3}{2})^{1/2} f + (\tfrac{3}{2})^{1/2} j_2\frac{E_0-2E}{M}\right]$$

$$+ \kappa_{u,v}(-)^s\frac{E_0-E}{2M}$$

$$\times \mathrm{Im}c_1^*\left[f + (\tfrac{3}{2})^{1/2} g\frac{E_0-2E}{M} \mp j_2(E_0/2M)\right]$$

$$\pm \epsilon_{u,v}\,\mathrm{Im}c_1^* j_3\frac{7EE_0 - 7E_0^2}{4M^2}$$

$F_{18}(E, u, v, s)$

$$= -\delta_{u,v}(3u^2 + 3u - 1)$$

$$\times [u/(u-1)(u+1)(u+2)(2u-1)(2u+3)]^{1/2}$$

$$\times (\tfrac{3}{5})^{1/2}\,\mathrm{Re}a_1^* j_3(5E^2/4M^2)$$

$$+ \frac{\rho_{u,v}}{u+1}\frac{5E^2}{4M^2}\mathrm{Re}c_1^*[\sqrt{3}g \pm \sqrt{2}j_2]$$

$$\pm \frac{\sigma_{u,v}}{u+1}(-)^s\,\mathrm{Re}c_1^* j_3(5E^2/2M^2)$$

$F_{19}(E, u, v, s)$

$$= -\delta_{u,v}(3u^2 + 3u - 1)$$

$$\times [u/(u-1)(u+1)(u+2)(2u-1)(2u+3)]^{1/2}$$

$$\times (\tfrac{3}{5})^{1/2}\,\mathrm{Re}a_1^* j_3\frac{10EE_0 - 5E^2}{4M^2}$$

$$+ \frac{\rho_{u,v}}{u+1}(E/M)\,\mathrm{Re}c_1^*\left[5\sqrt{2}f + \sqrt{3}g\frac{10E_0 - 5E}{4}\right.$$

$$\left.\pm \sqrt{2}j_2\frac{10E_0 - 15E}{4M}\right]$$

$$\pm \frac{\sigma_{u,v}}{u+1}\mathrm{Re}c_1^* j_3\frac{10EE_0 - 15E^2}{2M^2}$$

$F_{20}(E, u, v, s)$

$$= -\delta_{u,v}(3u^2 + 3u - 1)$$

$$\times [u/(u-1)(u+1)(u+2)(2u-1)(2u+3)]^{1/2}$$

$$\times (\tfrac{3}{5})^{1/2}\,\mathrm{Re}a_1^* j_3\frac{5E_0^2 - 5E^2}{4M^2}$$

$$+ \frac{\rho_{u,v}}{u+1}\mathrm{Re}c_1^*\left[5\sqrt{2}f\frac{E_0-E}{M} + \sqrt{3}g\frac{5E_0^2 - 5E^2}{4M^2}\right.$$

$$\pm 5\sqrt{2}j_2 \frac{E_0^2 - 4EE_0 + 3E^2}{4M^2}\Big]$$

$$\pm (-)^s \frac{\sigma_{u,v}}{u+1} \text{Re}c_1^* j_3 \frac{5E_0^2 - 20EE_0 + 15E^2}{2M^2}$$

$F_{21}(E, u, v, s)$

$$= -\delta_{u,v}(3u^2 + 3u - 1)$$
$$\times [u/(u-1)(u+1)(u+2)(2u-1)(2u+3)]^{1/2}$$
$$\times (\tfrac{3}{5})^{1/2} \text{Re}a_1^* j_3 \frac{5E_0^2 - 10EE_0 + 5E^2}{4M^2}$$
$$+ \frac{\rho_{u,v}}{u+1} \text{Re}c_1^*(\sqrt{3}g \mp \sqrt{2}j_2) \frac{5E_0^2 - 10EE_0 + 5E^2}{4M^2}$$
$$\mp (-)^s \frac{\sigma_{u,v}}{u+1} \text{Re}c_1^* j_3 \frac{5E_0^2 - 10EE_0 + 5E^2}{2M^2}$$

$F_{22}(E, u, v, s)$

$$= \pm\delta_{u,v}(3u^2 + 3u - 1)$$
$$\times [u/(u-1)(u+1)(u+2)(2u-1)(2u+3)]^{1/2}$$
$$\times (\tfrac{3}{5})^{1/2} \text{Im}a_1^* j_3 (5E^2/4M^2)$$
$$+ \frac{\rho_{u,v}}{u+1} \text{Im}c_1^*(\pm\sqrt{3}g + \sqrt{2}j_2)(5E^2/4M^2)$$
$$- (-)^s \frac{\sigma_{u,v}}{u+1} \text{Im}c_1^* j_3 (5E^2/4M^2)$$

$F_{23}(E, u, v, s)$

$$= \pm\delta_{u,v}(3u^2 + 3u - 1)$$
$$\times [u/(u-1)(u+1)(u+2)(2u-1)(2u+3)]^{1/2}$$
$$\times (\tfrac{3}{5})^{1/2} \text{Im}a_1^* j_3 \frac{EE_0 - E^2}{2M^2}$$
$$\pm \frac{\rho_{u,v}}{u+1} \sqrt{3} \text{ Im}c_1^* g \frac{5EE_0 - 5E^2}{2M^2}$$

$F_{24}(E, u, v, s)$

$$= \pm\delta_{u,v}(3u^2 + 3u - 1)$$
$$\times [u/(u-1)(u+1)(u+2)(2u-1)(2u+3)]^{1/2}$$
$$\times (\tfrac{3}{5})^{1/2} \text{Im}a_1^* j_3 \frac{E_0^2 - 2EE_0 + E^2}{2M^2}$$
$$+ \frac{\rho_{u,v}}{u+1} \text{Im}c_1^*(\pm\sqrt{3}g - \sqrt{2}j_2) \frac{5E_0^2 - 10EE_0 + 5E^2}{4M^2}$$
$$- (-)^s \frac{\sigma_{u,v}}{u+1} \text{Im}c_1^* j_3 \frac{5E_0^2 - 10EE_0 + 5E^2}{4M^2}$$

$F_{25}(E, u, v, s)$

$$= \tau_{u,v} \text{Re}c_1^* j_3 (35E^2/4M^2)$$

$F_{26}(E, u, v, s)$

$$= \tau_{u,v} \text{Re}c_1^* j_3 [(35EE_0 - 35E^2)/2M^2]$$

$F_{27}(E, u, v, s)$

$$= \tau_{u,v} \text{Re}c_1^* j_3 [(35E_0^2 - 70EE_0 + 35E^2)/4M^2)]. \tag{B6}$$

For convenience in discussion of the neutrino-averaged spectra we quote also

$H_0(E, u, v, s)$

$$= F_1(E, u, v, s)$$
$$= |a_1|^2 + 2\text{ Re}a_1^* a_2 (1/3M^2)[m_e^2 + 4EE_0$$
$$+ 2(m_e^2/E)E_0 - 4E^2] + |c_1|^2 + 2\text{ Re}c_1^* c_2 (1/9M^2)$$
$$\times [11m_e^2 + 20EE_0 - 2(m_e^2/E)E_0 - 20E^2]$$
$$- 2(E_0/3M) \text{ Re}c_1^*(c_1 + d \pm b)$$
$$+ (2E/3M)[3|a_1|^2 + \text{Re}c_1^*(5c_1 \pm 2b)]$$
$$- \tfrac{1}{3}(m_e^2/ME) \text{ Re}\{-3a_1^* e$$
$$+ c_1^*[2c_1 + d \pm 2b - h(E_0 - E)/2M]\}$$

$H_1(E, u, v, s)$

$$= F_4(E, u, v, s) + \tfrac{1}{3}F_7(E, u, v, s)$$
$$= \delta_{u,v}\left(\frac{u}{u+1}\right)^{1/2} \Big\{2\text{ Re}a_1^*[c_1 - (E_0/3M)(c_1 + d \pm b)$$
$$+ (E/3M)(7c_1 \pm b + d)]$$
$$- 4\text{ Re}(a_1^* c_2 + c_1^* a_2) \frac{EE_0 - E^2}{3M^2}\Big\}$$
$$\mp \frac{\gamma_{u,v}}{u+1} (-)^s \text{Re}c_1^*[c_1 + 4c_2(EE_0 - E^2)/3M^2$$
$$- (2E_0/3M)(c_1 + d \pm b)$$
$$+ (E/3M)(11c_1 - d \pm 5b)]$$
$$+ \frac{\lambda_{u,v}}{u+1} \text{Re}c_1^* \Big[-f(5E/M)$$
$$+ (\tfrac{3}{2})^{1/2} g \frac{E_0^2 - 11EE_0 + 6m_e^2 + 4E^2}{6M^2}$$
$$\pm 3j_2 \frac{8E^2 - 5EE_0 - 3m_e^2}{6M^2}\Big]$$

$$H_2(E,u,v,s) = F_{10}(E,u,v,s) + \tfrac{1}{3}F_{13}(E,u,v,s)$$
$$= \theta_{u,v}(E/2M)\ \mathrm{Re}c_1^*[c_1 + 8c_2(E_0 - E)/3M - d \pm b]$$
$$- \delta_{u,v}\left[\frac{u(u+1)}{(2u-1)(2u+3)}\right]^{1/2}(E/M)\ \mathrm{Re}a_1^*\left[(\tfrac{3}{2})^{1/2}f + g\,\frac{E+2E_0}{4M} \pm (\tfrac{3}{2})^{1/2}j_2(E_0-E)/2M\right]$$
$$+ (-)^s\kappa_{u,v}(E/2M)\ \mathrm{Re}c_1^*\left[\pm 3f \pm (\tfrac{3}{2})^{1/2}g\,\frac{E_0-E}{M} + j_2\,\frac{E_0-2E}{2M}\right]$$
$$+ \epsilon_{u,v}\ \mathrm{Re}c_1^* j_3(21E^2/8M^2)$$

$$H_3(E,u,v,s) = F_{18}(E,u,v,s)$$
$$= -\delta_{u,v}(3u^2+3u-1)\times[u/(u-1)(u+1)(u+2)(2u-1)(2u+3)]^{1/2}\times \mathrm{Re}a_1^* j_3[(15)^{1/2}E^2/4M^2]$$
$$+ \frac{\rho_{u,v}}{u+1}\mathrm{Re}c_1^*(\sqrt{3}g \pm \sqrt{2}j_2)(5E^2/4M^2)$$
$$\pm (-)^s\frac{\sigma_{u,v}}{u+1}\mathrm{Re}c_1^* j_3(5E^2/2M^2). \qquad \text{(B7)}$$

In the above the spin-dependent functions $\gamma_{u,v}$, $\lambda_{u,v}$, etc. are given by

$$\gamma_{u,v} = (-)^{u-v}[6u(u+1)(2u+1)]^{1/2}W(u1u1;v1) = \begin{cases} u+1 & u=v+1 \\ 1 & u=v \\ -u & n=v-1 \end{cases}$$

$$\lambda_{u,v} = (-)^{u-v}[u(u+1)(2u+1)/5]^{1/2}W(u1u2;v1) = \begin{cases} [3(u-1)(u+1)]^{1/2} & u=v+1 \\ [(2u-1)(2u+3)]^{1/2} & u=v \\ [3u(u+2)]^{1/2} & u=v-1 \end{cases}$$

$$\theta_{u,v} = (-)^{u-v}[30u(u+1)(2u+1)/(2u-1)(2u+3)]^{1/2}\times W(u1u;1v2) = \begin{cases} -(u+1)/(2u-1) & u=v+1 \\ 1 & u=v \\ -u/(2u+3) & u=v-1 \end{cases}$$

$$\kappa_{u,v} = (-)^{u-v}[5u(u+1)(2u+1)/(2u-1)(2u+3)]^{1/2}\times W(u1u2;v2) = (\tfrac{1}{2})^{1/2}\begin{cases} [(u-1)(u+1)]^{1/2}/(2u-1) & u=v+1 \\ [3/(2u-1)(2u+3)]^{1/2} & u=v \\ -[u(u+2)]^{1/2}/(2u+3) & u=v-1 \end{cases}$$

$$\epsilon_{u,v} = (-)^{u-v}[4u(u+1)(2u+1)/(2u-1)(2u+3)]^{1/2}\times W(u1u3;v2) = -2\,\frac{(10)^{1/2}}{35}\begin{cases} (2u-1)^{-1}\left[\dfrac{u(u-1)(2u-3)}{2u+3}\right]^{1/2} & u=v+1 \\ \left[\dfrac{3(u+2)(u-1)}{2(2u-1)(2u+3)}\right]^{1/2} & u=v \\ (2u+3)^{-1}\left[\dfrac{u(u+2)(2u+5)}{2u-1}\right]^{1/2} & u=v-1 \end{cases}$$

$$\rho_{u,v} = -(-)^{u-v}(3u^2+3u-1)\times\left[\frac{7u(u+1)(2u+1)}{5(u-1)(u+2)(2u-1)(2u+3)}\right]^{1/2}\times W(u1u2;v3) = -\frac{3u^2+3u-1}{5[(2u-1)(2u+3)]^{1/2}}\times\begin{cases} -\left[\dfrac{(u+1)(2u+3)}{(u-1)(2u-1)}\right]^{1/2} & u=v+1 \\ \sqrt{3} & u=v \\ -\left[\dfrac{u(2u-1)}{(u+2)(2u+3)}\right]^{1/2} & u=v-1 \end{cases}$$

$$\sigma_{u,v} = -(-)^{u-v}(3u^2+3u-1)\times\left[\frac{7u(u+1)(2u+1)}{5(u-1)(u+2)(2u-1)(2u+3)}\right]^{1/2}\times W(u1u3;v3) = \frac{3u^2+3u-1}{[10(u-1)(u+2)(2u-1)(2u+3)]^{1/2}}\times\begin{cases} \left[\dfrac{(2u-3)(u+1)(u+2)}{(2u-1)}\right]^{1/2} & u=v+1 \\ (6)^{1/2} & u=v \\ -\left[\dfrac{(2u-5)u(u-1)}{(2u+3)}\right]^{1/2} & u=v-1 \end{cases}$$

$$\phi_{u,v} = (-)^{u-v} \times \frac{9}{2}\left[\frac{3(u-1)(u+2)u(u+1)(2u+1)}{35(2u+5)(2u+3)(2u-1)(2u-3)}\right]^{1/2} \times W(u1u3; v4)$$
$$= \frac{9}{14(10)^{1/2}} \frac{(u-1)(u+2)}{(2u-1)(2u+3)} \times \begin{cases} -\left[\frac{(u+1)(u+2)}{(2u-1)(2u-3)}\right]^{1/2} & u = v+1 \\ (\frac{2}{3})^{1/2} & u = v. \\ -\left[\frac{u(u-1)}{(2u+3)(2u+5)}\right]^{1/2}. & u = v-1 \end{cases} \quad \text{(B8)}$$

In terms of these functions F_i the spectral functions $f_i(E)$ defined in Eq. (51) are given by

$$f_i(E) = F_i(E, J, J', 0) \quad \text{(B9)}$$

while the neutrino-averaged spectral functions $F_i(E)$ of Eq. (52) are given by

$$F_i(E) = H_i(E, J, J', 0. \quad \text{(B10)}$$

For the delayed α or photon emissions—Eqs. (53) and (57) we find

$$g_i(E) = F_i(E, J', J, 1) \quad \text{(B11)}$$

while for the neutrino averaged spectra—Eqs. (56) and (61)

$$G_i(E) = H_i(E, J', J, 1). \quad \text{(B12)}$$

The kinematic shift functions δ_i, Δ_i are given by

$$\delta_1(E, v^*, X) = -(2/Mv^*)\{E(|a|^2 + |c|^2) + [(E_0 - E)/100]\theta_{J'J}X|c|^2\}$$

$$\delta_2(E, v^*, X) = -(2E/Mv^*)[|a|^2 - \tfrac{1}{3}|c|^2 + \tfrac{1}{300}\theta_{J',J}X|c|^2]$$

$$\delta_3(E, v^*, X) = -(2/Mv^*)[(E_0 - E)(|a|^2 + |c|^2) + (p^2/100E)\theta_{J',J}X|c|^2]$$

$$\delta_4(E, v^*, X) = -\frac{2(E_0 - E)}{Mv^*}[|a|^2 - \tfrac{1}{3}|c|^2 + \tfrac{1}{300}\theta_{J',J}X|c|^2]$$

$$\delta_5(E, v^*, X) = -3X(E/Mv^*)\{\delta_{J,J'}[J(J+1)]^{1/2} \times 2\,\mathrm{Re}a^*c \pm \gamma_{J',J}|c|^2\}$$

$$\delta_6(E, v^*, X) = -3X(1/Mv^*)\{\delta_{J,J'}[J(J+1)]^{1/2} \times E_0 2\,\mathrm{Re}a^*c \pm \gamma_{J',J}(E_0 - 2E)|c|^2\}$$

$$\delta_7(E, v^*, X) = -3X[(E_0 - E)/Mv^*] \times \{\delta_{J,J'}[J(J+1)]^{1/2} 2\,\mathrm{Re}a^*c \mp \gamma_{J',J}|c|^2\}$$

$$\delta_8(E, v^*, X) = 4X(E/Mv^*)\theta_{J',J}|c|^2$$

$$\delta_9(E, v^*, X) = 4X[(E_0 - E)/Mv^*]\theta_{J',J}|c|^2$$

$$\Delta_1(E, v^*, X) = \delta_1(E, v^*, X) + \tfrac{1}{3}\delta_4(E, v^*, X)$$
$$= -\tfrac{2}{3}(E_0/Mv^*)[|a|^2 - \tfrac{1}{3}|c|^2(1 - \tfrac{1}{10}\theta_{J',J}X)] - \tfrac{4}{3}(E/Mv^*)[|a|^2 + \tfrac{5}{3}|c|^2(1 + \tfrac{1}{100}\theta_{J',J}X)]$$

$$\Delta_2(E, v^*, X) = \delta_5(E, v^*, X)$$
$$= -3X(E/Mv^*)\{\delta_{J',J}[J(J+1)]^{1/2} \times 2\,\mathrm{Re}ac^* \pm \gamma_{J',J}|c|^2\}. \quad \text{(B13)}$$

APPENDIX C

The modification of the spectral functions $F_i(E, u, v, s)$ due to Coulomb effects are found to be (B.R. Holstein, 1974a)

$$F_\mp(Z, E)\Delta F_1(E, u, v, s)$$
$$= |a|^2[|A|^2 + |B|^2 + |C|^2 + |D|^2 - 2\,\mathrm{Re}(A^*D + B^*C) + 2(m_e/E) \times \mathrm{Re}(A^*B + C^*D - A^*C - B^*D) - F_\mp(Z, E)]$$
$$+ |c|^2[|A|^2 + |B|^2 + |C|^2 + |D|^2 + \tfrac{2}{3}\mathrm{Re}(A^*D + B^*C) + 2(m_e/E)\,\mathrm{Re}(A^*B + C^*D + \tfrac{1}{3}A^*C + \tfrac{1}{3}B^*D) - F_\mp(Z, E)]$$

$$F_\mp(Z, E)\Delta F_2(E, u, v, s)$$
$$= |a|^2[|A|^2 - |B|^2 - |C|^2 + |D|^2 + 2\,\mathrm{Re}(B^*C - A^*D) - F_\mp(Z, E)]$$
$$- \tfrac{1}{3}|c|^2[|A|^2 - |B|^2 - |C|^2 + |D|^2 + 6\,\mathrm{Re}(A^*D - B^*C) - F_\mp(Z, E)]$$

$$F_\mp(Z, E)\Delta F_4(E, u, v, s)$$
$$= \delta_{u,v}\left(\frac{u}{u+1}\right)^{1/2} 2\,\mathrm{Re}a^*c[|A|^2 - |B|^2 + |C|^2 - |D|^2 - F_\mp(Z, E)]$$
$$\mp (-)^s[\gamma_{u,v}/(u+1)]|c|^2 \times [|A|^2 - |B|^2 + |C|^2 - |D|^2 - F_\mp(Z, E)]$$

$$F_{\mp}(Z,E)\Delta F_6(E,u,v,s)$$
$$= \delta_{u,v}\left(\frac{u}{u+1}\right)^{1/2} 2\,\mathrm{Re}a^*c[|A|^2+|B|^2+|C|^2+|D|^2$$
$$- 2\,\mathrm{Re}(A^*D+B^*C) + 2(m_e/E)$$
$$\times \mathrm{Re}(A^*B+C^*D-A^*C-B^*D) - F_{\mp}(Z,E)]$$
$$\pm (-)^s[\gamma_{u,v}/(u+1)]\,|c|^2$$
$$\times [|A|^2+|B|^2+|C|^2+|D|^2$$
$$+ 2\,\mathrm{Re}(A^*D+B^*C) + 2(m_e/E)$$
$$\times \mathrm{Re}(A^*B+C^*D+A^*C+B^*D) - F_{\mp}(Z,E)]$$

$$F_{\mp}(Z,E)\Delta F_7(E,u,v,s)$$
$$= \delta_{u,v}\left(\frac{u}{u+1}\right)^{1/2} 2\,\mathrm{Re}a^*c[-2|C|^2+2|D|^2$$
$$+ 2\,\mathrm{Re}(B^*C-A^*D)]$$
$$\pm (-)^s[\gamma_{u,v}/(u+1)]\,|c|^2[2|C|^2-2|D|^2$$
$$+ 2\,\mathrm{Re}(B^*C-A^*D)]$$

$$F_{\mp}(Z,E)\Delta F_{12}(E,u,v,s)$$
$$= \theta_{u,v}\,|c|^2[|A|^2-|B|^2+|C|^2-|D|^2-F_{\mp}(Z,E)]$$

$$F_{\mp}(Z,E)\Delta F_{14}(E,u,v,s)$$
$$= -\theta_{u,v}\,|c|^2[2\,\mathrm{Re}(A^*D+B^*C$$
$$+ 2(m_e/E)\,\mathrm{Re}(A^*C+B^*D)]$$

$$F_{\mp}(Z,E)\Delta F_{15}(E,u,v,s)$$
$$= -\theta_{u,v}\,|c|^2[2|C|^2-2|D|^2]. \qquad (C1)$$

For positron decay the lepton matrix element is replaced by

$$\bar{u}(k)\gamma^\lambda(1+\gamma_5)[\not{t}'+\not{s}'\gamma_0]v(p), \qquad (C2)$$

where

$$t' = -(B', -C'\mathbf{k}), \qquad D' = (A', -D'\mathbf{k}),$$

and A', B', C', D' are defined as in Eq. (88) except that $w^*(Z,r)\to w(-Z,r)$, $N(Z)\to N^*(-Z)$, etc., and η_κ is subject to the restriction

$$\kappa>0 \qquad \pi/2\le\eta_\kappa<\pi$$
$$\kappa<0 \qquad 0\le\eta_\kappa<\pi/2.$$

The changes in the spectral functions are found by replacing $A\to A'^*$, $B\to B'^*$, etc. in Eq. (C1) and using the lower sign.

The results given above are not extremely edifying even as to the direction of effects of these corrections, since they are in terms of functions A, B, C, D which are not intuitively known. However, if $Z\alpha\ll1$ it makes sense to expand these functions to first order in $Z\alpha$. Then if $F(q^2)$ is the weak form factor as defined previously and $G(k^2)$ is the charge form factor for the initial and final nuclei, we define

$$X = \int_0^\infty dkG(k^2)F'(k^2) \qquad Y = \int_0^\infty dkF(k^2)G'(k^2) \qquad (C3)$$

in terms of which the corrections to the spectral functions are[16]

$$\Delta F_1(E,u,v,s)$$
$$= \mp(8\alpha Z/3\pi)\{|a|^2[4E(X+Y)+E_0X$$
$$+ (m_e^2/E)(X+2Y)] + |c|^2[E(\tfrac{16}{3}X+4Y)$$
$$- \tfrac{1}{3}E_0X + (m_e/^2E)(X+2Y)]$$

$$\Delta F_2(E,u,v,s)$$
$$= \mp((8\alpha Z/3\pi)\{|a|^2[4E(X+Y)+E_0X]$$
$$- |c|^2[\tfrac{4}{3}E(2X+Y)-E_0X]\}$$

$$\Delta F_4(E,u,v,s)$$
$$= \mp\ (8\alpha Z/3\pi)\left[\delta_{u,v}\left(\frac{u}{u+1}\right)^{1/2} 2\,\mathrm{Re}a^*c\right.$$
$$\left.\mp (-)^s\frac{\gamma_{u,v}}{u+1}\,|c|^2\right]E(5X+4Y)$$

$$\Delta F_6(E,u,v,s)$$
$$= \mp\ (8\alpha Z/3\pi)\left\{\delta_{u,v}\left(\frac{u}{u+1}\right)^{1/2} 2\,\mathrm{Re}a^*c[4E(X+Y)\right.$$
$$+ E_0X + (m_e^2/E)(X+2Y)]$$
$$\pm (-)^s\frac{\gamma_{u,v}}{u+1}\,|c|^2[E(6X+4Y)-E_0X$$
$$\left.+ (m_e^2/E)(X+2Y)]\right\}$$

$$\Delta F_7(E,u,v,s)$$
$$= \mp\ (8\alpha Z/3\pi)\left[\delta_{u,v}\left(\frac{u}{u+1}\right)^{1/2} 2\,\mathrm{Re}a^*c\right.$$
$$\left.\pm (-)^s\frac{\gamma_{u,v}}{u+1}\,|c|^2\right](E_0-E)X$$

$$\Delta F_{12}(E,u,v,s)$$
$$= \mp(8\alpha Z/3\pi)\theta_{u,v}\,|c|^2E(5X+4Y)$$

$$\Delta F_{14}(E,u,v,s)$$
$$= \mp(8\alpha Z/3\pi)\theta_{u,v}\,|c|^2(E_0-E)X. \qquad (C4)$$

[16] Similar results for $J=1$, $J'=0$ transitions were derived by L. W. Armstrong and C. W. Kim (1972a), and by A. Bottino and G. Ciocchetti (1973).

We note that for uniform charge and weak charge density, $F(k^2) = G(k^2) = [3/(kR)^3](\sin kR - kR \cos kR)$, we have

$$X = Y = 9\pi R/140, \tag{C5}$$

while for a surface charge and weak charge distribution, $F(k^2) = G(k^2) = (\sin kR)/kR$

$$X = Y = \pi R/12. \tag{C6}$$

REFERENCES

Armstrong, L. W. and C. W. Kim, 1972a, Phys. Rev. C**5**, 672.
Armstrong, L. W. and C. W. Kim, 1972b, Phys. Rev. C**6**, 1924.
Bég, M. A. B. and J. Bernstein, 1972, Phys. Rev. D**5**, 714.
Bernstein, J. and R. R. Lewis, 1958, Phys. Rev. **112**, 232.
Bjorken, J. D. and S. D. Drell, 1964, *Relativistic Quantum Mechanics* (McGraw-Hill, New York).
Blin-Stoyle, R. J. and M. Rosina, 1965, Nucl. Phys. **70**, 321.
Blin-Stoyle, R. J. 1973, *Fundamental Interactions and the Nucleus* (North-Holland, Amsterdam, The Netherlands).
Bloom, S. D., L. G. Mann, and J. A. Miskel, 1962, Phys. Rev. **125**, 2021.
Bottino, A. and G. Ciocchetti, 1973, Phys. Lett. B**43**, 170.
Bottino, A., G. Ciocchetti, and C. W. Kim, 1974, Phys. Rev. C**9**, 2052.
Bouchiat, C. C., 1959, Phys. Rev. Lett. **3**, 516.
Brodine, J. C., 1970a, Phys. Rev. D**1**, 100.
Brodine, J. C., 1970b, Phys. Rev. D**2**, 2090.
Buhring, W., and L. Schülke, 1965, Nucl. Phys. **65**, 369.
Cabibbo, N. 1964, Phys. Rev. Lett. **12**, 137.
Calaprice, F. P. 1974, Bull. Am. Phys. Soc. **19**, 58.
Calaprice, F. P., E. D. Commins, H. M. Gibbs, G. L. Wick, and D. A. Dobson, 1969, Phys. Rev. **184**, 1117.
Callen, C. G., and S. B. Treiman, 1967, Phys. Rev. **162**, 1494.
Chen, H. H., 1969a, Phys. Rev. **185**, 2003.
Chen, H. H., 1969b, Phys. Rev. **185**, 2007.
Christenson, C. J., A. Nielson, A. Bohnson, W. K. Brown, and B. M. Rustad, 1967, Phys. Lett. B**26**, 11.
Christenson, C. J., A. Nielson, A. Bohnson, W. K. Brown, and B. M. Rustad, 1969, Phys. Lett. B**28**, 411.
Csonka, P. L., M. J. Moravcsik, and M. D. Scadron, 1966, Ann. Phys. (N.Y.) **40**, 100.
Delorme, J., and M. Rho, 1971a, Phys. Lett. B**34**, 238.
Delorme J., and M. Rho, 1971b, Nucl. Phys. B**34**, 317.
Delorme J., and M. Rho, 1972, Nucl. Phys. B**46**, 332.
Eichner, E., K. H. Lauterjung, H. Meinhardt, B. Schwimmer, and W. Schmidtohr, 1966, Z. Naturforsch. A**21**, 908.
Feynman, R. P., and M. Gell-Mann, 1958, Phys. Rev. **109**, 193.
Frazier, J., and C. W. Kim, 1968, Phys. Rev. **177**, 2568.
Garcia, A., 1971, Phys. Rev. D**3**, 2638.
Gell-Mann, M., 1958, Phys. Rev. **111**, 362.
Gell-Mann, M., and M. Levy, 1960, Nuovo Cimento **17**, 705.
Gerschtein, S. S., and J. B. Zeldovich, 1955, Zh. Eksp. Teor. Fiz. **29**, 698 [Sov. Phys.—JETP **2**, 76 (1957)].
Goldberger, M. L., and S. B. Treiman, 1958, Phys. Rev. **111**, 354.
Henley, E., 1969, Ann. Rev. Nucl. Sci. **19**, 367.
Henley E., and L. Wolfenstein, 1971, Phys. Lett. B**36**, 28.
Holstein B. R., and S. B. Treiman, 1971, Phys. Rev. C**3**, 1921.
Holstein, B. R., 1971a, Phys. Rev. C**4**, 740.
Holstein, B. R., 1971b, Phys. Rev. C**4**, 764.
Holstein, B. R., 1972, Phys. Rev. C**5**, 1529.
Holstein, B. R., W. Shanahan, and S. B. Treiman, 1972, Phys. Rev. C**5**, 1849.
Holstein, B. R., 1974a, Phys. Rev. C**9**, 1742.
Holstein, B. R., 1974b, Phys. Rev. C**9**, 1646.
Huffaker, J. N., and C. E. Laird, 1967, Nucl. Phys. A**92**, 584.
Huffaker, J. N., and E. Greuling, 1963, Phys. Rev. **132**, 738.
Kim, C. W., and S. L. Mintz, 1971, Nucl. Phys. B**27**, 621.
Kim, C. W., and H. Primakoff, 1965a, Phys. Rev. **139**, B1447.
Kim, C. W., and H. Primakoff, 1965b, Phys. Rev. **140**, B566.
Kim, C. W., and H. Primakoff, 1969, Phys. Rev. **180**, 1502.
Kim, C. W., 1971, Phys. Lett. B**34**, 383.
Kim, C. W., and T. Fulton, 1971, Phys. Rev. C**4**, 390.
Konopinski, E. J., 1966, *The Theory of Beta Radioactivity* (Clarendon, Oxford, England).
Krmpotic, F., and D. Tadic, 1969, Phys. Rev. **178**, 1804.
Kubodera, K., and C. W. Kim, 1973, Phys. Lett. B**43**, 275.
Kubodera, K., J. Delorme, and M. Rho, 1973, Nucl. Phys. B**66**, 253.
Lee, Y. K., L. W. Mo, and C. S. Wu, 1963, Phys. Rev. Lett. **10**, 253.
Lindquist, J., *et al.*, 1971, Phys. Rev. Lett. **27**, 612.
Maiani, L. 1968, Phys. Lett. B**26**, 538.
Morita, M. 1973, *Beta Decay and Muon Capture* 1973 (Benjamin, New York).
Nambu, Y., 1960, Phys. Rev. Lett. **4**, 380.
Nordberg, M. F., F. B. Morinigo, and C. A. Barnes, 1962, Phys. Rev. **125**, 321.
Ohtsubo, H., J. Fujita, and G. Takeda, 1970, Prog. Theor. Phys. **44**, 1596.
Preston, M. A., *Physics of the Nucleus* 1962, (Addison-Wesley, Reading, MA).
Primakoff, H., 1967, in *High Energy Physics and Nuclear Structure*, edited by G. Alexander (North-Holland, Amsterdam, The Netherlands).
Primakoff, H., 1970 in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser, M. Grisaru, and H. Pendleton (M.I.T. Press, Cambridge, MA, 1970).
Rose, M., 1957, *Elementary Theory of Angular Momentum* (Wiley, New York).
Rose, M. E., and R. K. Osborn, 1954, Phys. Rev. **93**, 1326.
Rosen, S. P., 1972, Phys. Rev. D**5**, 760.
Schopper, H., 1966, *Weak Interactions and Nuclear Beta Decay* (North-Holland, Amsterdam, The Netherlands).
Schülke, L., 1964, Z. Phys. **174**, 331.
Stech, B., and L. Schülke, 1964, Z. Phys. **179**, 314.
Tribble, R. E., and G. T. Garvey, 1974, Phys. Rev. Lett. **32**, 314.
Weinberg, S., 1958, Phys. Rev. **112**, 1375.
Wilkinson, D. H., 1970a, Nucl. Phys. A**158**, 476.
Wilkinson, D. H., 1970b, Phys. Lett. B**31**, 447.
Wilkinson, D. H., and D. E. Alburger, 1970a, Phys. Rev. Lett. **24**, 1134.
Wilkinson, D. H., and D. E. Alburger, 1970b, Phys. Lett. **32B**, 190..
Wilkinson, D. H., and D. E. Alburger, 1971, Phys. Rev. Lett. **26**, 1127.
Wu, C. S., 1964, Rev. Mod. Phys. **36**, 618.
Wu, C. S., 1966, in *Proceedings of International School of Physics Enrico Fermi Course XXXII* (Academic, New York).