Spontaneous symmetry breaking, gauge theories, the Higgs mechanism and all that

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A more or less self-contained introductory review is presented of the so-called Higgs phenomenon. This is the mechanism by which, in a certain class of gauge theories, the "photon" and would-be Goldstone scalar mesons conspire together to produce massive vector mesons via a "spontaneous" breaking of gauge invariance. It is conceivable that this is the way in which nature has chosen to unify weak and electromagnetic interactions. It is hoped that a reader of this review will come to understand the meaning of the first three sentences in this abstract and will then be able to proceed to confront a rapidly growing literature in the subject of gauge theories.

CONTENTS

I. INTRODUCTION

Elementary particle theory seems to proceed from fashion to fashion in intervals of two to three years. A few familiar names from the recent past will give the general idea: renormalizable field theories, dispersion relations, conserved and partially conserved currents, current algebras, Regge poles, etc., etc. At the moment when these specialities are at the height of their activity, most practitioners have neither the time nor inclination to go back and read the very early literature in the discipline, so that a group of standard references is arrived at and these become oft quoted and rarely read. When one actually does go back to read these early papers, one is often amazed by how much their authors knew or conjectured, and one comes to the conclusion that these papers rearranged and unified are probably the best introduction to the subject. That will be the spirit and methodology of this review. Anything novel on the part of the present author is unintentional.

The plan of attack in this review is as follows. We begin (in Sec. II) in 1960 when Nambu (Nambu, 1960; Nambu and Jona-Lasinio 1961) observed that the natural interpretation of a conserved $\Delta S = 0$ axial vector current in weak interactions is in the limiting case of a world in which the mass of the meson is set equal to zero. At essentially the same time, Gell-Mann and Lévy (1960) produced several field theoretic models in which this phenomenon was shown to occur as a consequence of the field equations appropriate to the model. Of these models, the so-called "σ" model is, in the present context, the most interesting. In its simplest version, there are three basic fields in the Lagrangian: an isotopic vector pion, an isotopic doublet nucleon, and an isotopic singlet, Lorentz scalar, σ meson. The latter has the quantum numbers of "a vacuum state" of the strong interactions. We use the phrase a vacuum state advisedly since in this class of theories there are in general an infinite number of orthogonal states with the property that

$$H|0\rangle = 0, \tag{1.1}$$

where H is the Hamiltonian of the theory, and $|0\rangle$ is one of the vacua. Since σ has the quantum numbers of a vacuum, its vacuum expectation value is not forced by any symmetry principle to vanish, i.e., we may have

$$\langle 0|\sigma(x)|0\rangle = \langle 0|\sigma(0)|0\rangle = \lambda \neq 0, \tag{1.2}$$

where λ is a real number and σ is an Hermitian operator with dimensions of a mass. We assume that

$$\sigma(x) = \exp[-i(Px)]\sigma(0)\exp[i(Px)], \tag{1.3}$$

with—note the metric convention +, +, +, − to be used throughout—

$$(Px) = \mathbf{P}\cdot\mathbf{x} - P_0 t, \tag{1.4}$$

where the P_μ are the generators of space–time displacements. We always assume that

$$P_\mu|0\rangle = 0. \tag{1.5}$$

If we want to give the σ field a particle interpretation, we are forced to redefine it in such a way that

$$\langle 0|\sigma'(0)|0\rangle = 0 \tag{1.6}$$

with

$$\sigma'(x) = \sigma(x) - \lambda$$

otherwise the "vacuum" and the one-particle state will not be orthogonal. As we shall see, it is possible to arrange the Lagrangian of the σ model in such a way that, at the outset the pion and σ have a "bare" mass. The nucleons appear to be massless but, in fact, acquire a mass that is proportional to λ, the vacuum expectation value of the σ. To achieve exact conservation of the axial vector current, A_μ, in this theory, it is necessary to give the pion zero bare mass. This, it turns out, corresponds to a limit in which the theory is invariant with respect to the group $SU(2)\times SU(2)$; exact chiral invariance. Hence, in this way of realizing the symmetry, zero mass bosons—in this case the pions—make their appearance. In 1961, Goldstone conjectured that such zero mass bosons would be an inevitable consequence of a symmetry realization of theories like the σ-model in which the Lagrangian would be fully invariant with respect to a continuous group, but in which the vacuum would not be invariant

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under the group. (In the σ model, as we shall see, the axial charges do not annihilate the vacuum, which allows $\langle 0|\sigma(0)|0\rangle \neq 0$ so that the nucleons acquire a mass.) This conjecture appeared to have been fully demonstrated in a paper by Goldstone, Salam and Weinberg, (1962)[1] who presented several "proofs" of same. In the next section, we shall present a proof due to Gilbert (1969) in which the assumptions that go into the argument are made especially clear.

Here is where things stood until 1964 when Higgs, responding to Gilbert's argument, showed that there was a loophole in it.[2] As we shall see, Gilbert's argument—essentially a restatement of one of the proofs in Goldstone *et al*—makes use of the "manifest covariance" of certain matrix elements. In essence this means that if we are considering the matrix element of a Lorentz four-vector operator J_μ we can construct this operator's matrix elements only out of the covariants at hand which, in this case, transform like four-vectors. A familiar example is the matrix element of a current between states of zero spin and four-momenta P and P';$|P\rangle$,$|P'\rangle$ in which, by covariance,

$$\langle P'|J_\mu(0)|P\rangle = (P+P')_\mu F_+(q^2) + (P-P')_\mu F_-(q^2) \tag{1.8}$$

with $F_\pm(q^2)$ being arbitrary "form factor" functions of

$$q^2 = (P-P')^2. \tag{1.9}$$

There are no other four-vectors one can construct out of the four-momenta P and P'.

In making this argument, one had better be sure that the J_μ in question is really a four-vector and not some object that has four components but a more complicated transformation property. Indeed, as Higgs stressed, the photon field $A_\mu(x)$ is just such an object. We shall go into this more carefully in Sec. III, but we remind the reader here that the "radiation gauge" condition

$$\nabla \cdot \mathbf{A}(x) = 0 \tag{1.10}$$

is clearly noncovariant which means that if we wish to maintain transversality of the photon in all Lorentz frames, the photon field $A_\mu(x)$ cannot transform like a four-vector.[3] This is no catastrophe, since the photon *field* is not an observable, and one can readily show that the S-matrix elements, which *are* observable have covariant structures. In his 1964 note, Higgs argued that in gauge theories one might arrange things so that one had a symmetry breakdown because of the noninvariance of the vacuum; but, because the Goldstone *et al.* proof breaks down, the zero mass, Goldstone mesons need not appear. In a subsequent note, Higgs (1964) constructed a Lagrangian model of a gauge theory—essentially charged scalar electrodynamics in which the scalar field has no bare mass, but rather self-interactions of a prescribed type—where one could see the would-be Goldstone mesons disappear from the theory as observable particles. In fact, what happens is exceedingly remarkable. The zero mass photon also disappears from the theory! More precisely speaking, one begins with the Higgs Lagrangian[4] which contains $A_\mu(x)$, the photon, and $\phi_1(x)$ and $\phi_2(x)$ the Hermitian fields which combine to yield the charged fields $\phi^\pm(x) = [\phi_1(x) \pm \phi_2(x)]/\sqrt{2}$. The bare mass of all these objects is taken to be zero so that the Lagrangian is invariant against transformations of the form (see Sec. IV for details)

$$A_\mu(x) \to A_\mu(x) + (1/e_0)(\partial/\partial x^\mu)\Lambda(x). \tag{1.11}$$

$$\phi_1(x) \to \phi_1(x)\cos(\Lambda(x)) + \phi_2(x)\sin(\Lambda(x)), \tag{1.12}$$

$$\phi_2(x) \to -\phi_1(x)\sin(\Lambda(x)) + \phi_2(x)\cos(\Lambda(x)), \tag{1.13}$$

where e_0 is the bare electric charge, and $\Lambda(x)$ is an arbitrary "gauge function" of space–time. A choice of gauge can be made so that

$$\langle 0|\phi_1(0)|0\rangle = 0 \tag{1.14}$$

and

$$\langle 0|\phi_2(0)|0\rangle = \eta \tag{1.15}$$

It then turns out that $\phi_2(x)$ acquires a mass, while $\phi_1(x)$ combines with A_μ to form a vector field

$$B_\mu(x) = A_\mu(x) - (1/e_0\eta)(\partial/\partial x^\mu)\phi_1(x), \tag{1.16}$$

which, in fact, satisfies the equations of motion of a *massive* vector field of mass $e_0\eta$. Indeed, as Higgs showed in a subsequent publication (1966), once one realizes that this is what is going to happen one may rewrite the original Lagrangian in such a way that the only fields that appear in it are a massive scalar field and a massive vector field with complicated nonlinear interactions between them. The theory written in this way shows no traces of its original gauge-invariant electromagnetic origins. The fact that the vector meson has acquired a mass in this way is referred to in the contemporary literature as the "Higgs mechanism" and the scalar fields with nonvanishing vacuum expectation values in such theories are now known, appropriately, as "Higgs fields."

Between 1964 and 1967, some interesting developments of a technical nature took place. In 1964 Englert and Brout discovered the Higgs mechanism independently by a different route, which we review in Sec. V. They again studied scalar electrodynamics with a broken gauge symmetry realized by nonvanishing vacuum expectation values of the scalar field. However, rather than dealing with the equations of motion, they quantized the theory using Feynman-like rules. The new feature here is that there are graphs in which the scalar meson is emitted and simply disappears into the vacuum. These graphs contribute to the vacuum polarization tensor of the photon which, as is well known, determines the propagator of the interacting photon. Englert and Brout showed, in lowest

[1] See also S. Bludman and A. Klein (1963).

[2] A suggestion of the role of covariance is found in A. Klein and B. W. Lee (1964). Professor A. Salam has pointed out to the author that P. W. Anderson (1958) should be regarded as the True Father of this subject since he noted that Goldstones could be avoided in the BCS electron model if long-range Coulomb interactions were included—a forerunner of the Higgs mechanism.

[3] One can use the covariant "Lorentz gauge" condition $(\partial/\partial x_\mu)A_\mu(x) = 0$. This introduces unwanted degrees of freedom and leads to its own difficulties with the Goldstone *et al.* (1962) argument. These have been analyzed by T. W. B. Kibble (1967) and will be discussed in Sec. IV of this review.

[4] The Lagrangian considered in this paper is a gauge invariant version of the original Goldstone (1961) Lagrangian.

nontrivial order, that these graphs shift the pole in the "photon" propagator from $q^2 = 0$ to $q^2 = -e_0^2\eta^2$ where the notation is the same as above. This is, of course, the Higgs phenomenon. They also generalized this result to the case in which the gauge symmetry is not the simple Abelian phase group $U(1)$ but some arbitrary compact Lie group. Here any generator which fails to annihilate the vacuum will, by the Higgs mechanism, induce a mass for the corresponding "photon" provided that the Higgs scalars are introduced accordingly. In the real world there is only one massless vector meson, the photon, and from the Higgs viewpoint it remains massless because the vacuum has no net electric charge.[5] The electric charge, Q_γ must annihilate the vacuum; i.e.,

$$Q_\gamma|0\rangle = 0. \tag{1.17}$$

Also in 1964 Guralnik, Hagen, and Kibble investigated the electrodynamic model of Englert and Brout and Higgs and made the important observation that in these models, the local symmetry condition

$$(\partial/\partial x_\mu)J_\mu(x) = 0 \tag{1.18}$$

does not imply the time independence of

$$Q = \int d^3\mathbf{x}\, J_0(\mathbf{x}, t). \tag{1.19}$$

In fact, even the mathematical existence of such global charges is a very delicate matter[6] in these theories. Finally, in 1967, Kibble presented an elegant mathematical treatment of the various models previously considered and, in particular, showed that the breakdown of the Goldstone theorem in the gauge theories could also, as one would expect, be demonstrated in the Lorentz gauge characterized by the manifestly covariant condition

$$(\partial/\partial x_\mu)A_\mu(x) = 0. \tag{1.20}$$

What happens here is that the Goldstone *et al.* argument goes through so that there are, apparently, Goldstone particles. But these, it turns out, are decoupled from the rest of the fields and can simply be factored out of the theory.

It is probably fair to say that while no one doubted the correctness of these arguments, no one quite believed that nature was diabolically clever enough to take advantage of them. In fact, as of this writing, no one can be sure that nature *has* been clever enough to take advantage of them. However, in 1967, Weinberg proposed an application of these ideas to weak and electromagnetic interactions which, subsequently, has convinced many theorists that nature *should* take advantage of the possibility of gauge theories with nongauge-invariant vacua. We shall discuss this model and its generalizations in Secs. VII and VIII, but here we wish to give some of the flavor. It is well known that when Fermi (1934) constructed the first field theoretic model of β decay, he employed an interaction modeled as closely as possible on quantum electrodynamics. In Fermi's theory the leptons interact with the neutron and proton weakly via "currents" of the form

$$\mathfrak{L}_\mu(x) = i\bar{\psi}_e(x)\gamma_\mu\psi_\nu(x), \tag{1.21}$$

while charged particles interact with hadrons via a conserved current of the form

$$J_\mu(x) = i\bar{\psi}_e(x)\gamma_\mu\psi_e(x). \tag{1.22}$$

There are, of course, several obvious distinctions between electromagnetism and β decay, only some of which Fermi could have known in 1934. We list a few:

(1) The photon is known to exist. Thus, the electromagnetic interaction, in fact, takes the form (in lowest nontrivial order)[7]

$$H_{\rm em} = e_0^2 \iint J_\mu^{\rm em}(x) D^{\mu\nu}(x-y) J_\nu^{\rm em}(y)\, d^4x\, d^4y, \tag{1.23}$$

where $J_\mu(x)$ is the conserved electromagnetic current, and $D^{\mu\nu}(x-y)$ is the photon propagator. All evidence on electromagnetic interactions is consistent with a photon of zero mass which means that two charged particles interact via the $1/|\mathbf{x}-\mathbf{y}|$ Coulomb potential in the low energy, static limit.

On the other hand the "weak photon," at least at this writing, is not known to exist. However, it is known that an effective weak interaction of the form

$$H_{\rm wk} = G \int J^{{\rm wk}_\mu}(x) J_\mu^{\rm wk}(x)\, d^4x \tag{1.24}$$

gives a good representation of all the data on weak interactions at relatively low energy. Taken literally, this would correspond to a current–current interaction mediated by a weak photon of infinite mass. Clearly at least some of the weak photons, if they exist, must be charged. $\mathfrak{L}_\mu(x)$, for example, changes charge by one unit, and all the data so far are perfectly consistent with a hypothetical, charged, weak photon with a mass of at least 5 BeV. Hence, a striking distinction between electromagnetic and weak interactions is the mass of the carrier.

(2) Weak interactions are "weak", and electromagnetic interactions are not. At first sight, this might appear to deal a death blow to all hopes of unifying the two couplings. However, just the difference in masses between the weak and electromagnetic photons may save the situation. The quantity $H_{\rm em}$, defined above, is dimensionless since $J_\mu^{\rm em}$ has dimensions of E^3 and $D^{\mu\nu}(x)$ has dimensions of E^2. (Here E stands for a dimension of energy or inverse length.) Thus, e_0^2 is dimensionless. However, the same argument shows that G has dimensions of $(1/E^2)$. In a theory with a weak photon it turns out that

$$G = c(g^2/m_w^2), \tag{1.25}$$

where g is the weak dimensionless charge, m_w is the mass of the weak photon, and c is a numerical constant depending on the theory. (In Weinberg's 1967 theory[8] $c = 1/4\sqrt{2}$). Since m_w is not known, we can always imagine finding an m_w such that $g^2 \simeq e_0^2$. All we know so

[5] If this were not so, no one particle state would have a well defined electric charge.

[6] These and many related questions are treated in the excellent review article by G. S. Guralnik *et al.* (1968).

[7] These are effective dimensionless elements of the S-matrix.

[8] See also A. Salam (1968). Both Weinberg and Salam conjectured that the theory would be renormalizable.

far from low-energy weak interaction experiments is G which is given experimentally by

$$Gm_p^2 \simeq 10^{-5}. \tag{1.26}$$

Here m_p is the mass of the proton. Typically, to arrange this we need $m_w \gtrsim 40$ BeV.

(3) Finally, of course, there is parity nonconservation in the weak interactions,[9] which means that the weak photon will have interactions with both axial vector and vector currents. This, as we shall see, can be arranged.

The big barrier to unification would appear, at first sight, to be the enormous discrepancy between the masses of the weak and electromagnetic photons. In a conventional broken symmetry, one would begin with a Hamiltonian that commutes with all the operators generating the symmetry. The solutions to this "unperturbed" problem would classify themselves into degenerate multiplets labeled by the quantum numbers of the symmetry group. Hence, we would recognize the validity of such a symmetry in nature by observing such degenerate or nearly degenerate—in mass—multiplets. The lack of complete degeneracy we would explain by adding a "small" term to the Hamiltonian which fails to commute with the symmetry generators. Clearly, a photon with zero mass and a weak photon with a mass of 40 BeV do not appear to fall into an approximately degenerate multiplet. It is just here that the Higgs mechanism may save us and allow a unification, although in a new and unconventional sense. In the gauge theories *all* photons have zero mass provided that the vacuum is invariant under the gauge group.[10] Once this requirement is dropped, the Higgs scalar fields can develop nonvanishing vacuum expectation values and all of the photons except the electromagnetic photon acquire mass. There is no reason why any of these masses need be small. The size of these masses, as we shall see, depends on the coupling constants characterizing the self-couplings of the Higgs fields and these can be large. One would not be able to detect the symmetry by looking for nearly degenerate multiplets but rather by relations among coupling constants and cross sections. Apart from the elegance of the idea, there is the additional payoff that such combined weak and electromagnetic theories can be found so that the resultant mixture is "renormalizable", which means that the convergence of the individual terms in the S-matrix is no worse than quantum electrodynamics, which we know—despite the problem of infinities—to be the most precise physical theory so far invented. This means that, for the first time, a weak interaction field theory has been found in which the weak interactions remain genuinely "weak" in higher orders in perturbation theory. Heretofore, what has happened is that the lowest order matrix elements are characterized by the weak constant G, but all of the higher orders diverge and must be cut off arbitrarily. In the gauge theories no cutoff is needed, and one can, in principle, find answers to all of the questions involving weak and electromagnetic couplings to arbitrary order. Even if none of the models so far produced survive experimental testing, the gauge theories are a new and remarkably subtle class of finite—after renormalization—field theories. All of these matters will be discussed in more detail at the end of this review. However, enough has been said so far perhaps to convince the reader that, in Einstein's phrase,[11] while gauge theories with vacuum broken symmetries may or may not be the "true Jacob," they demand "serious attention."

[9] As well as C and CP violation.

[10] In these theories, we always assume that the Lagrangian is gauge invariant so that none of the "photons" has a bare mass to begin with.

II. THE GOLDSTONE THEOREM

We imagine a theory described by a Hamiltonian H and we assume that there are one or more "currents"—operators that transform as four-vectors under proper Lorentz transformations—which are conserved in virtue of the equations of motion;[12] i.e.

$$(\partial/\partial x_\mu)J_\mu(x) = 0. \tag{2.1}$$

We call the "formal charge" associated with this current

$$Q(t) = \int d^3\mathbf{x}\, J_0(\mathbf{x}, t), \tag{2.2}$$

"formal" in the sense that this integral taken over all space may not exist. Even so, objects like

$$\int d^3\mathbf{x}[J_0(\mathbf{x},t), \phi(\mathbf{y},t)],$$

where ϕ is some field operator, may exist provided that we commute first and then do the integral. Where it is relevant, we shall be careful about this. Formally, we can write for a conserved current

$$\int d^3\mathbf{x}\nabla\cdot\mathbf{J}(\mathbf{x},t) = (d/dt)Q(t). \tag{2.3}$$

Hence, there are two possibilities,

$$(1) \qquad \int d^3\mathbf{x}\nabla\cdot\mathbf{J}(\mathbf{x},t) = 0 \tag{2.4}$$

or

$$(2) \qquad \int d^3\mathbf{x}\nabla\cdot\mathbf{J}(\mathbf{x},t) \neq 0. \tag{2.5}$$

The latter case may occur provided that there exists a pair of states $|P\rangle$ and $|P'\rangle$ such that

$$\begin{aligned}
&\langle P'|\int d^3\mathbf{x}\nabla\cdot\mathbf{J}(\mathbf{x},t)|P\rangle \\
&= i^{-1}\int d^3\mathbf{x}\langle P'|[\mathbf{P},\mathbf{J}(\mathbf{x},t)]|P\rangle \\
&= i^{-1}(\mathbf{P}'-\mathbf{P})\cdot\int\langle P'|\mathbf{J}(\mathbf{x},t)|P\rangle d^3\mathbf{x} \\
&= i^{-1}(\mathbf{P}'-\mathbf{P}) \\
&\qquad\cdot\int\langle P'|\exp[-i(Px)]\mathbf{J}(0)\exp[i(Px)]|P\rangle d^3\mathbf{x} \\
&= (2\pi)^3/i\,\delta^3(\mathbf{P}'-\mathbf{P})\exp[i(E_{P'}-E_P)t](\mathbf{P}'-\mathbf{P}) \\
&\qquad\cdot\langle P'|\mathbf{J}(0)|P\rangle \\
&\neq 0.
\end{aligned} \tag{2.6}$$

[11] Taken from a letter written by Einstein to Max Born in 1926 concerning the discovery of quantum mechanics. The letter is quoted in M. J. Klein (1970).

[12] In our metric $(\partial/\partial x_\mu)J_\mu(\mathbf{x},t) = \nabla\cdot\mathbf{J}(\mathbf{x},t) + (\partial/\partial it)J_0(\mathbf{x},t) = \nabla\cdot\mathbf{J}(\mathbf{x},t) + \dot{J}_0(\mathbf{x},t)$.

This can only happen if

$$\lim_{\mathbf{P}'\to\mathbf{P}}\langle P'|\mathbf{J}(0)|P\rangle = \infty \qquad (2.7)$$

which might occur in a theory with a zero mass boson coupled to $J_\mu(0)$.[13] We shall return to this case later and suppose, for the time being, that

$$\dot{Q} = i[H,Q] = 0. \qquad (2.8)$$

Let the eigenstates of H be designated as $|E_n\rangle$. Then, formally

$$HQ|E_n\rangle = E_n Q|E_n\rangle \qquad (2.9)$$

so that $Q|E_n\rangle$ is an eigenstate of H with energy E_n. If $|E_n\rangle$ is an eigenstate of Q with eigenvalue q, then this degeneracy is trivial, otherwise $|E_n\rangle$ and $Q|E_n\rangle$ may represent orthogonal states with distinct quantum numbers. We may apply these considerations to the "vacuum" state characterized by the requirement that

$$P_\mu|0\rangle = 0, \qquad (2.10)$$

where P_μ are the energy–momentum operators of the theory. We always assume this of the vacuum. In conventional theories

$$U = \exp(i\alpha Q), \qquad (2.11)$$

with Q Hermitian and α real, is a unitary operator which generates whatever symmetry of the Lagrangian gave rise to the conservation of J_μ. Hence, if the states are to reflect the symmetry, we must have

$$\exp(i\alpha Q)|0\rangle = |0\rangle \qquad (2.12)$$

or

$$Q|0\rangle = 0. \qquad (2.13)$$

As a concrete example, consider the statement that the electron is an eigenstate of electric charge with eigenvalue +1. This is a consequence of the gauge condition

$$[Q,\psi^\dagger(\mathbf{x},t)] = +\psi^\dagger(\mathbf{x},t), \qquad (2.14)$$

where Q is the conserved electric charge[14], and ψ is the electron field operator, provided that

$$Q|0\rangle = 0. \qquad (2.15)$$

We can see this from the equations

$$\begin{aligned} Q|P\rangle &= Q\lim_{t\to\pm\infty}\int d^3\mathbf{x}\psi^\dagger(\mathbf{x},t)\frac{\exp[i(Px)]}{(2\pi)^{3/2}}U(P)|0\rangle \\ &= \lim_{t\to\pm\infty}\int d^3\mathbf{x}[Q,\psi^\dagger(\mathbf{x},t)]\frac{\exp[i(Px)]}{(2\pi)^{3/2}}U(P)|0\rangle \\ &\quad + \lim_{t\to\pm\infty}\int d^3\mathbf{x}\psi^\dagger(\mathbf{x},t)\frac{\exp[i(Px)]}{(2\pi)^{3/2}}U(P)Q|0\rangle \\ &= |P\rangle, \end{aligned} \qquad (2.16)$$

where $U(P)$ is a solution of the Dirac equation and the limit process represents the creation of a one-electron state with energy–momentum P_μ.

In the theories we are about to consider, we sometimes have

$$\dot{Q} = 0 \qquad (2.17)$$

and

$$Q|0\rangle \neq 0. \qquad (2.18)$$

We might be tempted, then, to identify $Q|0\rangle$ as a second vacuum state. This interpretation can be maintained, but only with caution. To see what is involved,[15] consider

$$\langle 0|QQ|0\rangle = \int d^3\mathbf{x}\langle 0|J_0(\mathbf{x},t)Q(t)|0\rangle. \qquad (2.19)$$

Since

$$\exp[i(\mathbf{P}\cdot\mathbf{x})]Q(t)\exp[-i(\mathbf{P}\cdot\mathbf{x})] = Q(t), \qquad (2.20)$$

we have

$$\langle 0|QQ|0\rangle = \int d^3\mathbf{x}\langle 0|J_0(0)Q(0)|0\rangle. \qquad (2.21)$$

Since

$$Q|0\rangle \neq 0 \qquad (2.22)$$

and

$$\int d^3\mathbf{x} = \infty, \qquad (2.23)$$

the state $Q|0\rangle$ is not normalizable. This is difficult to live with but not impossible, since in all applications we will consider commutators involving $J_0(\mathbf{x},t)$ and then integrate safely later. However, it does mean that

$$U = \exp(i\alpha Q) \qquad (2.24)$$

is not a unitary operator in a simple sense—another manifestation of the broken symmetry. Nonetheless, expressions like $UAU^\dagger$ can be meaningful, since if the exponentials are expanded, the resulting commutators may be well defined. Clearly this is a subject in which common sense will have to guide the passage between the Scylla of mathematical Talmudism and the Charybdis of mathematical nonsense.

To study the Goldstone theorem, we imagine we have a theory with n conserved currents $J_\mu^1(x), \ldots, J_\mu^n(x)$, n formal Hermitian charges $Q_1(t) = \int d^3\mathbf{x}J_0^1(\mathbf{x},t), \ldots, Q_n(t) = \int d^3\mathbf{x}J_0^n(\mathbf{x},t)$, and n Hermitian scalar fields $\phi_1(x)\cdots\phi_n(x)$ satisfying equal time commutation relations of the form

$$[Q_i(t),\phi_j(\mathbf{x},t)] = \tau_{ijk}\phi_k(\mathbf{x},t), \qquad (2.25)$$

where τ_{ijk} is a pure imaginary totally antisymmetric function of the three indices.[16] We shall assume that

$$\langle 0|[Q_i(t),\phi_j(\mathbf{x},t)]|0\rangle = \tau_{ijk}\langle 0|\phi_k(\mathbf{x},t)|0\rangle \neq 0 \qquad (2.26)$$

for at least some values of the indices, which means that

[13] In such a theory the matrix element develops a pole at $q^2 = (P'-P)^2 = (\mathbf{P}'-\mathbf{P})^2 - (E_{P'}-E_P)^2 = 0$ which can occur at $\mathbf{P} = \mathbf{P}'$. I am grateful to M. A. B. Bég for discussions of this question.

[14] Actually, this is one of the cases in which $(\partial/\partial x_\mu)J_\mu(x) = 0$ does *not* imply that $\dot{Q}(t) = 0$ when higher order radiative corrections are included. The "bare" charge dresses itself. This is not relevant to what we are trying to illustrate here.

[15] This discussion follows E. Fabri and L. E. Picasso (1966) and G. S. Guralnik *et al.* (1968).

[16] We are allowing here for the possibility that the charges may be functions of time. Examples of all of this will be furnished in due course.

at least one charge fails to annihilate the vacuum and at least one scalar field develops a nonvanishing vacuum expectation value. Keeping this in mind, we choose such an i and j and consider the quantity[17]

$$M_\mu^{ij}(k) = \int d^4x \langle 0|[J_\mu^i(\mathbf{x},0), \phi_j(0)]|0\rangle \exp[i(kx)]. \quad (2.27)$$

We shall assume that the eigenstates of P_μ—the energy–momentum operator—are complete and have *positive definite norm.*[18] We label these states $|n\rangle$. Hence,

$$\begin{aligned} M_\mu^{ij}(k) &= \sum_n \int d^4x \exp[i(kx)] \\ &\quad \{\langle 0|J_\mu^i(x)|n\rangle\langle n|\phi_j(0)|0\rangle \\ &\quad - \langle 0|\phi_j(0)|n\rangle\langle n|J_\mu^i(x)|0\rangle\} \\ &= (2\pi)^4 \sum_n \{\delta^4(P_n + k)\langle 0|J_\mu^i(0)|n\rangle\langle n|\phi_j(0)|0\rangle \\ &\quad - \delta^4(P_n - k)\langle 0|\phi_j(0)|n\rangle\langle n|J_\mu^i(0)|0\rangle\}. \end{aligned} \quad (2.28)$$

Since $\phi_j(0)$ is a scalar operator, under proper Lorentz transformations, the only states $|n\rangle$ that can contribute to this sum have no intrinsic spin. This does not conflict with the four-vector character of J_μ since J_μ can connect two spinless states; i.e. J_μ can connect $|n\rangle$ and $|0\rangle$. However, it does imply the following[19]

$$\langle 0|J_\mu(0)|\mathbf{P}, E_P\rangle = a(P^2,(Pn))P_\mu + b(P^2,(Pn))\eta_\mu, \quad (2.29)$$

where η_μ is a constant, timelike, four-vector which, in a suitable Lorentz frame, we can write as

$$\eta = (0,0,0,1), \quad (2.30)$$

and a and b are arbitrary functions of the indicated variables. This is a consequence of the three-dimensional rotational covariance of the theory. We always assume that the vacuum is a scalar under the full Lorentz group. Hence, from the spin-zero character of $|\mathbf{P}, E_P\rangle$, it follows that for any three-vector operator $\mathbf{V}(0)$

$$\langle 0|\mathbf{V}(0)|\mathbf{P}, E_P\rangle = \mathbf{P}a(\mathbf{P}^2, E_P), \quad (2.31)$$

where $a(\mathbf{P}^2, E_P)$ is an arbitrary function of the indicated variables. There is no other direction available for $\langle 0|\mathbf{V}(0)|\mathbf{P}, E_P\rangle$ to point in. One may, of course, give a formal argument involving the rotation generator. In our case, $\mathbf{J}(0)$ *is* a three-vector and this argument goes through. As we see below, an extra assumption is needed to fix $\langle 0|J_0(0)|\mathbf{P}, E_P\rangle$.

However, to finish his proof of the Goldstone theorem, Gilbert (1964) did make an extra assumption which we may call "manifest covariance." He assumed that

$$\langle 0|J_\mu(0)|\mathbf{P}, E_P\rangle = P_\mu a(P^2). \quad (2.32)$$

This is, of course, consistent with the general form, but it is not always true. Whenever it is true, and this assumption is joined to the others above, we have, as we shall now see, the Goldstone theorem. The fact that it fails to be true in gauge theories, like electrodynamics, is the loophole through which the Higgs mechanism finds its way.

Given manifest covariance we may write

$$M_\mu^{ij} = \epsilon(k_0)k_\mu\rho_1^{ij}(k^2) + k_\mu\rho_2^{ij}(k^2), \quad (2.33)$$

where the $\rho^{ij}(k^2)$ are, as yet, arbitrary functions of k^2 and

$$\epsilon(k_0) = \begin{cases} 1 & k_0 \geq 0 \\ -1 & k_0 < 0. \end{cases} \quad (2.34)$$

The reason for this form is that $M_\mu(k)$ consists of two sums, one of which, because of the δ functions, contributes when $k_0 > 0$, and the other when $k_0 < 0$. The $\epsilon(k_0)$ function expresses this compactly. According to the Goldstone hypothesis

$$(\partial/\partial x_\mu)J_\mu^i(x) = 0. \quad (2.35)$$

Thus

$$\epsilon(k_0)k^2\rho_1^{ij}(k^2) + k^2\rho_2^{ij}(k^2) = 0 \quad (2.36)$$

so evaluating this for $k_0 > 0$ and $k_0 < 0$[20]:

$$k^2[\rho_1^{ij}(k^2) + \rho_2^{ij}(k^2)] = 0 \quad (2.37)$$

$$k^2[\rho_2^{ij}(k^2) - \rho_1^{ij}(k^2)] = 0. \quad (2.38)$$

The covariant solutions to these functional equations are

$$\rho_1(k^2) = c_1\delta(k^2) \quad (2.39)$$

and

$$\rho_2(k^2) = c_2\delta(k^2), \quad (2.40)$$

where c_1 and c_2 are as yet undetermined constants. If we can show that either of these numbers is not zero, we have the Goldstone theorem, for returning to the sum over states $|n\rangle$ which yields $M_\mu^{ij}(k)$ this would mean that there would be at least one such state of spin zero and $P^2 = 0$: the dreaded zero mass Goldstone boson![21]

We have not used, as yet, the condition

$$\langle 0|[Q_i(t), \phi_j(0)]|0\rangle_{t=0} = \tau_{ijk}\langle 0|\phi_k(0)|0\rangle \neq 0. \quad (2.41)$$

This, we shall now show, guarantees that c_1 does not vanish. Since $Q_i(t)$ may not exist, we had better write the commutation relation above as

$$\int d^3\mathbf{x}\langle 0|[J_0^i(\mathbf{x},t), \phi_2(0)]|0\rangle_{t=0} = \tau_{ijk}\langle 0|\phi_k(0)|0\rangle \neq 0. \quad (2.42)$$

This will certainly be equivalent to the first expression, provided that $J_0^i(\mathbf{x}, t)$ is a "local operator" so that

$$[J_0^i(\mathbf{x},0), \phi_j(0)] = \delta^3(\mathbf{x})\tau_{ijk}\phi_k(0). \quad (2.43)$$

[17] We follow here the argument of Gilbert (1964).

[18] For Lorentz gauge electrodynamics there exist an infinity of states with nonpositive norm; i.e., $\langle A \mid A\rangle = 0$ need not imply $\mid A\rangle = 0$. See, for example, J. M. Jauch and F. Rohrlich (1955) and T. W. B. Kibble (1967).

[19] This is not quite the most general form. There can be an additional term of the form $cn_\mu\delta^4(P)$ which enters the argument of the next section.

[20] Note that since $k^2 = \mathbf{k}^2 - k_0^2$, the two signs of k_0 correspond to the same k^2 which justifies the next equation.

[21] "Dreaded" because, experimentally, there is no such object, although the pion, as we shall see, appears to do its best to behave like a Goldstone.

This feature is preserved even in the gauge theories as we shall see.[22] Now, using the above

$$\begin{aligned} M_0^{ij}(k) &= \epsilon(k_0)k_0 c_1 \delta(k^2) + k_0 c_2 \delta(k^2) \\ &= \int d^4x \exp i(kx)[\langle 0|[J_0^i(\mathbf{x},t), \phi_j(0)]|0\rangle] \end{aligned} \tag{2.44}$$

and noting that

$$\delta(k^2) = \delta(|\mathbf{k}| - k_0)/2|\mathbf{k}| + \delta(|\mathbf{k}| + k_0)/2|\mathbf{k}| \tag{2.45}$$

we have[23]

$$\begin{aligned} \int_{-\infty}^{\infty} dk_0 M_0^{ij}(k)_{\mathbf{k}=0} &= \int d^4x 2\pi\delta(t)\langle 0|[J_0^i(\mathbf{x},t), \phi_j(0)]|0\rangle \\ &= 2\pi i \tau_{ijk}\langle 0|\phi_k(0)|0\rangle \\ &= c_1 \neq 0. \end{aligned} \tag{2.46}$$

Because of the $\epsilon(k_0)$ in $M_0^{ij}(k)$, the term involving c_1 cannot cancel against the term involving c_2 for all k_0. Hence the proof is done.

It will be appreciated that this argument made use only of the equal time commutation relations between $J_0^i(x)$ and $\phi_j(0)$. However, the following remark is interesting. Consider

$$\begin{aligned} \hat{M}_0^{ij}(t) &= \int_{-\infty}^{\infty} \exp(ik_0 t) M_0^{ij}(k)\, dk_0 \\ &= \int_{-\infty}^{\infty} \exp(ik_0 t)\{\epsilon(k_0)k_0 c_1 \delta(k^2) + k_0 c_2 \delta(k^2)\}\, dk_0 \\ &= c_1 \cos(|\mathbf{k}|t) + ic_2 \sin(|\mathbf{k}|t) \\ &= 2\pi \int d^3\mathbf{x} \exp[i(\mathbf{k}\cdot\mathbf{x})]\langle 0|[J_0^i(x), \phi_j(0)]|0\rangle. \end{aligned} \tag{2.47}$$

When $\mathbf{k} = 0$, we conclude that

$$\int d^3\mathbf{x}\langle 0|[J_0^i(\mathbf{x},t), \phi_j(0)]|0\rangle = c_1/2\pi \tag{2.48}$$

which would, of course, follow from current conservation. Here c_1 does not depend on t provided that

$$\int d^3\mathbf{x}\langle 0|[\nabla\cdot\mathbf{J}^i(\mathbf{x},t), \phi_j(0)]|0\rangle = 0. \tag{2.49}$$

Hence, the assumption of manifest covariance has ruled out the presence of poles at $\mathbf{P} = 0$ in the matrix element $\langle 0|\mathbf{J}(0)|P\rangle$ despite the presence in the theory of zero mass particles.[24] Put somewhat differently, assuming manifest covariance,

$$\langle 0|J_\mu(0)|P\rangle = P_\mu F(P^2). \tag{2.50}$$

Since $|P\rangle$ is a state, here, with $P^2 = 0$, and since J_μ is conserved, we have

$$P^2 F(0) = 0, \tag{2.51}$$

thus $F(0)$ cannot diverge like a pole. In the general case, e.g., the gauge theories, this result, as we shall see, is drastically altered.

To conclude this section, we shall present an example where all of the conditions of the theorem appear to be applicable and where the Goldstone bosons are a welcome consequence. The example we have in mind is the so-called "σ model" of Gell-Mann and Lévy (1960). We first write the Lagrangian for only pions and the σ, and then adjoin the nucleons. Consider

$$\begin{aligned} \mathfrak{L}(x) &= -\frac{1}{2}\left[\frac{\partial}{\partial x_\mu}\boldsymbol{\phi}(x)\cdot\frac{\partial}{\partial x^\mu}\boldsymbol{\phi}(x) + \frac{\partial}{\partial x_\mu}\sigma(x)\frac{\partial}{\partial x^\mu}\sigma(x)\right] \\ &\quad - \alpha^2[\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x) + \sigma(x)^2 - \beta]^2 \\ &= -\frac{1}{2}\left[\frac{\partial}{\partial x_\mu}\boldsymbol{\phi}(x)\cdot\frac{\partial}{\partial x^\mu}\boldsymbol{\phi}(x) - 4\beta\alpha^2\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x)\right] \\ &\quad - \frac{1}{2}\left[\frac{\partial}{\partial x_\mu}\sigma(x)\frac{\partial}{\partial x^\mu}\sigma(x) - 4\beta\alpha^2\sigma(x)^2\right] \\ &\quad - \alpha^2[\sigma(x)^2 + \boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x)]^2 - \alpha^2\beta^2. \end{aligned} \tag{2.52}$$

Here $\boldsymbol{\phi}(x)$ is the pion field with

$$\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x) = \sum_{i=1}^{3}\phi_i(x)^2, \tag{2.53}$$

where the ϕ's are Hermitian, as is $\sigma(x)$, and α and β are real numbers which may have either sign. Note that α is dimensionless, and β has dimensions of the square of a mass. This Lagrangian is invariant under the following infinitesimal transformations:

(1) *isotopic spin*

$$\begin{aligned} \boldsymbol{\phi} &\to \boldsymbol{\phi} + \boldsymbol{\Lambda}\times\boldsymbol{\phi}, \\ \sigma &\to \sigma, \end{aligned} \tag{2.54}$$

where $\boldsymbol{\Lambda}$ is an infinitesimal constant vector, and

(2) *chirality*

$$\begin{aligned} \boldsymbol{\phi} &\to \boldsymbol{\phi} - \boldsymbol{\Lambda}\sigma, \\ \sigma &\to \sigma + \boldsymbol{\Lambda}\cdot\boldsymbol{\phi}, \end{aligned} \tag{2.55}$$

which yield, respectively, the conserved currents

$$\mathbf{V}_\mu(x) = -\boldsymbol{\phi}(x)\times(\partial/\partial x^\mu)\boldsymbol{\phi}(x) \tag{2.56}$$

and

$$\mathbf{A}_\mu(x) = -\sigma(x)(\partial/\partial x^\mu)\boldsymbol{\phi}(x) + (\partial/\partial x^\mu)\sigma(x)\boldsymbol{\phi}(x), \tag{2.57}$$

from which we can construct the formal charges Q_i and Q_{5i}. Of interest to us in the sequel are the following commutation relations involving the charges and the fields which are a consequence of the canonical commutator for bosons[25]

$$[\phi_i(\mathbf{x},t), \dot{\phi}_j(\mathbf{x}',t)] = i\delta^3(\mathbf{x}-\mathbf{x}')\delta_{ij}, \tag{2.58}$$

$$\int d^3\mathbf{x}'[A_0^i(\mathbf{x}',t), \phi_j(\mathbf{x},t)] = -i\delta_{ij}\sigma(\mathbf{x},t) \tag{2.59}$$

and

$$\int d^3\mathbf{x}'[A_0^i(\mathbf{x}',t), \sigma(\mathbf{x},t)] = i\phi_i(\mathbf{x},t). \tag{2.60}$$

[22] This has to do with the Lorentz covariance of J_μ which requires that J_0 be a function of the fields and their conjugate momenta with no dependence on the gauge fields except as they occur in conjugate momenta. See, for example, L. S. Brown (1966).

[23] Keep in mind that while k_0 is an *odd* function of k_0, $\epsilon(k_0)k_0$ is an *even* function.

[24] This point was first emphasized to me by M. A. B. Bég. In fact, the argument can be turned around to give another proof of the Goldstone theorem.

[25] Keep in mind that ϕ_i and ϕ_j are independent fields for $i \neq j$ and

$$A_0^i(x) = -\phi_i(x)\dot{\sigma}(x) + \sigma(x)\dot{\phi}_i(x).$$

The commutation relations among the charges[26]

$$[Q_i, Q_j] = i\epsilon_{ijk} Q_k, \tag{2.61}$$

$$[Q_{5i}, Q_{5j}] = i\epsilon_{ijk} Q_k, \tag{2.62}$$

$$[Q_i, Q_{5j}] = i\epsilon_{ijk} Q_{5k}, \tag{2.63}$$

express the invariance of the Lagrangian against the $SU(2) \times SU(2)$ transformation group generated by this charge algebra. This invariance yields, as we now argue, two totally different views of the $\pi - \sigma$ universe depending on the sign of the constant β in $\mathfrak{L}(x)$:

(1) The conventional view—$\beta < 0$. An examination of $\mathfrak{L}(x)$, in this case, shows that

$$m_\pi^2 = m_\sigma^2 = -4\beta\alpha^2, \tag{2.64}$$

i.e , π and σ are degenerate states with distinct quantum numbers. In the real world, the π and σ are imagined to have opposite parities; i.e., if P is the parity operator we can choose

$$P\boldsymbol{\phi}(0)P^{-1} = -\boldsymbol{\phi}(0) \tag{2.65}$$

and

$$P\sigma(0)P^{-1} = \sigma(0). \tag{2.66}$$

Since these fields occur quadratically in $\mathfrak{L}(x)$, parity is conserved by this theory. However $A_\mu(x)$, defined above, is a pseudovector. If such a symmetry realization manifested itself in nature, we would expect to see opposite parity π's and σ's nearly degenerate in mass. There have been reports of scalar resonances in the literature, but these have masses of 700 MeV or more, while the pion's mass is only 140 MeV. To compound matters, for this point of view, if nucleons are added, as we do below, these must have zero mass in the symmetry limit and would also come in parity doublets. All of this makes a conventional $SU(2) \times SU(2)$ symmetry scheme look rather unattractive. Hence, we are led to try something else.

(2) The Goldstone view—$\beta > 0$. This choice of sign does not affect the symmetries of $\mathfrak{L}(x)$ with respect to the $SU(2) \times SU(2)$ group. However, the parameter $\beta\alpha^2$ no longer acts like a mass. In fact, the terms

$$2\beta\alpha^2[\sigma(x)^2 + \boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x)] - \alpha^2[\sigma(x)^2 + \boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x)]^2 - \alpha^2\beta^2 \tag{2.67}$$

act like a mesonic coupling potential. We may look for the classical fields σ and $\boldsymbol{\phi}$ that minimize this potential. The answer is any of the infinite set obeying

$$\sigma(x)^2 + \boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x) = \beta, \tag{2.68}$$

while in the $\beta < 0$ case the *unique* answer is

$$\sigma(x) = \pi(x) = 0. \tag{2.69}$$

In quantum theory we can translate this into a statement about expectation values of the Hamiltonian and, in particular, if we call $|0\rangle$ a state of lowest energy, we would expect that such a state would obey,

$$\langle 0|\sigma(0)^2 + \boldsymbol{\phi}(0)\cdot\boldsymbol{\phi}(0)|0\rangle = \beta. \tag{2.70}$$

We can satisfy this, in leading order, if we choose[27]

$$\langle 0|\sigma(0)|0\rangle = \sqrt{\beta} \neq 0 \tag{2.71}$$

and

$$\langle 0|\boldsymbol{\phi}(0)|0\rangle = 0. \tag{2.72}$$

This, however, makes a particle interpretation of the σ field impossible. However, if we call

$$\chi(x) = \sigma(x) - \sqrt{\beta}, \tag{2.73}$$

we can write an effective Lagrangian in which all the fields have a particle interpretation. In higher orders we would define

$$\chi(x) = \sigma(x) - \langle 0|\sigma(0)|0\rangle, \tag{2.74}$$

where $\langle 0|\sigma(0)|0\rangle$ would have to be computed to the given order. In fact,

$$\begin{aligned}\mathfrak{L}(x)_{\text{effective}} = &-\tfrac{1}{2}(\partial/\partial x_\mu)\boldsymbol{\phi}(x)\cdot(\partial/\partial x^\mu)\boldsymbol{\phi}(x) \\ &- \tfrac{1}{2}((\partial/\partial x_\mu)\chi(x)(\partial/\partial x^\mu)\chi(x) + 8\alpha^2\beta\chi^2(x)) \\ &- \alpha^2(\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x) + \chi^2(x))^2 \\ &+ 4\sqrt{\beta}\,\chi(x)(\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x) + \chi^2(x)) - \alpha^2\beta^2.\end{aligned} \tag{2.75}$$

The theory remains invariant under the symmetry generated by the transformed currents and charges, but notice that to this order

$$m_\chi^2 = 8\alpha^2\beta \tag{2.76}$$

and

$$m_\pi^2 = 0. \tag{2.77}$$

Higher order corrections will modify m_χ but will leave $m_\pi = 0$ alone, since all of the conditions of the Goldstone theorem are satisfied and the proof was independent of perturbation theory. From the Goldstone point of view, it is the fact that

$$\langle 0|\boldsymbol{\phi}(0)|0\rangle = 0 \tag{2.78}$$

which allows the χ to become massive.[28]

This picture looks like a better fit to the observed meson spectrum, and the "small" observed pion mass can be put in by hand, if necessary, to break chiral symmetry. Moreover, it is quite straightforward to add nucleons to the stew. To this end we adjoin

$$\begin{aligned}\mathfrak{L}_{\text{nnc}}(x) = &-\bar{\psi}(x)\gamma_\mu(\partial/\partial x_\mu)\psi(x) \\ &- g\bar{\psi}(x)(\sigma(x) + i\boldsymbol{\tau}\cdot\boldsymbol{\phi}(x)\gamma_5)\psi(x).\end{aligned} \tag{2.79}$$

Here

$$\psi(x) = \begin{pmatrix}\psi_P(x)\\ \psi_N(x)\end{pmatrix}. \tag{2.80}$$

[26] ϵ_{ijk} is the totally antisymmetric function of i, j, k characterized by the definition $\epsilon_{123} = 1$. Diligent application of the equal time commutation relations among the fields will yield these results. In these expressions we consider, where necessary, the charges at equal times and first commute the current densities and then integrate if the charges appear ill defined.

[27] $\langle 0|\boldsymbol{\phi}(0)|0\rangle \neq 0$ would violate parity conservation.

[28] Otherwise stated, $Q_i\,|\,0\rangle = 0$ while $Q_{5i}\,|\,0\rangle \neq 0$.

The τ are the isotopic spin matrices, and γ_5 has been inserted because of the pseudoscalar character of the ϕ. We must now enlarge the $SU(2)\times SU(2)$ transformations to include

$$\psi(x) \to (1 + i(\tau/2 \cdot \Lambda)\psi(x) \tag{2.81}$$

and

$$\psi(x) \to (1 + i(\tau/2 \cdot \Lambda\gamma_5)\psi(x). \tag{2.82}$$

The full Lagrangian is now invariant under the total isotopic and chiral transformations leading to the conserved currents

$$\mathbf{V}_\mu(x) = i\bar{\psi}(x)\gamma_\mu(\tau/2)\psi(x) - \phi(x)\times(\partial/\partial x^\mu)\phi(x) \tag{2.83}$$

and

$$\mathbf{A}_\mu(x) = i\bar{\psi}(x)\gamma_\mu\gamma_5(\tau/2)\psi(x) - (\sigma(x)(\partial/\partial x^\mu)\phi(x) - \phi(x)(\partial/\partial x^\mu)\sigma(x)). \tag{2.84}$$

We again have a choice:

(1) $\beta < 0$. This leads as before to, in leading order,

$$m_\pi^2 = m_\sigma^2 = -4\beta\alpha^2 \tag{2.85}$$

but

$$m_N = 0 \tag{2.86}$$

since no bare nucleon mass appears in $\mathfrak{L}(x)$.

(2) $\beta > 0$. In this case, as before,

$$m_\pi^2 = 0 \tag{2.87}$$

$$m_\chi^2 = 8\beta\alpha^2 \tag{2.88}$$

while

$$m_N = g\sqrt{\beta}. \tag{2.89}$$

After displacing σ in $\mathfrak{L}(x)$, all traces of the particle multiplets have disappeared, although $\mathfrak{L}(x)$ is invariant under the appropriately redefined symmetry generators. In the next section, we turn to the Higgs loophole which applies when electromagnetism is put into the game.

III. THE HIGGS LOOPHOLE

Before returning to the vacuum-broken symmetries, we wish to consider the question of how a theory can be Lorentz covariant but not "manifestly" Lorentz covariant.[29] Since this is the key to the Higgs loophole, it is worthwhile to study, carefully, an example. We consider free electrodynamics given by the Lagrangian

$$\mathfrak{L}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) = -\frac{1}{2}\left(\frac{\partial A_\mu(x)}{\partial x^\nu} - \frac{\partial A_\nu(x)}{\partial x^\mu}\right)\frac{\partial A^\mu(x)}{\partial x_\nu} \tag{3.1}$$

with

$$F_{\mu\nu}(x) = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}. \tag{3.2}$$

[29] In this treatment we have found Bjorken and Drell (1965) very useful. Note, however, that our metric conventions differ from theirs.

The equations of motion

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\nu}A_\nu(x) - \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x^\nu}A_\mu(x) = 0 \tag{3.3}$$

have no unique solution until the potentials are assigned a "gauge." We shall frequently choose to work in the "radiation gauge" defined by the condition that

$$\nabla \cdot \mathbf{A}(x) = 0 \tag{3.4}$$

which is, of course, the transversality condition of a free physical photon. It is just here that the noncovariance has slipped in. This is worth examining in rather pedantic detail. In matrix notation, we can write the equation for an infinitesimal Lorentz transformation of the coordinates as

$$x' = (1 + \epsilon)x, \tag{3.5}$$

where

$$\epsilon^T = -\epsilon. \tag{3.6}$$

For example, if the transformation is to a moving frame in the 1-direction

$$\epsilon = \begin{Vmatrix} 0 & 0 & 0 & i\beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i\beta & 0 & 0 & 0 \end{Vmatrix} \tag{3.7}$$

with

$$\beta^2 \simeq 0. \tag{3.8}$$

Thus, a four-vector field $V_\mu(0)$ will transform as

$$V'(0) = (1 + \epsilon)V(0). \tag{3.9}$$

Hence

$$V'(x) = \{\exp[-i(Px)]V(0)\exp[i(Px)]\}' = \exp[-i(P'x')]V'(0)\exp[i(P'x')] = V(x') + \epsilon V(x') \tag{3.10}$$

using the Lorentz invariance of (Px). If we Fourier transform this relationship we have in momentum space

$$V'(k') = V(k) + \epsilon V(k), \tag{3.11}$$

with

$$k' = (1 + \epsilon)k. \tag{3.12}$$

Thus

$$(k'V'(k')) = (k(1-\epsilon)(1+\epsilon)V(k)) = (kV(k)) \tag{3.13}$$

where, as usual,

$$(kV(k)) = \mathbf{k}\cdot\mathbf{V}(k) - k_0V_0(k). \tag{3.14}$$

Now, clearly, if the photon field $A(k)$ transformed like $V(k)$ we could never maintain the condition

$$\mathbf{k}'\cdot\mathbf{A}'(k') = \mathbf{k}\cdot\mathbf{A}(k) = 0. \tag{3.15}$$

Therefore, let us suppose that $A(k)$ has a different transformation property designed to maintain transversality. Later we shall see how this transformation emerges from the field theory. Hence, we assume

$$A'(k') = (1 + \epsilon)A(k) + E(k), \tag{3.16}$$

where $E(k)$ is the extra term of order β. We carry out this work for the free photon, as an example, where

$$\mathbf{k} \cdot \mathbf{A}(k) = 0, \tag{3.17}$$

$$A_0(k) = 0, \tag{3.18}$$

and

$$\mathbf{k}^2 = k_0^2. \tag{3.19}$$

Thus

$$A_0'(k') = \epsilon_{0i} A^i(k) + E_0(k). \tag{3.20}$$

Hence, to have

$$A_0'(k') = 0 \tag{3.21}$$

we must have

$$E_0(k) = -\epsilon_{0i} A^i(k). \tag{3.22}$$

Further,

$$0 = k'^i A_i'(k') = k_\nu(\delta^{\nu i} + \epsilon^{\nu i})(\delta_{ie} + \epsilon_{ie})A^e(k) + k^i E_i(k) \tag{3.23}$$

or

$$k^i E_i(k) + k_0 \epsilon^{0i} A_i(k) = 0. \tag{3.24}$$

Thus, all requirements are satisfied with

$$E_\nu(k) = -k_\nu k_0/|\mathbf{k}|^2 \epsilon_{0i} A^i(k) \tag{3.25}$$

which establishes the conflict between gauge and Lorentz transformations.

But how does this transformation arise in the context of a canonical field theory? To see this, we must first give the correct canonical commutation relations. The only nonvanishing commutator among the vector potentials is

$$[A_i(\mathbf{x}, t), \dot{A}_j(\mathbf{x}', t)] = i\delta_{ij}^{\mathrm{Tr}}(\mathbf{x} - \mathbf{x}') \tag{3.26}$$

where the transverse δ function $\delta_{ij}^{\mathrm{Tr}}$ is defined by

$$\delta_{ij}^{\mathrm{Tr}}(\mathbf{x}) = (1/(2\pi)^3) \int d^3\mathbf{k} \exp[i(\mathbf{k} \cdot \mathbf{x})][\delta_{ij} - (k_i k_j/|\mathbf{k}|^2)] \tag{3.27}$$

and has been introduced to maintain the transversality of $A_i(\mathbf{x}, t)$. Now if $|P\rangle$ and $|q\rangle$ are any two eigenstates of P_μ, the quantum mechanical covariance condition is, where $|P'\rangle$ and $|q'\rangle$ are the Lorentz-transformed states $U|P\rangle$ and $U|q\rangle$,

$$\langle P'|A_\mu(0)|q'\rangle = \langle P|UA_\mu(0)U^{-1}|q\rangle, \tag{3.28}$$

and[30]

$$U = 1 + (i/2)\epsilon M. \tag{3.29}$$

[30] We have put the space–time variable equal to zero in $A_\mu(x)$ to simplify the transformation.

Here ϵ is as above, and for Lorentz transformations

$$M^{0k} = \int d^3\mathbf{x} \left\{ t\dot{\mathbf{A}}(\mathbf{x}, t) \cdot \frac{\partial}{\partial x_k}\mathbf{A}(\mathbf{x}, t) - x_k/2[\dot{\mathbf{A}}(\mathbf{x}, t)^2 + (\nabla \times \mathbf{A}(\mathbf{x}, t))^2] \right\}, \tag{3.30}$$

Thus

$$UA_i(0)U^{-1} = A_i(0) + i\epsilon_{0k}[M^{0k}, A_i(0)]. \tag{3.31}$$

As M^{0k} is independent of time, we can set $t = 0$ and write

$$\begin{aligned} [M^{0k}, A_i(0)] &= -\int d^3\mathbf{x}[(x_k/2)\dot{\mathbf{A}}(\mathbf{x}, 0)^2, A_i(0)] \\ &= i\int d^3\mathbf{x}\, x_k \delta_{ij}^{\mathrm{Tr}}(\mathbf{x})\dot{A}_j(\mathbf{x}, 0) \\ &= -i\int d^3\mathbf{x} \int \frac{d^3\mathbf{k}}{(2\pi)^3} x_k \frac{k_i k_j}{|\mathbf{k}|^2}\dot{A}_j(\mathbf{x}, 0) \end{aligned} \tag{3.32}$$

which reduces to the expression above, Eq. (3.25), for $E_i(k)$ when one Fourier transforms and takes the transversality of $\mathbf{A}(k)$ into account. If $A_i(0)$ transformed like the spatial component of a four-vector we would have had simply

$$UA_i(0)U^{-1} = A_i(0), \tag{3.33}$$

since

$$A_0(0) = 0. \tag{3.34}$$

It is the extra term that maintains the gauge condition in all Lorentz frames.

If A_i has this complicated transformation property, one may ask how the S-matrix manages to be Lorentz covariant in the conventional sense in Coulomb gauge electrodynamics. The answer, as is well known, goes, in outline, as follows:

Suppose we express in momentum–space the Coulomb gauge condition in a given Lorentz frame as the statement that

$$(kA(k)) + (\eta k)(\eta A(k)) = 0, \tag{3.35}$$

where in this frame we can choose

$$\eta = (0, 0, 0, 1). \tag{3.36}$$

In writing the condition this way, we can also allow for the fact that for the interacting fields

$$A_0(k) \neq 0. \tag{3.37}$$

Now for a given k we can construct four orthonormal four-vectors; i.e., the two polarizations ϵ_1, ϵ_2; η and a vector $\hat{k}$ defined as

$$\hat{k}_\mu = [k_\mu + (k\eta)\eta_\mu]/|\mathbf{k}|^2 \tag{3.38}$$

Thus, we have the completeness statement

$$\sum_{\lambda=1}^{2} \epsilon_\mu^\lambda \epsilon_\nu^\lambda = g_{\mu\nu} + \eta_\mu\eta_\nu - \hat{k}_\mu\hat{k}_\nu, \tag{3.39}$$

where the sign of the last term is determined by the fact that $\bar{k}$ and η are orthogonal to ϵ. From this it follows that the free photon Green's function in the Coulomb gauge is

$$iD_F^{\text{Tr}}(x',x)_{\nu\mu} = \langle 0|T(A_\nu(x')A_\mu(x))|0\rangle$$
$$= i\int \frac{d^4k}{(2\pi)^4}\frac{\exp[-i(k(x'-x))]}{k^2+i\epsilon}\sum_{\lambda=1}^{2}\epsilon_\nu^\lambda(k)\epsilon_\mu^\lambda(k)$$
$$= i\int \frac{d^4k}{(2\pi)^4}\frac{\exp[-ik(x'-x)]}{k^2+i\epsilon}\left[g_{\mu\nu} + \frac{k^2\eta_\mu\eta_\nu - k_\mu k_\nu - (k\eta)(k_\mu\eta_\nu + k_\nu\eta_\mu)}{|\mathbf{k}|^2}\right]. \quad (3.40)$$

When, in the S-matrix calculations, this Green's function is taken between vertices corresponding to a *conserved* current, it can be shown that the terms proportional to k_μ or k_ν do not contribute. The remaining noncovariant term proportional to $\eta_\mu\eta_\nu$ is just the Fourier transform of the static Coulomb potential and is canceled by the explicit occurence of the Coulomb potential in the Coulomb gauge Hamiltonian. Hence, effectively, the Green's function acts like the covariant object $g_{\mu\nu}/(k^2+i\epsilon)$ which is the key to restoring the covariance.

The reason for bringing this up is that we now want to study, à la Higgs (1964), what happens to the Goldstone theorem in a theory in which there are n gauge fields $A_\mu^1(x)\cdots A_\mu^n(x)$, and n Hermitian scalar fields $\phi_1(x)\cdots\phi_n(x)$; gauge fields like the photon, but which obey the condition that for some i and j

$$M_\mu^{ij}(k)_A = \int d^4x\langle 0|[A_\mu^i(x),\phi_j(0)]|0\rangle \exp[i(kx)] \neq 0. \quad (3.41)$$

The notation is as before, except that $A_\mu^i(x)$ and the current $J_\mu^i(x)$ are connected by the Maxwell equation

$$J_\mu^i(x) = \frac{\partial}{\partial x_\nu}\left(\frac{\partial}{\partial x^\mu}A_\nu^i(x) - \frac{\partial}{\partial x^\nu}A_\mu^i(x)\right) \quad (3.42)$$

so that

$$(\partial/\partial x_\mu)J_\mu^i(x) = 0. \quad (3.43)$$

We shall suppose that the fields $A_\mu^i(x)$ have no bare mass and hence enter their sector of the Lagrangian as in Eq. (3.1). Hence, for this Maxwell equation to have a solution, we must pose a gauge condition which we take to be that of Eq. (3.35), thereby losing manifest covariance; i.e.,

$$(kA^i(k)) + (\eta k)(\eta A^i(k)) = 0 \quad (3.44)$$

with

$$\eta = (0,0,0,1). \quad (3.45)$$

There is no reason, now, why the vector η_μ should not enter the expression of $M_\mu^{ij}(k)_A$ along with k_μ. In fact, by three-dimensional rotational covariance, and in the absence of manifest Lorentz covariance,

$$M_\mu^{ij}(k)_A = a(k^2,(\eta k))k_\mu + b(k^2,(\eta k))\eta_\mu + c\eta_\mu\delta^4(k), \quad (3.46)$$

where we can dispense with the $\epsilon(k_0)$ since $a(k^2,(\eta k))$ etc. are general functions of k_0 and k^2.[31] Therefore,

$$M_\mu^{ij}(k)_J = \int d^4x\langle 0|[J_\mu^i(x),\phi_j(0)]|0\rangle\exp[i(kx)]$$
$$= k^2M_\mu^{ij}(k)_A - k_\mu k^\nu M_\nu^{ij}(k)_A$$
$$= b(k^2,(\eta k))[k^2\eta_\mu - k_\mu(\eta k)] \quad (3.47)$$

assuming that the various integrations by parts are legal. We may notice two very significant things about this expression:

$$(1) \qquad k^\mu M_\mu^{ij}(k)_J = 0 \quad (3.48)$$

identically. Hence, in contrast to the situation described in the last section Eq. (2.39) and Eq. (2.40), *no* conclusion about the function $b(k^2,(\eta k))$ can be drawn from the conservation of $M_\mu^{ij}(k)_J$. In other words, we are not forced to a form factor of the form $\delta(k^2)$, and hence *the proof of the Goldstone theorem has broken down*. Note, incidentally, that the gauge condition

$$\nabla\cdot\mathbf{A}(x) = 0 \quad (3.49)$$

implies that

$$a(k^2,(\eta k)) = c\delta(\mathbf{k}^2). \quad (3.50)$$

This is *not* the Goldstone meson because a particle of *finite* mass may have $\mathbf{k}^2 = 0$. It does mean that

$$M_l^{ij}(k)_A = 0 \quad (3.51)$$

for $l = 1,\ldots,3$, while

$$M_0^{ij}(k)_A \neq 0. \quad (3.52)$$

This is a rephrasing of the theorem that single photon cannot be emitted in a 0–0 transition.

(2) Because of the arbitrary dependence of $b(k^2,(\eta k))$ on k_0 it is no longer true that

$$\int d^3\mathbf{x}\langle 0|[J_0^i(\mathbf{x},t),\phi_j(0)]|0\rangle \quad (3.53)$$

is a constant independent of t. To see this, refer back to Eq. (2.48). Here is a case in which

$$(\partial/\partial x_\mu)J_\mu(x) = 0 \quad (3.54)$$

does *not* imply that

$$\dot{Q}_J(t) = 0. \quad (3.55)$$

We come back to this more fully later. For the moment, let us turn from the discussion of the breakdown of the Goldstone theorem—the Higgs loophole—to a discussion of the "Higgs mechanism" which supplants the Goldstone theorem in the gauge theories.

IV. THE HIGGS MECHANISM, OR WHERE HAVE ALL THE GOLDSTONE'S GONE?

The last section reproduces, essentially, the contents of the Higgs letter of 1964. The letter contained no particu-

[31] The last term in Eq. (3.46) could arise if there were a constant term in the matrix element Eq. (3.41). In any case, it drops out in what follows. For additional discussion, see Gilbert (1964) and Klein and Lee (1964).

lar gauge model and did not suggest what would happen in a theory if all the conditions described in the last section of this review were met. This was also clarified by Higgs in 1964 in terms of a model which can be described as scalar boson electrodynamics, where the boson is given a *pure imaginary* bare mass and a quartic selfcoupling. In fact, most of the models which exhibit the Higgs mechanism are variants on this theme, but where more "photons", bosons, and fermions are added. Once one understands the original model, the extension to the more complicated models such as Weinberg's 1967 model is not difficult to grasp, since the basic ideas are very similar. Our starting point is the Lagrangian (For the expert, we have indicated the "counter terms", which render the theory finite, by the $+\cdots$. These will not concern us here because we will not consider "closed loop" corrections.)

$$\begin{aligned}\mathfrak{L}(x) = {} & \tfrac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \tfrac{1}{2}F_{\mu\nu}(x)\left(\frac{\partial}{\partial x_\mu}A^\nu(x) - \frac{\partial}{\partial x^\nu}A^\mu(x)\right) \\ & + \boldsymbol{\phi}_\mu(x)\cdot\frac{\partial}{\partial x_\mu}\boldsymbol{\phi}(x) + \boldsymbol{\phi}^\mu(x)\cdot\boldsymbol{\phi}_\mu(x) \\ & - \boldsymbol{\phi}^\mu(x)\cdot ie_0 q\boldsymbol{\phi}(x)A_\mu(x) \\ & + \tfrac{1}{2}m_0^2\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x) - \tfrac{1}{8}f^2(\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x))^2 + \cdots . \end{aligned} \tag{4.1}$$

The writer hopes that the reader will not be put off by this slightly arcane looking notation which he will now explain. In order to avoid having more than first derivatives in the Lagrangian, it is convenient to write it as above à la Schwinger.[32] To recover the familiar field equations one varies $F_{\mu\nu}(x)$, $A_\mu(x)$, $(\partial/\partial x^\nu)A_\mu(x)$, $\boldsymbol{\phi}_\mu(x)$, $\boldsymbol{\phi}(x)$ and $(\partial/\partial x_\mu)\boldsymbol{\phi}(x)$ independently. To complete the notation, by $\boldsymbol{\phi}(x)$ we mean

$$\boldsymbol{\phi}(x) = \begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix}, \tag{4.2}$$

and

$$q = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}. \tag{4.3}$$

The first thing we wish to observe before we consider the general equations is that, despite appearances, the term $\frac{1}{2}m_0^2\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x)$ is *not* a mass term. To see this, let

$$e_0 = f = 0 \tag{4.4}$$

and vary with respect to $\boldsymbol{\phi}_\mu(x)$ and $\boldsymbol{\phi}(x)$. Thus

$$\boldsymbol{\phi}^\mu(x) = -(\partial/\partial x_\mu)\boldsymbol{\phi}(x) \tag{4.5}$$

and

$$(\partial/\partial x_\mu)\boldsymbol{\phi}^\mu(x) = m_0^2\boldsymbol{\phi}(x), \tag{4.6}$$

or

$$[\nabla^2 - (\partial^2/\partial t^2)]\boldsymbol{\phi}(x) = -m_0^2\boldsymbol{\phi}(x) \tag{4.7}$$

or, symbolically,

$$-\mathbf{P}^2 + E^2 = -m_0^2, \tag{4.8}$$

which is to say that m_0 acts like an *imaginary* mass. This reminds us of the $\beta > 0$ sector of the σ model discussed in Sec. II. In fact, with this example in mind, we are going to present an approximate solution to the equations of motion with the *ansatz*

$$ie_0\langle 0|\boldsymbol{\phi}(0)|0\rangle = e_0\left\langle 0\left|\begin{pmatrix}\phi_2(0)\\ -\phi_1(0)\end{pmatrix}\right|0\right\rangle \neq 0. \tag{4.9}$$

The consequences will emerge in what follows. We may derive the equations of motion. We list these below indicating which variations were used in each case.

$$\delta F_{\mu\nu}:\ F^{\mu\nu}(x) = (\partial/\partial x_\mu)A^\nu(x) - (\partial/\partial x_\nu)A^\mu(x) \tag{4.10a}$$

$$\delta\boldsymbol{\phi}^\mu:\ (\partial/\partial x^\mu)\boldsymbol{\phi}(x) + \boldsymbol{\phi}_\mu(x) - ie_0 q\boldsymbol{\phi}(x)A_\mu(x) = 0 \tag{4.10b}$$

$$\begin{aligned}\delta\boldsymbol{\phi}:\ (\partial/\partial x^\mu)\boldsymbol{\phi}^\mu(x) = {} & ie_0 q\boldsymbol{\phi}^\mu(x)A_\mu(x) + m_0^2\boldsymbol{\phi}(x) \\ & - \tfrac{1}{2}f^2\boldsymbol{\phi}(x)(\boldsymbol{\phi}(x))\cdot\boldsymbol{\phi}(x)\end{aligned} \tag{4.10c}$$

$$\delta A_\mu:\ (\partial/\partial x^\nu)F^{\mu\nu}(x) = -ie_0\boldsymbol{\phi}^\mu(x)\cdot q\boldsymbol{\phi}(x). \tag{4.10d}$$

Before we discuss the approximate solution to these equations, we shall consider the basic gauge symmetry of the Lagrangian. Clearly, in virtue of the last equation, the quantity

$$\begin{aligned}J^\mu(x) &= i\boldsymbol{\phi}^\mu(x)\cdot q\boldsymbol{\phi}(x) \\ &= -i\{(\partial/\partial x_\mu)\boldsymbol{\phi}(x) - ie_0 q\boldsymbol{\phi}(x)A^\mu(x))\cdot q\boldsymbol{\phi}(x)\} \\ &= -i[(\partial/\partial x_\mu)\boldsymbol{\phi}(x)\cdot q\boldsymbol{\phi}(x) + e_0\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x)A^\mu(x)]\end{aligned} \tag{4.11}$$

is a conserved current.[33] To see the role of the charge associated with this current as a symmetry generator, we find the conjugate "momentum" to $\boldsymbol{\phi}(x)$:

$$\boldsymbol{\Pi}_\mu(x) = \delta\mathfrak{L}(x)/\frac{\delta\partial\boldsymbol{\phi}(x)}{\partial x^\mu} = \boldsymbol{\phi}_\mu(x). \tag{4.12}$$

Thus, we can write

$$Q_J(t) = i\int\boldsymbol{\Pi}_0(\mathbf{x},t)\cdot q\boldsymbol{\phi}(\mathbf{x},t)\,d^3\mathbf{x}, \tag{4.13}$$

with

$$[\phi_i(\mathbf{x}',t), \Pi_{j0}(\mathbf{x},t)] = i\delta_{ij}\delta^3(\mathbf{x}'-\mathbf{x}), \tag{4.14}$$

Therefore,

$$[J_0(\mathbf{x},t), \boldsymbol{\phi}(\mathbf{x}',t)] = \delta^3(\mathbf{x}-\mathbf{x}')q\boldsymbol{\phi}(\mathbf{x}',t) \tag{4.15}$$

or defining, formally, with $\Lambda \ll 1$

$$U(\Lambda)_t \simeq (1 + i\Lambda Q_J(t)), \tag{4.16}$$

$$\begin{aligned}U(\Lambda)_t\begin{pmatrix}\phi_1(\mathbf{x},t)\\ \phi_2(\mathbf{x},t)\end{pmatrix}U(\Lambda)_t^{-1} &= \begin{pmatrix}\phi_1(\mathbf{x},t)\\ \phi_2(\mathbf{x},t)\end{pmatrix} + \Lambda\begin{pmatrix}\phi_2(\mathbf{x},t)\\ -\phi_1(\mathbf{x},t)\end{pmatrix} \\ &\simeq \begin{pmatrix}\cos(\Lambda)\phi_1(\mathbf{x},t) + \sin(\Lambda)\phi_2(\mathbf{x},t)\\ \cos(\Lambda)\phi_2(\mathbf{x},t) - \sin(\Lambda)\phi_1(\mathbf{x},t)\end{pmatrix}.\end{aligned} \tag{4.17}$$

[32] The author was first introduced to this technique in J. Schwinger's lectures in field theory in 1952. He recalls—still—the sinking feeling he experienced when Schwinger stated that the "natural" representation of a spin-zero field was the five-component object $(\phi_\mu(x), \phi(x))$. In the subsequent years, he has come to see that Schwinger had a point; computations with the Lagrangian are greatly simplified.

[33] Despite the explicit appearance of $A_\mu(x)$ in $J_\mu(x)$ it is a true four-vector in all gauges. See, for example, L. S. Brown (1966) or B. Zumino (1960).

That the Lagrangian is invariant under such rotations of the fields $\boldsymbol{\phi}$ is evident from its form. However, there is a wider invariance under "local" rotations defined in terms of a local "angle" $\Lambda(x)$, where $\Lambda(x)$ is an essentially arbitrary function of space–time. Let

$$\phi_1(x) \to \phi_1(x)\cos(\Lambda(x)) + \phi_2(x)\sin(\Lambda(x)), \quad (4.18)$$

$$\phi_2(x) \to -\phi_1(x)\sin(\Lambda(x)) + \phi_2(x)\cos(\Lambda(x)), \quad (4.19)$$

and

$$A_\mu(x) \to A_\mu(x) + (1/e_0)(\partial/\partial x^\mu)\Lambda(x). \quad (4.20)$$

Without this last transformation the Lagrangian would *not* be invariant since the derivative $\partial/\partial x_\mu$ has no simple transformation property when Λ is a function of space–time. However, the quantity

$$\boldsymbol{\phi}_\mu(x) = -(\partial/\partial x^\mu)\boldsymbol{\phi}(x) + ie_0 q\boldsymbol{\phi}(x)A_\mu(x) \quad (4.21)$$

is "covariant" under these transformations; i.e.,

$$\phi_{1\mu}(x) \to \phi_{1\mu}(x)\cos(\Lambda(x)) + \phi_{2\mu}(x)\sin(\Lambda(x)), \quad (4.22)$$

$$\phi_{2\mu}(x) \to \phi_{2\mu}(x)\cos(\Lambda(x)) - \phi_{1\mu}(x)\sin(\Lambda(x)), \quad (4.23)$$

while

$$F_{\mu\nu}(x) \to F_{\mu\nu}(x), \quad (4.24)$$

which makes the invariance of $\mathfrak{L}(x)$ evident.

At this point, one may proceed in various directions but to maintain contact with what has gone before we begin by following Higgs (1966). We can minimize the energy if we demand that the vacuum satisfies

$$0 = \langle 0|m_0^2\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x) - \tfrac{1}{2}f^2(\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x))^2|0\rangle. \quad (4.25)$$

In leading order we can choose

$$\langle 0|\phi_1(0)|0\rangle = 0 \quad (4.26)$$

$$\langle 0|\phi_2(0)|0\rangle = \sqrt{2}(m_0/f) \quad (4.27)$$

as one of an infinite set of equivalent possibilities allowed by the local $U(1)$ invariance discussed above.[34] In fact, we will later re-do this work by choosing a $\Lambda(x)$ so that after rotation

$$\phi_1'(x) = 0 \quad (4.28)$$

and

$$\phi_2'(x) = \sqrt{\phi_1^2(x) + \phi_2^2(x)} \quad (4.29)$$

or

$$\tan(\Lambda(x)) = -[\phi_1(x)/\phi_2(x)]. \quad (4.30)$$

This enables one to short circuit some computations, but at the expense of introducing a rather formal looking transformation which may hide the physics. Let us, following Higgs (1964) and Guralnik *et al.*, deal directly with the equations of motion by linearizing them. We let

$$\boldsymbol{\phi}(x) = \boldsymbol{\eta} + \boldsymbol{\phi}'(x), \quad (4.31)$$

where

$$\eta = \begin{pmatrix} 0 \\ \sqrt{2}(m_0/f) \end{pmatrix}, \quad (4.32)$$

and $\boldsymbol{\phi}'(x)$ the "quantum fluctuations" will be taken to be "small." We then write the equations for $\boldsymbol{\phi}'(x)$, dropping terms in $\boldsymbol{\phi}'^2$ and $e_0\boldsymbol{\phi}'$. Thus, we have the approximate set

$$F^{\mu\nu}(x) = (\partial/\partial x_\mu)A^\nu(x) - (\partial/\partial x_\nu)A^\mu(x), \quad (4.33)$$

$$\boldsymbol{\phi}_\mu(x) = -(\partial/\partial x^\mu)\boldsymbol{\phi}'(x) + ie_0 q\boldsymbol{\eta}A_\mu(x); \quad (4.34)$$

$$((\partial/\partial x^\mu) - ie_0 qA_\mu(x))\boldsymbol{\phi}^\mu(x) = -f^2\boldsymbol{\eta}(\boldsymbol{\phi}'(x)\cdot\boldsymbol{\eta}). \quad (4.35)$$

$$(\partial/\partial x^\nu)F^{\mu\nu}(x) = -ie_0\boldsymbol{\phi}^\mu(x)\cdot q\boldsymbol{\eta} = -m\phi_1^\mu(x)$$
$$= m^2((1/m)(\partial/\partial x_\mu)\phi_1'(x) - A^\mu(x)) \quad (4.36)$$

where we have defined

$$m = e_0\sqrt{2}(m_0/f). \quad (4.37)$$

We turn to Eq. (4.36) shortly, but first let us look at Eq. (4.35) in the limit $e_0 = 0$. It reads, separating the components,

$$(\partial/\partial x_\mu)(\partial/\partial x^\mu)\phi_1'(x) = 0 \quad (4.38)$$

and

$$(\partial/\partial x_\mu)(\partial/\partial x^\mu)\phi_2'(x) = 2m_0^2\phi_2'(x) \quad (4.39)$$

In the second of these equations, $+2m_0^2$ *does* have the correct sign to be a mass. Hence, $\phi_2'(x)$ represents a boson of mass $\sqrt{2}m_0$. But what of $\phi_1'(x)$? Is it the Goldstone boson? We can get an important insight if we consider, using Eq. (4.36),

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}F^{\mu\nu}(x) = 0 = \left(\frac{1}{m}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\phi_1'(x) - \frac{\partial}{\partial x^\mu}A^\mu(x)\right). \quad (4.40)$$

This equation is incomplete until we specify $(\partial/\partial x^\mu)A^\mu(x)$. First we shall consider the choice[35]

$$(\partial/\partial x^\mu)A^\mu(x) = 0, \quad (4.41)$$

the Lorentz gauge. In this gauge

$$(\partial/\partial x^\mu)(\partial/\partial x_\mu)\phi_1'(x) = 0. \quad (4.42)$$

In other words, in the Lorentz gauge $\phi_1'(x)$ *is* a Goldstone boson but it is *decoupled from the rest of the system.* This decoupling will, in fact, maintain itself to all orders so that in the Lorentz gauge there is a harmless Goldstone boson. We have found the Goldstone in the linearized theory by studying the equations of motion. If everything is consistent, we would naturally expect to be able to prove the Goldstone Theorem in the Lorentz gauge. To this end, consider $J_\mu(x)$ defined by

$$J_\mu(x) = \lambda(mA_\mu(x) - (\partial/\partial x^\mu)\phi_1'(x)) \quad (4.43)$$

Where we have included in the current the factor

$$\lambda = \sqrt{2}(m_0/f), \quad (4.44)$$

[34] The rotations of the Cartesian "coordinates" $\phi_1(x)$ and $\phi_2(x)$ are equivalent to phase transformations of the complex fields $\phi^\pm(x) = \phi_1(x) \pm i\phi_2(x)/\sqrt{2}$.

[35] A fuller treatment is given in T. W. B. Kibble (1967).

so that $J_\mu(x)$ will have dimensions of L^{-3}. Then

$$J_0(x) = \lambda(mA_0(x) + \dot{\phi}_1'(x)) \tag{4.45}$$

yielding

$$[J_0(\mathbf{x},t), \phi_1'(\mathbf{x}',t)] = -i\lambda\delta^3(\mathbf{x}-\mathbf{x}') \tag{4.46}$$

with, of course,

$$(\partial/\partial x_\mu)J_\mu(\mathbf{x},t) = 0, \tag{4.47}$$

so that the conditions of the Goldstone theorem are satisfied.

Indeed, if we look at the expression for $M_\mu(k)$ in the linearized approximation, we find

$$\begin{aligned} M_\mu(k) &= \int d^4x \exp i(kx)\lambda\langle 0|[mA_\mu(x) \\ &\qquad - (\partial/\partial x^\mu)\phi_1'(x), \phi_1'(0)]|0\rangle \\ &= i\lambda k_\mu \int d^4x \exp[i(kx)]\langle 0|[\phi_1'(x),\phi_1'(0)]|0\rangle \\ &= 2\pi i\lambda k_\mu \delta(k^2+\mu^2)\epsilon(k_0), \end{aligned} \tag{4.48}$$

where we have used the expression for the free-boson commutator

$$\begin{aligned} i\Delta(x,\mu^2) \cdot &\equiv \cdot [\phi_1'(x)_1\phi_1'(0)] \\ &= \int \frac{d^4k}{(2\pi)^3}\exp[-i(kx)]\epsilon(k_0)\delta(k^2+\mu^2) \end{aligned} \tag{4.49}$$

and allowed for the possibility that $\phi_1'(x)$ might be massive with mass μ. We see that Eq. (4.48) is only consistent with

$$k^\mu M_\mu(k) = 0 \tag{4.50}$$

if

$$\mu^2 = 0 \tag{4.51}$$

which is the Goldstone theorem.

This appears to dispose of $\phi_1'(x)$ and $\phi_2'(x)$, but what are we to make of $A_\mu(x)$? To obtain an insight, we look at the field equation

$$\begin{aligned} &\frac{\partial}{\partial x^\nu}\left[\frac{\partial}{\partial x_\mu}A^\nu(x) - \frac{\partial}{\partial x_\nu}A^\mu(x)\right] \\ &\qquad = m^2\left[\frac{1}{m}\frac{\partial}{\partial x_\mu}\phi_1'(x) - A^\mu(x)\right] \end{aligned} \tag{4.52}$$

which, in the Lorentz gauge, becomes

$$-\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x_\nu}A^\mu(x) = m^2\left[\frac{1}{m}\frac{\partial}{\partial x_\mu}\phi_1'(x) - A^\mu(x)\right]. \tag{4.53}$$

This is clearly *not* the equation of a massless photon. In fact, and in anticipation of later work, if we call

$$B^\mu(x) = A^\mu(x) - (1/m)(\partial/\partial x_\mu)\phi_1'(x) \tag{4.54}$$

we can write, in view of the equation satisfied by $\phi_1'(x)$,

$$(\partial/\partial x^\nu)(\partial/\partial x_\nu)B^\mu(x) = m^2B^\mu(x), \tag{4.55}$$

which is to say that $B^\mu(x)$ satisfies the equation of a free vector meson of mass m with the constraint

$$(\partial/\partial x^\mu)B^\mu(x) = 0. \tag{4.56}$$

In summary then, the linearized approximation to the Higgs Lagrangian solved in the Lorentz gauge shows:

1. a Goldstone theorem
2. an uncoupled Goldstone meson
3. a massive scalar meson
4. a massive vector meson.

Indeed the ratio of the masses of these last two objects is given by

$$\sqrt{2}\,m_0/m = e_0f, \tag{4.57}$$

yielding the remarkable result that these masses are given in terms of each other as a function of the two dimensionless coupling constants of the theory. No trace of any multiplet structure remains, and what is left of the gauge symmetry is expressed in the mass formula Eq. (4.57).

If the theory is to make sense, we would expect that the observable physics should be independent of the gauge of $A_\mu(x)$. To see how this happens, we shall re-do the work in the radiation gauge with

$$\nabla\cdot\mathbf{A}(\mathbf{x},t) = 0, \tag{4.58}$$

where we lose manifest covariance and where we expect that the Goldstone theorem will break down. In the limit $e_0 = 0$ the equation for $\phi_2'(x)$ is unchanged so that this field still corresponds to a boson of mass $\sqrt{2}\,m_0$. The current

$$e_0J_\mu(x) = m[mA_\mu(x) - (\partial/\partial x^\mu)\phi_1'(x)] \tag{4.59}$$

is still conserved in view of the antisymmetry of $F^{\mu\nu}(x)$; however, we now have a nontrivial connection between $A_\mu(x)$ and $\phi_1(x)$, namely

$$(\partial/\partial x_\mu)(\partial/\partial x^\mu)\phi_1'(x) = m\dot{A}_0(x), \tag{4.60}$$

so that $\phi_1'(x)$ is no longer a free mass zero boson. To this equation we can adjoin

$$\frac{\partial}{\partial x_\mu}\dot{A}_0(x) - \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x_\nu}A^\mu(x) = m^2\left[\frac{1}{m}\frac{\partial}{\partial x_\mu}\phi_1'(x) - A^\mu(x)\right] \tag{4.61}$$

or

$$\nabla\dot{A}_0(x) - \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x_\nu}\mathbf{A}(x) = m^2\left[\frac{1}{m}\nabla\phi_1'(x) - \mathbf{A}(x)\right] \tag{4.62}$$

and

$$\ddot{A}_0(x) + \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x_\nu}A_0(x) = -m^2\left[\frac{1}{m}\dot{\phi}_1'(x) - A_0(x)\right]. \tag{4.63}$$

This strange looking array of equations may be solved as follows: Since

$$\nabla\cdot\mathbf{A}(x) = 0, \tag{4.64}$$

$$\nabla^2\dot{A}_0(x) = m\nabla^2\phi_1'(x), \tag{4.65}$$

or

$$\dot{A}_0(x) = m\phi_1'(x) + F(\mathbf{x}), \tag{4.66}$$

where

$$\nabla^2 F(\mathbf{x}) = 0 \tag{4.67}$$

for all $\mathbf{x}$. Since A_0 is nonsingular as a function of $\mathbf{x}$, $F(\mathbf{x})$ is a constant which we will set equal to zero. Thus

$$\dot{A}_0(x) = m\phi_1'(x). \tag{4.68}$$

Hence, in the Coulomb gauge $\phi_1'(x)$ is *not* a Lorentz scalar but the time derivative of $A_0(x)$. But we have seen from the last section that in the Coulomb gauge $A_\mu(x)$ is *not* a four-vector. We carried out this work for a free photon, but more generally under a Lorentz transformation generated by $U(\epsilon)$

$$U(\epsilon)A^\mu(x)U^{-1}(\epsilon) = A^\mu(x') - \epsilon^{\mu\nu}A_\nu(x') + (\partial/\partial x_\mu')\Lambda(x,'\epsilon), \tag{4.69}$$

where $\Lambda(x,'\epsilon)$ is a complicated function of the fields which we computed explicitly only for the free field case.[36] Thus, if we call, as before

$$\begin{aligned} B^\mu(x) &= \left(-\frac{1}{m}\frac{\partial}{\partial x_\mu}\phi_1'(x) + A^\mu(x)\right) \\ &= \frac{1}{m^2}\left(\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x_\nu}A^\mu(x) - \frac{\partial}{\partial x_\mu}\dot{A}_0(x)\right) \\ &= \frac{1}{m^2}\frac{\partial}{\partial x^\nu}\left(\frac{\partial}{\partial x_\nu}A^\mu(x) - \frac{\partial}{\partial x_\mu}A^\nu(x)\right) \end{aligned} \tag{4.70}$$

this quantity *is* a four-vector, since the term involving $\Lambda(x,'\epsilon)$ will cancel out in the Lorentz transformation. Clearly

$$(\partial/\partial x^\mu)B_\mu(x) = 0 \tag{4.71}$$

which follows directly from the antisymmetry of $F^{\mu\nu}(x)$ or from the equation

$$(\partial/\partial x_\mu)(\partial/\partial x^\mu)\phi_1'(x) = m\dot{A}_0(x) = m^2\phi_1'(x). \tag{4.72}$$

Among other things, this last equation means that in this gauge $\phi_1'(x)$ no longer functions as a Goldstone boson. We shall return shortly to an examination of how the Goldstone theorem breaks down here. From Eq. (4.72) and its companion

$$(\partial/\partial x^\nu)(\partial/\partial x_\nu)A^\mu(x) = m^2A^\mu(x) \tag{4.73}$$

we conclude at once that

$$(\partial/\partial x^\nu)(\partial/\partial x_\nu)B_\mu(x) = m^2B_\mu(x). \tag{4.74}$$

Hence in the Coulomb gauge we have:

(1) no Goldstone theorem

(2) no Goldstone meson

(3) a massive scalar meson

(4) a massive vector meson

[36] See, for example, J. D. Bjorken and S. D. Drell (1965) p. 89. In the free-field case we wrote the extra term in momentum space as

$$E_\nu(k) = -(k_\nu k_0/|\mathbf{k}|^2)E_{0i}A^i(k)$$

which in coordinate space becomes

$$(\partial/\partial x_\mu')\Lambda(x',\epsilon) = -\epsilon_{0i}(\partial/\partial x_\mu')\int\frac{d^3\mathbf{x}}{4\pi|\mathbf{x}-\mathbf{x}'|}\dot{A}^i(\mathbf{x},t')$$

which is of the general form given in Eq. (4.69).

and the mass ratios remain the same. There are still some points to focus on. First, what has happened to the Goldstone theorem? To explore this we study, as usual,

$$[J_\mu(x),\phi_1'(0)] = (m/e_0)[mA_\mu(x) - (\partial/\partial x^\mu)\phi_1'(x),\phi_1'(0)]. \tag{4.75}$$

As we have seen, in the linearized approximation

$$J_\mu(x) = (m^2/e_0)B_\mu(x), \tag{4.76}$$

where in this approximation $B_\mu(x)$ is a free vector meson of mass m. Hence it is $B_\mu(x)$ that is to be quantised according to the statement

$$[B_\mu(x),B_\nu(y)] = -i\left(g_{\mu\nu} - \frac{1}{m^2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right)\Delta(x-y,m^2) \tag{4.77}$$

with $\Delta(x-y,m^2)$ defined as above. Thus in the linearized approximation the only state in the sum over states that contributes to

$$\begin{aligned} M_\mu(k) &= \int d^4x\,\exp[i(kx)]\langle 0|[J_\mu(x),\phi_1'(0)]|0\rangle \\ &= \int d^4x\,\exp[i(kx)](m^2/e_0)\langle 0|[B_\mu(x),\phi_1'(0)]|0\rangle \end{aligned} \tag{4.78}$$

is $|B(k)\rangle$, a state of one B vector meson.[37] If $\phi_1'(0)$ were a true Lorentz scalar we would have

$$\langle 0|\phi_1'(0)|B(k)\rangle = 0 \tag{4.79}$$

since B_μ is a spin-one particle, and hence $M_\mu(k)$ would vanish, contradicting the basic vacuum broken symmetry hypothesis. However $\phi_1'(0)$, as noted above, is *not* a scalar. In fact since

$$B_\mu(x) = A_\mu(x) - (1/m)(\partial/\partial x^\mu)\phi_1'(x) \tag{4.80}$$

and

$$\nabla\cdot\mathbf{A}(x) = 0 \tag{4.81}$$

we have

$$\nabla\cdot\mathbf{B}(x) = -(1/m)\nabla^2\phi_1'(x) \tag{4.82}$$

or, symbolically,

$$\phi_1'(x) = -(1/m)(1/\nabla^2)\nabla\cdot\mathbf{B}(x). \tag{4.83}$$

This $\phi_1'(0)$ *can* connect the vacuum to the spin-one B state; i.e.,

$$\langle 0|\phi_1(0)|B(k)\rangle \sim \boldsymbol{\epsilon}\cdot\mathbf{k}/|\mathbf{k}|^2, \tag{4.84}$$

where $\epsilon_\mu(k)$ is the B_μ polarization vector satisfying

$$(k\epsilon(k)) = 0. \tag{4.85}$$

It is just here, in this model, that the noncovariance which spoils the proof of the Goldstone theorem has slipped in.

[37] I am grateful to Dr. Hugh Osborn of Cambridge University for an enlightening correspondence on these matters.

We can make this discussion more precise by computing

$$[B_\mu(x), \phi_1'(y)] = -m[B_\mu(x), (1/\nabla^2)\nabla \cdot \mathbf{B}(y)]$$
$$= \frac{i}{m}\left(m^2 \frac{g_{\mu i}}{\nabla_y^2}\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y^\mu}\right)\Delta(x-y,m^2)$$
$$= \frac{i}{m}\left[\eta_\mu \frac{\partial}{\partial y^\nu}\frac{\partial}{\partial y_\nu} - \left(\eta\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y^\mu}\right] \Big/ \left[\frac{\partial}{\partial y^\nu}\frac{\partial}{\partial y_\nu} + \left(\eta\frac{\partial}{\partial y}\right)\left(\eta\frac{\partial}{\partial y}\right)\right]\left(\eta\frac{\partial}{\partial y}\right)\Delta(x-y,m^2), \quad (4.86)$$

where

$$\eta = (0,0,0,1), \quad (4.87)$$

and where we have used

$$[\nabla_y^2 - (\partial^2/\partial t^2)]\Delta(x-y,m^2) = m^2\Delta(x-y,m^2). \quad (4.88)$$

Hence

$$M_\mu(k) = \frac{m^2}{e_0}\frac{1}{m}\left(\frac{\eta_\mu k^2 - (\eta k)k_\mu}{k^2 + (\eta k)(nk)}\right) \times (\eta k)\int d^4x \exp[i(kx)]\Delta(x,m^2) \quad (4.89)$$
$$= -2\pi i\frac{m}{e_0}\left(\frac{\eta_\mu k^2 - (\eta k)k_\mu}{k^2 + (\eta k)(\eta k)}\right)(\eta k)\delta(k^2+m^2)\epsilon(k_0)$$

which is of the Higgs form derived in the last chapter, Eq. (3.47), as the escape from the Goldstone theorem. Note that since

$$m = \sqrt{2}(e_0 m_0/f), \quad (4.90)$$

if we set

$$e_0 = 0 \quad (4.91)$$

$$M_\mu(k) \to (\sqrt{2}m_0/f)2\pi i k_\mu \delta(k^2)\epsilon(k_0) \quad (4.92)$$

which is precisely the covariant form leading to a Goldstone theorem. This illustrates the essential role of the gauge fields in escaping the Goldstone boson. For $e_0 \neq 0$ we see that

$$M_0(k) = 2\pi i(m/e_0)k_0\epsilon(k_0)\delta(k^2+m^2). \quad (4.93)$$

Thus

$$\int_{-\infty}^{\infty} dk_0 \exp(ik_0 t)M_0(k) = 2\pi i(m/e_0)\cos(\sqrt{|\mathbf{k}|^2+m^2}\,t)$$
$$= 2\pi\int d^3\mathbf{x}\exp[i(\mathbf{k}\cdot\mathbf{x})]\langle 0|[J_0(x),\phi_1'(0)]|0\rangle. \quad (4.94)$$

If we set $\mathbf{k} = 0$ in this expression it becomes

$$\int d^3\mathbf{x}\langle 0|[J_0(\mathbf{x},t),\phi_1'(0)]|0\rangle = (\mathrm{im}/e_0)\cos(mt). \quad (4.95)$$

In other words, even though

$$(\partial/\partial x^\mu)J_\mu(x) = 0, \quad (4.96)$$

the "charge" is not a constant of the motion. When $e_0 \to 0$

$$\int d^3\mathbf{x}\langle 0|[J_0(\mathbf{x},t),\phi'(0)]|0\rangle = i\sqrt{2}(m_0/f) \quad (4.97)$$

and is independent of t.

From the equation of motion in the Coulomb gauge,

$$B_\mu(x) = \frac{1}{m^2}\left(\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x_\nu}A_\mu(x) - \frac{\partial}{\partial x^\mu}\dot{A}_0(x)\right), \quad (4.98)$$

we have

$$B_0(x) = \frac{1}{m^2}\nabla^2 A_0(\mathbf{x},t). \quad (4.99)$$

We might, naively, assume that

$$\int d^3\mathbf{x}[B_0(\mathbf{x},t),\phi_1'(0)] = \frac{1}{m^2}\int d^3\mathbf{x}\nabla^2[A_0(\mathbf{x},t),\phi_1'(0)] = 0. \quad (4.100)$$

However we easily show that

$$[A_0(\mathbf{x},t),\phi_1'(0)] = im(1/\nabla^2)(\partial/\partial t)\Delta(x,m^2) \quad (4.101)$$

so that surface terms *cannot* be neglected and the above integral does not vanish. This is, of course, a familiar feature of the Maxwell theory where the equation

$$(\partial/\partial x^\nu)F^{\mu\nu}(x) = -eJ^\mu(x) \quad (4.102)$$

does *not* imply that the electric charge is zero. Again, surface terms play an essential role because of the zero mass character of the photon.

The treatment so far has been confined to the lowest nontrivial order of perturbation theory. To see how these results generalize to all orders is no easy matter.[38] However, if one is not squeamish about somewhat obscure looking formal transformations one can get a feeling of how this will work out. To this end we are simply going to use the local $U(1)$ invariance to "rotate" $\phi_1(x)$ out of the game. As

$$\phi_1'(x) = \phi_1(x)\cos(\Lambda(x)) + \phi_2(x)\sin(\Lambda(x)), \quad (4.103)$$
$$\phi_2'(x) = -\phi_1(x)\sin(\Lambda(x)) + \phi_2(x)\cos(\Lambda(x)), \quad (4.104)$$

we may rotate away $\phi_1(x)$ with the choice

$$\Lambda(x) = -\tan^{-1}[\phi_1(x)/\phi_2(x)]. \quad (4.105)$$

Thus

$$\phi_1'(x) = 0 \quad (4.106)$$

and

$$\phi_2'(x) = \sqrt{\phi_1(x)^2 + \phi_2(x)^2}. \quad (4.107)$$

Moreover under this transformation

$$A_\mu'(x) = A_\mu(x) + \frac{1}{e_0}\frac{\partial}{\partial x^\mu}\Lambda(x)$$
$$= A_\mu(x) - \frac{1}{e_0}\frac{1}{\phi_1^2(x)+\phi_2^2(x)}\left(\frac{\partial}{\partial x^\mu}\phi_1(x)\phi_2(x) - \phi_1(x)\frac{\partial}{\partial x^\mu}\phi_2(x)\right), \quad (4.108)$$

[38] See, for example B. W. Lee (1972) for a discussion of the general S-matrix.

$$F'_{\mu\nu}(x) = F_{\mu\nu}(x), \tag{4.109}$$

$$\phi'_1(x)_\mu = -e_0\sqrt{\phi_1^2(x) + \phi_2^2(x)}\,A'_\mu(x), \tag{4.110}$$

$$\phi'_2(x)_\mu = -(\partial/\partial x^\mu)(\sqrt{\phi_1^2(x) + \phi_2^2(x)}). \tag{4.111}$$

Since $\mathcal{L}(x)$ is invariant under the local $U(1)$ group, the field equations are *covariant* and become

$$G^{\mu\nu}(x) = (\partial/\partial x_\mu)A'^\nu(x) - (\partial/\partial x_\nu)A'^\mu(x) \tag{4.112}$$

and

$$(\partial/\partial x^\nu)G^{\mu\nu}(x) = e_0^2\phi'_2(x)^2 A^{\mu\prime}(x) \tag{4.113}$$

while the remaining two equations are the identity

$$\frac{\partial}{\partial x^\mu}(e_0\phi'_2(x)A_\mu(x)) = \frac{\partial}{\partial x^\mu}(e_0\phi'_2(x)A_\mu(x)) \tag{4.114}$$

and

$$-\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\phi'_2(x) = m_0^2\phi'_2(x) - \frac{1}{2}f^2\phi'_2(x)^3 + e_0\frac{\partial}{\partial x^\mu}\phi'_2(x)A'_\mu(x). \tag{4.115}$$

Up to this point the local $U(1)$ symmetry has not been manifestly broken. We achieve this as before by demanding that

$$\langle 0|\phi'_2(0)|0\rangle = \eta \neq 0. \tag{4.116}$$

We shall now see in this more general context how this fixes the gauge and breaks the symmetry. To this end let

$$\chi(x) = \phi'_2(x) - \eta, \tag{4.117}$$

where it is understood that in each order in the coupling constants η is readjusted so that to that order

$$\langle 0|x(0)|0\rangle = 0, \tag{4.118}$$

and to conform with the notation of the linearized theory let

$$A'_\mu(x) \cdot \equiv \cdot B_\mu(x). \tag{4.119}$$

The exact equations now take the form

$$G^{\mu\nu}(x) = (\partial/\partial x_\mu)B^\nu(x) - (\partial/\partial x_\nu)B^\mu(x) \tag{4.120}$$

$$(\partial/\partial x^\nu)G^{\mu\nu}(x) = -e_0^2[\eta^2 + 2\chi(x)\eta + \chi^2(x)]B^\mu(x), \tag{4.121}$$

and

$$-\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\chi(x) = m_0^2[\eta + \chi(x)] - \frac{1}{2}f^2(\eta^3 + 3\eta^2\chi(x) + 3\eta\chi^2(x) + \chi^3(x)) + e_0\frac{\partial}{\partial x_\mu}\chi(x)B_\mu(x). \tag{4.122}$$

We reduce to the linearized theory if we let

$$\eta = \sqrt{2}(m_0/f) \tag{4.123}$$

and

$$m = \sqrt{2}(e_0 m_0/f) \tag{4.124}$$

and drop the coupling terms. The perturbation is to be taken in powers of e_0 and f while e_0/f is taken as order unity. Hence to lowest nontrivial order

$$(\partial/\partial x^\nu)G^{\mu\nu}(x) = -m^2B^\mu(x) \tag{4.125}$$

and

$$(\partial/\partial x^\mu)(\partial/\partial x_\mu)\chi(x) = -\left(m_0^2 - 3/2f^2\left(\frac{2m_0^2}{f^2}\right)\right)\chi(x) = 2m_0^2\chi(x). \tag{4.126}$$

We would then quantize these free fields by the conditions

$$[B_\mu(x), B_\nu(y)] = -i\left(g_{\mu\nu} - \frac{1}{m^2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right)\Delta(x-y, m^2), \tag{4.127}$$

$$[B_\mu(x), \chi(y)] = 0, \tag{4.128}$$

and

$$[\chi(x), \chi(y)] = -i\Delta(x-y, 2m_0^2). \tag{4.129}$$

In this gauge all trace of the massless fields we began with has disappeared. In fact we can now verify to lowest order that

$$\int d^3\mathbf{x}(\partial/\partial x^j)G^{j0}(x) = -m^2\int d^3\mathbf{x}\,B^0(\mathbf{x},t) = 0. \tag{4.130}$$

To see this, note that in view of the quantization

$$B_\mu(\mathbf{x},t) = \sum_{\lambda=1}^{3}\frac{1}{(2\pi)^{3/2}}\int\frac{d^3\mathbf{k}}{2w_k}\epsilon_\mu(k,\lambda) [A(k,\lambda)\exp[-i(kx)] + A^\dagger(k,\lambda)\exp[i(kx)]], \tag{4.131}$$

where

$$(k\epsilon(k,\lambda)) = 0, \tag{4.132}$$

and

$$w_k = \sqrt{|\mathbf{k}|^2 + m^2}. \tag{4.133}$$

We have therefore

$$\int d^3\mathbf{x}\,B_0(\mathbf{x},t) = \sum_{\lambda=1}^{3}(2\pi)^{3/2}\int\frac{d^3\mathbf{k}}{2w_k}\delta^3(\mathbf{k})\epsilon_0(\mathbf{k},\lambda) [A(\mathbf{k},\lambda)\exp[iw_kt] + A^\dagger(\mathbf{k},\lambda)\exp[-iw_kt]]. \tag{4.134}$$

But

$$\epsilon_0(0,\lambda) = 0. \tag{4.135}$$

Thus to this order, and in fact to all orders,

$$\int d^3\mathbf{x}(\partial/\partial x^j)G^{j0}(x) = 0. \tag{4.136}$$

In summary: While the physical particle content of the three gauges we have studied is the same, the mathematical structure is quite different. The last gauge we have studied is called the "unitary" or U-gauge since the S-matrix constructed from the χ and B fields will be manifestly unitary. The task of showing that all of the S-matrices belonging to different gauges lead to the same physics is exceedingly intricate. We give a hint of how this works in the next section.

We conclude this section by adding spin-1/2 particles to the amalgam in a preliminary way. In what follows we will consider $\psi(x)$, a spin-1/2 field, $A_\mu(x)$, the vector field, and the Higgs mesons $\phi(x)$ and $\phi^\dagger(x)$ where in terms of the old language

$$\phi(x) = [\phi_1(x) + i\phi_2(x)]/\sqrt{2} \qquad (4.137)$$

and

$$\phi^\dagger(x) = [\phi_1(x) - i\phi_2(x)]/\sqrt{2}. \qquad (4.138)$$

In this new language the local $U(1)$ invariance becomes phase invariance. It is convenient to do things this way because it unifies the treatment of the spin-1/2 particles and the Higgs mesons. Hence we write[39]

$$\begin{aligned}\mathfrak{L}(x) = &-\frac{1}{4}\left(\frac{\partial}{\partial x_\mu}A^\nu(x) - \frac{\partial}{\partial x_\nu}A^\mu(x)\right)^2 \\ &- \left(\frac{\partial}{\partial x^\mu} + ie_0A_\mu(x)\right)\phi^\dagger(x)\left(\frac{\partial}{\partial x_\mu} - ie_0A^\mu(x)\right)\phi(x) \\ &+ m_0^2\phi^\dagger(x)\phi(x) - \tfrac{1}{2}f^2(\phi^\dagger(x)\phi(x))^2 \\ &+ \bar{\psi}(x)\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + m_f\right)\psi(x) - ie_0\bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(x).\end{aligned} \qquad (4.139)$$

The first part of the Lagrangian is our old friend, while the last two terms are the "baryon" addition. We notice that $\mathfrak{L}(x)$ has the following invariance:

(1)

$$\begin{aligned}\psi(x) &\to e^{i\Lambda}\psi(x),\\ \phi(x) &\to \phi(x),\\ A_\mu(x) &\to A_\mu(x),\end{aligned} \qquad (4.140)$$

where Λ is a real number. This leads to

$$(\partial/\partial x_\mu)\mathfrak{J}_\mu(x) \cdot \equiv \cdot (\partial/\partial x_\mu)(i\bar{\psi}(x)\gamma_\mu\psi(x)) = 0. \qquad (4.141)$$

(2)

$$\begin{aligned}\psi(x) &\to \psi(x),\\ \phi(x) &\to e^{i\Lambda}\phi(x),\\ A_\mu(x) &\to A_\mu(x),\end{aligned} \qquad (4.142)$$

which leads to the conservation law

$$\begin{aligned}\frac{\partial}{\partial x_\mu}S_\mu(x) \cdot \equiv \cdot\, & i\frac{\partial}{\partial x_\mu}\left(\frac{\partial}{\partial x^\mu} + ie_0A_\mu(x)\right)\phi^\dagger(x)\phi(x) \\ &- \phi^\dagger(x)\left(\frac{\partial}{\partial x^\mu} - ie_0A_\mu(x)\phi(x)\right) = 0\end{aligned} \qquad (4.143)$$

and finally there is the local invariance

(3)

$$\begin{aligned}\psi(x) &\to e^{i\Lambda(x)}\psi(x),\\ \phi(x) &\to e^{i\Lambda(x)}\phi(x),\\ A_\mu(x) &\to A_\mu(x) + (1/e_0)(\partial/\partial x^\mu)\Lambda(x),\end{aligned} \qquad (4.144)$$

which is reflected in the conservation of the combined generators

$$J_\mu(x) = \mathfrak{J}_\mu(x) + S_\mu(x). \qquad (4.145)$$

If we wish to interpret

$$B = \int d^3\mathbf{x}\,\mathfrak{J}_0(\mathbf{x},t) \qquad (4.146)$$

as a conserved baryon number, we must insist that

$$B \neq 0, \qquad (4.147)$$

$$\dot{B} = 0, \qquad (4.148)$$

and

$$B|0\rangle = 0. \qquad (4.149)$$

Without this last condition no state would have a well defined "baryon number" and the whole concept would become meaningless. Clearly we also have

$$[B,\phi(x)] = 0, \qquad (4.150)$$

i.e., *no meson* has baryon number.

However we can formally define

$$S(t) = \int d^3\mathbf{x}\,S_0(\mathbf{x},t) \qquad (4.151)$$

so that, from the canonical commutation relations,

$$\int [S_0(\mathbf{x}',t),\phi(\mathbf{x},t)]\,d^3\mathbf{x}' = \phi(\mathbf{x},t). \qquad (4.152)$$

So we can say that ϕ has one unit of S number. On the other hand

$$[S,B] = 0 \qquad (4.153)$$

and

$$[S(t),\psi(\mathbf{x},t)] = 0. \qquad (4.154)$$

We can now carry through the Higgs analysis on the S current and ϕ fields. We look for a solution of the theory with

$$\langle 0|\int d^3\mathbf{x}'[S_0(\mathbf{x}',t),\phi(0)]|0\rangle \neq 0 \qquad (4.155)$$

so that

$$\langle 0|\phi(0)|0\rangle = \eta \neq 0. \qquad (4.156)$$

At this point we may bring time reversal into the discussion. The basic statement of time reversal invariance upon which we shall build is the condition on the current, where T is the time reversal operation,

$$T\mathbf{J}(\mathbf{x},t)T^{-1} = -\mathbf{J}(\mathbf{x},-t), \qquad (4.157)$$

$$TJ_0(\mathbf{x},t)T^{-1} = J_0(\mathbf{x},-t). \qquad (4.158)$$

We can achieve this if we choose the time reversal phases as follows:

$$T\phi_1(\mathbf{x},t)T^{-1} = \phi_1(\mathbf{x},-t), \qquad (4.159)$$

$$T\phi_2(\mathbf{x},t)T^{-1} = -\phi_2(\mathbf{x},-t), \qquad (4.160)$$

so that for the complex fields $\phi(x)$

$$T\phi(\mathbf{x},t)T^{-1} = \phi(\mathbf{x},-t). \qquad (4.161)$$

[39] This is a model of how to implement, say, baryon number conservation in the Higgs context. This Lagrangian form was given by S. Weinberg in a January 1972 unpublished lecture.

Hence

$$\langle 0|\phi(0)|0\rangle = \langle 0|\phi(0)|0\rangle^*. \qquad (4.162)$$

Since we assume

$$T|0\rangle = |0\rangle, \qquad (4.163)$$

we conclude that

$$\eta = \eta^*. \qquad (4.164)$$

This is consistent with the result we would obtain by rotating into the U gauge. Note that the U gauge transformation

$$\phi_1'(x) = 0 \qquad (4.165)$$

$$\phi_2'(x) = \sqrt{\phi_1^2(x) + \phi_2^2(x)} = \sqrt{2\phi^\dagger(x)\phi(x)} \qquad (4.166)$$

will preserve the time reversal character of the theory. Hence, if we rotate into the U gauge and make

$$\langle 0|\phi_2'(0)|0\rangle = \eta, \qquad (4.167)$$

this η is real. The U-gauge Lagrangian becomes, written in terms of fields with vanishing vacuum expectation values, [a constant has been dropped in $\mathcal{L}(x)$, and we have called $\psi'(x) = e^{i\Lambda(x)}\psi(x)$ as we must also transform the ψ:]

$$\begin{aligned}\mathcal{L}(x) &= \bar{\psi}'(x)\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m_f\right)\psi'(x) \\ &= \frac{1}{2}G^{\mu\nu}(x)\left(\frac{\partial}{\partial x^\mu}B_\nu(x) - \frac{\partial}{\partial x^\nu}B_\mu(x)\right) \\ &\quad + \frac{1}{4}G^{\mu\nu}(x)G_{\mu\nu}(x) - \frac{m^2}{2}B_\mu^2(x) - \frac{1}{2}\frac{\partial}{\partial x_\mu}\chi(x)\frac{\partial}{\partial x^\mu}\chi(x) \\ &\quad - \frac{e_0^2}{2}\chi^2(x)B_\mu^2(x) - e_0 m\chi(x)B_\mu^2(x) \\ &\quad + [m_0^2 - 3/2f^2\eta^2]\frac{\chi^2(x)}{2} \\ &\quad - \frac{f^2}{8}\chi(x)^4 + \eta\chi(x)\left(m_0^2 - \frac{f^2}{2}\eta^2\right) - \frac{\eta f^2}{2}\chi^3(x) \\ &\quad - ie_0\bar{\psi}'(x)\gamma_\mu\psi'(x)B^\mu(x) \qquad (4.168)\end{aligned}$$

with the notation as before. The current

$$\begin{aligned}J_\mu(x) &= ie_0\bar{\psi}'(x)\gamma_\mu\psi'(x) + m^2 B_\mu(x) + e_0^2\chi^2(x)B_\mu(x) \\ &\quad + 2e_0 m B_\mu(x)\chi(x) \qquad (4.169)\end{aligned}$$

is conserved and has a vanishing charge. However the "baryon" current

$$\mathcal{J}_\mu(x) = ie_0\bar{\psi}'(x)\gamma_\mu\psi'(x) \qquad (4.170)$$

is *separately* conserved. This follows from the invariance of $\mathcal{L}(x)$ under

$$\begin{aligned}\psi(x) &\to e^{i\Lambda}\psi(x), \\ \chi(x) &\to \chi(x), \qquad (4.171) \\ B_\mu(x) &\to B_\mu(x).\end{aligned}$$

We certainly do *not* want

$$\int d^3\mathbf{x}\,\mathcal{J}_0(\mathbf{x},t) = 0. \qquad (4.172)$$

Hence, there must be a cancellation between the two charges that make up $\int d^3\mathbf{x}\,J_0(\mathbf{x},t)$. It is interesting to see schematically how this cancellation[40] takes place in lowest nontrivial order. We must have in this order

$$m^2\int B_0(\mathbf{x},t)\,d^3\mathbf{x} + ie_0\int\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)\,d^3\mathbf{x} = 0.$$

This follows from the explicit Green's function solution to the equation for the Heisenberg field $B_\mu(x)$; i e.,

$$(\partial/\partial x_\lambda)(\partial/\partial x^\lambda)B_\mu(x) = m^2 B_\mu(x) + ie_0\bar{\psi}(x)\gamma_\mu\psi(x)$$

valid to order e_0, with $\psi(x)$ a free spinor.

V. THE S-MATRIX

In this section we are going to discuss the Higgs mechanism from the point of view of the S-matrix, which means in terms of Feynman diagrams. In most treatments[41] of this subject one usually begins with an unperturbed Lagrangian, which eventually defines the interaction representation and already contains the effects of the Higgs mechanism in the lowest nontrivial order. In this Lagrangian the "photon" has already become massive. One then studies the higher order corrections. However, following Englert and Brout (1964), we can begin one step earlier and trace, in terms of Feynman graphs, how the "photon" acquired its mass in the first place. This gives an additional insight into the Higgs mechanism which Englert and Brout discovered independently.

To begin with, we may ask how the usual field theory accommodates the fact that the photon—the real photon—has zero mass. The fact that the photon has no "bare mass" follows from gauge invariance, since a mass term in the Lagrangian

$$m_0^2 A_\mu(x)A^\mu(x) \qquad (5.1)$$

cannot be invariant under

$$A_\mu(x) \to A_\mu(x) + (1/e_0)(\partial/\partial x^\mu)\Lambda(x). \qquad (5.2)$$

If this were all there were to it, the Higgs mechanism could never operate, since the Higgs Lagrangian is also gauge invariant.

Of course, this is *not* all there is to it. We must, in fact, study the photon propagator to see if photonic interactions have, somehow, conspired to give the photon a mass. The photon propagator is defined to be—we first deal with the unrenormalized propagator—

$$\begin{aligned}D_f(k)_{\mu\nu} &= \frac{1}{(2\pi)^4}\int D_f(x)_{\mu\nu}\exp[i(kx)]\,d^4x \\ &= -i\frac{1}{(2\pi)^4}\int d^4x\langle 0|T(A_\mu(x)A_\nu(0))|0\rangle\exp[i(kx)], \qquad (5.3)\end{aligned}$$

[40] A reader who actually repeats this computation will find that the condition

$$\frac{d}{dt}\int d^3\mathbf{x}\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t) = 0$$

will come in in an essential way. He (or she) should also remember that for the free $B_\mu(x)$

$$\int d^3\mathbf{x}\,B_0(\mathbf{x},t) = 0.$$

[41] See, for example, B. W. Lee (1972).

where

$$T(A_\mu(x)A_\nu(0)) = \theta(t)A_\mu(\mathbf{x},t)A_\nu(0) + \theta(-t)A_\nu(0)A_\mu(\mathbf{x},t) \tag{5.4}$$

and the $A_\mu(x)$ are the unrenormalized photon fields. For the *free* photon—neglecting gauge terms—$D_f^0(k)_{\mu\nu}$ is simply

$$D_f^0(k)_{\mu\nu} = g_{\mu\nu}/(k^2 + i\epsilon). \tag{5.5}$$

In the limit $\epsilon \to 0$ Eq. 5.5 has a pole at $k^2 = 0$, the photon mass. If the *exact* Green's function had a pole somewhere else, the photon would have acquired a mass through interactions and this would manifest itself, for example, in the fact that the static limit of the interaction between two charged particles would not be the Coulomb potential but rather a Yukawa potential with an exponential falloff whose range would be the inverse of the reciprocal of the position of this pole. Hence, the full theory must arrange itself so that the position of the pole is *not* shifted and, indeed, in the Higgs theory this arrangement must break down somewhere.

To see how this happens we must study the structure of the *exact* photon propagator. This is a matter of summing, at least symbolically, the full set of Feynman graphs.[42] The exact photon propagator will be drawn as the blob in Fig. 1. This blob can be represented in perturbation theory as follows:

(a) Define a "proper" contribution to the blob as any diagram that cannot be rendered disjoint by removing a single photon line. See Fig. 2 for examples of "proper" and "improper" contributions.

(b) Remove the external photon legs and call $\Pi_{\mu\nu}(q)$ the sum of all "proper" graphs to arbitrary order in e_0, the unrenormalized charge. (The factor ie_0^2 in front is conventional.)

(c) Thus, the exact photon propagator—again ignoring gauge terms—is, dropping the $i\epsilon$ which defines the Feynman contour (See Fig. 3).

$$iD_f(q^2)_{\mu\nu} = i\frac{g_{\mu\nu}}{q^2} + \frac{i}{q^2}(ie_0^2\,\Pi_{\mu\nu}(q))\frac{1}{q^2} + \frac{i}{q^2}(ie_0^2\,\Pi_{\mu\lambda}(q^2)\frac{i}{q^2}(ie_0^2\,\Pi^\lambda_\nu(q^2))\frac{i}{q^2} + \cdots. \tag{5.6}$$

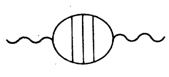

FIG. 1. The exact photon propagator. In this, and subsequent diagrams, wavy lines are photons, and solid lines are charged particles.

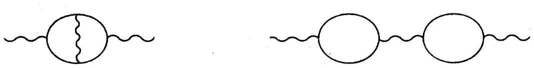

FIG. 2. Figure 2(a) is a typical "proper" diagram, while 2(b) is an "improper" diagram.

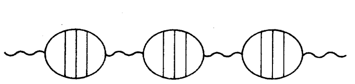

FIG. 3. The "bubble sum" for the exact photon propagator.

[42] Once again, we follow the exposition in J. D. Bjorken and S. D. Drell (1965), although not their metric conventions.

To make the next step we must remind the reader of a result from field theory, namely,

$$q^\mu \Pi_{\mu\nu}(q) = 0 \tag{5.7}$$

i.e., $\Pi_{\mu\nu}(q)$ is a *conserved* tensor. This is a consequence of the fact that A_μ interacts with a *conserved current*.[43] Hence, this result holds in the Higgs-type theories in which gauge invariance is broken, but the photon in all gauges interacts with a conserved current. Thus, since $\Pi_{\mu\nu}(q)$ is a tensor and depends only on $q_\mu q_\nu$ and $g_{\mu\nu}$, we must have

$$\Pi_{\mu\nu}(q) = (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2). \tag{5.8}$$

It is the properties[44] of $\Pi(q^2)$ that determine the functional form of $D_f(q^2)_{\mu\nu}$ and, in particular, the position of its poles. Hence, dropping terms in $q_\mu q_\nu$ since these do not contribute to the S-matrix,

$$\begin{aligned} iD_f(q^2)_{\mu\nu} &= i\frac{g_{\mu\nu}}{q^2} - \frac{i}{q^2}e_0^2 g_{\mu\nu}\Pi(q^2) + \frac{i}{q^2}e_0^4 g_{\mu\nu}\Pi^2(q^2) + \cdots \\ &= i\frac{g_{\mu\nu}}{q^2}\left(\frac{1}{1+e_0^2\Pi(q^2)}\right). \end{aligned} \tag{5.9}$$

All of this goes through for the Higgs theory. Why then is the photon massless, while in the Higgs theory it acquires mass? This has to do with the *analytic structure* of $\Pi(q^2)$ in the two theories. So long as $\Pi(0)$ is *finite*; i.e., has no pole at $q^2 = 0$, the photon will remain massless. However, if as $q^2 \to 0$

$$\Pi(q^2) \to \eta^2/q^2 \tag{5.10}$$

the "photon" will acquire a mass $e_0\eta$. In ordinary quantum electrodynamics, there is no way that $\Pi(q^2)$ can acquire a pole at $q^2 = 0$. This could happen only if there were a zero mass intermediate state in one of the proper diagrams. Since $\Pi(q^2)$, by definition, involves only proper photon graphs and since all other massless particles—the neutrinos—have no electric charge, all one-particle intermediate states in the normal theories have a nonzero rest mass. However, as we shall now see, the Higgs theories are something else. To carry out our program, we return to the Higgs Lagrangian which we write in its complex form as

$$\mathfrak{L}(x) = \mathfrak{L}_0(x) + \mathfrak{L}_I(x) \tag{5.11}$$

with

$$\mathfrak{L}_0(x) = -\frac{1}{4}\left(\frac{\partial}{\partial x_\mu}A^\nu(x) - \frac{\partial}{\partial x_\nu}A_\mu(x)\right)^2 - \frac{\partial}{\partial x^\mu}\phi^\dagger(x)\frac{\partial}{\partial x_\mu}\phi(x) \tag{5.12}$$

[43] Bjorken and Drell (1965) show this in quantum electrodynamics using a Ward identity. The Ward identity in Q.E.D. follows from *current conservation* alone, as is evident from the Takahashi proof (1957). Other proofs can be given using gauge invariance, but current conservation suffices.

[44] We are being rather sloppy here about renormalized and unrenormalized quantities. In general, the unrenormalized $\Pi(q^2)$ will be infinite for all q^2. To render it finite, it will have to be subtracted at, say, the physical mass of the "photon." This nicety will not trouble us in what follows, since we shall not go beyond the no-loop approximation. A discussion of the role of the singularities in $\Pi(q^2)$ and the "photon" mass was given first by Schwinger (1962).

and

$$\mathfrak{L}_I(x) = -ie_0\left[A_\mu(x)\left(\phi^\dagger(x)\frac{\partial}{\partial x_\mu}\phi(x) - \frac{\partial}{\partial x_\mu}\phi^\dagger(x)\phi(x)\right)\right] - e_0^2 A_\mu(x)A^\mu(x)\phi^\dagger(x)\phi(x) + m_0^2\phi^\dagger(x)\phi(x) - \frac{1}{2}f^2(\phi^\dagger(x)\phi(x))^2 + \text{counter terms}\,. \tag{5.13}$$

If this theory were quantized in a normal way with

$$\langle 0|\phi(0)|0\rangle = 0 \tag{5.14}$$

it would not exist, since everything is hopelessly infrared divergent because of the zero mass scalar mesons. However, let us assume the vacuum-broken symmetry condition and shift $\phi(x)$ so that

$$\phi'(x) = \phi(x) - (\eta/\sqrt{2}) \tag{5.15}$$

with

$$\langle 0|\phi'(0)|0\rangle = 0 \tag{5.16}$$

with η real. We leave $A_\mu(x)$ alone. Thus, in terms of the shifted variables

$$\mathfrak{L}_0(x) = -\frac{1}{4}\left(\frac{\partial}{\partial x_\mu}A^\nu(x) - \frac{\partial}{\partial x_\nu}A^\mu(x)\right)^2 - \frac{\partial}{\partial x^\mu}\phi'^\dagger(x)\frac{\partial}{\partial x_\mu}\phi'(x), \tag{5.17}$$

while

$$\begin{aligned}\mathfrak{L}_I(x) = &-ie_0\frac{\eta}{\sqrt{2}}A_\mu(x)\left(\frac{\partial}{\partial x_\mu}\phi'(x) - \frac{\partial}{\partial x_\mu}\phi'^\dagger(x)\right)\\ &- e_0^2\frac{\eta^2}{2}A_\mu(x)A^\mu(x)\\ &- ie_0A_\mu(x)\left(\phi'^\dagger(x)\frac{\partial}{\partial x_\mu}\phi'(x) - \frac{\partial}{\partial x_\mu}\phi'^\dagger(x)\phi'(x)\right)\\ &- e_0^2\Big(\phi'^\dagger(x)\phi'(x)\\ &\qquad + \frac{\eta}{\sqrt{2}}(\phi'^\dagger(x) + \phi'(x))\Big)A_\mu(x)A^\mu(x)\\ &+ m_0^2(\phi'(x) + \eta/\sqrt{2})^\dagger(\phi'(x) + \eta/\sqrt{2})\\ &- \frac{f^2}{2}((\phi'(x) + \eta/\sqrt{2})^\dagger(\phi'(x) + \eta/\sqrt{2}))^2.\end{aligned} \tag{5.18}$$

We may now proceed, following Englert and Brout (1964), to quantize the theory and derive the Feynman rules. The interaction representation involves a zero mass photon and a zero mass ϕ'. The leading order contributions to the "photon" propagator are given by the two diagrams in Fig. 4. Thus, in this approximation

$$\Pi_{\mu\nu}(q) = (q^2 g_{\mu\nu} - q_\mu q_\nu)(1/q^2), \tag{5.19}$$

where we have taken account of the fact that the interaction representation propagator of the ϕ' goes as $(1/q^2)$. Thus, the effective "photon" propagator goes, when summed over such point couplings, as

FIG. 4. These diagrams represent the leading corrections to the free photon propagator in the Higgs theory. The wavy line is the photon, and the dashed line is the $\phi'(0)$. The vertex in Fig. 4(a) is given by $e_0^2(\eta^2/2)$, while each vertex in Fig. 4(b) is given by $iq_\mu(\eta e_0/\sqrt{2})$, where q_μ is the "photon" four-momentum.

$$iD_f(q^2)_{\mu\nu} = i(g_{\mu\nu}/q^2)\left[1\Big/\left(1 + \frac{e_0^2\eta^2}{q^2}\right)\right], \tag{5.20}$$

which means that the "photon" has acquired a mass à la Higgs. The pole is achieved here because of the zero mass scalar bosons in the intermediate states.

We may develop an elegant appearing propagator, including gauge terms, if we adopt the Landau gauge for the free photon propagator, i.e.,

$$iD_f^0(q^2) = i[g_{\mu\nu} - (q_\mu q_\nu/q^2](1/q^2). \tag{5.21}$$

The tensor

$$P_{\mu\nu} = [g_{\mu\nu} - (q_\mu q_\nu/q^2)] \tag{5.22}$$

is idempotent; i.e.,

$$P_{\mu\nu}P^\nu_\lambda = P_{\mu\lambda}. \tag{5.23}$$

Hence, the sum can be performed as before and yields

$$iD_f(q^2)_{\mu\nu} = i\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{1}{q^2}\frac{1}{1 + \Pi(q^2)} \tag{5.24}$$

with the *same* $\Pi(q^2)$ as in Eq. (5.8). The important thing to note about $D_f(q^2)_{\mu\nu}$ is that it is *not* the propagator—even in the lowest nontrivial order—of a free vector meson with mass. This latter propagator would be

$$V_{\mu\nu}(q) = \left(g_{\mu\nu} + \frac{q_\mu q_\nu}{m^2}\right)\Big/(q^2 + m^2). \tag{5.25}$$

It is *not* transverse and the presence of the factor $q_\mu q_\nu/m^2$ causes severe ultraviolet problems.

What is very striking about the derivation of the broken symmetry propagator, Eq. (5.24), from the underlying electrodynamic model is that the theory remembers enough of its electrodynamic origins to produce the gauge term $q_\mu q_\nu/q^2$ which is not badly behaved at high energies. The remarkable fact that the "photon" in this theory acquires a mass in this special way enables the theory to escape the snares of both infrared and ultraviolet divergences. In the last analysis, it is, at least in this respect, no worse than quantum electrodynamics.[45]

[45] The qualification we have in mind here has to do with the so-called "Adler anomalies." See S. L. Adler (1969) and also J. S. Bell and R. Jackiw (1969). These are "anomalous" terms in Ward identities involving two-vector and one-axial-vector currents. The presence of these terms is, in some sense, the price one pays for unifying weak and electromagnetic interactions, since inevitably there must be a coupling to the nonconserved axial current. In the gauge theories these anomalies can spoil the renormalization program. It is possible to cancel out the anomalies in a large class of models–not including, by the way, the Weinberg 1967 model in its original form and this is sometimes regarded as an important criterion for selecting among gauge models. See, for example, Zumino (1972) for a summary of what is involved. On the other hand, these anomalous terms only enter the S-matrix in order g^6 so, from a practical point of view, they do not *influence agreement with low and medium energy experiments.*

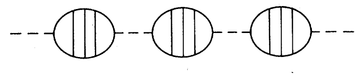

FIG. 5. This figure represents the "bubble sum" leading to the exact unrenormalized ϕ' propagator. Each bubble is the exact sum over "proper" diagrams. The dashed lines are the free propagators ϕ'.

Let us now discuss the ϕ' propagator in the same spirit. We can write the exact unrenormalized ϕ' propagator as a series (see Fig. 5),

$$iD_f(q^2) = \frac{i}{q^2} - \frac{i}{q^2}P(q^2) + \frac{i}{q^2}P^2(q^2) + \cdots = \frac{i}{q^2}\frac{1}{(1+P(q^2))}. \tag{5.26}$$

Here we have, in analogy to the photon situation, defined the "blob" function, say $F(q^2)$, to be of the form

$$iF(q^2) \cdot \equiv \cdot q^2 P(q^2) \tag{5.27}$$

so that the ϕ' acquires a mass if $P(q^2)$ has a pole at $q^2 = 0$. To see this pole develop we may consider the ϕ'-self-coupling terms in $\mathcal{L}_I(x)$; i.e.,

$$\begin{aligned} m_0^2\left(\phi'(x) + \frac{\eta}{\sqrt{2}}\right)^\dagger\left(\phi'(x) + \frac{\eta}{\sqrt{2}}\right) & \\ - \frac{1}{2}f^2((\phi'(x) + \eta/\sqrt{2})^\dagger(\phi'(x) + \eta/\sqrt{2}))^2 & \\ = m_0^2\left[\frac{\eta^2}{2} + \frac{\eta}{\sqrt{2}}(\phi'(x) + \phi'^\dagger(x)) + \phi'^\dagger(x)\phi'(x)\right] & \\ - \frac{f^2}{2}\left[\frac{\eta^4}{4} + \frac{\eta^2}{2}(\phi'(x) + \phi'^\dagger(x))^2 + \phi'^\dagger(x)^2\phi'^2(x)\right. & \\ + \frac{\eta^3}{\sqrt{2}}(\phi'^\dagger(x) + \phi'(x)) + \eta^2\phi'^\dagger(x)\phi'(x) & \\ \left. + \sqrt{2}\eta(\phi'(x) + \phi'^\dagger(x))\phi'^\dagger(x)\phi'(x)\right]. & \end{aligned} \tag{5.28}$$

The terms linear in $\phi'(x)$ and $\phi'^\dagger(x)$ represent "spurion" interactions in which ϕ' can disappear into the vacuum. These are inadmissable since ϕ' has been adjusted to have a *vanishing* vacuum expectation value. Hence, a consistency condition to leading order is

$$m_0^2(\eta/\sqrt{2}) = f^2\eta^3/2\sqrt{2}. \tag{5.29}$$

This equation has two solutions:

(1)

$$\eta = 0. \tag{5.30}$$

This solution would take us back to the unbroken symmetry situation with its insoluable infrared difficulties and is, therefore, unacceptable.

(2)

$$\eta = \sqrt{2}\,\frac{m_0}{f} \tag{5.31}$$

This returns us to the linearized theory of the previous section so that we adopt it.

FIG. 6. The leading contribution to the ϕ' propagator correction.

The leading contribution to the ϕ' corrected propagators are given by the $\phi'^\dagger(x)\phi'(x)$ terms in $\mathcal{L}_I(x)$ (Fig. 6.) These are proportional to

$$m_0^2 - f^2\eta^2 = -m_0^2. \tag{5.32}$$

Hence, we have

$$P(q^2) = 2m_0^2/q^2 \tag{5.33}$$

and we recover the result of the linearized theory.

Going beyond this no-loop ("tree") approximation brings us up against the infrared problem in all its glory. Clearly, what is required is an iterative procedure in which first the tree graphs are summed to give the ϕ' and "photon" a lowest order mass, and then the radiative corrections are added on. An approach to this is given in a paper of Coleman and Weinberg (1973).[46] One fascinating result of their work is that even if $m_0 = 0$, the limiting case between the real and imaginary mass situation, the ϕ' will still acquire a mass because of ordinary electromagnetic radiative corrections provided that the theory is solved with the vacuum broken symmetry conditions. This result appears to resolve the problem of what significance one is to attach to the notion of a charged particle of zero mass. In particular, what is the limit of scalar boson electrodynamics when the boson bare mass m_0 is allowed to tend towards zero? For $m_0^2 > 0$ we have ordinary quantum electrodynamics which is a renormalizable theory. For $m_0^2 < 0$ we must interpret the theory à la Higgs. We no longer have electrodynamics, but rather the theory of a massive vector meson interacting with a Higgs scalar of finite mass. As Coleman and Weinberg show, if $m_0^2 \to 0$ from either above or below, the theory retains its Higgs form provided that one includes corrections of at least order α. Zero mass "electrodynamics" is neither the theory of zero mass particles nor is it electrodynamics, but it does appear to be perfectly sensible. The interested reader should consult the paper by Coleman & Weinberg for details. We now resume the line of development that will lead us to the unified theories of weak and electromagnetic interactions.

VI. NON-ABELIAN GAUGE SYMMETRIES

Up to this point in the discussion, we have been concerned with gauge transformations that produce c-number—that is, numerical—changes of phase of the "charged" fields. These changes of phase can be "global", i.e.,

$$\psi(x) \to \exp(i\alpha)\psi(x), \tag{6.1}$$

where α is a number independent of x, or they can be "local", i.e.,

$$\psi(x) \to \exp[i\alpha(x)]\psi(x). \tag{6.2}$$

Let us recapitulate what we have learned about the invariances of suitable Lagrangians under these transfor-

[46] See, for example, S. Coleman and E. Weinberg (1973) for an updated and complete discussion of these questions within the Higgs framework.

mations by means of a very simple example which we will soon generalize. Consider

$$\mathcal{L}_0(x) = \bar{\psi}(x)\gamma_\mu(\partial/\partial x_\mu)\psi(x), \qquad (6.3)$$

the Lagrangian for a free massless fermion field. Clearly, it is invariant under the *global* gauge transformation given in Eq. (6.1) and we may readily show that this implies the conservation of[47]

$$J_\mu(x) = i\bar{\psi}(x)\gamma_\mu\psi(x) \qquad (6.4)$$

It is *not* invariant under the *local* gauge transformation. Indeed, if we call

$$\psi(x)' = \exp[i\alpha(x)]\psi(x), \qquad (6.5)$$

then

$$\mathcal{L}_0(x) = \bar{\psi}(x)\gamma_\mu\frac{\partial}{\partial x_\mu}\psi(x)$$
$$= \bar{\psi}(x)'\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - i\frac{\partial}{\partial x_\mu}\alpha(x)\right)\psi'(x) \qquad (6.6)$$

which exhibits the lack of invariance explicitly. However, we can restore the invariance if we couple the $\psi(x)$ to a photon, i.e., a vector field with no bare mass, in the usual way:

$$\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_\gamma(x) + \mathcal{L}_I(x) = \bar{\psi}(x)\gamma_\mu(\partial/\partial x_\mu)\psi(x) + F(A_\mu(x)) - ie_0A^\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x), \qquad (6.7)$$

where we now demand that, along with the local phase transformation of the $\psi(x)$,

$$A^{\mu\prime}(x) = A^\mu(x) + (1/e_0)(\partial/\partial x_\mu)\alpha(x). \qquad (6.8)$$

We also assume that the function $F(A_\mu(x))$ is invariant under the transformation given by Eq. (6.8), i.e.,

$$F'(A_\mu(x)) = F(A_\mu(x)). \qquad (6.9)$$

This will be assured if, as before, F is a function of

$$\partial A^\mu(x)/\partial x_\nu - \partial A^\nu(x)/\partial x_\mu. \qquad (6.10)$$

Hence, we *might* say that if the photon did not exist we *might* have invented it to restore local gauge symmetry.

We now propose—following Yang and Mills (1954)—to generalize these considerations to *operator* phase transformations which, in general, will no longer commute among themselves. In this review we shall mainly restrict our concern to a relatively simple generalization, the $SU(2)$ group. But enough of the basics are involved here so that one may confront the literature with its growing profusion of groups without excessive difficulty. We will then break this symmetry with the Higgs mechanism also suitably generalized.

Let us then suppose we have two massless spinor fields $\psi_1(x)$ and $\psi_2(x)$ which we combine into the "doublet" spinor

$$\psi(x) = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}. \qquad (6.11)$$

The free Lagrangian of this system is

$$\mathcal{L}(x) = \bar{\psi}(x)\gamma_\mu(\partial/\partial x_\mu)\psi(x). \qquad (6.12)$$

Now let S be any 2×2 Hermitian matrix. Clearly, $\mathcal{L}(x)$ is invariant under all global phase transformations of the form

$$\psi(x) \to \exp[i\lambda S]\psi(x), \qquad (6.13)$$

where λ is a real number. However, *any* 2×2 Hermitian matrix can be written as

$$S = \boldsymbol{\alpha}\cdot(\tau/2) + I\alpha_0, \qquad (6.14)$$

where $\alpha_0 \cdots \alpha_3$ are real numbers, the $\boldsymbol{\tau}$ are the Pauli 2×2 spin matrices, and I is the 2×2 identity matrix. Henceforth, we shall define

$$\mathbf{T} = \tau/2 \qquad (6.15)$$

where

$$[T_i, T_j] = i\epsilon_{ijk}T_k. \qquad (6.16)$$

Now clearly the part of S that is proportional to I commutes with everything else so that it can be factored out of the exponential; i.e.,

$$\exp(i\lambda S) = \exp(i\lambda\alpha_0 I)\exp(i\lambda\boldsymbol{\alpha}\cdot\mathbf{T}). \qquad (6.17)$$

The first factor is simply a $U(1)$ Abelian gauge transformation of the type previously studied. The new work is contained in the factor $\exp(i\lambda\boldsymbol{\alpha}\cdot\mathbf{T})$. These are the $SU(2)$ transformations represented by the traceless 2×2 matrices $\mathbf{T}$. To make the rest of the work tractable, we consider

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}(x) \qquad (6.18)$$

where each of these components is taken as *infinitesimal*. Incorporating the λ into the definition of the $\boldsymbol{\alpha}$ we write

$$S \simeq 1 + i\boldsymbol{\alpha}(x)\cdot\mathbf{T}. \qquad (6.19)$$

With

$$\psi'(x) = (1 + i\boldsymbol{\alpha}(x)\cdot\mathbf{T})\psi(x) \qquad (6.20)$$

we have

$$\bar{\psi}(x)\gamma_\mu(\partial/\partial x_\mu)\psi(x) = \bar{\psi}'(x)\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - i\frac{\partial}{\partial x_\mu}\boldsymbol{\alpha}(x)\cdot\mathbf{T}\right)\psi'(x). \qquad (6.21)$$

[47] This is a special case of the following observation which can be generalized to non-Abelian transformations: Let

$$\psi(x) \to (1 + \Lambda(x))\psi(x),$$

where $\Lambda(x)$ is a real c number such that

$$\Lambda(x) \ll 1.$$

Then if $\mathcal{L}(x)$ is invariant under

$$\psi(x) \to (1 + \Lambda)\psi(x)$$

for *constant* Λ, the quantity

$$J_\mu(x) \cdot \equiv \cdot \frac{\delta\mathcal{L}'(x)}{\delta\dfrac{\partial\Lambda(x)}{\partial x^\mu}}\Bigg|_{\Lambda(x)=0},$$

where $\mathcal{L}'(x)$ is the transformed Lagrangian, is a *conserved* current as a consequence of the Euler-Lagrange equations. See, for example, M. Gell-Mann and M. Lévy (1960) for a more detailed discussion.

Hence, to restore the invariance, the previous example would suggest that we introduce a triplet of "photons", $\mathbf{b}_\mu(x)$, i.e., the Yang–Mills fields, by the prescription

$$(\partial/\partial x_\mu)\psi(x) \to ((\partial/\partial x_\mu) - ig\mathbf{b}_\mu(x)\cdot\mathbf{T})\psi(x). \quad (6.22)$$

We must now find how the $\mathbf{b}_\mu(x)$ transform to keep the newly constructed Lagrangian invariant. As above, let us define the local $SU(2)$ unitary transformation $S(x)$ by the equation

$$\psi'(x) = S(x)\psi(x). \quad (6.23)$$

Further, let us call the "covariant derivative"

$$\begin{aligned} D_\mu &= (\partial/\partial x^\mu) - ig\mathbf{b}_\mu(x)\cdot\mathbf{T} \\ \cdot &\equiv \cdot (\partial/\partial x^\mu) - igB_\mu(x), \end{aligned} \quad (6.24)$$

where the $B_\mu(x)$ are defined to be

$$B_\mu(x) \cdot \equiv \cdot \mathbf{b}_\mu(x)\cdot\mathbf{T}, \quad (6.25)$$

and have been introduced to simplify the writing. Thus, the covariance condition of the D_μ derived from the statement

$$\bar\psi(x)D_\mu\psi(x) = \bar\psi'(x)D'_\mu(x)\psi'(x) \quad (6.26)$$

is

$$(\partial/\partial x^\mu) - igB_\mu(x) = S^{-1}(x)[(\partial/\partial x^\mu) - igB'_\mu(x)]S(x) \quad (6.27)$$

or

$$igS^{-1}(x)B'_\mu(x)S(x) = igB_\mu(x) + S^{-1}(x)[\partial S(x)/\partial x^\mu] \quad (6.28)$$

or

$$B'_\mu(x) = S(x)B_\mu(x)S^{-1}(x) + (1/ig)[\partial S(x)/\partial x^\mu]S^{-1}(x). \quad (6.29)$$

It is the last term of the right-hand side of this equation which reflect the *local* character of the transformation. It is now a matter of some algebra to discover how various functions of $B_\mu(x)$ transform.

A very important observation in this respect is that, unlike the Abelian case, the quantity $(\partial/\partial x^\mu)B_\nu(x) - (\partial/\partial x^\nu)B_\mu(x)$ is *not* convariant. Indeed, after some straightforward algebra one finds instead that the tensor[48]

$$\begin{aligned} F_{\mu\nu}(x) &= (\partial/\partial x^\mu)B_\nu(x) - \frac{\partial}{\partial x^\nu}B_\mu(x) \\ &\quad - ig(B_\mu(x)B_\nu(x) - B_\nu(x)B_\mu(x)) \end{aligned} \quad (6.30)$$

enjoys the property that

$$F'_{\mu\nu}(x) = S(x)F_{\mu\nu}(x)S(x)^{-1}. \quad (6.31)$$

[48] Recall that since the **T** do not commute

$$B_\mu(x)B_\nu(x) \neq B_\nu(x)B_\mu(x).$$

Furthermore, from the definition of $B_\mu(x)$ and the familiar properties of the 2×2 spin matrices, we can show that

$$F_{\mu\nu}(x) = \mathbf{T}\cdot\mathbf{f}_{\mu\nu}(x), \quad (6.32)$$

where

$$\mathbf{F}_{\mu\nu}(x) = (\partial/\partial x^\mu)\mathbf{b}_\nu(x) - (\partial/\partial x^\nu)\mathbf{b}_\mu(x) + g\mathbf{b}_\mu(x)\times\mathbf{b}_\nu(x). \quad (6.33)$$

Finally, as we now know the transformation properties of $B_\mu(x)$, we can infer directly those of $\mathbf{b}_\mu(x)$. For an infinitesimal transformation of the form

$$S(x) \simeq 1 + i\boldsymbol{\alpha}(x)\cdot(\boldsymbol{\tau}/2), \quad (6.34)$$

$$\mathbf{b}'_\mu(x) = \mathbf{b}_\mu(x) + \mathbf{b}_\mu(x)\times\boldsymbol{\alpha}(x) + (1/g)(\partial/\partial x^\mu)\boldsymbol{\alpha}(x), \quad (6.35)$$

which is to say that $\mathbf{b}_\mu$ transforms like a conventional isovector under global gauge transformations, but that local gauge transformations add an additional term proportional to $(\partial/\partial x^\mu)\boldsymbol{\alpha}(x)$. However $\mathbf{f}_{\mu\nu}(x)$, as defined above, Eq. (6.33), *does* transform like an isovector since the extra term drops out, i.e.,

$$\mathbf{f}'_{\mu\nu}(x) = \mathbf{f}_{\mu\nu}(x) + \mathbf{f}_{\mu\nu}(x)\times\boldsymbol{\alpha}(x). \quad (6.36)$$

Putting all of these remarks together, we can construct the full invariant—against local $SU(2)$ transformations—Lagrangian involving fermions and Yang-Mills fields,

$$\begin{aligned} \mathfrak{L}(x) &= -\frac{1}{4}\left(\frac{\partial}{\partial x^\mu}\mathbf{b}_\nu(x) - \frac{\partial}{\partial x^\nu}\mathbf{b}_\mu(x) + g\mathbf{b}_\mu(x)\times\mathbf{b}_\nu(x)\right)^2 \\ &\quad - \bar\psi(x)\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}^\mu(x)\right)\psi(x). \end{aligned} \quad (6.37)$$

We could have included a mass term $\bar\psi(x)\psi(x)$ in $\mathfrak{L}(x)$ without disturbing the invariance, but for reasons which will soon be clear, we prefer not to. On the other hand, a term of the form $\mathbf{b}_\mu(x)\cdot\mathbf{b}^\mu(x)$, while invariant under global $SU(2)$, is *not* invariant under *local* $SU(2)$ and hence is excluded in $\mathfrak{L}(x)$.

We may arrange to break this symmetry à la Higgs by adding Higgs fields in close analogy to the Abelian case. However, here we shall add an $SU(2)$ doublet

$$\phi(x) = \begin{pmatrix} \phi^+(x) \\ \phi^0(x) \end{pmatrix}. \quad (6.38)$$

The significance of the +, 0 superscripts is the following: So long as we add a Higgs Lagrangian of the form

$$\begin{aligned} \mathfrak{L}_H(x) &= -\left(\frac{\partial}{\partial x^\mu} + ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}_\mu(x)\right)\phi^\dagger(x) \\ &\quad \left(\frac{\partial}{\partial x_\mu} - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}^\mu(x)\right)\phi(x) \\ &\quad + m_0^2\phi^\dagger(x)\phi(x) - \frac{1}{2}f^2(\phi^\dagger(x)\phi(x))^2, \end{aligned} \quad (6.39)$$

where

$$\phi^\dagger(x)\phi(x) = \phi^+(x)^\dagger\phi^+(x) + \phi^0(x)^\dagger\phi^0(x), \quad (6.40)$$

we preserve the local $SU(2)$ invariance of the total Lagrangian. However, as we saw at the end of Sec. IV, there is also a global phase invariance under

$$\psi(x) \to \exp(i\lambda)\psi(x), \quad (6.41a)$$

$$\phi(x) \to \exp(i\lambda)\phi(x), \quad (6.41b)$$

$$\mathbf{b}_\mu(x) \to \mathbf{b}_\mu(x). \quad (6.41c)$$

This implies the conservation of the current

$$Y_\mu(x) = i\left[\bar{\psi}(x)\gamma_\mu\psi(x) + \left(\frac{\partial}{\partial x^\mu} + ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}_\mu(x)\right)\phi^\dagger(x)\phi(x) - \phi^\dagger(x)\left(\frac{\partial}{\partial x^\mu} - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}_\mu(x)\right)\phi(x)\right]. \quad (6.42)$$

In addition, there is the conserved isotopic spin current whose existence follows from the invariance under

$$\psi'(x) = [1 + i\boldsymbol{\alpha}\cdot(\boldsymbol{\tau}/2)]\psi(x) \quad (6.43a)$$

$$\phi'(x) = [1 + i\boldsymbol{\alpha}\cdot(\boldsymbol{\tau}/2)]\phi(x) \quad (6.43b)$$

$$\mathbf{b}'_\mu(x) = \mathbf{b}_\mu(x) + \mathbf{b}_\mu(x)\times\boldsymbol{\alpha}. \quad (6.43c)$$

Thus, we find for the conserved isotopic spin current

$$\mathbf{T}_\mu(x) = i\bar{\psi}(x)\gamma_\mu\frac{\boldsymbol{\tau}}{2}\psi(x) + i\left[\left(\frac{\partial}{\partial x^\mu} + ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}_\mu(x)\right)\phi^\dagger(x)\frac{\boldsymbol{\tau}}{2}\phi(x) - \phi^\dagger(x)\frac{\boldsymbol{\tau}}{2}\left(\frac{\partial}{\partial x^\mu} - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}_\mu(x)\right)\phi(x)\right] + \mathbf{f}_{\mu\nu}(x)\times\mathbf{b}^\nu(x). \quad (6.44)$$

The second term in $\mathbf{T}_\mu(x)$ is a generalization of the familiar fact that the Klein–Gordon electromagnetic current involves the electromagnetic field itself, while the last term reflects the fact that the Yang–Mills field $\mathbf{b}_\mu(x)$ also carries isospin and must be included in the isotopic spin current. If we call the "electric charge"

$$Q = \tfrac{1}{2}\int d^3\mathbf{x}\, Y_0(\mathbf{x},t) + \int d^3\mathbf{x}\, T_0(\mathbf{x},t)_3 = (Y/2) + T_3, \quad (6.45)$$

then $\phi^+(x)$ carries charge $+1$, while $\phi^0(x)$ carries zero charge.

Since the field momentum conjugates to $\phi(x)$ is

$$\Pi(x) = [(\partial/\partial t) + ig\boldsymbol{\tau}\cdot\mathbf{b}_0(\mathbf{x},t)]\phi^\dagger(\mathbf{x},t) \quad (6.46)$$

we have

$$\int d^3\mathbf{x}[T_0(\mathbf{x},0),\phi(0)] = (\mathbf{t}/2)\phi(0). \quad (6.47)$$

Hence, if we are to have vacuum broken $SU(2)$ symmetry, we must have

$$\langle 0|\phi(0)|0\rangle = \eta \neq 0, \quad (6.48)$$

where η is, in general, a complex two-component column vector. Hence, we can displace as usual and write

$$\chi(x) = \phi(x) - \eta. \quad (6.49)$$

However, before doing this it is worthwhile to rotate, say, $\phi^+(x)$ out of the game. This will define the passage to the U-gauge. Under an $SU(2)$ transformation,

$$\phi'(\gamma) = \exp[i\boldsymbol{\alpha}(x)\cdot\boldsymbol{\tau}/2]\phi(x) = \left(\cos\left(\frac{\alpha(x)}{2}\right) + i\hat{\boldsymbol{\eta}}\cdot\boldsymbol{\tau}\sin\left(\frac{\alpha(x)}{2}\right)\right)\phi(x), \quad (6.50)$$

where

$$\boldsymbol{\alpha}(x) = |\boldsymbol{\alpha}(x)|\hat{\eta}, \quad (6.51)$$

and we have called $|\boldsymbol{\alpha}(x)|$ simply $\alpha(x)$. Thus,

$$\phi^{+\prime}(x) = \left[\cos\left(\frac{\alpha(x)}{2}\right) + i\sin\left(\frac{\alpha(x)}{2}\right)\eta_z\right]\phi^+(x) + i(\eta_x - i\eta_y)\sin\left(\frac{\alpha(x)}{2}\right)\phi^0(x) \quad (6.52a)$$

and

$$\phi^{0\prime}(x) = \left[\cos\left(\frac{\alpha(x)}{2}\right) - i\sin\left(\frac{\alpha(x)}{2}\right)\eta_z\right]\phi^0(x) + i(\eta_x + i\eta_y)\sin\left(\frac{\alpha(x)}{2}\right)\phi^+(x). \quad (6.52b)$$

Hence, if we rotate about the y direction we can achieve

$$\phi^{+\prime}(x) = 0 \quad (6.53)$$

if

$$\tan\left(\frac{\alpha(x)}{2}\right) = -\frac{(\phi^+(x)\phi^0(x)^\dagger + \phi^0(x)\phi^+(x)^\dagger \pm i(\phi^+(x)\phi^0(x)^\dagger - \phi^0(x)\phi^+(x)}{2|\phi^0(x)|^2}, \quad (6.54)$$

which reduces to the work of Sec. IV, Eq (4.105) when ϕ^+ and ϕ^0 are Hermitian. After rotation we can set

$$\langle 0|\phi^{0\prime}(0)|0\rangle = \eta \neq 0 \quad (6.55)$$

and from the time reversal condition

$$T\phi^{0\prime}(0)T^{-1} = \phi^{0\prime}(0) \quad (6.56)$$

conclude, as above, that

$$\eta = \eta^*. \quad (6.57)$$

We can define[49]

$$\chi(x) = \phi^{0\prime}(x) - \eta \quad (6.58)$$

and work out $\mathcal{L}_H(x)$ in this gauge in terms of these variables. Thus,

[49] The reader will notice that $\chi(x)$ is not Hermitian. A reader who has studied Weinberg (1967) may wonder why his surviving Higgs field is Hermitian and ours is not. The difference, as we shall see in the next section, is that Weinberg's group is $SU(2)\times U(1)$; i.e., he has an extra *local* Abelian gauge symmetry corresponding to hypercharge conservation. Hence, Weinberg can rotate the phase of $\phi^0(x)$ away after he has made his $SU(2)$ rotation, without spoiling the equations of motion.

$$\begin{aligned}\mathcal{L}_H(x) = &-\frac{\partial}{\partial x^\mu}\chi^\dagger(x)\frac{\partial}{\partial x_\mu}\chi(x)\\ &+ i\frac{g}{2}\Big[(\chi^\dagger(x)+\eta)b_z'(x)_\mu\frac{\partial}{\partial x_\mu}\chi(x)\\ &\qquad -\frac{\partial}{\partial x_\mu}\chi^\dagger(x)b_z'(x)_\mu(\chi(x)+\eta)\Big]\\ &-\frac{g^2}{4}\mathbf{b}_\mu'(x)\\ &\cdot\mathbf{b}^{\mu\prime}(x)[\chi(x)\chi^\dagger(x)+(\chi(x)+\chi^\dagger(x))\eta+\eta^2]\\ &+m_0^2(\chi(x)+\eta)^\dagger(\chi(x)+\eta)\\ &-\tfrac{1}{2}F^2[(\chi(x)+\eta)(\chi(x)+\eta)^\dagger]^2. \qquad (6.59)\end{aligned}$$

It is clear from the expression for $\mathcal{L}_H(x)$ that both the "photons" and the χ have acquired a mass. From the relation

$$\mathbf{b}_\mu(x)\cdot\mathbf{b}^\mu(x) = 2\frac{(\mathbf{b}_{\mu1}(x)+ib_{\mu2}(x))}{\sqrt{2}}\frac{(b_1^\mu(x)-ib_2^\mu(x))}{\sqrt{2}} + b_{\mu3}^2(x), \qquad (6.60)$$

we see that the "charged" Yang–Mills fields

$$b_\mu^+(x) = [b_{\mu1}(x)+ib_{\mu2}(x)]/\sqrt{2}, \qquad (6.61)$$

$$b_\mu^-(x) = [b_{\mu1}(x)-ib_{\mu2}(x)]/\sqrt{2}, \qquad (6.62)$$

acquire the same bare mass $g\eta/\sqrt{2}$ as $b_{\mu z}$; the "neutral" field.

The reader, given the development of Sec V, should have no difficulty in generalizing the argument there to compute the no-loop $\mathbf{b}_\mu$ vacuum polarization tensor à la Englert and Brout to obtain the same masses as those in the U gauge. This was actually done by Englert and Brout for the general compact lie group in their 1964 letter. In fact, the work of this chapter goes through essentially unchanged if, instead of

$$(\mathbf{b}_\mu\times\mathbf{b}_\nu)_i = \epsilon_{ijk}b_{\mu j}b_{\nu k}, \qquad (6.63)$$

we replace ϵ_{ijk} by a general "structure function" f_{ijk} of the Lie group, and instead of the isotopic matrices

$$\mathbf{T} = \boldsymbol{\tau}/2 \qquad (6.64)$$

we have the corresponding generalized "isotopic spin" algebra. Needless to say, "photons" and Higgs; fields will proliferate like rabbits, but the essential mathematics goes through as above.

We turn next to the weak interactions and Weinberg's 1967 model.

VII. WEINBERG'S 1967 MODEL

All theories of the weak interactions must incorporate two basic facts about neutrinos. The neutrino is massless and left-handed.[50] If $\nu(x)$ is the neutrino free field (we make no distinction between electron and muon neutrinos since both appear to be massless and left handed) we can express these facts by the pair of equations

$$\gamma_\mu(\partial/\partial x_\mu)\nu(x) = 0, \qquad (7.1)$$

and

$$\gamma_5\nu(x) = \nu(x). \qquad (7.2)$$

(We shall always use Hermitian γ-matrices.)

This latter equation is equivalent in momentum–space to the statement[51]

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{P}})\nu(P) = -\nu(P). \qquad (7.3)$$

Any solution to the Dirac equation, $\psi(x)$, can be written

$$\begin{aligned}\psi(x) &= \tfrac{1}{2}(1+\gamma_5)\psi(x)+\tfrac{1}{2}(1-\gamma_5)\psi(x)\\ &\cdot\equiv\cdot\ \psi^L(x)+\psi^R(x).\end{aligned} \qquad (7.4)$$

Clearly, $\psi^L(x)$ and $\psi^R(x)$ are eigenfunctions of γ_5 with eigenvalues ± 1. However, it is only if the particles are strictly massless that these states are also eigen-states of $\boldsymbol{\sigma}\cdot\hat{\mathbf{P}}$, i.e., have a definite handedness.[52]

There is no hope of unifying the description of electrons and neutrinos unless we begin from a situation in which initially they are *both* massless. We must, then, arrange things so that the neutrino remains massless while the electron acquires a mass. In constructing such models one is, at present, handicapped by an incomplete knowledge of the lepton spectrum. Are there heavy leptons—particles with the quantum numbers of electrons, muons and neutrinos, but with masses of hadronic magnitude—one or more GeV? One simply does not know. The Weinberg 1967 model has, in any case, the virtue of being among the unified gauge models of weak and electromagnetic interactions in which *no* new leptons are needed. (It can be generalized still remaining within the domain of the known leptons by adding the μ meson and, possibly, its neutrino in a fundamental multiplet as, for example,

$$\psi(x) = \begin{pmatrix}\mu^+(x)\\ \nu(x)\\ e^-(x)\end{pmatrix}, \qquad (7.5)$$

where $\nu(x)$ is formed out of

$$\nu_L(x)\cdot\equiv\cdot\ \nu_e(x), \qquad (7.6)$$

and

$$\nu_R(x)\cdot\equiv\cdot\ \nu_\mu^C(x) \qquad (7.7)$$

the muon anti-neutrino. This leads to an $SU(3)\times SU(3)$ gauge model with 16 real gauge fields as opposed to the $SU(2)\times U(1)$ model with four real gauge fields. The consequences of doing this are studied in S. Weinberg

[50] Needless to say, these statements are true only within experimental error. One must be open-minded enough to accept the possibility in principle—however unattractive—that the neutrino might have a small mass and might not be exactly left-handed.

[51] Here P is the unit momentum three-vector.

[52] Free Dirac particle with mass m and momentum p, of course, satisfies

$$(i\gamma p + m)\psi(p) = 0.$$

The statement follows from the identity in the Hermitian representation

$$i\gamma_4\gamma_5\boldsymbol{\gamma} = \begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}.$$

(1972a), which the reader may consult for details. The group theoretical aspects of this model were already worked out in an early paper by A. Salam and J. C. Ward (1964). (This was done prior to the introduction of the Higgs mechanism, so that the various masses were put in by hand and the renormalizability was obscure. In fact, without the Higgs mesons, the theory *cannot* be made finite.) We consider, following Weinberg (1967), the following object:

$$L(x) = \tfrac{1}{2}(1+\gamma_5)\begin{pmatrix} \nu(x) \\ e(x) \end{pmatrix} \tag{7.8}$$

where both $e(x)$ and $\nu(x)$ are four-component Dirac fields corresponding to zero bare mass. Now following the work of the last section, we can write a Lagrangian which is invariant under local $SU(2)$ transformations as follows:

$$\mathfrak{L}(x) = -\frac{1}{4}\left(\frac{\partial}{\partial x^\mu}\mathbf{b}_\nu(x) - \frac{\partial}{\partial x^\nu}\mathbf{b}_\mu(x) + g\mathbf{b}_\mu(x)\times\mathbf{b}_\nu(x)\right)^2 - \bar{L}(x)\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}^\mu(x)\right)L(x). \tag{7.9}$$

The conserved "isotopic spin" current is then

$$\mathbf{T}_\mu(x) = i\bar{L}(x)\gamma_\mu\frac{\boldsymbol{\tau}}{2}L(x) + \mathbf{f}_{\mu\nu}(x)\times\mathbf{b}^\nu(x). \tag{7.10}$$

Because an expression like this may look unfamiliar, it is worthwhile to examine it somewhat more carefully. Let us look first at

$$\begin{aligned}\boldsymbol{\mathcal{J}}_\mu(x) &= \mathbf{f}_{\mu\nu}(x)\times\mathbf{b}^\nu(x)\\ &= [(\partial/\partial x^\mu)\mathbf{b}_\nu(x) - (\partial/\partial x^\nu)\mathbf{b}_\mu(x) + g\mathbf{b}_\mu(x)\times\mathbf{b}_\nu(x)]\\ &\quad\times\mathbf{b}^\nu(x).\end{aligned} \tag{7.11}$$

Of special interest is $\boldsymbol{\mathcal{J}}_4(x)$, i.e.,

$$\boldsymbol{\mathcal{J}}_4(x) = \left(\frac{\partial}{\partial x^4}\mathbf{b}^\nu(x) - \frac{\partial}{\partial x^\nu}\mathbf{b}_4(x) + g\mathbf{b}_4(x)\times\mathbf{b}_\nu(x)\right)\times\mathbf{b}^\nu(x). \tag{7.12}$$

If we quantize in the radiation gauge with

$$\boldsymbol{\nabla}\cdot\mathbf{b}(x) = 0, \tag{7.13}$$

then, noting that the canonical momentum to $\mathbf{b}_\mu(x)$ is

$$\boldsymbol{\Pi}_\mu(x) = \delta\mathfrak{L}(x)/\delta\dot{\mathbf{b}}_\mu = \dot{\mathbf{b}}_\mu(x) - g\mathbf{b}_\mu(x)\times\mathbf{b}_0(x), \tag{7.14}$$

we have the equal time commutation relation

$$[b_s(\mathbf{x},t)_i, \Pi_t(\mathbf{x}',t)_j] = i\delta_{ij}\delta^{\mathrm{Tr}}_{st}(\mathbf{x}-\mathbf{x}') \tag{7.15}$$

where $\delta^{\mathrm{Tr}}_{st}(\mathbf{x}-\mathbf{x}')$ is the transverse δ function obeying

$$\frac{\partial}{\partial x_s}\delta^{\mathrm{Tr}}_{st}(\mathbf{x}-\mathbf{x}') = 0. \tag{7.16}$$

Thus,

$$\begin{aligned}\boldsymbol{\tau}(t) &= \int d^3\mathbf{x}\,\boldsymbol{\mathcal{J}}_0(\mathbf{x},t)\\ &= \int d^3\mathbf{x}\,\boldsymbol{\Pi}_\nu(\mathbf{x},t)\times\mathbf{b}^\nu(\mathbf{x},t)\\ &\quad - \int d^3\mathbf{x}\left(\frac{\partial}{\partial x_i}\mathbf{b}_0(\mathbf{x},t)\times\mathbf{b}^i(\mathbf{x},t)\right).\end{aligned} \tag{7.17}$$

We may integrate by parts over space on the last term and use the radiation gauge condition to drop it. Thus, we have simply

$$\boldsymbol{\tau}(t) = \int d^3\mathbf{x}\,\boldsymbol{\Pi}_\nu(\mathbf{x},t)\times\mathbf{b}^\nu(\mathbf{x},t), \tag{7.18}$$

since

$$\partial/\partial x_\mu\boldsymbol{\mathcal{J}}_\mu(x) \neq 0$$

in the interacting case $\boldsymbol{\tau}(t)$ will be a function of time. From the canonical commutation relations we have

$$[\boldsymbol{\tau}(t)_i, \boldsymbol{\tau}(t)_j] = i\epsilon_{ijk}\boldsymbol{\tau}(t)_k. \tag{7.19}$$

We may next study

$$\boldsymbol{\mathcal{J}}'_\mu = i\bar{L}(x)\gamma_\mu\frac{\boldsymbol{\tau}}{2}L(x) = i(\bar{\nu}(x)\bar{e}(x))\gamma_\mu\frac{\boldsymbol{\tau}}{2}\left(\frac{1+\gamma_5}{2}\right)\begin{pmatrix}\nu(x)\\ e(x)\end{pmatrix}. \tag{7.20}$$

We shall want to consider

$$\begin{aligned}\boldsymbol{\tau}'(t) &= \int d^3\mathbf{x}\,\boldsymbol{\mathcal{J}}'_0(\mathbf{x},t)\\ &= \int d^3\mathbf{x}(\nu^\dagger(\mathbf{x},t)e^\dagger(\mathbf{x},t))\frac{\boldsymbol{\tau}}{2}\frac{(1+\gamma_5)}{2}\begin{pmatrix}\nu(\mathbf{x},t)\\ e(\mathbf{x},t)\end{pmatrix}.\end{aligned} \tag{7.21}$$

The following identity will come in handy. Let Γ and Γ' be any two 4×4 matrices—each is some linear combination of the 16 independent matrices in the Dirac algebra—and let $\psi(x)$ and $\psi'(x)$ be any four-component Dirac fields then using the equal-time anticommutation relations of the ψ and ψ'[53]

$$\begin{aligned}&[\psi^\dagger_\alpha(\mathbf{x},t)\Gamma_{\alpha\beta}\psi_\beta(\mathbf{x},t), \psi'^\dagger_{\alpha'}(\mathbf{x}',t)\Gamma'_{\alpha'\beta'}\psi'_{\beta'}(\mathbf{x}',t)]\\ &= \delta^3(\mathbf{x}-\mathbf{x}')\\ &\quad\{\psi^\dagger_\alpha(\mathbf{x},t)\psi_{\beta'}(\mathbf{x}',t)\Gamma_{\alpha\beta}\Gamma'_{\beta\beta'}\\ &\quad - \psi^\dagger_{\alpha'}(\mathbf{x}',t)\psi_\beta(\mathbf{x},t)\Gamma_{\alpha\beta}\Gamma'_{\alpha'\alpha}\},\end{aligned} \tag{7.22}$$

if ψ and ψ' are the *same* fields and otherwise the commutator is *zero*. If we let $\psi(x)$ be the eight-component field

$$\psi(x) = \begin{pmatrix}\nu(x)\\ e(x)\end{pmatrix}, \tag{7.23}$$

we then see that

$$\left[\psi^\dagger(\mathbf{x},t)\frac{(1+\gamma_5)}{2}\frac{\tau_i}{2}\psi(\mathbf{x},t), \psi^\dagger(\mathbf{x}',t)\frac{(1+\gamma_5)}{2}\frac{\tau_j}{2}\psi(\mathbf{x}',t)\right] \tag{7.24}$$

$$= \delta^3(\mathbf{x}'-\mathbf{x})\left[\psi^\dagger(\mathbf{x},t)\left(\frac{1+\gamma_5}{2}\right)\left(\frac{\tau_i\tau_j - \tau_j\tau_i}{2}\right)\psi(\mathbf{x}',t)\right] \tag{7.25}$$

or

$$[\boldsymbol{\tau}'(t)_i, \boldsymbol{\tau}'(t)_j] = i\epsilon_{ijk}\boldsymbol{\tau}'(t)_k. \tag{7.26}$$

[53] That is,

$$[\psi(\mathbf{x},t), \psi'^\dagger(\mathbf{x}',t)]_+ = \delta^3(\mathbf{x}-\mathbf{x}')$$

if ψ and ψ' are the same field and

$$[\psi(\mathbf{x},t), \psi'^\dagger(\mathbf{x}',t)]_+ = 0$$

if ψ and ψ' are distinct fields.

Hence, with

$$\mathbf{T} = \boldsymbol{\tau}(t) + \boldsymbol{\tau}'(t) \tag{7.27}$$

and remembering $\mathbf{T}$ *is* independent of time, we have

$$[T_i, T_j] = i\epsilon_{ijk} T_k. \tag{7.28}$$

Moreover, the *state*

$$|L(P)\rangle = \lim_{t\to\pm\infty} \int d^3\mathbf{x}\psi^\dagger(\mathbf{x},t) \times \frac{\exp[i(px)]}{(2\pi)^{3/2}} \begin{pmatrix} \left(\frac{1+\gamma_5}{2}\right)\nu(P) \\ \frac{(1+\gamma_5)}{2} e(P) \end{pmatrix} |0\rangle \tag{7.29}$$

has the property that

$$T_i|L(P)\rangle = \frac{\tau_i}{2}|L(P)\rangle \tag{7.30}$$

provided that

$$T_i|0\rangle = 0. \tag{7.31}$$

Hence, we see that $|L(P)\rangle$ is a doublet representation of $SU(2)$, while T_i is the $SU(2)$ generator.

Even if we were to confine ourselves to the weak interactions alone, we cannot stop here. In the first place, our $SU(2)$ invariant Lagrangian is both γ_5 invariant and parity conserving, so that, in particular, the electron would never acquire a mass. Therefore, we must give the electron a right-handed component to go along with its left hand. Thus, we define

$$|R(P)\rangle = \lim_{t\to\pm\infty} \int d^3\mathbf{x} e^\dagger(\mathbf{x},t) \frac{\exp[i(px)]}{(2\pi)^{3/2}} \frac{(1-\gamma_5)}{2} e(P)|0\rangle. \tag{7.32}$$

Using the assembled technology along with the identity

$$(1+\gamma_5)(1-\gamma_5) = 0, \tag{7.33}$$

we easily show that $|R(P)\rangle$ is an $SU(2)$ singlet, i.e.,

$$T_i|R(P)\rangle = 0. \tag{7.34}$$

Hence, the Lagrangian, with $R(x) \cdot \equiv \cdot [(1-\gamma_5)/2]e(x)$,

$$\begin{aligned}\mathfrak{L}(x) = &-\frac{1}{4}\left(\frac{\partial}{\partial x^\mu}\mathbf{b}_\nu(x) - \frac{\partial}{\partial x^\nu}\mathbf{b}_\mu(x) + g\mathbf{b}_\mu(x)\times\mathbf{b}_\nu(x)\right)^2 \\ &- \bar{L}(x)\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}^\mu(x)\right)L(x) \\ &- \bar{R}(x)\gamma_\mu\frac{\partial}{\partial x_\mu}R(x),\end{aligned} \tag{7.35}$$

is *also* invariant under *local* $SU(2)$ transformations. We could, of course, break this symmetry à la Higgs but we would not get electromagnetism. The reason is that all of the $\mathbf{b}^\mu(x)$ will acquire the same mass—there would be no massless photon. This can be traced back to the fact that the physical photon is *not* a member of an $SU(2)$ triplet. Put in a slightly different way, the electric charge is, in general, not simply proportional to the third component of the isotopic spin, but is rather a linear combination of the isospin and hypercharge. This suggests that we enlarge the local invariance group to include a local $U(1)$ invariance associated with the hypercharge. This will mean introducing an additional "photon" which we will call $a_\mu(x)$; it is not to be confused with the true photon $A_\mu(x)$ which will emerge, in due course, from the wash. Thus, the new Langrangian invariant under local $SU(2) \times U(1)$ transformations is

$$\begin{aligned}\mathfrak{L}(x) = &-\frac{1}{4}\left(\frac{\partial}{\partial x^\mu}\mathbf{b}_\nu(x) - \frac{\partial}{\partial x^\nu}\mathbf{b}_\mu(x) + g\mathbf{b}_\mu(x)\times\mathbf{b}_\nu(x)\right)^2 \\ &- \frac{1}{4}\left(\frac{\partial}{\partial x^\mu}a_\nu - \frac{\partial}{\partial x^\nu}a_\mu(x)\right)^2 \\ &- \bar{R}(x)\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - ig'a^\mu(x)\right)R(x) \\ &- \bar{L}(x)\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - i\frac{g'}{2}a^\mu(x) - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}^\mu(x)\right)L(x).\end{aligned} \tag{7.36}$$

There are several points to be made in connection with this Lagrangian and, in particular, we wish to understand the origin of the $g'/2$ which appears in the last term.

Let us call[55]

$$\begin{aligned}N_L &= \int d^3\mathbf{x}\, L^\dagger(\mathbf{x},t)L(\mathbf{x},t) \\ &= \int d^3\mathbf{x}\{e^\dagger(x)[(1+\gamma_5)/2]e(x) + \nu^\dagger(x)[(1+\gamma_5)/2]\nu(x)\},\end{aligned} \tag{7.37}$$

and

$$N_R = \int R^\dagger(\mathbf{x},t)R(\mathbf{x},t)\,d^3\mathbf{x} = \int d^3\mathbf{x} e^\dagger(x)[(1-\gamma_5)/2]e(x). \tag{7.38}$$

Note that the operator

$$\begin{aligned}&\tau_3' - N_R - \tfrac{1}{2}N_L \\ &= \int d^3\mathbf{x}\Big[\nu^\dagger(x)\left(\frac{1+\gamma_5}{4}\right)\nu(x) \\ &\quad - e^\dagger(x)\left(\frac{1+\gamma_5}{4}\right)e(x) - e^\dagger(x)\left(\frac{1-\gamma_5}{2}\right)e(x) \\ &\quad - \tfrac{1}{2}e^\dagger(x)\left(\frac{1+\gamma_5}{2}\right)e(x) - \tfrac{1}{2}\nu^\dagger(x)\left(\frac{1+\gamma_5}{2}\right)\nu(x)\Big]\end{aligned} \tag{7.39}$$

where, in the absence of the Yang–Mills coupling, Q is the total electric charge. If we call Y the leptonic hypercharge, then to conform to the notation of the last section

$$Q = \tau_3' + (Y/2), \tag{7.40}$$

[54] Here, $\nu(p)$ and $e(p)$ are solutions of the free-particle Dirac equation for massless particles.

[55] Note, incidentally, that

$$[R(\mathbf{x},t), L^\dagger(\mathbf{x}',t)]_+ = 0.$$

where[56]

$$y = -2N_R - N_L. \tag{7.41}$$

Thus, if the hypercharge "photon" $a_\mu(x)$ couples to $R(x)$ with a coupling $|g'|$, it must couple to $L(x)$ with a coupling $(|g'|/2)$. The signs and over-all scale have been chosen to conform to the notation of Weinberg (1967). It is, by the way, straightforward to show that

$$[Y(t), \boldsymbol{\tau}'(t)] = 0 \tag{7.42}$$

so that the $SU(2)$ and $U(1)$ groups are independent and commuting. Before we go on to break the $SU(2) \times U(1)$ symmetry, we would like to fish the real photon out of this soup so we can keep track of how it remains massless and how its interactions maintain parity conservation. To this end, we write out the coupling of $a_\mu(x)$ and $b_3(x)$ to $R(x)$ and $L(x)$ in detail; i.e.,

$$\begin{aligned} &ig'\bar{R}(x)\gamma_\mu a^\mu(x) R(x) \\ &\quad + i\bar{L}(x)\left[\gamma_\mu\left(\frac{g'}{2}a^\mu(x) + g\frac{t_3}{2}b_3^\mu(x)\right)\right]L(x) \\ &\quad = i\bar{e}(x)\gamma_\mu\left[g'a^\mu(x)\left(\frac{1-\gamma_5}{2}\right) + \frac{g'}{2}a^\mu(x)\left(\frac{1+\gamma_5}{2}\right)\right. \\ &\qquad \left. - \frac{g}{2}b_3^\mu(x)\left(\frac{1+\gamma_5}{2}\right)\right]e(x) \\ &\quad + i\bar{\nu}(x)\gamma_\mu\left[\frac{g'}{2}a^\mu(x) + \frac{g}{2}b_3^\mu(x)\right]\frac{(1+\gamma_5)}{2}\nu(x). \end{aligned} \tag{7.43}$$

To fish the photon out we shall demand that *only* its neutral partner, which after Weinberg (1967), we shall call $Z_\mu(x)$, couples to neutrinos so as to preserve both its electromagnetic neutrality and the $(1 + \gamma_5)/2$ character of the weak couplings; i.e., we must have, since we exclude a direct neutrino–photon coupling

$$Z^\mu(x) = \frac{c}{2}(g'a^\mu(x) + gb_3^\mu(x)), \tag{7.44}$$

where c is an over-all normalization factor. We can determine c from the requirement that $Z^\mu(x)$ be a canonical field obeying the equal time commutation relation—say, in the Coulomb gauge—

$$[Z_s(\mathbf{x}, t), \Pi_t(\mathbf{x}', t)] = i\delta_{st}^{\mathrm{Tr}}(\mathbf{x} - \mathbf{x}'). \tag{7.45}$$

Thus, using the canonical independence of a_μ and $\mathbf{b}_\mu$.

$$(c^2/4)(g'^2 + g^2) = 1. \tag{7.46}$$

Thus,

$$Z^\mu(x) = [1/(g'^2 + g^2)^{\frac{1}{2}}][g'a^\mu(x) + gb_3^\mu(x)]. \tag{7.47}$$

Let us call the photon—the real one—

$$A^\mu(x) = aa^\mu(x) + bb_3^\mu(x). \tag{7.48}$$

From the canonical requirements, including the canonical independence of A^μ and Z^μ, we have

$$a^2 + b^2 = 1 \tag{7.49}$$

and

$$g'a + bg = 0, \tag{7.50}$$

or, solving,

$$A^\mu(x) = [1/(g^2 + g'^2)^{\frac{1}{2}}](ga^\mu(x) - g'b_3^\mu(x)) \tag{7.51}$$

or, inverting,

$$(g^2 + g'^2)^{-\frac{1}{2}}(g'Z^\mu(x) + gA^\mu(x)) = a^\mu(x) \tag{7.53}$$

and

$$(g^2 + g'^2)^{-\frac{1}{2}}(gZ^\mu(x) - g'A^\mu(x)) = b_3^\mu(x). \tag{7.54}$$

Thus,

$$\begin{aligned} &ig'\bar{R}(x)\gamma_\mu a^\mu(x) R(x) \\ &\quad + i\bar{L}(x)\left[\gamma_\mu\left(\frac{g'}{2}a^\mu(x) + g\frac{\tau_3}{2}b_3^\mu(x)\right)\right]L(x) \\ &\quad = \frac{(g^2+g'^2)^{\frac{1}{2}}}{2}\, i\bar{\nu}(x)\gamma_\mu\frac{(1+\gamma_5)}{2}\nu(x)Z^\mu(x) \\ &\quad + \frac{gg'}{(g^2+g'^2)^{\frac{1}{2}}}\, i\bar{e}(x)\gamma_\mu e(x)A^\mu(x) \\ &\quad + i\bar{e}(x) \\ &\quad \left[\gamma_\mu\frac{(3g'^2 - g^2)}{4(g^2+g'^2)^{\frac{1}{2}}} - \gamma_\mu\gamma_5\frac{(g^2+g'^2)^{\frac{1}{2}}}{4}\right]e(x)Z^\mu(x) \\ &\quad \cdot \equiv \cdot -\mathfrak{L}_I^0(x). \end{aligned}$$

It is very interesting to note that, in this model, the requirements that the neutrino be electrically neutral and that A_μ and Z_μ by dynamically independent have yielded a photon coupling which depends only on γ_μ, i.e., is both charge conjugation and parity conserving. The rationalized bare electric charge is then given by

$$e = gg'/(g^2 + g'^2)^{1/2}. \tag{7.55}$$

In addition to $\mathfrak{L}_I^0(x)$, there is also $\mathfrak{L}_I^+(x)$ defined by

$$\begin{aligned} -\mathfrak{L}_I^+(x) \cdot \equiv \cdot\, & i\frac{g}{2}\bar{L}(x)\gamma_\mu[\tau_1 b_1^\mu(x) + \tau_2 b_2^\mu(x)]L(x) \\ = \, & i\frac{g}{2}\bar{L}(x)\gamma_\mu[\tau_+(b_1^\mu(x) - ib_2^\mu(x)) \\ & + \tau_-(b_1(x)^\mu + ib_2^\mu(x))]L(x), \end{aligned} \tag{7.56}$$

where

$$\tau_+ = \tau_-^\dagger = (\tau_1 + i\tau_2)/2 \tag{7.57}$$

or

$$\begin{aligned} -\mathfrak{L}_I^+(x) = i\frac{g}{2}\Big[&\bar{\nu}(x)\gamma_\mu\frac{(1+\gamma_5)}{2}e(x)(b_1^\mu(x) - ib_2^\mu(x) \\ &+ \bar{e}(x)\gamma_\mu\frac{(1+\gamma_5)}{2}\nu(x)(b_1^\mu(x) + ib_2^\mu(x))\Big]. \end{aligned} \tag{7.58}$$

If we call

$$W_\mu^+(x) = (W_\mu^-(x))^\dagger \cdot \equiv \cdot [b_{1\mu}(x) + ib_{2\mu}(x)]/\sqrt{2} \tag{7.59}$$

we have finally

[56] Our notation differs somewhat from Weinberg (1967). It is a matter of choice where one includes the signs and over-all factors. We prefer to use an integral hypercharge, whereas other authors call the hypercharge the mean multiplet charge. For doublets this is $\pm\frac{1}{2}$. So long as we are internally consistent, the final physics will be the same. Our coupling constants g and g' are chosen so that our final Lagrangian agrees with Weinberg (1967).

$$-\mathfrak{L}_I^+(x) = \frac{ig}{2\sqrt{2}}[\bar{\nu}(x)\gamma_\mu(1+\gamma_5)e(x)W^{-\mu}(x) + \bar{e}(x)\gamma_\mu(1+\gamma_5)\nu(x)W^{+\mu}(x)]. \quad (7.60)$$

We have been through this in such detail because otherwise the coupling constant factors in Weinberg (1967) appear very mysterious. In particular, the definition of g introduced above ultimately depends on the $SU(2)$ local gauge invariance. Indeed, if one goes back to the more ancient literature on this subject,[57] one will find the coupling constant g_A—"A" for "ancient"—introduced via the expression

$$g_A W^{+\mu}(x)\bar{e}(x)\gamma_\mu(1+\gamma_5)\nu(x) + \text{h.c.} \quad (7.61)$$

so that

$$g_A = g/2\sqrt{2}. \quad (7.62)$$

One must keep this in mind when comparing formulae.

We wish next to rewrite the vector meson part of the Lagrangian in terms of Z_μ, A_μ and $W_\mu^\pm$. This will fix the electromagnetic and weak interaction properties of the vector mesons themselves. Let us call

$$a_{\mu\nu}(x) \cdot \equiv \cdot (\partial/\partial x^\mu)a_\nu(x) - (\partial/\partial x_\nu)a_\mu(x) \quad (7.63)$$

$$A_{\mu\nu}(x) \cdot \equiv \cdot (\partial/\partial x_\mu)A_\nu(x) - (\partial/\partial x_\nu)A_\mu(x) \quad (7.64)$$

$$Z_{\mu\nu}(x) \cdot \equiv \cdot (\partial/\partial x^\mu)Z_\nu(x) - (\partial/\partial x^\nu)Z_\mu(x) \quad (7.65)$$

and

$$\mathbf{b}_{\mu\nu}(x) = (\partial/\partial x^\mu)\mathbf{b}_\nu(x) - (\partial/\partial x^\nu)\mathbf{b}_\mu(x) \quad (7.66)$$

which is, of course, not the complete Yang–Mills tensor but only that part valid when $g = 0$. We will produce the remainder later. Thus, we find

$$(a_{\mu\nu}(x))^2 = (g^2+g'^2)^{-1} [g'^2(Z_{\mu\nu}(x))^2 + g^2(A_{\mu\nu}(x))^2 + 2gg'Z_{\mu\nu}(x)A^{\mu\nu}(x)] \quad (7.67)$$

while

$$(\mathbf{b}_{\mu\nu}(x))^2 = 2\left(\frac{\partial}{\partial x^\mu}W_\nu^+(x) - \frac{\partial}{\partial x^\nu}W_\mu^+(x)\right) \times \left(\frac{\partial}{\partial x_\mu}W^{-\nu}(x) - \frac{\partial}{\partial x_\nu}W^{-\mu}(x)\right) + \frac{1}{(g^2+g'^2)}[g^2(Z_{\mu\nu}(x))^2 + g'^2(A_{\mu\nu}(x))^2 - 2gg'Z_{\mu\nu}(x)A^{\mu\nu}(x)]. \quad (7.68)$$

Hence, the kinetmatic part of the vector meson Lagrangian, $\mathfrak{L}_0^v(x)$, is given by

$$\mathfrak{L}_0^v(x) = -\tfrac{1}{4}[(a_{\mu\nu}(x))^2 + (\mathbf{b}_{\mu\nu}(x))^2] = -\tfrac{1}{4}\Big[(Z_{\mu\nu}(x))^2 + (A_{\mu\nu}(x))^2 + 2\left(\frac{\partial}{\partial x^\mu}W_\nu^+(x) - \frac{\partial}{\partial x^\nu}W_\mu^+(x)\right) \times \left(\frac{\partial}{\partial x_\mu}W^{-\nu}(x) - \frac{\partial}{\partial x_\nu}W^{-\mu}(x)\right)\Big], \quad (7.69)$$

[57] For example, R. P. Feynman and M. Gell-Mann (1958) or T. D. Lee and C. N. Yang (1960).

while a calculation using the antisymmetry of the $A_{\mu\nu}$ etc. to get the numerical factors straight shows

$$\begin{aligned}\mathfrak{L}_I^v = &-\frac{g}{4}\{2\mathbf{b}_\mu(x)\times\mathbf{b}_\nu(x)\cdot\mathbf{b}^{\mu\nu}(x) + g(\mathbf{b}_\mu(x)\times\mathbf{b}_\nu(x)\cdot\mathbf{b}^\mu(x)\times\mathbf{b}^\nu(x))\}\\
= &-\frac{ig}{(g^2+g'^2)^{1/2}}\Big\{g'A^\mu(x) \times\Big[W^{-\nu}(x)\left(\frac{\partial}{\partial x^\mu}W_\nu^+(x) - \frac{\partial}{\partial x^\nu}W_\mu^+(x)\right) - W^{+\nu}(x)\left(\frac{\partial}{\partial x^\mu}W_\nu^-(x) - \frac{\partial}{\partial x^\nu}W_\mu^-(x)\right)\Big]\\
&- g'A^{\mu\nu}(x)W_\mu^+(x)W_\nu^-(x) + gZ^\mu(x)\Big[W^{+\nu}(x)\left(\frac{\partial}{\partial x^\mu}W_\nu^-(x) - \frac{\partial}{\partial x^\nu}W_\mu^-(x)\right) - W^{-\nu}(x)\left(\frac{\partial}{\partial x^\mu}W_\nu^+(x) - \frac{\partial}{\partial x^\nu}W_\mu^+(x)\right)\Big]\\
&+ gZ^{\mu\nu}(x)W_\mu^+(x)W_\nu^-(x)\Big\} \qquad (7.70)\\
&-\frac{g^2}{g^2+g'^2}\{g^2[(Z_\mu(x))^2W_\nu^+(x)W^{-\nu}(x) - Z_\mu(x)W^{-\mu}(x)Z_\nu(x)W^{+\nu}(x)]\\
&+ g'^2[(A_\mu(x))^2W_\nu^+(x)W^{-\nu}(x) + A_\mu(x)W^{-\mu}(x)A_\nu(x)W^{+\nu}(x)]\\
&+ g'g[A_\mu(x)W^{-\nu}(x)Z_\nu(x)W^{+\nu}(x) + Z_\mu(x)W^{-\mu}(x)A_\nu(x)W^{+\nu}(x) - 2Z_\mu(x)A^\mu(x)W_\nu^+(x)W^{-\nu}(x)]\}\\
&-\frac{g^2}{2}[(W_\mu^-(x)W^{+\mu}(x))^2 - (W_\mu^-(x))^2(W_\nu^+(x))^2].\end{aligned}$$

Clearly, someone would have to be a True Believer before undertaking detailed computations—especially in higher orders—with a Lagrangian that is so complicated.[58] We shall elucidate some of its virtues after we have broken the symmetry—the $SU(2)\times U(1)$ symmetry—by the Higgs mechanism. Hence, we add to the rest, the Higgs Lagrangian $\mathfrak{L}_H(x)$ with

[58] Indeed, as far as the author knows, no one attempted such calculations until 1971 when Hooft (1971) derived the Feynman rules for a variety of gauge theories including the vacuum broken $SU(2)\times U(1)$. He attempted a proof, using Ward identities, that this model was renormalizable. His original argument neglected the problem of anomalies—see Footnote 46—which was clarified by a variety of authors later. See Zumino (1972) for a review and a full list of references. Professor A. Salam (1969) was certainly an early true believer, but did not, as far as the author knows, publish a detailed argument to demonstrate the renormalizability. Following Hooft's reopening of the renormalizability question, Weinberg (1971) showed how, in several physically interesting examples, his 1967 model resolved many of the problems of convergence and unitarity which had beset the prior weak interaction theories based on intermediate vector mesons. The renormalizability of the theory has now been demonstrated to most people's satisfaction to all orders in perturbation theory—see, for example, B. W. Lee (1972) and the references contained therein. The theories appear to be *consistent*. It now remains to be seen if they are *true*.

$$\mathcal{L}_H(x) = \left(\frac{\partial}{\partial x^\mu} + ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}_\mu(x) - i\frac{g'}{2}a_\mu(x)\right)\phi^\dagger(x) \times \left(\frac{\partial}{\partial x_\mu} - ig\frac{\boldsymbol{\tau}}{2}\cdot b^\mu(x) + i\frac{g'}{2}a^\mu(x)\right)\phi(x) + m_0^2\phi^\dagger(x)\phi(x) - \tfrac{1}{2}f^2(\phi^\dagger(x)\phi(x))^2, \quad (7.71)$$

where

$$\phi(x) = \begin{pmatrix}\phi^+(x)\\ \phi^0(x)\end{pmatrix}. \quad (7.72)$$

Here again we must clarify the origin of the new terms in $\mathcal{L}_H(x)$ which are proportional to g'. As before we let

$$Q = T_3 + Y/2 \quad (7.73)$$

so that the hypercharge assignments to ϕ^+ and ϕ^0 are $+1$. (Both members of the isotopic doublet have the *same* hypercharge since the $SU(2)$ and $U(1)$ generators commute.) In contrast, the $SU(2)$ doublet L has hypercharge -1 reflecting the fact that the negatively electrically charged electron is the "particle", while the positively charged positron is the antiparticle. The $SU(2)$ singlet R has hypercharge -2. The hypercharge of the vector mesons is zero, which is reflected in the fact that the coupling constant g' does not occur in the Yang–Mills Lagrangian. Hence, the covariant derivatives of the $U(1)$ gauge group become in the three cases, in terms of the coupling constant g' chosen to agree with Weinberg (1967),

$$D_\mu R(x) = ((\partial/\partial x^\mu) - ig'a_\mu(x))R(x), \quad (7.74)$$

$$D_\mu L(x) = ((\partial/\partial x^\mu) - i\frac{g'}{2}a_\mu(x))L(x) \quad (7.75)$$

and

$$D_\mu\phi(x) = ((\partial/\partial x^\mu) + i(g'/2)a_\mu(x))\phi(x). \quad (7.76)$$

The reason for exhibiting this in such detail is made evident in the next step when we ask which of the two neutral vector mesons acquires a mass in the U-gauge when the $SU(2)\times U(1)$ symmetry is vacuum broken. To this end, note from the structure of $\mathcal{L}_H(x)$ the full covariant derivative

$$\left(\frac{\partial}{\partial x^\mu} - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}_\mu(x) + i\frac{g'}{2}a_\mu(x)\right)\begin{pmatrix}\phi^+(x)\\ \phi^0(x)\end{pmatrix} = \left[\frac{\partial}{\partial x^\mu} - \frac{ig}{\sqrt{2}}(\tau_+W_\mu^-(x) + \tau_-W_\mu^+(x)) + \frac{i}{2}(g'a_\mu(x) - g\tau_3 b_{\mu 3}(x))\right]\begin{pmatrix}\phi^+(x)\\ \phi^0(x)\end{pmatrix}. \quad (7.77)$$

Following the steps of the last section, we can rid ourselves of $\phi^+(x)$ by making the $SU(2)$ transformation defined by

$$\tan\left(\frac{\alpha(x)}{2}\right) = -\frac{(\phi^+(x)\phi^0(x)^\dagger + \phi^0(x)\phi^+(x)^\dagger \pm i(\phi^+(x)\phi^0(x)^\dagger - \phi^0(x)\phi^+(x)^\dagger))}{2|\phi^0(x)|^2}. \quad (7.78)$$

Thus,

$$\begin{pmatrix}\phi^+(x)\\ \phi^0(x)\end{pmatrix} \to \begin{pmatrix}0\\ \phi^{0\prime}(x)\end{pmatrix}. \quad (7.79)$$

However, this $\phi^{0\prime}(x)$ is *not* Hermitian. We may write

$$\phi^{0\prime}(x) = \exp[i\theta(x)]R(x), \quad (7.80)$$

where $\theta(x)$ and $R(x)$ *are* Hermitian. To rid ourselves of $\theta(x)$, we may make a $U(1)$ local gauge transformation allowed by the local hypercharge gauge invariance. All of the other fields, L, R, a_μ and $\mathbf{b}_\mu$, are now to be thought of as taken in this gauge. We will understand this and not pollute the notation still further by festooning field variables with primes. Hence, in the U-gauge, the covariant derivative above takes the form—note the absence of $A_\mu(x)$—

$$D_\mu\phi(x) \to \left[\frac{\partial}{\partial x^\mu} - i\left(\frac{g}{\sqrt{2}}W_\mu^-(x) - \frac{(g^2+g'^2)^{1/2}}{2}Z_\mu(x)\right)\right]R(x) \quad (7.81)$$

so that $\mathcal{L}_H(x)$ is given by

$$\begin{aligned}\mathcal{L}_H(x) = &-\frac{\partial}{\partial x_\mu}R(x)\frac{\partial}{\partial x^\mu}R(x) + \frac{g(g^2+g'^2)}{2\sqrt{2}}(W_\mu^+(x)Z^\mu(x) + W_\mu^-(x)Z^\mu(x))R^2(x) \\ &- \left(\frac{g^2}{2}W_\mu^+(x)W^{-\mu}(x) + \frac{(g^2+g'^2)}{4}Z_\mu(x)Z^\mu(x)\right)R^2(x) \\ &+ m_0^2R^2(x) - \tfrac{1}{2}f^2R(x)^4. \end{aligned} \quad (7.82)$$

Hence, in the U-gauge we have the very important results that the electromagnetic photon does not couple to the Higgs field. We shall exploit this in a moment.

Up to this point, we have not given the leptons any kind of interaction which could generate a mass. We can trace this back to the basic Lagrangian in which right-handed leptons are coupled only to right-handed leptons and left-handed leptons only to left-handed leptons. Thus the leptonic terms have, so far, an invariance under global chiral phase transformations of the form $\exp i\lambda\gamma_5$ which, if maintained, would preclude any mass term from developing.[59] This can be remedied by introducing a chiral *non*invariant coupling of the form—before going to the U-gauge—

$$\mathcal{L}_H^I(x) = -G(\bar{L}(x)\phi(x)R(x) + \bar{R}(x)\phi^\dagger(x)L(x)), \quad (7.83)$$

where G is a new coupling constant that has no necessary relation to either g or g'. In this sense, parity nonconser-

[59]This is an ancient observation and can be demonstrated by the remark that an expression of the form $A + B\gamma_5$ has the opposite transformation properties from $(A' + B'\gamma_5)\gamma P$ under $\exp(i\lambda\gamma_5)$. Hence, in a chirally invariant theory the two cannot coexist in the Green's function—or its inverse—which means that no mass can develop. A bare mass is excluded from the Lagrangian by the same argument.

vation and, as we shall see, the electron mass are put in by hand from the beginning. Perhaps in the ultimate gauge theory—if there is one—these might be derived from something even more fundamental. Since $\overline{L}(x)$ and $\phi(x)$ belong to conjugate representations of $SU(2)$, while $R(x)$ is an $SU(2)$ singlet, $\mathfrak{L}_H^I(x)$ is *locally* invariant under $SU(2)$. Moreover, since the hypercharges of $\overline{L}(x)$ and $\phi(x)$ are $+1$, respectively, while $R(x)$ has a hypercharge of -2; $\mathfrak{L}_H^I(x)$ is also invariant under the local $U(1)$ group. Hence, we can rotate the whole shooting match to the U-gauge where

$$\mathfrak{L}_H^I(x) = -G\bar{e}(x)e(x)R(x). \tag{7.84}$$

and $R(x)$ is now to be understood as the Higgs field in the U-gauge. The entire Lagrangian—while expressed in the U-gauge, is still invariant under local $SU(2)\times U(1)$ gauge transformations. We now break this by letting, as before,

$$R(x) = \chi(x) + \eta, \tag{7.85}$$

where

$$\langle 0|\chi(0)|0\rangle = 0. \tag{7.86}$$

What happens?

We list the consequences numerically:

(1) The electron acquires a bare mass $m_e = G\eta$, which is undetermined in the theory, since G is arbitrary. At this point, we should mention that the muon and its neutrino have not yet entered the discussion. These can be incorporated in complete analogy to the electron terms. However in the Weinberg–1967 model the muon–electron mass ratio is put in by hand choosing the coupling constants suitably. It is to be hoped that some gauge theory will eventually be found where it would be forced by the structure of the model.

(2) The χ field acquires a mass since, as in the Abelian case,

$$\begin{aligned} m_0^2 R(x)^2 - \tfrac{1}{2}f^2 R(x)^4 &= (m_0^2 - \tfrac{1}{2}f^2\eta^4) \\ &\quad + \chi(x)2\eta(m_0^2 - f^2\eta^2) \\ &\quad + \chi^2(x)(m_0^2 - 3f^2\eta^2) - 2f\eta\chi^3(x) \\ &\quad - \tfrac{1}{2}f^2\chi^4(x). \end{aligned} \tag{7.87}$$

As before, we may drop the constant term since it produces a harmless uniform phase change in all the transition elements (Weinberg, 1972b). To leading order we must have

$$\eta = m_0/f \tag{7.88}$$

to rid ourselves of terms in which $\chi(x)$ can vanish into the vacuum, contradicting

$$\langle 0|\chi(0)|0\rangle = 0. \tag{7.89}$$

Hence, in this order

$$m_x = \sqrt{2}\, m_0 \tag{7.90}$$

as before, and the remaining terms are complicated self-interactions of the x.

(3) We have

$$\begin{aligned} &\left(\frac{g^2}{2} W_\mu^+(x)W^{-\mu}(x) + \frac{(g^2+g'^2)}{4} Z_\mu(x)Z^\mu(x)\right)R^2(x) \\ &\quad = \left(\frac{g^2}{2} W_\mu^+(x)W^{-\mu}(x) + \frac{(g^2+g'^2)}{4} Z_\mu(x)Z^\mu(x)\right) \\ &\qquad \times (\eta^2 + 2\eta\chi(x) + \chi^2(x)) \end{aligned} \tag{7.91}$$

or the bare masses of the W and Z are

$$m_{W^+}^2 = m_{W^-}^2 = g^2\eta^2/2, \tag{7.92}$$

while

$$m_Z^2 = (g^2 + g'^2)\eta^2/2, \tag{7.93}$$

so that

$$\frac{m_Z^2}{m_W^2} = 1 + \frac{g'^2}{g^2} \cdot \equiv \cdot \frac{1}{\cos^2(\theta)}, \tag{7.94}$$

where

$$\tan(\theta) \cdot \equiv \cdot g'/g \tag{7.95}$$

defines the Weinberg (1972b) mixing angle, which must be determined empirically. In the same notation, the electric charge is given by

$$e = g'/\left(1 + \frac{g'^2}{g^2}\right)^{\frac{1}{2}} = g\sin(\theta), \tag{7.96}$$

which is to say, the three constants g, g', and e are all of the same order of magnitude.

(4) The bare mass of

$$A_\mu(x) = 1/(g^2 + g'^2)^{\frac{1}{2}}(ga_\mu(x) - g'b_{\mu 3}(x)) \tag{7.97}$$

is *zero*, since in the U-gauge the photon is decoupled from the Higgs field so the Higgs mechanism does not work on it. Since there are no zero mass charged particles left in the U-gauge, we would expect to be able to pursue the general arguments on the vacuum polarization tensor to show that the photon remains massless to all orders in all the coupling constants. There are some tricky points in this argument however. Because all of the gauge theories with photons and heavy neutral vector mesons *both* coupled to electrons—as in the Weinberg (1967) model—have in common these difficult points, it is worth going into a bit more detail.[60]

The essential problem is that in these theories the photon and the massive neutral vector mesons can *mix* via the weak and electromagnetic interactions. Figure 7 illustrates such a mixing term as it would arise in the Weinberg—1967 model. Hence, one must make a renormalization perscription which defines the physical unmixed neutral vector meson fields. *These* fields will have the correct physical masses, i.e., mass zero for the photon. Hence, in the $SU(2)\times U(1)$ model we define the renormalized Z field $Z_\mu^r(x)$, and the photon field $A_\mu^r(x)$, so that

[60]The author's attention to the arguments given below was called by Professor J. C. Taylor in the spring of 1972. They arose in connection with a calculation—unpublished—of radiative corrections to elastic electron-neutrino scattering done in collaboration with the present author, in which we used the Weinberg (1967) model.

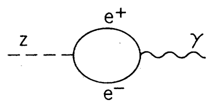

FIG. 7. A diagram illustrating the Z–γ mixing in the $SU(2)\times U(1)$ model. The Z couples to the electron–positron pair via the "weak" coupling, while γ couples to the pair via the electromagnetic coupling.

for the *physical* Z state $|Zk\epsilon\rangle$ with k the momentum and ϵ the polarization and for the *physical* γ state $|\gamma k\epsilon\rangle$ we have

$$\langle 0|A_\mu^r(x)|\gamma k\epsilon\rangle = \epsilon_\mu^\gamma \exp[i(kx)], \quad (7.98)$$

$$\langle 0|Z_\mu^r(x)|Zk\epsilon\rangle = \epsilon_\mu^Z \exp[i(kx)], \quad (7.99)$$

while

$$\langle 0|Z_\mu^r|\gamma k\epsilon\rangle = 0, \quad (7.100)$$

$$\langle 0|A_\mu^r|Zk\epsilon\rangle = 0. \quad (7.101)$$

The vacuum polarization tensor for the $Z-\gamma$ system is a 2×2 matrix whose diagonal terms are the γ–γ and Z–Z transitions and whose off-diagonal term—the tensor is symmetric—is the Z–γ transition. In writing down this expression, we bear in mind that the photon couples to a conserved current while the neutral vector meson does not. If we call m_Z the renormalized, i.e., the physical, mass of the Z, then the renormalized vacuum polarization tensor $M_{\mu\nu}(k)$ takes the form, in the Feynman gauge[61]

$$M_{\mu\nu}(k) = \begin{pmatrix} k^2 g_{\mu\nu} + (k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi_1(k^2) & (k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi_2(k^2) \\ (k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi_2(k^2) & (k^2 + m_Z^2) g_{\mu\nu} + \Pi_3(k)_{\mu\nu} \end{pmatrix} \quad (7.102)$$

The reader will notice that there are three independent polarization functions: $\Pi_1(k^2)$, $\Pi_2(k^2)$, and $\Pi_3(k)_{\mu\nu}$. The tensor covariants in front of these functions have been fixed by the requirement of current conservation when it is applicable. The fact that the off-diagonal matrix element is conserved reflects the fact that the γ couples to a conserved current even though the Z does not. The renormalization has been fixed by the requirements that

$$\Pi_1(k^2) \sim O(k^2) \quad (7.103)$$

$$\Pi_2(k^2) \sim O(k^2 + m_Z^2) \quad (7.104)$$

$$\Pi_3(k)_{\mu\nu} \sim O((k^2 + m_Z^2)^2). \quad (7.105)$$

These conditions imply that

$$M_{\mu\nu}(k)^{-1} = g_{\mu\nu}\begin{pmatrix} \dfrac{1}{k^2} & 0 \\ 0 & \dfrac{1}{k^2 + m_Z^2} \end{pmatrix} + \text{nonsingular terms} . \quad (7.106)$$

Hence, the poles will be correctly located. Therefore, to pass from the unrenormalized to the renormalized propagators $\Pi_1(k^2)$ must be subtracted at $k^2 = 0$ while $\Pi_2(k^2)$ and $\Pi_3(k)_{\mu\nu}$ must be subtracted at $k^2 = -m_Z^2$. This prescription plays an essential role in maintaining the masslessness of the physical photons in those gauge theories with both photons and massive neutral vector mesons.

[61]This is the gauge in which the free photon propagator $D_{\mu\nu}^0(k)$ takes the form

$$D_{F\mu\nu}^0(k) = g_{\mu\nu}/(k^2 + i\epsilon).$$

In the work that follows, we do not display the $i\epsilon$ term explicitly.

While the U-gauge argument for the vanishing of the photon mass has the advantage of a certain simplicity, it does not clearly illustrate the general discussion of the Higgs mechanism and the breakdown of the Goldstone theorem from which we began. Hence, we would like to examine this question afresh in the general gauge using the linearized theory. To this end, we can forget about the lepton couplings and consider a system described by the simplified Lagrangian

$$\begin{aligned} \mathfrak{L}(x) = &-\frac{1}{4}\left(\frac{\partial}{\partial x^\mu}\mathbf{b}_\nu(x) - \frac{\partial}{\partial x^\nu}\mathbf{b}_\mu(x) + g\mathbf{b}_\mu(x)\times\mathbf{b}_\nu(x)\right)^2 \\ &-\frac{1}{4}\left(\frac{\partial}{\partial x^\mu}a_\nu(x) - \frac{\partial}{\partial x^\nu}a_\mu(x)\right)^2 \\ &-\left(\frac{\partial}{\partial x^\mu} + ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}_\mu(x) - i\frac{g'}{2}a_\mu(x)\right)\phi^\dagger(x) \\ &\times\left(\frac{\partial}{\partial x_\mu} - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}^\mu(x) + i\frac{g'}{2}a^\mu(x)\right)\phi(x) \\ &+ m_0^2\phi^\dagger(x)\phi(x) - \tfrac{1}{2}f^2(\phi^\dagger(x)\phi(x))^2 \end{aligned} \quad (7.107)$$

where

$$\phi(x) = \begin{pmatrix}\phi^+(x)\\ \phi^0(x)\end{pmatrix}. \quad (7.108)$$

We have local $SU(2)\times U(1)$ invariance and hence the conserved currents. The notation is the same as in the last section,

$$\begin{aligned} \mathbf{T}_\mu(x) = &\,\mathbf{f}_{\mu\nu}(x)\times\mathbf{b}^\nu(x) \\ &+ i\Big[\left(\frac{\partial}{\partial x^\mu} + ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}_\mu(x) - i\frac{g'}{2}a_\mu(x)\right)\phi^\dagger(x)\frac{\boldsymbol{\tau}}{2}\phi(x) \\ &- \phi^\dagger(x)\frac{\boldsymbol{\tau}}{2}\left(\frac{\partial}{\partial x^\mu} - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}(x) + i\frac{g'}{2}a_\mu(x)\right)\phi(x)\Big] \end{aligned} \quad (7.109)$$

with

$$\int d^3\mathbf{x}[\mathbf{T}_0(\mathbf{x},0),\phi(0)] = (\boldsymbol{\tau}/2)\phi(0). \quad (7.110)$$

In addition, the $U(1)$ invariance implies the conservation of

$$\begin{aligned} Y_\mu(x) = i\Big[&\left(\frac{\partial}{\partial x^\mu} + ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}_\mu(x) - i\frac{g'}{2}a_\mu(x)\right)\phi^\dagger(x)\phi(x) \\ &- \phi^\dagger(x)\left(\frac{\partial}{\partial x^\mu} - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}_\mu(x) + i\frac{g'}{2}a_\mu(x)\right)\phi(x)\Big] \end{aligned} \quad (7.111)$$

and we have

$$\int d^3\mathbf{x}[Y_0(\mathbf{x},0),\phi(0)] = \phi(0). \quad (7.112)$$

If we form the electric charge

$$Q = T_3 + (Y/2), \quad (7.113)$$

we have the pair of equations

$$\int d^3\mathbf{x}[\mathcal{T}_{03}(\mathbf{x},0) + \mathcal{Y}_0(\mathbf{x},0)/2, \phi^+(0)] = \phi^+(0) \quad (7.114)$$

and

$$\int d^3\mathbf{x}[\mathcal{T}_{03}(\mathbf{x},0) + \mathcal{Y}_0(\mathbf{x},0)/2, \phi^0(0) = 0. \quad (7.115)$$

Whatever else happens *electric* charge must be conserved and all states must have well-defined electric charges. Hence, we must demand that

$$Q|0\rangle = 0, \quad (7.116)$$

It is this requirement, as we shall now see, that keeps the photon massless. In particular, in *all* gauges

$$\langle 0|\phi^+(0)|0\rangle = \langle 0|[Q,\phi^+(0)]|0\rangle = 0. \quad (7.117)$$

This is, of course, consistent with the U-gauge statement that

$$\phi^{+\prime} \cdot \equiv \cdot 0 \quad (7.118)$$

but it is more general. Since Q commutes with $\phi^0(x)$, no corresponding condition holds for $\langle 0|\phi^0(0)|0\rangle$, and we may break the symmetry by demanding that

$$\langle 0|\phi^0(0)|0\rangle = \eta \neq 0. \quad (7.119)$$

Again, we may invoke time reversal to prove that

$$\eta = \eta^*. \quad (7.120)$$

Now we can solve the theory in the linarized approximation in the general gauge to discover the various masses. The equations of motion for the vector fields are given by

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial}{\partial x_\mu}a^\nu(x) - \frac{\partial}{\partial x_\nu}a^\mu(x)\right) = i\frac{g'}{2}\left[\phi^\dagger(x)\left(\frac{\partial}{\partial x_\mu} - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}^\mu(x) + i\frac{g'}{2}a^\mu(x)\right)\phi(x) - \left(\frac{\partial}{\partial x_\mu} + ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}^\mu(x) - i\frac{g'}{2}a^\mu(x)\right)\phi^\dagger(x)\phi(x)\right] \quad (7.121)$$

and, dropping the self-coupling Yang–Mills terms,[62]

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial}{\partial x_\mu}\mathbf{b}^\nu(x) - \frac{\partial}{\partial x_\nu}\mathbf{b}^\mu(x)\right) = -i\frac{g}{2}\left[\boldsymbol{\tau}\phi^\dagger(x) \times\left(\frac{\partial}{\partial x_\mu} - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}^\mu(x) + i\frac{g'}{2}a^\mu(x)\right)\phi(x) - \left(\frac{\partial}{\partial x_\mu} + ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{b}^\mu(x) - i\frac{g'}{2}a^\mu(x)\right)\phi^\dagger(x)\boldsymbol{\tau}\phi(x)\right]. \quad (7.122)$$

We shall solve this set approximately by setting

$$\phi(x) = \eta + \phi'(x) \quad (7.123)$$

where

$$\eta = \begin{pmatrix}0\\ \eta\end{pmatrix} \quad (7.124)$$

and

$$\phi'(x) = \begin{pmatrix}\phi^{+\prime}(x)\\ \phi^{0\prime}(x)\end{pmatrix} \quad (7.125)$$

is treated as "small." Thus, approximately, for the neutral fields,

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial}{\partial x_\mu}a^\nu(x) - \frac{\partial}{\partial x_\nu}a^\mu(x)\right) = \frac{g'\eta}{2}i\frac{\partial}{\partial x_\mu}(\phi^{0\prime}(x) - \phi^{0\prime}(x)^\dagger) - \frac{\eta^2}{2}(g'^2a^\mu(x) + gg'b_3^\mu(x)) \quad (7.126)$$

and

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial}{\partial x_\mu}b_3^\nu(x) - \frac{\partial}{\partial x_\nu}b_3^\mu(x)\right) = \frac{g\eta}{2}i\frac{\partial}{\partial x_\mu}(\phi^{0\prime}(x) - \phi^{0\prime}(x)^\dagger) - \frac{\eta^2}{2}(gg'a^\mu(x) + g^2b_3^\mu(x)). \quad (7.127)$$

To find the neutral vector fields that propagate with a definite mass—clearly $a^\mu(x)$ and $b_3^\mu(x)$ do *not*—we first multiply Eq. (7.126) by g' and Eq. (7.127) by g and call the Hermitian field

$$\theta(x) \cdot \equiv \cdot i(\phi^{0\prime}(x) - \phi^{0\prime}(x)^\dagger). \quad (7.128)$$

Thus,

$$g'\frac{\partial}{\partial x^\nu}\left(\frac{\partial}{\partial x_\mu}a^\nu(x) - \frac{\partial}{\partial x_\nu}a^\mu(x)\right) = \frac{\eta g'^2}{2}\frac{\partial}{\partial x_\mu}\theta(x) - \frac{\eta^2 g'^2}{2}(g'a^\mu(x) + gb_3^\mu(x)) \quad (7.129)$$

and

$$g\frac{\partial}{\partial x^\nu}\left(\frac{\partial}{\partial x_\mu}b_3^\nu(x) - \frac{\partial}{\partial x_\nu}b_3^\mu(x)\right) = \frac{g^2\eta}{2}\frac{\partial}{\partial x_\mu}\theta(x) - \frac{g^2\eta^2}{2}(g'a^\mu(x) + gb_3^\mu(x)), \quad (7.130)$$

or, adding

$$\frac{\partial}{\partial x^\nu}\left[g'\left(\frac{\partial}{\partial x_\mu}a^\nu(x) - \frac{\partial}{\partial x_\nu}a^\mu(x)\right) + g\left(\frac{\partial}{\partial x_\mu}b_3^\nu(x) - \frac{\partial}{\partial x_\nu}b_3^\mu(x)\right)\right] = \frac{g^2 + g'^2}{2}\eta\frac{\partial}{\partial x_\mu}\theta(x) - \frac{\eta^2}{2}(g^2 + g'^2)(g'a^\mu(x) + gb_3^\mu(x)), \quad (7.131)$$

[62]They do not, in lowest order, contribute to the masses.

while a similar computation shows that

$$\frac{\partial}{\partial x^{\nu}}\left[g\left(\frac{\partial}{\partial x_{\mu}}a^{\nu}(x)-\frac{\partial}{\partial x^{\nu}}a^{\mu}(x)\right)-g'\left(\frac{\partial}{\partial x_{\mu}}b_3^{\nu}(x)-\frac{\partial}{\partial x_{\nu}}b_3^{\mu}(x)\right)\right]=0. \quad (7.132)$$

In view of what has gone before, it is evident how we are going to interpret these equations. But before doing so, we shall complete the set. From the approximate equations

$$\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial}{\partial x_{\mu}}\mathbf{b}^{\nu}(x)-\frac{\partial}{\partial x_{\nu}}\mathbf{b}^{\mu}(x)\right)=\frac{-ig}{2}\left[\phi^{\dagger}(x)\boldsymbol{\tau}\frac{\partial\phi(x)}{\partial x_{\mu}}-\frac{\partial}{\partial x_{\mu}}\phi^{\dagger}(x)\boldsymbol{\tau}\phi(x)\right]-\frac{g^2}{2}\phi^{\dagger}(x)\mathbf{b}^{\mu}(x)\phi(x), \quad (7.133)$$

we have

$$\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial}{\partial x_{\mu}}\frac{(b_1^{\nu}(x)+ib_2^{\nu}(x))}{\sqrt{2}}-\frac{\partial}{\partial x_{\nu}}\frac{(b_1^{\mu}(x)+ib_2^{\mu}(x))}{\sqrt{2}}\right)=i\frac{g\eta}{\sqrt{2}}\frac{\partial\phi'^{\dagger}(x)^{\dagger}}{\partial x_{\mu}}-\frac{g^2\eta^2}{2}\frac{(b_1^{\nu}(x)+ib_2^{\nu}(x))}{\sqrt{2}} \quad (7.134)$$

along with its conjugate partner.

Clearly, what is happening here is that, in the spirit of the earlier sections, the three scalar degrees of freedom, $\phi^{+}(x)$, $\phi^{+}(x)^{\dagger}$, and $\theta(x)=i(\phi^{0\prime}(x)-\phi^{0\prime}(x)^{\dagger})$ are combining with the vector fields $a_{\mu}(x)$ and $\mathbf{b}_{\mu}(x)$ to produce massive mesons W_{μ}^{+}, W_{μ}^{-}, and Z_{μ}, and the photon A^{μ} which is massless. However, before we can be sure of this interpretation we must look at the approximate equations of motion for $\phi^{\dagger}(x)$ and $\phi^{0}(x)$ to see whether everything is consistent and where the physical Higgs particle is buried. If we impose the consistency condition in leading order

$$f^2\eta^2=m_0^2 \quad (7.135)$$

which ensures that

$$\langle 0|\phi^{0\prime}(0)|0\rangle=0, \quad (7.136)$$

we find, after some manipulation, the approximate equations

$$\frac{\partial}{\partial x_{\mu}}\frac{\partial}{\partial x^{\mu}}\begin{pmatrix}\phi^{+\prime}(x)\\ \phi^{0\prime}(x)\end{pmatrix}+\frac{i}{2}\frac{\partial}{\partial x_{\mu}}(g'a_{\mu}(x)-g\boldsymbol{\tau}\cdot\mathbf{b}_{\mu}(x))\begin{pmatrix}0\\ \eta\end{pmatrix}=\begin{pmatrix}0\\ m_0^2(\phi^{0\prime}(x)^{\dagger}+\phi^{0\prime}(x))\end{pmatrix} \quad (7.137)$$

and the conjugate partner. We can rewrite these as

$$\frac{\partial}{\partial x_{\mu}}\frac{\partial}{\partial x^{\mu}}\phi^{+\prime}(x)-i\frac{g\eta}{\sqrt{2}}\frac{\partial}{\partial x_{\mu}}W_{\mu}^{-}(x)=0 \quad (7.138)$$

and

$$\frac{\partial}{\partial x_{\mu}}\frac{\partial}{\partial x^{\mu}}\phi^{0\prime}(x)+\frac{i}{2}(g^2+g'^2)^{1/2}\eta\frac{\partial}{\partial x_{\mu}}Z_{\mu}(x)=m_0^2(\phi^{0\prime}(x)^{\dagger}+\phi^{0\prime}(x)) \quad (7.139)$$

along with the conjugate partners. This last equation is better rewritten as—using its conjugate partner—

$$\frac{\partial}{\partial x_{\mu}}\frac{\partial}{\partial x^{\mu}}(\phi^{0\prime}(x)-\phi^{0\prime}(x)^{\dagger})+\frac{i}{2}(g^2+g'^2)^{\frac{1}{2}}\eta\frac{\partial}{\partial x_{\mu}}Z_{\mu}(x)=0 \quad (7.140)$$

and

$$\frac{\partial}{\partial x_{\mu}}\frac{\partial}{\partial x^{\mu}}(\phi^{0\prime}(x)+\phi^{0\prime}(x)^{\dagger})=2m_0^2(\phi^{0\prime}(x)+\phi^{0\prime}(x)^{\dagger}). \quad (7.141)$$

The significance of the last equation is evident, namely, in *all* gauges

$$\chi(x)=[\phi'^{0}(x)+\phi'^{0}(x)^{\dagger}]/2 \quad (7.142)$$

is the *physical* i.e., "observable," Higgs particle of bare mass $\sqrt{2}m_0$ in complete agreement, as it must be, with the U-gauge result. The significance of the remaining equations is evidently gauge-dependent, since they involve the divergences of the vector fields $a_{\mu}(x)$ and $\mathbf{b}_{\mu}(x)$. These equations are fully consistent—as they must be—with the equations derived simply by noticing that the vectors such as

$$\frac{\partial}{\partial x_{\nu}}a_{\mu\nu}(x)=\frac{\partial}{\partial x_{\nu}}\left(\frac{\partial}{\partial x^{\mu}}a_{\nu}(x)-\frac{\partial}{\partial x^{\nu}}a_{\mu}(x)\right) \quad (7.143)$$

are conserved. As we have seen before, the *observed* particle content of the theory is independent of the gauge. In the Lorentz gauge $\phi^{+\prime}(x)$, $\phi^{+\prime}(x)^{\dagger}$, and $i(\phi^{0\prime}(x)-\phi^{0\prime}(x)^{\dagger})$ are Goldstone mesons that factor out of the physical particle spectrum, while in the Coulomb gauge these objects are coupled to the spatial divergences of the vector fields and are not Lorentz scalars at all, thus defeating the Goldstone theorem. In *all* gauges W_{μ}^{+}, W_{μ}^{-}, Z^{μ} are observable massive vector mesons with bare masses identical to those found in the U-gauge, while A^{μ} is the massless photon.

In the next, and *final*, section we will comment briefly about generalizations of the Weinberg—1967 model, about its renormalizibility and experimental consequences. As a preliminary to this Herculean task we wish to discuss the subject of the static electromagnetic properties of the charged vector mesons in a theory like the Weinberg—1967 model or its successors. We wish to clarify the following two points:

(1) It is sometimes claimed that the "principle" of *minimal* electromagnetic couplings prohibits an *intrinsic* Pauli magnetic moment for elementary particles. This is true for spin–1/2 charged particles where, as we have seen, the electromagnetic Lagrangian is obtained from the free Lagrangian by the substitution

$$(\partial/\partial x_{\mu})\rightarrow(\partial/\partial x_{\mu})-ie_0A^{\mu}(x), \quad (7.144)$$

yielding the familiar electromagnetic coupling

$$\mathcal{L}_I(x)=-ie_0\bar{\psi}(x)\gamma_{\mu}\psi(x)A^{\mu}(x) \quad (7.145)$$

with no intrinsic Pauli coupling. But for a charged spin–

one principle there is an ambiguity in this "principle" as follows. Let

$$\mathcal{L}_0(x) = -\frac{1}{2}\left(\frac{\partial}{\partial x^\mu} W_\nu^+(x) - \frac{\partial}{\partial x^\nu} W_\mu^+(x)\right) \times \left(\frac{\partial}{\partial x_\mu} W^{-\nu}(x) - \frac{\partial}{\partial x_\nu} W^{-\mu}(x)\right) - m_w^2 W_\mu^+(x) W^{-\mu}(x) \quad (7.146)$$

yielding the free equations of motion. To $\mathcal{L}_0(x)$ we may add a *total* derivative and derive the same equations of motion. Let us add $(\partial/\partial x_\nu)\Lambda_\nu(x)$ where $\Lambda_\nu(x)$ is defined by the expression

$$\Lambda_\nu(x) = \frac{\kappa}{2}\left(\frac{\partial W_\mu^+(x)}{\partial x_\mu} W_\nu^-(x) - \frac{\partial}{\partial x_\mu} W_\nu^+(x) W_\mu^-(x) + \frac{\partial W_\mu^-(x)}{\partial x_\mu} W_\nu^+(x) - \frac{\partial}{\partial x_\mu} W_\nu^-(x) W_\mu^+(x)\right) \quad (7.147)$$

where κ is a free, i.e., arbitrary, parameter. Thus,

$$\frac{\partial}{\partial x_\nu}\Lambda_\nu(x) = \kappa\left(\frac{\partial}{\partial x_\mu} W_\mu^+(x) \frac{\partial}{\partial x_\nu} W_\nu^-(x) - \frac{\partial}{\partial x_\mu} W_\nu^+(x) \frac{\partial}{\partial x_\nu} W_\mu^-(x)\right). \quad (7.148)$$

We may now make the gauge substitution

$$\frac{\partial}{\partial x_\mu} W_\mu^+(x) \to \left(\frac{\partial}{\partial x_\mu} + ie_0 A^\mu(x)\right) W_\mu^+(x) \quad (7.149)$$

and so forth, to produce

$$\frac{\partial}{\partial x_\nu}\Lambda_\nu(x) \to \frac{\partial}{\partial x_\nu}\Lambda_\nu(x) + ie_0\kappa A_\mu(x) \frac{\partial}{\partial x^\nu}(W^{+\mu}(x) W^{-\nu}(x) - W^{+\nu}(x) W^{-\mu}(x)). \quad (7.150)$$

The quantity

$$M^\mu(x) = i\frac{\partial}{\partial x^\nu}(W^{+\mu}(x) W^{-\nu}(x) - W^{+\nu}(x) W^{-\mu}(x)) \quad (7.151)$$

is clearly a conserved current and the interaction

$$\mathcal{L}_I^M(x) = -e_0\kappa A_\mu(x) M^\mu(x) \quad (7.152)$$

is clearly as "minimal" as that with $\kappa = 0$.

We can also write this Lagrangian modulo a total derivative which cannot effect the equations of motion,

$$\mathcal{L}_I^M(x) = ie_0\kappa\left(\frac{\partial}{\partial x^\nu} A_\mu(x) - \frac{\partial}{\partial x^\mu} A_\nu(x)\right) W^{+\mu}(x) W^{-\nu}(x). \quad (7.153)$$

We shall see what this expression means physically shortly. For comparison, let us make the gauge substitution in $\mathcal{L}_0(x)$; i.e.,

$$\mathcal{L}_0(x) \to -\frac{1}{2}\left(\frac{\partial}{\partial x^\mu} W_\nu^+(x) - \frac{\partial}{\partial x^\nu} W_\mu^+(x)\right) \times \left(\frac{\partial}{\partial x_\mu} W^{-\nu}(x) - \frac{\partial}{\partial x_\nu} W^{-\mu}(x)\right) - m_w^2 W_\mu^+(x) W^{-\mu}(x) - ie_0 A_\mu(x) \times \left[W_\nu^+(x)\left(\frac{\partial}{\partial x_\nu} W^{-\nu}(x) - \frac{\partial}{\partial x_\nu} W^{-\mu}(x)\right) - W_\nu^-(x)\left(\frac{\partial}{\partial x_\mu} W^{+\nu}(x) - \frac{\partial}{\partial x^\nu} W^{+\mu}(x)\right)\right] - e_0^2[A^\mu(x) A_\mu(x) W^{+\nu}(x) W_\nu^-(x) - A_\mu(x) W^{+\mu}(x) A_\nu(x) W^{-\nu}(x)]. \quad (7.154)$$

We see, glancing back at the Weinberg—1967 Lagrangian, that terms of both the $\mathcal{L}_I^M(x)$ and the gauge displaced $\mathcal{L}^0(x)$ form occur.

(2) This brings us to the second point to clarify; i.e., what do these terms mean physically? In particular, what is the meaning of $\mathcal{L}_I^M(x)$, and what is κ in the Weinberg—1967 model? We approach these questions by examining the nonrelativistic limit to the equations of motion of $W_\mu^\pm(x)$ in an external classical field. We drop terms of order e_0^2. In this approximation, the equation of motion for, say, $W^{-\nu}(x)$ becomes

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial}{\partial x_\mu} W^{-\nu}(x) - \frac{\partial}{\partial x_\nu} W^{-\mu}(x)\right) - m_w^2 W^{-\nu}(x) = ie_0\frac{\partial}{\partial x_\mu}(A_\mu(x) W^{-\nu}(x) - A_\nu(x) W^{-\nu}(x)) + ie_0\left(\frac{\partial}{\partial x_\mu} W^{-\nu}(x) - \frac{\partial}{\partial x_\nu} W^{-\mu}(x)\right) A_\mu(x) - ie_0\kappa\left(\frac{\partial}{\partial x_\mu} A_\nu(x) - \frac{\partial}{\partial x_\nu} A_\mu(x)\right) W^{-\mu}(x). \quad (7.155)$$

We shall be specializing $A_\mu(x)$ to a constant uniform magnetic field with

$$A_0(x) = \dot{\mathbf{A}}(x) = (\partial/\partial x_\mu) A_\mu(x) = 0. \quad (7.156)$$

We also have the subsidiary condition, dropping terms of order e_0^2,

$$\left(\frac{\partial}{\partial x^\nu} - ie_0 A_\nu(x)\right) W^{-\nu}(x) = \frac{ie_0\kappa}{m_w^2}\frac{\partial}{\partial x^\nu}\left(\left(\frac{\partial}{\partial x_\mu} A_\nu(x) - \frac{\partial}{\partial x_\nu} A_\mu(x)\right) W^{-\mu}(x)\right). \quad (7.157)$$

In going to the static limit with an external magnetic field, we shall drop terms of order $(1/m_w^2)$ as compared to terms of order $1/m_w$. Using the subsidiary we have the approximate equation

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} W^{-\nu}(x) - m_w^2 W^{-\nu}(x) = ie_0\left(2A_\mu(x)\frac{\partial}{\partial x_\mu} W^{-\nu}(x) - (1+\kappa)\left(\frac{\partial}{\partial x_\mu} A_\nu(x) - \frac{\partial}{\partial x_\nu} A_\mu(x)\right) W^{-\mu}(x)\right). \quad (7.158)$$

If we take a uniform magnetic field

$$A_j = \epsilon_{jkl}(x_k H_l/2), \quad (7.159)$$

we have for $\nu = s = 1, 2, 3$

$$(\partial/\partial x^\mu)(\partial/\partial x_\mu)W^{-s}(x) - m_w^2 W^{-s}(x) = -ie_0(\mathbf{H}\cdot\mathbf{r}\times\nabla W^s(x) + (1+\kappa)(\mathbf{W}(x)\times\mathbf{H})_s). \quad (7.160)$$

Clearly, the first term on the right-hand side is an "orbital" interaction with $\mathbf{H}$ while we can readily show that the spin magnetic moment is

$$\mu = e_0(1+\kappa)/2m_w. \quad (7.161)$$

A glance at the Weinberg—1967 Lagrangian shows that in this theory $\kappa = 1$.[63] We can trace this back to the group property of the underlying Yang–Mills Lagrangian and it will vary from group to group.

A similar—albeit somewhat more complicated—computation can be done to test the response of the W^- to an external electric field. Here one looks for the coefficient of

$$\left.\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}A_0(\mathbf{x},0)\right|_{\mathbf{x}=0}$$

which gives the quadruple moment. Since the quadruple moment goes as $(1/m_w^2)$ one cannot neglect terms of this order in the equations of motion. If one works through the debris, one will find, with κ defined as above,

$$Q = -e_0\kappa/m_w^2 \quad (7.162)$$

so that Q is also determined by the group.

VIII. CONCLUSIONS

In evaluating gauge theories, Bjorken (1972) has introduced the useful distinction between "believable" gauge theories and those merely "not in contradiction with experiment." The former Bjorken defines to be those theories that deepen our understanding of "Great Questions." His list of same includes the origin of lepton mass and, in particular, why the ratio of electron to muon mass is of the order of the fine structure constant, what accounts for the value of the Cabbibo angle,[64] etc. To Bjorken's list we may also add the origin of CP violation[65] and perhaps even the value of the fine structure constant itself. Bjorken concludes, and in this we concur, that no gauge model so far devised is "believable"; the Great Questions are still Great Questions, but we now have a new context in which to think about them.

On the other hand, essentially none of the gauge models—suitably doctored—are, as far as the author knows, as yet in contradiction with experiment. As we have indicated in the beginning of the last section, the most significant handicap in model building is that we simply do not know the physical particle spectrum. We do not know where the lepton spectrum stops—in mass—and we do not know how many—if *any*—weakly interacting vector mesons there are. Even the existence of the "classical" charged $W^\pm$ weak vector mesons is, as of this writing, still not established. A similar and even more vexing question arises when we attempt to extend the gauge models to include hadrons. These generalizations involve "quark" currents which are constructed so that the approximate symmetries of the strong interactions—$SU(3)$ or $SU(3)\times SU(3)$—or, perhaps, other groups are respected. How many quarks are there, and where, if anywhere, are they observable as physical particles? We simply do not know. In contemplating these questions, one acquires a certain sympathy towards the physicists and chemists of the late nineteenth century who realized clearly that the atomic hypothesis produced marvelous regularities among the phenomena, but who were in doubt as to whether or not the atom existed.

Without getting too involved in the gauge group theoretic details we can say that all models so far proposed appear to fall into one of two classes, which may be overlapping:

(1) Models with neutral vector mesons (massive), in addition to the photon, of which Weinberg (1967) may serve as the canonical example.

(2) Models in which the only neutral vector meson that couples to leptons is the electromagnetic photon. These models are all characterized by the presence in them of heavy leptons.

As a brief illustrative aside, we give an example of a model of Class 2. This is a model due to Georgi and Glashow (1972), which has the feature that the *only* neutral vector meson in the theory is the electromagnetic photon. The only charged massive vector mesons are the $W^\pm$. The gauge group used by Georgi and Glashow is $SU(2)$ alone. As we have seen in the last section, the use of $SU(2)$ doublets *alone à la* Weinberg (1967) will not give electromagnetism *and* the weak interactions; we had to adjoin an independent $U(1)$ group. However, Georgi and Glashow propose to use the *triplet* representations of $SU(2)$. Here is where the heavy leptons come in. In particular, they define for the electronic leptons, E^+, E^0, e^-, and ν_e, where E^+ and E^0 are the hypothetical heavy electronic leptons, the triplets

$$\psi_L = \frac{(1+\gamma_5)}{2}\begin{pmatrix} E^+ \\ \nu_e\sin\beta + E^0\cos\beta \\ e^- \end{pmatrix} \quad (8.1)$$

$$\psi_R = \frac{(1-\gamma_5)}{2}\begin{pmatrix} E^+ \\ E^0 \\ e^- \end{pmatrix} \quad (8.2)$$

[63] This feature of the $SU(2)\times U(1)$ Yang–Mills Lagrangian was recognized in the important paper of Salam and Ward (1964) in which the particle masses are put in by hand rather than by the Higgs mechanism.

[64] For the uninitiated, this "angle" measures the weakness of $\Delta S = 1$ weak processes relative to the $\Delta S = 0$ processes we have been considering. Experimentally,

$$\sin(\theta) \sim 0.2.$$

The origin of this number only becomes a Great Question when the gauge theories are extended from leptons to hadrons. How to do this is also a Great Question.

[65] T. D. Lee (1973) has recently given a model of a spontaneously broken gauge theory in which a spontaneous, i.e., vacuum-broken CP violation of the right order of magnitude is a feature. The source of the CP violation in Lee's model is the presence of, say, two distinct Higgs fields coupled to each other which have vacuum expectation values that are relatively complex with respect to each other. The reader will recall how we used T-invariance in Sec. IV—Eq. (4.157) and what follows—to show that in this case

$$\langle 0|\phi(0)|0\rangle = \eta,$$

where η is real. In Lee's model this assumption is given up. The reader is advised to read Lee's paper for details.

and the corresponding singlets

$$S_R = [(1-\gamma_5)/2](E^0 \sin\beta - \nu_e \cos\beta). \quad (8.3)$$

Here β is an arbitrary mixing angle which could it turns out, in principle be determined from the relation that holds in their theory

$$m_w = 53.0 \text{ BeV} \sin\beta, \quad (8.4)$$

setting for them an upper limit of 53 BeV on the $W^\pm$ mass. This mass is, needless to say, generated by a Higgs mechanism. Indeed, in their model they introduce a Higgs $SU(2)$ triplet

$$\boldsymbol{\phi} = \begin{pmatrix} \phi^+ \\ \phi^0 \\ \phi^- \end{pmatrix}. \quad (8.5)$$

Since electric charge is conserved, only the neutral Higgs field ϕ^0 acquires a real nonvanishing vacuum expectation value, which we may call η. Thus, the $SU(2)$ invariant vector meson mass term will take the form

$$\boldsymbol{\phi}(x) \cdot (\mathbf{W}_\mu(x) \times \mathbf{W}^\mu(x)), \quad (8.6)$$

where

$$\mathbf{W}_\mu = \begin{pmatrix} W_\mu^+ \\ W_\mu^0 \\ W_\mu^- \end{pmatrix}, \quad (8.7)$$

which means that only $W^\pm$ will acquire a mass à la Higgs. We will identify W_μ^0 with the electromagnetic photon. It couples to

$$\psi = \psi_L + \psi_R \quad (8.8)$$

in the usual parity and charge-conserving fashion. The scheme is very elegant, but as Bjorken (1972) remarks, "the believability rating of the model plummets toward zero as one notices that m_e is given as the difference of two terms, one a bare mass term $\simeq (m_E/2)$ (m_E is the mass of the heavy electronic lepton), the other emerging from the spontaneous breakdown $\bar{\psi} \times \psi \cdot \langle\boldsymbol{\phi}\rangle$; the notation as above. No rationale for the miraculous cancellation is given." It is probably fair to say that if by an equivalent miracle all of the Glashow–Georgi particles should turn up at the right sorts of masses, the right sort of "rationale" would also turn up.

All Class 1 models have in common the predicted existence of neutral *weak* currents transmitted by the massive Z^0 of Weinberg (1967) or its equivalents. These currents modify all of the predictions of the usual weak interaction theories. For example,[66] the Weinberg (1967) model gives a definite prediction for the "diagonal" process $\bar{\nu}_e + e^- \to \bar{\nu}_e + e^-$ in terms of the Weinberg mixing angle θ. [See Eq. (7.96) for the definition.] This process also occurs in lowest order in the conventional weak interaction theories via the perturbative chain

$$\bar{\nu}_e + e^- \to W^- \to \bar{\nu}_e + e^-.$$

[66] See, for example, Weinberg (1972b) for a discussion of many of these predictions, and B. W. Lee (1972b) for a recent survey of the experimental situation.

However, in Class 1 theories there is the additional perturbative chain, in the same order, which is derived from the sequence

$$\bar{\nu}_e + \nu_e \to Z^0 \to e^+ + e^-.$$

One may "flip" the ν_e and the e^+, taking particle into antiparticle to obtain the elastic scattering. It is claimed (see B. W. Lee 1972 b) that the very difficult experiments of Gurr *et al.* (1972) set a limit on the Weinberg angle of

$$\sin^2\theta \lesssim 0.4.$$

(No such elastic electron–neutrino scattering events have actually been seen.)

A more characteristic prediction of Class 1 theories is that the reaction (ν_μ is the muon neutrino)

$$\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-,$$

which is "forbidden" i.e., occurs only in higher orders, in the conventional weak interaction theories ($\bar{\nu}_\mu + e^- \to W^-$ violates the conservation of "muon number"), is "allowed" in the Class 1 theories. The perturbation chain in lowest order is

$$\bar{\nu}_\mu + \nu_\mu \to Z^0 \to e^+ + e^-$$

which can be "flipped" to yield

$$\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-.$$

As of this writing, no such events have been seen. (See B. W. Lee, 1972 b, and references cited therein.) There exists also a class of processes involving hadrons such as

$$\nu + P \to \nu + P$$

and

$$\nu + P \to \nu + N + \pi^+$$

in which neutral currents—Z^0 induced processes—play a role. These are confounded, in their interpretation, by details of the strong interactions, but so far no unambiguous neutral current effects have surfaced. If this continues, Class 1 theories will, sooner or later, be definitively ruled out.[67] This will not do in the gauge theories, since a plethora of Class 2 models remain.

The fact that no "believable" unified gauge model of weak and electromagnetic processes has, as yet, been found should not be allowed to obscure what even the imperfect models that *have* been found accomplish. Therefore, it is appropriate to close this review by giving a few examples of how some of the longstanding difficulties in leptonic weak interaction physics appear to have found their resolution in the gauge theories. We shall work in the context of the Weinberg (1967) model.

[67] An ingenious variant of the Weinberg (1967) model has been produced by Bég and Zee (1973). This model, based on $SU(2) \times U(1)$, has *both* a Z^0 and a heavy lepton, as well as innumerable quarks—nine, to be exact. It turns out that a choice of mixing angle can be made in their model so that the Z^0 couples *only* to neutrinos. With this choice, needless to say, there is no conflict with any of the present experiments. The reader is urged to consult their note for details.

Similar results are forthcoming in the other models. We begin with the discussion of a reaction which, although far from being directly observable, has served as a useful theoretical laboratory for several years (Gell-Mann *et al.*, 1969), i.e.,

$$\nu + \bar{\nu} \rightarrow W^+ + W^-.$$

In conventional theories this process takes place via the diagram given in Fig. 8. The essential physics which illuminates the difficulty with this process can be stated as follows: a massive W meson, unlike the photon, has *three* states of polarization reprsented by the normalized, orthogonal four-vectors:

$$\epsilon^{(+)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix}, \tag{8.9}$$

$$\epsilon^{(-)} = \epsilon^{(+)\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix}, \tag{8.10}$$

$$\epsilon^{(0)} = \begin{pmatrix} \dfrac{\sqrt{|\mathbf{P}|^2 + m_W^2}}{m_W} \dfrac{\mathbf{P}}{|\mathbf{P}|} \\ i\dfrac{|\mathbf{P}|}{m_W} \end{pmatrix}. \tag{8.11}$$

If $\mathbf{P}$ is taken in the z direction, then these three polarizations correspond to spins in the z direction or "helicities" of +1, −1, and 0 respectively. Of interest for the rest of the discussion is the somewhat weird, but perfectly permissible reaction in which both the W^+ and W^- are longitudinally polarized, i.e., have zero helicity. The neutrino and antineutrino have opposite helicities. (See Fig. 9 where the reaction is drawn in the barycentric system.) The interest in considering this special reaction is that the total angular momentum of the final W^+, W^- system is *fixed* in the limit in which the neutrino momenta tend to infinity. This comes about because the neutrino and antineutrino have equal and opposite helicities. Hence, for longitudinally polarized W's, the whole reaction must vanish in the forward direction, which means that if θ is the scattering angle, the angular distribution must contain $\sin(\theta)$ as a factor to conserve angular momentum. For finite energies there are additional $\cos(\theta)$ terms in the angular distribution, but these tend towards unity as P, the neutrino energy, tends toward infinity. Following Weinberg (1971) we can write the scattering amplitude in the form, where E, P, θ, ϕ all refer to the W^+,

$$f_e(\theta,\phi) = -\frac{ig^2}{m_W^2}\frac{1}{8\pi\sqrt{2}}\frac{P^{3/2}\sin(\theta)e^{-i\phi}}{(2E)^{1/2}}\Big\{\Big[2E^2(1 - \frac{E}{P}\cos(\theta)) - m_W^2\Big]\Big/\Big[2E^2(1-\frac{P}{E}\cos(\theta)) - m_W^2 + m_e^2\Big]\Big\} \xrightarrow[P\to\infty]{} \frac{-ig^2}{m_W^2}\frac{1}{16\pi}P\sin(\theta)e^{-i\phi}. \tag{8.12}$$

This amplitude and hence the differential crosssection tend, inadmissably, to infinity, with the energy in violation of the P-wave unitary limit. Within the conventional theory the only remedy was to arbitrarily cut the theory off at some finite energy. However, in Class 1 theories such as Weinberg (1967), there are additional diagrams involving the neutral vector mesons. (See Fig. 10; Figure 11 shows the kinds of additional diagrams in Class 2 theories with heavy leptons which serve to keep these theories finite and unitary). It must be strongly emphasized that the mere presence of these extra diagrams does *not* guarantee that they will fix up the unitarity problem. The crucial point is that these diagrams in the gauge theories are hooked together by the group structure, as the Weinberg Lagrangian, displayed in the last section, makes clear. (See Fig. 10, where the coupling constants are explicitly indicated.) If these interactions were introduced with arbitrary coupling constants, no cancellations would take place, and if one imposed conditions to enforce such cancellations, one would be led back to the group. Again following Weinberg (1971), the diagram of Fig. 10 leads to an additional contribution to $\nu_e + \bar{\nu}_e \rightarrow W^+ + W^-$ of the form

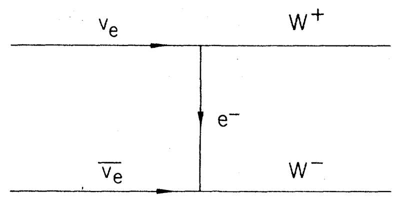

FIG. 8. The conventional weak interaction Feynman diagram giving the reaction $\nu_e + \bar{\nu}_e \rightarrow W^+ + W^-$.

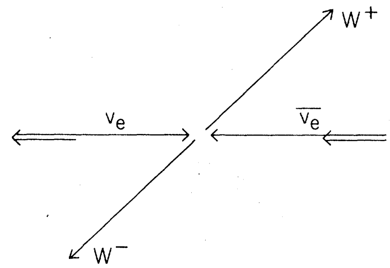

FIG. 9. A kinematical diagram in the barycentric system for $\nu_e + \bar{\nu}_e \rightarrow W^+ + W^-$. The double arrows on the neutrino lines show the directions of the neutrino spins relative to their momenta. The neutrino is "left handed", while the antineutrino is "right handed."

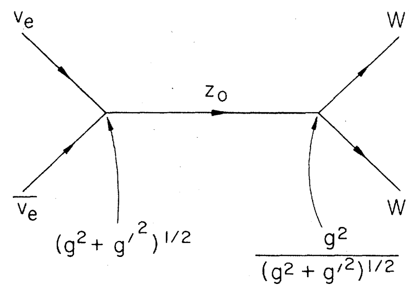

FIG. 10. The Z^0 contribution to $\nu_e + \bar{\nu}_e \rightarrow W^+ + W^-$. Note the coupling constants whose combination is fixed by the group structure.

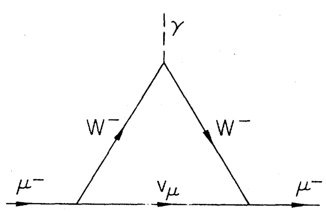

FIG. 11. A heavy electron contribution to $\nu_e + \bar{\nu}_e \rightarrow W^+ + W^-$ in Class 2 theories.

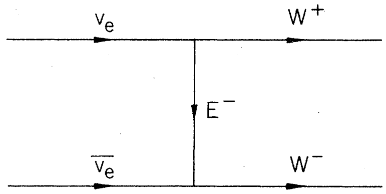

FIG. 12. The W meson contribution to the muon electromagnetic structure. Particle lines have been suitably labeled. A similar diagram holds for neutrino scattering where the roles of the muon and neutrino are interchanged.

$$f_z(\theta,\phi) = \frac{ig^2}{m_w^2}\frac{1}{8\pi\sqrt{2}}\frac{P^{3/2}\sin(\theta)\exp(-i\phi)}{(2E)^{1/2}}\{[4E^2 + 2m_w^2]/[4E^2 - m_z^2]\}. \quad (8.13)$$

If we consider the limit in which E is much larger than all the masses in the game, we find

$$f_e(\theta,\phi) + f_z(\theta,\phi) \to \frac{ig^2}{16\pi}\sin(\theta)\cos(\theta)\exp(-i\phi)/[E(1 - \frac{P}{E}\cos(\theta))]. \quad (8.14)$$

Not only is this remarkable expression consistent with unitarity—note the $1/E$ falloff—but the angular distribution has the physically reasonable feature that it peaks in the forward direction as $P/E \to 1$.

The example we have just given is a very nice illustration of how the gauge theories work in the cure of diseases that have plagued the weak interactions at the "no loop" level. At this level there are no infinities, since none of the diagrams involve integrations which might diverge. Here, as we have seen, the difficulties are associated with unitarity and vanish in the gauge theories. We would like to end this review with an illustration of how the gauge theories deal with infinities arising from loop integrations. We shall focus on the diagram of Fig. 12, which has a long history beginning with Bernstein and Lee (1963). These authors worked in what corresponds to the U- gauge, using a gauge-invariant regularization procedure to be discussed shortly. We have drawn Fig. 12 with external muons—a contribution to the electromagnetic structure of the muon. If the role of muons and neutrinos are interchanged in Fig. 12, one has instead a contribution to the electromagnetic structure of the neutrino, which was, in fact, the concern of Bernstein and Lee (1963). Later we shall comment about this quantity in connection with the Weinberg 1967 model.

Before introducing any regularization, the W propagator in the U gauge takes the form

$$iW_{\mu\nu}(q) = \left(g_{\mu\nu} + \frac{q_\mu q_\nu}{m_w^2}\right)/(q^2 + m_w^2). \quad (8.15)$$

The W-W-γ vertex for arbitrary κ—the "anomalous" moment—can be written (T. D. Lee, 1962) for the transition

$$W_\alpha^-(P) + \gamma_\mu(q) \to W_\beta(P'),$$

with

$$p + q = p' \quad (8.16)$$

$$V_{\beta\mu\alpha} = ie_0[g_{\alpha\beta}(p+p')_\mu + \kappa(g_{\alpha\mu}p_\beta + p'_\alpha g_{\mu\beta}) - (1+\kappa)(g_{\alpha\mu}p'_\beta + p_\alpha g_{\beta\mu})]. \quad (8.17)$$

We do not, fortunately, intend to calculate anything with this vertex except to count the degree of divergence in Fig. 12. Naively counting powers would lead to the conclusion that this graph diverges as

$$\int d^4k \sim \Lambda^4,$$

where Λ is a cutoff. However, it turns out that careful use of gauge invariance reduces the degree of divergence by two—the graph is "only" quadratically divergent. This means that some sort of regulator method must be introduced even to give the graph a unique meaning. If not, various momentum routes are inequivalent. We can do this á la T. D. Lee (1962) by defining the following regulated propagators and vertex which satisfy the Ward identity and hence preserve ordinary gauge invariance:

$$iW_{\mu\nu}(q)_\xi = \frac{1}{q^2 + m_w^2}\left(g_{\mu\nu} + \frac{(1-\xi)q_\mu q_\nu}{\xi q^2 + m_w^2}\right) \quad (8.18)$$

and

$$V_{\beta\mu\alpha}^\xi = ie_0[g_{\alpha\beta}(p+p')_\mu + (\xi + \kappa)(g_{\alpha\mu}p_\beta + p_\alpha{}'g_{\mu\beta}) - (1+\kappa)(g_{\alpha\mu}p'_\beta + p_\alpha g_{\beta\mu})]. \quad (8.19)$$

Clearly, as $\xi \to 0$, these expressions reduce to their predecessors. Now at least for $\xi \neq 0$ the graphs have meaning. Let us define the muon electromagnetic vertex in terms of form factors $F_1(q^2)$ and $F_2(q^2)$ as follows:[68]

$$\mathcal{J}_\mu(q) = ie_0\bar{u}(p')[F_1(q^2)\gamma_\mu + (1/2m_\mu)\sigma_{\mu\nu}q^\nu F_2(q^2)]u(p), \quad (8.20)$$

where $u(p)$ is a free-muon spinor, $\sigma_{\mu\nu} = (1/2i)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$, and $q = p' - p$. Written this way, $F_1(0)$ is the charge renormalization and

$$F_2(0) = (g_\mu - 2)/2, \quad (8.21)$$

the anomalous muon magnetic moment. It turns out that Fig. 12 leads to a quadratic and logarithmic divergence in $F_1(0)$, which in the Weinberg theory—or any other—must be absorbed in the charge renormalization. If e in Eq. (8.20) is the observable charge, then the renormalization must be defined so that

$$F_1(0) = 1. \quad (8.22)$$

On the other hand, $F_2(0)$ computed from Fig. 12 is only logarithmically divergent. This can be traced back to the extra factor of q^ν in front of $F_2(q^2)$ which, it can be shown, reduces the degree of divergence by one. (Loosely speaking, one differentiates $\mathcal{J}_\mu(q)$ to obtain $F_2(0)$ and this produces an extra power in the propagator denominators in Fig. 12, thus improving the convergence.) After a number of different attempts by several authors who produced mutually contradictory results, the expression

[68] There is an additional conserved parity-violating term that is proportional to

$$\gamma_5(\gamma_\mu q^2 - (\gamma q)q_\mu)$$

that can come in because we are considering weak corrections to the electromagnetic vertex. It is not relevant to the present story.

for the W meson contribution to $F_2(0)$ was definitively settled by Brodsky and Sullivan (1967), who programmed the rather intricate trace algebra on a computer. They found, using the regularization technique described above (we have rewritten their result using the modern definition of g [see Eq. (7.62)], that

$$F_2(0) = \frac{G_F m_\mu^2}{8\sqrt{2}\pi^2}\left\{2(1-\kappa)\log(\xi) + \frac{10}{3}\right\}_{\xi=0}$$

$$\simeq 1.16 \times 10^{-9}\left\{2(1-\kappa)\log(\xi) + \frac{10}{3}\right\}_{\xi=0}. \quad (8.23)$$

The most significant feature of this expression is, of course, the fact that it diverges for all values of κ except $\kappa = 1$, which is precisely the value of κ fixed by the $SU(2) \times U(1)$ group structure in the Weinberg theory. [See Eq. (7.161) and the discussion that precedes it.] This is another beautiful example of how these gauge models cure pathologies in the weak interactions. One may also note two other properties of Eq. (8.23).

1) $F_2(0) \to 0$ as $m_\mu \to 0$. This is a special case of the theorem, to be discussed below, that the helicity conserving weak and electromagnetic interactions will not produce a magnetic moment for a "neutrino" which is an eigen-state of helicity.

2) The Weinberg theory produces finite corrections to $F_2(0)$ for the muon, which in order of magnitude are $\lesssim 10^{-8}$. In addition to Fig. 12, there are other *finite* graphs involving Higgs fields or Z^0's which have been computed (see, for example, Jackiw and Weinberg, 1972) and yield numerical results that depend on unknown Higgs masses and coupling constant ratios, but are expected to be of a similar order of magnitude. The present experimental value for $F_2(0)$ (Brodsky, 1972) is given for the muon as

$$F_2(0) = (11661.6 \pm 3.1) \times 10^{-7}$$

so that these weak interaction corrections are below the level of both experimental error and the difficult to estimate strong interaction corrections. The experimental value is at present consistent with ordinary quantum electrodynamics. It has been pointed out (Primack, 1972) that Class 2 theories with suitable choices of the masses of the heavy leptons etc. could produce weak interaction corrections large enough to approach present experimental error. Any improvement in the experiments on the muon magnetic moment could help to constrain such models.

Finally, we turn to a discussion of the "electromagnetic properties" of the neutrino, Fig. 12 with the muon and neutrino interchanged. In the early theories such as Bernstein and Lee (1963), only the conventional particle spectrum—only charged W's and the observed leptons—was considered. In such theories the quantity—J_μ^γ is the electric current—

$$\langle \nu' | J_\mu^\gamma(0) | \nu \rangle = i\bar{u}(\nu')\gamma_\mu F_1(q^2) u(\nu), \quad (8.24)$$

with $u(\nu)$ a neutrino spinor satisfying

$$\gamma_5 u(\nu) = u(\nu) \quad (8.25)$$

is an "observable." Note that, as mentioned above, Eq. (8.25) eliminates the magnetic moment term in Eq. (8.24). Since the neutrino is chargeless, we must have

$$F_1(0) = 0. \quad (8.26)$$

In these theories $F_1(q^2)$ could in principle be determined in a reaction like

$$\nu + e \to \nu + e. \quad (8.27)$$

Hence, if these theories were to be consistent, $F_1(q^2)$ had to be *finite*. In fact Fig. 12 produces—as mentioned above—a quadratic and logarithmically divergent expression, and much effort was expended to try to give these a meaning. In Class 1 models, however, to every graph with a photon exchange one must add a similar graph with a Z^0 exchange. These occur in the same order in e—the electric charge—since the various charges and masses are hooked together via the gauge group. Hence, photon exchange diagrams by themselves are *not observable*, and need not be and indeed are *not* finite, even after renormalization. What must be finite is the S-matrix for neutrino–electron scattering, including *all* radiative corrections. To show this in the U gauge in Weinberg's model involves detailed calculations which have been carried out (Bernstein and Taylor, 1972, unpublished; and especially S. Y. Lee, 1972). It turns out that if enough graphs are included, cancellations of the infinities do take place leaving small finite radiative corrections which might some day be measurable.

It is clear from what has been said in this review that the gauge theories are leading us into a new domain of large masses and small distances. What is lacking is experimental guidance to see if the whole idea is true as well as beautiful. Of theoretical speculations there are plenty.

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