Two-fluid hydrodynamic description of ordered systems

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A unified description of systems with a condensed phase in terms of hydrodynamic equations of motion is given. These equations are of two kinds: First those equations obeyed by the thermal excitations (the "first fluid") typically are local conservation equations of mass, energy and momentum. Second the equation obeyed by the condensed phase (the "second fluid") is an equation of motion related to the order parameter of the broken symmetry. These equations are established on phenomenological grounds making use of irreversible thermodynamics. The eigenmodes of the linearized form of these equations, typically first and second sound, are discussed in particular with respect to their manifestation in inelastic light and neutron scattering. The systems considered are homogenous superfluids, superconductors, dielectric crystals and magnetic systems. Except for superfluid ^{4}He the critical behavior at the phase transition to the ordered state is not systematically discussed.

CONTENTS

INTRODUCTION

In many-body theory physical understanding has often been overshadowed by the difficulties with Green's function techniques. We think this to be one of the reasons for the relatively recent realization that the underlying phenomenological structure of a number of systems of condensed matter is of astonishing similarity. Here we consider systems which are characterized by a phase transition into an ordered state in which some kind of condensed phase can be defined.

It is the purpose of this review to show that the phenomenological structure of such ordered systems can be described by two-fluid hydrodynamic equations, the first fluid consisting of the *thermal excitations* and the second being the *condensed phase*. In this derivation of two-fluid hydrodynamics we exclude voluntarily any microscopic Green's function techniques and transport theory. The hydrodynamics of the first fluid is obtained within the framework of irreversible thermodynamics [see, for example, de Groot and Mazur (1962)]. Here our point of view is similar to that of the review by Kadanoff and Martin (1963).

In the description of the condensed phase, on the other hand, we rely on the microscopic dynamics to the extent that we derive from it the equation of motion of the second fluid. [The dielectric crystals are an exception insofar as the phase transition is of first order and the equation of motion is given by elasticity theory (see Sec. III).] The existence of a condensed phase is intimately related to the *breakdown of symmetry* occurring at the phase transition (Wagner, 1966; Schneider and Meier, 1973.) [See also Bogoliubov (1960, 1962)]. Symmetry breaking, of course, does not affect the microscopic Hamiltonian but appears only on the level of the state (statistical ensemble). This means that below the phase transition the system is described by a symmetry-breaking density matrix.

In other words, below the transition the density matrix is invariant under a smaller ("broken") group of symmetry operations (the gauge group for superfluids, the translation group for dielectric crystals, the spin-rotation group for magnetic crystals) than above (Wagner, 1966; Schneider and Meier, 1973). Hence the system (ensemble) exhibits a higher degree of order below the transition than above, and develops a condensed phase of long-range order at the transition.

As a consequence there exist quantities which, when averaged with the density matrix of the ordered state, yield a nonzero value, even though their average with the density matrix of the state above the transition vanishes. If such a quantity is, in addition, specific to the system (the field operator of superfluid helium, the Cooper pair operator of superconductors, the displacement operator of dielectric crystals, the spin-raising operator of magnetic crystals), its average is called the *order parameter* of the system [see, for example, Stanley (1971); Fisher (1967); Heller (1967)].

The order parameter describes the dynamics of the condensate and, consequently, brings in a new degree of freedom, the *condensate velocity*. This new degree of freedom has two distinct aspects, a dynamical and a hydrodynamical. Evidently, the dynamical aspect must persist down to zero temperature T. On the other hand, the hydrodynamical aspect is characterized by local thermal equilibrium. Since the density of thermal excitations (the first fluid) vanishes in the limit $T \rightarrow 0$, the establishment of local thermal equi-

librium becomes increasingly slower as $T \to 0$. In terms of a relaxation time τ_{eq} this means that $\tau_{eq} \to \infty$ for $T \to 0$.

Therefore a *dynamical mode* of excitation of frequency ω persists into the *collisionless domain* [see, for example, Kadanoff and Baym (1962)] defined by $\omega\tau_{eq} \gg 1$. On the other hand, the long-range order of the condensate causes the response of the order parameter to external perturbations to be dominated by the long-wavelength contributions [i.e., the associated response function has a pole at zero wave number q (Wagner, 1966; Schneider and Meier, 1973)]. These long-wavelength dynamical modes are: (1) collisionless isothermal first sound, [also called zero sound e.g., Kadanoff and Baym (1962)] in superfluid helium, (2) the plasmon (i.e., the oscillation due to the long-range Coulomb force) in superconductors, (3) collisionless isothermal first sound [i.e., elastic waves of the lattice, also called zero sound (Cowley, 1967)] in dielectric crystals, and (4) collisionless isothermal spin waves (i.e., magnetization waves which could also be called zero magnons) in magnetic crystals.

These dynamical modes are often referred to as *Goldstone bosons* which expresses the belief that they are the modes required by the Goldstone theorem (Wagner, 1966; Schneider and Meier, 1973; Klein and Lee, 1964). This theorem, which states that a broken symmetry gives rise to a gapless mode in the limit $q \to 0$, can be traced back to Heisenberg's program of generating all elementary particles from a Lagrangian of higher symmetry than that of most particle interactions (Heisenberg, 1957). Since the presence of long-range forces (the Coulomb interaction in superconductors and the dipolar interaction in magnetic crystals) invalidates a crucial step in the proof of the Goldstone theorem (Lange, 1965), the dynamical modes with a gap (the plasmon and the magnons in the presence of dipolar interactions) may, in a generalized sense, also be called Goldstone bosons.

At this point the question arises of the relevance of Goldstone bosons to the hydrodynamical description of this paper, since the *hydrodynamic domain* is defined by $\omega\tau_{eq} \ll 1$. The point is that the dynamical modes enumerated above are *all* exptrapolations into the collisionless domain of *hydrodynamic modes*. The converse, however, is not true since in each of the systems described in this paper there is one more hydrodynamic mode which disappears for $T \to 0$, namely *second sound*. On the other hand, second sound resembles a Goldstone boson in that it is a gapless long-wavelength mode which appears as a consequence of symmetry breaking; it is missing above the phase transition. The reason for this is that the additional degree of freedom brought in by the order parameter (the condensate velocity) also gives rise to a new hydrodynamic equation of motion, thus leading to *two-fluid hydrodynamics*.

It is interesting to note that the two-fluid idea may be traced back to a paper by Gorter and Casimir (1934a,b) in which two phases were introduced to describe the thermodynamics of superconductors. But it was only Tisza (1938) and Landau (1941) who generalized this concept to hydrodynamics by introducing the velocities of the two phases.

In analogy to zero sound one may also ask whether there is a collisionless second sound. More precisely the question is, where the second sound pole moves in the transition from the hydrodynamic to the collisionless domain. This question is legitimate although somewhat academic since, while the residue of the relevant response function vanishes for $T \to 0$, its pole cannot just disappear. There have been some speculations on this question [see Enz and Müller (1970)], but the answer is not known.

The hydrodynamic modes discussed in this paper have yet another aspect related to broken symmetry, also shared by the Goldstone bosons. In fact, their long-wavelength limit can be viewed as having a *"symmetry restoring" effect.* This is most easily seen in the case of first sound in dielectric crystals which for $q \to 0$ describes a uniform and arbitrary displacement of the whole lattice. One can say, therefore, that it "restores" the continuous translation group of the melted system. Similarly spin waves in the limit $q \to 0$ describe a uniform and arbitrary rotation of magnetization and hence "restore" the continuous spin-rotation group of the paramagnet. Likewise isothermal first sound in superfluid helium and the plasmon in superconductors describe, in the limit $q \to 0$, a uniform and arbitrary phase of the order parameter, so that one can say that these modes "restore" the continuous gauge group of the normal systems (see Sections I.C and II.B). In the gapless case the fluctuation of the order parameter is of the same magnitude as the order parameter itself, while in the case of excitations with gap (plasmon, magnon in the presence of dipolar interactions) this fluctuation oscillates with the gap frequency.

Second sound in the superfluids has the same "symmetry restoring" effect as first sound, while in dielectric and magnetic crystals this is only true to the extent that there is a coupling between thermal and nonthermal variables. This difference is due to the fact that in superfluids the *sum* of the densities of the first and second fluids obeys the continuity equation, while in the other systems such an equation is obeyed only by the density of the second fluid (the mass density of the lattice in dielectric crystals and the magnetization in magnetic crystals).

An outline of these ideas has already been presented in an earlier publication (Enz, 1972b) where, in addition to the systems described in this review, nematic liquid crystals were also considered. Although the latter are of interest with respect to the second fluid [the director field, see Enz, (1972b)], the fact that the first fluid is the liquid crystal itself classifies nematics too far apart from the systems to be discussed here (see below). A review of the hydrodynamics of liquid crystals very similar in its point of view to the present work has been written by Martin, Parodi, and Pershan (1972). [See also Jähnig and Schmidt (1972).]

The close analogy of superconductors, dielectric, and magnetic crystals with superfluid helium established in this review leaves no doubt that the heat waves found in dielectric crystals are indeed second sound. It also strongly supports the prediction of the same effect in magnetic crystals, while for superconductors our conclusion with respect to the realizability of second sound is quantitatively negative.

Quite generally, second sound is a wave propagation of temperature and entropy excitations, and hence is quite different from the ordinary diffusive propagation by heat conduction. Since entropy is carried by the thermal excita-

tions (the first fluid) alone, one can also say that second sound is *sound in the excitation fluid* [Khalatnikov (1965), p. 69]. Thus a necessary condition for second sound propagation is that the thermal excitations have the characteristics of a fluid. This implies that these excitations can support a flow or *drift*.

If the dominant excitations can be approximated by noninteracting, nonconserved *bosons* with spectrum ω_k a necessary condition for the existence of a drift $\mathbf{v}_n$ is that $x \equiv (\omega_k - \mathbf{v}_n \cdot \mathbf{k})/k_B T > 0$ (k_B is Boltzmann's constant) for all wave vectors $\mathbf{k}$ in the range of this approximation. The reason is that otherwise the Bose distribution function $(e^x - 1)^{-1}$ of these excitations does not exist. Since this approximation is good for the hydrodynamic (longwave) excitations (phonons and magnons, see Sec. I.C, III.C, and IV.C) the condition $x > 0$ means that, at $k = 0$, ω_k must either have a finite slope or a gap. But this is just Landau's criterion for superfluidity [e.g., Landau and Lifshitz (1958)], restricted to long wavelengths.

This criterion is fulfilled in superfluid ^{4}He, in dielectric crystals and in many magnetic crystals but also in normal liquids. However, normal liquids do not permit second sound because the first fluid is the liquid itself since it obviously carries the entropy. Therefore, there is no second fluid (condensed phase), and the superfluid mass density ρ_s is zero. Hence, both the number of hydrodynamic degrees of freedom and the number of hydrodynamic equations are reduced (see Sec. I.E). This leaves ordinary or first sound as the only hydrodynamic mode of excitation of normal liquids (apart from the viscosity mode and the Mountain mode mentioned in Sec. I.E). The same argument excludes second sound in nematic liquid crystals, as was already pointed out by Enz (1972b), the second fluid being here of a different nature (see above).

The above criterion does not apply for excitations which involve *fermions* (Particle–hole pairs in Fermi liquids, normal metals, ^{3}He–^{4}He mixtures and gapless superconductors, Bogoliubov quasiparticles in superconductors with gap). Indeed, the Fermi distribution function $(e^x + 1)^{-1}$ obviously exists for positive as well as negative x, and a drift $\mathbf{v}_n$ just shifts the Fermi sphere away from the origin. Thus, the criterion for second sound (and for superfluidity) is more subtle. In the case of the superconductor with gap Δ the criterion is that the superfluid (Cooper pair) density ρ_s is nonzero. But $\rho_s \neq 0$ if and only if $\Delta \neq 0$ (see Sec. II.A) and hence there cannot be second sound in normal metals (or other normal Fermi liquids).

For crystalline systems, the presence of a *lattice* gives rise to intrinsic dissipation of flow due to Umklapp processes (and also to imperfections). If this dissipation is too strong the excitations cannot support a flow and second sound is not possible. This supplementary condition gives rise to a *frequency window* (see Sec. II, III, and IV) and is the general reason for the difficulties with the experimental realization, or for the overdamped character, of second sound in crystals.

Although the full nonlinear form of the hydrodynamic equations is derived in the subsequent sections, the typical nonlinear features of periodic inhomogeneities in space (Bénard–Rayleigh effect) and time (vorticity, turbulence) will not be discussed here. The first type of inhomogeneity has so far been studied only for ordinary fluids [e.g., Velarde (1972)] while the second type has also given rise to an extended literature in the case of superfluid helium [see, for example, Putterman (1972)]. For the other systems discussed in this review nonlinear hydrodynamics has, to our knowledge, never been seriously considered, but might lead to some interesting new effects in the future.

In recent years, the hydrodynamic modes of the linearized equations have attracted much attention in connection with the critical behavior near the phase transition [see, for example, Stanley (1971); Fisher (1967); Heller (1967)]; in particular, they are the ingredients of mode–mode coupling theory [e.g., Stanley (1971); Kawasaki (1970)]. Although the critical behavior of first and second sound is discussed in detail for the case of superfluid helium (Sec. I), this problem actually lies outside of the hydrodynamic domain and therefore is not systematically pursued in later sections.

We use units such that $\hbar = 1$ throughout; k_B designates Boltzmann's constant.

I. NEUTRAL SUPERFLUIDS: HELIUM II

In spite of the large number of texts on superfluid helium already in existence (London, 1954; Bogoliubov, 1963; Khalatnikov, 1965; Hohenberg and Martin, 1965; Wilks, 1967; Galasiewicz, 1970, 1971) some selected properties of this substance are treated in considerable detail here. The reason is that this section serves as a guideline for the two-fluid description of the systems treated in the subsequent sections. Emphasis is put, therefore, on the symmetry breaking at the λ transition and on the phenomenological derivation of the hydrodynamic equations. In the latter derivation we follow essentially the line of Khalatnikov's (1965) well known treatise. But in view of some more recent developments the hydrodynamic modes, namely, first and second sound, are also discussed with respect to their critical behavior at the λ transition as well as their possible excitation in neutron and Brillouin scattering.

The two-fluid model was a natural way of describing superfluid helium once the existence of a condensate had been recognized. This recognition came rather hesitantly; in fact, it was only in 1938 that Fritz London (1938a,b) pointed out the analogy of the λ transition with Bose–Einstein condensation (eg., London, 1954; Galasiewicz, 1971). The two-fluid theory developed by Tisza (1938) and independently by Landau (1941) was directly influenced by London's (1938a,b) idea. This theory at last brought some order into the long list of "super" properties discovered in the course of the years. But the most important success was the prediction both by Tisza and by Landau of the existence of temperature waves. These waves, which Landau called "second sound," were observed for the first time in 1944 by Peshkov (1944) in the form of standing waves.

A. Two-fluid hydrodynamics

The crucial feature of superfluidity is the existence, below the λ transition temperature $T = T_\lambda(p)$, of a *condensate* described by a nonvanishing average of the boson field operator $\Psi(\mathbf{r})$. Below T_λ the average is defined with a

density matrix ρ which commutes with the effective Hamiltonian $\mathcal{H} - m\mu N$ [see Eq. (3.38) of Hohenberg and Martin (1965)] but not with the number operator N. Here m is the atomic mass of ^{4}He, and μ the chemical potential per unit mass. Since N is the generator of the gauge group (i.e., $e^{i\alpha N}$ is the general element of this group) this noncommutativity describes the gauge symmetry breaking of the λ transition. Since Ψ^+ raises the particle number by one, $N\Psi^+ = \Psi^+(N+1)$, the relations

$$[\rho, \mathcal{H} - m\mu N] = 0, \qquad [\rho, N] \neq 0, \qquad [\mathcal{H}, N] = 0 \quad (1.1)$$

together with the Heisenberg representation

$$\Psi^+(\mathbf{r}, t) = \exp(i\mathcal{H}t)\Psi^+(\mathbf{r}) \exp(-i\mathcal{H}t)$$

imply the existence of a nonvanishing complex order parameter $\langle\Psi^+\rangle = \mathrm{Tr}(\rho\Psi^+)$ with the time evolution

$$\langle\Psi^+(\mathbf{r}, t)\rangle = \exp(+im\mu t)\langle\Psi^+(\mathbf{r})\rangle = (n_c)^{1/2} \exp[+i\varphi(t)], \quad (1.2)$$

n_c being the number density of the condensate. It follows from Eq. (1.2) that the phase evolves as

$$\varphi(t) = \varphi(0) + m\mu t \quad (1.3)$$

Physically a density matrix ρ with the properties (1.1) may be constructed by coupling the system to an external particle source described by $\mathcal{H}_{\text{ext}}$, N_{ext}, and μ_{ext} via a tunneling Hamiltonian $\mathcal{H}'$ analogous to the case of the Josephson effect in superconductors (see Sec. II.A), for which $[\mathcal{H}', N] \neq 0$ but $[\mathcal{H}', \mathcal{H} - m\mu N] = 0$. [Anderson and Dayem (1964); Anderson (1966a,b). For the latest experimental situation, however, see Musinski and Douglass (1972), where the earlier experiments are also quoted.] The last relation means that the transfer of particles to the system requires no energy (defined as eigenvalue of $\mathcal{H} - m\mu N$). Then

$$\rho = \exp[(F - \mathcal{H} - \mathcal{H}_{\text{ext}} - \mathcal{H}' + m\mu N + m\mu_{\text{ext}}N_{\text{ext}})/k_BT]$$

indeed commutes with $\mathcal{H}$ and $\mathcal{H}_{\text{ext}}$, but not with N and N_{ext}, and Eq. (1.2) is just the Kubo formula for the linear response of the system to $\mathcal{H}'$.

In local equilibrium the phase φ defined by Eq. (1.2) determines the superfluid velocity

$$\mathbf{v}_s = -(1/m)\nabla\varphi. \quad (1.4)$$

From Eqs. (1.3) and (1.4) then follows the equation of motion

$$d\mathbf{v}_s/dt = \dot{\mathbf{v}}_s + (\mathbf{v}_s\cdot\nabla)\mathbf{v}_s = -\nabla(\mu + \mu'), \quad (1.5)$$

where μ' is the dissipative part of μ.

The remaining hydrodynamic equations are the usual local conservation laws of mass and energy and the local balance equation of momentum,

$$\dot{\rho} + \nabla\cdot(\mathbf{j} + \mathbf{j}') = 0,$$
$$\dot{\mathbf{j}} + \nabla(\Pi + \Pi') = -\rho\nabla\psi, \quad (1.6)$$

$$(\rho\epsilon)^{\cdot} + \nabla\cdot(\mathbf{J}_\epsilon + \mathbf{J}_\epsilon') = 0. \quad (1.7)$$

Here ρ, $\mathbf{j}$, Π, ϵ, and $\mathbf{J}_\epsilon$ are, respectively, the mass density, the momentum density or mass flux, the momentum flux, the energy per unit mass, and the energy flux. Here also $\mathbf{j}'$, Π', and $\mathbf{J}_\epsilon'$ are the associated dissipative parts, $-\rho\nabla\psi$ is the external force density, and the second Eq. (1.6) is just Newton's law per unit volume.

In the local frame of reference moving with $\mathbf{v}_s$ the total current is carried by the normal phase of excitations alone,

$$\mathbf{j}_0 = \rho_n(\mathbf{v}_n - \mathbf{v}_s). \quad (1.8)$$

This defines the normal or excitation mass density ρ_n. In the laboratory frame the flux of ρ_n defines the normal current $\mathbf{j}_n = \rho_n\mathbf{v}_n$, while the total current is increased by the flux of ρ due to the Galilei transformation,

$$\mathbf{j} = \mathbf{j}_0 + \rho\mathbf{v}_s = \rho_n\mathbf{v}_n + \rho_s\mathbf{v}_s. \quad (1.9)$$

Here

$$\rho_s = \rho - \rho_n \quad (1.10)$$

is the superfluid mass density and the flux of ρ_s defines the supercurrent $\mathbf{j}_s = \rho_s\mathbf{v}_s$.

It follows from the above definitions that $\rho_n\mathbf{v}_n$ describes the current response of a sample with $\mathbf{v}_s = 0$ contained in an infinitely long tube whose walls move with velocity $\mathbf{v}_n$ along the tube axis. From this results the general microscopic expression (Baym, 1969)

$$\rho_n = \lim_{q\to 0}\int\frac{d\omega}{\pi}\frac{C_t(\mathbf{q}, \omega)}{\omega} \quad (1.11)$$

where $C_t(\mathbf{q}, \omega)$ is the transverse part of the Fourier transform of the current dissipation function,

$$C_{ij}(\mathbf{r} - \mathbf{r}', t - t') = \tfrac{1}{2}\langle[j_i(\mathbf{r}, t), j_j(\mathbf{r}', t')]\rangle. \quad (1.11a)$$

If, on the other hand, the tube has closed ends, its motion with $\mathbf{v}_n$ carries the whole fluid along so that $\mathbf{v}_s = \mathbf{v}_n$ and the response is $\rho\mathbf{v}_n$. This leads to an expression (1.11) with ρ_n replaced by ρ, and C_t by the longitudinal part C_l (Baym, 1969; Hohenberg and Martin, 1965).

The reason for the occurrence of the transverse part of C_{ij} in Eq. (1.11) is that this expression also describes the more physical situation where a cylindrical sample is rotated around its axis. In this case $\mathbf{v}_n = \boldsymbol{\omega} \times \mathbf{r}$ where $\boldsymbol{\omega}$ is the angular velocity. Since this velocity field is closely analogous to the vector potential of a superconductor in a magnetic field $\mathbf{H}$, $\mathbf{A} = \frac{1}{2}(\mathbf{H} \times \mathbf{r})$, it is not surprising that Eq. (1.11) also holds in the latter case [Pines (1965); Nozieres (1966). See Sec. II.A.]

The superfluid mass density ρ_s differs from the condensate mass density $\rho_c = mn_c$ defined in Eq. (1.2) because of interactions between the helium atoms. Since according to Eq. (1.4) the supercurrent $\mathbf{j}_s = \rho_s \mathbf{v}_s$ depends on the phase of the order parameter $\langle \Psi^+ \rangle$, $\mathbf{j}_s$ responds to variations of an external particle source. Comparison of this response with that of $\langle \Psi^+ \rangle$ leads to the following general microscopic relation (Baym, 1969), between ρ_s and ρ_c, first derived by Hohenberg and Martin (1965). [See their Eqs. (4.24), (4.29)] and independently by Josephson (1966) [see also Galasiewicz (1968). A more explicit but only approximate expression for ρ_c/ρ_s has also been given by Haug and Weiss (1972)],

$$\frac{\rho_c}{\rho_s} = \lim_{q\to 0} \frac{q^2}{2m} \int \frac{d\omega}{\pi} \frac{A(q,\omega)}{\omega}, \tag{1.11b}$$

where $A(q,\omega)$ is the Fourier transform of the field dissipation function $\frac{1}{2}\langle[\Psi(\mathbf{r},t), \Psi^+(\mathbf{r}',t')]\rangle$. It is easy to see that for free particles $A(q,\omega) = \pi\delta(\omega - q^2/2m)$, and hence $\rho_c = \rho_s$.

This free-particle approximation is expected to be valid for helium in the limit of large q where the interaction energy is small compared to the kinetic energy. As a consequence neutron scattering at large momentum and energy transfers should exhibit a peak proportional to $(\rho_c/\rho)\delta(\omega - q^2/2m)$ due to the condensate (Hohenberg and Platzmann, 1966). Although this peak has not been seen, its presence may be infered from the sharpening of the neutron scattering spectrum when the temperature is lowered through T_λ (Cowley and Woods, 1968). From this, as well as from other determinations (Penrose and Onsager, 1956; McMillan, 1965; Harling, 1970; Kerr *et al.*, 1970), ρ_c/ρ is found to be of the order of 10%. The most recent value, $(2.4 \pm 1)\%$ (Mook *et al.*, 1972), only shows the considerable uncertainty of this determination which is due to final-state interactions (Hohenberg and Platzmann, 1966).

The hydrodynamic equations (1.5), (1.6), (1.7) have to be supplemented by expressions for μ, Π, and $\mathbf{J}_\epsilon$. The extensive character of the thermodynamic potentials serves to define the pressure p such that (Hohenberg and Martin, 1965)

$$d\mu = -s\,dT + (1/\rho)\,dp + d\psi - \tfrac{1}{2}(\rho_n/\rho)d(\mathbf{v}_n - \mathbf{v}_s)^2, \tag{1.12}$$

where ψ is an external potential per unit mass. In the local frame moving with $\mathbf{v}_s$ the total momentum flux is

$$\Pi_0 = p1 + \rho_n(\mathbf{v}_n - \mathbf{v}_s) \otimes (\mathbf{v}_n - \mathbf{v}_s).$$

In the laboratory frame this leads to (Hohenberg and Martin, 1965)

$$\begin{aligned}\Pi &= \Pi_0 + \mathbf{j}_0 \otimes \mathbf{v}_s + \mathbf{v}_s \otimes \mathbf{j}_0 + \rho \mathbf{v}_s \otimes \mathbf{v}_s \\ &= p1 + \rho_n \mathbf{v}_n \otimes \mathbf{v}_n + \rho_s \mathbf{v}_s \otimes \mathbf{v}_s \\ &= \Pi_n + \rho_s \mathbf{v}_s \otimes \mathbf{v}_s \end{aligned} \tag{1.13}$$

where use has been made of Eq. (1.8).

The energy flux $\mathbf{J}_\epsilon$ is obtained by transforming the energy conservation equation (1.7) into the entropy balance equation

$$(\rho s)^\cdot + \nabla\cdot(\mathbf{J}_s + \mathbf{J}_s') = \sigma, \tag{1.14}$$

where $\mathbf{J}_s$ is the entropy flux, $\mathbf{J}_s'$ its dissipative part, and σ the entropy production density. For this transformation we make use of the thermodynamic relation expressing $d(\rho\epsilon)$ linearly in terms of the differentials of the quantities satisfying the hydrodynamic equations of motion (1.5), (1.6), and (1.14) for $\mathbf{v}_s$, ρ, $\mathbf{j}$, and ρs (Hohenberg and Martin, 1965),

$$d(\rho\epsilon) = \nu\, d\rho + \boldsymbol{\lambda}\cdot d\mathbf{v}_s + \boldsymbol{\omega}\cdot d\mathbf{j} + T d(\rho s). \tag{1.15}$$

In order to determine the coefficients ν, $\boldsymbol{\lambda}$, $\boldsymbol{\omega}$ we note that the superfluid phase does not carry entropy. Hence entropy must have a simple form in the frame in which the superfluid is locally at rest, namely (Hohenberg and Martin, 1965),

$$T d(\rho s) = d(\rho\epsilon_0) - \mu\, d\rho - (\mathbf{v}_n - \mathbf{v}_s)\cdot d\mathbf{j}_0.$$

Here ϵ_0 and $\mathbf{j}_0$ are the energy and momentum densities in this frame. Since in the laboratory frame

$$\rho\epsilon = \rho\epsilon_0 + \mathbf{j}_0\cdot\mathbf{v}_s + \tfrac{1}{2}\rho\mathbf{v}_s^2,$$

comparison with Eq. (1.15) and use of Eq. (1.8) yields (Hohenberg and Martin, 1965)

$$\begin{aligned}\nu &= \mu + \tfrac{1}{2}(\mathbf{v}_n - \mathbf{v}_s)^2 - \tfrac{1}{2}\mathbf{v}_n^2 \\ \boldsymbol{\lambda} &= \rho_s(\mathbf{v}_s - \mathbf{v}_n) = \mathbf{j} - \rho\mathbf{v}_n \\ \boldsymbol{\omega} &= \mathbf{v}_n. \end{aligned} \tag{1.16}$$

Equation (1.15) implies that

$$T(\rho s)^\cdot = (\rho\epsilon)^\cdot - \nu\dot{\rho} - \boldsymbol{\lambda}\cdot\dot{\mathbf{v}}_s - \boldsymbol{\omega}\cdot\dot{\mathbf{j}}.$$

Inserting here the equations of motion (1.5), (1.6), (1.7), and the expressions (1.13) and (1.16) we find, to second order in the velocities

$$\begin{aligned}T(\rho s)^\cdot = &-\nabla\cdot(\mathbf{J}_\epsilon + \mathbf{J}_\epsilon') + \mu\nabla\cdot(\mathbf{j} + \mathbf{j}') \\ &+ \boldsymbol{\lambda}\cdot\nabla(\mu + \mu') + \mathbf{v}_n\cdot(\nabla p + \rho\nabla\psi) + \mathbf{v}_n\cdot\nabla\Pi'.\end{aligned}$$

Eliminating ∇p with the aid of Eq. (1.12), we find, after some rearrangements,

$$\begin{aligned}(\rho s)^\cdot = &-\nabla\cdot[(1/T)(\mathbf{J}_\epsilon + \mathbf{J}_\epsilon' - \mu\mathbf{j} - \mu\mathbf{j}' - \mu'\boldsymbol{\lambda} - \Pi'\mathbf{v}_n)] \\ &+ (\mathbf{J}_\epsilon + \mathbf{J}_\epsilon' - \mu\mathbf{j} - \mu\mathbf{j}' - T\rho s\mathbf{v}_n - \mu'\boldsymbol{\lambda} - \Pi'\mathbf{v}_n)\cdot\nabla(1/T) \\ &- (1/T)[\mathbf{j}'\cdot\nabla\mu + \mu'\nabla\cdot\boldsymbol{\lambda} + (\Pi'\nabla)\cdot\mathbf{v}_n].\end{aligned} \tag{1.17}$$

Since in the absence of dissipation the entropy is conserved, the right-hand side of Eq. (1.17) with $\mathbf{J}_\epsilon' = 0$, $\mathbf{j}' = 0$, $\mu' = 0$,

and $\Pi' = 0$ must be a divergence. This determines the nondissipative energy flux to second order in the velocities,

$$\mathbf{J}_\epsilon = \mu\mathbf{j} + T\rho s\mathbf{v}_n. \tag{1.18}$$

Equation (1.17) now takes the form (1.14) with

$$\mathbf{J}_s = (1/T)(\mathbf{J}_\epsilon - \mu\mathbf{j}) = \rho s\mathbf{v}_n, \tag{1.19}$$

$$\mathbf{J}_s' = (1/T)(\mathbf{J}_\epsilon' - \mu\mathbf{j}' - \mu'\boldsymbol{\lambda} - \Pi'\mathbf{v}_n), \tag{1.20}$$

and

$$\sigma = T\mathbf{J}_s'\cdot\nabla(1/T) - (1/T)[\mathbf{j}'\cdot\nabla\mu + \mu'\nabla\cdot\boldsymbol{\lambda} + (\Pi'\nabla)\cdot\mathbf{v}_n]. \tag{1.21}$$

For $\psi = \text{const}$, the hydrodynamic equations of motion (1.5), (1.6), and (1.12) express $\dot{\rho}$, $\dot{\mathbf{v}}_s$, $\dot{\mathbf{j}}$, and $(\rho s)^{\cdot}$, respectively, as divergences of the fluxes $\mathbf{j}$, μ, Π, and $\mathbf{J}_s$ and their dissipative parts. The associated forces are defined as gradients of the coefficients in Eq. (1.15), that is, respectively, as $\nabla\nu$, $\nabla\otimes\boldsymbol{\lambda}$, $\nabla\otimes\boldsymbol{\omega}$, and ∇T. Now to lowest order in the fluctuations and in the spacial derivatives the dissipative fluxes $\mathbf{j}'$, μ', Π', and $\mathbf{J}_s'$ are linear functions of these forces. In order to lead to irreversibility these dissipative fluxes must transform under time reversal with opposite sign as compared to the corresponding nondissipative (reactive) fluxes. This, together with rotational symmetry, restricts the linear functions of the forces to the following expressions, valid to the second-lowest order in the velocities (Khalatnikov, 1965):

$$\begin{aligned}
\mathbf{j}' &= -D_1\nabla\mu - D_2\nabla T\\
\mu' &= -\zeta_3\nabla\cdot\boldsymbol{\lambda} - \zeta_4\nabla\cdot\mathbf{v}_n\\
\Pi_{ij}' &= -\delta_{ij}[\zeta_1\nabla\cdot\boldsymbol{\lambda} + \zeta_2\nabla\cdot\mathbf{v}_n]\\
&\quad - \eta[\nabla_i v_{nj} + \nabla_j v_{ni} - \tfrac{2}{3}\delta_{ij}\nabla\cdot\mathbf{v}_n]\\
\mathbf{J}_s' &= -D_3\nabla\mu - (\kappa/T)\nabla T.
\end{aligned} \tag{1.22}$$

Here η is the first viscosity, ζ_1 to ζ_4 are the second viscosities, κ is the heat conductivity, and D_1, D_2, D_3 are diffusion constants. The symmetry of the coefficient matrix in (1.22) implies the Onsager relations

$$\zeta_1 = \zeta_4, \qquad D_2 = D_3. \tag{1.23}$$

Insertion of Eqs. (1.22) and (1.23) into (1.21) leads to the expression (Khalatnikov, 1965)

$$\begin{aligned}
T\sigma &= (\kappa/T)(\nabla T)^2 + 2D_2\nabla T\cdot\nabla\mu + D_1(\nabla\mu)^2\\
&\quad + \zeta_3(\nabla\cdot\boldsymbol{\lambda})^2 + 2\zeta_1\nabla\cdot\boldsymbol{\lambda}\nabla\cdot\mathbf{v}_n + \zeta_2(\nabla\cdot\mathbf{v}_n)^2\\
&\quad + \tfrac{1}{2}\eta\sum_{ij}(\nabla_j v_{ni} + \nabla_i v_{nj} - \tfrac{2}{3}\delta_{ij}\nabla\cdot\mathbf{v}_n)^2
\end{aligned} \tag{1.24}$$

which is positive definite under the conditions that all coefficients are positive and that (Khalatnikov, 1965)

$$\zeta_1^2 < \zeta_2\zeta_3, \qquad D_2^2 < (\kappa/T)D_1. \tag{1.25}$$

Since experimentally D_1 is negligible, we infer from Eqs. (1.23) and (1.25) that $D_1 = D_2 = D_3 = 0$, or that

$$\mathbf{j}' = 0, \qquad \mathbf{J}_s' = -(\kappa/T)\nabla T. \tag{1.26}$$

In the approximation by free quasiparticles the normal phase consists of the fluid of phonon–roton excitations with the Landau spectrum ω_k [see, for example, Wilks (1967), Fig. 12 of Chap. 5]. In drifting local equilibrium the distribution function of these excitations is

$$n_{\mathbf{k}} = \{\exp[(\omega_k - \mathbf{v}_n\cdot\mathbf{k})/k_BT] - 1\}^{-1}. \tag{1.27}$$

Here the quasiparticle chemical potential is zero because of nonconservation of the excitations. ρ_n is given as the coefficient of the linear term in the expansion of

$$\mathbf{j}_n = V^{-1}\sum_{\mathbf{k}}\mathbf{k}n_{\mathbf{k}} \tag{1.28}$$

in powers of $\mathbf{v}_n$. Inserting Eq. (1.27) into (1.28), one finds

$$\rho_n = (3k_BTV)^{-1}\sum_{\mathbf{k}}\mathbf{k}^2 n_k^0(n_k^0 + 1), \tag{1.29}$$

where $n_k^0 = [\exp(\omega_k/k_BT) - 1]^{-1}$ is the global equilibrium distribution function.

The result (1.29) is also obtained from the general expression (1.11) if the current operator in (1.11a) is expressed in terms of free quasiparticle operators,

$$\mathbf{j}(\mathbf{q}) = V^{-1/2}\sum_{\mathbf{k}}\mathbf{k}a_{\mathbf{k}-\mathbf{q}/2}{}^{+}a_{\mathbf{k}+\mathbf{q}/2}.$$

Indeed, with $n_k^0 = \langle a_{\mathbf{k}}{}^{+}a_{\mathbf{k}}\rangle$ one finds for the Fourier transform of (1.11a)

$$C_{ij}(\mathbf{q},\omega) = V^{-1}\sum_{\mathbf{k}}\pi\delta(\omega + \omega_k - \omega_{|\mathbf{k}+\mathbf{q}|})\cdot(k_i + q_i/2)(k_j + q_j/2)(n_k^0 - n_{|\mathbf{k}+\mathbf{q}|}{}^0)$$

which, inserted into (1.11), leads immediately to Eq. (1.29).

On the other hand, it follows from the energy per unit mass

$$\epsilon = (\rho V)^{-1}\sum_{\mathbf{k}}\omega_k n_{\mathbf{k}} \tag{1.30}$$

that the specific heat per unit mass is given by

$$c_V = \left(\frac{\partial\epsilon}{\partial T}\right)_\rho = (\rho k_BT^2V)^{-1}\sum_{\mathbf{k}}\omega_k^2 n_k^0(n_k^0 + 1). \tag{1.31}$$

For sufficiently low temperatures, only the linear phonon branch

$$\omega_k = c_1 k \tag{1.32}$$

is excited. Insertion into Eqs. (1.29) and (1.31) then yields

$$\rho_n = (2\pi^2/45c_1^5)(k_BT)^4 \tag{1.33}$$

and

$$c_V = (2\pi^2 k_B/15\rho c_1^3)(k_B T)^3. \tag{1.34}$$

Equation (1.34) gives the dominant contribution to c_V at $T \lesssim 0.6°K$, while at higher temperatures the roton contribution becomes dominant [see Wilks (1967), Fig. 18 of Chap. 5]. Equation (1.33) for ρ_n is valid only well below 0.6°K; at $T \cong 0.6°K$ the roton part is already of the same order of magnitude as the phonon part [see Wilks (1967), p. 137). Near the lambda point Eqs. (1.29) and (1.31) cease to be good approximations because of interactions between the excitations which, in particular, give rise to a temperature dependence of the spectrum ω_k [see Wilks (1967), Sec. 5.5].

B. Incompressible fluid: second sound

Incompressibility means that $\rho = \text{const}$. Retaining only linear terms in the fluctuations around the global equilibrium values we then have from the first Eq. (1.6), making use of Eq. (1.9),

$$\nabla\cdot\mathbf{j} = \rho_n \nabla\cdot\mathbf{v}_n + \rho_s \nabla\cdot\mathbf{v}_s = 0. \tag{1.35}$$

On the other hand, the second Eq. (1.6), together with (1.13), the third Eq. (1.22), and the first Eq. (1.26), imply the linearized Navier–Stokes equation

$$\dot{\mathbf{j}} = -\nabla[p - \zeta_1 \nabla\cdot\boldsymbol{\lambda} - (\zeta_2 + \tfrac{4}{3}\eta)\nabla\cdot\mathbf{v}_n] + \eta[\nabla^2\mathbf{v}_n - \nabla\nabla\cdot\mathbf{v}_n] \tag{1.36}$$

where the last term has divergence zero. Since $\nabla\cdot\mathbf{j} = 0$ means that $\mathbf{j}$ derives from a vector potential $\mathbf{j} = \nabla \times \mathbf{A}$, it follows that

$$p = p_0 + \zeta_1 \nabla\cdot\boldsymbol{\lambda} + (\zeta_2 + \tfrac{4}{3}\eta)\nabla\cdot\mathbf{v}_n \tag{1.37}$$

with constant p_0. If we start from an initial situation where $\mathbf{v}_n = \mathbf{v}_s = 0$, it follows from Eq. (1.9) that initially $\dot{\mathbf{j}} = 0$ and from Eqs. (1.36) and (1.37) that $\mathbf{j} = 0$ at all times. Hence

$$\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s = 0$$

for all times. This means that in thermal excitation of the fluid which is initially at rest the normal phase and the superfluid phase move in opposite directions.

In linear approximation in the fluctuations the entropy production (1.21) is zero so that Eq. (1.14), together with (1.19) and the second Eq. (1.26), implies

$$\dot{s} + s\nabla\cdot\mathbf{v}_n = (\kappa/\rho T)\nabla^2 T. \tag{1.38}$$

On the other hand, we obtain from Eq. (1.5), with the help of (1.12), the second Eq. (1.22), the first Eq. (1.23), and (1.37), for $\psi = \text{const}$,

$$\dot{\mathbf{v}}_s = s\nabla T - [(1/\rho)\zeta_1 - \zeta_3]\nabla\nabla\cdot\boldsymbol{\lambda} - [(1/\rho)(\zeta_2 + \tfrac{4}{3}\eta) - \zeta_1]\nabla\nabla\cdot\mathbf{v}_n.$$

From the second Eq. (1.16), together with Eq. (1.35), we have in linear approximation

$$\nabla\cdot\boldsymbol{\lambda} = -\rho\nabla\cdot\mathbf{v}_n. \tag{1.39}$$

Replacing $\nabla\cdot\mathbf{v}_n$ and $\nabla\cdot\boldsymbol{\lambda}$ with the help of Eqs. (1.38) and (1.39), we obtain in linear approximation

$$\dot{\mathbf{v}}_s = s\nabla T - (1/\rho s)[2\rho\zeta_1 - \zeta_2 - \rho^2\zeta_3 - \tfrac{4}{3}\eta]\nabla[\dot{s} - (\kappa/\rho T)\nabla^2 T]. \tag{1.40}$$

Taking the time derivative of Eq. (1.38) and replacing $\nabla\cdot\mathbf{v}_n$ with the help of (1.35) we find

$$\ddot{s} - (\rho_s/\rho_n)s\nabla\cdot\dot{\mathbf{v}}_s = (\kappa/\rho T)\nabla T^2.$$

Substituting here $\dot{\mathbf{v}}_s$ from Eq. (1.40) and making use of the definition of the specific heat at constant volume, $c_V \delta T = T\delta s$, we finally find [see Eq. (12-8) of Khalatnikov (1965)]

$$\ddot{T} - c_2^2\nabla^2 T = (\kappa/\rho c_V)\nabla^2\dot{T} + (\rho_s/\rho_n)(1/\rho)[-2\rho\zeta_1 + \zeta_2 + \rho^2\zeta_3 + \tfrac{4}{3}\eta][\nabla^2\dot{T} - (\kappa/\rho c_V)\nabla^4 T], \tag{1.41}$$

where

$$c_2 = [(\rho_s/\rho_n)(Ts^2/c_V)]^{1/2} \tag{1.42}$$

is the *velocity of second sound*. Note that third-order spatial derivatives in Eq. (1.22) would lead to a term $\nabla^4\dot{T}$ in Eq. (1.41), so that it is justified to keep the term $\nabla^4 T$.

For a plane-wave excitation $\sim\exp[i(\mathbf{q}\cdot\mathbf{r} - \omega t)]$ we find the dispersion law [see Eq. (12-8) of Khalatnikov (1965)]

$$\omega^2/q^2 = c_2^2(1 - i\omega\tau_2 + \lambda_2^2 q^2) \tag{1.43}$$

with [$c_2^2\tau_2$ is identical with D_2, Eq. (4.22) of Hohenberg and Martin (1965)]

$$\tau_2 = c_2^{-2}\{D_T + (\rho_s/\rho_n)(1/\rho)[-2\rho\zeta_1 + \zeta_2 + \rho^2\zeta_3 + \tfrac{4}{3}\eta]\},$$
$$\lambda_2^2 = D_T(\tau_2 - c_2^{-2}D_T). \tag{1.44}$$

Here

$$D_T = \kappa/\rho c_V$$

is the thermal diffusion constant. For low frequencies and long wavelengths,

$$\omega\tau_2 \ll 1, \qquad q\lambda_2 \ll 1, \tag{1.45}$$

the excitation propagates as a second-sound wave with velocity $c_2(1 + \frac{1}{2}\lambda_2^2 q^2)$ and damping $\frac{1}{2}\tau_2 c_2^2 q^2$.

Hydrodynamics, quite generally, is a low-frequency long wavelength approximation valid under the conditions of local thermal equilibrium,

$$\omega\tau_{\text{eq}} \ll 1, \qquad q\lambda_{\text{eq}} \ll 1. \tag{1.46}$$

Here τ_{eq} and λ_{eq} are, respectively, the relaxation time and the mean free path assuring thermal equilibrium at every macroscopic space-time point and are determined in principle by the Boltzmann collision operator for the phonon–roton excitations.

Since the conditions (1.45) and (1.46) are always compatible for sufficiently small ω and q, detection of second sound in superfluid helium imposes no further conditions on the system. This is markedly different from the situation reigning in the solid systems to be discussed in the subsequent sections. There momentum dissipation due to the presence of a rigid lattice imposes severe restrictions on the propagation of second sound and is the reason for the difficulties in detecting this mode experimentally (see the Introduction).

Next let us examine the motion of the order parameter, Eq. (1.2), in a second-sound wave

$$\delta T = |\delta T| \exp[i(\mathbf{q}\cdot\mathbf{r} - c_2 q t - \delta_2)].$$

Neglecting dissipation, we have from Eqs. (1.40) and (1.43)

$$\mathbf{v}_s = -(s/c_2)\,\hat{q}\delta T,$$

where $\hat{q} = \mathbf{q}/q$. From Eq. (1.4) we find at $\mathbf{r} = 0$

$$\begin{aligned}\delta\varphi(t) &= -\operatorname{Re}[i(ms/c_2 q)\delta T] \\ &= -(ms/c_2 q)\,|\delta T|\sin(c_2 q t + \delta_2).\end{aligned} \tag{1.47}$$

The fluctuation of the order parameter due to the second-sound wave then is, according to Eq. (1.2),

$$\begin{aligned}(\delta\langle\Psi^+\rangle)^2 &= |\langle\Psi^+\rangle_{\delta\varphi} - \langle\Psi^+\rangle|^2 \\ &= 2n_c[1 - \cos\delta\varphi(t)],\end{aligned} \tag{1.48}$$

where $\langle\Psi^+\rangle_{\delta\varphi}$ is the average in the presence of the second-sound wave, i.e., with phase $\varphi(t) + \delta\varphi(t)$. For sufficiently small q the amplitude $ms\,|\delta T|/c_2 q$ of $\delta\varphi(t)$ can be made larger than π. Then the fluctuation $\delta\langle\Psi^+\rangle$ is of the same magnitude as the order parameter $\langle\Psi^+\rangle$ itself. In the limit $q \to 0$ the phase (1.47) becomes a constant of arbitrary value, so that one may say that the second-sound excitation "restores" the continuous gauge group (see the Introduction). For this reason second sound has sometimes been identified as the Goldstone boson associated with gauge symmetry breaking in the superfluid (Ferrell, 1969). However, since a Goldstone boson is a collisionless excitation persisting to $T = 0$ (see the Introduction), this identification is not justifiable, as is evident from the following consideration of the zero-temperature limit.

In the limit $T \to 0$, we have from Eq. (1.33) $\rho_n \to 0$, and hence $\rho_s \to \rho$. In addition $s \to \frac{1}{3}c_V$ so that it follows from Eqs. (1.33) and (1.34) that

$$\rho_n/sT\rho = c_1^{-2}, \qquad T \to 0. \tag{1.49}$$

From Eqs. (1.49) and (1.42) we also find

$$c_2 = c_1/\sqrt{3}, \qquad T \to 0. \tag{1.50}$$

Because of $\rho_n \to 0$ the phonon gas becomes infinitely dilute, so that collisions between phonons become negligible as compared to collisions with the walls. Hence, thermal equilibrium no longer holds, and the conditions (1.46) are violated. Therefore the Landau limit (1.50) cannot be reached. Instead energy is carried by individual phonons and thus propagates with c_1. In fact one observes experimentally that heat pulses propagate with velocity c_1 (de Klerk *et al.*, 1954; Peshkov, 1960; Dynes *et al.*, 1973). [See also Wilks (1967), Section 8.9]. This is the collisionless situation discussed in the Introduction.

C. Isothermal fluid: first sound

We now assume $T = \text{const}$ and $\beta = (\partial p/\partial T)_\rho = -\rho^2(\partial s/\partial\rho)_T = 0$ [here the second equality follows from Eq. (1.12)], that is, vanishing of the tension coefficient βp. Then we also have $s = \text{const}$. β is related to the thermal expansion coefficient $\alpha = -(1/\rho)(\partial\rho/\partial T)_p$ and the isothermal compressibility $\gamma = (1/\rho)(\partial\rho/\partial p)_T$ by the identity $\beta = -\alpha/\gamma$. Since for low temperatures $\alpha = -1.09 \times 10^{-3} T^3 K^{-1}$ [see Table A1, p. 666 of Wilks (1967)], the assumption $\beta = 0$ is fulfilled in the limit $T \to 0$.

With $T = \text{const}$ and $s = \text{const}$ Eq. (1.14) combined with Eqs. (1.19), (1.21), and (1.26) becomes in linear approximation in the fluctuations

$$\dot{\rho} + \rho\nabla\cdot\mathbf{v}_n = 0. \tag{1.51}$$

Subtracting from this equation the first Eq. (1.6) combined with Eqs. (1.9) and (1.26) we find

$$\nabla\cdot(\mathbf{v}_n - \mathbf{v}_s) = 0. \tag{1.52}$$

In the same approximation, combination of (1.5) and the second Eq. (1.6), together with Eqs. (1.9), (1.10), (1.12), and (1.13), leads to

$$\rho_n(\dot{\mathbf{v}}_n - \dot{\mathbf{v}}_s) = -\nabla\Pi' + \rho\nabla\mu'. \tag{1.53}$$

Neglecting dissipation, the last equation together with Eq. (1.52) implies that $\mathbf{v}_n - \mathbf{v}_s = \text{const}$, or that the two fluids move in phase, $\mathbf{v}_n = \mathbf{v}_s$, if the excitation starts from equilibrium, $\mathbf{v}_n = \mathbf{v}_s = 0$.

Combination of the two equations (1.6) yields, with Eq. (1.13), the third Eq. (1.22), and (1.26),

$$\ddot{\rho} - \nabla^2 p = -(\zeta_2 + \tfrac{4}{3}\eta)\nabla^2\nabla\cdot\mathbf{v}_n. \tag{1.54}$$

Here we have made use of $\nabla\cdot\boldsymbol{\lambda} = 0$ which follows from the second Eq. (1.16) and (1.52). Substituting in Eq. (1.54) $\nabla\cdot\mathbf{v}_n$ from Eq. (1.51) and making use of the definition of the isothermal compressibility $\delta\rho = \rho\gamma\delta p$, we find

$$\ddot{\rho} - c_1^2\nabla^2\rho = (1/\rho)(\zeta_2 + \tfrac{4}{3}\eta)\nabla^2\dot{\rho}, \tag{1.55}$$

where

$$c_1 = (\rho\gamma)^{-1/2} \tag{1.56}$$

is the velocity of *isothermal* first sound.

In plane-wave representation, $\rho \propto \exp[i(\mathbf{q}\cdot\mathbf{r} - \omega t)]$, we obtain the dispersion law [see Eq. (12-7) of Khalatnikov (1965)]

$$\omega^2/q^2 = c_1^2(1 - i\omega\tau_{10}), \tag{1.57}$$

with [$c_1^2\tau_{10}$ is identical with D_1, Eq. (4.21) of Hohenberg and Martin (1965)]

$$\tau_{10} = (1/\rho c_1^2)(\zeta_2 + \tfrac{4}{3}\eta). \tag{1.58}$$

From Eqs. (1.5), (1.12), (1.56), (1.57), and $\delta\rho = \rho\gamma\delta p$ we find in plane-wave approximation, neglecting dissipation,

$$\mathbf{v}_s = \hat{q}(c_1/\rho)\delta\rho$$

so that according to Eq. (1.4)

$$\delta\varphi(t) = \mathrm{Re}[i(mc_1/\rho q)\delta\rho] = (mc_1/\rho q)\,|\,\delta\rho\,|\sin(c_1 q t + \delta_1)$$

in a first-sound excitation at $\mathbf{r} = 0$ with complex amplitude $|\,\delta\rho\,|\exp(-i\delta_1)$. Equation (1.48) then shows that for sufficiently small q the fluctuation of the order parameter due to a first-sound wave is of the same magnitude as the order parameter itself. As was the case with the expression (1.47), $\delta\varphi(t)$ becomes a constant of arbitrary value for $q \to 0$, so that one may say that the first-sound excitation "restores" the continuous gauge group. This suggests the identification of isothermal first sound as the Goldstone boson associated with gauge symmetry breaking which is legitimate in the sense of the collisionless limit of c_1 (see the Introduction).

If, on the other hand, the isothermal compressibility γ is singular at T_λ [see Fig. 17 of Ahlers (1973)], it follows from Eq. (1.56) that $c_1 \to 0$, i.e., *isothermal* first sound is a soft mode (Schneider *et al.*, 1972). This is not the case, however, for the *adiabatic* first sound which is directly measurable (see below).

D. Thermal expansion coupling of first and second sound

We now relax all the restrictions on ρ, ψ, and T, and allow $\beta \neq 0$, that is, nonvanishing thermal expansion α. Then Eqs. (1.51) to (1.54) generalize to the following form, again valid in linear approximation

$$\rho\dot{s} + \dot{\rho}s + \rho s\nabla\cdot\mathbf{v}_n = (\kappa/T)\nabla^2 T, \tag{1.59}$$

$$\rho\dot{s} + \rho_s s\nabla\cdot(\mathbf{v}_n - \mathbf{v}_s) = (\kappa/T)\nabla^2 T, \tag{1.60}$$

$$\rho_n(\dot{\mathbf{v}}_n - \dot{\mathbf{v}}_s) + \rho s\nabla T = \rho_s(-\zeta_1 + \rho\zeta_3)\nabla\nabla\cdot(\mathbf{v}_n - \mathbf{v}_s) + (-\rho\zeta_1 + \zeta_2 + \tfrac{1}{3}\eta)\nabla\nabla\cdot\mathbf{v}_n + \eta\nabla^2\mathbf{v}_n, \tag{1.61}$$

$$\ddot{\rho} - \nabla^2 p = \rho\nabla^2\psi + \rho_s\zeta_1\nabla^2\nabla\cdot(\mathbf{v}_n - \mathbf{v}_s) - (\zeta_2 + \tfrac{4}{3}\eta)\nabla^2\nabla\cdot\mathbf{v}_n. \tag{1.62}$$

Here the first two equations can be used to eliminate $\nabla\cdot\mathbf{v}_n$ and $\nabla\cdot\mathbf{v}_s$ from the second two equations. This elimination does not, however, determine the transverse parts $\mathbf{v}_{nt}$ and $\mathbf{v}_{st}$ which have zero divergence. Now Eq. (1.4) implies that $\mathbf{v}_{st} = 0$ so that the transverse projection of Eq. (1.61) becomes the diffusion equation

$$\rho_n\dot{\mathbf{v}}_{nt} = \eta\nabla^2\mathbf{v}_{nt}. \tag{1.61a}$$

This equation is completely decoupled from the longitudinal projection (divergence) of Eq. (1.61), which now has to be combined with Eqs. (1.59), (1.60), and (1.62).

Choosing as independent variables $\delta\rho$ and δT, we eliminate δp and δs with the help of the relations (which because of time reversal invariance do not contain the velocities to first order)

$$\delta p = (1/\rho\gamma)\delta\rho + \beta\delta T$$
$$\delta s = -(\beta/\rho^2)\delta\rho + (c_V/T)\delta T. \tag{1.63}$$

Here, as before, $\beta = (\partial p/\partial T)_\rho = -\rho^2(\partial s/\partial\rho)_T$ defines the tension coefficient βp.

Insertion of Eqs. (1.59), (1.60), and (1.63) into Eq. (1.62) leads to the equation

$$\ddot{\rho} - c_1^2\nabla^2\rho - \beta\nabla^2 T = \rho\nabla^2\psi + c_1^2\tau_1\nabla^2\dot{\rho} + \beta(\tau_1'\nabla^2\dot{T} - \lambda_1'^2\nabla^4 T), \tag{1.64}$$

and combining the time derivative of Eq. (1.60) with the divergence of (1.61) we obtain, inserting Eqs. (1.59), (1.60), and (1.63),

$$\ddot{T} - c_2^2\nabla^2 T - (c_1^2\vartheta/\beta)\ddot{\rho} = c_2^2(\tau_2\nabla^2\dot{T} - \lambda_2^2\nabla^4 T) - (c_1^2 c_2^2\vartheta/\beta)\tau_2'\nabla^2\dot{\rho}. \tag{1.65}$$

Equation (1.65) shows that the temperature does not couple directly to an external potential. The quantities c_1, c_2, τ_2, λ_2 in Eqs. (1.64) and (1.65) are defined, respectively, by Eqs. (1.56), (1.42), and (1.44). In addition we have introduced the definitions

$$\tau_1 = \tau_{10} - \vartheta\tau_1' \tag{1.66}$$

where τ_{10} is given by Eq. (1.58),

$$\tau_1' = (c_V/\beta s T)(-\rho\zeta_1 + \zeta_2 + \tfrac{4}{3}\eta)$$
$$\tau_2' = \tau_2 - c_2^{-2}D_T - \tau_1'$$
$$\lambda_1'^2 = D_T\tau_1' \tag{1.67}$$

and

$$c_1^2\vartheta = \beta^2 T/\rho^2 c_V. \tag{1.68}$$

In plane-wave representation Eqs. (1.64) and (1.65) become

$$[(\omega^2/q^2) - c_1^2(1 - i\omega\tau_1)]\delta\rho - \beta[1 - i\omega\tau_1' + \lambda_1'^2 q^2]\delta T = \rho\delta\psi \tag{1.69}$$

$$[(\omega^2/q^2) - c_2^2(1 - i\omega\tau_2 + \lambda_2^2 q^2)]\delta T - (c_1^2\vartheta/\beta)[(\omega^2/q^2) + c_2^2 i\omega\tau_2']\delta\rho = 0, \tag{1.70}$$

where $\delta\rho$, δT, and $\delta\psi$ are the variations of ρ, T, and ψ around the global equilibrium values, respectively. Because of Eq. (1.68) this system of equations is decoupled for $\beta = 0$ and $\psi = \text{const}$, and the previous results (1.43) and (1.57) are recovered. Note that the diffusion equation (1.61a) leads to the viscosity mode

$$\omega = -i(\eta/\rho_n)q^2.$$

Retaining only the lowest powers of q and ω, but separately for real and imaginary parts, insertion of Eq. (1.70) into (1.69) yields the *density-density correlation function*

$$\chi_{\rho\rho}(\mathbf{q},\omega) = -\delta\rho/\delta\psi = -G(\mathbf{q},\omega)[\omega^2 - c_2^2(1 - i\omega\tau_2)q^2]\rho q^2 \tag{1.71}$$

where

$$G^{-1}(\mathbf{q},\omega) = [\omega^2 - \tilde{c}_1^2(1 - i\omega\tilde{\tau}_1)q^2] \times[\omega^2 - c_2^2(1 - i\omega\tau_2)q^2] - c_1^2c_2^2\vartheta(1 - i\omega\tau_1') \times[1 - i\omega(\tau_2 - \tau_2')]q^4. \tag{1.72}$$

Here we have

$$\tilde{c}_1^2 = c_1^2(1 + \vartheta) \tag{1.73}$$

and

$$\tilde{\tau}_1 = \tau_{10}(1 + \vartheta)^{-1}. \tag{1.74}$$

For $\vartheta = 0$, Eqs. (1.71) to (1.74), with Eqs. (1.42), (1.66), (1.58), and (1.44), are identical with Eqs. (4.32) and (4.19) to (4.22) of Hohenberg and Martin (1965).

$\tilde{c}_1$, as defined by Eq. (1.73), is the *adiabatic* first-sound velocity. Indeed, according to Eq. (1.63), the adaibatic compressibility $\tilde{\gamma} = (1/\rho)\,(\partial\rho/\partial p)_s$ satisfies the relation

$$\tilde{\gamma}^{-1} - \gamma^{-1} = \beta^2T/\rho c_V.$$

Inserted together with Eqs. (1.56) and (1.68) into Eq. (1.73) this gives

$$\tilde{c}_1 = (\rho\tilde{\gamma})^{-1/2}.$$

From the thermodynamic identity

$$T\beta^2\gamma/\rho = c_p - c_V, \tag{1.75}$$

we further obtain

$$1 + \vartheta = c_p/c_V. \tag{1.76}$$

At low temperatures $\beta = -\rho^2(\partial s/\partial\rho)_T \propto c_V \propto T^3$, so that according to Eq. (1.68) $\vartheta \propto T^4$. Hence the second term in Eq. (1.72) vanishes in the limit $T \to 0$, so that first and second sound become decoupled and first sound becomes isothermal.

In the general case where $\vartheta \neq 0$ the second term in Eq. (1.72) describes the *thermal expansion coupling* of first and second sound. While these two modes remain the physical excitations of the system, their velocities and relaxation times are renormalized by the thermal expansion coupling. Calling these renormalized values c_{I}, c_{II}, and τ_{I}, τ_{II}, respectively, Eq. (1.72) may be rewritten in the factorized form

$$G^{-1}(\mathbf{q},\omega) = [\omega^2 - c_{\mathrm{I}}^2(1 - i\omega\tau_{\mathrm{I}})q^2] \times[\omega^2 - c_{\mathrm{II}}^2(1 - i\omega\tau_{\mathrm{II}})q^2] + O(\omega^2q^4). \tag{1.72a}$$

valid to order ω^2q^4. By comparison with Eq. (1.72) the renormalized quantities are found to be given by

$$c_{\mathrm{I}}^2 = \tilde{c}_1^2 + u, \qquad c_{\mathrm{II}}^2 = c_2^2 - u$$
$$c_{\mathrm{I}}^2\tau_{\mathrm{I}} = \tilde{c}_1^2\tilde{\tau}_1 + v, \qquad c_{\mathrm{II}}^2\tau_{\mathrm{II}} = c_2^2\tau_2 - v, \tag{1.72b}$$

where, expressed by the renormalized quantities [except for the small quantity $(\tau_1' - \tau_2')\vartheta$],

$$u = \tfrac{1}{2}(c_{\mathrm{I}}^2 - c_{\mathrm{II}}^2)z[1 + (1 - z)^{1/2}]^{-1}$$
$$v = \{c_{\mathrm{II}}^2\tau_{\mathrm{II}}(u + c_{\mathrm{I}}^2\vartheta) - c_{\mathrm{I}}^2\tau_{\mathrm{I}}u(1 + \vartheta) + (c_{\mathrm{I}}^2 - u)(c_{\mathrm{II}}^2 + u)(\tau_1' - \tau_2')\vartheta\} \times[c_{\mathrm{I}}^2 - c_{\mathrm{II}}^2(1 + \vartheta) - u(2 + \vartheta)]^{-1}. \tag{1.72c}$$

Here

$$z = 4c_{\mathrm{I}}^2c_{\mathrm{II}}^2\vartheta(c_{\mathrm{I}}^2 - c_{\mathrm{II}}^2), \tag{1.77}$$

ϑ being given by Eq. (1.76). Likewise we can decompose the numerator in Eq. (1.71),

$$\omega^2 - c_2^2(1 - i\omega\tau_2)q^2 = r(1 + i\omega\tau_r) \times[\omega^2 - c_{\mathrm{I}}^2(1 - i\omega\tau_{\mathrm{I}})q^2] + [1 - r(1 + i\omega\tau_r)] \times[\omega^2 - c_{\mathrm{II}}^2(1 - i\omega\tau_{\mathrm{II}})q^2] + O(\omega^2q^2),$$

where

$$r = (z/2)[1 + (1 - z)^{1/2}]^{-1}, \tag{1.78}$$

and

$$\tau_r = (c_{\mathrm{I}}^2\tau_{\mathrm{I}} - c_{\mathrm{II}}^2\tau_{\mathrm{II}} - v/r)(c_{\mathrm{I}}^2 - c_{\mathrm{II}}^2)^{-1}. \tag{1.78a}$$

With this decomposition Eq. (1.71) can be written in the form

$$-\frac{1}{\rho}\chi_{\rho\rho}(q,\omega) = \frac{[1 - r(1 + i\omega\tau_r)]q^2}{\omega^2 - c_{\mathrm{I}}^2(1 - i\omega\tau_{\mathrm{I}})q^2} + \frac{r(1 + i\omega\tau_r)q^2}{\omega^2 - c_{\mathrm{II}}^2(1 - i\omega\tau_{\mathrm{II}})q^2} \tag{1.79}$$

valid to order ω^2q^4.

Up to this order the above result is identical with a recent calculation by Hohenberg (1973). More explicitly, our Eqs. (1.71) and (1.72a) are the same as Eq. (2) of Hohenberg (1973), and our Eqs. (1.72b) and (1.72c) correspond

exactly to Eqs. (9) through (12) of this reference. In these equations our quantities $\tilde{c}_1^2$, c_2^2, $\tilde{c}_1^2\tilde{\tau}_1$, $c_2^2\tau_2$, c_I^2, c_{II}^2, $c_I^2\tau_I$, $c_{II}^2\tau_{II}$, $c_1^2\tau_1'\vartheta$, and $c_2^2(\tau_1'+\tau_2')$ coincide respectively with Hohenberg's c_{10}^2, γc_{20}^2, D_1^0, $D_2^0+D_K(\gamma-1)$, c_1^2, c_2^2, D_1, D_2, $a(D_1^0-\zeta_1)$, and $D_2^0-D_K$. Note that Hohenberg's parameter a is our $\beta/\rho s$ and that he does not make use of the Onsager relation $\zeta_1=\zeta_4$. Also Hohenberg does not make the physically natural decomposition (1.79) which separates the first- and second-sound poles.

From Eq. (1.79) it is easy to calculate the spectral intensity or *dynamic structure factor* which, up to a constant, is given by

$$S(q,\omega) = (1/\pi\rho)(1/\omega)\,\mathrm{Im}\chi_{\rho\rho}(q,\omega)$$
$$= S_I(q,\omega)+S_{II}(q,\omega)+O(\omega^2q^4). \tag{1.79a}$$

One finds

$$\pi S_I(q,\omega) = \frac{(1-r)c_I^2\tau_I q^4 + r(\omega^2-c_I^2q^2)\tau_r q^2}{(\omega^2-c_I^2q^2)^2+(\omega q^2 c_I^2\tau_I)^2}$$

$$\pi S_{II}(q,\omega) = r\,\frac{c_{II}^2\tau_{II}q^4-(\omega^2-c_{II}^2q^2)\tau_r q^2}{(\omega^2-c_{II}^2q^2)^2+(\omega q^2c_{II}^2\tau_{II})^2}. \tag{1.79b}$$

Equations (1.79a) and (1.79b) describe directly the response measured in inelastic neutron scattering with small momentum transfer $\mathbf{q}$ and in Brillouin scattering. It shows that this response consists in a two-peak structure describing coupled first and second sound with amplitudes $1-r$ and r, respectively. The two peaks contribute fractions I_I and I_{II}, respectively, to the integrated intensity,

$$\int_{-\infty}^{+\infty}\frac{d\omega}{\omega}S(q,\omega) = I_I+I_{II} = \frac{1-r}{c_I^2}+\frac{r}{c_{II}^2} \tag{1.80}$$

[note that all three relaxation times τ_I, τ_{II}, and τ_r occurring in Eq. (1.79) drop out in this integration]. As noted before, ϑ goes to zero as T^4 in the limit $T\to 0$. Consequently the intensity ratio for $T\ll T_\lambda$ is found from Eqs. (1.77), (1.78), and (1.80) to be given by

$$(I_{II}/I_I)_{T\to 0} = c_I^4\vartheta(c_I^2-c_{II}^2)^{-2}\propto T^4.$$

This shows that far below T_λ the second-sound peak is too weak to be seen in neutron or Brillouin scattering.

On the other hand, in the limit $c_{II}/c_I\to 0$ Eqs. (1.77), (1.78), (1.80), and (1.76) yield

$$(I_{II}/I_I)_{c_{II}/c_I\to 0} = (c_p/c_V)-1$$

which is the well known Landau–Placzek ratio (Landau and Placzek, 1934). [See also Landau and Lifshitz (1960); Mountain (1966b).] This expression is valid for normal fluids where the second-sound doublet merges into the Rayleigh peak at $\omega=0$. But it is also valid for superfluid helium near T_λ provided that two-fluid hydrodynamics is still valid (see below).

In fact, $c_{II}/c_I\to 0$ in the limit $T\to T_\lambda$ (see below), so that we obtain from Eqs. (1.72b), (1.72c), (1.77), (1.78), and (1.78a)

$$c_I^2\to\tilde{c}_1^2,\qquad c_{II}^2\to c_2^2(1+\vartheta)^{-1}$$
$$\tau_I\to\tilde{\tau}_1,\qquad \tau_{II}\to\tfrac{1}{2}(1+\vartheta)(\tilde{\tau}_1+\tau_2-\tau_1'+\tau_2')$$
$$r\to(c_2^2/\tilde{c}_1^2)\vartheta/(1+\vartheta)$$
$$\tau_r\to\tilde{\tau}_1+\tfrac{1}{2}(1+\vartheta)(\tilde{\tau}_1-\tau_2-\tau_1'+\tau_2'). \tag{1.80a}$$

Inserting these expressions into Eqs. (1.79b), we find

$$S_I(q,0)\to\tilde{\tau}_1/\tilde{c}_1^2$$
$$S_{II}(q,0)\to\{(2+\vartheta)\tilde{\tau}_1-(1+\vartheta)(\tau_1'-\tau_2')\}\vartheta/\tilde{c}_1^2. \tag{1.80b}$$

Since ϑ diverges in the limit $T\to T_\lambda$ (see below), this shows that $S_{II}(q,0)$ indeed gives rise to a Rayleigh peak. Hohenberg (1973) has made a detailed analysis of $S(q,0)$ in this limit. His expression (24) coincides exactly with the sum of the two terms in (1.80b). This must of course be so, since for $\omega=0$ Eq. (1.79a) is exact.

Recently Winterling, Holmes, and Greytak (1973) [See also Vaughan *et al.* (1972)] have succeeded, for the first time, in seeing the second-sound peak in Brillouin scattering on pure superfluid ^{4}He. This success is due precisely to the choice of temperatures very near to T_λ so that the intensity ratio I_{II}/I_I is given by the Landau–Placzek value which diverges at T_λ. Further enhancement of this ratio is obtained by choosing high pressures. The experimental results for I_{II}/I_I at 25.1 atm and $q\cong 1.4\times 10^5$ cm^{-1} which are plotted in Fig. 2 of Winterling *et al.* (1973) show good agreement with $(c_p/c_V)-1$ up to $T_\lambda-T\cong 0.4$ m deg. For smaller values of $T_\lambda-T$ a marked deviation from the Landau–Placzek value is observed which indicates that two-fluid hydrodynamics is no longer valid in the critical region (Winterling *et al.*, 1973).

The critical region is defined by $q\xi(T)>1$, where

$$\xi(T) = \xi_0\tau^{-\nu'} \tag{1.81}$$

is the correlation length, $\tau=(T_\lambda-T)/T_\lambda$, and the critical exponent $\nu'\cong 2/3$ (Kadanoff *et al.*, 1967). In this region the temperature dependence of the thermodynamic and hydrodynamic quantities is dominated by critical fluctuations which modify the singularities of these quantities. The specific heat which is best known experimentally is that measured at saturated vapor pressure, c_{svp}. Defining its critical exponent α' by

$$c_{svp} = A'[(\tau^{-\alpha'}-1)/\alpha'] + B' \tag{1.82}$$

the crucial experiments by Buckingham and Fairbank (1961) were compatible with $\alpha'=0$. This result is confirmed by very recent measurements of the second-sound velocity in the vicinity of T_λ by Greywall and Ahlers (1973), [see below] who find $\alpha'=-0.01\pm 0.03$.

With $\alpha'=0$ the identity

$$\lim[(\tau^{-\alpha'}-1)/\alpha'] = \log(1/\tau)$$

implies that Eq. (1.82) becomes

$$c_{svp} = A'\log(1/\tau)+B', \tag{1.82a}$$

which is the form given by Ahlers (1969, 1973) and by Kadanoff *et al.* (1967). Because the saturated vapor pressure varies little with temperature, Eq. (1.82a) then also holds for c_p (Ahlers, 1969, 1973; Greywall and Ahlers, 1973). Since, on the other hand, α and γ have the same singularity as c_p [see Eqs. (4) and (6) of Ahlers (1969)] $\beta = -\alpha/\gamma$ is nonsingular and therefore also c_V [see Eq. (37) of Ahlers (1969)]. Note, however, that in the temperature region $|T - T_\lambda| \gtrsim 10^{-6}$K explored up to date the asymptotic region is not yet reached. This is the reason why the form (1.82a) could also be used to fit the nonsingular quantities β and c_V (Ahlers, 1969, 1973). It is also the reason why Hohenberg (1973) obtains almost constant slopes in his plot of $\beta/\rho s$ (his Fig. 2).

The present experimental situation is actually even more complicated. As shown by the detailed analysis of Ahlers (1973), agreement with the theoretical predictions of scaling can be obtained only if higher-order terms in τ are included in the expression (1.82) for c_p and if $-0.04 \leq \alpha' < 0$. This would mean that c_p and hence also c_1 stay finite at the λ line so that the *isothermal* first-sound mode would not become completely soft.

From the above discussion it follows that $\vartheta = (c_p/c_V) - 1$ diverges as c_p [see Fig. 2 of Winterling *et al.*, 1973]. And we conclude from Eqs. (1.56) and (1.73) that the *adiabatic* sound velocity $\tilde{c}_1$ reaches a *finite* value c_λ at the λ point. Precision measurements of $\tilde{c}_1$ in the temperature range $10^{-6}\text{K} \lesssim |T - T_\lambda| \lesssim 0.35\text{K}$ at saturated vapor pressure and at small $q(\tilde{c}_1 q/2\pi = 22$ kc/sec) have been performed by Barmatz and Rudnick (1968), who obtain a value $c_\lambda = 217.3$ *m*/sec. A thermodynamic analysis of these results has been made by Ahlers (1969), who also showed the importance of the gravitational pressure gradient for these experiments.

The superfluid density vanishes as (Greywall and Ahlers, 1973; Kadanoff *et al.*, 1967; Ferrell *et al.*, 1968)

$$(\rho_s/\rho) = r\tau^\zeta \tag{1.83}$$

with $\zeta \cong \frac{2}{3}$. Since c_V is nonsingular at T_λ Eqs. (1.42) and (1.83) imply that the second-sound velocity vanishes [see Fig. 22 on p. 71 of Wilks (1967)] and that $c_2 \propto \tau^{(\zeta+\alpha)/2}$. Now for $\omega/q \cong c_2 \ll \tilde{c}_1$ we can neglect the term ω^2 in the first bracket of Eq. (1.72). Retaining only lowest powers of q and ω the second-sound pole of G^{-1} thus is located at

$$\omega^2/q^2 = \tilde{c}_2^2(1 - i\omega\tilde{\tau}_2), \tag{1.84}$$

where

$$\tilde{c}_2^2 = c_2^2(1 + \vartheta)^{-1} \tag{1.85}$$

and

$$\tilde{\tau}_2 = \tau_2 + \vartheta\tau_2' + [\vartheta/(1 + \vartheta)](\tau_1 - \tau_1'). \tag{1.86}$$

With Eqs. (1.42), (1.78), (1.82a), and (1.83), Eq. (1.85) becomes (Ferrell *et al.*, 1968)

$$\tilde{c}_2^2 = c_2^2 \frac{c_V}{c_p} = \frac{\rho_s}{\rho_n}\frac{s^2 T}{c_p} = \frac{r\tau^\zeta s_\lambda^2 T_\lambda}{A' \log(1/\tau) + B'}. \tag{1.87}$$

Since according to Eqs. (1.69) and (1.70) second sound is coupled to first sound, the measurable second-sound velocity is c_{II}. From Eqs. (1.72c) and (1.77) we find to first order in c_{II}^2/c_I^2

$$u = [1 + (1 + \vartheta)c_{II}^2/c_I^2]c_{II}^2\vartheta.$$

Inserted into Eqs. (1.72b) this leads to

$$c_{II}^2 = \tilde{c}_2^2[1 - \vartheta\tilde{c}_2^2/\tilde{c}_1^2]. \tag{1.87a}$$

Precision measurements of c_{II} have recently been made by Greywall and Ahlers (1973) in the temperature range $2 \times 10^{-5}K \lesssim T_\lambda - T \lesssim 10^{-2}K$ and at various pressures by a resonator technique. They find that the difference $c_{II} - \tilde{c}_2$ given by Eq. (1.87a) [which is their Eq. (6)] is less than the experimental resolution over the complete range of data. Using their values of $\tilde{c}_2$ to deduce ρ_s/ρ according to Eq. (1.87), they find the parameters in Eq. (1.83) in the limit $p \to 0$, $\tau \to 0$ to be $r = 2.40$ and $\zeta = 0.67 \pm 0.01$. However, in their temperature range they find a non-negligible correction factor $1 + a\tau^y$ with $a = 0.65$, $y = 0.5 \pm 0.1$, and r and a are both pressure dependent.

The solution $\omega = \omega' - i\omega''$ of Eq. (1.84) is

$$\omega'' = \tfrac{1}{2}\tilde{c}_2^2\tilde{\tau}_2 q^2$$
$$\omega'^2/\tilde{c}_2^2 q^2 = 1 - (\tfrac{1}{4}\tilde{c}_2^2\tilde{\tau}_2^2 - \tilde{\lambda}_2^2)q^2. \tag{1.88}$$

Ferrell *et al.* (1968) argue that $D_T = \tilde{c}_2^2\tilde{\tau}_2$ has a singularity $\propto (\xi/c_{svp})^{1/2}$. However, since the hydrodynamic equations, strictly speaking, are valid only in the hydrodynamic region $q\xi(T) < 1$, the solution (1.88) still represents a soft mode in the sense that the complex frequency $\omega = \omega' - i\omega''$ goes to zero at the phase transition (Schneider *et al.*, 1972). From Eqs. (1.81), (1.82a), (1.83), (1.87), and (1.88) we find

$$\omega''/\omega' \propto \tau^{(\nu'-\zeta)/2} \cong \text{const} \tag{1.89}$$

so that $\omega \to 0$ along a line with finite slope.

E. Normal liquid

Above the λ transition, $\rho_c = 0$, $\rho_s = 0$, $\mathbf{v}_s = 0$, and $\boldsymbol{\lambda} = 0$. Hence Eq. (1.5) does not hold, and both the number of variables and the number of equations is reduced by three. Writing $\mathbf{v}_n = \mathbf{v}$ Eqs. (1.6) and (1.14) become in linear approximation with $\psi = \text{const}$

$$\dot{\rho} + \rho\nabla\cdot\mathbf{v} = 0$$
$$\rho\dot{\mathbf{v}} + \nabla p = (\zeta_2 + \tfrac{1}{3}\eta)\nabla\nabla\cdot\mathbf{v} + \eta\nabla^2\mathbf{v}$$
$$\rho\dot{s} = (\kappa/T)\nabla^2 T. \tag{1.90}$$

The transverse projection of the second Eq. (1.90) leads to the diffusion equation

$$\rho\dot{\mathbf{v}}_t = \eta\nabla^2\mathbf{v}_t. \tag{1.90a}$$

In the longitudinal projection (divergence) of this equation $\nabla\cdot\mathbf{v}$ can be eliminated with the help of the first Eq. (1.90).

Making use of Eqs. (1.58), (1.63), (1.68) and the third equation (1.44), we find the two equations describing hydrodynamic excitations in a *normal liquid*,

$$\ddot{\rho} - c_1^2\nabla^2\rho - \beta\nabla^2 T = c_1^2\tau_{10}\nabla^2\dot{\rho}$$
$$\dot{T} - (c_1^2/\beta)\dot{\rho} = D_T\nabla^2 T. \tag{1.91}$$

In plane-wave representation this leads to the dispersion relation

$$\frac{\omega^2}{q^2} = \mathcal{c}_1^2\,(1 - i\omega\tilde{\tau}_1) - c_1^2\frac{iD_Tq^2}{\omega + iD_Tq^2} \tag{1.92}$$

where $\mathcal{c}_1$ and $\tilde{\tau}$, are given by Eqs. (1.73) and (1.74), respectively.

For small q this equation has the three solutions

$$\omega_\pm \cong \pm\,\mathcal{c}_1 q - \tfrac{1}{2}i\left(\frac{\zeta_2 + \frac{4}{3}\eta}{\rho} + \frac{c_p - c_V}{c_V}\frac{\kappa}{\rho c_p}\right)q^2 \tag{1.93}$$

and, using (1.76),

$$\omega_3 \cong -\,i(\kappa/\rho c_p)\,q^2 \tag{1.94}$$

which describe damped adiabatic sound and heat conduction in a compressible normal liquid. The diffusion equation (1.90a) leads to the viscosity mode

$$\omega_4 = -\,i(\eta/\rho)\,q^2. \tag{1.95}$$

In *molecular liquids* an energy exchange between the internal vibrational modes and the translational modes takes place. This exchange gives rise to a new relaxation mechanism and adds a new term, ξ, to the dissipative part Π' of the second Eq. (1.6). In the simplest case ξ just satisfies an equation of the form [compare Landau and Lifshitz (1959); Mountain (1968)]

$$\dot{\xi} + (1/\tau)\xi = -\lambda^2\nabla\cdot\mathbf{j} = \lambda^2\dot{\rho} \tag{1.91a}$$

which is manifestly not time reversal invariant so that ξ is indeed dissipative. When combined with Equations (1.91), where on the right of the first equation appears now the additional term $\nabla^2\xi$, Eq. (1.91a) gives rise to the dispersion relation

$$\frac{\omega^2}{q^2} = \mathcal{c}_1^2(1 - i\omega\tilde{\tau}_1) - c_1^2\vartheta\,\frac{iD_Tq^2}{\omega + iD_Tq^2} + \frac{\lambda^2\omega}{\omega + i/\tau} \tag{1.92a}$$

which differs from Eq. (1.92) in the last term. This term becomes dominant if the window condition

$$D_Tq^2 \ll \omega \ll \tilde{\tau}_1^{-1}$$

is satisfied. In this case the dispersion relation (1.92a) reduces to Eq. (78.8) of Landau and Lifshitz (1959) where $c_0 = \mathcal{c}_1$ and $c_\infty^2 = c_0^2 + \lambda^2$. The three solutions of this equation are first and zero sound (see the Introduction),

$$\omega_\pm = \begin{cases} \pm\,c_0q - \tfrac{1}{2}i\tau\lambda^2q^2; & \text{if } c_0q\tau \ll 1 \\ \pm c_\infty q - \tfrac{1}{2}i(\lambda^2/\tau c_\infty^2); & \text{if } c_0q\tau \gg 1 \end{cases} \tag{1.93a}$$

respectively, and a new diffusive mode

$$\omega_3 = \begin{cases} -\,i[(1/\tau) - \tau\lambda^2q^2]; & \text{if } c_0q\tau \ll 1 \\ -\,i[c_0^2/\tau c_\infty^2) + (c_0^4\lambda^2/\tau^3c_\infty^8q^2)]; & \text{if } c_0q\tau \gg 1 \end{cases} \tag{1.94a}$$

whose existence has been emphasized by Mountain (1966a, 1968). This *Mountain mode* which was also detected experimentally in Brillouin scattering (Gornall *et al.*, 1966; Mountain, 1966a) has recently been invoked in relation with the problem of a central peak in structural phase transitions [see Sec. III.D and Pytte (1973)].

F. Helium II in fine pores: fourth sound

The effect of the porous substrate is to exchange any amount of momentum with the normal phase of the fluid. Consequently, momentum conservation as expressed by the second equation (1.6) with ψ = const is invalid, and $\mathbf{v}_n = 0$. Thus both the number of variables and the number of equations is reduced by three. Energy conservation (1.7) still holds, and the entropy balance (1.14), with (1.18) and (1.26), becomes

$$(\rho s)^{\cdot} = (\kappa/T)\nabla^2 T. \tag{1.96}$$

In linear approximation Eq. (1.5), with (1.12), the first Eq. (1.22), and the second Eq. (1.16), takes the form

$$\dot{\mathbf{v}}_s = s\nabla T - (1/\rho)\nabla p + \rho_s\zeta_3\nabla\nabla\cdot\mathbf{v}_s, \tag{1.97}$$

and the first Eq. (1.6) with (1.9) simplifies to

$$\dot{\rho} + \rho_s\nabla\cdot\mathbf{v}_s = 0. \tag{1.98}$$

Replacing in Eq. (1.97) $\nabla\cdot\mathbf{v}_s$ from (1.98) and inserting Eq. (1.97) back into the time derivative of (1.98), we obtain

$$\ddot{\rho} - (\rho_s/\rho)\nabla^2 p + \rho_s s\nabla^2 T = \rho_s\zeta_3\nabla^2\dot{\rho}.$$

Eliminating from this equation δp and from Eq. (1.96) δs with the help of Eqs. (1.63) one finds, with (1.56) and the third equation (1.44),

$$\ddot{\rho} - \frac{\rho_s}{\rho}c_1^2\nabla^2\rho - \rho_s s\left(\frac{\beta}{\rho s} - 1\right)\nabla^2 T = \rho_s\zeta_3\nabla^2\dot{\rho}$$
$$\dot{T} - \frac{Ts}{\rho c_V}\left(\frac{\beta}{\rho s} - 1\right)\dot{\rho} = D_T\nabla^2 T. \tag{1.99}$$

These equations have the same form as Eqs. (1.91) for the normal fluid.

Insertion of plane waves, making use of Eq. (1.42), leads to the dispersion relation (Khalatnikov, 1965)

$$\frac{\omega^2}{q^2} = c_4^2(1 - i\omega\tau_4) - \frac{\rho_n}{\rho}c_2^2\left(\frac{\beta}{\rho s} - 1\right)^2\frac{iD_Tq^2}{\omega + iD_Tq^2}, \tag{1.100}$$

where (Rudnick and Shapiro, 1962)

$$c_4 = \left[\frac{\rho_s}{\rho} c_1^2 + \frac{\rho_n}{\rho}\left(\frac{\beta}{\rho s} - 1\right)^2 c_2^2\right]^{1/2} \tag{1.101}$$

and

$$\tau_4 = \rho_s \zeta_3 / c_4^2. \tag{1.102}$$

Here c_4 is the velocity of fourth sound, a terminology which is due to Atkins (1959). In analogy to Eq. (1.92), the three solutions of Eq. (1.100) for small q are

$$\omega_\pm \cong \pm c_4 q - \tfrac{1}{2} i \rho_s \zeta_3 q^2 \tag{1.103}$$

and, using Eq. (1.102)

$$\omega_3 \cong -i(\rho_s/\rho)(c_1^2/c_4^2) D_T q^2. \tag{1.104}$$

The fourth-sound mode (1.103), with $\beta = 0$, as well as the diffusive mode (1.104), were first proposed by Pellam (1948). The first experimental detection of fourth sound, as well as the exact formula (1.101), are due to Rudnick and Shapiro (1962), who later also made a precision measurement of c_4 in the temperature range from 1.1 K to T_λ (Shapiro and Rudnick, 1965). More recently Kriss and Rudnick (1970) have measured c_4 for various pore size distributions in order to deduce ρ_s/ρ as well as size effects.

In the limit $T \to 0$ the quantity $\beta/\rho s \cong -8.94$ is best calculated with the aid of the relation

$$\beta/\rho s = -\alpha c_1^2/s$$

which is obtained from Eq. (1.56) and the thermodynamic identity $\beta = -\alpha/\gamma$. Hence fourth sound approaches first sound in this limit and therefore (see Sec. I.C) may be identified as a Goldstone boson (Schneider and Meier, 1973). We notice that α changes sign at 1.14K, so that according to the above relation the same is true for $\beta/\rho s$. At a temperature of about 1.7K, $-\alpha c_1^2/s$ reaches a maximum of approximately 0.79, so that $(\beta/\rho s) - 1 \lesssim -0.21$. In the vicinity of this temperature the equations of motion (1.99) for $\delta\rho$ and δT are approximately decoupled and Eqs. (1.103) and (1.104) are fairly realistic.

II. CHARGED SUPERFLUIDS: SUPERCONDUCTORS

As mentioned in the Introduction, superconductors were the first systems for which a two-fluid idea was advanced by Gorter and Casimir (1934a,b). They adopted an idea due to Kronig according to which a superconductor is considered as a two-phase system. In the view of Gorter and Casimir the electron gas starts condensing into a "crystal phase" (the superfluid) below the critical temperature T_c. They determine the fraction of electrons in the "gas phase" (the normal fluid)

$$x = \rho_n/\rho = (T/T_c)^{2/(1-\alpha)}$$

from the condition $(\partial f/\partial x)_T = 0$, where $f = \epsilon_s + \epsilon_n - sT$ is the free energy, $\epsilon_s = -\beta(1-x)$ and $\epsilon_n = \frac{1}{2}\gamma T^2 x^\alpha$ are the energies of the two phases, $s = (\partial \epsilon_n/\partial T)_x$ is the entropy (which is carried by the "gas phase" alone), all per unit mass, and $T_c = (2\beta/\alpha\gamma)^{1/2}$. Gorter and Casimir (1934a) conclude that $\alpha = \frac{1}{2}$ if the critical field has the form $H_c = H_{c0}[1 - (T/T_c)^2]$ which follows from $\rho(s_n - s) = -(1/8\pi)\,\partial H_c^2/\partial T$ where $s_n = \gamma T$ is the entropy per unit mass of the normal state. They used this two-fluid idea to explain the thermodynamic properties only and did not extend it to hydrodynamic phenomena. This means that they did not introduce currents for the two phases.

Persistent currents, on the other hand, are the most striking hydrodynamic feature of superconductors. According to Landau's criterion for superfluidity (see the Introtion) this phenomenon requires a gap in the excitation spectrum (except for dirty superconductors or thin films in a parallel magnetic field). Daunt and Mendelssohn (1946) [see also Welker (1939); Landau (1941)] proposed such an energy gap of the order of 10^{-4} eV to explain their experimental result that the Thomson heat of a persistent current is zero. This means that reversible energy exchanges between superconducting electrons and the lattice are forbidden which is true if there is a gap in the excitation spectrum.

Daunt and Mendelssohn (1946) also discussed the analogy between superconductors and superfluid helium with respect to their thermodynamic and hydrodynamic properties. Theoretically this analogy is not so obvious, however, because London's (1938a,b) explanation of the condensed phase of superfluid helium as Bose condensation does not immediately apply to superconductors. It was Schafroth (1954) who pointed out that superconductivity could eventually be understood as Bose condensation of metastable resonance states of electron pairs.

This pair idea was taken up by Cooper (1956), who showed that with an attractive interaction between the electrons the Fermi sea is unstable against formation of quasibound pairs of electrons of opposite momentum and spin. Finally Bardeen, Cooper, and Schrieffer (1957) derived an energy gap by constructing a superconducting ground state containing a distribution of Cooper pairs at the Fermi energy which is self-consistently determined by the gap. This redistribution of the electrons into a state of lower energy which was independently derived by Bogoliubov (1958) and Valatin (1958) describes the condensed phase in close analogy to the case of superfluid helium.

Now the two-fluid description with condensed Cooper pairs and excited normal electrons also becomes analogous to the one for superfluid helium, and one can define currents for the two fluids. While the description of the motion of these currents in external electromagnetic fields was developed by F. London and H. London (1935) [see also London (1950); Ginzburg (1944)] the first application of the two-fluid model to the question of second sound in superconductors was made by Bardeen (1958). He gave the expression for the normal density ρ_n and pointed out that second sound would be difficult to observe in superconductors.

There are, however, two important differences between superconductors and superfluid helium. The first is the

charge of the electron which, through the long-range character of the Coulomb force, transforms the first-sound mode into the plasmon.

The second difference is the presence of the crystal lattice which breaks the continuous translation group. As a consequence, momentum conservation is valid only up to Umklapp processes. As in the case of dielectric or magnetic crystals (Sections III and IV), this gives rise to an intrinsic momentum dissipation from which results a window condition for the frequency domain in which second sound can propagate. It turns out that for superconductors the large value of the (Fermi) velocity of the electrons as compared to the second-sound velocity excludes a frequency window, except deep in the critical region, that is, extremely close to the transition temperature where two-fluid hydrodynamics is not valid anymore (see Sec. II.D).

A. Two-fluid hydrodynamics

In superconductors the condensate which develops below the critical temperature T_c is formed by Cooper pairs near the Fermi surface. The creation operator of a Cooper pair is (Josephson, 1962, 1965, 1969)

$$S^+ = \sum_k \alpha_k{}^+ \alpha_{-k}{}^+, \tag{2.1}$$

where $\alpha_{\pm k}{}^+$ creates a Bogoliubov quasiparticle of momentum $\pm\mathbf{k}$, $|\mathbf{k}| \sim k_F$, and spin $\pm\frac{1}{2}$. The condensate average is defined by a density operator ρ which commutes with the effective Bogoliubov Hamiltonian (compare Section IA)

$$\mathcal{H} - m\mu N = \sum_k E_k \alpha_k{}^+ \alpha_k. \tag{2.2}$$

where $E_k = (\epsilon_k{}^2 + \Delta^2(T))^{1/2}$, $\epsilon_k = (\mathbf{k}^2/2m) - m\mu$, $\Delta(T)$ is the energy gap, and μ the chemical potential per unit electronic mass. But ρ does not commute with the number operator $N = \sum_k \alpha_k{}^+ \alpha_k$. Since, on the other hand, $[\mathcal{H}, N] = 0$, Eqs. (1.1) again hold, which, together with the Heisenberg representation

$$S^+(t) = \exp(i\mathcal{H}t)\, S^+ \exp(-i\mathcal{H}t)$$

and the fact that S^+ creates a Cooper pair, then leads to a nonvanishing order parameter $\langle S^+ \rangle$ and to the following time evolution of the Gorkov pair function at zero separation (Gor'kov, 1958; Anderson *et al.*, 1965):

$$\begin{aligned} F_{\uparrow\downarrow}(r = 0, t) &= \langle S^+(t) \rangle = \langle S^+ \rangle \exp(+2m\mu t) \\ &= (n_c)^{1/2} \exp[+i\varphi(t)]. \end{aligned} \tag{2.3}$$

Here n_c is the density of Cooper pairs and

$$\varphi(t) = \varphi(0) + 2m\mu t. \tag{2.4}$$

Physically a density matrix ρ leading to Eq. (2.3) may be constructed by coupling the system to an external Cooper pair source via a tunneling Hamiltonian $\mathcal{H}' = MS_{\text{ext}}{}^+ S + H.c.$ which transfers Cooper pairs through a Josephson junction. Since from Eqs. (2.1) and (2.2) it follows that $[\mathcal{H}', \mathcal{H} - m\mu N] = 0$ this transfer requires no energy, then

$$\begin{aligned} \rho = \exp[(&F - \mathcal{H} - \mathcal{H}_{\text{ext}} - \mathcal{H}' + m\mu N \\ &+ m\mu_{\text{ext}} N_{\text{ext}})/k_B T] \end{aligned}$$

indeed commutes with $\mathcal{H}$ and $\mathcal{H}_{\text{ext}}$ but not with N and N_{ext} and Eq. (2.3) is just the Kubo formula for the linear response of the system to $\mathcal{H}'$.

In an external potential V this leads immediately to the ac Josephson current (Josephson, 1962, 1965, 1969). Indeed, to lowest order in $\mathcal{H}'$

$$N(t) = N + i \int_0^t dt' [\mathcal{H}'(t'), N],$$

where $\mathcal{H}'(t)$ is the time evolution with $\mathcal{H} + \mathcal{H}_{\text{ext}} - m\mu N - m\mu_{\text{ext}} N_{\text{ext}}$ (interaction representation with respect to $\mathcal{H}'$). The Josephson current is

$$\langle J(t) \rangle = e\langle \dot{N}(t) \rangle = 2Re[\exp(-i2eVt) K],$$

where

$$K = ieM\langle S^+ [S, N] \rangle,$$

and

$$V = (m/e)(\mu_{\text{ext}} - \mu)$$

is the applied voltage, e being the true electron charge.

In a local description the phase φ defined in Eq. (2.3) determines the superfluid velocity

$$\mathbf{v}_s = -(1/2m)\nabla\varphi \tag{2.5}$$

so that from Eq. (2.4) the equation of motion analogous to (1.5) (London, 1950; Ginzburg, 1944),

$$d\mathbf{v}_s/dt = \dot{\mathbf{v}}_s + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = -\nabla(\mu + \mu'), \tag{2.6}$$

is obtained where μ' is the dissipative part of μ.

Apart from the factor 2 in Eq. (2.5) the analogy with superfluid helium is complete: The electron fluid consists of the superfluid phase of Cooper pairs moving with velocity $\mathbf{v}_s$ and of the normal phase of single quasiparticle excitations moving with drift velocity $\mathbf{v}_n$. The excitation mechanism consists in breaking a Cooper pair by supplying a minimum energy 2Δ.

In the local frame moving with $\mathbf{v}_s$ the total current is carried by the normal phase alone, $\mathbf{j}_{e0} = \rho_n(\mathbf{v}_n - \mathbf{v}_s)$, defining the excitation mass density ρ_n (Bardeen, 1958). In the frame of the crystal lattice we have as in superfluid helium

$$\mathbf{j}_e = \mathbf{j}_{e0} + \rho_e \mathbf{v}_s = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s, \tag{2.7}$$

where

$$\rho_s = \rho_e - \rho_n \tag{2.8}$$

is the superfluid mass density.

In the presence of electromagnetic fields the normal current obeys Ohm's law

$$\mathbf{j}_n = \rho_n \mathbf{v}_n = (m/e)\sigma_e \mathbf{E} \tag{2.7a}$$

while the supercurrent $\mathbf{j}_s = \rho_s \mathbf{v}_s$ satisfies the London equation (F. London and H. London, 1935). The latter follows most easily from the analogy mentioned in Sec. I.A between a superconductor in a magnetic field $\mathbf{H}$ and a neutral superfluid rotating with angular velocity $\boldsymbol{\omega}$. Assuming $\mathbf{E} = 0$ so that, according to Eq. (2.7a), $\mathbf{j}_n = 0$, the analogue of the normal velocity $\mathbf{v}_n = \boldsymbol{\omega} \times \mathbf{r}$ in the neutral superfluid is now

$$\mathbf{v}_n = (e/mc)\mathbf{A} = (e/2mc)(\mathbf{H} \times \mathbf{r}).$$

The response to this $\mathbf{v}_n$ is the paramagnetic current

$$\mathbf{j}_{\text{para}} = \rho_n \mathbf{v}_n$$

which defines the normal density ρ_n in the general microscopic form [Eq. (1.11)] of Pines (1965) and Noziéres (1966). The supercurrent response is

$$\mathbf{j}_s = \mathbf{j}_{\text{para}} + \mathbf{j}_{\text{dia}}$$

where

$$\mathbf{j}_{\text{dia}} = -\rho_e(e/mc)\mathbf{A}$$

is the diamagnetic current (F. London and H. London, 1935; Nozières, 1966). We find by insertion, making use of Eq. (2.8),

$$\mathbf{j}_s = \rho_s \mathbf{v}_s = -\rho_s(e/mc)\mathbf{A}, \tag{2.7b}$$

the curl of which is the London equation.

Since $\mathbf{E} = 0$ we have in addition $\mathbf{j}_s = \mathbf{j}_e$ so that from Maxwell's equations for the stationary case

$$\nabla \times \mathbf{H} = (4\pi/c)(e/m)\mathbf{j}_e; \qquad \nabla \cdot \mathbf{j}_e = 0,$$

we deduce

$$\nabla^2 \mathbf{j}_e = (4\pi e^2 \rho_s/m^2c^2)\mathbf{j}_e.$$

This equation describes the Meissner effect (Meissner and Ochsenfeld, 1933): The current is contained in a layer of width

$$\lambda_L = (m^2c^2/4\pi e^2\rho_s)^{1/2}$$

below the surface of the superconductor, λ_L is the London penetration depth (F. London and H. London, 1935; London, 1950), which varies as $\rho_s^{-1/2}$, and hence becomes infinite at T_c while at $T = 0$ it is of the order of 10^{-5} cm.

Due to the electric charge, the thermodynamic relation analogous to Eq. (1.12),

$$d\mu = -s\,dT + (1/\rho_e)\,dp_e + (e/m)\,d\phi - \tfrac{1}{2}(\rho_n/\rho_e)\,d(\mathbf{v}_n - \mathbf{v}_s)^2 \tag{2.9}$$

now contains a potential $(e/m)\phi$ due to the fluctuating electric field $\mathbf{E}$ which is determined by the Poisson equation

$$-\nabla^2\phi = \nabla \cdot \mathbf{E} = 4\pi(e/m)\delta\rho_e. \tag{2.10}$$

Here s is the entropy per unit electronic mass, p_e the pressure of the electron fluid, and $\delta\rho_e = \rho_e - (\rho_e)_{eq}$ the electronic density fluctuation.

Equation (2.6) determines the motion of the superfluid phase. The remaining hydrodynamic equations are analogous to Eqs. (1.6) and (1.7)

$$\begin{aligned} &\dot{\rho}_e + \nabla \cdot (\mathbf{j}_e + \mathbf{j}_e') = 0, \\ &\dot{\mathbf{j}}_e + \nabla(\Pi_e + \Pi_e') = (e/m)\rho_e \mathbf{E} - (1/\tau_s)\rho_n \mathbf{v}_n, \\ &(\rho_e \epsilon)^{\cdot} + \nabla \cdot (\mathbf{J}_\epsilon + \mathbf{J}_\epsilon') = 0, \end{aligned} \tag{2.11}$$

where $\mathbf{j}_e', \Pi_e'$, and $\mathbf{J}_\epsilon'$ are, respectively, the dissipative parts of the momentum density $\mathbf{j}_e$, the momentum flux Π_e, and the energy flux $\mathbf{J}_\epsilon$, and ϵ is the energy per unit electronic mass.

The first of these equations (2.11) expresses charge conservation and the second momentum balance. Due to the terms on the right-hand side the latter is not a conservation law: Electron momentum changes reversibly due to the Lorentz force $(e/m)\mathbf{E}$ and dissipates due to Umklapp processes or (elastic) impurity scattering. This dissipation is here described by the relaxation time τ_s in the superconducting state [see Eq. (2.83) below]. In superfluid helium $\tau_s = \infty$ since momentum is strictly conserved, and the Lorentz force is replaced by $-\nabla\psi$ [see the second Eq. (1.6)]. If, in particular, $\mathbf{E}$ is a constant external field, a stationary and homogeneous situation develops so that $\dot{\mathbf{j}}_e = 0$ and $\nabla(\Pi_e + \Pi_e') = 0$, and the second Eq. (2.11) describes Ohm's law (2.7a) with $\sigma_e = \tau_s(e^2/m^2)\rho_e$. In what follows we omit external fields and consider $\mathbf{E}$ to be the fluctuating field determined by Eq. (2.10).

In an isotropic approximation the momentum flux is given by the analogue of Eq. (1.13),

$$\Pi_e = p_e 1 + \rho_n \mathbf{v}_n \otimes \mathbf{v}_n + \rho_s \mathbf{v}_s \otimes \mathbf{v}_s. \tag{2.12}$$

Note that in writing Eqs. (2.7), (2.8), and (2.12) we have made use of the same Galilean invariance argument as in the case of superfluid helium. While in a crystal this is of course not a rigorous argument, it is still justified for wave numbers q which are close to the center of the Brillouin zone. For such q's the excitations do not notice the periodicity of the lattice and a hydrodynamic description is justified. In particular, the velocities are defined by Eq. (2.5) and, for free quasiparticles, by Eq. (2.19) below. With this last equation the normal parts of the hydrodynamic quantities could actually be derived explicitly, in complete analogy with the derivation given in Sec. III.A for the phonons in dielectric crystals. An example is given by Eq. (2.20) below or, in the general case, by Eq. (1.11) which was discussed earlier in this section.

The third equation (2.11), finally, expresses energy conservation, energy transfer to the phonons being neglected. It is useful again to convert energy conservation into an

entropy balance equation. This can be done with the aid of the thermodynamic relation analogous to (1.15)

$$d(\rho_e\epsilon) = \nu\, d\rho_e + \boldsymbol{\lambda}\cdot d\mathbf{v}_s + \boldsymbol{\omega}\cdot d\mathbf{j}_e + Td(\rho_e s)$$

where, in analogy to Eq. (1.16)

$$\begin{aligned}\nu &= \mu + \tfrac{1}{2}(\mathbf{v}_n - \mathbf{v}_s)^2 - \tfrac{1}{2}\mathbf{v}_n^2,\\ \boldsymbol{\lambda} &= \rho_s(\mathbf{v}_s - \mathbf{v}_n) = \mathbf{j}_e - \rho_e\mathbf{v}_n\\ \boldsymbol{\omega} &= \mathbf{v}_n.\end{aligned} \tag{2.13}$$

It follows that

$$T(\rho_e s)^{\cdot} = (\rho_e\epsilon)^{\cdot} - \nu\dot{\rho}_e - \boldsymbol{\lambda}\cdot\dot{\mathbf{v}}_s - \boldsymbol{\omega}\cdot\dot{\mathbf{j}}e.$$

Substituting here the time derivatives from Eqs. (2.6) and (2.11), making use of Eqs. (2.12) and (2.13), and eliminating ∇p_e with the aid of Eq. (2.9), we find, to second order in the velocities,

$$\begin{aligned}(\rho s)^{\cdot} = {} & -\nabla\cdot[(1/T)(\mathbf{J}_\epsilon + \mathbf{J}_\epsilon' - \mu\mathbf{j}_e - \mu\mathbf{j}_e' - \mu'\boldsymbol{\lambda}\\ & - \Pi_e'\mathbf{v}_n)] + (\mathbf{J}_\epsilon + \mathbf{J}_\epsilon' - \mu\mathbf{j}_e - \mu\mathbf{j}_e' - T\rho_e s\mathbf{v}_n\\ & - \mu'\boldsymbol{\lambda} - \Pi_e'\mathbf{v}_n)\cdot\nabla(1/T) - (1/T)[\mathbf{j}_e'\cdot\nabla\mu + \mu'\nabla\cdot\boldsymbol{\lambda}\\ & + (\Pi_e'\nabla)\cdot\mathbf{v}_n] + (\rho_n/T\tau_s)\mathbf{v}_n^2.\end{aligned}$$

This equation, which is analogous to Eq. (1.17), can again be written in the form (1.14),

$$(\rho_e s)^{\cdot} + \nabla\cdot(\mathbf{J}_s + \mathbf{J}_s') = \sigma. \tag{2.14}$$

Identification of the nondissipative and dissipative terms leads to the entropy fluxes

$$\begin{aligned}\mathbf{J}_s &= (1/T)(\mathbf{J}_\epsilon - \mu\mathbf{j}_e) = \rho_e s\mathbf{v}_n\\ \mathbf{J}_s' &= (1/T)(\mathbf{J}_\epsilon' - \mu\mathbf{j}_e' - \mu'\boldsymbol{\lambda} - \Pi_e'\mathbf{v}_n)\end{aligned} \tag{2.15}$$

and to the entropy production density

$$\begin{aligned}\sigma = {} & T\mathbf{J}_s'\cdot\nabla(1/T) - (1/T)[\mathbf{j}_e'\cdot\nabla\mu + \mu'\nabla\cdot\boldsymbol{\lambda}\\ & + (\Pi_e'\nabla)\cdot\mathbf{v}_n] + (\rho_n/T\tau_s)\mathbf{v}_n^2\end{aligned} \tag{2.16}$$

in analogy to Eqs. (1.19), (1.20), and (1.21).

Turning now to the dissipative fluxes μ', $\mathbf{j}_e'$, Π_e', and $\mathbf{J}_s'$, we note that these are, to lowest order in the fluctuations and in the spatial derivatives, linear functions of the gradients of the coefficients (2.13) of the expression for $d(\rho_e\epsilon)$, that is, of $\nabla\otimes\boldsymbol{\lambda}$, $\nabla\nu$, $\nabla\otimes\boldsymbol{\omega}$, and ∇T. Under time reversal these dissipative fluxes must transform with opposite sign as compared to the corresponding nondissipative fluxes, in order to lead to irreversibility. In an isotropic approximation we then have, in analogy to Eq. (1.22) and taking into account the Onsager relations (1.23),

$$\begin{aligned}\mathbf{j}_e' &= -D_1\nabla\mu - D_2\nabla T\\ \mu' &= -\zeta_{e3}\nabla\cdot\boldsymbol{\lambda} - \zeta_{e1}\nabla\cdot\mathbf{v}_n\\ \Pi_{eij} &= -\delta_{ij}(\zeta_{e1}\nabla\cdot\boldsymbol{\lambda} + \zeta_{e2}\nabla\cdot\mathbf{v}_n)\\ &\quad - \eta_e(\nabla_i v_{nj} + \nabla_j v_{ni} - \tfrac{2}{3}\delta_{ij}\nabla\cdot\mathbf{v}_n)\\ \mathbf{J}_s' &= -D_2\nabla\mu - (\kappa_{es}/T)\nabla T.\end{aligned} \tag{2.17}$$

Here η_e and ζ_{e1}, ζ_{e2}, ζ_{e3} are viscosities of the electron fluid and κ_{es} is the electronic heat conductivity in the superconductive state. Inserting Eq. (2.17) into (2.16) we find an expression analogous to Eq. (1.24) but with the additional term $(\rho_n/\tau_s)\mathbf{v}_n^2$. Its positive definiteness implies relations analogous to Eq. (1.25). As in (1.26) we will put $D_1 = D_2 = 0$.

In the approximation by free quasiparticles as defined by Eq. (2.2) the normal current density is

$$\mathbf{j}_n = (2/V)\sum_{\mathbf{k}}\mathbf{k}f_{\mathbf{k}} \tag{2.18}$$

where the factor 2 accounts for the spin states and

$$f_{\mathbf{k}} = \{\exp[(E_k - \mathbf{v}_n\cdot\mathbf{k})/k_BT] + 1\}^{-1} \tag{2.19}$$

is the drifting local equilibrium distribution function. The excitation mass density ρ_n is the coefficient of $\mathbf{v}_n$ in the expansion of Eq. (2.18) and is found to be (Bardeen, 1958)

$$\rho_n = (1/3k_BT)(2/V)\sum_{\mathbf{k}}\mathbf{k}^2 f_k^0(1 - f_k^0), \tag{2.20}$$

where $f_k^0 = [\exp(E_k/k_BT) + 1]^{-1}$. For $m\mu \gg k_BT$, one finds

$$\frac{\rho_n}{\rho_e} = 2\int_0^\infty \frac{e^y\,dx}{(e^y+1)^2} \tag{2.21}$$

where $y \equiv E_k/k_BT = \{x^2 + [\Delta(T)/k_BT]^2\}^{1/2}$. It follows from Eq. (2.21) that $\rho_s = \rho_e - \rho_n \geq 0$, and the equality sign holds if and only if $\Delta = 0$, *i.e.*, above T_c. The entropy s and specific heat in the superconductive state c_{Vs} are given by [e.g., Rickayzen (1965)]

$$\begin{aligned}s = {} & -(k_B/\rho_e)(2/V)\sum_{\mathbf{k}}\{(1 - f_k^0)\ln(1 - f_k^0)\\ & + f_k^0\ln f_k^0\}\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}c_{Vs} = T(\partial s/\partial T)_V = {} & (1/k_BT^2)(2/V)\sum_{\mathbf{k}}(E_k^2 - T\Delta\Delta')\\ & \times f_k^0(1 - f_k^0).\end{aligned} \tag{2.23}$$

For $m\mu \gg k_BT$ this becomes

$$s = \frac{6k_B^2T}{k_F^2}\int_0^\infty\left\{\frac{y}{e^y+1} + \ln(1 + e^{-y})\right\}dx \tag{2.24}$$

and

$$c_{Vs} = \frac{6k_B^2T}{k_F^2}\int_0^\infty\frac{e^y}{(e^y+1)^2}\left(y^2 - \frac{\Delta\Delta'}{k_B^2T}\right)dx. \tag{2.25}$$

Here the second term in the last bracket produces the discontinuity of c_V at $T = T_c$ since $(\Delta\Delta'/k_B^2T)_{T\to T_c} = -4.59$.

The two-fluid equations (2.6) and (2.17) have been discussed by Ginzburg (1961). A derivation from microscopic theory using the Gorkov formulation was given by Stephen (1965). [See also Bardeen and Schrieffer (1961); Stephen

and Suhl (1965); Meservey (1965); Jakeman and Pike (1966).] The existence of a finite coherence length ξ for the Cooper pairs which is of the order of 10^{-4} cm sets a lower limit for the mean free path λ_{eq} responsible for local thermal equilibrium (Bardeen, 1958),

$$q\lambda_{eq} \ll 1, \qquad \lambda_{eq} \gtrsim \xi. \tag{2.26}$$

It has been shown by Ginzburg (1961) that the Coulombic electron–electron interaction is far too slow to bring the electrons to thermal equilibrium. The corresponding mean free path $l_{e-e} \propto T^{-2}$ at $T = 10$ K is $l_{e-e} \cong 0.3$ cm (Ginzburg, 1961; Ginzburg and Silin, 1955). Hence the electrons thermalize either via their interaction with the phonons (for pure samples and at moderately low temperatures) or via their interaction with nonmagnetic impurities (in the domain of the residual resistance). In the former case, electrons and phonons have the same temperature and drift velocity so that they form a two-component fluid. This system might have interesting hydrodynamic properties which, however, will not be analyzed further here. Anyhow, according to Ginzburg and Silin (1955) the electron–phonon mean free path $l_{e-p} \propto T^{-5}$ at $T = 10$ K is $l_{e-p} \simeq 0.1$ cm, so that this case is not very realistic [see also Fig. 7.3 of Rickayzen (1965)].

In the case of thermalization by impurities the decoupling of electrons and phonons is a reasonable approximation. The electron-impurity mean free path is, for a pure specimen, $l_{e-i} \gtrsim 10^{-3}$ cm (Ginzburg and Silin, 1955). For not too pure specimens the drift velocity $\mathbf{v}_n$ may relax to zero so quickly that a situation analogous to superfluid helium in fine pores develops (see Sec. I.F). In the following sections this decoupled electronic two-fluid system is investigated further.

B. Undamped hydrodynamic modes

We first discuss the hydrodynamic equations (2.6), (2.11), and (2.14) in the absence of dissipation. From the first two Eqs. (2.11) combined with Eqs. (2.10) and (2.12) we obtain

$$\ddot{\rho}_e - \nabla^2 p_e + \omega_{p1}^2 \delta\rho_e = 0, \tag{2.27}$$

where

$$\omega_{p1} = [4\pi(e^2/m^2)\rho_e]^{1/2} \tag{2.28}$$

is the *plasma frequency*. Combination of the second Eq. (2.11) with (2.6) and making use of Eqs. (2.9), (2.7), (2.8), and (2.12) leads to

$$\rho_n(\dot{\mathbf{v}}_n - \dot{\mathbf{v}}_s) + \rho_e s \nabla T = 0. \tag{2.29}$$

From the first Eq. (2.11) combined with (2.14) and using Eqs. (2.15), (2.7), and (2.8) we obtain

$$\rho_e \dot{s} + \rho_s s \nabla\cdot(\mathbf{v}_n - \mathbf{v}_s) = 0. \tag{2.30}$$

Elimination of $\mathbf{v}_n - \mathbf{v}_s$ from the last two equations yields

$$\ddot{s} - (\rho_s/\rho_n) s^2 \nabla^2 T = 0. \tag{2.31}$$

Substituting the variables p_e and s with the help of the equations

$$\begin{aligned} \delta p_e &= (1/\rho_e\gamma_e)\delta\rho_e + \beta_e \delta T, \\ \delta s &= -(\beta_e/\rho_e^2)\delta\rho_e + (c_{Vs}/T)\delta T, \end{aligned} \tag{2.32}$$

we obtain the coupled wave equations

$$\begin{aligned} \ddot{\rho}_e - c_{e1}^2\nabla^2\rho_e + \omega_{p1}^2\delta\rho_e &= \beta_e\nabla^2 T, \\ \ddot{T} - c_{e2}^2\nabla^2 T &= (T\beta_e/\rho_e^2 c_{Vs})\ddot{\rho}_e, \end{aligned} \tag{2.33}$$

where the velocities of propagation

$$\begin{aligned} c_{e1} &= (\rho_e\gamma_e)^{-1/2} \\ c_{e2} &= [(\rho_s/\rho_n)(Ts^2/c_{Vs})]^{1/2} \end{aligned} \tag{2.34}$$

are the analogues of the first- and second-sound velocities [Eqs. (1.56) and (1.42)] in superfluid helium. Note that c_{e1} is the *isothermal* sound velocity in the electron fluid and not in the host crystal and that, therefore, it is not a directly measurable quantity.

Insertion of plane waves $\propto \exp[i(\mathbf{q}\cdot\mathbf{r} - \omega t)]$ into Eqs. (2.33) leads to the secular determinant

$$(\omega^2 - c_{e1}^2q^2 - \omega_{p1}^2)(\omega^2 - c_{e2}^2q^2) - \frac{T\beta_e^2}{\rho_e^2 c_{Vs}}\omega^2q^2 = 0.$$

Introducing the analogue of the *adiabatic* first-sound velocity (1.73) by

$$\tilde{c}_{e1}^2 = c_{e1}^2(1 + \vartheta_e), \tag{2.35}$$

where

$$\vartheta_e = \beta_e^2 T/\rho_e^2 c_{Vs} c_{e1}^2, \tag{2.36}$$

the above dispersion law can be written as

$$(\omega^2 - \omega_{p1}^2 - \tilde{c}_{e1}^2q^2)(\omega^2 - c_{e2}^2q^2) - c_{e1}^2c_{e2}^2\vartheta_e q^4 = 0. \tag{2.37}$$

Equation (2.37) has the solutions

$$\begin{aligned} \omega_1^2 &= \omega_{p1}^2 + \tilde{c}_{e1}^2q^2 + (c_{e1}^2c_{e2}^2\vartheta_e/\omega_{p1}^2)q^4 + O(q^6) \\ \omega_2^2 &= c_{e2}^2q^2 - (c_{e1}^2c_{e2}^2\vartheta_e/\omega_{p1}^2)q^4 + O(q^6) \end{aligned} \tag{2.38}$$

corresponding to the *plasmon* and *second-sound* excitations which for any q are well separated in energy.

As has been shown by Anderson (1958) and, more generally, by Ambegaokar and Kadanoff (1961), the collective plasmon mode is essential in order to have a gauge invariant and charge conserving microscopic formalism. It is well known that the original BCS theory (Bardeen *et al.*, 1957) gives a non-gauge invariant Meissner effect. Gauge invariance is restored by including in the electromagnetic fields acting on the quasiparticles the effect of the motion of the Cooper pairs on these fields, which then become self-consistent.

This restoration of gauge invariance in the Meissner effect by the plasmon mode can also be seen in our hydrodynamic approach. We start from the stationary situation described in the previous section, where

$$\mathbf{E} = 0, \qquad \phi = 0$$

$$\mathbf{H} = \nabla \times \mathbf{A}, \qquad \dot{\mathbf{A}} = 0$$

and where the currents (2.7a) and (2.7b) imply that $\mathbf{v}_n = 0$ but $\mathbf{v}_s \neq 0$.

A plane-wave plasmon excitation then introduces a density variation $\delta\rho_e$ which according to Eq. (2.10) gives rise to the field disturbances

$$\delta\mathbf{E} = -\nabla\delta\phi = -(1/c)\delta\dot{\mathbf{A}}' = -i(\hat{q}/q)4\pi(e/m)\delta\rho_e, \tag{2.10a}$$

where $\hat{q} = \mathbf{q}/q$, and

$$\delta\mathbf{H} = \nabla \times \delta\mathbf{A}' = 0.$$

Here we have introduced two gauges; in the unprimed gauge $\delta\mathbf{A} = 0$, while in the primed gauge $\delta\phi' = 0$. From Eqs. (2.6) and (2.9) we deduce for the supercurrent disturbance in linear approximation, leaving out dissipative terms,

$$\delta\dot{\mathbf{j}}_s = \rho_s\delta\dot{\mathbf{v}}_s = -(\rho_s/\rho_e)\nabla\delta p_e - \rho_s(e/m)\nabla\delta\phi.$$

Introducing the simplifying assumption $\beta_e = 0$ so that the plasmon does not couple to the thermal mode and $\delta T = 0$, the last equation can be written, with Eq. (2.28), the first Eq. (2.32), the first Eq. (2.34), and Eq. (2.10a) above,

$$\delta\dot{\mathbf{j}}_s = -\rho_s\frac{e}{mc}\left(1 - \frac{{c_{e1}}^2}{{\omega_{p1}}^2}\nabla^2\right)\delta\dot{\mathbf{A}}'.$$

For small q such that $c_{e1}q \ll \omega_{p1}$ we obtain with Eqs. (2.10a) and (2.28) and with $\omega = \omega_{p1}$

$$\delta\mathbf{j}_s = \rho_s\delta\mathbf{v}_s = -\rho_s\frac{e}{mc}\delta\mathbf{A}' = \frac{\rho_s}{\rho_e}\omega_{p1}\frac{\hat{q}}{q}\delta\rho_e \tag{2.10b}$$

which has the same form as Eq. (2.7b).

On the other hand, with $\delta T = 0$, $\delta s = 0$ it follows from Eqs. (2.29) and (2.30) that

$$\delta\mathbf{v}_n = \delta\mathbf{v}_s$$

and hence, making use of Eqs. (2.7), (2.8), (2.10a), (2.10b), (2.28), and $\omega = \omega_{p1}$,

$$\delta\mathbf{j}_e = \omega_{p1}(\hat{q}/q)\delta\rho_e = -(m/4\pi e)\delta\dot{\mathbf{E}}.$$

The last equation and (2.10a) imply that the Maxwell equation

$$\nabla \times (\mathbf{H} + \delta\mathbf{H}) = \frac{1}{c}\delta\dot{\mathbf{E}} + \frac{4\pi}{c}\frac{e}{m}(\mathbf{j}_e + \delta\mathbf{j}_e)$$

reduces to the unperturbed form of the last section,

$$\nabla \times \mathbf{H} = (4\pi/c)(e/m)\mathbf{j}_e.$$

This means that the Meissner effect is unperturbed by the plasmon. On the other hand, we see from Eq. (2.10a) that

$$\chi = \frac{ic}{\omega}\delta\phi = -i\frac{\hat{q}}{q}\cdot\delta\mathbf{A}' = \frac{ic}{\omega q^2}4\pi\frac{e}{m}\delta\rho_e$$

is the gauge field transforming from unprimed to primed potentials. Since $\omega = \omega_{p1}$ and q is arbitrarily small, χ can be made of arbitrary size by varying amplitude and wave number of the plasmon. We can say, therefore, that the plasmon "restores" gauge invariance in the Meissner effect.

The plasmon also restores the broken gauge symmetry. Indeed, it follows from Eq. (2.5) and Eq. (2.10b) above that the phase disturbance at $\mathbf{r} = 0$ due to a plasmon wave

$$\delta\rho_e = |\,\delta\rho_e\,|\exp[i(\mathbf{q}\cdot\mathbf{r} - \omega_{p1}t - \delta_{p1})]$$

is

$$\delta\varphi(t) = \mathrm{Re}[i(2m\omega_{p1}/\rho_e q^2)\delta\rho_e] = \frac{2m\omega_{p1}}{\rho_e q^2}|\,\delta\rho_e\,|\sin(\omega_{p1}t + \delta_{p1}).$$

This gives rise to a fluctuation of the order parameter (2.3) analogous to Eq. (1.48),

$$(\delta\langle S^+\rangle)^2 = |\,\langle S^+\rangle_{\delta\varphi} - \langle S^+\rangle\,|^2 = 2n_c[1 - \cos\delta\varphi(t)],$$

where $\langle S^+\rangle_{\delta\varphi}$ is the average in the presence of the plasmon wave. As in Sec. I.B and I.C, a sufficiently small q gives rise to a $\delta\varphi(t)$ with amplitude larger than π, so that the amplitude of the fluctuation $\delta\langle S^+\rangle$ becomes of the same magnitude as the order parameter itself. Here, however, the fluctuation oscillates with the plasma frequency. This means that the order parameter is essentially zero and gauge symmetry breaking in the sense $[\rho, N] \neq 0$ of Eq. (1.1) is essentially healed. The large value of ω_{p1} guarantees a collissionless situation, $\omega_{p1}\tau_{eq} \gg 1$. Therefore the plasmon at $q \to 0$ is also the Goldstone boson associated with gauge symmetry breaking in the isothermal superconductor (see the Introduction). The fact that the plasma frequency is nonzero at $q = 0$ has been shown by Lange (1965) to be irrelevant for the nonrelativistic Goldstone theorem.

With respect to second sound, the conditions reigning in superconductors are very unfavorable. In order to understand the reasons we have to reinstate the dissipative terms in the hydrodynamic equations.

C. Inclusion of dissipation

The dissipative terms introduce the following modification of Eqs. (2.27), (2.29), and (2.30):

$$\ddot{\rho}_e - \nabla^2 p_e + \omega_{p1}{}^2\delta\rho_e = \rho_s\zeta_{e1}\nabla^2\nabla\cdot(\mathbf{v}_n - \mathbf{v}_s) - (\zeta_{e2} + \tfrac{4}{3}\eta_e)\nabla^2\nabla\cdot\mathbf{v}_n + (\rho_n/\tau_s)\nabla\cdot\mathbf{v}_n \tag{2.39}$$

$$\rho_n(\dot{\mathbf{v}}_n - \dot{\mathbf{v}}_s) + \rho_e s\nabla T = \rho_s(-\zeta_{e1} + \rho_e\zeta_{e3}) \times\nabla\nabla\cdot(\mathbf{v}_n - \mathbf{v}_s) + (-\rho_e\zeta_{e1} + \zeta_{e2} + \tfrac{1}{3}\eta_e)\nabla\nabla\cdot\mathbf{v}_n + \eta_e\nabla^2\mathbf{v}_n - (\rho_n/\tau_s)\mathbf{v}_n, \tag{2.40}$$

and

$$\rho_e\dot{s} + \rho_s s\nabla\cdot(\mathbf{v}_n - \mathbf{v}_s) = (\kappa_{es}/T)\nabla^2 T, \tag{2.41}$$

where we have made use of Eqs. (2.17) with $D_1 = D_2 = 0$, and have neglected the second-order term σ in Eq. (2.14). In order to eliminate the velocities, these three equations have to be supplemented by Eq. (2.14) which, with the aid of the first Eq. (2.15) and the last Eq. (2.17) with $D_2 = 0$, can be written in the form

$$\nabla\cdot\mathbf{v}_n = -(\dot{s}/s) - (\dot{\rho}_e/\rho_e) + (\kappa_{es}/\rho_e sT)\nabla^2 T. \tag{2.42}$$

Insertion of Eqs. (2.41), (2.42), (2.32), and (2.34) into (2.39) leads to the analogue of Eq. (1.64),

$$\ddot{\rho}_e + \tilde{\tau}_s^{-1}\dot{\rho}_e - c_{e1}^2\nabla^2\rho_e + \omega_{p1}^2\delta\rho_e + (\rho_n c_{Vs}/sT\tau_s)\dot{T} - \beta_e\nabla^2 T = c_{e1}^2\tau_{e1}\nabla^2\dot{\rho}_e + \beta_e(\tau_{e1}'\nabla^2\dot{T} - \lambda_{e1}'^2\nabla^4 T). \tag{2.43}$$

Here

$$\tilde{\tau}_s^{-1} = (\rho_n/\rho_e)(1 - \beta_e/\rho_e s)\tau_s^{-1}, \tag{2.44}$$

$$\tilde{\beta}_e = \beta_e + \rho_n\kappa_{es}/\rho_e sT\tau_s, \tag{2.45}$$

and, in analogy to Eqs. (1.66) to (1.69),

$$\tau_{e1} = (1/\rho_e c_{e1}^2)(\zeta_{e2} + \tfrac{4}{3}\eta_e) - \xi_e\tau_{e1}',$$
$$\tau_{e1}' = (c_{Vs}/\beta_e sT)(-\rho_e\zeta_{e1} + \zeta_{e2} + \tfrac{4}{3}\eta_e),$$
$$\lambda_{e1}'^2 = (\kappa_{es}/\rho_e c_{Vs})\tau_{e1}' = D_{sT}\tau_{e1}'. \tag{2.46}$$

D_{sT} is the electronic thermal diffusion constant in the superconducting phase. Combining the time derivative of Eq. (2.41) with the divergence of Eq. (2.40) we obtain, inserting Eqs. (2.41), (2.42), (2.32), (2.34), and (2.36), the analogue of Eq. (1.65),

$$\ddot{T} + \tau_J^{-1}\dot{T} - \tilde{c}_{e2}^2\nabla^2 T + \frac{c_{e2}^2}{\rho_e s\tilde{\tau}_s}\dot{\rho}_e - \frac{c_{e1}^2\vartheta_e}{\beta_e}\ddot{\rho}_e = c_{e2}^2(\tau_{e2}\nabla^2\dot{T} - \lambda_{e2}^2\nabla^4 T) - (c_{e1}^2 c_{e2}^2\vartheta_e/\beta_e)\tau_{e2}'\nabla^2\dot{\rho}_e. \tag{2.47}$$

Here

$$\tilde{c}_{e2}^2 = c_{e2}^2 + (\rho_s/\rho_e)(D_{sT}/\tau_s), \tag{2.48}$$

$$\tau_J = (\rho_e/\rho_s)\tau_s, \tag{2.49}$$

and, in analogy to Eqs. (1.44) and (1.67),

$$\tau_{e2} = (1/c_{e2}^2)\{D_{sT} + (\rho_s/\rho_n\rho_e)[-2\rho_e\zeta_{e1} + \zeta_{e2} + \rho_e^2\zeta_{e3} + \tfrac{4}{3}\eta_e]\},$$
$$\lambda_{e2}^2 = D_{eT}(\tau_{e2} - D_{sT}/c_{e2}^2),$$
$$\tau_{e2}' = \tau_{e2} - (D_{sT}/c_{e2}^2) - \tau_{e1}'. \tag{2.50}$$

In plane-wave representation $\sim\exp[i(\mathbf{q}\cdot\mathbf{r} - \omega t)]$ Eqs. (2.43) and (2.47) lead to a complicated secular equation. Here, however, we are interested only in second-sound-type excitations,

$$\omega \cong \tilde{c}_{e2}q \ll \omega_{p1}. \tag{2.51}$$

If in addition

$$\omega_{p1}\tau_s \gg 1 \tag{2.52}$$

terms proportional to $\omega^2\tau_s^{-2}$, $\omega q^2\tau_s^{-1}$, $\omega^3\tau_s^{-1}$, etc., are negligible as compared to terms proportional to $\omega^2\omega_{p1}^2$, $q^2\omega_{p1}^2$, $\omega^2\omega_{p1}^2$, etc., respectively, in the secular equation. This amounts to putting $\delta\rho_e \cong 0$, i.e., the density is not coupled to a second-sound-type excitation characterized by Eq. (2.51). Then Eq. (2.47) leads to the dispersion relation

$$\frac{\omega^2}{q^2} = \tilde{c}_{e2}^2\left\{1 - \frac{i\omega}{\tilde{c}_{e2}^2 q^2\tau_J} - \frac{c_{e2}^2}{\tilde{c}_{e2}^2}(i\omega\tau_{e2} - \lambda_{e2}^2 q^2)\right\}. \tag{2.53}$$

Neglecting the last term in the bracket, we see then that Eq. (2.53) describes second sound if the window condition

$$1/\tau_J \ll \omega \ll 1/\tau_{eq} \tag{2.54}$$

as well as the condition

$$\omega\tau_{e2} \ll 1 \tag{2.55}$$

are fulfilled. In analogy to Eq. (1.46) τ_{eq} in Eq. (2.54) is the relaxation time responsible for thermal equilibrium of the electron fluid. These conditions are exactly the same as those in the case of dielectric crystals [Eqs. (3.70) and (3.75) below].

In order to have heat diffusion we multiply Eq. (2.53) by $\tau_J q^2$ and obtain, neglecting again the last term in the bracket,

$$-i\omega(1 - i\omega\tau_J + q^2 c_{e2}^2\tau_J\tau_{e2}) + \tilde{c}_{e2}^2\tau_J q^2 = 0.$$

In the low-frequency, small-wave-number domain

$$\omega\tau_J \ll 1, \qquad q^2 c_{e2}^2\tau_J\tau_{e2} \ll 1, \tag{2.56}$$

we then obtain heat diffusion

$$-i\omega + D_{sT}'q^2 = 0, \tag{2.57}$$

where D_{sT}' is defined in analogy to the case of dielectric crystals [Eq. (3.101) below], by

$$D_{sT}' = \tilde{c}_{e2}^2\tau_J = D_{sT} + (\rho_e/\rho_s)c_{e2}^2\tau_s, \tag{2.58}$$

where D_{sT} is defined by the third equation (2.46).

In the case of strong impurity scattering $\mathbf{v}_n = 0$ and momentum balance is invalid, while energy conservation still holds. This is the analogous situation to helium in fine pores where the porous substrate can absorb any quantity

of momentum but not energy (Sec. I.E). Hence the second Eq. (2.11) is invalid, while the third is again replaced by Eq. (2.14). In the form (2.42) the latter now reads

$$\rho_e\dot{s} + \dot{\rho}_e s = (\kappa_{es}/T)\nabla^2 T. \tag{2.59}$$

Eq. (2.6), with (2.9) and the first Eq. (2.17) with the second Eq. (2.13), becomes

$$\dot{\mathbf{v}}_s = s\nabla T - (1/\rho_e)\nabla p_e + (e/m)\mathbf{E} + \rho_s\zeta_{e3}\nabla\nabla\cdot\mathbf{v}_s, \tag{2.60}$$

while the first Eq. (2.11) combined with Eq. (2.7) leads to

$$\dot{\rho}_e + \rho_s\nabla\cdot\mathbf{v}_n = 0. \tag{2.61}$$

Combination of the divergence of Eq. (2.60) and the time derivative of (2.61), making use of (2.32), leads to

$$\ddot{\rho}_e - (\rho_s/\rho_e)c_{e1}^2\nabla^2\rho_e + (\rho_s/\rho_e)\omega_{p1}^2\delta\rho_e + \rho_s s(1 - \beta_e/\rho_e s)\nabla^2 T = \rho_e\zeta_{e3}\nabla^2\dot{\rho}_e \tag{2.62}$$

while with Eq. (2.32) Eq. (2.59) takes the form

$$\dot{T} + (sT/\rho_e c_{Vs})(1 - \beta_e/\rho_e s)\dot{\rho}_e = D_{sT}\nabla^2 T. \tag{2.63}$$

These two equations are analogous to Eqs. (1.99). In plane-wave representation they lead to the secular equation

$$\omega^2 = (\rho_s/\rho_e)\omega_{p1}^2 + c_{e4}^2(1 - i\omega\tau_{e4})q^2 - (\rho_n/\rho_e)c_{e2}^2 \times\left(1 - \frac{\beta_e}{\rho_e s}\right)^2\frac{iD_{sT}q^4}{\omega + iD_{sT}q^2}, \tag{2.64}$$

where, in analogy to Eqs. (1.101) and (1.102)

$$c_{e4} = \left[\frac{\rho_s}{\rho_e}c_{e1}^2 + \frac{\rho_n}{\rho_e}\left(1 - \frac{\beta_e}{\rho_e s}\right)^2 c_{e2}^2\right]^{1/2},$$
$$\tau_{e4} = \rho_s\zeta_{e3}/c_{e4}^2. \tag{2.65}$$

For small q such that

$$c_{e4}q \ll (\rho_s/\rho_e)^{1/2}\omega_{p1}, \tag{2.66}$$

Eq. (2.64) has the approximate solutions

$$\omega_1^2 \cong (\rho_s/\rho_e)\omega_{p1}^2 + c_{e4}^2q^2(1 - i\omega\tau_{e4}) \tag{2.67}$$

and

$$\omega_2 \cong -iD_{sT}q^2. \tag{2.68}$$

In the particular case

$$\beta_e/\rho_e s = 1 \tag{2.69}$$

Eqs. (2.62) and (2.63) are decoupled and Eqs. (2.67) and (2.68) become exact. [The realizability of Eq. (2.69) is hard to estimate since β_e is not directly accessible to experiment.]

D. The question of second sound in superconductors

In the last section we found that second sound occurs under the conditions (2.54) and (2.55). There is, however, a much more stringent condition imposed by Eq. (2.26). Indeed, since the electrons move with the Fermi velocity v_F this inequality becomes

$$q^{-1} = \tilde{c}_{e2}/\omega \gg \lambda_{eq} = v_F\tau_{eq} \tag{2.70}$$

which is much stronger than $\omega\tau_{eq} \ll 1$ since, as we will show, $\tilde{c}_{e2}$ is much smaller than v_F. This fact was emphasized in a paper by Enz (1966b) which we follow in the discussion below.

We first calculate c_{e2} from Eq. (2.34) in the two limits $T \lesssim T_c$ and $T \ll T_c$. From BCS theory [see for example Rickayzen (1965)] one finds

$$u \equiv \frac{\Delta(T)}{k_B T} \cong \begin{cases} 3.07\epsilon^{1/2}; & \epsilon \ll 1; \\ 1.76t^{-1} \gg 1; & t \ll 1 \end{cases} \tag{2.71}$$

where $t = 1 - \epsilon = T/T_c$. Applied to Eq. (2.21) this gives

$$\frac{\rho_n}{\rho_e} \cong \begin{cases} 1 - u^2\int_0^\infty \frac{e^x(e^x - 1)}{x(e^x + 1)^3}dx \cong 1 - 1.98\epsilon; & \epsilon \ll 1 \\ (2\pi u)^{1/2}\exp(-u) \cong 3.33t^{-1/2}\exp(-1.76/t); & t \ll 1. \end{cases} \tag{2.72}$$

The entropy (2.24) becomes

$$\frac{s}{s_c} \cong \begin{cases} (1 - 3u^2/2\pi^2)t \cong 1 - 2.43\epsilon; & \epsilon \ll 1 \\ (6/\pi^2)(\pi/2)^{1/2}tu^{3/2}\exp(-u) \\ \quad \cong 1.79t^{-1/2}\exp(-1.76/t); & t \ll 1, \end{cases} \tag{2.73}$$

where

$$s_c = \pi^2 k_B^2 T_c/k_F^2 = \gamma T_c \tag{2.74}$$

is the entropy at T_c and $c_{Vn} = \gamma T$ is the specific heat in the normal phase. From Eq. (2.73) we obtain for the specific heat

$$\frac{c_{Vs}}{s_c} = t\frac{\partial}{\partial t}\left(\frac{s}{s_c}\right) \cong \begin{cases} 2.43(1 - \epsilon); & \epsilon \ll 1 \\ 3.16t^{-3/2}\exp(-1.76/t); & t \ll 1. \end{cases} \tag{2.75}$$

With Eqs. (2.72), (2.73), and (2.75) the second-sound velocity (2.34) becomes

$$\frac{c_{e2}^2}{T_c s_c} = \left(\frac{\rho_e}{\rho_n} - 1\right)\frac{t(s/s_c)^2}{c_{Vs}/s_c} \cong \begin{cases} 0.816\epsilon; & \epsilon \ll 1 \\ 0.304t^2; & t \ll 1 \end{cases}. \tag{2.76}$$

By interpolation one sees that $(c_{e2}^2/T_c s_c)_{max} < 0.2$. Now

from Eq. (2.74) we find

$$\frac{T_c s_c}{v_F^2} = \left(\frac{\pi}{2}\frac{k_B T_c}{\mu}\right)^2 \cong 1.2 \times 10^{-5}, \tag{2.77}$$

where we have used most favorable values, $v_F = 0.5 \times 10^8$ cm/sec, $\mu = 0.70$ eV, $T_c = 18$ K, $k_B T_c = 1.55 \times 10^{-3}$ eV. This shows that $c_{e2}/v_F < 1.6 \times 10^{-3}$. From (2.54) and (2.70) we obtain the decisive condition

$$\tau_{eq}/\tau_J \ll \bar{c}_{e2}/v_F. \tag{2.78}$$

Now the mechanism which brings the electrons to equilibrium is described by the relaxation time τ_s, so that $\tau_{eq} = \tau_s$ and, according to Eqs. (2.49), (2.72), (2.76), (2.77), and (2.78),

$$(\rho_s v_F/\rho_e c_{e2})^2 \cong 4.0 \times 10^5 \epsilon \ll 1. \tag{2.79}$$

This shows that second sound is possible only at temperatures which are less than 10^{-5} away from T_c. This, however, is already in the critical region where ρ_s and c_{e2} must go to zero in analogy to Eqs. (1.82) and (1.87), respectively. We conclude, therefore, that it is highly unlikely to find second sound in superconductors.

Finally we show the connection of the relaxation time τ_s with the heat conductivity κ_{es}. We have for the dissipative energy current density in a superconductor (the factor 2 accounts for the spin states)

$$\mathbf{J}_\epsilon' = (2/V)\sum_{\mathbf{k}} E_k(\partial E_k/\partial k)\delta f_k \tag{2.80}$$

where $\delta f_k = f_k - f_k^0$ is obtained from the Boltzmann equation namely, for $\omega = 0$,

$$\delta f_k = \tau_{sk}(\partial f_k^0/\partial E_k) E_k(\partial E_k/\partial k)(\nabla T/T). \tag{2.81}$$

The relaxation time τ_{sk} in the superconducting state is related to the relaxation time τ_n in the normal state [e.g. Rickayzen (1965)] by

$$\tau_{sk} = (E_k/\epsilon_k)\tau_n. \tag{2.82}$$

Note that the relaxation time τ_s introduced in the second Eq. (2.11) is the following average over k of τ_{sk}, valid for $m\mu \gg k_B T_c$,

$$\frac{\tau_n}{\tau_s} = \left\langle\frac{1}{\tau_{sk}}\right\rangle \tau_n = -\frac{\tau_n}{3\pi^2}\int_0^\infty \frac{1}{\tau_{sk}}\frac{\partial f_k^0}{\partial E_k} k^4\, dk$$

$$= 2\frac{\rho_e}{\rho_n}\int_0^\infty \frac{x}{y}\frac{e^y}{(e^y+1)^2}\, dx = \frac{\rho_e}{\rho_n}\frac{2}{1+e^u}. \tag{2.83}$$

Inserting Eqs. (2.81) and (2.82) into (2.80) and performing the angular integration we find

$$\mathbf{J}_\epsilon' = \frac{\tau_n}{3T}\frac{2}{V}\sum_{\mathbf{k}}\frac{E_k^3}{\epsilon_k}\left(\frac{\partial E_k}{\partial k}\right)^2\frac{\partial f_k^0}{\partial E_k}\nabla T = -\kappa_{es}\nabla T. \tag{2.84}$$

This leads to the expression, valid for $m\mu \gg k_B T_c$,

$$\kappa_{es} = \frac{2\tau_n \rho_e k_B^2 T}{m^2}\int_0^\infty \frac{xye^y}{(e^y+1)^2}\, dx \tag{2.85}$$

which was first obtained by Bardeen, Rickayzen, and Tewordt (1959). [See also Rickayzen (1965).] This equation shows that the diffusion constant D_{sT}' defined in Eq. (2.58) is indeed related to heat diffusion since according to Eqs. (2.83) and (2.85) τ_s is proportional to κ_{es}.

III. DIELECTRIC CRYSTALS

In 1946, two years after his first detection of second sound in superfluid helium, Peshkov (1947) made the remark that "a gas of thermal quanta capable of performing vibrations similar to those of sound should exist in a perfect crystal." Since a dielectric crystal is primarily an elastic medium, this remark of Peshkov can be considered as the first application of the two-fluid idea to and the prediction of second sound in such systems. Indeed, in an anharmonic crystal the two fluids have to be identified as the classical displacement field $\mathbf{u}(\mathbf{r}, t)$ of the lattice and the fluid of the thermal phonons.

This is so because $\mathbf{u}$ describes the dynamics of the condensed phase, the elastic medium, and hence plays the role of the order parameter. This analogy is not perfect, however, since the phase transition is of first order. In addition, as Peshkov's remark already indicates, second sound in dielectric crystals exists independently of the elastic medium. This difference with the superfluids is due to the fact that the mass density of the two fluids do not mix in a crystal. Indeed, since phonon number is not conserved, only the atomic density of the lattice satisfies a continuity equation.

Peshkov's idea was taken up by Ward and Wilks (1951). [see also London (1954), Sec. 16] who mention in a footnote that Nernst (1918) had already remarked that "good thermal conductors at low temperature might have sufficient inertia to give rise to an oscillatory discharge".[1] Subsequently Dingle (1952) applied the two-fluid equations for second sound to nonideal gases, insulators, conductors, superconductors and ferromagnets. But the clear realization that phonon drift is vital for second sound only came when Süssmann and Thellung (1963) and Gurzhi (1964) independently showed that, under conditions such that the momentum conserving (normal) phonon scatterings dominate the momentum dissipating (Umklapp) processes, heat transport is mainly convective and due to a Poiseuille type flow of the phonons. This new type of heat transport, first suggested by Peierls [see the acknowledgment by Süssmann and Thellung (1963)] which was already known in superfluid helium (Whitworth, 1958), has subsequently been found experimentally in solid ^{4}He by Mezhov-Deglin (1964, 1965, 1967).

Both Poiseuille flow and second sound depend on the condition that momentum dissipating processes be neg-

[1] The original wording of Nernst (1918, footnote 1, p. 21) is: "Da höchstwahrscheinlich die Wärme Trägheit besitzt, könnte sogar bei sehr tiefen Temperaturen und dadurch bedingter sehr guter Leitfähigkeit unter Umständen eine *oszillatorische Entladung thermischer Potentialdifferenzen* möglich werden."

ligible or, expressed in terms of a relaxation time τ_J, that $\omega\tau_J \gg 1$. Together with the restriction $\omega\tau_{eq} \ll 1$ of the hydrodynamic domain this leads to the window condition for the frequency ω already encountered in Sec. II, which was first discussed in the papers by Krumhansl and his collaborators (Guyer and Krumhansl, 1964; Prohowski and Krumhansl, 1964).

Based on the analysis of the window condition by Guyer and Krumhansl (1966b), in which the data of Mezhov–Deglin (1964, 1965) were taken into account and on other evaluations [see the historical account in the review by Ackerman and Guyer (1968)] Ackerman, Bertman, Fairbank, and Guyer concluded that second sound was feasible in solid ^{4}He. And in 1966 they published the first successful observation of second sound in a dielectric crystal (Ackerman *et al.*, 1966) which was achieved by the heat pulse method [see Ackerman and Guyer (1968); H. Beck and R. Beck (1973)] already in use with superfluid helium. Subsequently second sound has also been found in ^{3}He (Ackerman and Overton, 1969), in NaF (Jackson and Walker, 1970, 1971; McNelly *et al.*, 1970; Rogers, 1971) and in the semimetal Bi (Narayanamurti and Dynes, 1972) in which electron (hole)–phonon scattering is weak. In all these cases high chemical and isotopic purity and high crystal perfection were essential to have a sufficiently large relaxation time τ_J. [For more details on second sound and Poiseuille flow experiments see the review by Beck *et al.* (1974).]

The first formal connection between the hydrodynamic excitations in dielectric crystals and in superfluid helium was made by Kwok and Martin (1966) [see also Griffin (1968)], who treated the phonons in the crystal as a continuous scalar field with constant sound velocity. Götze and Michel (1967a) then introduced a two-fluid description of dielectric crystals in which the phonons were described as a Bose liquid in analogy to Landau's theory of Fermi liquids. While Götze and Michel (1967b), Niklasson (1970), Meier (1969), and others [see Beck *et al.* (1974)] used sophisticated Green's function techniques, we will in this section follow the works by Götze and Michel (1967a) and by Niklasson (1970) in their phenomenological content [see also Gurzhi (1965)] and derive all the physically significant results including the response measured in neutron and light scattering experiments.

A. Two-fluid hydrodynamics

We start again with a description of the "condensed phase." The displacement field $\mathbf{u}$ at the lattice positions $\mathbf{R}$ is determined as an average of the displacement operator $\mathbf{x}(\mathbf{R})$, $\mathbf{u}(\mathbf{R}) = \langle \mathbf{x}(\mathbf{R}) \rangle$. This average is defined, as in superfluids, by a density matrix ρ which commutes with the effective Hamiltonian (compare Sec. I.A) but not with the generator of the broken symmetry group. In the liquid–solid transition it is the continuous translation (and rotation) group which is broken and the group generator is the momentum operator $\mathbf{P}$ (i.e., $e^{i\boldsymbol{\varphi}\cdot\mathbf{P}}$ is the general element of the translation group). Thus we have in analogy to Eq. (1.1)

$$[\rho, \mathcal{H} - \mathbf{v}_L\cdot\mathbf{P}] = 0, \qquad [\rho, \mathbf{P}] \neq 0, \qquad [\mathcal{H}, \mathbf{P}] = 0,$$

where $\mathbf{v}_L$ is a fixed velocity.

The time evolution of $\mathbf{u}(\mathbf{R})$ is determined by the Heisenberg representation

$$\mathbf{x}(\mathbf{R}, t) = \exp(i\mathcal{H}t)\mathbf{x}(\mathbf{R}) \exp(-i\mathcal{H}t)$$

We first consider a *homogeneous* displacement $\mathbf{u}$ as obtained with an $\mathbf{R}$-independent operator $\mathbf{x}$. In this case the momentum operator $\mathbf{P}$ is canonically conjugate to $\mathbf{x}$ so that

$$\exp(i\mathbf{v}_L\cdot\mathbf{P}t)\mathbf{x} \exp(-i\mathbf{v}_L\cdot\mathbf{P}t) = \mathbf{x} + \mathbf{v}_L t$$

With this relation together with $[\rho, \mathcal{H} - \mathbf{v}_L\cdot\mathbf{P}] = 0$ the time evolution of $\mathbf{u} = Tr(\rho\mathbf{x})$ takes a form analogous to Eq. (1.3),

$$\mathbf{u}(t) = \mathbf{u}(0) + \mathbf{v}_L t.$$

This determines the "lattice velocity" in analogy to Eq. (1.4) as

$$\mathbf{v}_L = \dot{\mathbf{u}}. \tag{3.1}$$

It is worth noting that $\mathbf{u}$ can also be described by coherent states [see Appendix A of Beck *et al.* (1974)] and that such a description is valid for any order parameter.

Locally the displacement field $\mathbf{u}(\mathbf{r}, t)$ satisfies the equation of motion of classical elasticity theory. With Eq. (3.1) this equation can be written in a form analogous to the equation of motion of the superfluid velocity [Eq. (1.5)], but without a flow term $(\mathbf{v}_L\cdot\nabla)\mathbf{v}_L$,

$$\dot{\mathbf{v}}_L = (1/\rho)\nabla(\Sigma + \Sigma') - \nabla\psi \tag{3.2}$$

Thus the classical displacement $\mathbf{u}$ plays a role analogous to the phase φ of the superfluid order parameter, except that the lattice velocity is defined by the *time* derivative of $\mathbf{u}$ while the superfluid velocity is defined by the *space* derivative of φ.

In Eq. (3.2) ρ is the mass density of the crystal, ψ an external potential per unit mass, and $\Sigma_{ij} = \Sigma_{ji}$ the stress tensor which satisfies the thermodynamic relation

$$d\Sigma_{ij} = \sum_{kl} C_{ij,kl} d\theta_{kl} - \beta_{ij} dT. \tag{3.3}$$

Here

$$\theta_{ij} = \tfrac{1}{2}(\nabla_j u_i + \nabla_i u_j) \tag{3.4}$$

is the strain tensor,

$$C_{ij,kl} = (\partial\Sigma_{ij}/\partial\theta_{kl})_T \tag{3.5}$$

are the isothermal elastic constants,

$$\beta_{ij} = -(\partial\Sigma_{ij}/\partial T)_\theta \tag{3.6}$$

is the tensor of the tension coefficients, and Σ' is the dissipative part of Σ which will be discussed later.

With Eq. (3.3), Eq. (3.2) may be written

$$\dot{\mathbf{v}}_{Li} = (1/\rho)\sum_j \Big(\sum_{kl} C_{ij,kl}\nabla_j \theta_{kl} - \beta_{ij}\nabla_j T + \nabla_j \Sigma_{ij}'\Big) - \nabla_i \psi. \tag{3.7}$$

Mass conservation in the lattice is expressed by

$$\dot{\rho} + \nabla\cdot\mathbf{j}_L = 0, \tag{3.8}$$

where

$$\mathbf{j}_L = \rho \mathbf{v}_L \tag{3.9}$$

is the mass flux or momentum density of the lattice. Note that a dissipative term in $\mathbf{j}_L$ which would describe mass diffusion in the lattice is absent in an ideal crystal. Equation (3.8) is different from the case of the superfluid where the *sum* of superfluid plus normal mass is conserved.

From Eqs. (3.2), (3.8), and (3.9) one finds

$$\dot{\mathbf{j}}_L + \nabla(\Pi_L + \Pi_L') = -\rho\nabla\psi - \mathbf{v}_L(\nabla\cdot\mathbf{j}_L) \tag{3.10}$$

with

$$\Pi_L = -\Sigma, \qquad \Pi_L' = -\Sigma'. \tag{3.11}$$

The absence of a flow term $(\mathbf{v}_L\cdot\nabla)\mathbf{v}_L$ in Eq. (3.2) has the effect that even without external forces, $\nabla\psi = 0$, momentum is conserved in the lattice only in linear approximation in $\mathbf{v}_L$. This is consistent with the lack of full translational invariance. But the fact that Eq. (3.10) was derived from Eqs. (3.2), (3.8), and (3.9) shows that $\mathbf{j}_L$ is not an independent variable anyway.

The lattice energy per unit mass is

$$\epsilon_L = \tfrac{1}{2}\mathbf{v}_L^2 + f_L + Ts_L, \tag{3.12}$$

where s_L is the lattice entropy per unit mass and the lattice free energy per unit mass f_L satisfies the thermodynamic relation

$$d(\rho f_L) = \sum_{ij} \Sigma_{ij}\, d\theta_{ij} + \psi\, d\rho - \rho s_L\, dT. \tag{3.13}$$

Because of the symmetry $\theta_{ij} = \theta_{ji}$ it follows from Eq. (3.13) that

$$\Sigma_{ij} = \tfrac{1}{2}(1+\delta_{ij})\,(\partial(\rho f_L)/\partial\theta_{ij})_T \tag{3.14}$$

so that from Eqs. (3.13), (3.11), and (3.6) we obtain the Maxwell relation

$$\frac{1+\delta_{ij}}{2}\left(\frac{\partial(\rho s_L)}{\partial\theta_{ij}}\right)_T = -\frac{1+\delta_{ij}}{2}\frac{\partial^2(\rho f_L)}{\partial T\,\partial\theta_{ij}}$$

$$= -(\partial\Sigma_{ij}/\partial T)_\theta = \beta_{ij}. \tag{3.15}$$

The tensor of thermal expansion

$$\alpha_{ij} = (\partial\theta_{ij}/\partial T)_\Sigma \tag{3.16}$$

satisfies, combined with Eqs. (3.5) and (3.6), the identity

$$\beta_{ij} = \sum_{kl} C_{ij,kl}\,\alpha_{kl}. \tag{3.17}$$

Hence neglecting thermal expansion, $\alpha_{ij} = 0$, it follows that $\beta_{ij} = 0$ and Eq. (3.7) becomes a closed, purely dynamical (i.e. nonthermal) equation [provided dissipation to the phonon fluid is neglected in Σ_{ij}', see Eq. (3.42) below]. Therefore the "lattice fluid" has a vanishing specific heat per unit mass,

$$c_{VL} = T(\partial s_L/\partial T)_\theta = 0. \tag{3.18}$$

Note that we assume here, as in Eq. (3.8), an ideal lattice. Lattice imperfections of course give rise to a nonvanishing lattice specific heat.

On the other hand, ρ depends on T only through thermal expansion [Eq. (3.16)]. In fact, according to Eq. (3.4) the change of volume per unit mass due to deformation is $\nabla\cdot\mathbf{u} = \sum_i \theta_{ii}$, so that

$$d\rho = -\rho\sum_i d\theta_{ii}. \tag{3.19}$$

Hence

$$(\partial(\rho s_L)/\partial T)_\theta = 0, \tag{3.20}$$

so that Eq. (3.15) yields [see Eq. (4.7) of Gurevich and Efros (1966)]

$$d(\rho s_L) = \sum_{ij}\beta_{ij}\, d\theta_{ij}. \tag{3.21}$$

From Eqs. (3.12), (3.13), (3.8), (3.9), (3.2), (3.4), (3.1), and (3.21) one deduces the energy balance equation

$$(\rho\epsilon_L)^{\cdot} + \nabla\cdot(\mathbf{J}_{\epsilon L} + \mathbf{J}_{\epsilon L}') = (\rho/2)\mathbf{v}_L\cdot\nabla\mathbf{v}_L^2 + [(T\beta - \Sigma')\nabla]\cdot\mathbf{v}_L, \tag{3.22}$$

where

$$\mathbf{J}_{\epsilon L} = [-\Sigma + (\rho/2)\mathbf{v}_L^2 + \rho\psi]\mathbf{v}_L$$
$$\mathbf{J}_{\epsilon L}' = -\Sigma'\,\mathbf{v}_L \tag{3.23}$$

are the lattice energy flux and its dissipative part. Hence even to second order in small quantities energy is not conserved in the "lattice fluid" because of the last term on the right-hand side of Eq. (3.22). The nondissipative part of it, $(T\beta\nabla)\cdot\mathbf{v}_L = T(\rho s_L)^{\cdot}$, acts as an energetic coupling of the "lattice fluid" to the phonon fluid through thermal expansion.

For the phonon fluid particle number is not conserved so that there is no analogue to Eq. (3.8). The energy per unit mass, ϵ_p, and the energy flux $\mathbf{J}_{\epsilon p}$ of the phonon fluid are defined, respectively, by $\rho\epsilon_p = \langle h(\mathbf{r}, t)\rangle$ and $\mathbf{J}_{\epsilon p} = \langle \mathbf{s}(\mathbf{r}, t)\rangle$ where h and $\mathbf{s}$ are the corresponding operator densities and the average is that of drifting local thermal equilibrium (Enz, 1968).

The analogous definitions for the momentum density $\mathbf{j}_p$ and the momentum flux Π_p are less certain because of an inherent difficulty in the definition of the phonon number density operator (Enz, 1968). Physically, however, these quantities must be well defined. In the approximation of the phonons by noninteracting quasiparticles these definitions are straightforward, as will be seen below.

Hydrodynamically the phonon fluid is characterized by a local drift velocity $\mathbf{v}_p$ analogous to the normal velocity $\mathbf{v}_n$ of the superfluid. The phonon mass density ρ_p (Enz, 1968) is then defined by

$$\mathbf{j}_p = \rho_p \mathbf{v}_p. \tag{3.24}$$

Note that there is no dissipative part $\mathbf{j}_p'$ because the phonon mass density ρ_p is not conserved. For simplicity we will treat ρ_p as a scalar.

Since because of phonon number nonconservation the chemical potential of the phonons vanishes, thermodynamics tells us that the pressure of the phonon fluid is $-\rho f_p$, where f_p is the phonon free energy per unit mass. Hence the momentum flux is, in analogy to Π_n in Eq. (1.13),

$$\Pi_p = -\rho f_p + \rho_p \mathbf{v}_p \otimes \mathbf{v}_p + O(v_p^4), \tag{3.25}$$

where ρ_p is the phonon mass density defined by Eq. (3.24).

Thermodynamically the energy density $\rho\epsilon_p$ is given by

$$\rho\epsilon_p = \mathbf{v}_p \cdot \mathbf{j}_p + \rho(f_p + Ts_p), \tag{3.26}$$

where s_p is the phonon entropy per unit mass. The relation analogous to Eq. (3.13) is

$$d(\rho f_p) = -\mathbf{j}_p \cdot d\mathbf{v}_p - \rho s_p \, dT. \tag{3.27}$$

It is obvious from Eq. (3.22) that energy cannot be conserved in the phonon fluid. On the other hand, momentum is also not conserved, and this in spite of the fact that to linear approximation in $\mathbf{v}_L$ without external forces momentum is conserved in the "lattice fluid." The reason is that due to the presence of the lattice the continuous translation group is broken so that the phonons satisfy momentum conservation only modulo Umklapp processes. In imperfect crystals other momentum dissipating processes also contribute. As is discussed by Enz (1968), Umklapp processes give rise to a force density $\mathbf{f}_p$ exerted by the lattice on the phonon fluid, and $\mathbf{f}_p$ occurs as an inhomogeneous term in the momentum balance equation of the phonons. But the reaction $-\mathbf{f}_p$ of this force on the lattice fluid does not occur in the momentum balance equation (3.10) for the lattice. This is because the periodic boundary conditions introduced in order to have a representation by propagating normal coordinates (phonons) prevent the description of recoil of the lattice (recoil can only be described in a representation by standing-wave normal coordinates).

From a Boltzmann equation point of view (Götze and Michel, 1967a; Hardy, 1970) the lattice force is due to relaxation of phonon momentum, $\mathbf{f}_p = -(1/\tau_J)\mathbf{j}_p$, so that the phonon momentum balance equation takes the form (Enz, 1968; Hardy, 1970)

$$\dot{\mathbf{j}}_p + \nabla(\Pi_p + \Pi_p') = -(1/\tau_J)\mathbf{j}_p, \tag{3.28}$$

where Π_p' is the dissipative part of Π_p. The tensor τ_J will be treated as a scalar in order to simplify the problem. Note that in distinction to the second Eq. (1.7) the external force density $-\rho\nabla\psi$ is here contained in Eqs. (3.2) and (3.10).

The approximation of the phonons by noninteracting quasiparticles is identical to the case of superfluids. In drifting local equilibrium the phonon distribution function is, with $x \equiv \omega_k/k_BT$, $y \equiv \mathbf{v}_p \cdot \mathbf{k}/k_BT$, and $n_k^0 = (e^x - 1)^{-1}$,

$$n_k = [\exp(x - y) - 1]^{-1} = n_k^0 + n_k^0(n_k^0 + 1)y + n_k^0(n_k^0 + 1)(n_k^0 + \tfrac{1}{2})y^2 + \cdots \tag{3.29}$$

Here ω_k is the acoustic phonon spectrum, taken to be isotropic, and $k = (\mathbf{k}, \mu)$ where μ is the polarization index. The momentum and energy densities are, respectively,

$$\mathbf{j}_p = V^{-1} \sum_k \mathbf{k} n_k \tag{3.30}$$

and

$$\epsilon_p = (\rho V)^{-1} \sum_k \omega_k n_k. \tag{3.21}$$

With Eq. (3.29) one finds from Eq. (3.30) that the phonon mass density defined by Eq. (3.24) is given by

$$\rho_p = (3k_BTV)^{-1} \sum_k \mathbf{k}^2 n_k^0(n_k^0 + 1) + O(v_p^2). \tag{3.32}$$

The specific heat of the crystal [remember that the "lattice fluid" has a vanishing specific heat, Eq. (3.18)] is given by

$$c_V = c_{Vp} = (\partial\epsilon_p/\partial T)_V = (k_BT^2\rho V)^{-1} \sum_k \omega_k^2 \times n_k^0(n_k^0 + 1) + O(v_p^2). \tag{3.33}$$

At low temperatures dispersion is negligible in ω_k so that

$$\omega_{\mathbf{k},\mu} = c_\mu |\mathbf{k}|; \qquad \mu = 1, 2, 3. \tag{3.34}$$

This leads to the expressions

$$c_V = (2\pi^2 k_B^4/15\rho) \sum_\mu c_\mu^{-3} T^3 \tag{3.35}$$

and (Enz, 1968)

$$\rho_p = (\rho T c_V/3)\left(\sum_\mu c_\mu^{-5} / \sum_\mu c_\mu^{-3}\right) = (2\pi^2 k_B^4/45) \sum_\mu c_\mu^{-5} T^4. \tag{3.36}$$

The free energy per unit mass is

$$f_p = -k_BT(\rho V)^{-1} \sum_k \ln(1 + n_k) = f_p^0 - \tfrac{1}{2}(\rho_p/\rho)\mathbf{v}_p^2 + O(v_p^4), \tag{3.37}$$

where the second equality is obtained with the help of Eqs. (3.29) and (3.32) and where $f_p^0 = f_p|_{v_p=0}$. With Eqs. (3.27) and (3.37) the relation (3.26) is readily verified by making use of (3.30) and (3.31). On the other hand, defining the nondissipative momentum flux by

$$\Pi_{pij} = V^{-1}\sum_k (\partial\omega_k/\partial k_i) k_j n_k, \tag{3.38}$$

the relation (3.25) is verified, making use, for the zeroth-order term, of a partial integration of Eq. (3.37). With Eq. (3.29) one first finds

$$\rho_p = \tfrac{2}{15}(k_BT)^{-2}V^{-1}\sum_k |\mathbf{k}|^3 |\partial\omega_k/\partial\mathbf{k}| n_k^0 \times (n_k^0+1)(n_k^0+\tfrac{1}{2}).$$

That this is identical with Eq. (3.32) follows by partial integration, making use of the identity

$$(k_BT)^{-1}(\partial\omega_k/\partial\mathbf{k}) n_k^0(n_k^0+1)(n_k^0+\tfrac{1}{2}) = -\tfrac{1}{2}\partial/\partial\mathbf{k}[n_k^0(n_k^0+1)].$$

Finally the nondissipative energy flux is given in this approximation by

$$\mathbf{J}_{\epsilon p} = V^{-1}\sum_k \omega_k(\partial\omega_k/\partial\mathbf{k}) n_k = \rho s_p T\mathbf{v}_p + O(v_p^3). \tag{3.39}$$

Here the second equality follows with the help of Eq. (3.29) and of the identity $(\partial\omega_k/\partial\mathbf{k}) n_k^0(n_k^0+1) = -k_BT\partial n_k^0/\partial\mathbf{k}$ by partial integration, making use of Eqs. (3.25), (3.26), (3.31), and (3.38) for $\mathbf{v}_p = 0$.

The dissipative parts of Π_L and Π_p are due to the viscosities of the lattice and of the phonon fluid. This means that they are linear functions of the deformations

$$D_{Lij} \equiv \tfrac{1}{2}(\nabla_j v_{Li} + \nabla_i v_{Lj}) = \dot\theta_{ij} \tag{3.40}$$

and

$$D_{pij} \equiv \tfrac{1}{2}(\nabla_j v_{pi} + \nabla_i v_{pj}). \tag{3.41}$$

In general, there will be cross terms, so that [see Eq. (4.14) of Gurevich and Efros (1966)]

$$\Pi_{Lij}' = -\Sigma_{ij}' = -\sum_{kl}(\eta_{ij,kl}D_{Lkl} + \mu_{ij,kl}D_{pkl}) \tag{3.42}$$

and [see Eq. (4.13) of Gurevich and Efros (1966)]

$$\Pi_{pij}' = -\sum_{kl}(\mu_{ij,kl}D_{Lkl} + \gamma_{ij,kl}D_{pkl}), \tag{3.43}$$

where the equality of the cross viscosities, $\mu_{ij,kl} = \mu_{kl,ij}$, in the two equations, as well as the symmetry of η, μ, γ in the two pairs of indices, express Onsager's reciprocity principle (Gurevich and Efros, 1966; Götze and Michel, 1967a). These cross terms constitute a dissipative coupling between the two fluids.

Two-fluid hydrodynamics of an anharmonic dielectric crystal is thus contained in the equations of motion (3.2), (3.8), (3.28), (3.40), and in the equation of motion for the total energy density per unit mass which is a sum of Eqs. (3.12) and (3.26),

$$\epsilon = \epsilon_L + \epsilon_p = \tfrac{1}{2}\mathbf{v}_L^2 + (\rho_p/\rho)\mathbf{v}_p^2 + f + Ts. \tag{3.44}$$

Here

$$f = f_L + f_p, \qquad s = s_L + s_p, \tag{3.45}$$

are the total free energy and entropy per unit mass, respectively. While according to Eq. (3.21) the lattice part of the entropy density only depends on θ, the phonon part only depends on T; in linear approximation in $\mathbf{v}_p$ one finds from Eqs. (3.26) and (3.27)

$$d(\rho s_p) = (\rho c_V/T)\, dT. \tag{3.46}$$

From Eqs. (3.13), (3.27), (3.44), (3.45), and (3.9) we find the thermodynamic relation analogous to Eqs. (1.15) and (1.16),

$$d(\rho\epsilon) = (\tfrac{1}{2}\mathbf{v}_L^2 + \psi)\, d\rho + \sum_{ij}\Sigma_{ij}\, d\theta_{ij} + \mathbf{j}_L\cdot d\mathbf{v}_L + \mathbf{v}_p\cdot d\mathbf{j}_p + T\, d(\rho s). \tag{3.47}$$

Over-all energy conservation implies

$$(\rho\epsilon)^{\cdot} + \nabla\cdot(\mathbf{J}_\epsilon + \mathbf{J}_\epsilon') = 0. \tag{3.48}$$

As in the case of superfluids, we convert the energy conservation into an entropy balance equation which will simultaneously give us an expression for the nondissipative energy flux $\mathbf{J}_\epsilon$. From Eq. (3.47) we get

$$T(\rho s)^{\cdot} = (\rho\epsilon)^{\cdot} - (\tfrac{1}{2}\mathbf{v}_L^2 + \psi)\dot\rho - \sum_{ij}\Sigma_{ij}\dot\theta_{ij} - \mathbf{j}_L\cdot\dot{\mathbf{v}}_L - \mathbf{v}_p\cdot\dot{\mathbf{j}}_p. \tag{3.49}$$

Inserting from the equations of motion (3.48), (3.8), (3.40), (3.2), and (3.28), and making use of Eqs. (3.9), (3.11), (3.24), (3.25), (3.27), we find, to second order in the velocities, in analogy to Eq. (1.17),

$$\begin{aligned}(\rho s)^{\cdot} = {} & -\nabla\cdot[(1/T)(\mathbf{J}_\epsilon + \mathbf{J}_\epsilon' - \mathbf{J}_{\epsilon L} - \mathbf{J}_{\epsilon L}' - \Pi_p'\mathbf{v}_p)] \\ & + (\mathbf{J}_\epsilon + \mathbf{J}_\epsilon' - \mathbf{J}_{\epsilon L} - \mathbf{J}_{\epsilon L}' - T\rho s_p\mathbf{v}_p - \Pi_p'\mathbf{v}_p)\cdot\nabla(1/T) \\ & - (1/T)[(\Pi_L'\nabla)\cdot\mathbf{v}_L + (\Pi_p'\nabla)\cdot\mathbf{v}_p] + (\rho_p/T\tau_J)\mathbf{v}_p^2,\end{aligned} \tag{3.50}$$

where we have made use of the definitions (3.23). Without dissipation entropy must be conserved. This determines the nondissipative energy flux from the condition that the nondissipative contributions to the second term on the right-hand side of Eq. (3.50) must add up to zero,

$$\mathbf{J}_\epsilon - \mathbf{J}_{\epsilon L} \equiv \mathbf{J}_{\epsilon p} = T\rho s_p\mathbf{v}_p. \tag{3.51}$$

This is the energy flux of the phonon fluid [Eq. (3.39)]. Equation (3.50) now takes the form, valid to second order

in the velocities,

$$(\rho s)^{\cdot} + \nabla\cdot(\mathbf{J}_s + \mathbf{J}_s') = \sigma. \tag{3.52}$$

Here

$$\mathbf{J}_s = (1/T)\mathbf{J}_{\epsilon p} = \rho s_p \mathbf{v}_p \tag{3.53}$$

is the entropy flux,

$$\mathbf{J}_s' = (1/T)(\mathbf{J}_\epsilon' - \mathbf{J}_{\epsilon L}' - \Pi_p'\mathbf{v}_p) \tag{3.54}$$

its dissipative part, where we made use of Eqs. (3.11), (3.23), and (3.51), and

$$\sigma = T\mathbf{J}_s'\cdot\nabla(1/T) - (1/T)[(\Pi_L'\nabla)\cdot\mathbf{v}_L + (\Pi_p'\nabla)\cdot\mathbf{v}_p] + (\rho_p/T\tau_J)\mathbf{v}_p^2 \tag{3.55}$$

the entropy production density which has the same form as Eq. (2.16).

With Eqs. (3.40) to (3.43) and

$$\mathbf{J}_s' \equiv (1/T)\mathbf{J}_{\epsilon p}' = -(\kappa/T)\nabla T \tag{3.56}$$

where κ is the heat conductivity tensor, Eq. (3.55) takes the form analogous to Eq. (1.24),

$$T\sigma = \nabla T\cdot[(\kappa/T)\nabla T] + \sum_{ijkl}\{\eta_{ij,kl}D_{Lij}D_{Lkl} + 2\mu_{ij,kl}D_{Lij}D_{pkl} + \gamma_{ij,kl}D_{pij}D_{pkl}\} + (\rho_p/\tau_J)\mathbf{v}_p^2. \tag{3.57}$$

Here the coefficients have to be such as to make σ a positive definite form.

Note that the form of the dissipative fluxes Π_L', Π_p', and $\mathbf{J}_s'$ can also be obtained by the general method discussed in Sec. I.A. For ψ = const the fluxes associated with the equations of motion (3.8), (3.40), (3.2), (3.28), and (3.52) are, respectively, $\mathbf{j}_L$, $\mathbf{v}_L$, $\Sigma = -\Pi_L$, Π_p, and $\mathbf{J}_s$. Their dissipative parts $\mathbf{j}_L' = 0$, $\mathbf{v}_L' = 0$, $\Sigma' = -\Pi_L'$, Π_p', and $\mathbf{J}_s'$ are linear functions of the forces which are defined as the gradients of the coefficients in Eq. (3.47), that is, respectively, as $\nabla\Psi = 0$, $\nabla \otimes \Sigma$, $\nabla \otimes \mathbf{j}_L$, $\nabla \otimes \mathbf{v}_p$ and ∇T (to first order in the velocities). Under time reversal $\mathbf{j}_L$, $\mathbf{v}_p$, and $\mathbf{J}_s$ transform with negative sign, Σ, Π_p, and T with positive sign. The condition that Σ', Π_p', and $\mathbf{J}_s'$ have opposite sign under time reversal compared to Σ, Π_p, and $\mathbf{J}_s$, respectively (irreversibility), then determines the linear relations (3.42), (3.43), and (3.56), where in Eq. (3.56) a term proportional to $\nabla \otimes \Sigma$ is neglected. The last three equations together with (3.23) and (3.54) finally determine $\mathbf{J}_\epsilon'$.

B. Rigid lattice: Second sound and Poiseuille flow

Rigidity of the lattice means that $\mathbf{u}$ = const or that $\theta_{ij} = 0$ and $\mathbf{v}_L = 0$. The elastic equation of motion (3.7) then reduces to a static equilibrium condition in which the external force is matched by internal forces of thermal origin,

$$\sum_{jkl}\mu_{ij,kl}\nabla_j\nabla_k v_{pl} - \sum_j \beta_{ij}\nabla_j T = \rho\nabla_i\Psi.$$

In this section we put $\mu_{ij,kl} = 0$, $\beta_{ij} = 0$ and Ψ = const, so that the last equation is trivially satisfied. Since, according to Eq. (3.21), $(\rho s)^{\cdot} = 0$ it follows from Eqs. (3.45) and (3.46) that

$$(\rho s)^{\cdot} = (\rho c_V/T)\dot{T}. \tag{3.58}$$

Treating all fluctuations in linear approximation the equation of motion for the entropy (3.52) with Eqs. (3.53) and (3.56) then becomes

$$(\rho s)^{\cdot} + \rho s_p\nabla\cdot\mathbf{v}_p - [(\kappa/T)\nabla]\cdot\nabla T = 0, \tag{3.59}$$

while Eq. (3.28) with Eqs. (3.24), (3.25), (3.27), (3.41) and (3.43) becomes

$$\rho_p\dot{v}_{pi} + \rho s_p\nabla_i T - \sum_{jkl}\gamma_{ij,kl}\nabla_j\nabla_k v_{pl} + (1/\tau_J)\rho_p v_{pi} = 0. \tag{3.60}$$

These equations are the same as Eq. (3.6) of Gurevich and Efros (1966), Eqs. (57) and (58) of Götze and Michel (1967a) and Eqs. (6.41), (5.11), and (6.44) of Niklasson (1970). The equations of Götze and Michel (1967a) and of Niklasson (1970) also contain the terms coupling the phonon fluid to the "lattice fluid."

We now introduce plane waves $\sim\exp[i(\mathbf{q}\cdot\mathbf{r} - \omega t)]$ and define the longitudinal heat conductivity by

$$\kappa_l(\mathbf{q},\omega) = \sum_{ij}\kappa_{ij}(\mathbf{q},\omega)\hat{q}_i\hat{q}_j, \tag{3.61}$$

where $\hat{q} = \mathbf{q}/|\,\mathbf{q}\,|$. We also introduce the matrix

$$(\lambda_N^2(\mathbf{q},\omega))_{ij} = (\tau_J/\rho_p)\sum_{mn}\gamma_{im,nj}(\mathbf{q},\omega)\hat{q}_m\hat{q}_n \tag{3.62}$$

which, because of Onsager's principle, is symmetric, and because of the definite sign of dissipation has positive eigenvalues λ_{Ni}^2 $(i = 1, 2, 3)$. It is obvious from their definition that the λ_{Ni} are related to normal, i.e., momentum conserving, phonon processes. In Eqs. (3.61) and (3.62) we have made the usual assumption that the linear relations (3.56) and (3.43) are local in $\mathbf{q}$, ω- space.

With Eqs. (3.58) and (3.62), Eqs. (3.59) and (3.60) go over, respectively, into

$$-i\omega(\rho c_V/T)\delta T + i\mathbf{q}\cdot\mathbf{v}_p\rho s_p + q^2(\kappa_l/T)\delta T = 0 \tag{3.63}$$

and

$$[-i\omega + (1/\tau_J) + (1/\tau_J)\lambda_N^2 q^2]\mathbf{v}_p + i\mathbf{q}(\rho/\rho_p)s_p\delta T = 0, \tag{3.64}$$

where δs and δT are the variations of s and T around the global equilibrium values, respectively.

Multiplying Eq. (3.64) with the reciprocal of the matrix in the bracket we obtain

$$\mathbf{v}_p = -(i/T\rho s_p)(\varphi\mathbf{q})\delta T, \tag{3.65}$$

where the tensor φ is defined by

$$\varphi(\mathbf{q},\omega) = (T\rho^2 s_p^2/\rho_p)[-i\omega + (1/\tau_J) + (1/\tau_J)\lambda_N^2 q^2]^{-1}. \tag{3.66}$$

Inserting Eq. (3.65) into (3.63) we obtain the dispersion relation for the thermal mode

$$-i\omega + (1/\rho c_V)\{\varphi_l(\mathbf{q},\omega) + \kappa_l(\mathbf{q},\omega)\}q^2 = 0, \tag{3.67}$$

where φ_l is the longitudinal projection of φ defined in analogy to Eq. (3.61).

With Eqs. (3.53) and (3.65), written in x space, the energy flux due to phonon drift, i.e., the *convective* heat flux, may be put into the form:

$$\mathbf{J}_{ep} = T\rho s_p \mathbf{v}_p = -\varphi\nabla T. \tag{3.68}$$

Comparing with Eqs. (3.54) and (3.56) we conclude that φ may be interpreted as *heat convectivity tensor* (Enz, 1966a, 1968).

Equation (3.67) is the same as Eq. (61) of Götze and Michel (1967a) if there we replace T_0 by our T, C_v/N by c_V, S/N by s_p, ρ by ρ_p, $\mathbf{s}$ by $\mathbf{u}$, ω^k by $1/\tau_J$, d_{ij} by $(\rho/\rho_p)^{1/2}Ts_p\delta_{ij}$, π_{il}^{-1} by $-i(\rho_p/T\rho^2 s_p^2)\varphi_{il}$, λ_{jk}' by $(\rho c_V)^{-1}\kappa_{jk}$, $v\beta_{jk}$ by $\rho^{-1}\beta_{jk}$, $\pi_{kj,mi}'$ by $\rho_p^{-1}\gamma_{kj,mi}$ and $\pi_{lk,nm}''$ by $(\rho\rho_p)^{-1/2}\mu_{lk,nm}$ and if we neglect the last two terms of Eq. (61) of Götze and Michel (1967a) which are due to the coupling with the "lattice fluid."

The dispersion relation (3.67) has three main domains of physical interest which are determined by the assumption that one of the three terms in the bracket of (3.66) is dominant.

In this way the *second-sound domain* is defined by

$$\omega\tau_J \gg 1, \qquad \omega\tau_J \gg q^2\lambda_{Ni}^2. \tag{3.69}$$

The first condition (which has no analogue in superfluid helium because of full translational invariance), combined with the local equilibrium condition $\omega\tau_{eq} \ll 1$, gives rise to the well known criterion of a frequency window analogous to Eq. (2.54), which was first emphasized by Guyer and Krumhansl (1964) and Prohowski and Krumhansl (1964),

$$1/\tau_J \ll \omega \ll 1/\tau_{eq}. \tag{3.70}$$

In this domain we obtain from Eq. (3.66), treating ρ_p as a scalar,

$$\varphi_l(\mathbf{q},\omega) = \frac{i}{\omega}\frac{T\rho^2 s_p^2}{\rho_p}\left[1 - i\frac{1+(\lambda_N^2)_l q^2}{\omega\tau_J}\right], \tag{3.71}$$

where $(\lambda_N^2)_l$ means the longitudinal projection analogous to Eq. (3.61) [i.e., in the coordinate system in which λ_N^2 is diagonal, $(\lambda_N^2)_l = \sum_i \lambda_{Ni}^2 \hat{q}_i^2$].

Inserting this into Eq. (3.67) we obtain

$$\frac{\omega^2}{q^2} = c_2^2\left[1 - i\frac{1+(\lambda_N^2)_l q^2}{\omega\tau_J} - i\omega\tau_2\right], \tag{3.72}$$

where the velocity of second sound

$$c_2 = (\rho T s_p^2/\rho_p c_V)^{1/2} \tag{3.73}$$

is defined in exactly the same way as in the superfluid case [Eq. (1.42)], and

$$c_2^2\tau_2(\mathbf{q},\omega) = (1/\rho c_V)\kappa_l(\mathbf{q},\omega) \equiv D_T(\mathbf{q},\omega) \tag{3.74}$$

is the thermal diffusion coefficient, and τ_2 the corresponding relaxation time. Equation (3.72) shows that in addition to (3.69) and (3.70), propagation of second sound requires that

$$\omega\tau_2 \ll 1. \tag{3.75}$$

From Eqs. (3.65) and (3.66) it is seen that a finite drift velocity $\mathbf{v}_p$ is essential for propagation of second sound.

As in the case of the superfluid, the limit $q \to 0$ of the second-sound mode is a zero-energy translational motion of the excitation fluid (the phonons) relative to the condensate (the lattice). However, the first condition (3.69), or $q \gg 1/c_2\tau_J$, prevents this limit from being taken. The existence of a finite τ_J is due to breaking of translational symmetry at the liquid–solid phase transition (Umklapp processes) and hence is an intrinsic property of a crystal. On the other hand, the Goldstone bosons of the dielectric crystal are the collisionless first-sound modes $\mathbf{u}(\mathbf{R},t) = \langle\mathbf{x}(\mathbf{R},t)\rangle$, also called zero sound (Cowley, 1967). In fact, in the limit $q \to 0$ they describe zero-energy translational motions of the lattice and hence "restore" the translational symmetry and momentum conservation (see the Introduction).

The second main domain is defined by the conditions

$$\omega\tau_J \ll q^2\lambda_{Ni}^2, \qquad q^2\lambda_{Ni}^2 \gg 1. \tag{3.76}$$

Here combination with the local equilibrium condition $q\lambda_{eq} \ll 1$ gives rise to a wave-number window condition

$$1/\lambda_{Ni} \ll q \ll 1/\lambda_{eq}. \tag{3.77}$$

We now have from Eqs. (3.66) and (3.73), keeping only the leading term,

$$(1/\rho c_V)\varphi_l(\mathbf{q},\omega)q^2 = c_2^2\tau_J(\lambda_N^{-2})_l \equiv \tau_N^{-1}, \tag{3.78}$$

[where, in the coordinate system in which λ_N^2 is diagonal, $(\lambda_N^{-2})_l = \sum_i \hat{q}_i^2/\lambda_{Ni}^2$].

Inserting this into Eq. (3.67) we get, with Eq. (3.74),

$$-i\omega + (1/\tau_N) + D_T(\mathbf{q},\omega)q^2 = 0. \tag{3.79}$$

We now assume that the λ_{Ni} are so large that the q values permitted by Eq. (3.77) are negligibly small with respect to the variation of $\tau_2(\mathbf{q}, \omega)$. Then writing the stationary diffusion constant as

$$\tau_2(0, 0) = \tau_s c_2'^2/c_2^2 \tag{3.80}$$

a relaxation time approximation of the phonon Boltzmann equation to be discussed below yields

$$\tau_2(0, \omega) = [\tau_s/(1 - i\omega\tau_s)](c_2'^2/c_2^2). \tag{3.81}$$

Note that in the literature there is very often no distinction made between the relaxation times τ_S and τ_J [Guyer and Krumhansl (1966a,b) and Ackerman and Guyer (1968) write $\tau_S = \tau_J = \tau_R^z = \tau_z$]. This is due to the common assumption of a proportionality between the momentum operator $\mathbf{J}$ and the energy flux operator $\mathbf{S}$. Microscopically τ_J and τ_S are determined by the momentum and the energy flux autocorrelation functions,

$$\langle \mathbf{J}(0) \otimes \mathbf{J}(t) \rangle \simeq \langle \mathbf{J} \otimes \mathbf{J} \rangle \exp(-t/\tau_J)$$

and

$$\langle \mathbf{S}(0) \otimes \mathbf{S}(t) \rangle \simeq \langle \mathbf{S} \otimes \mathbf{S} \rangle \exp(-t/\tau_S),$$

respectively. However, as was pointed out by Enz (1968), there is in general no reason to expect a simple proportionality between $\mathbf{J}$ and $\mathbf{S}$ for a dielectric crystal. Therefore τ_J and τ_S may very well be different. They have indeed been treated as different by Niklasson (1970). [There τ_u is our τ_J and τ^{00} our $(c_2'^2/c_2^2)\tau_S$].

Inserting Eq. (3.81) into Eq. (3.79) we find

$$\omega^2/q^2 = (1 + i/\omega\tau_N)^{-1}(1 + i/\omega\tau_S)^{-1}c_2'^2. \tag{3.82}$$

Thus if

$$\omega\tau_N \gg 1, \qquad \omega\tau_S \gg 1 \tag{3.83}$$

we have again a damped soundlike mode with velocity c_2',

$$\omega^2/q^2 = c_2'^2[1 - (i/\omega)(\tau_N^{-1} + \tau_S^{-1})]. \tag{3.84}$$

The first condition (3.83) means, according to Eqs. (3.78) and (3.79), that the convectivity, that is, the phonon drift, is negligible. Therefore, this hypothetic mode has been called *driftless second sound* by Enz (1968). Physically phonon drift is small if the phonon fluid is strongly viscous, i.e., if the $\gamma_{ij,kl}$ are large. This is in agreement, according to Eq. (3.62), with the above assumption of large λ_{Ni}.

The physical difficulty with driftless second sound stems from the fact that when the first condition (3.83) is combined with the local equilibrium condition $\omega\tau_{eq} \ll 1$, it gives rise to a window condition

$$1/\tau_N \ll \omega \ll 1/\tau_{eq}. \tag{3.85}$$

Now in the literature (Süssmann and Thellung, 1963; Gurzhi, 1964; Guyer and Krumhansl, 1964, 1966a,b; Ackerman and Guyer, 1968) τ_N is usually identified with τ_{eq}. In order to see this equality, let us consider the *isotropic case*. Then the viscosity tensor may be written as

$$\gamma_{ij,kl} = \gamma[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + (\nu - 1)\delta_{ij}\delta_{kl}] \tag{3.86}$$

with $\nu > 1$, Equation (3.62) leads to the matrix

$$(\rho_p/\gamma\tau_J)(\lambda_N^2)_{ij} = (\delta_{ij} + \nu\hat{k}_i\hat{k}_j) = \left(\delta_{ij} - \frac{\nu}{1+\nu}\hat{k}_i\hat{k}_j\right)^{-1} \tag{3.87}$$

from which it follows that

$$(\lambda_N^{-2})_l = (\lambda_N^2)_l^{-1} = \rho_p/(1 + \nu)\gamma\tau_J \tag{3.88}$$

or, with Eq. (3.78), that

$$\gamma = \rho_p c_2^2 \tau_N/(1 + \nu). \tag{3.89}$$

The viscosity term in Eq. (3.60) may now be written as

$$\sum_{jkl} \gamma_{ij,kl}\nabla_j\nabla_k v_{pl} = \frac{\rho_p c_2^2 \tau_N}{1+\nu}(\nabla^2 v_{pi} + \nu\nabla_i\nabla\cdot\mathbf{v}_p). \tag{3.90}$$

On the other hand, a Boltzmann equation treatment of τ_{eq} introduces a term $-\rho_p c_2^2\tau_{eq}(1 + \nu)^{-1}(\nabla^2 v_{pi} + \nu\nabla_i\nabla\cdot\mathbf{v}_p)$ into Eq. (3.60). This is the procedure of Süssmann and Thellung (1963), Gurzhi (1964, 1965), Guyer and Krumhansl (1964, 1966a,b), H. Beck and R. Beck (1973), Götze and Michel (1967a), Meier (1969), Niklasson (1970), Hardy (1970) [in this paper $\tau_J = \tau_{eq} = \tau$], and Beck *et al.* (1974), while Gurevich and Efros (1966) start from the assumption (3.43). This shows the formal origin of the viscosity term in Eq. (3.60) and leads to the equality $\tau_N = \tau_{eq}$. However, in the more detailed treatment by Niklasson (1970) a distinction is made between τ_N and τ_{eq} (there τ^{11} is our τ_N and τ our τ_{eq}) but the two differ only by a numerical factor of order unity [see Eqs. (6.14) and (9.30) of Niklasson (1970)].

Thus in view of Eq. (3.85) driftless second sound is very unlikely to be a physically realizable mode of excitation [see, however, the work of Hardy (1970)].

A more realistic physical situation occuring in the main domain (3.76) is *Poiseuille flow* (Süssmann and Thellung, 1963; Gurzhi, 1964; Guyer and Krumhansl, 1966a; Ackerman and Guyer, 1968; Meier, 1969; Beck *et al.*, 1974), which is a well known phenomenon in superfluid helium (Whitworth, 1958). Keeping only the leading term, the convectivity tensor (3.66) becomes, making use of Eq. (3.73),

$$\varphi = \rho c_V c_2^2 \tau_J \lambda_N^{-2} q^{-2}. \tag{3.91}$$

When this is inserted into Eq. (3.65) we obtain in x space

$$\nabla^2 v_{pi} = (c_V c_2^2 \tau_J/T s_p) \sum_j (\lambda_N^{-2})_{ij}\nabla_j T, \tag{3.92}$$

that is, we have a stationary situation.

Let us go to the coordinate system (x, y, z) in which $\lambda_N{}^2$ is diagonal. The eigenvalues of the matrix (3.87) are easily found to be 1, 1, $1+\nu$ so that

$$\nabla^2\mathbf{v}_p = (c_V/Ts_p\tau_N)[(1+\nu)\nabla_x T, (1+\nu)\nabla_y T, \nabla_z T]. \tag{3.93}$$

Assuming that the sample has the shape of a spherical cylinder of radius R with axis along the z direction, that $T = T_0 + az$, and that $\mathbf{v}_p|_{r=R} = 0$, we find $v_{px} = v_{py} = 0$.

On the other hand, it follows from Eq. (3.59) for the stationary case, since $\nabla T = \text{const}$, that $\nabla\cdot\mathbf{v}_p = \nabla_z v_{pz} = 0$. By symmetry v_{pz} does not depend on the azimuthal angle, hence $v_{pz} = v(r)$ and Eq. (3.93) goes over into

$$(rv'(r))'/r = (c_V/Ts_p\tau_N)a$$

which together with the boundary condition $v(R) = 0$ leads to

$$\mathbf{v}_p = -\frac{c_V(R^2 - r^2)}{4Ts_p\tau_N}\nabla T. \tag{3.94}$$

According to Eqs. (3.51) and (3.56) the corresponding heat flux is

$$\mathbf{J}_{\epsilon p} + \mathbf{J}_{\epsilon p}' = -\left[\frac{\rho c_V(R^2 - r^2)}{4\tau_N} + \kappa(0,0)\right]\nabla T \tag{3.95}$$

or, integrated over the cross section and making use of Eq. (3.74),

$$2\pi\int_0^R (\mathbf{J}_{\epsilon p} + \mathbf{J}_{\epsilon p}')r\,dr = -\pi R^2\rho c_V[(R^2/8\tau_N) + D_T(0,0)]\nabla T. \tag{3.96}$$

In order to have Poiseuille flow, the first (convective) term has to dominate over the second (conductive) term, which leads to the Poiseuille flow condition

$$D_T(0,0) = c_2{}'^2\tau_s \ll R^2/8\tau_N. \tag{3.97}$$

Poiseuille flow has been predicted independently by Süssmann and Thellung (1963) and by Gurzhi (1964). Its first observation is due to Mezhov-Deglin (1964, 1965, 1967) in experiments on solid ^{4}He. As is clear from Eqs. (3.96) and (3.97), such an experiment gives information about τ_N. In fact the Poiseuille flow determinations of τ_N were used to estimate the window condition in the first second-sound experiment [see Ackerman and Guyer (1968)]. For other experiments see Beck *et al.* (1974).

Finally, the third main domain is the *diffusion domain* defined as the low-frequency, low-wave number limit,

$$\omega\tau_J \ll 1, \qquad q^2\lambda_{Ni}{}^2 \ll 1. \tag{3.98}$$

In this domain we have from Eqs. (3.66) and (3.73), keeping only the leading term,

$$(1/\rho c_V)\varphi_l(\mathbf{q},\omega) = c_2{}^2\tau_J \tag{3.99}$$

so that Eq. (3.67) describes heat diffusion

$$-i\omega + D_T'q^2 = 0, \tag{3.100}$$

where, making use of Eqs. (3.74), (3.80), and (3.99),

$$D_T' = \lim_{\omega\to 0}\lim_{q\to 0}(1/\rho c_V)\{\varphi_l(\mathbf{q},\omega) + \kappa_l(\mathbf{q},\omega)\} = c_2{}^2\tau_J + c_2{}'^2\tau_S. \tag{3.101}$$

We close this section by deriving the expressions for c_2 and c_2' in the free quasiparticle approximation.

From Eqs. (3.73) and (3.36) we obtain immediately Dingle's expression (Dingle, 1952; Sussmann and Thellung, 1963; Enz, 1966a, 1968),

$$c_2{}^2 = \rho T c_V/9\rho_p = \tfrac{1}{3}\sum_\mu c_\mu{}^{-3}/\sum_\mu c_\mu{}^{-5}. \tag{3.102}$$

In order to derive c_2' we notice that in analogy to Eq. (3.39) the dissipative energy flux may be written as

$$\mathbf{J}_{\epsilon p}' = V^{-1}\sum_k \omega_k(\partial\omega_k/\partial\mathbf{k})\delta n_k, \tag{3.103}$$

where δn_k is determined by a Boltzmann equation with relaxation time τ_s,

$$(n_k + \delta n_k)^{\cdot} + \nabla(n_k + \delta n_k)\cdot(\partial\omega_k/\partial\mathbf{k}) = -(1/\tau_s)\delta n_k. \tag{3.104}$$

Calculating $\nabla(n_k + \delta n_k)$ from Eq. (3.29) with $\mathbf{v}_p = 0$ and with a time-independent local temperature we get to lowest order and after Fourier transformation in time

$$\delta n_k = -\frac{\tau_s/k_BT}{1 - i\omega\tau_s} n_k{}^0(n_k{}^0 + 1)\omega_k\frac{\partial\omega_k}{\partial\mathbf{k}}\cdot\frac{\nabla T}{T}. \tag{3.105}$$

Inserting into Eq. (3.103) and making use of Eqs. (3.35), (3.56), and (3.74) one readily finds Eq. (3.81) with [see Griffin (1965); Enz (1966a, 1968)]

$$c_2{}'^2 = (3k_BT^2c_V\rho V)^{-1}\sum_k \omega_k{}^2\left(\frac{\partial\omega_k}{\partial\mathbf{k}}\right)^2 n_k{}^0(n_k{}^0 + 1) = \tfrac{1}{3}\sum_\mu c_\mu{}^{-1}/\sum_\mu c_\mu{}^{-3}. \tag{3.106}$$

C. Isothermal crystal: first sound

We now assume $T = \text{const}$, $\psi = \text{const}$, and $\beta_{ij} = 0$ and again treat all fluctuations in linear approximation. Then Eqs. (3.21) and (3.46) imply $d(\rho s_L) = 0$ and $d(\rho s_p) = 0$, respectively, so that, according to Eq. (3.45),

$$(\rho s)^{\cdot} = 0, \tag{3.107}$$

and Eq. (3.59) becomes

$$\rho s_p\nabla\cdot\mathbf{v}_p = 0. \tag{3.108}$$

For $T = \text{const}$ it follows from Eq. (3.60) that $\mathbf{v}_p = 0$,

except for transient effects. Hence Eq. (3.108) is satisfied. On the other hand, the condition $\beta_{ij} = 0$ is satisfied only at $T = 0$.

Now the problem is reduced to the elastic equation of motion (3.7) which, together with Eqs. (3.4), (3.1), (3.43), (3.40), and (3.41), takes the form

$$\ddot{u}_i = (1/\rho) \sum_{jkl} (C_{ij,kl}\nabla_j\nabla_k u_l + \eta_{ij,kl}\nabla_j\nabla_k \dot{u}_l). \tag{3.109}$$

Going over to plane waves $\sim\exp[i(\mathbf{q}\cdot\mathbf{r} - \omega t)]$ and defining the symmetric positive matrices

$$(c_1^2(\hat{q}))_{ij} = (1/\rho) \sum_{mn} C_{im,nj}\hat{q}_m\hat{q}_n \tag{3.110}$$

and

$$(c_1^2(\hat{q})\tau_1(\hat{q}))_{ij} = (1/\rho) \sum_{mn} \eta_{im,nj}\hat{q}_m\hat{q}_n \tag{3.111}$$

Eq. (3.109) leads to the wave equation of first sound

$$(\omega^2/q^2)\mathbf{u} = c_1^2(1 - i\omega\tau_1)\mathbf{u}. \tag{3.112}$$

At zero temperature the eigenvalues $c_{1\mu}(\hat{q})$ of the matrix c_1 are the velocities c_μ of the acoustic phonons (3.34) [zero sound, see Cowley (1967)].

D. Thermal expansion coupling of first and second sound

We now relax all restrictions on $\mathbf{u}$, T, and ψ and allow $\beta_{ij} \neq 0$, that is, according to Eq. (3.17), nonvanishing thermal expansion. In order to keep complications within limits we still neglect the cross-coupling terms in Eqs. (3.42) and (3.43), $\mu_{ij,kl} = 0$. Then Eqs. (3.21) and (3.46) with (3.4) and (3.45) lead to

$$(\rho s)^{\cdot} = (\rho c_V/T)\dot{T} + (\beta\nabla)\cdot\dot{\mathbf{u}}, \tag{3.113}$$

while Eqs. (3.59) and (3.60) are unchanged. Equation (3.7) with (3.4), (3.41), and (3.42) becomes

$$\ddot{u}_i = (1/\rho) \sum_{jkl} (C_{ij,kl}\nabla_j\nabla_k u_l + \eta_{ij,kl}\nabla_j\nabla_k \dot{u}_l) - (1/\rho) \sum_j \beta_{ij}\nabla_j T - \nabla_i\psi.$$

In plane-wave representation the solution of Eq. (3.60) is given by Eq. (3.65), which we insert into (3.59) together with (3.113). This leads to the following generalization of Eq. (3.67):

$$-i\omega + (1/\rho c_V)(\varphi_l + \kappa_l)q^2]\delta T + \omega(T/\rho c_V)(\beta\mathbf{q})\cdot\mathbf{u} = 0. \tag{3.115}$$

Equation (3.114) gives rise to the generalized form of Eq. (3.112),

$$[\omega^2 - c_1^2(1 - i\omega\tau_1)q^2]\,\mathbf{u} - (i/\rho)(\beta\mathbf{q})\delta T = i\mathbf{q}\delta\psi \equiv -\mathbf{f}, \tag{3.116}$$

where $\mathbf{f}$ is the external force per unit mass.

We discuss the last two equations in the three main domains of $\varphi_l(\mathbf{q}, \omega)$ defined in Sec. III.B.

Starting with the domain (3.76), insertion of Eqs. (3.74) and (3.78) into (3.115) leads to

$$(-i\omega + \tau_N^{-1} + D_T q^2)(i/\rho)(\beta\mathbf{q})\delta T = -i\omega c_1^2\vartheta q^2\mathbf{u}, \tag{3.117}$$

where we have defined the matrix ϑ in analogy to Eq. (1.68) by

$$c_1^2\vartheta = (T/\rho^2 c_V)(\beta\hat{q}) \otimes (\beta\hat{q}). \tag{3.118}$$

Equation (3.117) shows that δT has the pole (3.79). Substituting this equation into Eq. (3.116) and retaining only lowest powers of q and ω, but separately for real and imaginary parts, we find the *displacement-displacement correlation function matrix*

$$\chi_{ij}(\mathbf{q}, \omega) \equiv \delta u_i/\delta f_j = -[\omega^2 - c_1^2(1 - i\omega\tau_1')q^2]_{ij}^{-1} \tag{3.119}$$

where

$$\tau_1' = \tau_1 + \tau_N\vartheta. \tag{3.120}$$

Equation (3.119) shows that in this domain $\mathbf{u}$ propagates with the *isothermal* velocities $c_{1\mu}(\hat{q})$. Retention of the lowest powers of q and ω under the conditions (3.76) means that the limit "first $\omega \to 0$, then $q \to 0$" has been taken. This is indeed the usual definition of the isothermal limit [e.g., Wehner and Klein (1971)]. From Eqs. (3.117) and (3.119) we see that in this limit $\delta T = 0$ and, according to (3.46), also $\delta(\rho s_p) = 0$. But from Eq. (3.21) we find with (3.4) and (3.119) that in this limit $\nabla(\rho s_L) = -\mathbf{q}(\mathbf{q}, \beta\mathbf{u}) = \hat{q}(\hat{q}, \beta c_1^{-2}\mathbf{f}) \neq 0$. This confirms the isothermal nature of this limit.

In the domain (3.98) consideration of Eqs. (3.99), (3.74), and (3.80) shows that we have to insert (3.101) into (3.115), so that with Eq. (3.118)

$$(-i\omega + D_T' q^2)(i/\rho)(\beta\mathbf{q})\delta T = -i\omega c_1^2\vartheta q^2\mathbf{u}. \tag{3.121}$$

This has a pole at the heat diffusion mode (3.100). Substituting Eq. (3.121) into (3.116) we find for the correlation function matrix defined in Eq. (3.119)

$$\chi(\mathbf{q}, \omega) = -\{\omega^2 - c_1^2(1 - i\omega[\tau_1 + (-i\omega + D_T' q^2)^{-1}\vartheta])q^2\}^{-1}, \tag{3.122}$$

This equation is equivalent to Eqs. (3.56) and (3.57) of the review by Götze and Michel (1974), which contains the most compact and systematic derivation of transport coefficients in dielectric crystals from microscopic theory, but which neglects phonon drift. The occurrence of the diffusion pole in the damping term of Eq. (3.122) has the important consequence that the two limits "first $\omega \to 0$, then $q \to 0$" and "first $q \to 0$, then $\omega \to 0$," which are both compatible with the conditions (3.98), lead to different

results (Cowley, 1967; Götze, 1967; Götze and Michel, 1967a, 1969; Wehner and Klein, 1971; Paszkiewicz, 1974). This difference is best seen in the dynamic structure factor as defined in Eq. (1.79a).

Let us first assume $D_T'q^2 \ll \omega$ and neglect terms $(D_T'q^2/\omega)^2$. Then Eq. (3.122) becomes

$$\chi(\mathbf{q}, \omega) = -\{\omega^2 - \tilde{c}_1^2(1 - i\omega\tilde{\tau}_1)q^2\}^{-1} \tag{3.122a}$$

where

$$\tilde{c}_1^2 = c_1^2(1 + \vartheta) \tag{3.123}$$

defines the *adiabatic sound velocity matrix* $\tilde{c}_1(\hat{q})$ in analogy to Eq. (1.73) and

$$\tilde{\tau} = (1 + \vartheta)^{-1}[\tau_1 + \vartheta D'_T q^2/\omega^2]. \tag{3.123a}$$

Equation (3.122a) shows that in the domain (3.98) $\mathbf{u}$ propagates with the *adiabatic* velocities $\tilde{c}_{1\mu}(\hat{q})$. According to the assumption $D_T'q^2 \ll \omega$ this is the case "first $q \to 0$ then $\omega \to 0$," which is the usual definition of the adiabatic limit [e.g., Wehner and Klein (1971)]. From Eqs. (3.121) and (3.122a) we see that again $\delta T = 0$ and $\delta(\rho s_p) = 0$. But now Eq. (3.21) with (3.4) and (3.122a) also leads to $(\rho s_L)^{\cdot} = \omega(\mathbf{q}, \beta\mathbf{u}) = 0$ which is indeed the adiabatic condition.

With the opposite assumption, $\omega \ll D_T'q^2 \ll c_1q$, we find from Eq. (3.122)

$$\chi(\mathbf{q}, \omega) = \{1 - i\omega(\tau_1 + \vartheta/D_T'q^2) + \omega^2[(\vartheta/D_T'^2q^4) - (c_1q)^{-2}] + O(\omega^3)\}^{-1}(c_1q)^{-2}. \tag{3.122b}$$

In order to obtain simple expressions for the dynamic structure factor we assume $\hat{q}$ to be a symmetry direction such that $c_1(\hat{q})$ and $\tau_1(\hat{q})$ are simultaneously diagonal and have eigenvalues c_{1l}, τ_{1l} with eigenvector $\mathbf{u}_l \parallel \hat{q}$ and c_{1t}, τ_{1t} with eigenvectors $\mathbf{u}_t \perp \hat{q}$. Assuming cubic symmetry so that the tension tensor β is diagonal it follows from Eq. (3.118) and the identity (1.75) that ϑ is also diagonal and has eigenvalues $\vartheta_l = (c_p/c_V) - 1$ and $\vartheta_t = 0$. In this representation $\chi_{\mu\nu} = \chi_\mu\delta_{\mu\nu}$ and we find from Eq. (3.122b)

$$\pi S_\mu(\mathbf{q}, \omega) = \frac{q^2}{\omega}\,\mathrm{Im}\chi_\mu(\mathbf{q}, \omega) = c_{1\mu}{}^{-2}\left(\tau_{1\mu} + \frac{\vartheta_\mu}{D_T'q^2}\right) \times\left\{1 - \omega^2\left[\left(\tau_{1\mu} + \frac{\vartheta_\mu}{D'q^2}\right)^2 + 2\left(\frac{\vartheta_\mu}{D'^2q^4} - \frac{1}{c_{1\mu}{}^2q^2}\right)\right] + O(\omega^4)\right\} \tag{3.124}$$

This equation shows that $S_l(\mathbf{q}, \omega)$ has a *peak* at $\omega = 0$ which in the limit $q \to 0$ grows as q^{-2} and narrows as q^{-4} while $S_t(\mathbf{q}, \omega)$ has a constant *dip* narrowing as q^{-2}. This *heat diffusion peak* of S_l is well known from Rayleigh and neutron scattering. Equation (3.122a), on the other hand, gives rise to an S_μ as given by S_I in Eqs. (1.79b) for $r = 0$, and has peaks at $\omega = \pm\tilde{c}_{1\mu}q$ for both $\mu = l$ and t. These are the *first-sound peaks* well known from Billouin and neutron scattering.

The exciting new fact in the situation just described is that in neutron scattering a peak (and not a dip!) has recently been discovered in S_t near the structural phase transition of Nb_3Sn at $T = 45$ K (Shirane and Axe, 1971). [See also Axe and Shirane (1973a,b)]. This phase transition is due to a soft transverse acoustic mode for which $c_{1t} \to 0$ as $T \to T_c$ [see Fig. 1 of Shirane and Axe (1971)]. Similar "central peaks" of S_t have recently been discovered near phase transitions of $SrTiO_3$ and other perovskites (Riste *et al.*, 1971; Shapiro *et al.*, 1972; Kjems *et al.*, 1973; Müller *et al.*, 1974). In these cases, however, the soft mode responsible for the transition is an optic transverse mode at the R-corner of the Brillouin zone.

All these central peaks can be parametrized by an equation which has exactly the form of Eq. (3.122). In the notation of Eq. (1) of Pytte (1973) the parameter values of Eq. (3.122) are $\omega_0 = c_1q$, $\omega_\infty = \tilde{c}_1q$, $\Gamma = \tilde{\tau}_1\tilde{c}_1^2q^2$, $\gamma = D_T'q^2$ and $\delta^2 = \vartheta c_1^2q^2$. It is clear therefore that these central peaks are also due to a pole in the damping term of Eq. (3.122). But the residue of this pole cannot be due to thermal expansion coupling of the solft mode to the heat diffusion mode. Pytte (1973) [see also Axe and Shirane (1973b)] has emphasized the analogy of this pole with the Mountain mode $\omega \cong -i/\tau$, of Sec. I.E. However, the fact that in general $\tau^{-1} \gg D_T'q^2$ is difficult to reconcile with the extreme narrowness of the central peaks. So far none of the formal analyses of the problem (Feder, 1971, 1973; Schwabl, 1972a,b, 1973; Enz, 1972c; Silberglitt, 1972; Schneider, 1973; Meier, 1973; Beck and Meier, 1973) has been successful in explaining the physical origin of this pole. From a recent quantitative analysis of the problem in terms of the hydrodynamic description developed in this section it appears, however, that the relevant mechanism is a coupling of the solft mode to the heat diffusion mode due to the strong temperature dependence of the soft-mode frequency (Enz, 1974).

After this digression into central peaks let us, finally, discuss the domain (3.69). Insertion of Eqs. (3.71), (3.73), and (3.118) into (3.115) yields

$$\left[\omega^2 - c_2^2\left(1 - i\frac{1 + (\lambda_N{}^2)_l\,q^2}{\omega\tau_J} - i\omega\tau_2\right)q^2\right] \times (i/\rho)(\beta\mathbf{q})\delta T = \omega^2c_1^2\vartheta q^2\mathbf{u}. \tag{3.125}$$

Thus δT has a pole at the second-sound mode (3.72). Substituting Eq. (3.125) into (3.116) we find for the correlation function matrix of Eq. (3.119), neglecting the term $(\lambda_N{}^2)_l\,q^2$,

$$\chi(\mathbf{q}, \omega) = -G(\mathbf{q}, \omega)\{\omega^2 - c_2^2[1 - (i/\omega\tau_J) - i\omega\tau_2]q^2\} \tag{3.126}$$

where

$$G^{-1}(\mathbf{q}, \omega) = [\omega^2 - c_1^2(1 - i\omega\tau_1)q^2] \times\{\omega^2 - c_2^2[1 - (i/\omega\tau_J) - i\omega\tau_2]q^2\} - c_1^2\vartheta q^2\omega^2. \tag{3.127}$$

These equations are essentially the same as Eqs. (2.24) and (2.26) of Wehner and Klein (1972). These authors point out that in light scattering there is a *direct* coupling of light to thermal fluctuations, via the temperature derivative of

the dielectric constant, in addition to the *indirect* coupling via thermal expansion.

Equations (3.126) and (3.127) are closely analogous to Eqs. (1.71) and (1.72), except for the important difference that here the second-sound damping contains the term $i/\omega\tau_J$ which gives rise to the window condition (3.70). Because of this term the factorization of G^{-1} takes the form

$$G^{-1}(\mathbf{q},\omega) = [\omega^2 - c_{\rm I}^2(1 - i\omega\tau_{\rm I} - i/\omega\tau_{J\rm I})q^2] \times[\omega^2 - c_{\rm II}^2(1 - i\omega\tau_{\rm II} - i/\omega\tau_{J\rm II})q^2] \equiv A_{\rm I}A_{\rm II}, \quad (3.128)$$

valid to first order in the small quantities $\omega\tau_{\rm I}$, $\omega\tau_{\rm II}$, $(\omega\tau_{J\rm I})^{-1}$, $(\omega\tau_{J\rm II})^{-1}$, etc. Comparison with Eq. (3.127) yields, in analogy to the superfluid case,

$$c_{\rm I}^2 = \tilde{c}_{\rm I}^2 + u, \qquad c_{\rm II}^2 = c_2^2 - u,$$
$$c_{\rm I}^2\tau_{\rm I} = c_1^2\tau_1 + v, \qquad c_{\rm II}^2\tau_{\rm II} = c_2^2\tau^2 - v,$$
$$c_{\rm II}^2\tau_{J\rm II}^{-1} = c_2^2\tau_J^{-1} - c_{\rm I}^2\tau_{J\rm I}^{-1},$$

where the matrices, u, v and $\tau_{J\rm I}^{-1}$ are determined, respectively, by the equations

$$c_1^2c_2^2 = c_{\rm I}^2c_{\rm II}^2,$$
$$c_1^2c_2^2(\tau_1 + \tau_2) = c_{\rm I}^2\tau_{\rm I}c_{\rm II}^2 + c_1^2c_{\rm II}^2\tau_{\rm II},$$

and

$$c_1^2c_2^2\tau_J^{-1} = c_{\rm I}^2\tau_{J\rm I}^{-1}c_{\rm II}^2 + c_1^2c_{\rm II}^2\tau_{J\rm II}^{-1}.$$

Since at low temperatures $\beta \propto c_V \propto T^3$ and $s_p \propto T^3$ it follows from Eq. (3.118) that $\vartheta \propto T^4$ as in the superfluid case. Since, on the other hand, the window condition (3.70) is only satisfied at low temperatures, we can assume ϑ to be small. One then finds to first order in ϑ

$$u \cong (c_1^2 - c_2^2)^{-1}c_1^2c_2^2\vartheta,$$
$$v \cong (c_1^2 - c_2^2)^{-1}[c_2^2\tau_2(c_1^2\vartheta + u) - c_1^2\tau_1u],$$
$$\tau_{J\rm I}^{-1} \cong (c_1^2 - c_2^2)^{-1}c_1^{-2}(c_1^2\vartheta + u)c_2^2\tau_J^{-1}.$$

The analogous decomposition of the numerator of Eq. (3.126) is

$$a \equiv \omega^2 - c_2^2(1 - i\omega\tau_2 - i/\omega\tau_J)q^2 = A_{\rm II}(1 - R) + RA_{\rm I} \quad (3.129)$$

with

$$R = r(1 + i\omega\tau_r + i/\omega\tau_{Jr}), \quad (3.130)$$

valid to first order in the small quantities $\omega\tau_r$, $(\omega\tau_{Jr})^{-1}$, etc. The matrices r, τ_r, and τ_{Jr} are determined, respectively, by the equations

$$c_2^2 = c_{\rm II}^2(1 - r) + rc_{\rm I}^2,$$
$$c_2^2\tau_2 = c_{\rm II}^2\tau_{\rm II}(1 - r) + c_{\rm II}^2r\tau_r + rc_{\rm I}^2\tau_{\rm I} - r\tau_rc_{\rm I}^2,$$

and

$$c_2^2\tau_J^{-1} = c_{\rm II}^2\tau_{J\rm II}^{-1}(1 - r) + c_{\rm II}^2r\tau_{Jr}^{-1} + rc_{\rm I}^2\tau_{J\rm I}^{-1} - r\tau_{Jr}^{-1}c_{\rm I}^2.$$

To first order in ϑ one finds

$$r \cong (c_1^2 - c_2^2)^{-1}c_1^2c_2^2\vartheta(c_1^2 - c_2^2)^{-1} \quad (3.131)$$

which is analogous to Eqs. (1.78) and (1.77), taken for small ϑ,

$$\tau_r \cong (c_1^2\tau_1 - c_2^2\tau_2 - r^{-1}v)(c_1^2 - c_2^2)^{-1}$$

and

$$\tau_{Jr}^{-1} \cong -(c_2^2\tau_J^{-1} + r^{-1}c_1^2\tau_{J\rm I}^{-1})(c_1^2 - c_2^2)^{-1}.$$

Now Eqs. (3.126), (3.128), and (3.129) can be written in the form of Eq. (1.79),

$$-\chi(\mathbf{q},\omega) = A_{\rm II}^{-1}aA_{\rm I}^{-1} = (1 - R)A_{\rm I}^{-1} + A_{\rm II}^{-1}R \quad (3.132)$$

in which the first- and second-sound denominators $A_{\rm I}$ and $A_{\rm II}$ are separated, and where R is given by Eq. (3.130). This form implies that in inelastic neutron scattering with small momentum transfer $\mathbf{q}$ and in Brillouin scattering there exist peaks due to both first and second sound. However, due to the smallness of the amplitude ratio r, Eq. (3.131), mechanical second-sound excitation is too weak to be seen in such spectra. This is different from superfluid helium where the absence of a window condition allows the choice of temperatures very close to the λ transition where ϑ becomes large (see Sec. I.D).

The calculation of the integrated intensity analogous to Eqs. (1.79a) and (1.80) is complicated both by the matrix character of Eq. (3.132) and by the occurrence of the damping terms $i/\omega\tau_{J\rm I}$, $i/\omega\tau_{J\rm II}$, and $i/\omega\tau_{Jr}$ in this equation. We therefore refrain from deriving an expression for it.

We can now also define adiabatic elastic constants in analogy to Eq. (3.110) by

$$(\tilde{c}_{\rm I}^2(\hat{q}))_{ij} = (1/\rho)\sum_{mn}\tilde{C}_{im,nj}\hat{q}_m\hat{q}_n. \quad (3.133)$$

Making use of Eqs. (3.123) and (3.118) we find [compare Eq. (3.22) of Leibfried and Ludwig (1961); Eq. (4.11) of Gurevich and Efros (1966); and Eq. (27) of Götze and Michel (1967a). See also Paszkiewicz (1974)]

$$\tilde{C}_{ij,kl} = C_{ij,kl} + (T/\rho c_V)\beta_{ij}\beta_{kl}. \quad (3.134)$$

IV. MAGNETIC CRYSTALS

The role of the exchange interaction in the transition to an ordered state of the spins in a magnetic crystal was first discussed by Heisenberg (1928), Frenkel (1928), and Dorfman (Dorfman and Jaanus, 1929) in 1928. Two years later Bloch (1930) showed that this coupling gives rise to

wave excitations of the spin orientation. These spin waves or magnons were subsequently derived by Landau and Lifshitz (1935) in the framework of a purely phenomenological description of magnetism. But it was only the hydrodynamic description by Halperin and Hohenberg (1969) which established, for the planar and isotropic ferro- and antiferromagnets, the close analogy with the phenomenological description of superfluid helium. [For a microscopic analogy see Matsubara and Matsuda (1956). See also Kawasaki (1970)].

Halperin and Hohenberg (1969) obtained for the isotropic (Heisenberg) ferromagnet the well known quadratic magnon spectrum $\omega = Dq^2$, originally derived, in the collisionless case and for small q, by Bloch (1930). But for the planar ferromagnet in which the preferred direction of magnetization is restricted to an "easy plane" they found a sound-like spectrum, $\omega = c_1 q$ for small q. In the collisionless case such a linear dispersion relation had been discovered for antiferromagnets by Hulthén (1936), while for planar ferromagnets it was derived by Matsubara and Matsuda (1956). Halperin and Hohenberg also rederived this Hulthén mode for the planar antiferromagnet in which the sublattice magnetization is in an "easy plane" and for the isotropic (Heisenberg) antiferromagnet. They also obtained new predictions for the magnon dampings. A derivation of these results from the point of view of irreversible thermodynamics was subsequently given by Enz (1971).[2]

In analogy to dielectric crystals the two fluids in a magnetic crystal have to be identified as the classical magnetization field $\mathbf{M}(\mathbf{r}, t)$ and the fluid of the thermal magnons. Indeed, $\mathbf{M}$ describes the dynamics of the condensed phase of the ordered spins. It is also the order parameter of the ferromagnet, while in the antiferromagnet the sublattice (staggered) magnetization $\mathbf{N}$ plays this role. The magnetization satisfies a local conservation law analogous to the continuity equation, whereas this is not the case for the mass density of the magnon fluid since magnon number is not conserved. This difference with superfluids already encountered with dielectric crystals in the last section justifies the omission of the "first fluid" of thermal magnons in the treatment of Halperin and Hohenberg (1969).

The hydrodynamics of the magnon fluid was first considered in a note by Gulayev (1965) [see also Dingle (1952) and Gurzhi (1965)] in which the name "second spin waves" was introduced for the excitations of this fluid. Subsequently Reiter (1968) derived a Boltzmann equation for the magnon fluid of a Heisenberg ferromagnet. A generalized form of this Boltzmann equation was used by Michel and Schwabl (1969, 1970a) to derive coupled hydrodynamic equations for the two fluids in close analogy to the treatment of dielectric crystals by Götze and Michel (1967a).

This analogy extends, in particular, to the introduction of a magnon drift (quasimomentum) which is essential for the fluid properties of the magnons and hence for the existence of second sound or second spin waves (see the Introduction), called second magnons by Michel and Schwabl (1969, 1970a). In subsequent papers Michel and Schwabl (1970b,c; Schwabl and Michel, 1970) have derived this hydrodynamics from microscopic equations of motion. As was shown by Enz (1972a), the addition of the local balance equation for the magnon momentum to the hydrodynamic equations of Halperin and Hohenberg (1969; Enz, 1971) then also leads to second sound.

As discussed in the Introduction, for Bosonic excitations the existence of second sound depends on Landau's criterion for superfluidity. This means that a Bloch mode, $\omega = Dq^2$, does not give rise to second sound, while a Hulthén mode, $\omega = c_1 q$, does. In insulating magnetic crystals, however, the long-range dipolar interaction between the spins gives rise to a gap of the Bloch mode analogous to the plasmon and also transforms a Hulthén mode into a Bloch mode with gap. In fact, to date neutron scattering which measures directly the magnon spectrum has, to our knowledge, not revealed any case of a purely sound-like dispersion relation. The search for second sound in insulating magnetic crystals is therefore a quite realistic problem. The quantitative conditions for the realizability of a second magnon have been analyzed by Michel and Schwabl (1971a), Forney (1972), and Forney and Jäckle (1973), the conclusions being quite encouraging.

In this section we follow essentially the approach of Enz (1972a) except for two generalizations. The first is a description of the response measured by neutron scattering which is done in analogy to previous sections. The second generalization concerns the cases of the axial and isotropic ferro- and antiferromagnets, which are treated in much greater detail, including, in particular, the effects of the dipole–dipole interaction and a comparison with the phenomenological approach of Kittel (1963; Herring and Kittel, 1951). This generalization allows a direct comparison with earlier results for the magnon spectra (Clogston *et al.*, 1956; Loudon and Pincus, 1963; Brooks Harris, 1966; Anda, 1973) but, in addition, also yields expressions for the damping of all the magnon modes.

Thus we arrive at a general and unified hydrodynamic description of magnetic crystals within the classification into planar, isotropic, and axial systems. This classification, of course, is not exhaustive, as can be seen from more specialized reviews (Keffer, 1966; Nagamiya *et al.*, 1955). It is also evident that lattice displacements introduce variations in the coupling functions of the exchange interaction, giving rise to magnon–phonon interactions and hence to a coupling of the two fluids of this section with the two fluids of the preceding section.

A. Hydrodynamics of the planar ferromagnet

A planar ferromagnet is described by a Hamiltonian

$$\mathcal{H} = -\sum_{ij} \{J_{ij}^z S_i^z S_j^z + J_{ij}^\perp (S_i^x S_j^x + S_i^y S_j^y)\} \qquad (4.1)$$

with coupling constants $J_{ij}^\perp > J_{ij}^z > 0$, for the dominant pairs of neighbors i, j.

[2] This paper contains a confusion between magnon drift and "superfluid" velocity which is corrected as follows: Put $\tau = \infty$, discard the comparison of Eqs. (31) to (33) and (46) with Eqs. (4.14a) to (4.14c) and (4.20 a,b), respectively, of Michel and Schwabl (1970a). In addition the condition for the planar case should read $J_{ij}^z < J_{ij}^\perp$. See also Michel and Schwabl (1971b).

The crucial feature of a planar ferromagnetic state is the existence of a nonvanishing average of the spin raising operator $S_i^+ = S_i^x + iS_i^y$. The ferromagnetic average has the property that it is taken with a density matrix which, in analogy to Eq. (1.1), commutes with the effective Hamiltonian $\mathfrak{H} - m\mu \sum_i S_i^z$ (see Sec. I.A) but not with $\sum_i S_i^z$. Here $m\mu$ is the microscopic magnetic field seen by the spins (we use units such that the Bohr magneton $\mu_B = 1$). Since S_i^+ increases the value of S_i^z by one and since $[\sum_i S_i^z, \mathfrak{H}] = 0$ the Heisenberg representation

$$S_i^+(t) = \exp(i\mathfrak{H}t) S_i^+ \exp(-i\mathfrak{H}t),$$

together with Eq. (4.1), implies the existence of a nonvanishing order parameter $(1/v)\langle S_i^+\rangle$ with the time evolution

$$\begin{aligned}(1/v)\langle S_i^+(t)\rangle &= (1/v)\exp(+im\mu t)\langle S_i^+\rangle \\ &= M_\perp \exp[+i\varphi(t)] = M_x + iM_y. \end{aligned} \tag{4.2}$$

Here v is the volume of the unit cell, $M_\perp$ the perpendicular magnetization, and $\varphi(t)$ the precession angle. From Eq. (4.2) we have

$$\varphi(t) = \varphi(0) + m\mu t. \tag{4.3}$$

The magnetization energy per unit mass is

$$\epsilon_s = (\rho V)^{-1}\langle \mathfrak{H}\rangle. \tag{4.4}$$

The index s refers to the identification of the magnetization with the "superfluid" phase. ρ is the mass density of the crystal, which here is a constant, and V the volume of the crystal. Since ϵ_s and the parallel magnetization

$$M_z = (1/v)\langle S_i^z\rangle \tag{4.5}$$

are conserved quantities, $M_\perp$ must be a function of ϵ_s and M_z. As pointed out by Halperin and Hohenberg (1969), this supposes the existence of a relaxation mechanism which brings $M_\perp$ to its value $M_\perp(\epsilon_s, M_z)$ in a microscopic time.

In thermal equilibrium at a temperature T below the Curie point T_c, $\epsilon_s = \epsilon_s(M_z, T)$ so that

$$M_\perp = M_\perp(M_z, T) \tag{4.6}$$

and the thermodynamic field is, for $M_z \ll M_\perp$,

$$h_z(M_z, T) = (\partial(\rho\epsilon_s)/\partial M_z)_{ss} = \chi^{-1}(T) M_z, \tag{4.7}$$

where $s_s = s_s(M_z, T)$ is the magnetization entropy per unit mass and χ is the isothermal longitudinal susceptibility. Thermal equilibrium also means that the precession is zero and that the external field H_z equals the thermodynamic field (4.7), that is,

$$\dot\varphi_{eq} = m\mu_{eq} = 0, \qquad (H_z)_{eq} = h_z. \tag{4.8}$$

For $\mu \neq 0$ Eq. (4.3) tells us that

$$\dot\varphi = m\mu. \tag{4.9}$$

On the other hand, for an external field H_z so small that $M_z = 0$ is maintained, we see from Eq. (4.1) that we must have $m\mu = H_z$ (this is a metastable state!), hence

$$\mu = (1/m)(H_z - h_z). \tag{4.10}$$

Here μ plays the role of a chemical potential per unit mass where m is the mass of the magnetic ions.

Going over to local variables $\epsilon_s(\mathbf{r}, t)$, etc. defined as averages of density operators [see Halperin and Hohenberg (1960)], we may again define a velocity field

$$\mathbf{v}_s = -(1/m)\nabla\varphi \tag{4.11}$$

which is exactly analogous to the superfluid velocity (1.4). According to Eq. (4.9) $\mathbf{v}_s$ satisfies the equation of motion analogous to Eq. (1.5),

$$\dot{\mathbf{v}}_s = -\nabla(\mu + \mu'), \tag{4.12}$$

where μ' is the dissipative part of μ.

Equations (4.12) and (4.10) are the same, respectively, as Eqs. (2.15) and (2.10) of Halperin and Hohenberg (1969) with the identification of $\mathbf{v}/m$ with our $\mathbf{v}_s$ and of ψ/m with our μ.

In view of Eq. (4.7) and because $\mathbf{v}_s$ is an independent thermodynamic variable we have the thermodynamic relation

$$d(\rho\epsilon_s) = h_z\, dM_z + \rho_s \mathbf{v}_s \cdot d\mathbf{v}_s + T d(\rho s_s). \tag{4.13}$$

The coefficient ρ_s, which for simplicity we treat as a scalar, must be positive. It has the meaning of a stiffness constant [Halperin and Hohenberg (1969). See below.] and the dimension of a mass density and is the analogue of the superfluid mass density of Sec. I.

Local conservation of parallel magnetization is expressed in analogy to the first Eq. (1.6) and to Eq. (3.8) by

$$m\dot M_z + \nabla\cdot(\mathbf{j}_s + \mathbf{j}_s') = 0, \tag{4.14}$$

where the dissipative part $\mathbf{j}_s'$ describes spin diffusion [note that the corresponding terms in the first Eq. (1.6) and in Eq. (3.8) were missing]. Apart from a factor m, Eq. (4.14) is the same as Eq. (2.16) of Halperin and Hohenberg (1969).

In analogy to the case of phonons in dielectric crystals [Eq. (3.24)], the thermal magnons are described by a momentum density $\mathbf{j}_M$ and a local drift $\mathbf{v}_M$ which define the magnon mass density ρ_M through

$$\mathbf{j}_M = \rho_M \mathbf{v}_M. \tag{4.15}$$

Again we treat ρ_M as a scalar for the sake of simplicity. The local magnon momentum balance equation is, in analogy to Eq. (3.28),

$$\dot{\mathbf{j}}_M + \nabla(\Pi_M + \Pi_M') = -(1/\tau_J)\mathbf{j}_M. \tag{4.16}$$

Here, in analogy to Eq. (3.25),

$$\Pi_M = -\rho f_M 1 + \rho_M \mathbf{v}_M \otimes \mathbf{v}_M \tag{4.17}$$

is the momentum flux and f_M the magnon free energy per unit mass which satisfies the thermodynamic relation analogous to Eq. (3.27)

$$d(\rho f_M) = -\mathbf{j}_M \cdot d\mathbf{v}_M - \rho s_M \, dT. \tag{4.18}$$

Here s_M is the magnon entropy per unit mass, and Π_M' the dissipative part of Π_M.

In analogy to Eq. (3.26) the magnon energy density is given by

$$\rho\epsilon_M = \mathbf{v}_M \cdot \mathbf{j}_M + \rho(f_M + T s_M). \tag{4.19}$$

The total energy is the sum of the magnetization energy, the magnon energy, and the Zeeman energy of an arbitrary external field $\mathbf{H}$,

$$d(\rho\epsilon) = d(\rho\epsilon_s) + d(\rho\epsilon_M) - \mathbf{H} \cdot d\mathbf{M}. \tag{4.20}$$

Since according to Eq. (4.2),

$$\mathbf{M} = (M_\perp \cos\varphi, M_\perp \sin\varphi, M_z) \tag{4.21}$$

we can choose $\mathbf{H} = (0, H_y, H_z)$. Making use of Eqs. (4.13), (4.19), and (4.18), Eq. (4.20) can then be written in analogy to Eq. (3.47) as

$$\begin{aligned} d(\rho\epsilon) = {} & (h_z - H_z)\, dM_z - H_y \, dM_y \\ & + \rho_s \mathbf{v}_s \cdot d\mathbf{v}_s + \mathbf{v}_M \cdot d\mathbf{j}_M + T \, d(\rho s) \end{aligned} \tag{4.22}$$

where $s = s_s + s_M$.

The equilibrium condition for the total free energy at constant T and $\mathbf{j}_M$ is

$$d\int(\rho\epsilon - T\rho s)\, d^3r = 0, \qquad \delta T = 0, \qquad \delta\mathbf{j}_M = 0. \tag{4.23}$$

Remembering Eqs. (4.21) and (4.6) and making use of (4.11) and of a partial integration to eliminate $\mathbf{v}_s$, the conditions (4.23) with (4.22) yield

$$\begin{aligned} \int\{ & [h_z - H_z - H_y(\partial M_\perp/\partial M_z)_T \sin\varphi]\delta M_z \\ & + [-H_y M_\perp \cos\varphi - (\rho_s/m^2)\nabla^2\varphi]\delta\varphi\}\, d^3r = 0. \end{aligned}$$

Since δM_z and $\delta\varphi$ are independent variations, we find for the Fourier components, for small H_y,

$$\varphi = (m^2 M_\perp/q^2\rho_s) H_y + O(H_y^3) \tag{4.24}$$

and

$$h_z = H_z + \frac{m^2 M_\perp}{q^2\rho_s}\left(\frac{\partial M_\perp}{\partial M_z}\right)_T H_y^2 + O(H_y^4).$$

In equilibrium, $H_y = 0$, so that $\varphi_{eq} = 0$ and we recover Eq. (4.8). Inserting Eq. (4.24) into (4.21) we also obtain

$$\begin{aligned} M_x &= M_\perp + O(H_y^2), \\ M_y &= \chi_\perp H_y + O(H_y^3), \end{aligned} \tag{4.25}$$

where

$$\chi_\perp = m^2 M_\perp^2/q^2\rho_s \tag{4.26}$$

is the isothermal transverse susceptibility [see Eq. (2.30) of Halperin and Hohenberg (1969)].

From Eqs. (4.21), (4.24), and (4.11) it also follows that for $H_y = 0$

$$\nabla M_y = -m M_\perp \mathbf{v}_s \tag{4.27}$$

so that

$$\rho_s \mathbf{v}_s \cdot d\mathbf{v}_s = (\rho_s/m^2 M_\perp^2)\nabla M_y \cdot d\nabla M_y \tag{4.28}$$

is recognized as an exchange energy contribution (Kittel, 1973; Keffer, 1966) to Eq. (4.13). It is of the same form as the elastic energy contribution

$$\sum_{ijkl} C_{ij,kl}\nabla_i u_j \, d\nabla_k u_l$$

to Eq. (3.13), expressed with the aid of Eqs. (3.3) and (3.4). Thus we see that $\rho_s/m^2 M_\perp^2$ plays the same role as the "elastic stiffness constants" $C_{ij,kl}$ (Kittel, 1963).

As in the former sections, we convert the energy conservation

$$(\rho\epsilon)^\cdot + \nabla\cdot(\mathbf{J}_\epsilon + \mathbf{J}_\epsilon') = 0 \tag{4.29}$$

into the entropy balance

$$(\rho s)^\cdot + \nabla\cdot(\mathbf{J}_s + \mathbf{J}_s') = \sigma, \tag{4.30}$$

thus obtaining expressions for the magnetization flux $\mathbf{j}_s$, the energy flux $\mathbf{J}_\epsilon$, and the entropy flux $\mathbf{J}_s$. The dissipative parts μ', Π_M', $\mathbf{j}_s'$, $\mathbf{J}_\epsilon'$, and $\mathbf{J}_s'$ will be constructed afterwards.

From Eq. (4.22) we obtain with Eq. (4.10)

$$T(\rho s)^\cdot = (\rho\epsilon)^\cdot + m\mu\dot{M}_z + H_y\dot{M}_y - \rho_s\mathbf{v}_s\cdot\dot{\mathbf{v}}_s - \mathbf{v}_M\cdot\dot{\mathbf{j}}_M, \tag{4.31}$$

which apart from the magnon drift term is the same, for $\mathbf{H} = 0$, as Eq. (2.31) of Halperin and Hohenberg (1969). In order to compensate for the term $H_y\dot{M}_y$ in Eq. (4.31) the equation of motion (4.14) has to be modified in the presence of a transverse field H_y. This is done by writing

$$\begin{aligned} \dot{M}_z + (1/m)\nabla\cdot(\mathbf{j}_s + \mathbf{j}_s') &= -M_x H_y, \\ \dot{M}_x - \gamma M_x\dot{M}_z &= -M_y(H_z - h_z), \\ \dot{M}_y - \gamma M_y\dot{M}_z &= M_x(H_z - h_z), \end{aligned} \tag{4.32}$$

where the last two equations follow from Eqs. (4.21) and

(4.9) with

$$\gamma = (1/M_\perp)(\partial M_\perp/\partial M_z)_T.$$

Since according to Eq. (4.26) $\chi_\perp \gg \chi$ for small q, the right-hand side of Eq. (4.32) is just the precession term $\mathbf{M} \times (\mathbf{h} - \mathbf{H})$ with $\mathbf{h} = (0, 0, h_z)$, which brings Eq. (4.32) into the form of the Bloch equation (Kittel, 1963) or Landau–Lifshitz equation (Keffer, 1966).

Substituting for the right-hand side of Eq. (4.31) successively from the equations of motion (4.29), the first Eq. (4.32), (4.12), and (4.16), making use, in the last equation, of Eqs. (4.15), (4.17), and (4.18), we find

$$\begin{aligned}(\rho s)^\cdot = &-\nabla\cdot[(1/T)(\mathbf{J}_\epsilon + \mathbf{J}_\epsilon' + \mu(\mathbf{j}_s + \mathbf{j}_s') - \mu'\rho_s\mathbf{v}_s \\ &- \Pi_M'\mathbf{v}_M)] + [\mathbf{J}_\epsilon + \mathbf{J}_\epsilon' + \mu(\mathbf{j}_s + \mathbf{j}_s') - \mu'\rho_s\mathbf{v}_s \\ &- \Pi_M'\mathbf{v}_M - T\rho s_M\mathbf{v}_M]\cdot\nabla(1/T) + (1/T)(\mathbf{j}_s + \mathbf{j}_s' \\ &+ \rho_s\mathbf{v}_s)\cdot\nabla\mu - (1/T)[\mu'\nabla\cdot(\rho_s\mathbf{v}_s) + (\Pi_M'\nabla)\cdot\mathbf{v}_M] \\ &+ (\rho_M/T\tau_J)\mathbf{v}_M^2.\end{aligned}$$

Since the nondissipative terms on the right-hand side must add to a divergence we find

$$\mathbf{j}_s = -\rho_s\mathbf{v}_s \tag{4.33}$$

and

$$\mathbf{J}_\epsilon = \mu\rho_s\mathbf{v}_s + T\rho s_M\mathbf{v}_M \tag{4.34}$$

[Compare Eqs. (2.22), (2.35), and (2.21), (2.36) of Halperin and Hohenberg (1969)]. Comparison with Eq. (4.30) then yields the entropy flux

$$\mathbf{J}_s = \rho s_M\mathbf{v}_M, \tag{4.35}$$

its dissipative part

$$\mathbf{J}_s' = (1/T)(\mathbf{J}_\epsilon' + \mu\mathbf{j}_s' - \mu'\rho_s\mathbf{v}_s - \Pi_M'\mathbf{v}_M), \tag{4.36}$$

and the entropy production density

$$\begin{aligned}\sigma = &T\mathbf{J}_s'\cdot\nabla(1/T) + (1/T)[\mathbf{j}_s'\cdot\nabla\mu + \mu'\nabla\cdot\mathbf{j}_s \\ &+ (\Pi_M'\nabla)\cdot\mathbf{v}_M] + (\rho_M/T\tau_J)\mathbf{v}_M^2.\end{aligned} \tag{4.37}$$

[Note that the partition of dissipative terms into $\mathbf{J}_s'$ and σ is dictated by the positiveness of σ. In the paper by Enz (1972a) this condition was overlooked, without serious consequences, however.]

For small perturbations and for $H_y = 0$, the dissipative parts $\mathbf{j}_s'$, μ', Π_M', and $\mathbf{J}_s'$ must be linear functions of the gradients of the coefficients in Eq. (4.22), that is, of $-m\nabla\mu$, $\nabla \otimes (\rho_s\mathbf{v}_s)$, $\nabla \otimes \mathbf{v}_M$, and ∇T. Irreversibility requires that under time reversal they transform with opposite sign as their respective nondissipative parts. Since $\mathbf{j}_s$ and Π_M transform with positive sign, μ, $\mathbf{v}_M$, and $\mathbf{J}_s$ (and also $M_\perp$!) with negative sign, we find, with Eq. (4.33),

$$\begin{aligned}\mathbf{j}_s' &= +m^2\xi\nabla\mu, \\ \mu' &= +(\zeta\nabla)\cdot\mathbf{j}_s, \\ \Pi_{Mij}' &= -\sum_{kl}\gamma_{ij,kl}\nabla_k v_{Ml}, \\ \mathbf{J}_s' &= -(\kappa/T)\nabla T.\end{aligned} \tag{4.38}$$

Here we have neglected cross-terms coupling magnetic and thermal variables, that is, respectively, terms proportional to $\nabla \otimes \mathbf{v}_M$, ∇T, $\nabla\mu$, and $\nabla \otimes (\rho_s\mathbf{v}_s)$ in $\mathbf{j}_s'$, μ', Π_M', and $\mathbf{J}_s'$. [Note that the last Eq. (4.38) differs from Eq. (30) of Enz (1972a) which through Eq. (27) led to additional damping terms]. ξ is the spin diffusion tensor, ζ a second viscosity tensor analogous to ζ_3 in superfluid helium, $\gamma_{ij,kl}$ is the magnon viscosity tensor, and κ the magnon heat conductivity tensory. With Eq. (4.38), Eq. (4.37) becomes

$$\begin{aligned}T\sigma = &\nabla T\cdot[(\kappa/T)\nabla T] + m^2\nabla\mu\cdot(\xi\nabla\mu) + [\nabla\cdot(\rho_s\mathbf{v}_s)](\zeta\nabla) \\ &\cdot(\rho_s\mathbf{v}_s) + \sum_{ijkl}\nabla_i v_{Mj}\gamma_{ij,kl}\nabla_k v_{Ml} + (\rho_M/\tau_J)\mathbf{v}_M^2.\end{aligned} \tag{4.39}$$

B. Hydrodynamic modes of the planar ferromagnet

The hydrodynamic equations of motion (4.12), (4.14), (4.16), and (4.30) are, in linear approximation with $H_y = 0$, inserting from Eqs. (4.10), (4.15), (4.17), (4.18), (4.33), (4.35), and (4.38),

$$\dot{\mathbf{v}}_s - (1/m)\nabla h_z - \rho_s\nabla[(\zeta\nabla)\cdot\mathbf{v}_s] = -(1/m)\nabla H_z \tag{4.40}$$

$$\dot{M}_z - (\rho_s/m)\nabla\cdot\mathbf{v}_s - (\xi\nabla)\cdot\nabla h_z = -(\xi\nabla)\cdot\nabla H_z \tag{4.41}$$

$$\begin{aligned}&\dot{v}_{Mi} + (\rho/\rho_M)s_M\nabla_i T - (1/\rho_M)\sum_{jkl}\gamma_{ij,kl}\nabla_j\nabla_k v_{Ml} \\ &\quad + (1/\tau_J)v_{Mi} = 0\end{aligned} \tag{4.42}$$

$$\dot{s} + s_M\nabla\cdot\mathbf{v}_M - (1/\rho T)(\kappa\nabla)\cdot\nabla T = 0. \tag{4.43}$$

Choosing as independent variables δT, δM_z, $\mathbf{v}_s$, and $\mathbf{v}_M$ we have, in analogy to Eq. (1.63),

$$\begin{aligned}\delta s &= (c_M/T)\delta T - (\beta/\rho)\delta M_z, \\ \delta h_z &= \beta\delta T + (1/\chi)\delta M_z,\end{aligned} \tag{4.44}$$

where c_M is the specific heat at constant magnetization per unit mass, χ is defined by Eq. (4.7), and

$$\beta = (\partial h_z/\partial T)_{M_z} = -\rho(\partial s/\partial M_z)_T \tag{4.45}$$

is the magnetic tension coefficient. Here the second equality follows from Eq. (4.22).

Going over to plane waves and introducing the analogue of Eq. (3.62)

$$[\lambda_N^2(\mathbf{q}, \omega)]_{ij} = (\tau_J/\rho_M)\sum_{mn}\gamma_{im,nj}(\mathbf{q}, \omega)\hat{q}_m\hat{q}_n, \tag{4.46}$$

where $\hat{q} = \mathbf{q}/q$ Eqs. (4.40) to (4.43) become, after inser-

tion of (4.44),

$$\omega \mathbf{v}_s + i\mathbf{q}\rho_s \mathbf{q}\cdot\mathbf{v}_s + \mathbf{q}(1/m\chi)\delta M_z + \mathbf{q}(\beta/m)\delta T = \mathbf{q}(1/m)\delta H_z \tag{4.47}$$

$$\left(\omega + iq^2\frac{\xi_l}{\chi}\right)\delta M_z + \frac{\rho_s}{m}\mathbf{q}\cdot\mathbf{v}_s + iq^2\beta\xi_l\delta T = iq^2\xi_l\delta H_z \tag{4.48}$$

$$\left(-i\omega + \frac{1}{\tau_J} + \frac{1}{\tau_J}\lambda_N{}^2q^2\right)\mathbf{v}_M + i\mathbf{q}\frac{\rho}{\rho_M}s_M\delta T = 0 \tag{4.49}$$

and

$$\left(\omega + iq^2\frac{\kappa_l}{\rho c_M}\right)\delta T - \frac{Ts_M}{c_M}\mathbf{q}\cdot\mathbf{v}_M - \omega\frac{T\beta}{\rho c_M}\delta M_z = 0. \tag{4.50}$$

As in Eq. (3.61), $\xi_l = (\hat{q}\cdot\xi\hat{q})$ and $\kappa_l = (\hat{q}\cdot\kappa\hat{q})$. For $\delta H_z = 0$ Eqs. (4.48), (4.49), and (4.50) are the same, respectively, as Eqs. (4.14b), (4.14c), and (4.14a) of Michel and Schwabl (1970a) [see also Footnote (2)] if we put $\mathbf{v}_s = 0$ in Eq. (4.48), if the last term in Eq. (4.50) is substituted from (4.48), and if in the paper by Michel and Schwabl (1970a) the external temperature and field variations and the coupling coefficient of the magnon drift to the magnetization are put equal to zero [$\theta = 0$, $h = 0$, and $c_{ij}{}^1 = 0$. The latter could have been included by adding a term $-\rho_M c^1\chi dh_z$ to Eq. (4.18) and a term $+\rho_M c^1\chi h_z$ to Eq. (4.19) which, however, would lead to an additional coupling between magnetic and thermal variables and to an additional term in Eq. (4.33), namely, $\mathbf{j}_s = -\rho_s\mathbf{v}_s + c^1\chi\rho_M\mathbf{v}_M$.] Note that in the paper by Michel and Schwabl (1970a) the equivalent of Eq. (4.47) is absent. The resulting hydrodynamic modes therefore are modified in the general case (see below).

To solve the system (4.47) to (4.50) we first express the velocities from (4.47) and (4.49)

$$\mathbf{v}_s = -\frac{\mathbf{q}}{\omega m}\left[\frac{1}{\chi}\delta M_z + \beta\delta T - \delta H_z\right]\left[1 + i\frac{q^2}{\omega}\rho_s\zeta_l\right]^{-1} \tag{4.51}$$

and

$$\mathbf{v}_M = -(\varphi/T\rho s_M)i\mathbf{q}\delta T, \tag{4.52}$$

where, in analogy to Eq. (3.66),

$$\varphi(\mathbf{q},\omega) = \frac{T\rho^2 s_M{}^2}{\rho_M}\left[-i\omega + \frac{1}{\tau_J} + \frac{1}{\tau_J}\lambda_N{}^2q^2\right]^{-1} \tag{4.53}$$

is the heat convectivity tensor and ζ_l is the longitudinal projection of ζ.

As in Sec. I and III, we first examine the special case of complete decoupling of the magnetic variables $\mathbf{v}_s$, δM_z, and the thermal variables $\mathbf{v}_M$, δT. This decoupling is obtained under the assumption

$$\beta = 0. \tag{4.54}$$

There now exists a purely magnetic mode ($\mathbf{v}_M = 0$, $\delta T = 0$) and a purely thermal mode ($\mathbf{v}_s = 0$, $\delta M_z = 0$). Indeed, with Eq. (4.54) and $\mathbf{v}_M = 0$, $\delta T = 0$, Eqs. (4.49) and (4.52) are identically satisfied and Eqs. (4.48) and (4.51) yield, for $\delta H_z = 0$, force-free damped isothermal spin waves,

$$\omega^2/q^2 = c_1{}^2[1 + i(q^2/\omega)\rho_s\zeta_l]^{-1} - i\omega(\xi_l/\chi) \tag{4.55}$$

with the *longitudinal isothermal magnon velocity*

$$c_1 = (\rho_s/m^2\chi)^{1/2}. \tag{4.56}$$

This shows that ρ_s has the character of a stiffness constant. Assuming all the imaginary parts to be small so that they need be retained only linearly, Eq. (4.55) simplifies to

$$\omega = c_1 q - \tfrac{1}{2}i(\rho_s\zeta_l + \xi_l/\chi)q^2 \tag{4.57}$$

which is the same as Eqs. (2.48) and (2.60) of Halperin and Hohenberg (1969), who were the first to predict such a weakly damped *soundlike* magnon mode in the hydrodynamic domain.

The thermal mode is obtained with Eq. (4.54) and $\mathbf{v}_s = 0$, $\delta M_z = 0$, in which case Eqs. (4.48) and (4.51) are identically satisfied. Equations (4.50) and (4.52) take the form analogous to Eq. (3.67),

$$-i\omega + (1/\rho c_M)[\varphi_l(\mathbf{q},\omega) + \kappa_l(\mathbf{q},\omega)]q^2 = 0. \tag{4.58}$$

There exists a domain of second spin waves given by the conditions analogous to Eqs. (3.69) and (3.70),

$$1/\tau_J \ll \omega \ll 1/\tau_{eq}, \qquad \omega\tau_J \gg q^2\lambda_N{}^2. \tag{4.59}$$

The first is the "window condition" familiar from Sec. II and III, τ_{eq} being the relaxation time responsible for thermal equilibrium of the magnon fluid. In the domain (4.59) we have from Eq. (4.53)

$$\varphi_l(\mathbf{q},\omega) = \frac{i}{\omega}\frac{T\rho^2 s_M{}^2}{\rho_M}\left(1 - i\frac{1 + (\lambda_N{}^2)_l\, q^2}{\omega\tau_J}\right), \tag{4.60}$$

and inserting this into Eq. (4.58),

$$\frac{\omega^2}{q^2} = c_2{}^2\left(1 - i\frac{1 + (\lambda_N{}^2)_l\, q^2}{\omega\tau_J} + i\omega\tau_2\right) \tag{4.61}$$

which is identical with Eq. (3.72). Here

$$c_2 = (\rho T s_M{}^2/\rho_M c_M)^{1/2} \tag{4.62}$$

is the *second-magnon velocity*, which has exactly the same

form as the second-sound velocity in the phonon case, Eq. (3.73), and, as in Eq. (3.74),

$$\tau_2(\mathbf{q},\omega) = (1/\rho c_M c_2^2)\,\kappa_l(\mathbf{q},\omega) \tag{4.63}$$

is the thermal conduction relaxation time which, for a propagating second magnon, must satisfy the additional condition

$$\omega\tau_2 \ll 1. \tag{4.64}$$

Since with the decoupling assumption (4.54) Eq. (4.47) does not influence the thermal mode, our second magnon is the same as that derived by Michel and Schwabl (1970a) for $c^1 = 0$. As shown by these authors, $c^1 \neq 0$ modifies c_2 to become the *adiabatic* second-magnon velocity.

The window condition of Eq. (4.59) has been analyzed by Michel and Schwabl (1971a) for the cases of the axial antiferromagnet MnF_2 and the planar antiferromagnet K_2NiF_4. Their conclusions are favorable to the existence of a second magnon at temperatures of the order of 30°K for MnF_2, and of 90°K for K_2NiF_4 (see Sec. IV.D below). Detailed calculations of the window condition for the insulating ferromagnet EuS have been carried out by Forney (1972) and Forney and Jäckle (1973), who find an open window below approximately 2°K. This calculation is based on a Heisenberg plus a dipole–dipole coupling, the first giving rise to 4-magnon processes, the second to 3-magnon processes.

It is interesting to note that the conditions analogous to Eq. (3.76), $\omega\tau_J \ll q^2\lambda_{Ni}^2$, $q^2\lambda_{Ni}^2 \gg 1$, which define the domain of Poiseuille flow in the phonon case, are also realizable for magnetic crystals of high magnon viscosity. To our knowledge this effect has never been looked for yet.

Heat diffusion by magnons is obtained in the domain analogous to (3.98),

$$\omega\tau_J \ll 1, \qquad q^2\lambda_{Ni}^2 \ll 1 \tag{4.65}$$

where Eq. (4.50) reduces to the analogue of Eqs. (3.100) and (3.101),

$$-i\omega + D_T' q^2 = 0 \tag{4.66}$$

with

$$D_T' = \lim_{\omega\to 0}\lim_{q\to 0}\frac{1}{\rho c_M}[\varphi_l(\mathbf{q},\omega) + \kappa_l(\mathbf{q},\omega)]. \tag{4.67}$$

Neglecting here the convective contribution φ_l we recover Eq. (2.49) of Halperin and Hohenberg (1969).

We now discuss the solutions of the complete equations (4.48), (4.50), and (4.51), (4.52), that is, including an external field variation δH_z and dropping the decoupling assumption (4.54). This discussion is of interest in view of the possibility mentioned above of detecting a second-magnon peak, in addition to the first-magnon peak, in a neutron scattering experiment.

Insertion of Eq. (4.51) into (4.48) yields, with (4.56),

$$\left\{\frac{\omega^2}{q^2} - c_1^2\left[\left(1 + i\frac{q^2}{\omega}D_1\right)^{-1} - i\omega\tau_1\right]\right\}\delta M_z - c_1^2\beta\chi\left[\left(1 + i\frac{q^2}{\omega}D_1\right)^{-1} - i\omega\tau_1\right]\delta T = -c_1^2\chi\left[\left(1 + i\frac{q^2}{\omega}D_1\right)^{-1} - i\omega\tau_1\right]\delta H_z, \tag{4.68}$$

where we have introduced the definitions

$$D_1 = \rho_s\zeta_l; \qquad \xi_l/\chi = c_1^2\tau_1. \tag{4.69}$$

Insertion of Eq. (4.52) into (4.50) yields

$$c_1^2\beta\chi[-i\omega + (1/\rho c_M)(\varphi_l + \kappa_l)q^2]\delta T = -i\omega c_1^2\vartheta\delta M_z \tag{4.70}$$

where, in analogy to Eqs. (1.68), (2.36), and (3.118),

$$c_1^2\vartheta = \rho_s T\beta^2/m^2\rho c_M. \tag{4.71}$$

Equations (4.68) and (4.70) are closely analogous to Eqs. (3.116) and (3.115), respectively, so that the discussion of the three main domains of φ [Eq. 4.53)] can be taken over.

In the domain (3.76),

$$\omega\tau_J \ll q^2\lambda_{Ni}^2, \qquad q^2\lambda_{Ni}^2 \gg 1, \tag{4.72}$$

we can write, in analogy to Eq. (3.78),

$$(1/\rho c_M)\varphi_l q^2 = \tau_N^{-1}. \tag{4.73}$$

Inserting this into Eq. (4.68) and eliminating δT from (4.68) and (4.70), retaining only lowest powers in q and ω, but separately in real and imaginary parts, we obtain the *correlation function*

$$\chi(q,\omega) \equiv \delta M_z/\delta H_z = -\chi[\omega^2 - c_1^2(1 - i\omega\tau_1')q^2]^{-1}c_1^2q^2, \tag{4.74}$$

where, in analogy to Eq. (3.120),

$$\tau_1' = \tau_1 + \vartheta\tau_N + D_1/c_1^2. \tag{4.75}$$

Equation (4.74) shows that in the domain (4.72) δM_z propagates with the *isothermal* magnon velocity c_1.

In the domain (3.98),

$$\omega\tau_J \ll 1, \qquad q^2\lambda_{Ni}^2 \ll 1, \tag{4.76}$$

insertion of Eq. (4.67) into (4.70) and elimination of δT from (4.68) and (4.70) gives for the correlation function defined in Eq. (4.74)

$$\chi(q,\omega) = \chi\{1 - i\omega\vartheta(-i\omega + D_T'q^2)^{-1} - i\omega(c_1q)^{-2}[\tau_1 + (-i\omega + D_1q^2)^{-1}]^{-1}\}^{-1}. \quad (4.77)$$

The properties of this equation are analogous to those of Eq. (3.122). In the case $D_T'q^2 \ll \omega$, $D_1q^2 \ll \omega$, $\omega\tau_1 \ll 1$ we find

$$\chi(q,\omega) = -\chi[\omega^2 - \mathcal{c}_1^2(1 - i\omega\tilde{\tau}_1)q^2]^{-1}c_1^2q^2 \quad (4.77a)$$

where, in analogy to Eqs. (1.73), (2.35), and (3.123),

$$\mathcal{c}_1^2 = c_1^2(1 + \vartheta) \quad (4.78)$$

and

$$\tilde{\tau}_1 = \tau_1 + \left(D_1 + \frac{\vartheta}{1+\vartheta}D_T'\right)\Big/\mathcal{c}_1^2. \quad (4.79)$$

$\mathcal{c}_1$ is the *longitudinal adiabatic first-magnon velocity* which is connected with the adiabatic longitudinal susceptibility $\tilde{\chi}$ by the analogue of Eq. (4.56),

$$\tilde{\chi} = \rho_s/m^2\mathcal{c}_1^2 = \chi(1+\vartheta)^{-1}. \quad (4.80)$$

In the opposite case $\omega \ll D_T'q^2$, $\omega \ll D_1q^2$ one finds from Eq. (4.77)

$$\chi(q,\omega) = \chi\left\{1 - i\omega\left[\frac{\vartheta}{D_T'q^2} + \frac{D_1}{c_1^2(1+\tau_1D_1q^2)}\right] + \omega^2\left[\frac{\vartheta}{D_T'^2q^4} - \frac{1}{c_1^2q^2(1+\tau_1D_1q^2)^2}\right] + O(\omega^3)\right\}^{-1}. \quad (4.77b)$$

This expression shows that in the limit $q \to 0$ the heat diffusion pole in Eq. (4.77) dominates the response and gives rise to the Rayleigh peak discussed in Sec. III.D.

In the second-magnon domain (4.59) insertion of Eqs. (4.60), (4.62), and (4.63) into (4.68) and (4.70) yields for the correlation function of Eq. (4.74), after elimination of δT and neglecting damping terms proportional to q^2/ω,

$$\chi(q,\omega) = -\chi G(q,\omega)(1 - i\omega\tau_1)\{\omega^2 - c_2^2[1 - (i/\omega\tau_J) - i\omega\tau_2]q^2\}c_1^2q^2 \quad (4.81)$$

where

$$G^{-1}(q,\omega) = [\omega^2 - c_1^2(1 - i\omega\tau_1)q^2] \times\{\omega^2 - c_2^2[1 - (i/\omega\tau_J) - i\omega\tau_2]q^2\} - c_1^2\vartheta q^2\omega^2(1 - i\omega\tau_1). \quad (4.82)$$

Note that the isothermal limit yields correctly

$$\chi = \lim_{q\to 0}\lim_{\omega\to 0}\chi(q,\omega)$$

in agreement with Eq. (3.9a) of Halperin and Hohenberg (1969).

Apart from the damping in the second term of (4.82), this is the same as Eq. (3.127). Therefore G^{-1} can be factorized in the form (3.128),

$$G^{-1}(q,\omega) = [\omega^2 - c_{\rm I}^2(1 - i\omega\tau_{\rm I} - i/\omega\tau_{J\rm I})q^2] \times[\omega^2 - c_{\rm II}^2(1 - i\omega\tau_{\rm II} - i/\omega\tau_{J\rm II})q^2] \equiv A_{\rm I}A_{\rm II}, \quad (4.83)$$

valid to first order in the damping terms. Since the numerator of Eq. (4.81) is also the same as that of Eq. (3.126), apart from a factor $\sim(1 - i\omega\tau_1)$, Eqs. (3.129), (3.130) and (3.131) can be taken over. Hence

$$\chi(q,\omega) = -\chi\left(\frac{1-R}{A_{\rm I}} + \frac{R}{A_{\rm II}}\right)c_1^2q^2 \quad (4.84)$$

with

$$R = r(1 + i\omega\tau_r + i/\omega\tau_{Jr}) \quad (4.85)$$

and, to first order in ϑ,

$$r \cong c_1^2c_2^2(c_1^2 - c_2^2)^{-1}\vartheta. \quad (4.86)$$

As in the case of dielectric crystals, $\beta \propto c_M \propto T^3$ and $s_M \propto T^3$ at low temperatures so that, according to Eq. (4.71), $\vartheta \propto T^4$. Since the window condition restricts the occurrence of the second magnon to low temperatures, the second-magnon peak will be much smaller than the first-magnon peak in a magnetic excitation by inelastic neutron scattering.

The integrated intensity can be obtained from Eq. (4.84) in analogy to Eqs. (1.79a) and (1.80). However, the damping terms $i/\omega\tau_{J\rm I}$, $i/\omega\tau_{J\rm II}$, and $i/\omega\tau_{Jr}$ make the integration more complicated; we refrain from giving the details.

C. The axial and isotropic ferromagnets

The *axial* ferromagnet is characterized by $J_{ij}{}^z > J_{ij}{}^\perp > 0$, for the dominant pairs of neighbors i, j. This has the effect that without external field the magnetization aligns in the easy z direction and $M_\perp = 0$. Thus the precession angle is not a dynamic variable and the analogy with superfluid helium is lost. But a two-fluid description in terms of magnetic and thermal variables still exists.

The relation (4.7) is invalid in this case since $M_z \neq 0$ but $h_z = 0$. On the other hand, for $|M_i| \ll M_z$ $(i = x, y)\chi_\perp$ is not given by Eq. (4.26) but is finite and

$$M_i = \chi_\perp h_i, \qquad i = x, y. \quad (4.87)$$

Choosing the external field as $\mathbf{H} = (H_z, H_y, 0)$ we have

equations of motion analogous to Eq. (4.32),

$$\begin{aligned}
&\dot{M}_z + (1/m)\nabla\cdot\mathbf{j}_{sz} = -(\mathbf{M}\times m\boldsymbol{\mu})_z \\
&\quad = -M_xH_y + M_yH_x \cong 0, \\
&\dot{M}_x + (1/m)\nabla\cdot(\mathbf{j}_{sx} + \mathbf{j}_{sx}') = -(\mathbf{M}\times m\boldsymbol{\mu})_x, \\
&\dot{M}_y + (1/m)\nabla\cdot(\mathbf{j}_{sy} + \mathbf{j}_{sy}') = -(\mathbf{M}\times m\boldsymbol{\mu})_y,
\end{aligned} \tag{4.88}$$

where

$$\mu_i = (1/m)(H_i - h_i), \qquad i = x, y; \qquad \mu_z = 0. \tag{4.89}$$

The energy can be written in a form analogous to Eqs. (4.13), (4.20), (4.28),

$$\begin{aligned}
d(\rho\epsilon) &= \mathbf{h}\cdot d\mathbf{M} - \mathbf{H}\cdot d\mathbf{M} + \sum_{i=x,y}(\rho_s{}^i/m^2M_z{}^2)\nabla M_i\cdot d\nabla M_i \\
&\quad + \mathbf{v}_M\cdot d\mathbf{j}_M + Td(\rho s)
\end{aligned} \tag{4.90}$$

where $\rho_s{}^x = \rho_s{}^y = \rho_s < \rho_s{}^z$. The second and third terms are, respectively, the Zeeman and exchange energy contributions (Kittel, 1963; Keffer, 1966).

From the equilibrium condition (4.23), supplemented by the further requirement that

$$-\mathbf{j}_{sz} \propto \nabla M_z = 0 \tag{4.91}$$

[note that with this assignment the first Eq. (4.88) gives rise to a diffusive mode] we obtain, with Eq. (4.90), for the Fourier components, after a partial integration,

$$H_i = h_i + (\rho_s q^2/m^2M_z{}^2)M_i, \qquad i = x, y. \tag{4.92}$$

Thus, according to Eq. (4.87), $H_i = h_i$ in the limit $q \to 0$.

Writing in analogy to Eq. (4.27)

$$\begin{aligned}
\mathbf{v}_{sx} &= (1/mM_z)\nabla M_x \\
\mathbf{v}_{sy} &= -(1/mM_z)\nabla M_y
\end{aligned} \tag{4.93}$$

and using Eqs. (4.89) and (4.91), the energy expression (4.90) takes the form

$$\begin{aligned}
d(\rho\epsilon) &= -m\boldsymbol{\mu}\cdot d\mathbf{M} + \rho_s\sum_{i=x,y}\mathbf{v}_{si}\cdot d\mathbf{v}_{si} \\
&\quad + \mathbf{v}_M\cdot d\mathbf{j}_M + Td(\rho s).
\end{aligned} \tag{4.94}$$

Here $\mathbf{v}_{sx}$ and $\mathbf{v}_{sy}$ are introduced mainly to exhibit the analogy to the planar case, which is even more evident from their equations of motion. In fact, from Eqs. (4.88) and (4.93) we obtain

$$\begin{aligned}
\dot{\mathbf{v}}_{sx} &= +\nabla(\mu_y + \mu_y') - (1/m^2M_z)\nabla\nabla\cdot\mathbf{j}_{sx}, \\
\dot{\mathbf{v}}_{sy} &= +\nabla(\mu_x + \mu_x') + (1/m^2M_z)\nabla\nabla\cdot\mathbf{j}_{sy},
\end{aligned} \tag{4.95}$$

where

$$\begin{aligned}
\mu_x' &= +(1/m^2M_z)\nabla\cdot\mathbf{j}_{sy}', \\
\mu_y' &= -(1/m^2M_z)\nabla\cdot\mathbf{j}_{sx}'.
\end{aligned} \tag{4.96}$$

Equations (4.95) are of the form (4.12). The analogue of Eq. (4.33) can also be established by working out the transition from (4.29) to (4.30).

From Eq. (4.94) we have

$$T(\rho s)^{\cdot} = (\rho\epsilon)^{\cdot} + m\boldsymbol{\mu}\cdot\dot{\mathbf{M}} - \rho_s\sum_{i=x,y}\mathbf{v}_{si}\cdot\dot{\mathbf{v}}_{si} - \mathbf{v}_M\cdot\dot{\mathbf{j}}_M. \tag{4.97}$$

Inserting from Eqs. (4.29), (4.88), (4.95), and (4.16) we obtain, with (4.15), (4.17), and (4.18),

$$\begin{aligned}
(\rho s)^{\cdot} &= -\nabla\cdot\{(1/T)[\mathbf{J}_\epsilon + \mathbf{J}_\epsilon' + \sum_i\mu_i(\mathbf{j}_{si} + \mathbf{j}_{si}') \\
&\quad + \rho_s(\mu_y'\mathbf{v}_{sx} + \mu_x'\mathbf{v}_{sy}) - (\rho_s/m^2M_z)(\mathbf{v}_{sx}\nabla\cdot\mathbf{j}_{sx} - \mathbf{v}_{sy}\nabla\cdot\mathbf{j}_{sy}) \\
&\quad - \Pi_M'\mathbf{v}_M]\} + [\mathbf{J}_\epsilon + \mathbf{J}_\epsilon' + \sum_i\mu_i(\mathbf{j}_{si} + \mathbf{j}_{si}') + \rho_s(\mu_y'\mathbf{v}_{sx} \\
&\quad + \mu_x'\mathbf{v}_{sy}) - (\rho_s/m^2M_z)(\mathbf{v}_{sx}\nabla\cdot\mathbf{j}_{sx} - \mathbf{v}_{sy}\nabla\cdot\mathbf{j}_{sy}) - \Pi_M'\mathbf{v}_M \\
&\quad - T\rho s_M\mathbf{v}_M]\cdot\nabla(1/T) + (1/T)(\mathbf{j}_{sx} + \mathbf{j}_{sx}' - \rho_s\mathbf{v}_{sy})\cdot\nabla\mu_x \\
&\quad + (1/T)(\mathbf{j}_{sy} + \mathbf{j}_{sy}' - \rho_s\mathbf{v}_{sx})\cdot\nabla\mu_y - (1/Tm^2M_z)[\nabla \\
&\quad \cdot(\rho_s\mathbf{v}_{sx})\nabla\cdot\mathbf{j}_{sx} - \nabla\cdot(\rho_s\mathbf{v}_{sy})\nabla\cdot\mathbf{j}_{sy}] + (1/T)[\mu_y'\nabla \\
&\quad \cdot(\rho_s\mathbf{v}_{sx}) + \mu_x'\nabla\cdot(\rho_s\mathbf{v}_{sy}) - (\Pi_M'\nabla)\cdot\mathbf{v}_M] \\
&\quad + (\rho_M/T\tau_J)\mathbf{v}_M{}^2.
\end{aligned}$$

The form (4.30) then implies that

$$\begin{aligned}
\mathbf{j}_{sx} &= +\rho_s\mathbf{v}_{sy} = -(\rho_s/mM_z)\nabla M_y, \\
\mathbf{j}_{sy} &= +\rho_s\mathbf{v}_{sx} = -(\rho_s/mM_z)\nabla M_x,
\end{aligned} \tag{4.98}$$

where we have used Eq. (4.93), and

$$\begin{aligned}
\mathbf{J}_\epsilon &= -\sum_{i=x,y}\mu_i\mathbf{j}_{si} - (1/m^2M_z)\nabla\times(\mathbf{j}_{sx}\times\mathbf{j}_{sy}) \\
&\quad + T\rho_s s_M\mathbf{v}_M.
\end{aligned} \tag{4.99}$$

This leads again to Eq. (4.35) for $\mathbf{J}_s$ and to

$$\mathbf{J}_s' = (1/T)[\mathbf{J}_\epsilon' + \sum_{i=x,y}(\mu_i\mathbf{j}_{si}' + \mu_i'\mathbf{j}_{si}) - \Pi_M'\mathbf{v}_M] \tag{4.100}$$

and

$$\begin{aligned}
\sigma &= T\mathbf{J}_s'\cdot\nabla(1/T) + (1/T)\sum_{i=x,y}(\mathbf{j}_{si}'\cdot\nabla\mu_i \\
&\quad + \mu_i'\nabla\cdot\mathbf{j}_{si}) - (1/T)(\Pi_M'\nabla)\cdot\mathbf{v}_M + (\rho_M/T\tau_J)\mathbf{v}_M{}^2.
\end{aligned} \tag{4.101}$$

Note that addition of an independent dissipative part of the form of the second Eq. (4.38) to $\boldsymbol{\mu}$ on the right-hand side of Eqs. (4.88) would, according to Eqs. (4.30) and (4.97), violate the positive definite character of σ. Equations (4.98) are the same as Eqs. (7.3a,b) and (7.8) of Halperin and Hohenberg (1969).

The form of the dissipative terms Π_M' and $\mathbf{J}_s'$ is again given by Eq. (4.38). For $\mathbf{j}_{si}'$ rotational symmetry in the (x, y) plane implies the form

$$\mathbf{j}_{si}' = -m^2(\xi\nabla)\nabla_i \sum_{j=x,y} \nabla_j\mu_j, \qquad i = x, y. \tag{4.102}$$

With Eqs. (4.102) and (4.96) the second term of $T\sigma$, Eq. (4.101), becomes, making use of (4.87), (4.89), and (4.98) and assuming $\nabla H_i = 0$,

$$\begin{aligned}\sum_{i=x,y} (\mathbf{j}_{si}'\cdot\nabla\mu_i + \mu_i'\nabla\cdot\mathbf{j}_{si}) &= -\sum_{i,j=x,y} \nabla_i\{(\nabla h_i)\cdot(\xi\nabla)\nabla_j h_j \\ &+ (\rho_s\chi_\perp/m^2M_z^2)(\nabla^2 h_i)(\xi\nabla)\cdot\nabla\nabla_j h_j\} \\ &+ \sum_{i,j=x,y}\{(\nabla\nabla_i h_i)\cdot(\xi\nabla)\nabla_j h_j \\ &+ (\rho_s\chi_\perp/m^2M_z^2)(\nabla^2\nabla_i h_i)(\xi\nabla)\cdot\nabla\nabla_j h_j\}.\end{aligned}$$

Here the first term is a divergence and can be absorbed into $\nabla\cdot\mathbf{J}_s'$, and the second term is a positive quadratic form.

The hydrodynamic modes are obtained from the equations of motion (4.88) combined with (4.42) and (4.43). Equations (4.44) are modified as follows

$$\begin{aligned}\delta s &= (c_M/T)\delta T - \sum_{i=x,y} (\beta_i/\rho)\delta M_i, \\ \delta h_i &= \beta_i\delta T + (1/\chi_\perp)\delta M_i, \qquad i = x, y,\end{aligned} \tag{4.103}$$

where

$$\beta_i = (\partial h_i/\partial T)_M = -\rho(\partial s/\partial M_i)_T, \qquad i = x, y. \tag{4.104}$$

Excluding precessional motions the perpendicular fields and the magnetization vary around zero averages, $H_i = \delta H_i$, $h_i = \delta h_i$, $M_i = \delta M_i$. Then Eqs. (4.88), combined with (4.98), (4.102), (4.89), and the second Eq. (4.103), become

$$\begin{aligned}&\dot{M}_x + (M_z/\chi_\perp)\delta M_y - (\rho_s/m^2M_z)\nabla^2 M_y + (1/\chi_\perp)(\xi\nabla) \\ &\cdot\nabla\nabla_x \sum_{i=x,y} \nabla_i M_i + \beta_y\delta T + (\xi\nabla)\cdot\nabla\nabla_x \sum_{i=x,y} \beta_i\nabla_i T \\ &= +M_z\delta H_y + (\xi\nabla)\cdot\nabla\nabla_x \sum_{i=x,y} \nabla_i H_i \\ &\dot{M}_y - (M_z/\chi_\perp)\delta M_x + (\rho_s/m^2M_z)\nabla^2 M_x + (1/\chi_\perp)(\xi\nabla) \\ &\cdot\nabla\nabla_y \sum_{i=x,y} \nabla_i M_i - \beta_x\delta T + (\xi\nabla)\cdot\nabla\nabla_y \sum_{i=x,y} \beta_i\nabla_i T \\ &= -M_z\delta H_x + (\xi\nabla)\cdot\nabla\nabla_y \sum_{i=x,y} \nabla_i H_i.\end{aligned} \tag{4.105}$$

Eliminating here δT with the aid of Eqs. (4.42), (4.43), and the first Eq. (4.103) gives us the response of the system to an external field variation δH_i as obtained in an inelastic neutron scattering experiment. Rather than treating this general case, which gives rise to complicated expressions, we treat here only decoupled free spin waves. That is, we assume

$$\beta_i = 0, \qquad \delta H_i = 0, \qquad i = x, y. \tag{4.106}$$

In plane-wave representation Eqs. (4.105) then lead to

$$\begin{aligned}&[\omega + i(\xi_l/\chi_\perp)q^2q_x^2]\delta M_x = -i[H_e + Dq^2 \\ &\quad + (\xi_l/\chi_\perp)q^2q_xq_y]\delta M_y \\ &[\omega + i(\xi_l/\chi_\perp)q^2q_y^2]\delta M_y = i[H_e + Dq^2 \\ &\quad - (\xi_l/\chi_\perp)q^2q_xq_y]\delta M_x,\end{aligned} \tag{4.107}$$

where we have introduced the definitions

$$H_e = M_z/\chi_\perp, \qquad D = \rho_s/m^2M_z. \tag{4.108}$$

Equations (4.107) describe spin waves in the (x, y) plane with the dispersion relation

$$\omega = H_e + Dq^2 - \tfrac{1}{2}i(\xi_l/\chi_\perp)q^4\sin^2\vartheta_q + O(q^8), \tag{4.109}$$

where $\cos\vartheta_q = q_z/q$.

Such a quadratic dispersion relation is typical for axial and for isotropic ferromagnets [Kittel (1963); Keffer (1966). See below]. The gap H_e is important for the existence of a magnon drift and hence for the existence of a second magnon. Indeed, without a gap, the magnon distribution function

$$n_q = \{\exp[(\omega_q - \mathbf{v}_M\cdot\mathbf{q})/k_BT] - 1\}^{-1}$$

is negative for sufficiently small q with $\mathbf{v}_M\cdot\mathbf{q} > 0$. With a dispersion ω_q given by Eq. (4.109), however, the magnon drift exists for $|\mathbf{v}_M| < 2(H_eD)^{1/2}$. (Forney, 1972; Forney and Jäckle, 1973). The gap H_e arose here from the precessional (Bloch) terms in the equations of motion (4.88). As we shall see, in an ideal isotropic ferromagnet this is no longer the case.

Before discussing the isotropic case we wish to generalize the above equations by taking into account the effect of the dipole–dipole interaction between the spins.

Phenomenologically [see Section 30 of Keffer (1966)] this effect is described by a *dipolar field* $\mathbf{H}_d$ produced by the smeared-out spin distribution $\mathbf{M}$. The discreteness of the spins can be taken into account by the field produced in a small sphere centered at $\mathbf{r}$ by the magnetic moment in this sphere,

$$\mathbf{H}_d'(\mathbf{r}) = \left[\int_{\text{sphere}} d^3r'\mathbf{M}(\mathbf{r}')\cdot\nabla\right]\nabla|\,\mathbf{r} - \mathbf{r}'\,|^{-1} = -\tfrac{4}{3}\pi\mathbf{M}(\mathbf{r}).$$

The sum must satisfy (Herring and Kittel, 1951)

$$\nabla\cdot(\mathbf{H}_d + \mathbf{H}_d' + 4\pi\mathbf{M}) = 0, \qquad \nabla\times(\mathbf{H}_d + \mathbf{H}_d') = 0$$

which is equivalent to

$$\nabla^2(\mathbf{H}_d + \mathbf{H}_d') = -4\pi\nabla(\nabla\cdot\mathbf{M}).$$

Hence

$$\mathbf{H}_d = (4\pi/3)\mathbf{M} - 4\pi(\nabla^2)^{-1}\nabla(\nabla\cdot\mathbf{M}) - \alpha\mathbf{M}_0, \qquad (4.110a)$$

where the last term is an integration constant which depends on the shape of the sample [see Section 16 of Keffer (1966)]. Physically $\mathbf{M}_0 = (0, 0, M_z)$ and α is the *demagnetizing factor*.

Then Eqs. (4.88) are generalized by substituting for Eqs. (4.89) [see Eq. (30.5) of Keffer (1966)]

$$m\boldsymbol{\mu} = \mathbf{H} - \mathbf{h} + \mathbf{H}_d + \mathbf{H}_A, \qquad (4.110)$$

where $\mathbf{H}_A = (0, 0, H_A)$ is the *anisotropy field* which is due to higher-order perturbation effects of the dipole–dipole (and higher pole) interactions [see Sections 22 to 25 of Keffer (1966)]. Note that insertion of this modified form (4.110) of $m\boldsymbol{\mu}$ into Eqs. (4.88) does not give rise to modifications of Eq. (4.97). Hence the connection (4.98) is still valid. We therefore find, in plane-wave representation with $H_i = 0$, $M_i = \delta M_i$ $(i = x, y)$, the following generalization of Eqs. (4.107):

$$\begin{aligned}[\omega + i(\xi_l/\chi_\perp)q^2q_x^2 + 4\pi iM_z\hat{q}_x\hat{q}_y]\delta M_x &= -i[H_e + H_A \\ &+ Dq^2 + (\xi_l/\chi_\perp)q^2q_xq_y - \alpha M_z + 4\pi M_z\hat{q}_y^2]\delta M_y \\ [\omega + i(\xi_l/\chi_\perp)q^2q_y^2 - 4\pi iM_z\hat{q}_x\hat{q}_y]\delta M_y &= i[H_e + H_A \\ &+ Dq^2 - (\xi_l/\chi_\perp)q^2q_xq_y - \alpha M_z + 4\pi M_z\hat{q}_x^2]\delta M_x\end{aligned} \qquad (4.107a)$$

where $\hat{q} = \mathbf{q}/q$. Apart from the damping terms, Eqs. (4.107a) are the same as Eqs. (30.7) of Keffer (1966). They lead to the secular equation

$$\omega^2[1 + (i/\omega)(\xi_l/\chi_\perp)q^4\sin^2\vartheta_q] = \omega^2(\mathbf{q}),$$

where

$$\begin{aligned}\omega^2(\mathbf{q}) = (H_e + H_A - \alpha M_z + Dq^2)(H_e + H_A - \alpha M_z \\ + 4\pi M_z\sin^2\vartheta_q + Dq^2)\end{aligned} \qquad (4.111)$$

which has the solution

$$\omega = \omega(\mathbf{q}) - \tfrac{1}{2}i(\xi_l/\chi_\perp)q^4\sin^2\vartheta_q + O(q^8). \qquad (4.112)$$

The dispersion relation (4.111) was first derived by Clogston *et al.* (1956). [See also Herring and Kittel (1951)]. The merit of our hydrodynamic derivation is that, in addition to the $\mathbf{q}$ dependence of $\omega(\mathbf{q})$, it also determines the damping of the spin wave.

In the *isotropic*, or Heisenberg, ferromagnet $J_{ij}{}^{\perp} = J_{ij}{}^{z}$ and $\rho_s{}^x = \rho_s{}^y = \rho_s{}^z = \rho_s$ in Eq. (4.90). Assuming that in equilibrium without external field the magnetization points in the z direction we have the same situation as with the axial ferromagnet, except that $\mathbf{h} = 0$. This follows from the fact that in the planar case $\mathbf{h} = (0, 0, h_z)$, in the axial case $\mathbf{h} = (h_x, h_y, 0)$, and that the isotropic case is the limit between the two. $\mathbf{h} = 0$ has the effect that Eq. (4.92) now becomes, in analogy to Eqs. (4.25) and (4.26),

$$M_i = \chi_\perp H_i, \qquad i = x, y \qquad (4.113)$$

with

$$\chi_\perp = m^2M_z{}^2/\rho_sq^2. \qquad (4.114)$$

Since $\mathbf{h} = 0$ the second Eq. (4.103) implies that $\beta_i = 0$ and $1/\chi_\perp = 0$. With this modification the equations of motion (4.105) also hold in the isotropic case. However, $1/\chi_\perp = 0$ eliminates the damping of δM_x and δM_y. Now since M_i and μ_i transform with the same sign under time reversal we can add a term to Eq. (4.102) in which μ_i is replaced by M_i. While in the axial case this did not affect the form of $\mathbf{j}_{si}'$ it now gives rise to the leading term,

$$\mathbf{j}_{si}' = +2m(\eta\nabla)\nabla_i\sum_{j=x,y}\nabla_jM_j, \qquad i = x, y. \qquad (4.115)$$

With this expression the dispersion relation (4.109) is modified as follows:

$$\omega = Dq^2 - i\eta_lq^4\sin^2\vartheta_q + O(q^6), \qquad (4.116)$$

where D is given by the second Eq. (4.108). Here the real part is the same as Eq. (7.9) of Halperin and Hohenberg (1969). Equation (4.116) is the well known dispersion relation of the isotropic ferromagnet (Kittel, 1963; Keffer, 1966) first derived, for spin-$\frac{1}{2}$, by Bloch (1930). The phenomenological form (4.108) of D was first given by Landau and Lifshitz (1935) in their pioneering work on the macroscopic description of magnetic crystals. The dispersion relation (4.116) has been observed by neutron scattering in Fe, Co, Ni, and their alloys (Shirane *et al.*, 1968).

Even in an ideally isotropic ferromagnet a gap arises through dipole–dipole interaction [see, for example, Forney (1972); Forney and Jäckle (1973)]. In our phenomenological approach this effect is taken care of by substituting Eqs. (4.110) and (4.110a) for Eq. (4.89). This means that the dispersion relation (4.116) is replaced by Eqs. (4.111) and (4.112) with $H_e = 0$ which, to lowest order, is again of the form (4.109). Hence the second magnon should also occur in an ideally isotropic ferromagnet due to dipole–dipole interaction (Forney, 1972; Forney and Jäckle, 1973).

D. Hydrodynamics of the antiferromagnets

The antiferromagnet [see the review by Nagamiya *et al.* (1955)] is characterized by the existence of two sublattices

A and B such that the coupling constants J_{ij}^{z} and $J_{ij}^{\perp}$ are positive or negative if the positions i and j are on the same or on different sublattices, respectively. Defining numbers (Halperin and Hohenberg, 1969)

$$\begin{aligned}\eta_i &= +1; \text{ if } i \text{ on } A\\ &= -1; \text{ if } i \text{ on } B,\end{aligned} \tag{4.117}$$

the operators $\eta_i S_i^x$, $\eta_i S_i^y$, S_i^z satisfy the same commutation relations as S_i^x, S_i^y, S_i^z, and $\eta_i\eta_j J_{ij}^{\perp}$ is ferromagnetic. Hence the Hamiltonian (4.1) written in these variables describes ferromagnetism in the (x, y) plane. Defining the sublattice magnetizations by

$$\begin{aligned}\mathbf{M}_A &= (1/v)\langle \mathbf{S}_i\rangle; \qquad i \text{ on } A\\ \mathbf{M}_B &= (1/v)\langle \mathbf{S}_i\rangle; \qquad i \text{ on } B,\end{aligned} \tag{4.118}$$

the total magnetization is

$$\mathbf{M} = (1/v)\langle \sum_i \mathbf{S}_i\rangle = \mathbf{M}_A + \mathbf{M}_B, \tag{4.119}$$

and

$$\mathbf{N} = (1/v)\langle \sum_i \eta_i \mathbf{S}_i\rangle = \mathbf{M}_A - \mathbf{M}_B \tag{4.120}$$

is the staggered magnetization. This means for the Fourier components that $\mathbf{N}(\mathbf{q}) = \mathbf{M}(\mathbf{q}_R - \mathbf{q})$ where $\mathbf{q}_R$ is the vector to the (1, 1, 1)-corner of the Brillouin zone. Hence, near $q = 0$, N_x and N_y obey the same equations of motion as M_x and M_y in the ferromagnetic case. From Eqs. (4.119) and (4.120) we conclude that in thermal equilibrium

$$\mathbf{M}\cdot\mathbf{N} = \mathbf{M}_A^2 - \mathbf{M}_B^2 = 0, \tag{4.121}$$

since the two sublattices are equivalent except for reflections along symmetry or field directions. This is still true for hydrodynamic variations for which $\mathbf{q}$ is small compared to the border vectors of the Brillouin zone.

In the *planar* antiferromagnet $\mathbf{N}$ is in the easy (x, y) plane and

$$N_x + iN_y = N_\perp \exp(i\varphi) \tag{4.122}$$

is equivalent to Eq. (4.2), the motion being given by Eqs. (4.9) and (4.10). M_z [Eq. (4.5)] is again conserved, together with ϵ [Eq. (4.4)]. Thus Eq. (4.7) again holds.

On the other hand, N_z is not conserved but is supposed to relax to zero in a microscopic time. The orthogonality (4.121), together with (4.122), then implies that

$$\begin{aligned}\mathbf{M} &= (-M_\perp \sin\varphi, M_\perp \cos\varphi, M_z)\\ \mathbf{N} &= (N_\perp \cos\varphi, N_\perp \sin\varphi, 0).\end{aligned} \tag{4.123}$$

Here $M_\perp$ and M_z are small compared to $N_\perp$, and $N_\perp$ is supposed to relax to a value

$$N_\perp = N_\perp(M_z, T) \tag{4.124}$$

in a microscopic time.

Defining a velocity $\mathbf{v}_s$ by Eq. (4.11), the energy expression (4.22) is again valid and the equilibrium condition (4.23) yields,

$$\int\{(h_z - H_z)\delta M_z + [-H_y M_\perp \sin\varphi - (\rho_s/m^2)\nabla^2\varphi]\delta\varphi\}\, d^3r = 0.$$

Hence, for $H_y \neq 0$,

$$h_z = H_z, \qquad \varphi = 0 \tag{4.125}$$

and

$$M_\perp = M_y = \chi_\perp H_y \tag{4.126}$$

where the transverse susceptibility $\chi_\perp$ is now finite for $q \to 0$.

From Eqs. (4.11) and (4.125) we find, in analogy to (4.27),

$$\mathbf{v}_s = -(1/mN_\perp)\nabla N_y \tag{4.127}$$

so that the exchange energy contribution is now supplied by the staggered field,

$$\rho_s \mathbf{v}_s \cdot d\mathbf{v}_s = (\rho_s/m^2 N_\perp^2)\nabla N_y \cdot d\nabla N_y. \tag{4.128}$$

Thus we see that the dynamics of the planar antiferromagnet is exactly the same as that of the planar ferromagnet described in Sec. IV.B, if there $M_\perp$ is replaced by $N_\perp$. In particular, in the decoupling approximation (4.54) the spin-wave mode (4.57) with velocity (4.56) and the thermal mode (4.58) are recovered. The existence of a soundlike spin-wave dispersion in antiferromagnets was discovered by Hulthén (1936).

In the *axial* antiferromagnet, $\mathbf{N}$ is in the easy z direction. As noted above, the hydrodynamic (small q) motion of N_x and N_y is the same as the hydrodynamic motion of M_x and M_y in the ferromagnet. But here we need in addition the hydrodynamic motion of M_x and M_y.

Because of the condition (4.121) the motions of $\mathbf{M}$ and $\mathbf{N}$ are not independent but satisfy, at least for small q,

$$\mathbf{N}\cdot\dot{\mathbf{M}} + \mathbf{M}\cdot\dot{\mathbf{N}} = 0. \tag{4.129}$$

Leaving out for the moment gradient terms, these equations of motion then are

$$\dot{\mathbf{M}} = -\mathbf{M}\times m\boldsymbol{\mu} \tag{4.130}$$

and

$$\dot{\mathbf{N}} = -\mathbf{N}\times m\boldsymbol{\mu}. \tag{4.131}$$

As in the axial ferromagnet, $h_z = 0$ and Eq. (4.87) is valid. We choose again $\mathbf{H} = (H_x, H_y, 0)$ so that $m\boldsymbol{\mu}$ is given by Eqs. (4.89). Then the initial conditions are

$$\mathbf{M}_0 = (M_x, M_y, 0),$$
$$\mathbf{N}_0 = (0, 0, N_z), \tag{4.132}$$

since

$$M_z = \chi H_z, \tag{4.133}$$

with finite longitudinal susceptibility χ. Equations (4.129) and (4.132) together ensure that the orthogonality condition (4.121) is satisfied at all times.

Completing Eq. (4.130) by divergence terms in analogy to Eqs. (4.88), we have, in view of Eqs. (4.132) and (4.89),

$$\begin{aligned}
&\dot{M}_z = -m(M_x\mu_y - M_y\mu_x) \cong 0,\\
&\dot{M}_x + (1/m)\nabla\cdot(\mathbf{j}_{sx} + \mathbf{j}_{sx}') = mM_z\mu_x \cong 0,\\
&\dot{M}_y + (1/m)\nabla\cdot(\mathbf{j}_{sy} + \mathbf{j}_{sy}') = -mM_z\mu_y \cong 0.
\end{aligned} \tag{4.134}$$

Similarly we complete Eq. (4.131) by divergence terms

$$\begin{aligned}
&\dot{N}_z = -m(N_x\mu_y - N_y\mu_x) \cong 0\\
&\dot{N}_x + (1/m)\nabla\cdot(\mathbf{l}_{sx} + \mathbf{l}_{sx}') = mN_z(\mu_y + \mu_y')\\
&\dot{N}_y + (1/m)\nabla\cdot(\mathbf{l}_{sy} + \mathbf{l}_{sy}') = -mN_z(\mu_x + \mu_x').
\end{aligned} \tag{4.134a}$$

Here $\mathbf{l}_{si}$ and $\mathbf{l}_{si}'$ $(i = x, y)$ are new currents and their dissipative parts and μ_x', μ_y' are new dissipative parts of μ_x and μ_y. The latter were not allowed on the right-hand side of Eqs. (4.134) and (4.88) because they would have violated the positive definite character of the entropy production density σ [see the remark after Eq. (4.101)]. Since obviously the energy differential Eq. (4.94) does not contain a term $-m\boldsymbol{\mu}\cdot d\mathbf{N}$ these dissipative parts μ_x' and μ_y' in Eqs. (4.134a) do not contribute to Eq. (4.97) and hence do not influence σ. They are in fact important since they give the leading contribution to the damping (see below).

Writing the energy differential again in the form (4.94), $\mathbf{v}_{sx}$ and $\mathbf{v}_{sy}$ must now be related to N_x and N_y, respectively, since the dominant exchange energy obviously comes from the staggered field, as in Eq. (4.128). Writing in analogy to (4.93)

$$\begin{aligned}
&\mathbf{v}_{sx} = +(1/mN_z)\nabla N_x\\
&\mathbf{v}_{sy} = -(1/mN_z)\nabla N_y
\end{aligned} \tag{4.135}$$

we deduce from Eqs. (4.134a) that, in analogy to Eqs. (4.95),

$$\begin{aligned}
&\dot{\mathbf{v}}_{sx} = +\nabla(\mu_y + \tilde{\mu}_y') - (1/m^2N_z)\nabla\nabla\cdot\mathbf{l}_{sx}\\
&\dot{\mathbf{v}}_{sy} = +\nabla(\mu_x + \tilde{\mu}_x') + (1/m^2N_z)\nabla\nabla\cdot\mathbf{l}_{sy}.
\end{aligned} \tag{4.136}$$

Here

$$\begin{aligned}
&\tilde{\mu}_x' = \mu_x' + (1/m^2N_z)\nabla\cdot\mathbf{l}_{sy}'\\
&\tilde{\mu}_y' = \mu_y' - (1/m^2N_z)\nabla\cdot\mathbf{l}_{sx}'
\end{aligned} \tag{4.136a}$$

in analogy to Eqs. (4.96).

Inserting Eqs. (4.134) and (4.136) into (4.97), the entropy balance equation (4.30) follows again if we put

$$\begin{aligned}
&\mathbf{j}_{sx} = +\rho_s\mathbf{v}_{sy} = -(\rho_s/mN_z)\nabla N_y\\
&\mathbf{j}_{sy} = +\rho_s\mathbf{v}_{sx} = +(\rho_s/mN_z)\nabla N_x
\end{aligned} \tag{4.137}$$

and

$$\nabla\cdot\mathbf{l}_{sx}\nabla\cdot\mathbf{j}_{sy} - \nabla\cdot\mathbf{l}_{sy}\nabla\cdot\mathbf{j}_{sx} = 0. \tag{4.137a}$$

Then the energy current (4.99) is obtained but with $-(m^2M_z)^{-1}\nabla\times(\mathbf{j}_{sx}\times\mathbf{j}_{sy})$ replaced by

$$-(m^2N_z)^{-1}(\mathbf{j}_{sx}\nabla\cdot\mathbf{l}_{sy} - \mathbf{j}_{sy}\nabla\cdot\mathbf{l}_{sx})$$

while Eq. (4.35) for $\mathbf{J}_s$, (4.100) for $\mathbf{J}_s'$, and (4.101) for σ are unchanged.

The new currents $\mathbf{l}_{sx}$ and $\mathbf{l}_{sy}$ are not determined by this procedure. This is because we have omitted an exchange term $\rho_s'\sum_{i=x,y}\mathbf{w}_{si}\cdot d\mathbf{w}_{si}$ in the energy differential (4.94) where, in analogy to Eq. (4.135), $\mathbf{w}_{sx} = (mN_z)^{-1}\nabla M_x$, $\mathbf{w}_{sy} = -(mN_z)^{-1}\nabla M_y$ (there could also be cross-terms between $\mathbf{v}_{si}$ and $\mathbf{w}_{sj}$). For the sake of simplicity we leave out such additional exchange terms and put

$$\mathbf{l}_{si} = 0, \qquad \mathbf{l}_{si}' = 0, \quad i = x, y. \tag{4.138}$$

This choice is compatible with the condition (4.129) which, after insertion of Eqs. (4.134) and (4.134a), becomes

$$\sum_{i=x,y} N_i\nabla\cdot(\mathbf{j}_{si} + \mathbf{j}_{si}') = \sum_{i=x,y} M_i\nabla\cdot(\mathbf{l}_{si} + \mathbf{l}_{si}').$$

In fact, the initial conditions (4.132) immediately lead to (4.138).

In analogy to the second Eq. (4.38) we write

$$\mu_i' = +(\zeta\nabla)\cdot\mathbf{j}_{si}, \qquad i = x, y. \tag{4.139}$$

This is indeed compatible with rotational symmetry in the (x, y) plane. On the other hand, $\mathbf{j}_{sx}'$ and $\mathbf{j}_{sy}'$ are given by Eqs. (4.102). Excluding precessional motions, the perpendicular fields and magnetizations vary again around zero averages, $H_i = \delta H_i$, $h_i = \delta h_i$, $M_i = \delta M_i$, $N_i = \delta N_i$, while $M_z \cong 0$, $N_z \cong$ const. Then Eqs. (4.134) and (4.134a) become in plane-wave representation, inserting from Eqs.

(4.89), (4.102), (4.103), (4.137), (4.138), (4.139),

$$
\begin{aligned}
&(-i\omega+\sigma_{xx})\delta M_x+\sigma_{xy}\delta M_y+\Omega\delta N_y\\
&\quad=\sum_{i=x,y}\sigma_{xi}(\delta H_i-\beta_i\delta T)\\
&\sigma_{xy}\delta M_x+(-i\omega+\sigma_{yy})\delta M_y-\Omega\delta N_x\\
&\quad=\sum_{i=x,y}\sigma_{yi}(\delta H_i-\beta_i\delta T)\\
&2H_e\delta M_y+(-i\omega+\lambda)\delta N_x=N_z(\delta H_y-\beta_y\delta T)\\
&-2H_e\delta M_x+(-i\omega+\lambda)\delta N_y=-N_z(\delta H_x-\beta_x\delta T).
\end{aligned}
\tag{4.140}
$$

Here we have introduced the abbreviations

$$H_e=N_z/2\chi_\perp, \tag{4.141}$$

$$\lambda=\rho_s\zeta_l q^2, \tag{4.142}$$

and

$$\Omega=(\rho_s/m^2N_z)q^2;\qquad \sigma_{ij}=(\xi_l/\chi_\perp)q^4\hat{q}_i\hat{q}_j. \tag{4.142a}$$

Posing $\delta H_i=0$ and $\beta_i=0$, the solution is given by the following secular equation:

$$[\omega^2-c_1^2q^2+i\omega(\lambda+\tfrac{1}{2}\sigma_\perp)]^2+O(q^6\omega)=0 \tag{4.143}$$

where in analogy to Eq. (4.56)

$$c_1=(\rho_s/m^2\chi_\perp)^{1/2} \tag{4.144}$$

is the *transverse isothermal magnon velocity* and $\sigma_\perp=\sigma_{xx}+\sigma_{yy}$. The solution of Eq. (4.143) is, to lowest order in q, inserting (4.142),

$$\omega=c_1q-\tfrac{1}{2}i\rho_s\zeta_l q^2. \tag{4.145}$$

It is again of the form (4.57), valid also for the planar antiferromagnet.

As in the case of the axial ferromagnet, we now generalize the above equations by taking into account the effect of the dipole–dipole interaction. Leaving out gradient terms for the moment, this is done by replacing Eqs. (4.130) and (4.131) by

$$\dot{\mathbf{M}}=-\mathbf{M}\times m\boldsymbol{\mu}-\mathbf{N}\times\mathbf{H}_A \tag{4.146}$$

and

$$\dot{\mathbf{N}}=-\mathbf{N}\times m\boldsymbol{\mu}-\mathbf{M}\times\mathbf{H}_A. \tag{4.147}$$

Here, in distinction to Eq. (4.110),

$$m\boldsymbol{\mu}=\mathbf{H}-\mathbf{h}+\mathbf{H}_d \tag{4.148}$$

since the anisotropy field $\mathbf{H}_A=(0,0,H_A)$ couples differently. In analogy to Eq. (4.110a) the dipolar field is given by

$$\mathbf{H}_d=(4\pi/3)\mathbf{M}-4\pi(\nabla^2)^{-1}\nabla(\nabla\cdot\mathbf{M})-\alpha\mathbf{N}_0 \tag{4.148a}$$

where α is the demagnetizing factor and $\mathbf{N}_0=(0,0,N_z)$ [see (4.132)]. Equations (4.146) and (4.147) satisfy the condition (4.129). Together with Eqs. (4.148) and (4.148a) they are equivalent to Eqs. (17) of Anda (1973) if we identify our quantites N_z, αN_z and $N_z\mathbf{h}$ respectively with Anda's $2m_0$, $2N_z\chi_{||}B_0$ and $2H_e\mathbf{M}$.

The combination of Eqs. (4.146), (4.147) with (4.134), (4.134a) leads to the general equations valid, as before, for the situation without precession ($H_z=0$, $H_i=\delta H_i$, $h_i=\delta h_i$, $M_z\cong 0$, $M_i=\delta M_i$, $N_z\cong$ const, $N_i=\delta N_i$)

$$
\begin{aligned}
&\dot{M}_x+(1/m)\nabla\cdot(\mathbf{j}_{sx}+\mathbf{j}_{sx}')=-m\mu_z\delta M_y-H_A\delta N_y\\
&\dot{M}_y+(1/m)\nabla\cdot(\mathbf{j}_{sy}+\mathbf{j}_{sy}')=m\mu_z\delta M_x+H_A\delta N_x
\end{aligned}
\tag{4.149}
$$

and

$$
\begin{aligned}
&\dot{N}_x+(1/m)\nabla\cdot(\mathbf{l}_{sx}+\mathbf{l}_{sx}')=mN_z(\mu_y+\mu_y')\\
&\quad-m\mu_z\delta N_y-H_A\delta M_y\\
&\dot{N}_y+(1/m)\nabla\cdot(\mathbf{l}_{sy}+\mathbf{l}_{sy}')=-mN_z(\mu_x+\mu_x')\\
&\quad+m\mu_z\delta N_x+H_A\delta M_x.
\end{aligned}
\tag{4.149a}
$$

In plane-wave representation and with the choice (4.138) this leads to the following generalization of Eqs. (4.140)

$$
\begin{aligned}
&(-i\omega+\tilde{\sigma}_{xx})\delta M_x-(\alpha N_z-\tilde{\sigma}_{xy})\delta M_y+\tilde{\Omega}\delta N_y\\
&\quad=\sum_{i=x,y}\tilde{\sigma}_{xi}(\delta H_i-\beta_i\delta T)\\
&(\alpha N_z+\tilde{\sigma}_{xy})\delta M_x+(-i\omega+\tilde{\sigma}_{yy})\delta M_y-\tilde{\Omega}\delta N_x\\
&\quad=\sum_{i=x,y}\tilde{\sigma}_{yi}(\delta H_i-\beta_i\delta T)\\
&2B_{xy}\delta M_x+(A+2B_{yy})\delta M_y+(-i\omega+\lambda)\delta N_x\\
&\quad-\alpha N_z\delta N_y=N_z(\delta H_y-\beta_y\delta T)\\
&-(A+2B_{xx})\delta M_x-2B_{xy}\delta M_y+\alpha N_z\delta N_x\\
&\quad+(-i\omega+\lambda)\delta N_y=-N_z(\delta H_x-\beta_x\delta T).
\end{aligned}
\tag{4.150}
$$

Here we have introduced the abbreviations

$$A=2H_e+H_A-(4\pi/3)N_z;\qquad B_{ij}=2\pi N_z\hat{q}_i\hat{q}_j \tag{4.151}$$

and

$$\tilde{\Omega}=H_A+Dq^2;\qquad \tilde{\sigma}_{ij}=(\xi_l/\chi_\perp)[1+(8\pi/3)\chi_\perp]q^4\hat{q}_i\hat{q}_j \tag{4.151a}$$

which are consequences of Eqs. (4.148), (4.148a) and (4.141). The definitions (4.151a) are generalizations of (4.142a) and

$$D=\rho_s/m^2N_z. \tag{4.152}$$

Eqs. (4.152) and (4.141) are the analogue of Eqs. (4.108) of the axial ferromagnet. H_e is the *effective exchange field* [see Section 39 of Keffer (1966)].

We consider again only the homogeneous equations (4.150), posing $\delta H_i = 0$, $\beta_i = 0$. The associated secular equation is found, after considerable algebra, to be exactly given by

$$(\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2) + i\omega^3(2\lambda + \tilde{\sigma}_\perp) - \omega^2\lambda(\lambda + 2\tilde{\sigma}_\perp) - i\omega\{2\tilde{\Omega}(A + B_\perp)\lambda + \tilde{\Omega}A\tilde{\sigma}_\perp + (\alpha N_z)^2(2\lambda + \tilde{\sigma}_\perp) + \lambda^2\tilde{\sigma}_\perp\} + \tilde{\Omega}A\lambda\tilde{\sigma}_\perp + (\alpha N_z)^2\lambda^2 = 0, \tag{4.153}$$

where $B_\perp = B_{xx} + B_{yy}$, $\tilde{\sigma}_\perp = \tilde{\sigma}_{xx} + \tilde{\sigma}_{yy}$ and

$$\omega_\pm^2 = \tilde{\Omega}(A + B_\perp) + (\alpha N_z)^2 \pm \{\tilde{\Omega}^2 B_\perp^2 + 4\tilde{\Omega}(\alpha N_z)^2(A + B_\perp)\}^{1/2}. \tag{4.154}$$

For $q = 0$, Eq. (4.154) is essentially the same as Eq. (23) of Anda (1973) [see also Loudon and Pincus (1963)]; for $\alpha = 0$ it simplifies to

$$\omega_+^2 = \tilde{\Omega}(A + 2B_\perp) = (H_A + Dq^2)[2H_e + H_A - (4\pi/3)N_z + 4\pi N_z \sin^2\vartheta_q],$$
$$\omega_-^2 = \tilde{\Omega}A = (H_A + Dq^2)[2H_e + H_A - (4\pi/3)N_z], \tag{4.154a}$$

where $\cos\vartheta_q = \hat{q}_y$. To order q^2 this result is the same as Eqs. (13a), (13b) of Brooks Harris (1966). Apart from the local field correction $-(4\pi/3)N_z$ formulas (4.154a) were already given by Loudon and Pincus (1963).

However, the merit of our hydrodynamic treatment is not only to be able to reproduce the quoted results but, even more, to yield the damping terms. Indeed, the solutions of the secular equation (4.153) are, up to order q^4,

$$\omega_1 = \omega_+ + \frac{\omega_+^2 - (\alpha N_z)^2}{2\omega_+(\omega_+^2 - \omega_-^2)}\lambda^2 - \frac{i}{2}\left\{\lambda + \left(\frac{1}{2} + \frac{\tilde{\Omega}B_\perp}{\omega_+^2 - \omega_-^2}\right)\tilde{\sigma}_\perp\right\},$$
$$\omega_2 = \omega_- - \frac{\omega_-^2 - (\alpha N_z)^2}{2\omega_-(\omega_+^2 - \omega_-^2)}\lambda^2 - \frac{i}{2}\left\{\lambda + \left(\frac{1}{2} - \frac{\tilde{\Omega}B_\perp}{\omega_+^2 - \omega_-^2}\right)\tilde{\sigma}_\perp\right\}. \tag{4.155}$$

For $\alpha = 0$, these expressions reduce to

$$\omega_1 = [\tilde{\Omega}(A + 2B_\perp)]^{1/2}\left(1 + \frac{\lambda^2}{4\Omega B_\perp}\right) - \frac{i}{2}(\lambda + \tilde{\sigma}_\perp),$$
$$\omega_2 = (\tilde{\Omega}A)^{1/2}\left(1 - \frac{\lambda^2}{4\Omega B_\perp}\right) - \frac{i}{2}\lambda, \tag{4.155a}$$

valid for $\tilde{\Omega}B_\perp \neq 0$. If in addition $H_A = 0$ we find, to order q^2, inserting (4.151), (4.151a) and making use of (4.142), (4.144) and (4.152)

$$\omega_1 = c_1 q[1 - (1 - 3\sin^2\vartheta_q)(4\pi/3)\chi_\perp]^{1/2} - \tfrac{1}{2}i\rho_s\zeta_1 q^2$$
$$\omega_2 = c_1 q[1 - (4\pi/3)\chi_\perp]^{1/2} - \tfrac{1}{2}i\rho_s\zeta_1 q^2. \tag{4.155b}$$

We notice that away from the easy axis the dipolar field correction splits the mode (4.145).

We also see that a sound-like Hulthén mode is a general feature both of planar and, for $H_A = 0$, of axial antiferromagnets and hence, in particular, of *isotropic* antiferromagnets [see Halperin and Hohenberg (1969), p. 910, and Keffer (1966), p. 107]. On the other hand, neutron scattering has revealed that the quadratic dispersion relation with gap as described by Eqs. (4.155) and (4.155a) is valid not only for the *axial* antiferromagnets MnF_2 (Okazaki *et al.*, 1964, 1965) and FeF_2 (Guggenheim *et al.*, 1968), but also for the *planar* antiferromagnets K_2NiF_4 (Skalyo *et al.*, 1969) and K_2MnF_4 (Birgeneau *et al.*, 1973). This shows that the dipole–dipole interaction acts similarly in the latter cases.

Thermodynamically there is of course an important difference between the quadratic dispersion laws (4.109), (4.112), (4.116), (4.155), (4.155a), on the one hand, and the linear laws (4.57), (4.145), (4.155b) on the other. Indeed, if the gap is small compared to k_BT, quadratic dispersion gives rise to the well known $T^{3/2}$-dependence of the magnon specific heat c_M instead of the T^3-dependence valid for linear dispersions [for k_BT small compared to the gap ω_0 the specific heat behaves as $\exp(-\omega_0/k_BT)$]. This $T^{3/2}$-dependence has the important consequence that the quantity ϑ defined in Eq. (4.71) varies as $T^{5/2}$, and not as T^4, at low temperatures. This slower decrease makes the ratio r between the second- and the first-magnon peaks, as defined in Eq. (4.86), more favorable. It is therefore advantageous to look for the second-magnon peak in magnetic excitation experiments on systems with quadratic dispersion law with gap (because of drift, see above), that is for axial ferro- and antiferromagnets (Michel and Schwabl, 1971a; Forney, 1972; Forney and Jäckle, 1973).

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