The renormalization group in the theory of critical behavior*

Michael E. Fisher

Baker Laboratory and Materials Science Center, Cornell University, Ithaca, New York 14850

The renormalization group approach to the theory of critical behavior is reviewed at an introductory level with emphasis on magnetic systems. Among recent results reported are the dependence of critical exponents above T_c on dimensionality $d = 4 - \epsilon$; on the symmetry index or number of spin components, n; on the range and anisotropy of exchange couplings; and on dipolar interactions and lattice anisotropies, in ferro- and antiferromagnets. Calculations of the scaling functions for the equation of state and critical scattering are summarized.

CONTENTS

I. INTRODUCTION

This article reviews the recently developed ideas, in large part due to K. G. Wilson of Cornell University, of the renormalization group approach to the theory of critical phenomena. It also describes the application of these new ideas particularly to the theory of critical magnetic behavior, where the variety of interactions occurring in real magnetic materials present a strong challenge to our theoretical understanding. No attempt will be made to explain fully the calculational details or to expound all the results; the aim is rather to give a general flavor of the new approach and its scope, and an impression of the applications made and the results achieved. At the very least, the reader should gain a notion of the importance of the spatial dimensionality, d, of a physical system, and of the number, n, of components of the vector order parameter (e.g., of the spontaneous magnetization vector). The dimensionality will be generalized to *continuous* values and written $d = 4 - \epsilon$, where ϵ will be used as a small parameter. Similarly we will first let n, which might be termed the symmetry index or *degree of isotropy*, become infinite, and then use $1/n$ as a small parameter. In addition, the reader should become acquainted with some of the renormalization group terminology, particularly the concept of a "fixed point Hamiltonian", $\mathcal{H}^*$, and its relative stability.

At the outset, let us draw attention to some of the basic papers. Wilson (1971a) introduced the fundamental concepts of the renormalization group approach (see also Wilson and Kogut, 1974). (It should be remarked that the relation to the original renormalization group in field theory is not quite as close as the use of the same name might suggest.) The idea of an expansion in powers of $\epsilon = 4 - d$ was introduced by Wilson and Fisher (1972). In work by Wegner (1972b) and Pfeuty and Fisher (1972), anisotropic exchange was considered and the cross-over exponent ϕ was calculated for the first time. More recently, Amnon Aharony and Fisher considered dipolar interactions (Fisher and Aharony, 1973; see also Aharony and Fisher, 1973; Aharony, 1973c and 1973d). Long range isotropic power law interactions had been discussed previously with Ma and Nickel (Fisher, Ma, and Nickel, 1973) and by Suzuki (1972; Suzuki, Yamasaki, and Igarashi, 1972).[1] The early paper by Larkin and Khmel'nitski (1969; see also Aharony, 1973a) anticipated some of the ideas and, in particular, analyzed the case of dipolar interactions for Ising-like ($n = 1$) systems in $d = 3$. Wegner's general reformulation and discussion of the corrections to scaling should be especially noted (Wegner, 1972a; see also Wegner and Riedel, 1973). The method of $1/n$ expansions was developed independently by Abe (1972, 1973; Abe and Hikami, 1973) in Japan, without explicit reference to the renormalization group, and, effectively, by Ferrell and Scalapino (1972).

In this short survey, the literature cannot be further reviewed, but a list of names of some of those (not mentioned above) who have contributed significantly to the field will give an idea of the amount of work underway and already accomplished: Brézin, Wallace, Zinn-Justin, Le Guillou, Amit, de Dominicis, de Gennes, Balian, Toulouse, Migdal, Avdeeva, Hikami, Igarashi, Yamasaki, Kadanoff, Grover, Houghton, Riedel, Halperin, Hohenberg, Lubensky, Rubin, Baker, Liu, Chang, Stanley, Niemeijer, van Leeuwen, Sak, Nelson, Bruce, and Kosterlitz. Reference to the work of some of these authors will be made below. Other more detailed theoretical reviews now or soon available are by Wilson and Kogut (1974), by Ma (1973b), and by Wegner (1974a). A conference proceedings, with lively discussion reports and papers concerning the connection to field theory, has recently appeared (*Renormalization Group*, 1973).

* Revised and updated version of a lecture prepared at the Aspen Center for Physics, Colorado, and delivered at the International Conference on Magnetism, Moscow, U.S.S.R., 22 August 1973.

[1] See also S.-K. Ma, (1973a), expecially for details of the $1/n$ expansions.

II. CRITICAL BEHAVIOR

Before discussing the renormalization group method, it is appropriate to define some of the standard critical exponents and to sketch the phenomenological scaling theory of critical behavior.[2] This theory has been very successful, but it is only the recent developments that give it a more fundamental basis and offer a way of calculating the critical exponents and scaling functions, which remain incompletely specified by scaling postulates.

Let us consider a ferromagnet in equilibrium at temperature T and under the action of a uniform magnetic field H. We will use the reduced temperature variable

$$t = (T - T_c)/T_c, \tag{2.1}$$

where T_c is the critical temperature (or Curie or Néel point) and consider properties as $t \rightarrow 0$ with $H = 0$. The analogies with other physical systems are well developed[2] and will only be mentioned occasionally and in passing. In this regime, the initial susceptibility (or, for a fluid, the compressibility on the critical isochore) diverges as

$$\chi_0(T) \approx C/t^{\gamma}, \tag{2.2}$$

where the *critical exponent* γ is observed to have values in the vicinity of 1.36 for ferromagnets like Ni and Fe, but near 1.22 for anisotropic magnetic materials such as $CrBr_3$ and for fluids.[3] (For antiferromagnets and alloys, similar remarks apply to the staggered susceptibility χ_0' observable in neutron scattering.) The mean-field or "classical" prediction, $\gamma = 1$, is clearly incorrect. The specific heat in zero field (or at constant volume for a fluid) displays a critical anomaly which may be characterized by

$$C_{H=0}(T) \approx A/t^{\alpha}, \tag{2.3a}$$

or, more usefully in practice, by

$$C_{H=0}(T) \approx \tilde{A}(t^{-\alpha} - 1)/\alpha, \tag{2.3b}$$

where $\alpha \simeq 0.1$ for anisotropic magnets and for fluids, as we now know from the pioneering work of Voronel' (Bagatskii *et al.*, 1962; Voronel' *et al.*, 1963), while $\alpha \simeq 0$ for the lambda transition in liquid helium four, and $\alpha \simeq -0.1$ for Ni and other isotropic magnets. Note from Eq. (2.3b) that $\alpha = 0$ generally implies a logarithmic divergence while $\alpha < 0$ implies a finite cusp at which, however, (for $\alpha > -1$) dC_H/dt diverges to ∞.[2] Classical theory predicts only a jump discontinuity in $C(T)$.

The scaling theory of critical behavior now asserts that the singular part of the free energy $F(T, H)$ varies asymptotically as

$$f(T, H) = -(k_BT)^{-1}F_{\text{sing}}(T, H) \approx t^{2-\alpha}Y(H/t^{\Delta}), \tag{2.4}$$

where the gap exponent Δ is determined in terms of α and γ by

$$\Delta = \tfrac{1}{2}(2 - \alpha + \gamma). \tag{2.5}$$

For a fluid the magnetic field H is to be replaced by the chemical potential difference $\mu - \mu_\sigma(T)$, where $\mu_\sigma(T)$ is the value on the vapor pressure curve and its linear extension.[2] The *scaling function* $Y(y)$ depends only on a single variable, but is not otherwise given explicitly by the theory. From Eq. (2.4) we find that the spontaneous magnetization (or, for a fluid, the density discontinuity) vanishes when $t \rightarrow 0$ as

$$M_0(T) \approx B\,|\,t\,|^{\beta}, \tag{2.6a}$$

where the exponent is predicted by the *exponent relation*

$$\beta = \tfrac{1}{2}(2 - \alpha - \gamma). \tag{2.6b}$$

Experimental observation confirms this relation with values of β from about 0.31 for alloys to 0.36 for magnets. In addition, it follows that the equation of state, $M = \mathfrak{M}(T, H)$, can be written asymptotically in the reduced or scaled form

$$M/t^{\beta} \approx W(H/t^{\Delta}), \tag{2.7}$$

where $W(y)$ is again a single-variable scaling function. This relation has been strikingly verified in a number of experiments.[3] The work of Comly and Kouvel, and of Ho and Litster on $CrBr_3$ might especially be cited.[2,3]

In addition to thermodynamic behavior, the variation of the scattering intensity with wave vector $\mathbf{q}$, and temperature, is of particular interest in the critical region. This is proportional to

$$\hat{G}(\mathbf{q}, T) = \sum_{\mathbf{x}} \exp(i\mathbf{q}\cdot\mathbf{x})G(\mathbf{x}, T), \tag{2.8}$$

where the basic two-point correlation function is

$$G(\mathbf{x}, T) = \langle \vec{S}_0 \cdot \vec{S}_{\mathbf{x}} \rangle, \tag{2.9}$$

in which $\vec{S}_{\mathbf{x}}$ denotes a (localized) spin at site $\mathbf{x}$. [In a fluid, the pair density function $g_2(\mathbf{x})$ replaces $G(\mathbf{x})$.] At the critical point itself, one has

$$G_c(\mathbf{x}) \approx D_c/x^{d-2+\eta}, \quad \text{or} \quad \hat{G}_c(\mathbf{q}) \approx \hat{D}_c/q^{2-\eta},$$
$$\text{as} \quad x \rightarrow \infty, \quad \text{or} \quad q \rightarrow 0, \tag{2.10}$$

where the exponent η, which is hard to measure experimentally, is observed to lie in the range 0.03–0.1 (in disagreement with the classical, Ornstein–Zernike value $\eta = 0$). As $t \rightarrow 0$ (in zero field) scaling predicts the form

$$G(\mathbf{x}, T) \approx x^{-d+2-\eta}D(x/\xi), \qquad \xi \sim t^{-\nu}, \tag{2.11}$$

[2] See, e.g., M. E. Fisher, 1967, Rept. Prog. Phys. **30,** 615, and Fisher, M. E., 1970, *Critical Phenomena, Proceedings of the International School of Physics: Enrico Fermi Summer School Course Varenna, 1970, No. 51,* edited by M. S. Green (Academic, New York, 1971.) A general formal discussion of scaling, including hyperscaling and conformal covariance, etc., is given by M. E. Fisher, 1973, in *Proceedings,* the Nobel Symposium No. 24, *Collective Properties of Physical Systems,* edited by B. Lundqvist and S. Lundqvist (Academic, New York, 1973.)

[3] For recent reviews of experimental data, see M. Vicentini-Missoni, (1970) and L. J. de Jongh and A. R. Mie-dema, (1974).

where ξ is the correlation length or, equivalently,

$$\hat{G}(\mathbf{q}, T) \approx Ct^{-\gamma}\hat{D}(q^2/t^{2\nu}), \tag{2.12}$$

The correlation length exponent is given by

$$\nu = \gamma/(2-\eta). \tag{2.13}$$

The scaling function $\hat{D}(z^2)$ represents the scattering "line shape" near T_c. When $\eta > 0$ it must necessarily deviate from a simple Lorentzian (or OZ) form close to the critical point, although this may be hard to detect (as will be seen further below).

III. THEORY

The task of theory is now clear: firstly, it should show how to calculate the exponents, α, γ, and η, as functions of whatever essential physical parameters are needed to determine them. (To find what these are, is, of course, part of the problem!) Secondly, theory should predict or justify scaling (or, perhaps, reveal its failure). Thirdly, it should lead to explicit calculations of the scaling functions $Y(y)$, $W(y)$, $\hat{D}(z^2)$, etc. Lastly, a complete theory ought to yield a description of the *corrections* to the asymptotic scaling laws and concrete estimates of their magnitude.

To build a thoery let us start with a description of space. We may consider a lattice structure of spacing a, generated by a set of nearest neighbor vectors $\boldsymbol{\delta}$. The lattice sites will have coordinate vectors $\mathbf{x} = (x_i)$ with $i = 1,2, \ldots d$, where d is the *spatial dimensionality*. By habit we normally consider only $d = 3$, 2, and 1. However, d enters theoretical calculations in an essential way only through space and momentum (or wave number) integrals such as

$$\int d^d x \equiv C_d \int_0^\infty x^{d-1}\, dx, \tag{3.1}$$

$$\int_{\mathbf{q}} \equiv \int \frac{d^d q}{(2\pi)^d} \equiv \hat{C}_d \int_0^\infty q^{d-1}\, dq, \tag{3.2}$$

where $C_d = (2\pi)^d \hat{C}_d = 2\pi^{d/2}/\Gamma(\frac{1}{2}d)$. Since the gamma function $\Gamma(z)$ is definable for arbitrary z, we may likewise extend the definition of dimensionality to *continuous values* of d. [More thought is needed when the integrand depends on scalar products, such as $\mathbf{k}\cdot\mathbf{x}$, $\mathbf{q}\cdot\mathbf{r}$, etc., but this may also be dealt with (Wilson, 1972).] It will turn out later that, having defined continuous d, the difference

$$\epsilon = 4 - d, \tag{3.3}$$

forms a natural and important small parameter.

Once we have a lattice with sites $\mathbf{x}$ we must populate it with spins $\vec{S}_{\mathbf{x}}$. (We apparently consider localized spins but, in reality, all that is assumed is a "spin density field"; more generally we would take a scalar density field, or a complex, second-quantized wave-mechanical field, etc.) We suppose the spin vector has *n components*, i.e., $S_{\mathbf{x}} = (S_{\mathbf{x}}^\mu)$ with $\mu = 1, 2, \ldots n$, which enter equally into interactions. The basic cases for this *symmetry index* are

(a) $n = 3$, ordinary or Heisenberg spins, $\vec{S} = (S^x, S^y, S^z)$;

(b) $n = 2$, XY or "planar" spins, $\vec{S} = (S^x, S^y)$.

(c) $n = 1$, uniaxial or Ising spins, $\vec{S} = S^z$.

The last, Ising-like case also describes classical density fields as appropriate to fluids, alloys, etc. The $n = 2$ or XY-like case includes quantal fields since the wave function $\psi = (\psi', \psi'')$ has independent but equivalent, real and imaginary components.

We may also consider (as observed by Stanley, 1968) the limit $n \to \infty$; this corresponds precisely to the exactly soluble "spherical model" invented by Berlin and Kac. For this limit one finds that scaling is obeyed with exponents

$$\alpha = (4-d)/(d-2), \qquad \beta = \tfrac{1}{2}, \qquad \gamma = 2/(d-2),$$
$$\eta = 0, \qquad \text{for } 2 < d \leq 4 \ (n = \infty), \tag{3.4}$$

but

$$\alpha = 0, \qquad \beta = \tfrac{1}{2}, \qquad \gamma = 1, \qquad \text{and} \quad \eta = 0,$$
$$\text{for} \qquad d \geq 4 \ (n = \infty). \tag{3.5}$$

Note the sharp boundary in critical behavior at $d = 4$ which turns out to be quite general.

Now one may (Balian and Toulouse, 1973; Fisher, 1973) further generalize to *continuous n*. On considering *negative* values of n one discovers that $n = -2$ is also a soluble case; namely the so-called Gaussian model for which scaling is again obeyed with exponents.

$$\alpha = 2 - \tfrac{1}{2}d = \tfrac{1}{2}\epsilon, \qquad \beta = \tfrac{1}{4}d - \tfrac{1}{2} = \tfrac{1}{2} - \tfrac{1}{4}\epsilon,$$
$$\gamma = 1, \qquad \eta = 0, \qquad \text{for} \quad d \leq 4 \quad (n = -2). \tag{3.6}$$

The required statistical mechanical analysis may also be performed exactly for one-dimensional models (Balian and Toulouse, 1974; Nelson and Fisher, 1974a); here one finds $\eta = 1$ and $\beta = 0$. Finally, following the hint of the spherical model, one may anticipate that for $d > 4$ and *all* values of n, critical behavior will be classical with $\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = 1$ and $\eta = 0$. (This is confirmed by the renormalization group calculations.)

With this information, it is instructive to make a plot of the (d,n) plane for $1 \leq d \leq 4$ and $-2 \leq n \leq \infty$; see Fig. 1. On the boundaries of this region the critical behavior is known; unfortunately, we are seriously interested only in the interior! Specifically, the values $n = 1$, 2, and 3 can be realized in magnetic materials. Superfluid helium, and helium three/four mixtures near T_λ, are described by $n = 2$; normal fluids, fluid mixtures, and alloys by $n = 1$. The case $n = 0$ turns out to describe the statistics, size, and shape of a long self-avoiding walk, or polymer chain in solution, as pointed out by de Gennes (1972).[4] Regarding spatial dimensionality, $d = 3$ is always available, while $d = 2$ applies to films, monolayers, and submonolayers. Two-dimensional systems can also be well-approximated physically by carefully chosen layered magnetic systems. One dimension can also be modeled with surprising accuracy by suitably "designed" crystals with linear chains of magnetic ions (see de Jongh and Miedema, 1974).

[4] See also R. G. Bowers and A. McKerrell, (1973), J. de Cloiseaux (to be published); and P. G. Gerger and M. E. Fisher (1974).

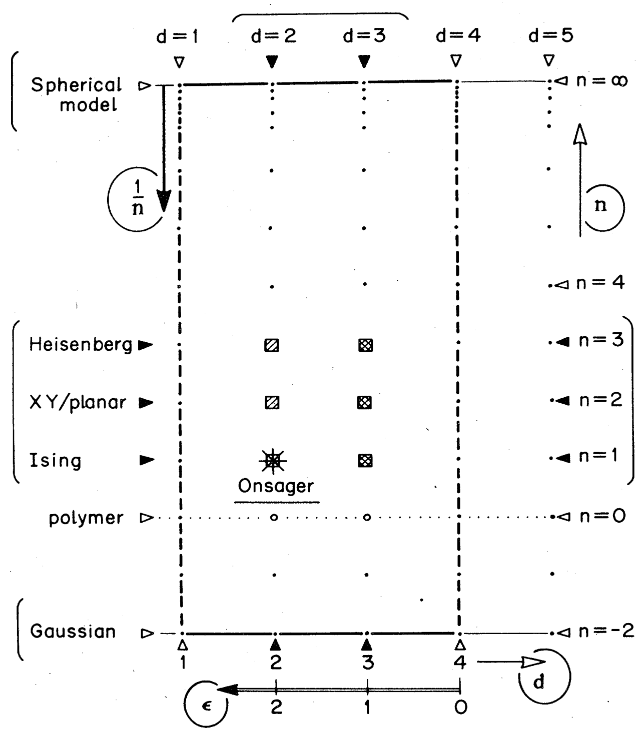

FIG. 1. Diagram of the (d, n) plane showing the expansion variables $\epsilon = 4 - d$ and $1/n$, the boundaries at $n = \infty$ and -2, and $d = 1$ and 4, and various physically relevant cases.

In the interior region of Fig. 1, the only exact results are for $n = 1$ and $d = 2$, where Onsager's famous solution of the two-dimensional Ising model leads to the answers.[2] However, the renormalization group approach will enable us to penetrate in from the boundaries at $d = 4$ and $n = \infty$, by expansions in ϵ and $1/n$, respectively. Hence we will gain a fairly accurate picture of the variation of critical exponents over the whole (d,n) plane. Indeed, for systems with *short range*, *isotropic* coupling, i.e., an interaction Hamiltonian of the form

$$\mathcal{H}\{\vec{s}_{\mathbf{x}}\} = \mathcal{H}_{\text{iso. exch.}} = -\tfrac{1}{2}\sum_{\mathbf{x},\mathbf{x}'} J(\mathbf{x} - \mathbf{x}')\vec{s}_{\mathbf{x}}\cdot\vec{s}_{\mathbf{x}'} \tag{3.7}$$

(as we have assumed implicitly above), the parameters d and n seem to be the only ones determining the critical exponents. As the soluble examples indicate, we may expect the exponents to vary continuously with n and d (at least over the main regions of interest).

In writing the above Hamiltonian, the normalized spins or local variables

$$\vec{s} = \vec{S}/[S(S+1)]^{1/2} \tag{3.8}$$

have been introduced. (For a fluid, s is a scalar ($n = 1$) and represents the density.) The Hamiltonian (3.7) then represents attractive pair interactions (Fisher, 1967, 1970, 1973). If one considers $S \to \infty$, the spins $\vec{s}$ become classical vectors of unit length. As far as one can see theoretically, at present, the quantal effects implied by finite spin values S do not play a role in determining critical exponents or scaling functions. (The same goes for quantal effects in normal fluids and alloys.)

For many theoretical purposes it is also convenient to allow the spin length $|\vec{S}|$ to vary continuously. It is then essential, however, to include in the statistical probability (or Boltzmann factor) a *spin weighting function* which restricts the fluctuations in spin length. The partition function for a system of N spins is then

$$Z_N[\bar{\mathcal{H}}] = \mathrm{Tr}_N\{\exp \bar{\mathcal{H}}\}, \tag{3.9}$$

where the reduced Hamiltonian $\bar{\mathcal{H}}$ is defined by

$$\bar{\mathcal{H}} = -(\mathcal{H}/k_BT) + \mathcal{W}, \tag{3.10}$$

in which $\mathcal{W}$ represents the spin weighting function: this will normally be a sum of identical terms $-w(\vec{s}_j)$ for each spin which become very large and negative as $|\vec{s}|$ becomes large (specific forms are mentioned below). For a fluid system, the weighting function represents the limitations on density imposed by the presence of hard repulsive cores in the pair interaction. Note that specification of $\bar{\mathcal{H}}$ determines the temperature, as well as the external magnetic field, pressure, etc., and all the interactions, which are presumed to be translationally invariant (although inequivalent crystal sites, etc., may be allowed). Finally the thermodynamics follow as usual from the free energy per spin

$$f[\bar{\mathcal{H}}] = -F(T,H)/k_BT = \lim_{N\to\infty} N^{-1}\ln Z_N[\bar{\mathcal{H}}]. \tag{3.11}$$

The thermodynamic limit, $N \to \infty$, is of course essential if true critical behavior is to be investigated.

IV. THE RENORMALIZATION GROUP APPROACH

The general ideas underlying the renormalization group approach may be formulated as follows (Wilson, 1971; Wilson and Kogut, 1974; Wegner 1972a).

(a) The given (or initial) Hamiltonian $\bar{\mathcal{H}}$ is transformed or *renormalized* to obtain a new Hamiltonian $\bar{\mathcal{H}}'$; formally we write

$$\bar{\mathcal{H}} \Rightarrow \bar{\mathcal{H}}' = \mathbf{R}[\bar{\mathcal{H}}]. \tag{4.1}$$

(b) The renormalization group operator $\mathbf{R}$ acts to reduce the number of degrees of freedom (i.e. spin variables) from N to

$$N' = N/b^d, \tag{4.2}$$

where d is again the dimensionality, while the *spatial rescaling factor*, b, exceeds unity. Different types of renormalization operator may be constructed. Frequently $\mathbf{R}$ is defined via a *partial trace* over $(N - N')$ of the N spin variables, or over suitably *transformed* variables (see below), so that the renormalized Hamiltonian itself is then given by

$$\exp \bar{\mathcal{H}}' = \mathrm{Tr}'_{N-N'}\{\exp \bar{\mathcal{H}}\}. \tag{4.3}$$

(c) The essential condition to be satisfied by $\mathbf{R}$ is that the *partition function must be preserved*, that is,

$$Z_{N'}[\bar{\mathcal{H}}'] = Z_N[\bar{\mathcal{H}}]. \tag{4.4}$$

This follows from (4.3), for example, merely by taking the remaining part of the trace over N' variables, to calculate $Z_{N'}[\bar{\mathcal{H}}']$. We also assume implicitly that the renormalized Hamiltonian $\bar{\mathcal{H}}'$ again displays translational invariance.

(d) In order to preserve the spatial density of degrees of freedom (i.e., of spins), all *spatial vectors*, entering into correlation functions, etc., are *rescaled* by the factor b according to

$$\mathbf{x} \Rightarrow \mathbf{x}' = \mathbf{x}/b. \tag{4.5}$$

Momenta are, naturally, rescaled reciprocally by

$$\mathbf{q} \Rightarrow \mathbf{q}' = b\mathbf{q}.$$

(e) Finally, in order to preserve the basic spin fluctuation magnitude, the renormalized *spin vectors are rescaled* as

$$\vec{s}_{\mathbf{x}} \Rightarrow \vec{s}'_{\mathbf{x}'} = \vec{s}_{\mathbf{x}}/c, \tag{4.6}$$

where the spin rescaling factor c depends on $\bar{\mathcal{H}}$, i.e., $c = c[\mathcal{H}]$. In more complex cases different rescaling factors for different classes of spins may be needed.[5]

Strictly this last step is an essential feature only of *linear* renormalization groups in which the basic local variable (or spin) is transformed into an equivalent renormalized variable to which it is linearly related. Not all renormalization groups are of this character[6] although we will only discuss linear groups.

Evidently the renormalization group operator depends on b and c. Under its action, it follows from Eqs. (3.10) and (4.4) that the free energy transforms according to

$$f[\bar{\mathcal{H}}'] = b^d f[\bar{\mathcal{H}}]. \tag{4.7}$$

Similarly from Eq. (4.6) one finds, for a linear renormalization group, that the basic spin–spin correlation function transforms as

$$G[\mathbf{x};\bar{\mathcal{H}}] = c^2 G[\mathbf{x}/b;\bar{\mathcal{H}}']. \tag{4.8}$$

These two relations ultimately lead to scaling properties.

The exact construction of a renormalization group is in general a very difficult task. For this reason Wilson, in his original work (Wilson, 1971a), devised an *approximate* renormalization group (as a nonlinear integral operator) which later, however, turned out (Wilson and Fisher, 1972; Wilson, 1972) to be exact to first order in $\epsilon = 4 - d$ (although approximate in higher orders). Baker (1972; Baker and Golner, 1973) then showed that Wilson's approximate renormalization transformation was actually exact for a special class of essentially one-dimensional spin models with hierarchical interactions of the general type first introduced by Dyson (1969). These models, however, are rather unrealistic and cannot be solved exactly (although see Bleher and Sinai, 1973). Much simpler renormalization groups can be constructed in exact closed form (as finite dimensional algebraic recursion relations) for one-dimensional Ising models with additional variables such as staggered magnetic fields, second neighbor interactions, etc.[7] In these cases the critical behavior (at $T_c = 0$) can be found exactly and thus the full apparatus of the renormalization group approach can be studied analytically in explicit and instructive detail.[7]

Having defined a renormalization group $\mathbf{R}$, the theory takes the following steps:

(i) The transformation is iterated:

$$\bar{\mathcal{H}}' = \mathbf{R}[\bar{\mathcal{H}}], \qquad \bar{\mathcal{H}}'' = \mathbf{R}[\bar{\mathcal{H}}'], \qquad \cdots. \tag{4.9}$$

(ii) One then attempts, by varying the parameters of the initial Hamiltonian, to locate a *fixed point Hamiltonian*, $\mathcal{H}^*$, which is approached under iteration. The fixed point Hamiltonian is defined by its invariance under $\mathbf{R}$, namely,

$$\mathbf{R}[\mathcal{H}^*] = \mathcal{H}^*. \tag{4.10}$$

(iii) Examination of the transformation relation (4.8) for correlations reveals that a fixed point Hamiltonian is *critical* in the sense that its basic pair correlation function is long ranged. Specifically one finds from (4.8) a functional equation for $G[\mathbf{x};\mathcal{H}^*]$ with unique solution

$$G[\mathbf{x};\mathcal{H}^*] \sim 1/x^{2\omega}, \qquad c^* = c[\mathcal{H}^*] = b^{-\omega}. \tag{4.11}$$

Comparison with (2.9) shows that the exponent η is thus determined via

$$c^* = b^{-(d-2+\eta)/2}. \tag{4.12}$$

All Hamiltonians $\bar{\mathcal{H}}$ that approach $\mathcal{H}^*$ under iteration lie on the *surface of criticality* in the space of Hamiltonians and are similarly critical. [The relation (4.12) demonstrates that in general one also has to vary the spin scaling factor dependence $c = c[\bar{\mathcal{H}}]$ in order to find a nontrivial fixed point.]

(iv) To discuss the approach to criticality the renormalization operator is linearized about $\mathcal{H}^*$. This yields

$$\bar{\mathcal{H}}' = \mathbf{R}[\bar{\mathcal{H}}] = \mathbf{R}[\mathcal{H}^* + hQ] = \bar{\mathcal{H}}^* + h\mathbf{L}Q + O(h^2), \tag{4.13}$$

where, now, $\mathbf{L}$ is simply a *linear* operator (on Hamiltonians).

(v) Having a linear renormalization operator, one can ask for its *eigenoperators* Q_j, and *eigenvalues* Λ_j, defined as usual through

$$\mathbf{L}Q_j = \Lambda_j Q_j. \tag{4.14}$$

[5] See Nelson and Fisher (1974b), Nelson, Kosterlitz, and Fisher, (1974). Applications to displacive transitions in peorvskites under ansiotropic stress have been discussed along similar lines by Aharony and Bruce Phys. Rev. **33**, 427 (1974).

[6] Bell and Wilson (to be published); Niemeijer and van Leeuwen, (1973), and (1974) describe a nonlinear renormalization group for Ising spins, $s = \pm 1$, and apply it to the numerical solution of the two-dimensional Ising model; see also Nelson and Fisher 1974a.

[7] Nelson and Fisher (1974a). An exact renormalization group for the one-dimensional Ising model was studied independently by L. P. Kadanoff, 1973, in *Renormalization Group* (1973), pp. 21–23.

The operator **L** and, hence, its eigenvalues depend explicitly on b, but from the semigroup (or iterable) property of **R** one sees that one should have

$$\Lambda_j = b^{\lambda_j}, \tag{4.15}$$

where the λ_j are independent of the original choice of b. The Q_j form a *spectrum* of *eigenoperators* or critical variables. One may generally identify one of these Q_1 ($\approx\mathcal{E}$) as the energy density (or, at least, its even or symmetrical part) and another Q_2 ($\approx M$) as the order parameter, or magnetization. The corresponding eigenvalue exponents λ_1 and λ_2 will be positive for normal critical behavior. [The constant or spin-independent term in the Hamiltonian may always be identified (Wegner, 1972a) as an eigenoperator Q_0 with $\lambda_0 = d$. This then plays a special but, for many purposes, largely ancillary role, so we will not discuss it here (Wegner 1972a; Ma, 1973b; Nelson and Fisher, 1974a. In general there are also *redundant* or nonphysical operators, with spurious exponents depending on the choice of **R**, which do not affect physical observables (Wegner, 1974b).]

(vi) Finally, for a near-critical Hamiltonian $\bar{\mathcal{H}}$, one expands about the fixed point $\bar{\mathcal{H}}^*$ in terms of the Q_j by writing

$$\bar{\mathcal{H}} = \mathcal{H}^* + \sum_j h_j Q_j. \tag{4.16}$$

(From a rigorous viewpoint the completeness of the Q_j may well be questioned, but we will expect at least "asymptotic completeness" as regards the description of the behavior of thermodynamic expectation values near criticality.) The free energy $f[\mathcal{H}]$ may then be viewed as a function of the coefficients h_j which, in turn, represent the (linear) *scaling fields*. In particular, h_1 will vary linearly with T, so that essentially one has $T - T_c \propto t \propto h_1$. Similarly h_2 will be the ordering field, i.e., proportional to the direct magnetic field H for a ferromagnet). On inserting the expansion (4.16) into (4.13) and using (4.14), one finds

$$\mathbf{R}[\bar{\mathcal{H}}] = \bar{\mathcal{H}}' = \mathcal{H}^* + \sum_j h_j \Lambda_j Q_j + O(h^2). \tag{4.17}$$

On neglecting the nonlinear terms, this may be re-expressed as the diagonalized *recursion relations*

$$h_j' \approx \Lambda_j h_j. \tag{4.18}$$

On using (4.14) and the basic free energy renormalization relation (4.7), we finally obtain

$$f(h_1, h_2, h_3, \cdots) \approx b^{-d} f(b^{\lambda_1}h_1, b^{\lambda_2}h_2, b^{\lambda_3}h_3, \cdots). \tag{4.19}$$

The rescaling factor b is now effectively arbitrary since, by iterating (4.18), we find

$$h_j^{(l)} = \Lambda_j^l h_j = b^{l\lambda_j} h_j = (b_l)^{\lambda_j} h_j, \tag{4.20}$$

in which $b_l = b^l$ can be made as large as we please.

(vii) The relation (4.19) is, in fact, an *asymptotic homogeneity relation* which implies scaling. To see this we choose b so that $b^{\lambda_1}t = 1$ and recall that $h_1 \approx k_1 t$ and $h_2 \approx k_2 H$, where k_1 and k_2 are constants. We then obtain

$$f(t, H, h_3, \cdots) \approx t^{d/\lambda_1} f(k_1, k_2 H/t^{\lambda_2/\lambda_1}, h_3/t^{\lambda_3/\lambda_1}, \cdots). \tag{4.21}$$

Comparison with the phenomenological scaling relation (2.4) now yields the identifications, first, of the exponents as

$$2 - \alpha = d/\lambda_1, \qquad \Delta = \lambda_2/\lambda_1, \tag{4.22}$$

and then of the scaling function as

$$Y(y) \propto f(k_1, k_2 y), \tag{4.23}$$

where we have dropped the dependence on $h_3, h_4, \cdots$ which, however, we will return to shortly. (The nonsingular parts of the total free energy are looked after by the spin-independent field h_0 which we are neglecting.)

Similar arguments applied to the correlation function renormalization relation (4.8) using (4.12), lead to scaling of the correlation functions in agreement with (2.11) and yield the identification $\nu = 1/\lambda_1$.

Four principal conclusions follow from the analysis sketched above:

(A) The existence of a fixed point Hamiltonian about which the renormalization group is linearizable implies *critical point scaling* of the free energy and of the correlations for all Hamiltonians which lie sufficiently close to $\mathcal{H}^*$.

(B) The values of the *critical point exponents* follow from the eigenvalues of the linearized renormalization group; specifically one has the formula (4.12) for η, the relations (4.22), and

$$\gamma = (2 - \eta)/\lambda_1 = (2\lambda_2 - d)/\lambda_1, \tag{4.24}$$

$$\nu = 1/\lambda_1 = (2 - \alpha)/d. \tag{4.25}$$

(C) Over large classes of Hamiltonian, $\mathcal{H}$, one has *universality*, in that the values of the critical exponents *and* the character of the scaling functions do not depend on the "details" of the Hamiltonian. The *universality class* encompasses all Hamiltonians on or near the critical surface (or subspace) which flow into the given fixed point. More concretely, operators Q_k for which $\Lambda_k < 1$, or $\lambda_k < 0$, are *irrelevant*: their addition to a Hamiltonian near $\mathcal{H}^*$ cannot change critical exponents. This is evident from (4.20) which shows that the corresponding fields h_k decay rapidly under iteration of the renormalization group and simply "relax" back to the fixed point values. The fields h_k can then asymptotically be set equal to zero in (4.21). On the other hand, irrelevant operators do contribute to the *corrections* to asymptotic scaling, which are nonuniversal in character (see Wegner, 1972a; Wegner and Riedel, 1973; and also below).

(D) In addition, and most significantly, the fixed point formalism yields a description of *crossover phenomena*. In particular, there may be extra operators Q_j (over and above $Q_1 \approx \mathcal{E}$ and $Q_2 \approx M$) which have $\Lambda_j > 1$, or $\lambda_j > 0$, and so are *relevant*: as follows from Eq. (4.20), the effects of a relevant operator grow unstably under iteration and so carry $\bar{\mathcal{H}}$ *away* from the fixed point $\mathcal{H}^*$ to a *new* fixed point with new (in general) different eigenvalues and, hence, with *different critical exponents*. (The instability associated with Q_1 and Q_2 is removed by adjusting T and H, or t and h_1, to their critical values.)

To appreciate the significance of this behavior we will, in Sec. V, review briefly the scaling theory of crossover effects and show how it relates to the renormalization group analysis. This section may, however, be skipped on a first reading.

V. CROSSOVER AND MULTICRITICAL PHENOMENA

The simplest example of a crossover phenomenon is probably that occuring in a weakly anisotropic magnetic system.[8] Consider a system with Hamiltonian

$$\mathcal{H} = \mathcal{H}_{\text{iso. exch.}} + g\mathcal{H}_{\text{aniso. exch.}}, \tag{5.1}$$

where the leading isotropic part has the form of Eq. (3.7), which involves only rotationally invariant $\vec{s}_{\mathbf{x}} \cdot \vec{s}_{\mathbf{x}'}$ coupling between spins. For concreteness let the spins be $n = 3$ component, or Heisenberg spins and suppose $J(\mathbf{x} - \mathbf{x}')$ vanishes except for nearest neighbor pairs, $(\mathbf{x},\mathbf{x}')$, on a three-dimensional lattice. When $g = 0$ the total Hamiltonian $\mathcal{H}$ thus describes a Heisenberg system whose susceptibility diverges as

$$\chi_0(T,g = 0) \approx Ct^{-\gamma}, \tag{5.2}$$

where $\gamma = \gamma(n = 3) \simeq 1.38$ is the Heisenberg critical exponent (see e.g. Ritchie and Fisher, 1972). But suppose now there is also anisotropic coupling present of the form, say,

$$\mathcal{H}_{\text{aniso. exch.}} = -J \sum_{(\mathbf{x},\mathbf{x}')} [s_{\mathbf{x}}^z s_{\mathbf{x}'}^z - \tfrac{1}{2}(s_{\mathbf{x}}^x s_{\mathbf{x}'}^x + s_{\mathbf{x}}^y s_{\mathbf{x}'}^y)]. \tag{5.3}$$

Note we include the $s^x s^x$ and $s^y s^y$ terms in the way exhibited so that $\langle \mathcal{H}_{\text{aniso. exch.}} \rangle_{g=0} = 0$; this choice is dictated by considerations of symmetry [and to do otherwise can yield very misleading results (Fisher and Pfeuty, 1972)].

Now for positive g the strength of the $s^z s^z$, or parallel coupling, is increased to $J(1 + g)$ but the perpendicular, $s^x s^x$ and $s^y s^y$, coupling is decreased to $J(1 - \frac{1}{2}g)$. Hence one expects the system to order into a uniaxial or Ising-like state with a *reduced symmetry index* of only $m = 1$. The transverse or perpendicular fluctuations may become quite large, but they will not go critical. Correspondingly the critical behavior will become

$$\chi_0(T,g > 0) \approx \dot{C}(g)\dot{t}^{-\dot{\gamma}}, \tag{5.4}$$

where owing to the shift in the critical temperature to $T_c(g)$, we have defined the shifted reduced temperature variable

$$\dot{t} = [T - T_c(g)]/T_c(g). \tag{5.5}$$

Correspondingly $\dot{\gamma}$ denotes the new, anisotropic critical exponent which will have the Ising value $\gamma(m = 1) \simeq 1.25$ (see Pfeuty, Fisher, and Jasnow, 1973, 1974).

The situation for negative g is similar, except that now the perpendicular, xx, and yy spin components dominate so that the ordered state and critical behavior are expected to be XY-like with reduced symmetry index $m = 2$ and $\dot{\gamma}$ in (5.4) replaced by $\ddot{\gamma} = \gamma(m = 2) \simeq 1.31$ (see e.g. Pfeury, Fisher, and Jasnow, 1974).

In the renormalization group context we thus expect that at the Heisenberg, or $n = 3$, fixed point there will be an additional relevant critical variable, say, Q_3, proportional to $\mathcal{H}_{\text{aniso. exch.}}$ [in the symmetry-adapted form given in Eq. (5.3)]. Small but nonzero values of the corresponding field $h_3 \approx k_3 g$ then build up unstably and carry the system either to an Ising-like, $n = 1$ fixed point, or, for opposite sign of $h_3 \sim g$, to an XY-like, $m = 2$ fixed point. Around each of these new fixed points there will again be scaling.

However, the two forms (5.2) and (5.4) (and, similarly, for the XY critical behavior) can be combined for small g, (Riedel and Wegner, 1969) by the scaling hypothesis

$$\chi_0(T, g) \approx Ct^{-\gamma}X(g/t^{\phi}), \tag{5.6}$$

where $X(z)$ is the crossover scaling function [normalized by $X(0) = 1$] and ϕ is the *crossover exponent*, first introduced by Riedel and Wegner (1969, 1970; Riedel, 1971). (The hypothesis (5.6) has actually been written in an "extended" form using t in place of $\dot{t}$ (Fisher and Jasnow, 1974; Pfeuty, Jasnow, Fisher, 1973, 1974).) Divergence of the crossover function at particular positive and negative arguments, $\dot{z}$ and $\ddot{z}$, then reproduces the anisotropic behavior (5.4) and its analog for $g < 0$. Furthermore, (5.6) leads straightforwardly (Riedel and Wegner, 1969; Pfeuty, Jasnow, and Fisher, 1973) to predictions for the g dependence of the anisotropic amplitude $\dot{C}(g)$ and of the critical temperature shift, $\Delta T_c(g) = T_c(g) - T_c(0)$. A moment's reflection shows that while g/t^{ϕ} remains small compared to unity, the observed critical behavior will closely follow the isotropic, $n = 3$ form (5.2). However, as t becomes smaller, g/t^{ϕ} becomes larger and, for $g > 0$, approaches $\dot{z}$. The observed behavior must then "cross over" to the characteristically anisotropic form of divergence with exponent $\dot{\gamma}$. The actual shape of the crossover scaling function, $X(z)$, determines how rapidly or how gradually the actual crossover occurs.

Now the crossover hypothesis (5.6) can clearly be extended to the whole free energy by postulating

$$f(t, H, g) \approx t^{2-\alpha}Y(H/t^{\Delta}, g/t^{\phi}), \tag{5.7}$$

which reduces to the original scaling hypothesis (2.4) when $g = 0$. Finally, comparison with the general renormalization group prediction (4.21) yields the same exponent identifications (4.22) as before but *also* gives the crossover exponent as

$$\phi = \lambda_3/\lambda_1 \tag{5.8}$$

and identifies the generalized free energy scaling function as

$$Y(y, z) = f(k_1, k_2 y, k_3 z). \tag{5.9}$$

[8] Fisher, (1966); Riedel and Wegner, (1969); Riedel and Wegner, (1970); Riedel, (1971); Fisher and Pfeuty, (1972); Fisher and Jasnow, (1974). For a numerical study of anisotropy crossover, using series extrapolations, see Pfeuty, Jasnow, and Fisher (1974).

In this last formula we have again assumed that all other variables are irrelevant (or, if relevant, absent for reasons of further symmetry, etc.). More generally, if there are other relevant variables present they will have crossover exponents given by

$$\phi_j = \lambda_j/\lambda_1. \tag{5.10}$$

This aspect of the theory is particularly applicable to magnetic critical behavior since, as will be indicated, there are many small, but nonzero terms in real magnetic Hamiltonians that are possible candidates for relevant operators.

The general formalism of crossover scaling and the renormalization group approach apply equally to behavior near *multicritical points*, which are characterized physically by the meeting or ending of lines of critical points. The simplest examples of such points are *tricritical points* (Griffiths, 1970, 1973), which have been observed as magnetic transitions in dysprosium aluminum garnet (DAG) and $FeCl_2$, as superfluid/fluid transitions in He^3–He^4 mixtures, as displacive transitions in ammonium halides, and as normal fluid transitions in multicomponent mixtures (Griffiths and Widom, 1973). In the space of field variables normally accessible physically, a critical line ends at the tricritical point and continues as a first order transition line (Griffiths, 1970, 1973). The first renormalization group analysis of a tricritical point was made by Riedel and Wegner (1972).

More recently it has been pointed out Fisher and Nelson (1974) that the *spin-flop point* in the $(H_{||},T)$ phase diagram of a weakly anisotropic uniaxial antiferromagnet, such as MnF_2, can be regarded as a *bicritical* point and also discussed in scaling terms. A renormalization group analysis of bicritical and related tetracritical points, has been presented by Nelson, Kosterlitz, and Fisher (1974). We will not, however, in this short review, enter further into these interesting but intricate phenomena.

It is clear from the discussion of crossover scaling that a small crossover exponent ϕ_i implies a very slow crossove so that it may be very hard experimentally to see the new exponents $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, etc. In practice with a restricted range of $\log |\dot{t}|$, one may instead observe an effective, or apparent critical exponent taking some intermediate value which varies continuously with the coupling parameter g between, say, γ and $\dot{\gamma}$.[9] In most real cases one must recognize that such continuous variation of exponents with parameters is *only* an observational artifact. However in the case of a *marginal operator*, Q_i, which has $\Lambda_i \equiv 1$ or $\lambda_i = 0$ so that $\phi_i \equiv 0$, the renormalization group does, in fact, allow such *continuous variation of exponents* with a field g_i in the Hamiltonian. [However, the existence of a marginal operator does not by itself imply continuously variable exponents (Kadanoff and Wegner, 1971).][10] A celebrated example of this so-called "nonuniversal" behavior occurs in the exactly soluble two-dimensional eight-vertex, or Baxter model (Baxter, 1971, 1972). However, the existence of a marginal operator in the model seems to depend crucially on the existence of a special symmetry [the identity of two Ising sublattices into which the system can be decomposed (Kadanoff and Wegner, 1971)]. Breaking this symmetry most probably leads back to characteristic two-dimensional Ising behavior. For similar reasons, it seems unlikely that critical exponents which truly vary continuously will be found in real physical systems; rapid or slow crossovers from one discrete exponent value to another one, should rather be the order of the day!

The discussion of crossover effects we have given, relates to relevant operators. Formally, however, one may take it over for an *irrelevant* operator with, say, a field g_k, which by (5.10) then has a *negative* crossover exponent ϕ_k. In this case the scaling relation (5.7) still applies (with g_k replacing g) but the combination $g_k/t^{\phi_k} = g_k t^{|\phi_k|}$ now vanishes as $t \to 0$ *even* for $g_k \neq 0$. Thus no crossover, in fact, occurs; this simply confirms the irrelevance of g_k as before. However, if one expands the scaling function $Y(y, z)$ with respect to z for small z (as will often be justifiable) one in fact generates *corrections to asymptotic scaling* (Wegner, 1972a; Wegner and Riedel, 1973) which vary as $t^{|\phi_k|}$. If $|\phi_k| < 1$, these corrections are quite singular and may, in fact, interfere with the observation of true asymptotic power laws (see e.g. Fisher, 1970). [Of course analytic corrections varying as t, t^2, etc., must always be expected; these are generated through the nonlinear corrections to the scaling fields which were neglected in Eq. (4.18).]

VI. PRACTICAL RENORMALIZATION GROUP CALCULATIONS

Most renormalization group calculations so far carried through have followed the lines of the original works. The main steps are:

(a) *Adoption of a continuous local variable* or spin $\vec{s}$, with a magnitude constrained by a weight factor

$$\exp[-w(\vec{s})] = \exp(-\tfrac{1}{2}|\vec{s}|^2 - \bar{u}|\vec{s}|^4 - \cdots), \quad (\bar{u} > 0), \tag{6.1}$$

for each individual variable $\vec{s}_{\mathbf{x}}$. The $\bar{u}|\vec{s}|^4$ term here plays a vital role—it approximates the sharp spin-magnitude cutoff in real systems. But except in special circumstances [certain types of tricritical point (Riedel and Wegner, 1972), etc.] higher order contributions to the initial $\bar{\mathcal{H}}$ (proportional to $|\vec{s}|^6$, etc.) seem to be inessential. (Ultimately, however, this point may warrant further detailed investigation especially in relation to spin $\frac{1}{2}$ Ising models where the strong constraint $s = \pm 1$ might still play a special role.)

(b) *Introduction of momentum space variables* $\vec{\sigma}_{\mathbf{q}} = (\sigma_{\mathbf{q}}^{\mu})_{\mu=1,\ldots,n}$, normalized to remove all temperature and other variation from the dominant q dependence of the reduced Hamiltonian $\bar{\mathcal{H}}$. This dependence thus takes the form

$$-\tfrac{1}{2}\int_{\mathbf{q}} q^2 \vec{\sigma}_{\mathbf{q}} \cdot \vec{\sigma}_{-\mathbf{q}}, \tag{6.2}$$

[9] Graphical examples of such slow crossover in the context of critical exponent renormalization by constrained variables (Fisher, 1968) have been exhibited by Scesney and Fisher, (1970); The problem has also been discussed by Riedel and Wegner, (1974) in an interesting paper based on "model" recursion relations, including the essential nonlinear terms dropped in Eq. (4.18).

[10] See also *Renormalization Group* especially concerning the field-theoretic Thirring model which also displays continuously variable exponents or "anomalous dimensions" (Wilson, 1970, 1971).

where the useful notation $\int_{\mathbf{q}}$ was defined in Eq. (3.2); in a finite system we have $\int_{\mathbf{q}} = a^{-d}N^{-1}\sum_{\mathbf{q}}$, where the sum runs over the appropriate discrete Brillouin zone.

To see explicitly what is involved we may first introduce the Fourier transformed spins

$$\hat{s}_{\mathbf{q}} = \sum_{\mathbf{x}} \exp(i\mathbf{q}\cdot\mathbf{x})\vec{s}_{\mathbf{x}}. \tag{6.3}$$

If, for illustrative purposes, we then start with the isotropic exchange Hamiltonian (3.7), the full reduced Hamiltonian becomes, using (6.1) in truncated form,

$$\begin{aligned}\bar{\mathcal{H}} &= -\mathcal{H}_{\text{iso. exch.}}/k_BT - \sum_{\mathbf{x}} (\tfrac{1}{2}\,|\,\vec{s}_{\mathbf{x}}\,|^2 - \tilde{u}\,|\,\vec{s}_{\mathbf{x}}\,|^4) \\ &= -\tfrac{1}{2}\int_{\mathbf{q}} [1 - \hat{J}(\mathbf{q})/k_BT]\hat{s}_{\mathbf{q}}\cdot\hat{s}_{-\mathbf{q}} \\ &\quad -\tilde{u}\int_{\mathbf{q}}\int_{\mathbf{q}'}\int_{\mathbf{q}''} (\hat{s}_{\mathbf{q}}\cdot\hat{s}_{\mathbf{q}'})(\hat{s}_{\mathbf{q}''}\cdot\hat{s}_{-\mathbf{q}-\mathbf{q}'-\mathbf{q}''})\end{aligned} \tag{6.4}$$

where

$$\hat{J}(\mathbf{q}) = \sum_{\mathbf{x}} \exp(i\mathbf{q}\cdot\mathbf{x})J(\mathbf{x}) = \hat{J}(0) - ja^2q^2 + O(q^4), \tag{6.5}$$

the expansion being always possible for *short range* interactions. Now, if we define the $\vec{\sigma}_{\mathbf{q}}$ through

$$\hat{s}_{\mathbf{q}} = (k_BT/ja^{d+2})^{1/2}\vec{\sigma}_{\mathbf{q}}, \tag{6.6}$$

the reduced Hamiltonian becomes

$$\begin{aligned}\bar{\mathcal{H}} &= -\tfrac{1}{2}\int_{\mathbf{q}} [r + q^2 + O(q^4)]\vec{\sigma}_{\mathbf{q}}\cdot\vec{\sigma}_{-\mathbf{q}} \\ &\quad - u\int_{\mathbf{q}}\int_{\mathbf{q}'}\int_{\mathbf{q}''} (\vec{\sigma}_{\mathbf{q}}\cdot\vec{\sigma}_{\mathbf{q}'})(\vec{\sigma}_{\mathbf{q}''}\cdot\vec{\sigma}_{-\mathbf{q}-\mathbf{q}'-\mathbf{q}''}),\end{aligned} \tag{6.7}$$

which exhibits the form (6.2). The parameter r is

$$r = (k_B/ja^2)(T - T_0), \qquad k_BT_0 = \hat{J}(0), \tag{6.8}$$

so that it becomes now the basic temperature variable. Second we have

$$u = a^{d-4}(k_BT/j)^2\tilde{u} > 0, \tag{6.9}$$

which represents the quartic part of the weighting function $w(\vec{s})$.

We have for this example now completed the third step for the simple Hamiltonian (3.7), namely:

(c) *Construction of the reduced Hamiltonian in momentum space.* More generally the reduced Hamiltonian takes the form

$$\begin{aligned}\bar{\mathcal{H}} &= -\tfrac{1}{2}\sum_{\mu,\nu}\int_{\mathbf{q}} V_2^{\mu\nu}(\mathbf{q})\sigma_{\mathbf{q}}^{\mu}\sigma^{\nu}_{-\mathbf{q}} \\ &\quad -\tfrac{1}{24}\sum_{\mu,\nu}\int_{\mathbf{q}}\int_{\mathbf{q}'}\int_{\mathbf{q}''} V_4^{\mu\nu}(\mathbf{q},\mathbf{q}',\mathbf{q}'')\sigma_{\mathbf{q}}^{\mu}\sigma_{\mathbf{q}'}^{\mu}\sigma^{\nu}_{\mathbf{q}''}\sigma^{\nu}_{-\mathbf{q}-\mathbf{q}'-\mathbf{q}''} \\ &\quad - \cdots.\end{aligned} \tag{6.10}$$

The first term, quadratic in the $\sigma_{\mathbf{q}}^{\mu}$, represents the zero order, Gaussian, or "free-field" Hamiltonian which embodies the original pair interactions and the $\frac{1}{2}\,|\,\vec{s}\,|^2$ weighting terms. As seen above and discussed further below, most of the details of the physical Hamiltonian go into the specification of the corresponding two-point potential $V_2^{\mu\nu}(\mathbf{q})$. The second, quartic, or four-point term derives, as seen in Eq. (6.4), primarily from the nonquadratic part of the spin weighting function. However, under iteration of the renormalization group, new quartic terms may be generated and must thus be allowed for. The most important of these, which covers many practical cases, is a term of cubic symmetry: thus one may usually take

$$\tfrac{1}{24}V_4^{\mu\nu}(\mathbf{q},\mathbf{q}',\mathbf{q}'') = u + v\delta_{\mu\nu}, \tag{6.11}$$

where v is proportional to the magnitude of the cubic contribution. Details of the lattice structure, etc., would enter into q-dependent terms here, and into the $O(q^4)$ terms in Eq. (6.7); however, such contributions turn out to be irrelevant. More generally one might have to consider a $V_4^{\kappa\lambda\mu\nu}$ with four distinct spin indices (Brezin, Le Guillou, and Zinn-Justin, 1974; Nelson and Fisher, 1974; Nelson, Kosterlitz, and Fisher, 1974). External magnetic fields will call for a linear term in (6.10) and cubic, or three-point terms may then be induced as well. The next step is:

(d) *Definition of a renormalization group by a partial trace over high momentum variables.* Owing to the assumed lattice structure of spacing a, the momenta $\mathbf{q}$ are essentially restricted by $|\,\mathbf{q}\,| \leq \pi/a$. (This neglects details of the shape of the Brillouin zone which, however, will also prove irrelevant.) More generally the physics of condensed matter will always dictate some form of fixed, high-momentum cutoff. (This contrasts with quantum field theory where there is normally no natural cutoff, so that the problem is to take the cutoff to infinity, or the lattice spacing to zero.) Following (4.3) we thus define **R** by taking a trace over all variables $\vec{\sigma}_{\mathbf{q}}$ satisfying

$$b^{-1} < |\,q\,|a/\pi \leq 1. \tag{6.12}$$

Evidently this introduces the spatial rescaling factor b.

A little thought shows that we have defined a linear renormalization group [see Sec. IV.e] since any spin components $\vec{\sigma}_{\mathbf{q}}$ with $|\mathbf{q}|a/\pi < b$ remain untransformed by the partial trace. Accordingly a spin rescaling factor, $c[\bar{\mathcal{H}}]$, must be introduced as in (4.6). This factor will normally be chosen to keep the coefficient of $q^2\vec{\sigma}_{\mathbf{q}}\cdot\vec{\sigma}_{-\mathbf{q}}$ in the renormalized Hamiltonian, $\bar{\mathcal{H}}'$, *constant*, in accord with the original normalization (6.2). It is found that this prescription leads to a nontrivial fixed point describing critical behavior for short range interactions [as embodied in Eq. (6.7)]; more generally, however, other forms of rescaling may be needed to reach an appropriate fixed point.[11]

(e) The final and crucial computational step is to realize **R** by a *perturbation expansion* treating u and v as small parameters. As might be expected, this leads to a graphical

[11] See, for example, the discussion by Sak (1973), of the crossover from long range to short range behavior Fisher, Ma, and Nickel (1973); Suzuki, (1972); and Nelson and Fisher (1974) and Nelson, Kosterlitz, and Fisher (1974).

formulation abounding in Feynman type integrals. The essential feature, however, is the discovery that the small parameter is not, in fact, u (which in field-theoretic language would be regarded as the "coupling constant") but rather, as announced earlier, the *dimensionality difference*, $\epsilon = 4 - d$.

For readers interested in some of the technical details a few equations should give the feel for what is involved and, in particular, demonstrate how and why ϵ enters as a parameter—however, those interested only in the results are strongly advised to skip this material. As usual the free graphical "propagator," $G_0^{\mu\nu}(\mathbf{q})$, derives from the quadratic potential. Diagramatically one may write

$$(\text{—}) = \tfrac{1}{2}V_2^{\mu\nu}(\mathbf{q}) \equiv \tfrac{1}{2}[G_0^{\mu\nu}(\mathbf{q})]^{-1}. \tag{6.13}$$

The factor $\frac{1}{2}$ reflects the same factor in Eq. (6.10); however propagator lines in other graphs do not carry this factor.

For isotropic short range forces one may by Eq. (6.7), simply take

$$G_0^{\mu\nu}(\mathbf{q}) = \delta_{\mu\nu}/(r + q^2), \tag{6.14}$$

where we recall from (6.8) that, the temperature appears essentially only in $r \propto (T - T_0)$, with T_0 being the mean field critical temperature. The perturbation "vertex" may be denoted

$$(\rangle\!\text{-}\!\langle) \equiv (u + v\delta_{\mu\nu}); \tag{6.15}$$

for simplicity we will describe only the $v \equiv 0$ situation. Each in-going or out-going line carries a momentum and a spin-component index. To leading order in u the renormalization group equation (4.3) now becomes a set of nonlinear recursion relations [compare with (4.18)] which may be written schematically as

$$
\begin{aligned}
(\text{—})' &= b^d c^2 \{(\text{—}) + u[(\text{♀}) + (\Omega)] + \cdots\},\\
(\rangle\!\text{-}\!\langle)' &= b^d c^4 \{(\rangle\!\text{-}\!\langle) - u[(\rangle\!\text{-}\!\circ\!\text{-}\!\langle) + (\rangle\!\text{-}\!\propto\!\langle) + (\asymp\!\!\propto\!\langle)] + \cdots\},\\
&\ \vdots \qquad \vdots \qquad\qquad \vdots\\
(\cdots)' &= b^d c^k \{(\cdots) - \cdots\},
\end{aligned} \tag{6.16}
$$

where, as before, we do not need the constant (spin-independent or "vacuum expectation" term).

As usual, internal lines in diagrams carry propagator factors $G_0^{\mu\nu}(\mathbf{q})$ and imply momentum integrations and spin index summations, but the momentum integrations run only over the "outer zone" (6.12). Owing to the spin summations, each closed (solid line) loop in a diagram contributes a factor n. In this way one finds that the coefficients of u in the first two recursion relations are proportional to $(n + 2)$ and $(n + 8)$, respectively. This is the only way, to this order, in which the spin symmetry index enters. The exponents of b and c in the prefactors follow from simple dimensional considerations: each spin $\vec{\sigma}_\mathbf{q}$ yields a factor $b^d c$, and each integral $\int_\mathbf{q}$ a factor b^{-d}. As mentioned, the value of c is to be determined by normalization of the dominant q dependence; here this simply means that G_0' should be of the form $\delta_{\mu\nu}/(r' + q^2)$, i.e., the coefficient of q^2 remains unity after renormalization in order to retain (6.2) for $\mathcal{H}'$. Now there is no first order contribution to the q dependence of G_0', so this yields

$$c^2 = b^{2-d+O(u^2)}. \tag{6.17}$$

Comparison with (4.12) reveals that, if some fixed point is found with $r = r^*$ and $u = u^*$, then the exponent η is given by $\eta = O(u^{*2})$. It now follows that to order u, the prefactors in (6.16) become simply b^2, and $b^{4-d} = b^\epsilon$, respectively. The origin of the parameter ϵ is thus seen as resulting essentially from a competition between the $q^2 |\vec{s}_\mathbf{q}|^2$ and $u |\vec{s}_\mathbf{x}|^4$ terms in $\mathcal{H}$, or, picturesquely, between the delocalizing effects of the "dynamical" interactions between sites, and the localizing tendency of the "kinetic" restrictions on the spin magnitudes. In any event, when ϵ is small one has $b^\epsilon \approx 1 + \epsilon \ln b$ and, as an anzatz, we assume that both r and u are of order ϵ in the fixed point region. This anzatz effectively decouples the recursion relations for r and u from those for any other parameters in the Hamiltonian. To leading order these recursion relations are then found to be

$$r' = b^2[r + 4(n + 1)uA_1(r, b)], \tag{6.18}$$

$$u' = u[1 + \epsilon \ln b - 4(n + 8)uA_2(r, b)], \tag{6.19}$$

where, with $d = 4 + O(\epsilon)$, the graphical integrals lead to

$$A_1(r, b) = \int_\mathbf{q}^{>} (r + q^2)^{-1} = K_1(1 - b^{-2})/a^2 + O(r, \epsilon), \tag{6.20}$$

$$A_2(r, b) = \int_\mathbf{q}^{>} (r + q^2)^{-2} = K_2 \ln b + O(r, \epsilon), \tag{6.21}$$

where K_1 and K_2 are constants and the superscript $>$ on the integrals denotes the outer zone of integration (6.12).

The u equation clearly has a trivial fixed point $u^* = 0$ which then implies $r^* = 0$. This fixed point corresponds to simple Gaussian behavior, as might be expected, and leads, for all n, to the corresponding exponents given in (3.6).

However, there is also a *nontrivial fixed point* with, to leading order in ϵ,

$$u^* = \frac{\epsilon \ln b}{4(n + 8)A_2(0, b)} = \frac{\epsilon}{4K_2(n + 8)}, \tag{6.22}$$

which is independent of b. Correspondingly one finds

$$r^* = -\frac{K_1}{K_2}\frac{(n + 2)}{(n + 8)}\epsilon + O(\epsilon^2). \tag{6.23}$$

For $\epsilon > 0$ (or $d < 4$) this fixed point turns out to be the important one. Indeed if one starts out with small u near the Gaussian fixed point, one merely crosses over to the nontrivial fixed point; the Gaussian fixed point is thus unstable. The converse happens for $\epsilon < 0$, which explains why the critical exponents in more than four dimensions stick at their classical values for all n. The fact that both u^* and r^* are of order ϵ confirms the anzatz and allows one to linearize to successive orders in ϵ and thereby carry through the general program outlined in the previous section.

From the argument following (6.17) we already see that

$$\eta = O(\epsilon^2), \tag{6.24}$$

although, of course, one needs to know the coefficient (see below). To complete the analysis to first order in ϵ, we linearize the recursion relations about the nontrivial fixed point, noting that $(\partial A_1/\partial r) = -A_2$. In terms of the deviations $\Delta r = r - r^* \sim t$ and $\Delta u = u - u^*$, we find

$$\begin{bmatrix} \Delta r' \\ \Delta u' \end{bmatrix} = \begin{bmatrix} b^2(1 - p_n\,\epsilon \ln b) & 4K_1(n+2)(b^2-1) \\ 0 & 1 - \epsilon \ln b \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta u \end{bmatrix},$$

$$\text{with} \quad p_n = \frac{n+2}{n+8} \tag{6.25}$$

[Contributions of order ϵ to the upper right hand matrix element also come from terms of order u^2 neglected in (6.18).] Thus we conclude that

$$\Lambda_1 \approx b^2\{1 - [(n+2)/(n+8)]\epsilon \ln b\},$$
$$\Lambda_2 = \Lambda_u \approx 1 - \epsilon \ln b, \tag{6.26}$$

and so by (4.15) we obtain

$$\lambda_1 \approx 2 - \frac{n+2}{n+8}\epsilon + O(\epsilon^2),$$
$$\lambda_2 = \lambda_u = -\epsilon + O(\epsilon^2). \tag{6.27}$$

Note that these results are *independent* of b as anticipated in the general discussion of Sec. IV.vi.

Finally we may apply (4.25) to find the correlation exponent

$$2\nu = 2/\lambda_1 = 1 + \frac{n+2}{2(n+8)}\epsilon + O(\epsilon^2). \tag{6.28}$$

By (5.8) we also find the crossover exponent for the irrelevant variable $|\vec{s}\,|^4$, namely,

$$\phi_u = \lambda_1/\lambda_u = -\tfrac{1}{2}\epsilon + O(\epsilon^2). \tag{6.29}$$

The fact that $|\phi_u|$ is probably less than unity for $\epsilon \simeq 1$ indicates that significant *singular* corrections to asymptotic scaling may well be encountered in real three-dimensional systems (as against just analytic corrections varying as t, t^2, etc., see Sec. V).

We may note that from (6.28) and (6.24), all the other standard critical exponents can be found to $O(\epsilon)$ via (4.25) and the usual scaling relations for exponents (discussed in Sec. II).

In the analysis just sketched, we have followed the full renormalization group formalism; but, as a matter of fact, if one knows, or is prepared to assume, enough about the form of the answers (e.g., scaling forms, etc.), one may side-step the recursion relations and use purely graphical methods of computation (Wilson, 1971a, 1972; Wilson and Kogut, 1974; Tsuneto and Abrahams, 1973). However, the actual justification of such techniques seems best accomplished via the renormalization group approach. This is especially so in the case of crossover phenoemna (Saks, 1973; Nelson and Fisher, 1974; Nelson, Kosterlitz, and Fisher, 1974).

The appearance of the factor n associated with closed loops suggests that a graphical resummation to collect up the leading contributions of diagrams with many loops might yield a systematic expansion. The procedure (which is appreciably more complicated than the ϵ expansion) can be carried through and does indeed yield the desired $1/n$ expressions Abe, 1972, 1973; Fisher, Ma, Nickel, 1972; Suzuki, 1972; Abe and Hikami, 1973; Ma, 1973a).

VII. EXPONENTS FOR ISOTROPIC SHORT RANGE EXCHANGE

One of the most important outputs of the practical renormalization group calculations has been expansions for the critical exponents. In leading order in $\epsilon(>0)$ the deviations from classical behavior for systems with isotropic short range exchange are revealed by

$$\gamma = 1 + \frac{n+2}{2(n+8)}\epsilon + \cdots, \tag{7.1}$$

$$\beta = \tfrac{1}{2} - \frac{3}{2(n+8)}\epsilon + \cdots, \tag{7.2}$$

$$\alpha = \frac{4-n}{2(n+8)}\epsilon + \frac{(n+2)^2(n+28)}{4(n+8)^3}\epsilon^2 + \cdots, \tag{7.3}$$

$$\eta = \frac{n+2}{2(n+8)^2}\epsilon^2 + \cdots, \qquad \delta = 3 + \epsilon + \cdots, \tag{7.4}$$

where δ is the exponent for the critical isotherm where $H \sim M^{\delta}$.[2] The first significant feature is simply the sign of the deviations from the classical values as d falls below 4; the second is the characteristic, but weaker, dependence on n, the degree of isotropy. Both features accord perfectly with experimental evidence and with the analysis of exact series expansions (see e.g., Fisher, 1970, 1973; Pfeuty, Jasnow, and Fisher, 1973, 1974; and especially Wortis, 1973).

Higher order terms have been calculated, to ϵ^2 by Wilson (1972), and more recently, to third order by Brézin, LeGuillou, Zin-Justin, and Nickel (1973; Brézin, LeGuillou, and Zinn-Justin, 1974d; see also Ketley and Wallace, 1973). (The number of coauthors is indicative of the labor needed to get the correct answer!) In each case one further power of ϵ becomes available for η (and δ) than calculated for α, γ, and ν. The expansion for α when truncated at second order and evaluated at $\epsilon = 1$ (i.e., $d = 3$) yields the estimates $\alpha \simeq 0.08$, -0.02, and -0.10 for $n = 1$, 2 and

TABLE I. Third order ϵ expansion[a] for the exponent γ.

$$\gamma = 1 + \frac{(n+2)}{2(n+8)}\epsilon + \frac{(n+2)(n^2+22n+52)}{4(n+8)^3}\epsilon^2 + \frac{(n+2)}{8(n+8)^3}\Big[(n+2)^2 + 24\frac{(n+2)(n+3)-(10n+44)\zeta(3)}{(n+8)} + 4\frac{55n^2+268n+424}{(n+8)^2}\Big]\epsilon^3 + O(\epsilon^4).$$

[a] From E. Brézin, J. C. LeGuillou, J. Zinn-Justin, and B. G. Nickel (1973).

TABLE II. Fourth order expansion[a] for the exponent η.

$$\eta = \frac{n+2}{2(n+8)^2}\epsilon^2 + \frac{n+2}{8(n+8)^2}\Big[\frac{24(3n+14)}{(n+8)^2} - 1\Big]\epsilon^3 + \frac{n+2}{2(n+8)^2}\Big[\frac{-5n^2+234n+1076}{16(n+8)^2} - 8\frac{3n^2+53n+160+3(5n+22)\zeta(3)}{(n+8)^3} + 45\frac{(3n+14)^2}{(n+8)^4}\Big]\epsilon^4 + O(\epsilon^5).$$

[a] From E. Brézin, J. C. LeGuillou, J. Zinn-Justin, and Nickel (1973).

3, respectively. These results correlate surprisingly well numerically with the experimental observations summarized in Section II.

Because of its impressive complexity, the third order expansion for γ is quoted in Table I! Perhaps the most interesting technical feature is the appearance of a transcendental number, namely $\zeta(3) = \sum_1^\infty k^{-3}$, in the third order term. Note also the factor $(n+2)$ in each coefficient which confirms that $n = -2$ yields the Gaussian value $\gamma = 1$. The denominators are all powers of $(n+8)$.[12] From a purely numerical point of view, however, the results are a little disappointing. Thus truncation of the series for $n = 1$ and $d = 3$ at order ϵ, ϵ^2, and ϵ^3 yields $\gamma \simeq 1.17$, 1.245, and 1.195, respectively. These figures may be compared with the best series estimates for the $d = 3$ Ising model (see e.g., Fisher, 1967), which gives $\gamma \simeq 1.250$. However, one should not be too surprised since various arguments (Wilson, 1971a, 1972) indicate that the ϵ expansion is probably only asymptotic for finite n [although for $n = \infty$ it evidently converges to the exact spherical model result $\gamma = (1 - \frac{1}{2}\epsilon)^{-1}$]. Nevertheless, if appreciably longer series could be obtained, Padé approximant and other summation techniques would probably be successful.

The fourth order expansion for η is exhibited in Table II. In conjunction with Table I and the exponent scaling relations (2.12), (2.25), (2.5), (2.6b), and

$$d(\delta - 1)/(\delta + 1) = 2 - \eta, \tag{7.5}$$

all exponents can then be calculated to third order in ϵ.

The $1/n$ expansions are also instructive (see Table III) although for the values of n of practical interest they seem less accurate numerically (and they are much harder to calculate). However, convergence may apparently be improved, as suggested by Suzuki (1973a, and to be published), by noting that $1/(n+8)$ rather than $1/n$ is the basic expansion variable.

Despite the limited numerical accuracy presently obtainable from the ϵ and $1/n$ expansions, they serve, in combination with other evidence already mentioned, to give us an excellent picture of the over-all variation of the exponents with d and n. Figure 2 is a plot of contours of constant values of the exponent α which illustrates this. Note in particular, the locus on which $\alpha = 0$ (corresponding to a logarithmic divergence), which passes through the "Onsager point" ($d = 2, n = 1$) and near the "helium point" ($d = 3$, $n = 2$). Above this locus, which region includes three-dimensional Heisenberg systems, the exponent α is negative and the specific heat rises only to a finite cusp.[2] Below the $\alpha = 0$ locus the specific heat diverges to infinity as a power law. For $d > 4$ the mean field discontinuity represents the dominant behavior. It is interesting that the one-dimensional Ising point ($d = n = 1$) seems to be a point of confluence of the contours.

As shown in Figs. 3–5 similar global contour diagrams may be drawn for other exponents. The uniform increase of the susceptibility exponent γ from unity at $d = 4$ and $n = -2$, to ∞ for $d \simeq 2$ and $n \gg 1$ is striking. Again the one-dimensional Ising model is a point of confluence. The dashed lines around $n = 2$ and 3 for $d = 2$, indicate a basic gap in our current knowledge! If, in fact, transitions occur at nonzero temperatures in such isotropic two-dimensional

TABLE III. First order $1/n$ expansions for exponents for $2 < d < 4$.[a]

$$\gamma = \frac{2}{d-2}\Big[1 - \frac{3A_d}{n} + O\Big(\frac{1}{n^2}\Big)\Big]$$

$$\alpha = -\frac{4-d}{d-2}\Big[1 - \frac{4(d-1)}{(4-d)}\frac{A_d}{n} + O\Big(\frac{1}{n^2}\Big)\Big]$$

$$\eta = \frac{2(4-d)}{d}\frac{A_d}{n} + O\Big(\frac{1}{n^2}\Big)$$

where

$$A_d = \frac{2\Gamma(d-2)\sin(\frac{1}{2}d-1)\pi}{[\Gamma(\frac{1}{2}d-1)]^2(\frac{1}{2}d-1)}, \qquad A_3 = 4/\pi^2,$$

and

$$A_{4-\epsilon} \approx \tfrac{1}{2}\epsilon \text{ as } \epsilon \to 0, \qquad A_{2+\theta} \approx \tfrac{1}{2}\theta \text{ as } \theta \to 0$$

[a] From M. E. Fisher, S. -K. Ma, and B. G. Nickel (1972); M. Suzuki (1972), S. -K. Ma, (1973); R. Abe, (1972, 1973); R. Abe and S. Hikami, (1973).

[12] Although one can see that all coefficients in the expansion will be rational functions of n for isotropic short range forces, the coefficients become *nonanalytic* functions of n for *biconical* fixed points: see Fisher and Nelson; Nelson, Kosterlitz, and Fisher (1974).

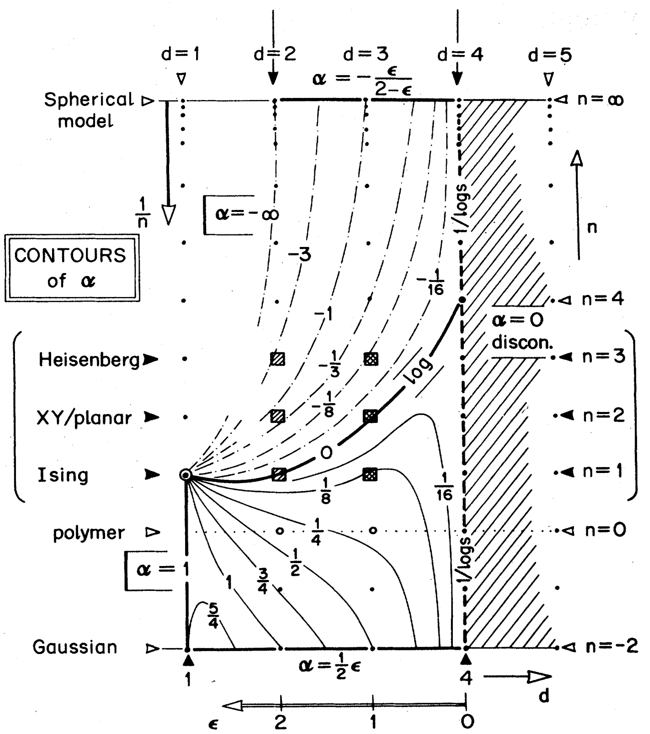

FIG. 2 Diagram showing contours of constant exponent α in the (d, n) plane. The dash–dot contours indicate negative α; the solid contours are for $\alpha \geq 0$.

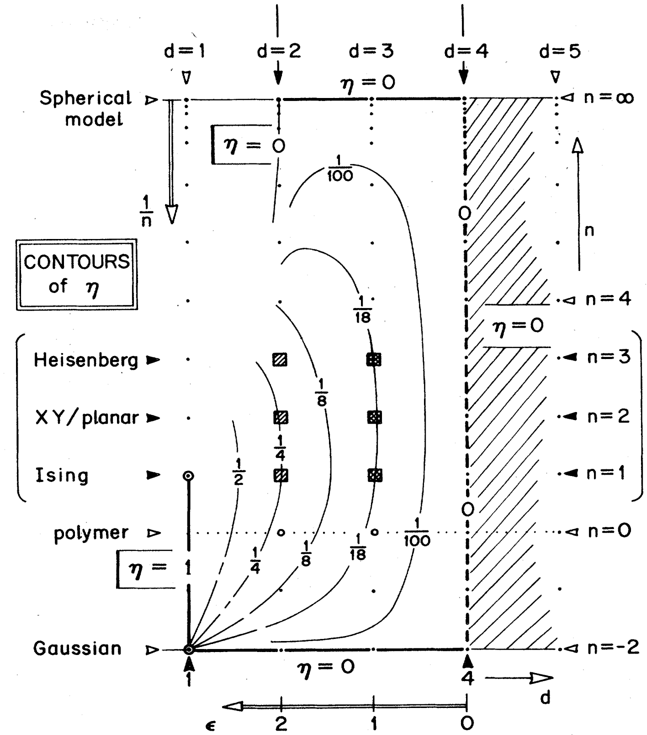

FIG. 4 Contours of constant exponent η in the (d, n) plane. Note the smoothly rising trend and the near vertical contours for $d = 3$, $n = 1$, 2, and 3.

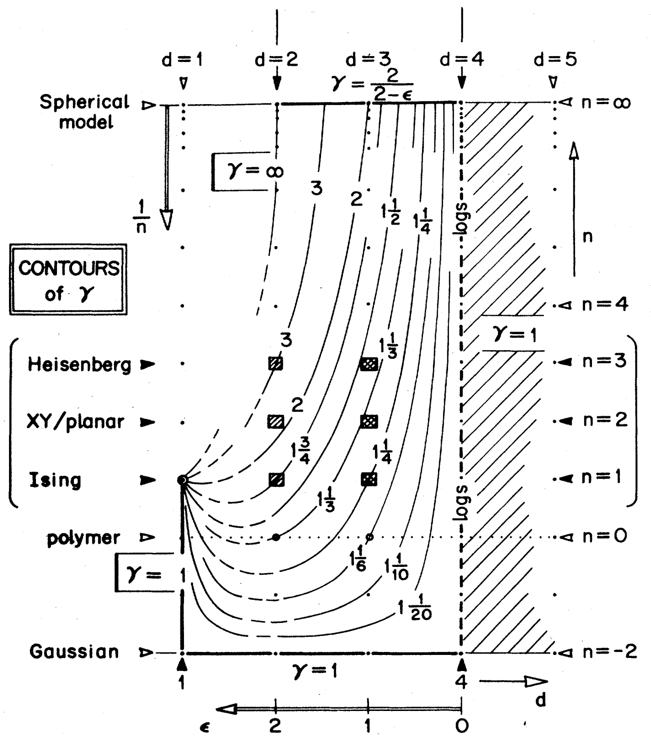

FIG. 3 Contours of constant susceptibility exponent γ in the (d, n) plane. The location of the $\gamma = \infty$ and other contours is uncertain in the region $d \simeq 2$, $n \geq 2$.

systems[13] then the values of γ indicated by the contours are probably fairly accurate. However, if the corresponding susceptibilities remain finite right down to zero temperature,[13] then the $\gamma = \infty$ contour probably descends vertically at least to $n = 2$.

Similar uncertainties and remarks apply to contours for η shown in Fig. 4. However, for $d = 3$ and $n = 1$, 2, and 3, the smooth and steady variation of the contours seems strong evidence for a positive, albeit quite small, value of η. The difficulty of observing a nonzero value of η in real three-dimensional systems, and the problems associated with the apparent violation of the "hyperscaling" relation (7.5) in three-dimensional Ising models,[2] can scarcely cause one to believe that a "Death Valley" appears in the η landscape, which just descends to "sea level" at $n = 1$, $d = 3$! (These questions may, perhaps, find their resolution in a detailed renormalization group calculation of the corrections to scaling.)

The contours for β in Fig. 5 are also instructive. Although not as steep as the η contours, the β contours rise quite rapidly near $d = 3$, so that β does not vary very strongly with n. Theoretically (but not practically!) it is interesting to note a region for large n and $d < 1\frac{1}{2}$ where β *exceeds* the classical value $\frac{1}{2}$. Furthermore, for $n < 1$ and $d < 2$, the

[13] Well known theorems prove the nonexistance of spontaneous order or long range correlations in short range two- or one-dimensional systems for $n \geq 2$, see Hohenberg, (1967), Mermin and Wagner, (1966), Chester, Fisher and Mermin, (1969). Jasnow and Fisher (1971); but these theorems do not so far restrict the susceptibility or rule out some new sort of critical behavior at a nonzero temperature as argued, at least for $n = 2$, by V. L. Berezinskii (1971) and Kosterlitz and Thouless (1972).

TABLE IV. Important classes of magnetic interactions.

Type of spin–spin interaction	Form of interaction	Dominant contribution to $V_2^{\mu\nu}(q)$
(1) *Iso*tropic, short range n-vector exchange	$-J(x)s_0\cdot s_x$	$(r+q^2)\delta_{\mu\nu}$
lattice structure effects		$+e_4q^4\delta_{\mu\nu}+\cdots$
(2) Single-ion *aniso*tropy with *reduced symmetry index* $m<n$	$-D_\mu(s_0^\mu)^2$	$r_\mu\delta_{\mu\nu}$
*Aniso*tropic short range exchange	$-D_\mu(x)[s_0^\mu s_x^\mu+s_0^\nu s_x^\nu]$	$(r_\mu+r_\nu)\delta_{\mu\nu}$
(3) Cubic short range (dipolar induced)	$\sum_\mu F(x_\mu^2)s_0^\mu s_x^\mu$	$f(q_\mu)^2\delta_{\mu\nu}$
(4) Pseudodipolar (short range) $n=d$	$-K(x)(\vec{x}\cdot\vec{s}_0)(\vec{x}\cdot\vec{s}_x)$	$hq_\mu q_\nu$
(5) Dzyaloshinskii-Moriya $n=d$	$\vec{A}(x)\cdot\vec{s}_0\wedge\vec{s}_x$	$ia_\lambda q_\lambda\epsilon_{\mu\nu\lambda}+$ c.c.[a]
(6) Isotropic dipolar $n=d$	$-[(g_s\mu_B)^2/x^d][d(x_\mu x_\nu/x^2)-\delta_{\mu\nu})]s_0^\mu s_x^\nu$	$g(q_\mu q_\nu/q^2)$
(7) Anisotropic dipolar symmetry index $m<n$	$g_s^2\Longrightarrow g_s^\mu g_s^\nu$	$g^\mu g^\nu q_\mu q_\nu/q^2$
(8) Long range isotropic exchange $0<\sigma<2$	$-(J_\infty/x^{d+\sigma})\vec{s}_0\cdot\vec{s}_x$	$j\|\mathbf{q}\|^\sigma\delta_{\mu\nu}$

[a] Note that $\epsilon_{\lambda\mu\nu}$ is the totally antisymmetric Levi-Civita symbol.

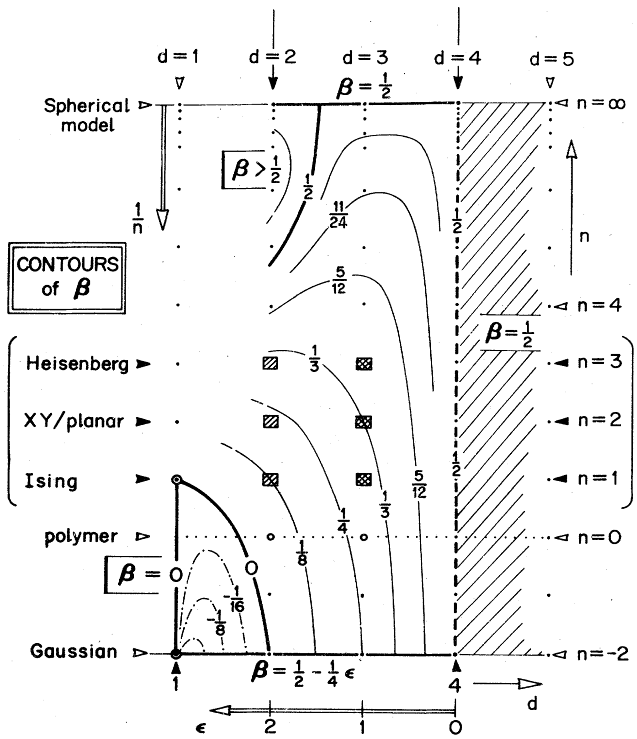

FIG. 5 Diagram of the (d, n) plane showing contours of constant β. There is a region where $\beta > \frac{1}{2}$, and a nonphysical region of negative β.

exponent β apparently becomes negative. This feature, already implied by the Gaussian results at $n = -2$, is, of course, quite unphysical; so are the values of α (exceeding unity) in the same region of the (d, n) plane. Perhaps this behavior means that the contours do not continue analytically into this (unphysical) domain; one may speculate that there is a locus, $n = 1 - 3(d-1) = 8 + 3\epsilon$, running from $(d, n) = (1, 1)$ to $(2, -2)$ on which $\alpha = 1$, $\beta = 0$, $\delta = \infty$, and $\eta = 2 - d$, and which represents a special borderline. But in any event, this issue does not seem to matter at all as regards realistic systems!

VIII. CROSSOVER EFFECTS IN MAGNETIC SYSTEMS

As pointed out earlier, much of the charm and complexity of magnetic phenomena arises from the many terms of different symmetry, spatial form, and magnitude which enter realistic Hamiltonians for magnetic materials. The renormalization group approach offers a systematic method for studying the effects of such terms on critical behavior and for distinguishing between them—something that was sorely missing in previous theories of critical behavior. Some of the most important classes of magnetic interactions and effects are listed in Table IV together with the type of term to which they give rise in the general Fourier transformed two-point interaction, $V_2^{\mu\nu}(\mathbf{q})$, which appears in Eq. (6.10).

The interaction types (1) to (4) in Table IV are all of short spatial range and exhibit high spatial symmetry so they give rise, in leading order, only to quadratic momentum dependence. Of course details of lattice structure lead to fourth and higher order momentum dependence as indicated under (1). However, the various interactions differ significantly in their spin symmetry. Thus, under (2), uniaxial single-ion anisotropy and planar or XY-like exchange anisotropy are illustrated. The Dzyaloshinskii–Moriya cross-product interaction, (5), is short ranged but of lower spatial symmetry and gives rise to spiral magnetic structures, etc. A renormalization group treatment has been presented by Liu (1973).

On the other hand the dipole–dipole interactions, (6), are of very long range and, as indicated, yield a very singular low-momentum dependence. Because of the direct coupling of spin and space, one must take $n = d$ (Fisher and Aharony, 1973; Aharony and Fisher, 1973; Aharony, 1973c, d). Actual dipolar interactions also generate (Aharony and Fisher,

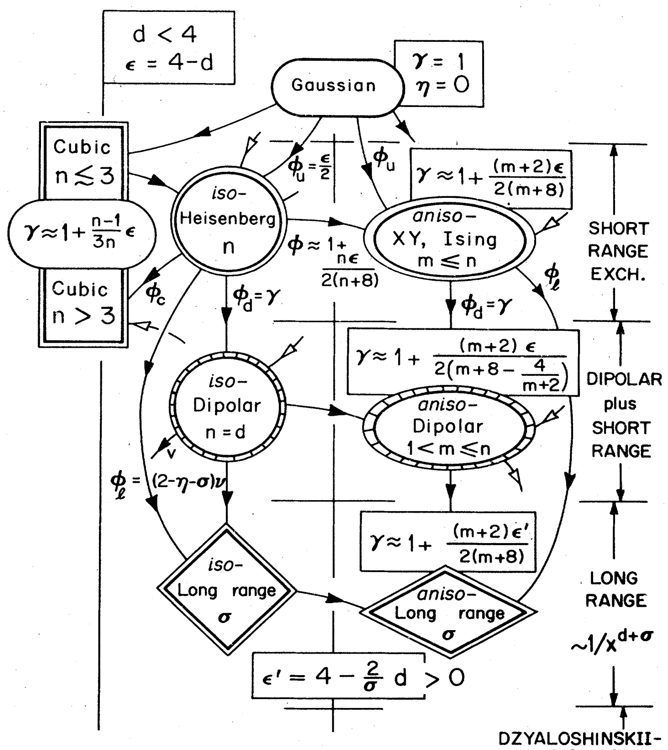

FIG. 6 Crossover map for magnetic fixed points for $d<4$. The arrows show the direction of instability under the perturbations characteristic of the more stable fixed point. The crossover exponents ϕ_c, ϕ_d, and ϕ_l refer to cubic, dipolar, and long range perturbations to the isotropic short range fixed point, respectively. See text for further explanations.

1973) short range cubic and pseudodipolar couplings as listed under (3) and (4) in Table IV; in antiferromagnets these terms arising from dipolar coupling may thus play a role even if the long-range character does not affect critical behavior (which, of course, is associated with a superlattice momentum vector in an antiferromagnet, rather than with $\mathbf{q}=0$). Anisotropic dipolar interactions (7) may enter when the gyromagnetic ratio g_S is replaced by a vector g_S^μ (or, more generally, by a tensor $g_S^{\mu\nu}$). Finally (8) we include very long range interactions decaying as $1/x^{d+\sigma}$. Such interactions are not particularly realistic (although they may be approximated in some materials) but they are most instructive theoretically (Fisher, Ma, and Nickel, 1973; Suzuki, 1972): indeed, in the spherical model they already lead to distinct exponents (Joyce, 1966). The various coupling parameters e_4, r_μ, f, g, h, $\cdots$ appearing in the column for $V_2^{\mu\nu}$ are proportional to the magnitude of the corresponding terms in the original Hamiltonian [but they are effectively normalized by the strength of the short range isotropic coupling which is assumed always to be present: see Sec. VI.b]. As before, the parameter r is proportional to $(T-T_0)$, where T_0 is the isotropic mean field critical temperature [see Eq. 6.8)].

To the list of important two-point interactions in Table IV we should add the expression (6.11) for the four-point interaction, namely,

$$V_4^{\mu\nu}(\mathbf{q},\mathbf{q}',\mathbf{q}'')\propto u+v\delta_{\mu\nu}. \tag{8.1}$$

The isotropic term u is always present. The *cubic* term proportional to v can arise from single-ion terms of the sort $D_4\Sigma_\mu(s_0^\mu)^4$ in the original spin Hamiltonian or from anisotropic biquadratic exchange such as $J_2^\mu(s_0^\mu s_x^\mu)^2$. But it may also be generated via dipolar, pseudodipolar, and two-point cubic interactions. The cubic term is of particular significance in displacive transitions in perovskites and similar materials (Cowley and Bruce, 1973; Bruce and Aharony, 1974). A further discussion of the various types of interaction term has been presented by Aharony (1974a), who also lists various results for crossover exponents, etc., which we will not present. In special cases one may have to consider more general fourth order interactions as mentioned in Sec. VI.c (Nelson and Fisher, 1974b; Nelson, Kosterlitz, and Fisher (1974); Brezin, LeGuillou, and Zinn-Justin, 1974c).

A. Gaussian and Heisenberg, XY, and Ising fixed points

Under the influence of the various terms exhibited in Eq. (8.1) and Table IV, a variety of fixed points may occur. Some of these are shown schematically in Fig. 6. The topmost is the trivial, or Gaussian fixed point (oval oblong) with $u=v=0$, i.e., no four-spin terms. This fixed point has exponents $\gamma=1$ and $\eta=0$ [see Eq. (3.6)] but, for $d<4$ (or $\epsilon>0$) it is unstable with respect to the four-spin terms and rapidly crosses over with exponent $\phi_u=\frac{1}{2}\epsilon$ (exactly) to some other fixed point, as indicated by the arrows. At the next level of stability are the short range exchange fixed points: on the left, the isotropic "Heisenberg" fixed point (circle) with degree of isotropy n; on the right, the anisotropic, "XY" or "Ising" fixed point (ellipse) with *reduced* symmetry index $m<n$, i.e., the *dominant* exchange interactions still have m-fold rotational symmetry. (Formally we may allow $m=n$.)

B. Anisotropic crossover

As indicated in the figure, the anisotropic exponents γ, α, η, etc., have identically the same form as do the isotropic exponents (discussed in the last section) but with m replacing n (Fisher and Pfeuty, 1972). The isotropic fixed point is unstable with respect to anisotropic single ion or exchange terms, as indicated by the right-going arrow. To second order the corresponding anisotropy crossover exponent is (Wilson, 1972)

$$\phi=1+\frac{n\epsilon}{2(n+8)}+\frac{(n^2+24n+68)n\epsilon^2}{4(n+8)^3}+\cdots, \tag{8.2}$$

which accords well with numerical estimates of $\phi\simeq 1.18$ and 1.25 for $n=2$ and 3 in three dimensions (Fisher and Pfeuty, 1972; Pfeuty, Jasnow, and Fisher, 1973, 1974). To order $1/n$ one finds (Hikami and Abe, 1974)

$$\phi=2(d-2)^{-1}[1-(4A_d/n)+\cdots], \tag{8.3}$$

where A_d is defined in Table III. We may mention that the analysis of the spin-flop bicritical point in anisotropic antiferromagnets (Fisher and Nelson, 1974; Nelson and Fisher, 1974b; Nelson, Kosterlitz, and Fisher, 1974) indicates that ϕ should be directly observable in such systems via the scaling variable $(H_{||} - H_{||,b})/t^{\phi}$, where $H_{||,b}$ is the value of the parallel applied field $H_{||}$ *at* the spin-flop point.

C. Cubic critical behavior

The cubic fixed points (squares on the top left) are associated directly with the parameter v in Eq. (8.1). Their stability with respect to the isotropic Heisenberg fixed point changes as n passes through the value

$$n^{\times}(d) = 4 - 2\epsilon + c^{\times}\epsilon^2 + O(\epsilon^3)$$
$$\simeq (4 + 3.176\epsilon)/(1 + 1.294\epsilon), \tag{8.4}$$

where $c^{\times} = (5/12)[6\zeta(3) - 1]$ (Aharony, 1974c; Wallace, 1973; Fisher and Nelson, 1974; Nelson, Kosterlitz, and Fisher, 1974). For $d = 3$ we find $n^{\times} \simeq 3.13$, so that the ordinary $n = 3$ Heisenberg fixed point is probably stable with respect to cubic perturbations, i.e., the exponents do not change. Since the cubic critical exponents (Aharony, 1974c; Wallace, 1973, Ketley and Wallace, 1973) are also numerically similar to the isotropic exponents for $n \simeq 3$, it seems likely that the effects of any cubic instability would be very hard to see experimentally [although effects might appear in the phase diagram (Bruce and Aharony, 1974a)]. For completeness, however, we record the cubic crossover exponent (Aharony, 1974c; see also Wallace, 1973) from the isotropic short range fixed point:

$$\phi_{\text{cub.}} = \frac{n-4}{2(n+8)}\epsilon + \frac{n^3 + 16n^2 + 4n + 240}{2(n+8)^3}\epsilon^2 + \cdots, \tag{8.5}$$

and the two cubic exponents

$$\eta_{\text{cub.}} = \frac{(n+2)(n-1)}{54n^2}\epsilon^2 + \cdots, \tag{8.6}$$

$$2\nu_{\text{cub.}} = 1 + \frac{n-1}{3n}\epsilon + \frac{(n-1)}{324n^3}(17n^2 + 290n - 424)\epsilon^2 + \cdots, \tag{8.7}$$

from which all the other exponents can be obtained through the scaling relations as explained in Sec. VII.

D. Dipolar effects

The addition of dipole–dipole interactions to short range exchange forces is, of course, essential to describe ferromagnetic materials; it results in another crossover (Fisher and Aharony, 1973; Aharony and Fisher, 1973; Aharony, 1973c and 1973d) with crossover exponent $\phi_{\text{dip.}} = \gamma$. This relation can be understood as crossover occurring when the inverse susceptibility, $\chi^{-1} \sim t^{\gamma}$, becomes comparable with the fixed demagnetization factor. The new dipolar fixed points have, to first order in ϵ, exponents $\gamma_{\text{dip.}}$ and ϕ which are obtained from the exchange forms (6.1) and (7.2) merely by replacing the factor $(n + 8)$ by

$$[n + 8 - 4/(n + 2)]$$

and similarly in m for anisotropic dipolar interactions (provided $m > 1$). For this reason these exponents for n, $m = 2$ or 3 change by a surprisingly small amount; furthermore the direction of the change is an increase *away* from the classical values. When $n = d = 4 - \epsilon$ (fully isotropic case) the exponent $\eta_{\text{dip.}}$ becomes $(20/867)\epsilon^2$, which, to this order, also represents an increase (by about 11%) compared with $\eta = (1/48)\epsilon^2$ for short range critical behavior.

Recently the dipolar exponents have been calculated to second order in ϵ (Bruce and Aharony, 1974b; Aharony and Bruce, 1974a). The specific heat exponent becomes

$$\alpha_{\text{dip.}} = -\frac{\epsilon}{34} - \frac{6223\epsilon^2}{58956} + \cdots \quad \text{(iso. dipolar)}, \tag{8.8}$$

which yields $\alpha \simeq -0.135$ at $\epsilon = 1$. This is to be compared with the corresponding isotropic short range result written with $n = d = 4 - \epsilon$, which is $\alpha \approx -\epsilon^2/8 = -0.125$ at $\epsilon = 1$. It is doubtful that a difference as small as 0.01 in a magnetic specific heat exponent can be determined experimentally (or, that one can trust the ϵ expansion to this accuracy). Nevertheless one should, in principle, see some differences between ferromagnetic specific heats and *anti*ferromagnetic specific heats since, as mentioned, dipolar interactions do not affect antiferromagnetic critical points. Furthermore, the induced f and h perturbations in Table IV turn out to be quite stable (as indicated in Fig. 6 by the ingoing arrows with free ends). Some experimental evidence does point in this direction (Kornblit, Ahlers, and Buehler, 1973; Salamon, 1973; Lederman, Salamon, and Shacklette, 1974).

From Eq. (8.8) and the value of η mentioned, one can find other dipolar exponents to order ϵ^2. For the sake of comparison, we mention

$$\gamma_{\text{dip.}} = 1 + \frac{9}{34}\epsilon + \frac{2111}{19652}\epsilon^2 + \cdots, \tag{8.9}$$

(Bruce and Aharony, 1974b; Aharony and Bruce, 1974a). which yields $\gamma_{\text{dip.}} \simeq 1.372$ at $\epsilon = 1$ (or $n = d = 3$). The corresponding short range result written with $n = d = 4 - \epsilon$ is

$$\gamma = 1 + \tfrac{1}{4}\epsilon + \frac{11}{96}\epsilon^2 + \cdots \tag{8.10}$$

which yields $\gamma \simeq 1.365$. Again the difference is probably undetectably small.

Finally, we should mention that the dipolar fixed point appears weakly unstable with respect to cubic perturbations with parameter v; but it is not known to what fixed point the instability leads (Fisher and Aharony, 1973; Aharony and Fisher, 1973; Aharony, 1973c, Bruce and Aharony, 1974b). This is not too serious, however, since the instability seems too weak to be experimentally detectable (even if it survives down to $\epsilon = 1$ or $d = 3$).

E. Uniaxial dipolar systems, etc.

It should be noted that dipolar forces in uniaxial, Ising-like systems ($m = 1$) are an interesting special case. Loosely, one may say that dipolar forces suppress longitudinal fluctuations; but an Ising-like system has no transverse spin components to pick up the fluctuation strength! For $d = 3$, this suppression of the critical fluctuations leads to classical exponent values, but with logarithmic corrections (Larkin and Khmel'nitskii, 1969; see also Aharony, 1973a and d). Specifically, the susceptibility is predicted to vary as

$$\chi_0(T) \approx C \,|\ln t\,|^{1/3}/t, \qquad \text{as } t \to 0. \tag{8.11}$$

Recent experiments by Als-Nielsen, Holmes, and Guggenheim (1974) bear out this striking prediction although, not surprisingly, the subtle logarithmic factor could not be isolated. Nonclassical exponents are predicted for $d < 3$; but this seems of academic interest only.

It may be remarked in passing that similar logarithmic factors, but with exponent $(n + 2)/(n + 8)$ for general n, arise on the *borderline* $d = 4$ for *short range* isotropic systems, as indicated in Figs. 2 and 3 (see Fisher, Ma, and Nickel 1973). A borderline at three dimensions with logarithmic factors, as in the anisotropic dipolar Ising model, also occurs in the analysis of tricritical phenomena (Riedel and Wegner, 1972). In that case, however, it arises from a competition between the usual $u\,|\vec{s}\,|^4$ term (which may go *negative* under renormalization group iteration) and the higher order $u_6\,|\vec{s}\,|^6$ term. Again, nonclassical tricritical exponents are predicted for two-dimensional systems and might eventually be observable there.

F. Long range interactions

Finally, all the previous fixed points are unstable with respect to long range interactions of the type indicated in Table IV and Fig. 6, provided $\sigma < 2 - \eta$.[14] The crossover exponent from the short range isotropic fixed point (Aharony, 1974a) is $\phi_{\text{long}} = (2 - \eta - \sigma)\nu$. Although such forces are not very realistic, it is interesting that the borderline dimensionality changes from $d = 4$ to $d = 2\sigma$ which can go down to $d = 1$ (or lower). In leading order, the results (7.1)–(7.3) for γ, β, and α then remain valid with ϵ replaced by $\epsilon' = 4 - (2d/\sigma)$; however, the second order expressions are much more complex (Fisher, Ma, and Nickel, 1973; Suzuki, 1972; Suzuki, Yamasaki, and Igarashi, 1972; Ma, 1973a). The exponent η is always equal to $2 - \sigma$ when long range forces act.

IX. FURTHER CALCULATIONS

As pointed out above, a good theory should not only provide the critical exponent values but also yield the various scaling functions. Such calculations have so far been made both for the *equation of state*[15] and for the *correlations* for short range exchange interactions in zero field[16] and in general fields.[17,18]

[14] See, for example, the discussion by J. Sak (1973) of the crossover from long range to short range behavior

[15] Brézin, Wallace, and Wilson (1972, 1973); Brezin and Wallace (1973) have made a $1/n$ calculation; Avdeeva and Migdal (1972); Bruce and Aharony (1974); Aharony and Bruce (1974).

[16] Fisher and Aharony (1973b). Aharony (1973b, 1974b) has made a $1/n$ expansion of the correlation scaling function for zero field.

[17] Brézin, Le Guillou, and Zinn-Justin (1974a, 1974b); Combescot, Droz, and Kosterlitz (1974).

[18] Earlier calculations had tested the exponent relation $\gamma = (2 - \eta)\nu$ within the Feymann graph expansion method (Wilson, 1972; Amit, 1972; Amit and Shcherbakov, 1973; Abe and Hikami, 1973).

A. Equations of state

The work of Brézin, Wallace, and Wilson (1972, 1973) and, independently, of Avdeeva and Migdal (1972), for $n = 1$ yields the scaling function $W(y)$ for the equation of state [as defined in Eq. (2.7)], to order ϵ^2 (for short range interactions). The most striking result is that the so-called "linear model" of Schofield (1969; see also Fisher, 1970) is exact to this order. Specifically, this means the asymptotic equation of state can be written parametrically in terms of a "radius" R, measuring the distance from the critical point, and an "angle" θ, as

$$H = c_1 R^{\Delta}\theta(1 - \theta)^2, \tag{9.1}$$

$$t = c_2 R(1 - b_0^2\theta^2), \tag{9.2}$$

$$M = c_3 R^{\beta}\theta, \tag{9.3}$$

where $-1 \leq \theta \leq 1$; one can choose $c_i = 1$ (all i) provided b_0 is specified correctly (to order ϵ^2) and M is suitably normalized. This form agrees quite well with experiment (Schofield, Lister and Ho, 1969; Ho and Litster, 1970; Vicentini-Misson, 1970) although it is not quite consistent with the best numerical evidence for the three-dimensional Ising model.[19] The linear model is also found to be incompatible with the third order ϵ calculations performed by Wallace and Zia (1974).

For isotropic XY and Heisenberg-like systems ($n \geq 2$), however, the linear model definitely does *not* apply (Brézin, Wallace and Wilson, 1973; Brézin and Wallace, 1973). In particular the differential susceptibility $\chi_T(T, H) = (\partial M/\partial H)_T$ below T_c is found to diverge as $H^{1-(d/2)}$ when $H \to 0$. This prediction follows at low T from spin wave theory but had also been anticipated to hold more generally.[20] At present, however, it is lacking proper experimental verification in the critical region.

The equation of state has also been calculated by Aharony and Bruce (1974a; Bruce and Aharony, 1974b) for dipolar interactions. Numerically the results are again close to the short range expressions. However, the ratio A^+/A^- of the specific heat amplitudes above and below T_c is (to order ϵ^0) predicted to be 20% larger for dipolar coupling; this may perhaps provide an experimental test of the difference. The cubic equation of state has also been obtained (Aharony, 1974c; Wallace, 1973); the numerical changes are once again small. Generally we may conclude that even though critical exponents and other universal parameters can depend on subtle features such as lack of full isotropic symmetry and dipolar interactions, the prime determinants numerically are just the dimensionality d and the underlying isotropic symmetry index n.

B. Correlations and scattering

The calculation of the scaling function $D(z^2)$ for the correlations or critical scattering [see Eq. (2.11)] has also

[19] Gaunt and Domb (1970); Tarko and Fisher (1973, 1974); these latter articles report a numerical study of the equation of state and correlations of the three-dimensional Ising model for general fields and temperatures.

[20] See, for example, Fisher, Barber, and Jasnow (1973).

been carried to second order in ϵ (and to first order in $1/n$) (Fisher and Aharony, 1973, 1974; Aharony, 1973b, 1974c) but the general results cannot be expressed as simply as the equation of state for $n = 1$. For small z, i.e., low q at fixed $t > 0$, and with an appropriate normalization so that $z = q\xi_1(T)$ (Fisher and Aharony, 1973; Tarko and Fisher, 1973, 1974), one finds

$$\hat{D}(z^2) = 1/[1 + z^2 - z^4\Sigma_4(z)], \tag{9.4}$$

where

$$\Sigma_4(z) = \frac{2(n+2)}{(n+8)^2} Q(z^2)\epsilon^2 + O(\epsilon^3), \tag{9.5}$$

in which $Q(y)$ is given by an explicit but complicated integral. This demonstrates unambiguously that the Lorentzian or Ornstein–Zernike approximation fails, but only in order ϵ^2 (or order $1/n$). In addition $Q(0) \simeq 0.0038$ is remarkably small numerically. However, in the opposite limit of $z \gg 1$, which corresponds to $t \to 0$, one finds more singular behavior. Specifically, as already conjectured on general grounds,[21] the T dependence of $G(q, T)$ has[22] leading terms of order $t^{1-\alpha}$ and t, which combine to give a *maximum* in the scattering (at fixed q) *above* T_c. In fact, if one sets

$$\tau = t/(f_1 qa)^{1/\nu}, \tag{9.6}$$

where the correlation length varies as $\xi_1(T) \approx f_1 a t^{-\nu}$ (see e.g. Tarko and Fisher, 1973, 1974), one can, to order ϵ^2, write

$$\hat{G}(\mathbf{q}, T) \approx \hat{G}_c(q)\{1 + (\gamma - 1)\tau(\tau^{-\alpha} - 1)/\alpha - \tau + \cdots\} \quad \text{as } \tau \to 0. \tag{9.7}$$

A similar result is found in the $1/n$ expansion (Aharony, 1973b, 1974b). Note that as $\alpha \to 0$ the second term correctly reproduces the expected logarithmic singularity $\tau \ln\tau \sim t \ln t$. From Eq. (9.7), the position of the scattering maximum is readily estimated. The results confirm the somewhat uncertain predictions based on series expansion studies (Fisher and Burford, 1967; Ritchie and Fisher, 1972; Tarko and Fisher, 1974) and are in accord with the only available magnetic measurements (Bally, *et al.*, 1968; Als-Nielsen, 1970; Popovici, 1971) which reveal this intriguing phenomenon, although it has also been seen in observations on beta brass (Als-Nielsen, 1969). Further experiments would be valuable.

The calculations by Brézin, LeGuillou, and Zinn-Justin, (1974a, b) and by Combescot, Droz, and Kosterlitz, (1974) of the correlation functions to order ϵ^2 *in a field* yield the scaling function $\Sigma_4(y, z)$ with $y = H/y^{\Delta}$, which generalizes (9.5). The corrections to the Lorentzian or O–Z form are found to be an order of magnitude larger below T_c and in a field near T_c than above T_c; but numerically the deviations, for moderate values of $z = q\xi_1$, are still quite small. These conclusions agree with recent series extrapolation studies for three-dimensional Ising models (Tarko and Fisher, 1973, 1974). Various important universal ratios such as f_1^+/f_1^-, the ratio of correlation lengths above and below T_c, can also be calculated and compare well with Ising model results. Experimental data are presently quite scarce but the predictions are now ready to be tested! It is worth remarking that the correlation scaling functions for $n \geq 2$ have an interesting singularity structure in a field, which involves the anisotropy crossover exponent ϕ (Brézin, LeGuillou, and Zinn-Justin, 1974a).

[21] See, for example, Fisher and Burford (1967); Ritchie and Fisher (1972); and Fisher and Langer (1968).

[22] Fisher and Aharony (1973 and 1974); Aharony (1973 and 1974). See also Brézin, Amit, Zinn-Justin (1974).

C. Other aspects

The calculation of full *crossover* scaling functions, such as $X(z)$ in (5.6), has not yet been achieved but there is hope that methods will shortly be found to overcome the difficulties associated with the necessity of taking proper account of *two* or more fixed points at the same time.

Successful renormalization group calculations have also been undertaken (Aharony, 1973e; Wegner, 1974c; Sak, to be published) to elucidate the problem of the interaction between magnetic and lattice degrees of freedom, and the effects on the critical behavior of a compressible ferromagnet. Long range Coulomb interactions have been considered in the context of charged Bose gases (Ma, 1972). An interesting extension of the renormalization technique by Lubensky and Rubin (1973) enables one to study semi-infinite systems and to calculate the crucial *surface scaling* critical exponent Δ_1.[23] The ϵ and $1/n$ expansion technique has also been generalized by Suzuki and Igarashi (1973, 1974; Suzuki, 1973b), and by Halperin, Hohenberg, Ma, and Siggia[24] to deal with *time-* and *frequency-dependent critical phenomena*.[25] The results are fascinating and quite intricate, revealing unsuspected corrections to mode–mode coupling calculations and also circumstances where $d = 6$ is the borderline demensionality for dynamic critical behavior [above which "classical" Van Hove type theory applies (Halperin, 1973)].

Lastly in our chronicle we must ask: "What about calculations which yield the corrections to scaling and reliably indicate the *size* of the asymptotic critical region?" The renormalization group techniques clearly lead the way to such calculations even though none can be reported at this time. We must however, hope, that before too long someone sufficiently courageous and energetic will undertake such work and see it through! Only then can we consider the theory fully developed and fully testable.

ACKNOWLEDGMENTS

I am indebted to Kenneth G. Wilson, Amnon Aharony, J. Michael Kosterlitz and David R. Nelson for fruitful interactions and comments. The support of the National Science Foundation, partly through the Material Science Center at Cornell University, has sustained all phases of the author's work described here.

[23] See Barber (1973); Binder and Hohenberg (1973); see also the review by Fisher (1973).

[24] Halperin, Hohenberg, and Ma (1972, 1974); Halperin, Hohenberg, and Siggia (1974).

[25] See review by Halperin (1973) and also Abe and Hakami (to be published).

REFERENCES

Abe, R., 1972, Prog. Theor. Phys. **48,** 1414.
Abe, R., 1973, Prog. Theor. Phys. **49,** 113.
Abe, R. and S. Hikami, 1973, Prog. Theor. Phys. **49,** 442.
Abe, R., and S. Hikami, to be published.
Aharony, A., 1973a, Phys. Lett. **44A,** 313.
Aharony, A., 1973b, Phys. Lett. **46A,** 287.
Aharony, A., 1973c, Phys. Rev. **B8,** 3342, 3349, 3358.
Aharony, A., 1973d, Phys. Rev. **B8,** 3363; **B9**, Erratum.
Aharony, A., 1973e, Phys. Rev. **B8,** 4314.
Aharony, A., 1974a, in *Proceedings of the 19th Annual Conference on Magaetism,* AIP Conf. Proc. **18,** 863.
Aharony, A., 1974b, Phys. Rev. **B9,** 2107.
Aharony, A., 1974c, Phys. Rev. **B8,** 4270.
Aharony, A. and A. D. Bruce, 1974a, Phys. Rev. **B10,** 2973.
Aharony, A. and A. D. Bruce, 1974b, (to be published).
Aharony, A. and M. E. Fisher, 1973, Phys. Rev. **B8,** 3323.
Als-Nielsen, J., 1969, Phys. Rev. **185,** 664.
Als-Nielsen, J., 1970, Phys. Rev. Lett. **25,** 730.
Als-Nielsen, J., L. M. Holmes, and H. J. Guggenheim, 1974, Phys. Rev. Lett. **32,** 610.
Amit, D. J., 1972, Phys. Lett. **42A,** 299.
Amit, D. J. and A. Shcherbakov, 1973, J. Stat. Phys. **9,** 317.
Avdeeva, G. M. and A. A. Migdal, 1972, Zh. Eksp. Teor. Fiz. Pis'ma Red. **16,** 253, [Sov. Phys. JETP Lett. **16,** 178 (1972)].
Bagatskii, M. I., A. V. Voronel', and B. G. Gusak, 1962, Zh. Eksp. Teor. Fiz. **43,** 728 [Sov. Phys. JETP **16,** 517 (1963)].
Baker, Jr., G. A., 1972, Phys. Rev. **B 5,** 2622.
Baker, Jr., G. A. and G. R. Golner, 1973, Phys. Rev. Lett. **31,** 22.
Balian, R. and G. Toulouse, 1973, Phys. Rev. Lett. **30,** 544.
Balian, R. and G. Toulouse, 1974, Ann. Phys. **83,** 28.
Bally, D., M. Popovici, M. Totia, B. Grabčev, and A. M. Lungu, 1968, Phys. Lett. **26A,** 396.
Barber, M. N., 1973, Phys. Rev. **B8,** 407.
Baxter, R. J., 1971, Phys. Rev. Lett. **26,** 832.
Baxter, R. J., 1972, Ann. Phys. **70,** 193.
Bell, T. L. and K. G. Wilson, 1974, to be published.
Berezinskii, V. L., 1971, Zh. Eksp. Teor. Fiz. **61,** 1144 [Sov. Phys. JETP **32,** 493 (1972)].
Binder, K. and P. C. Hohenberg, 1974, Phys. Rev. B **9,** 2194.
Bleher, P. M. and Ya. G. Sinai, 1973, Commun. Math. Phys. **33,** 23.
Bowers, R. G. and A. McKerrell, 1973, J. Phys. C **6,** 2721.
Brézin, E., D. Amit and J. Zinn-Justin, 1974, Phys. Rev. Lett. **32,** 151.
Brézin, E., J. C. LeGuillou, and J. Zinn-Justin, 1974a, Phys. Rev. Lett. **32,** 473.
Brézin, E., J. C. Le Guillou, and J. Zinn-Justin, 1974b, Phys. Lett. **47A,** 285.
Brézin, E., J. C. Le Guillou and J. Zinn-Justin, 1974c, Phys. Rev. **9,** 1121.
Brézin, E., J. C. Le Guillou, J. Zinn-Justin and B. G. Nickel, 1973, Phys. Lett. **44A,** 227.
Brézin, E. and D. J. Wallace, 1973, Phys. Rev. B **7,** 1967.
Brézin, E., D. J. Wallace, and K. G. Wilson, 1972, Phys. Rev. Lett. **29,** 591.
Brézin, E., D. J. Wallace, and K. G. Wilson, 1973, Phys. Rev. B **7,** 232.
Bruce, A. D. and A. Aharony, 1974a, Phys. Rev. Lett. **33,** 427.
Bruce, A. D. and A. Aharony, 1974b, Phys. Rev. B **10,** 2078.
Chester, G. V., M. E. Fisher, and N. D. Mermin, 1969, Phys. Rev. **185,** 760.
Combescot, M., M. Droz, and J. M. Kosterlitz, 1974, Phys. Rev. Lett. **33,** 705.
Cowley, R. A. and A. D. Bruce, 1973, J. Phys. C **6,** L191.
de Cloiseaux, J., 1974, Phys. Rev. B (to be published).
de Gennes, P. G., 1972, Phys. Lett. **38A,** 339.
De Jongh, L. J. and A. R. Miedema, 1974, Adv. Phys. **23,** 1.
Dyson, F. J., 1969, Commun. Math. Phys. **12,** 91.
Ferrell, R. A. and J. Scalapino, 1972, Phys. Rev. Lett. **29,** 413.
Fisher, M. E., 1966, Phys. Rev. Lett. **16,** 11.
Fisher, M. E., 1967, Rep. Prog. Phys. **30,** 615.
Fisher, M. E., 1968, Phys. Rev. **176,** 257.
Fisher, M. E., 1970, in *Critical Phenomena: Proceedings of the International School of Physics, "Enrico Fermi", Varenna, 1970,* No. **51,** edited by M. S. Green (Academic, New York, 1971), p. 1.
Fisher, M. E., 1973a, J. Vac. Sci. Tech. **10,** 665.
Fisher, M. E., 1973b, Phys. Rev. Lett. **30,** 679.
Fisher, M. E., 1973, in *Collective Properties of Physical Systems, Proceedings of the Twenty-fourth Nobel Symposium,* edited by B. Lundqvist and S. Lundqvist (Academic, New York, 1973.)
Fisher, M. E. and A. Aharony, 1973a, Phys. Rev. Lett. **30,** 559.
Fisher, M. E. and A. Aharony, 1973b, Phys. Rev. Lett. **31,** 1238, 1537.
Fisher, M. E. and A. Aharony, 1974, Phys. Rev. B (to be published).
Fisher, M. E., M. N. Barber, and D. Jasnow, 1973, Phys. Rev. A **8,** 1111.
Fisher, M. E. and R. J. Burford, 1967, Phys. Rev. **156,** 583.
Fisher, M. E. and D. Jasnow, 1974, "Theory of Correlations in the Critical Region" (to be published).
Fisher, M. E., and J. S. Langer, 1968, Phys. Rev. Lett. **20,** 665.
Fisher, M. E., S.-K. Ma, and B. G. Nickel, 1972, Phys. Rev. Lett. **29,** 917.
Fisher, M. E., and D. R. Nelson, 1974, Phys. Rev. Lett. **32,** 1350.
Fisher, M. E. and P. Pfeuty, 1972, Phys. Rev. B **6,** 1889.
Gaunt, D. S., and C. Domb, 1970, J. Phys. C **3,** 1442.
Gerber, P. G. and M. E. Fisher, 1974, Phys. Rev. B (in press).
Griffiths, R. B. 1970, Phys. Rev Lett **24,** 715.
Griffiths, R. B. 1973, Phys. Rev. B **7,** 545.
Griffiths, R. B. and B. Widom, 1973, Phys. Rev. A **8,** 2173.
Halperin, B. I., 1973, in *Collective Properties of Physical Systems, Proceedings of the Twenty-fourth Nobel Symposium,* edited by B. Lundqvist and S. Lundqvist (Academic, New York, 1973), 54.
Halperin, B. I., P. C. Hohenberg, and S.-K. Ma, 1972, Phys Rev. Lett. **29,** 1548.
Halperin, B. I., P. C. Hohenberg, and S.-K. Ma, 1974, Phys. Rev. B (to be published).
Halperin, B. I., P. C. Hohenberg, and E. D. Siggia, 1974, Phys. Rev. Lett. **32,** 1289.
Hikami, S. and Abe, A., 1974, Prog. Theor. Phys. **52,** No. 2.
Ho, J. T. and J. D. Litster, 1970, Phys. Rev. B **2,** 4523.
Hohenberg, P. C., 1967, Phys. Rev. **158,** 383.
Jasnow, D. and M. E. Fisher, 1971, Phys. Rev. B **3,** 895, 907.
Joyce, G. S., 1966, Phys. Rev. **146,** 349.
Kadanoff, L. P. and F. J. Wegner, 1971, Phys. Rev. B **4,** 3989.
Kadanoff, L. P., 1973, in *Renormalization Group* (1973) p. 21.
Ketley, I. J. and D. J. Wallace, 1973, J. Phys. A **6,** 1667.
Kornblit, A., G. Ahlers, and E. Buehler, 1973, Phys. Lett. **43A,** 531.
Kosterlitz, J. M. and D. J. Thouless, 1972, J. Phys. C **5,** 124.
Larkin, A. I. and D. E. Khmel'nitskii, 1969, Zh. Eksp. Teor. Fiz. **56,** 2087; [Sov. Phys. JETP **29,** 1123].
Lederman, F. L., M. B. Salamon, and L. W. Shacklette, 1974, Phys. Rev. B **9,** 2981.
Liu, L. L., 1973, Phys. Rev. Lett. **31,** 459.
Lubensky, T. C. and M. H. Rubin, 1973, Phys. Rev Lett **31,** 1469.
Ma., S.-K., 1972, Phys. Rev. Lett. **29,** 1311.
Ma, S.-K., 1973a, Phys. Rev. A **7,** 2172.
Ma, S.-K., 1973b, Rev. Mod. Phys. **45,** 589.
Mermin, N. D. and H. Wagner, 1966, Phys. Rev. Lett. **17,** 1133.
Niemeijer, Th. and J. M. J. Van Leeuwen, 1973, Phys. Rev. Lett. **31,** 1411.
Niemeijer, Th. and J. M. J. van Leeuwen, 1974, Physica **71,** 17 (1974).
Nelson, D. R. and M. E. Fisher, 1974a, in *Proceedings of 19th Annual Conference on Magnetism,* AIP Conf. Proc. **18,** 888 and full paper (to be published).
Nelson, D. R. and M. E. Fisher, 1974b, Phys. Rev. B (to be published).
Nelson, D. R., J. M. Kosterlitz, and M. E. Fisher, 1974, Phys. Rev. Lett. **33,** 813.
Pfeuty, P., D. Jasnow, and M. E. Fisher, 1973, AIP Conf. Proc. **10,** 817.
Pfeuty, P., D. Jasnow, and M. E. Fisher, 1974, Phys. Rev. B (to be published).
Popovici, M., 1971, Phys. Lett. **34A,** 319.
Renormalization Group in Critical Phenomena and Quantum Field Theory: Proceedings of a Conference, 1973, edited by J. D. Gunton and M. S. Green (Department of Physics, Temple University, Philadelphia, 1973).
Riedel, E. K., 1971, J. Appl. Phys. **42,** 1383.
Riedel, E. K. and F. Wegner, 1969, Z. Physik **225,** 195.
Riedel, E. K., and F. J. Wegner, 1970, Phys. Rev. Lett. **24,** 730, 930.
Riedel, E. K. and F. J. Wegner, 1972, Phys. Rev. Lett. **29,** 349.
Riedel, E. K. and F. J. Wegner, 1974, Phys. Rev. B **9,** 294.
Ritchie, D. S. and M. E. Fisher, 1972, Phys. Rev. B **5,** 2668.
Sak, J., 1973, Phys. Rev. B **8,** 281.
Sak, J., to be published.
Salamon, M. B., 1973, Solid State Commun. **13,** 1741.
Scesney, P. E. and M. E. Fisher, 1970, Phys. Rev. A **2,** 825.

Schofield, P., 1969, Phys. Rev. Lett. **22,** 606.
Schofield, P., J. D. Litster, and J. T. Ho, 1969, Phys. Rev. Lett. **23,** 1098.
Stanley, H. E., 1968, Phys. Rev. **176,** 718.
Suzuki, M., 1972, Phys. Lett. **42A,** 5.
Suzuki, M., 1973a, in *Renormalization Group*, p. 55.
Suzuki, M., 1973b, Prog. Theor. Phys. **50,** 1767.
Suzuki, M., to be published.
Suzuki, M. and G. Igarashi, 1973, Prog. Theor. Phys. **49,** 1070.
Suzuki, M. and G. Igarashi, 1974, Phys. Lett. **47A,** 361.
Suzuki, M., Y. Yamasaki, and G. Igarashi, 1972, Phys. Lett. **42A,** 313.
Tarko, H. B. and M. E. Fisher, 1973, Phys. Rev. Lett. **31,** 926.
Tarko, H. B. and M. E. Fisher, 1974, Phys. Rev. B (to be published).
Tsuento, T. and E. Abrahams, 1973, Phys. Rev. Lett. **30,** 217.
Vincentini-Missoni, M. 1970, in *Critical Phenomena: Proceedings of the International School of Physics "Enrico Fermi", Varenna 1970*, No. **51,** edited by M. S. Green (Academic, New York, 1971), p. 157.
Voronel' A. V., Yu. R. Chaskhin, V. A. Popov, and V. G. Simkin, 1963, Zh. Eksp. Teor. Fiz. **45,** 828 [Sov. Phys. JETP **18,** 568 (1964)].
Wallace, D. J., 1973, J. Phys. C **6,** 1390 (and Erratum).
Wallace, D. J. and R. K. P. Zia, 1974, Phys. Lett. **46A,** 261.
Wegner, F. J., 1972a, Phys. Rev. B **5,** 4529.
Wegner, F. J., 1972b, Phys. Rev. B **6,** 1891.
Wegner, F. J., 1974a, at the Eighth Finnish Summer School in Theoretical Solid State Physics, 1973 (to be published).
Wegner, F. J., 1974b, J. Phys. C **7,** 2098.
Wegner, F. J., 1974c, J. Phys. C **7,** 2109.
Wegner, F. J. and E. K. Riedel, 1973, Phys. Rev. B **7,** 248.
Wilson, K. G., 1970, Phys. Rev. D **2,** 1438.
Wilson, K. G., 1971a, Phys. Rev. B **4,** 3174, 3184.
Wilson, K. G., 1971b, Phys. Rev. D **3,** 1818.
Wilson, K. G., 1972, Phys. Rev. Lett. **28,** 540.
Wilson, K. G. and M. E. Fisher, 1972, Phys. Rev. Lett. **28,** 248.
Wilson, K. G., and J. Kogut, 1974, Phys. Rept. **12C,** 75.
Wortis, M., 1973, *Renormalization Group* (1973) p. 96.