Chiral symmetry breaking and meson-nucleon sigma commutators: A review

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Chiral symmetries are discussed in considerable detail, and their importance and consequences for the study of elementary particle physics are reviewed. For the study of broken symmetries we concentrate mainly on meson–nucleon sigma terms, $\pi\pi$ scattering, and K_{l3} decays. In addition to giving a critical and detailed review of most of the "experimental" estimates for pion–nucleon and kaon–nucleon sigma terms done so far, we also outline the most common chiral symmetry-breaking schemes at present, such as $(3,\bar{3})+(\bar{3},3)$, $(6,\bar{6})+(\bar{6},6)$, $(8,8)$ and $(1,8)+(8,1)$ representations of $SU(3)\times SU(3)$ and possible mixtures of them, and discuss and compare their predictions with experiment. Nonlinear effective Lagrangians are briefly discussed, and the connection between broken scale invariance and chiral symmetry breaking is outlined, with emphasis on our present knowledge of meson–nucleon sigma terms.

CONTENTS

I. INTRODUCTION

The concept of approximate symmetries and partially conserved quantum numbers has occupied an increasingly central role in particle physics since the fifties (Gell-Mann, 1969). After the discovery (Gell-Mann, 1962; Ne'eman, 1961) of the approximate $SU(3)$ invariance of strong interactions, it became apparent (Cabibbo, 1963) that $SU(3)$ furnishes a unified basis for describing both the electromagnetic and weak interactions. [The genesis of this unified picture is described in Gell-Mann and Ne'eman (1964).] More recently, Gell-Mann (1964) suggested that strong interactions are nearly symmetrical under the bigger chiral group $SU(3)\times SU(3)$, generated by the algebra of vector and axial-vector currents of the hadrons. This very elegant interpretation of an approximate $SU(3)\times SU(3)$ symmetry of strong interactions has been strongly indicated by the recent joint successes[1] of current algebra and partially conserved axial-vector current (PCAC), and appears (Weinberg, 1966a, 1968a; Dashen, 1969, 1971a; Dashen and Weinstein, 1969a) to be the only rational way in which one can understand the successful current algebra and PCAC results. In addition, there appears to be good evidence from experiment that the weak and electromagnetic currents of the hadrons indeed generate[2] the algebra of $SU(3)\times SU(3)$. The hypothesis that strong interactions are invariant under this bigger chiral group, except for some small symmetry-breaking piece, clearly provides a beautiful connection between the symmetry of hadrons and their weak and electromagnetic interactions.

All the so-called internal symmetries of hadrons are, in reality, broken symmetries. In general we believe that the $SU(2)$ (isospin) group is a very good symmetry of strong interactions and is only broken by the more or less understood electromagnetic interaction. However, as long as we are faced with the problems of how to calculate, for instance, the proton–neutron (and in general any $\Delta I=1$) mass difference or the $\eta\to 3\pi$ decay in this scheme, one cannot rule out the possibility that there is some small, purely hadronic, interaction that also breaks isospin ["tadpoles" (Coleman and Glashow, 1964)].

A far-reaching classification of hadrons became possible by introducing (Gell-Mann, 1962; Ne'eman, 1961; Gell-Mann and Ne'eman, 1964) the unitary symmetry scheme $SU(3)$. Exact $SU(3)$ symmetry means multiplets of particles degenerate in mass, the pseudoscalar meson and baryon octets for example. Thus, in the real world, $SU(3)$ must be broken to the extent required to obtain the experimentally observed mass spectrum. Although we do not know, on a fundamental level, the source of the symmetry breaking, we do have a very good phenomenology for $SU(3)$ breaking insofar as we know the group theoretical transformation property of the violation. There is good evidence that the part of the strong Hamiltonian which breaks $SU(3)$ transforms like the eighth component of an octet representation of $SU(3)$. Although the size of the symmetry violation is considerable (typically of the order of 20%), to first order in symmetry breaking the result is the marvellous Gell-Mann–Okubo mass formula (Gell-Mann and Ne'eman, 1964).

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[1] See, for example, Adler and Dashen (1968), and Renner (1968).
[2] See, for example, Adler and Dashen (1968), and Marshak, Riazuddin, and Ryan (1969).

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On the other hand, exact $SU(3) \times SU(3)$ symmetry does *not* mean $SU(3) \times SU(3)$ multiplets of particles. Rather, the exact $SU(3) \times SU(3)$ symmetry limit would imply (Dashen, 1969, 1971a; Weinstein, 1971a) the existence of $SU(3)$ multiplets of particles degenerate in mass (baryons, vector mesons, etc.) and eight massless pseudoscalar mesons (Goldstone bosons), π, K and η. The consequences of this symmetry are, of course, more than just the presence of massless pseudoscalar mesons. The symmetry also tells us that these mesons satisfy low-energy theorems (Adler and Dashen, 1968; Renner, 1968; Weinberg, 1966a, 1968a; Dashen and Weinstein, 1969a) which lead to predictions such as generalized (Nieh, 1968; Dashen and Weinstein, 1969b) Goldberger–Treiman relations (relations between baryon masses, meson–baryon coupling constants, the weak axial-vector couplings and the meson decay constants), the Adler–Weisberger sum rule (renormalization of the weak axial-vector coupling constant by strong interactions) and the Callan–Treiman relation [which relates a certain combination of form factors for $K \rightarrow \pi + l + \nu_l$ $(l = e, \mu)$ to the K_{l2} decay constants]. In addition, chiral $SU(3) \times SU(3)$ symmetry does, in fact, have implications (Dashen, 1969, 1971a) other than soft-meson theorems.

For the real world chiral symmetry breaking has to occur in order to generate approximate (not mass degenerate) $SU(3)$ multiplets of particles and, also, eight low-mass mesons π, K, and η which satisfy an approximate PCAC condition. In this case, the above mentioned low-energy theorems hold only in an approximate sense (Dashen and Weinstein, 1969b; Dashen, 1971b) when regarded as statements about one-shell hadronic processes. Evidently, there are two sources of information about $SU(3) \times SU(3)$ breaking. First, the entire mass of a pseudoscalar meson comes from the symmetry-breaking interaction. From this we cannot learn very much about the nature of symmetry-breaking interactions, except, from fitting the mass spectrum of pseudoscalar mesons, the magnitude of the free parameters in the symmetry-violating piece of the strong Hamiltonian. Secondly, and by far more sensitive to the symmetry-breaking mechanism, are corrections to low-energy theorems. These low-energy theorems, which relate the symmetry-breaking part of the total Hamiltonian to the scattering amplitude of zero-mass particles, would become exact in a limit where the pseudoscalar meson masses vanish and the axial-vector currents are conserved. Thus, the soft-meson theorems may be thought of as consequences of approximate symmetry, which has been especially stressed by Weinberg (1966a, 1968a). Most important tests of theories of chiral symmetry breaking come, therefore, from low-energy theorems of meson–baryon scattering, since most of the present experimental data on meson–meson scattering are by far more controversial and less accurate. (Whereas accepted current algebra predicts the low-energy values of the crossing-odd amplitudes, chiral symmetry breaking predicts the low-energy values of the crossing-even ones.) A detailed study of these low-energy theorems, namely the calculation of the meson–baryon σ terms, i.e., the nucleon expectation value of the equal-time (sigma) commutator of the axial-vector current with its divergence, provides crucial information about the nature of chiral symmetry breaking, to what extent chiral symmetry must be broken, and what symmetry-breaking mechanisms (models) should be used. In addition, a reliable evaluation of the σ terms takes on further importance, as it may be useful in providing an understanding of the mechanism by which scale invariance is broken (Gell-Mann, 1969; Fritzsch and Gell-Mann, 1971).

In the past few years various attempts have been made to determine the magnitude of the σ term from πN scattering data as well as from the exceedingly more complicated reaction of kaon–nucleon scattering. Since low-energy theorems are valid outside the physical energy region and for zero-mass mesons, it is certainly not a trivial problem how to extrapolate off the mass shell and then to physical situations: Either one has to use one sort or another of off-mass-shell extrapolations of various scattering amplitudes, or the σ term can be extracted from the measurable on-mass-shell meson–baryon scattering amplitudes extrapolated to an unphysical point in energy, provided lowest (first)-order calculations in chiral symmetry breaking are sufficient. Although the latter method, extrapolating in energy, is clearly difficult and requires a careful use of dispersion relations, it is not subject to the host of ambiguities inherent in any off-mass-shell extrapolation procedure. The most common techniques used up to now are, for example, off-mass-shell dispersion relations, broad-area subtracted dispersion relations, threshold subtracted fixed-t dispersion relations, linear expansions of scattering amplitudes making use of Weinberg's smoothness hypothesis and the Adler consistency conditions (PCAC), the Fubini–Furlan extrapolation technique applied to pion–nucleus scattering, and some other methods to be discussed in more detail later on.

Some of these estimates of the πN σ term yielded rather contradictory results and are in violent disagreement (by about one order of magnitude) with the theoretical predictions of the original $(3, \bar{3}) + (\bar{3}, 3)$ breaking (Gell-Mann, Oakes, and Renner, 1968; Glashow and Weinberg, 1968) of chiral $SU(3) \times SU(3)$. However, most of the other πN calculations are roughly, within a factor of two or three, in agreement with the $(3, \bar{3}) + (\bar{3}, 3)$ symmetry-breaking scheme. More recently, similar results have been obtained by using kaon–nucleon scattering data. Although these estimates favor the $(3, \bar{3}) + (\bar{3}, 3)$ model, a definite enhancement of the σ terms, with respect to this model, by about a factor of two or three, persists. Since the data on πN-, $K^{\pm}N$-, $\pi\pi$-scattering and K_{l3} decays, for example, are still not accurate enough to draw definite conclusions, the σ terms estimated so far leave the possibility open that the symmetry-breaking Hamiltonian either requires further admixtures in addition to the $(3, \bar{3}) + (\bar{3}, 3)$ transforming part, which transform for instance like $(8, 8)$ or $(1, 8) + (8, 1)$ representations of $SU(3) \times SU(3)$, or has transformation properties other than the $(3, \bar{3}) + (\bar{3}, 3)$ representation, like, for example the $(6, \bar{6}) + (\bar{6}, 6)$ or $(8, 8)$ symmetry-breaking scheme. On the basis of "theoretical" arguments we shall see later that the latter possibility is at least not favored, although the present experimental situation cannot definitely discriminate between these different symmetry-breaking schemes.

For the sake of clarity and in an attempt to make this paper relatively self-contained, we briefly summarize in the first part of Sec. II the main content of chiral groups and their unitary subgroups; in the second part we discuss the central role σ commutators play in applying these symmetries to the real world, namely in the study of approximate symmetries of the strong interactions, and are mainly concerned with pseudoscalar meson masses and low-energy theorems for meson–meson and meson–nucleon scattering. In Sec. III, we critically review and discuss the various calculations of the πN σ term performed, so far, whereas various estimates of the kaon–nucleon σ term are presented in Sec. IV. In Sec. V we discuss the purely theoretical aspect of chiral symmetry breaking namely models for the symmetry-violating piece of the total Hamiltonian transforming like $(3,\bar{3})+(\bar{3},3)$, $(6,\bar{6})+(\bar{6},6)$, $(8,8)$ and $(1,8)+(8,1)$ representations of $SU(3)\times SU(3)$ and possible mixtures of them. In addition we briefly discuss the more unified approach of nonlinear effective Lagrangians as means of treating chiral symmetry breaking, and conclude this section with some remarks concerning the relation between broken scale invariance and chiral symmetry. Finally, our conclusions are summarized in Sec. VI.

II. CHIRAL SYMMETRY, LOW-ENERGY THEOREMS, AND THE SIGMA-COMMUTATOR

A. Chiral symmetry

As discussed in the Introduction, chiral symmetry, by which we mean $SU(3)\times SU(3)$ or its subgroup $SU(2)\times SU(2)$, grew naturally out of the large body of work[3] on current algebra and PCAC. In current algebra one assumes eight $SU(3)$ vector currents $V_a{}^\mu(x)$ and eight axial-vector currents $A_a{}^\mu(x)$, with $a=1,\cdots,8$ and $x\equiv(x_0,\mathbf{x})$, which satisfy the famous Gell-Mann local equal time commutation relations (summing over repeated indices):

$$\delta(x_0-y_0)[V_a{}^0(x),V_b{}^0(y)]=if_{abc}V_c{}^0(x)\delta^4(x-y),$$
$$\delta(x_0-y_0)[V_a{}^0(x),A_b{}^0(y)]=if_{abc}A_c{}^0(x)\delta^4(x-y),$$
$$\delta(x_0-y_0)[A_a{}^0(x),A_b{}^0(y)]=if_{abc}V_c{}^0(x)\delta^4(x-y), \tag{2.1}$$

where the f_{abc} are the $SU(3)$ structure constants.[3] The generalized charges associated with these currents

$$F_a(x_0)=\int d^3x\, V_a{}^0(x) \tag{2.2a}$$
$$F_a{}^5(x_0)=\int d^3x\, A_a{}^0(x) \tag{2.2b}$$

generate the Lie algebra of $SU(3)\times SU(3)$, namely

$$[F_a(x_0),F_b(x_0)]=if_{abc}F_c(x_0), \tag{2.3a}$$
$$[F_a(x_0),F_b{}^5(x_0)]=if_{abc}F_c{}^5(x_0), \tag{2.3b}$$
$$[F_a{}^5(x_0),F_b{}^5(x_0)]=if_{abc}F_c(x_0), \tag{2.3c}$$

which are assumed valid independently of the extent to which the symmetry is broken, since the (measurable) vector and axial-vector octet charges are to be identified with the unitary generators, even though the strong-interaction Hamiltonian does not commute with all those generators. The integrated commutation relations of Eqs. (2.3) are of course a less restrictive version of Gell-Mann's hypothesis than Eq. (2.1), since even if total derivative terms were present on the right-hand side of Eq. (2.1) in addition to the δ function terms, Eqs. (2.3) would still be valid. We refer here to the so-called Schwinger terms, i.e., terms proportional to the gradient of the three-dimensional δ function, which must appear on the right-hand side of local commutation relations other than the "time–time" components in Eq. (2.1). Defining the well known chiral combinations

$$F_a{}^\pm(x_0)=\tfrac{1}{2}[F_a(x_0)\pm F_a{}^5(x_0)] \tag{2.4}$$

the commutation relations (2.3) can be rewritten as

$$[F_a{}^\pm(x_0),F_b{}^\pm(x_0)]=if_{abc}F_c{}^\pm(x_0)$$
$$[F_a{}^+(x_0),F_b{}^-(x_0)]=0 \tag{2.5}$$

which shows that the F_a and $F_a{}^5$ indeed generate the direct product group $SU(3)\times SU(3)$. [The word "chiral" (handedness) applies, in particular, to the transformations generated by the axial charges $F_a{}^5$, since the factor $(1\pm\gamma_5)$ implicit in $F_a{}^\pm$ projects definite helicity states.] The two sets of "left-handed" and "right-handed" F-spin operators $F_a{}^-$ and $F_a{}^+$, respectively, are then connected by the parity operator P for strong interactions:

$$PF_a{}^\pm P^{-1}=F_a{}^\mp. \tag{2.6}$$

The electromagnetic and weak currents of hadrons are supposed to be built out of the sixteen currents $V_a{}^\mu$ and $A_a{}^\mu$. We will not be particularly interested in this aspect of the subject, but let us briefly comment on it in view of what we said in the introductory remarks about the fascinating interrelationships between the strong-interaction symmetries of hadrons and their weak and electromagnetic interactions.

The nonstrange vector currents $V_a{}^\mu$, $a=1,2,3$ are the currents of isospin, which means that the charges F_a for $a=1,2,3$ are the generators of the isospin group $SU(2)$, the oldest and still best of the hadron symmetries realized by isospin multiplets. Furthermore, the remaining vector charges F_a for $a=4,5,6,7,8$ fill out the generators of $SU(3)$, with Eq. (2.3a) forming the Lie algebra for this group. In the exact symmetry limit this group is again realized by multiplets of particles degenerate in mass. Symmetry violation (of the order of 20%) accounts (Gell-Mann and Ne'eman, 1964) then for the experimentally observed mass spectrum within a given multiplet, the baryon octet for example. By now we generally believe that $SU(3)$ is a *true* approximate symmetry of the hadronic Hamiltonian, not simply a regularity like the regularities associated with, say, the shell model of nuclei. The fact that the strong interactions do conserve isospin and hypercharge ($2F_8/\sqrt{3}$ represents the hypercharge operator) implies that F_1, F_2, F_3, and F_8 are time-independent and that the corresponding vector currents are conserved (neglecting electromagnetic and weak interactions). This is the famous conserved vector current (CVC) hypothesis, namely

$$\partial_\mu V_a{}^\mu(x)=0 \qquad \text{for } a=1,2,3,8 \tag{2.7}$$

with $\partial_\mu=\partial/\partial x^\mu$. The situation is different for the strange-

[3] See, for instance, Adler and Dashen (1968).

ness-carrying vector currents $V_a{}^\mu$, $a = 4$ to 7. The fact that $SU(3)$ breaking transforms like the eighth member of an octet representation implies that the divergences $\partial_\mu V_a{}^\mu$, $a = 4$ to 7, are proportional (Adler and Dashen, 1968) to the $SU(3)$ symmetry-breaking parameter λ, say. Therefore, whereas Eq. (2.7) holds in the exact symmetry limit as well as for the (realistic) broken symmetry, the strangeness-carrying vector currents are only conserved in the exact ($\lambda = 0$) $SU(3)$ symmetry limit. The electromagnetic four current is constructed from the vector octet as

$$J_{em}{}^\mu = V_3{}^\mu + (1/\sqrt{3}) V_8{}^\mu \qquad (2.8)$$

(as long as no confusion occurs the explicit space–time dependence of currents will be suppressed). As an illustration of theoretical predictions resulting from the form (2.8) of the electromagnetic current, we mention the magnetic moments of Λ and Σ hyperons which can be calculated from those of the proton and neutron with an uncertainty of some 20%, characteristic of the strength of $SU(3)$ violation. The Λ moment is predicted to be half of the neutron moment, whereas experimentally $\mu_\Lambda/\mu_n = 0.37 \pm 0.04$.

Including the eight axial-vector charges $F_a{}^5$ we finally arrive at the chiral $SU(3) \times SU(3)$ group, generated by the Lie algebra Eqs. (2.3), with its subgroup $SU(2) \times SU(2)$ generated by the charges $F_{1,2,3}$ and $F_{1,2,3}{}^5$. This latter symmetry is expected to be a much better symmetry than $SU(3) \times SU(3)$ or $SU(3)$ itself, which is suggested but not required by the smallness of the pion mass (see below). The $SU(3) \times SU(3)$ symmetry is different from $SU(3)$ in that if it were not explicitly broken (by some more or less understood terms in the total Hamiltonian), it would be "spontaneously" broken (Goldstone, 1961; Goldstone, Salam, and Weinberg, 1962; Weinstein, 1971a). That is, if we could turn off the $SU(3) \times SU(3)$ breaking piece of the strong interactions, the hadron vacuum would not be an eigenstate of the $F_a{}^5$, i.e., it would not be invariant (symmetric) under transformations generated by $F_a{}^5$, but still would be invariant under $SU(3)$. This has the consequence that in the broken-symmetry world one does not get approximate $SU(3) \times SU(3)$ multiplets but rather, in addition to the $SU(3)$ multiplets, a set of low mass (π, K, η) particles (Goldstone bosons in the exact symmetry limit), which satisfy PCAC relations. This somewhat explains the successes of PCAC and current algebra in dealing with soft-meson processes. More explicitly, the PCAC hypothesis can be stated in the form

$$\partial_\mu A_a{}^\mu(x) = m_a{}^2 f_a \phi_a(x) \qquad (2.9)$$

where the pseudoscalar meson fields are denoted by $\phi_a(x)$, representing particles of mass m_a, and the f_a are the semileptonic K_{l2} decay constants, i.e., f_π and f_K are measured in $\pi \to \mu + \nu_\mu$ and $K \to \mu + \nu_\mu$, respectively. (As usual, the fields and currents are classified by their third component of isospin and their hypercharge, i.e., $A_{\pi^\pm} = (1/\sqrt{2})(A_1 \mp iA_2) \equiv A_{(1\mp i2)/\sqrt{2}}$, $A_{\pi^0} = A_3$, $A_{K^\pm} = A_{(4\mp i5)/\sqrt{2}}$, etc.). The total strong-interaction Hamiltonian density can be written as

$$\mathcal{H} = \mathcal{H}_0 + \epsilon\mathcal{H}', \qquad (2.10)$$

where $\mathcal{H}_0$ is invariant under the full chiral $SU(3) \times SU(3)$ group and $\epsilon\mathcal{H}'$ is the symmetry-breaking part. As a purely formal device for keeping track of powers of symmetry breaking, we introduced the "small" scale parameter ϵ. The total Hamiltonian is then given by $H = \int d^3x \mathcal{H}(x_0, \mathbf{x})$. [Of course, this decomposition into a symmetry-conserving and symmetry-breaking part means nothing until we add to it the assumption that an expansion about the limit $\epsilon \to 0$ makes sense, a hypothesis strongly suggested (Dashen and Weinstein, 1969a; Weinstein, 1971a) by any Lagrangian field theory.] Since $[F_a{}^\pm, H_0] = 0$, we get

$$\partial_\mu V_a{}^\mu(x) = i[\epsilon\mathcal{H}'(x), F_a(x_0)] \qquad (2.11a)$$

$$\partial_\mu A_a{}^\mu(x) = i[\epsilon\mathcal{H}'(x), F_a{}^5(x_0)] \qquad (2.11b)$$

as a local generalization (Glashow and Weinberg, 1968) of $\int d^3x \partial_\mu V_a{}^\mu = i[H, F_a]$, etc. Furthermore, because of the experimentally observed $\pi \to l + \nu_l$ and $K \to l + \nu_l$ decays ($l = e, \mu$), the single pseudoscalar meson states $|M_a(q)\rangle$ have to be coupled to the vacuum by the axial-vector current, i.e.,

$$\langle 0 | A_a{}^\mu(0) | M_a(q) \rangle = iq^\mu f_a, \qquad (2.12)$$

where q^μ is the four momentum of the meson. Taking the divergence of Eq. (2.12) gives, strictly from translation invariance, the identity

$$\langle 0 | \partial_\mu A_a{}^\mu(0) | M_a(q) \rangle = m_a{}^2 f_a. \qquad (2.13)$$

Since we obviously do not wish to decouple the current, we have to keep $f_a \neq 0$. From Eqs. (2.11b) and (2.13) it is now clear that the mechanism by which the symmetry is broken is also responsible for the masses of the mesons being nonvanishing: In the exact symmetry limit, $\epsilon = 0$, it is clear from Eq. (2.11b) that $\partial_\mu A_a{}^\mu$ must vanish and hence, according to Eq. (2.13), m_a must be zero (leaving us with an octet of massless Goldstone bosons). Thus, only in the symmetry limit do we have exact conservation of the axial-vector current, and the actual pseudoscalar masses are a measure of how badly the symmetry is broken. This then suggests that the $\Delta I = 1$, $\Delta S = 0$ (I = isospin, S = strangeness) pion currents $A_{1,2,3}{}^\mu$ are conserved to a good approximation, whereas the remaining axial-vector currents have larger divergences $\partial_\mu A_a{}^\mu$ corresponding to the larger masses of K and η mesons, compared to the pion mass. [In the usual dispersion-theoretic language, the PCAC hypothesis in Eq. (2.9) means that all matrix elements of $\partial_\mu A_a{}^\mu$ obey unsubtracted dispersion relations in the momentum transfer variable, and that these relations are dominated by the respective meson pole (π, K, or η) contribution.] Since we also have exact $SU(3)$ when $\epsilon = 0$, all the f_a's in Eq. (2.12) are the same in this limit. Thus we expect f_π and f_K to differ by not more than about 20%, which is typical of $SU(3)$ breaking. Evidently, this is borne out by the data.

As is well known (Cabibbo, 1963; Gell-Mann, 1964), another interesting part of the $SU(3) \times SU(3)$ symmetry is that its generators also form the basis for the hadronic weak current, which may be written as

$$J_W{}^\mu = (V_{1+i2}{}^\mu - A_{1+i2}{}^\mu)\cos\theta_C + (V_{4+i5}{}^\mu - A_{4+i5}{}^\mu)\sin\theta_C, \qquad (2.14)$$

where θ_C, the so-called Cabibbo angle, is supposed to be a universal constant of about 15°, taking into account that the rates for strangeness-changing (described by the $4 \pm i5$ components) semileptonic baryon decays are suppressed by at least an order of magnitude compared to the strangeness-conserving processes. (The empirical selection rules $\Delta I = 1$ for $\Delta S = 0$ semileptonic decays and $\Delta I = \frac{1}{2}$, $\Delta S = \Delta Q$ for $\Delta S = \pm 1$ decays are guaranteed by the use of the $1 \pm i2$ and $4 \pm i5$ components of the currents in $J_W{}^\mu$, respectively.) This assumption that the weak currents and the isovector part of the electromagnetic current belong to the same unitary octet, is now believed to hold on the basis of a number of very successful predictions.[4] The CVC hypothesis, then, in turn implies that there are relations between weak and electromagnetic processes which can be used to test the soundness of the theory. In particular, the CVC predictions for $\Delta S = 0$ processes are very well satisfied experimentally,[4] as in for example the absence of strong renormalization effects for $V_{1\pm i2}{}^\mu$, the relation between the weak and electromagnetic form factors, or the branching ratio for the pion β-decay $\pi^- \rightarrow \pi^0 + e^- + \bar{\nu}_e$ and the $\pi \rightarrow \mu + \nu_\mu$ decay. It is also of great interest to discuss the (approximate) conservation of $A_{1+i2}{}^\mu$ in the matrix element for neutron decay $n \rightarrow p + e^- + \bar{\nu}_e$ or neutrino reactions $\nu_l + n \rightarrow l + p$ $(l = e$ or $\mu)$. The matrix element has the form

$$\langle p \mid A_{1+i2}{}^\mu(0) \mid n \rangle = \bar{u}_p(q_p)[\gamma^\mu\gamma_5 G_1(k^2) + k^\mu\gamma_5 G_2(k^2)]u_n(q_n), \qquad (2.15)$$

where q_n and q_p are the four momenta of neutron and proton, u_n and u_p their Dirac spinors, and $k^2 = (q_p - q_n)^2$. From β decay one knows that $G_1(0) = g_A$ has the value $g_A \simeq 1.2$. Taking the divergence of Eq. (2.15), we obtain

$$\langle p \mid \partial_\mu A_{1+i2}{}^\mu(0) \mid n \rangle = i[2M_N G_1(k^2) + k^2 G_2(k^2)] \times \bar{u}_p(q_p)\gamma_5 u_n(q_n) \qquad (2.16)$$

M_N being the nucleon mass, so that *exact* axial-vector current conservation requires

$$G_2(k^2) = -2M_N G_1(k^2)/k^2. \qquad (2.17)$$

Neither $G_1(0)$ nor the nucleon mass vanish. Hence G_2 has a pole at $k^2 = 0$ corresponding to an exchange of a zero-mass pseudoscalar meson. Thus, exact conservation of $A_{1+i2}{}^\mu$ implies not only the vanishing of the pion mass but even the very existence of this zero mass particle. One can therefore speculate that the pion exists as a consequence of chiral $SU(2) \times SU(2)$ symmetry, and that similarly the K and η mesons exist because of the $SU(3) \times SU(3)$ symmetry. If these symmetries were strictly valid and the mesons strictly massless, the vacuum state would be, as already emphasized, degenerate with states containing any number of zero-momentum mesons, just as the ground state of an isotropic ferromagnet is degenerate for simultaneous rotations of all spins; the pseudoscalar mesons would be analogous to the magnons of ferromagnetism, the two cases being examples of a general situation first recognized by Goldstone (1961). The chiral symmetry is broken, of course, i.e., $\partial_\mu A_a{}^\mu \neq 0$, and this corresponds to the pseudoscalar mesons' having nonvanishing masses.

[4] See, for example, Marshak, Riazuddin, and Ryan (1969).

There are two good experimental confirmations that $SU(2) \times SU(2)$ is nearly an exact symmetry. One is the Goldberger–Treiman (GT) relation[1] which expresses the ratio g_A/f_π, containing the weak-interaction quantities of the nucleon and pion, in terms of the pure strong-interaction quantities M_N and the pion–nucleon coupling constant g:

$$g_A/f_\pi = g/M_N. \qquad (2.18)$$

This relation is obtained by assuming that the left-hand side of Eq. (2.16) is dominated by the pion pole, using Eq. (2.13) to calculate this pole term, and then evaluating Eq. (2.16) at $k^2 = 0$. [Similar generalized GT relations can be written down (Nieh, 1968; Dashen and Weinstein, 1969b; Gell-Mann, 1969) by taking into account hyperons and kaons.] The second relation establishing that $SU(2) \times SU(2)$ is nearly an exact symmetry expresses g_A in terms of the πN scattering amplitudes, again a pure strong-interaction property, and is called the Adler–Weisberger sum rule.[1] Both relations are very well satisfied.

Although by now it appears that we partly understand, from a purely phenomenological point of view, how and to what extent (chiral) symmetries are broken, we do not have any fundamental idea why the symmetry of hadrons and their weak and electromagnetic interactions are related by $SU(3) \times SU(3)$. Certainly $SU(3)$ does not need weak interactions for its existence. The mystery is even further deepened by the fact that the weak interactions, which violate essentially every symmetry known, choose, when coupling to hadrons, a set of approximately conserved currents.

B. More on meson masses

In order to establish a close relation between the pseudoscalar meson masses and the symmetry-breaking Hamiltonian density $\mathcal{H}'(x)$, Eq. (2.10), let us consider the following vacuum expectation value (Dashen, 1969, 1971a) (no sum over a)

$$i\int d^4x \,\langle 0 \mid T(\partial_\mu A_a{}^\mu(x)\partial_\nu A_a{}^\nu(0)) \mid 0 \rangle = m_a{}^2 f_a{}^2 + \int_{9m_a{}^2}^\infty d\mu^2 \frac{\rho_a(\mu^2)}{\mu^2}, \qquad (2.19)$$

where the first term on the right is the contribution of the single pseudoscalar meson state and ρ_a is the spectral function of the higher states, e.g., three-meson, baryon–antibaryon, etc. Since ρ_a is the spectral function of two divergences, it is clear, by Eq. (2.11b), that it is of order ϵ^2 in chiral symmetry breaking. However, the first term on the right-hand side of Eq. (2.19) is proportional to $m_a{}^2$ and hence of order ϵ rather than ϵ^2. (This comes about because of a factor $m_a^{-2} \sim \epsilon^{-1}$ from the denominator of the meson pole.) A straightforward current algebra calculation gives

$$-\langle 0 \mid [F_a{}^5(0), [F_a{}^5(0), \epsilon\mathcal{H}']] \mid 0\rangle = m_a^2 f_a^2 + \int_{9m_a^2}^{\infty} d\mu^2 \frac{\rho_a(\mu^2)}{\mu^2} \tag{2.20}$$

where the so-called sigma commutator (it derives its name from the σ model,[3] where it simply reduces to the canonical σ field)

$$\Sigma_{ab}(x) \equiv [F_a{}^5(x_0), [F_b{}^5(x_0), \epsilon\mathcal{H}'(x)]] \tag{2.21}$$

arises through the process of moving derivatives outside the time-ordered product[3] in Eq. (2.19). By Eq. (2.11b), the sigma commutator in Eq. (2.21) is nothing else but the equal-time commutator of the axial-vector current with its divergence:

$$\delta^4(x-y)\,\Sigma_{ab}(x) = i\delta(x_0 - y_0)[A_a{}^0(y), \partial_\mu A_b{}^\mu(x)]. \tag{2.22}$$

The expectation value of this commutator is usually called the "σ term." Equation (2.20) is still exact. To get something useful, we drop the spectral integral which is second order in ϵ to obtain

$$m_a^2 f_a^2 = -\langle 0 \mid [F_a{}^5(0), [F_a{}^5(0), \epsilon\mathcal{H}'(0)]] \mid 0\rangle + O(\epsilon^2) \tag{2.23}$$

which, via the σ commutator, relates the pseudoscalar meson masses directly to the symmetry-breaking Hamiltonian. According to Eq. (2.23), as outlined already in the previous section but on a different footing, the eight pseudoscalar meson masses vanish in the exact symmetry limit $\epsilon \rightarrow 0$.

As we shall see later, the meson masses in Eq. (2.23) can be fit with $\epsilon\mathcal{H}'$ belonging to almost any representation of $SU(3) \times SU(3)$. [In fact, Eq. (2.23) usually fixes the parameters of the model considered.] Thus, in order to check various symmetry-breaking models, i.e., different forms of $\mathcal{H}'$, we have to look elsewhere.

C. π–π scattering

Soft-pion calculations, based on the $SU(2) \times SU(2)$ algebra, were first performed by Weinberg (1966a, b, 1968a) and Tomozawa (1966); see also, for instance, Adler and Dashen (1968). Subsequently, a compact generalization for $SU(3) \times SU(3)$ was given by Dashen and Weinstein (1969a), and we shall use the general results of these works. Following Weinberg's (1966b) analysis, the scattering amplitude for the reaction $\pi_a(q_1) + \pi_b(q_2) \rightarrow \pi_c(q_3) + \pi_d(q_4)$ is defined by

$$\begin{aligned} i(2\pi)^4\delta^4(q_1 + q_2 - q_3 - q_4)\langle q_4 d; q_3 c \mid M \mid q_2 b; q_1 a\rangle \\ = -(q_1^2 - m_\pi^2)(q_2^2 - m_\pi^2)(q_3^2 - m_\pi^2)(q_4^2 - m_\pi^2) \\ \times (m_\pi^2 f_\pi)^{-4} \int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4 \\ \times \exp[i(q_4\cdot x_4 + q_3\cdot x_3 - q_2\cdot x_2 - q_1\cdot x_1)] \\ \times \langle 0 \mid T(\partial_\mu A_d{}^\mu(x_4)\partial_\nu A_c{}^\nu(x_3)\partial_\lambda A_b{}^\lambda(x_2)\partial_\sigma A_a{}^\sigma(x_1)) \mid 0\rangle \\ = i(2\pi)^4\delta^4(q_1 + q_2 - q_3 - q_4) \\ \times \{\delta_{ab}\delta_{cd}[A + B(t+u) + Cs] \\ + \delta_{ac}\delta_{bd}[A + B(s+u) + Ct] \\ + \delta_{ad}\delta_{bc}[A + B(s+t) + Cu] + O(q^4)\}, \end{aligned} \tag{2.24}$$

where $s = (q_1 + q_2)^2$, $t = (q_1 - q_3)^2$, and $u = (q_1 - q_4)^2$. Since in this section, we are only dealing with π–π scattering, the $SU(3)$ indices a, b, c, and d are just the pion isovector indices (running over 1, 2, 3). The second line in Eq. (2.24) results from the usual Lehmann–Symanzik–Zimmermann (1955) (LSZ) reduction, whereas the third line is the well known Weinberg expansion (Weinberg, 1966b) with constant coefficients A, B, and C. The physical threshold is at $s = 4m_\pi^2$, $t = u = 0$, so that the s-wave π–π scattering lengths (defined to be proportional to the S-matrix at threshold) are given by

$$\begin{aligned} a_0{}^{(0)} &= -(32\pi m_\pi)^{-1}(5A + 8m_\pi^2 B + 12m_\pi^2 C) \\ a_0{}^{(2)} &= -(32\pi m_\pi)^{-1}(2A + 8m_\pi^2 B), \end{aligned} \tag{2.25}$$

where the notation $a_l{}^{(I)}$ has been used. Among the constants A, B, and C there are two model-independent relations:

$$A + m_\pi^2(2B + C) = 0 \tag{2.26}$$

and

$$B - C = f_\pi^{-2}. \tag{2.27}$$

Equation (2.26) follows from the Adler self-consistency condition[3] [which states that M in Eq. (2.24) vanishes when any one of the four pion momenta vanish and the other three are on the mass shell, i.e., $M = 0$ when $s = t = u = m_\pi^2$], and can be obtained from Eq. (2.24) by extracting all derivatives and letting any $q_i{}^\mu \rightarrow 0$ in the resulting expression. Similarly we obtain Eq. (2.27) if we let $q_1{}^\mu$, $q_3{}^\mu \rightarrow 0$. Finally, taking all the pion momenta to zero, $q_1{}^\mu = q_2{}^\mu = q_3{}^\mu = q_4{}^\mu = 0$, Eq. (2.24) yields, using Eq. (2.11b), the following low-energy theorem:

$$\begin{aligned} A(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \\ = -f_\pi^{-4}\{\langle 0 \mid [F_d{}^5, [F_c{}^5, [F_a{}^5, [F_b{}^5, \epsilon\mathcal{H}'(0)]]]] \mid 0\rangle \\ + (dbac) + (cbad) + O(\epsilon^2)\}, \end{aligned} \tag{2.28}$$

with $F_i^5 = F_i^5(0)$. Inserting Eqs. (2.26) and (2.27) into (2.25), we get

$$a_0^{(0)} = \frac{1}{96\pi m_\pi}\left(5A + \frac{16m_\pi^2}{f_\pi^2}\right)$$

$$a_0^{(2)} = \frac{1}{96\pi m_\pi}\left(2A - \frac{8m_\pi^2}{f_\pi^2}\right). \tag{2.29}$$

For a definite isospin channel we can write for simplicity, according to Eq. (2.28),

$$A = -f_\pi^{-4}\langle 0 \mid [F_\pi^5, [F_\pi^5, [F_\pi^5, [F_\pi^5, \epsilon\mathcal{H}'(0)]]]] \mid 0\rangle, \tag{2.30}$$

(no sum over $\pi = 1, 2, 3$), which we shall use in our future applications. Again we found, Eq. (2.29), that the σ commutator provides a direct connection between the symmetry-breaking interaction and experimentally measurable quantities. The sigma term in Eq. (2.28) involving four commutators with the F_a^5's is of course just a generalization of Eq. (2.21), since by a LSZ reduction we have

$$\langle \pi_d \mid [F_c^5, [F_b^5, \epsilon\mathcal{H}']] \mid \pi_a\rangle$$
$$\propto \langle 0 \mid [F_d^5, [F_a^5, [F_c^5, [F_b^5, \epsilon\mathcal{H}']]]] \mid 0\rangle.$$

For completeness let us mention that Eq. (2.29) implies the famous Weinberg relation (Weinberg, 1966b)

$$2a_0^{(0)} - 5a_0^{(2)} = (3/4\pi)(m_\pi/f_\pi^2) \tag{2.31}$$

which is model-independent, since only the value of A depends on $\mathcal{H}'$.

In contrast to the meson masses in Eq. (2.23), the $\pi\pi$ scattering lengths in Eq. (2.29) would provide very sensitive and selective tests for various symmetry-breaking models, i.e., various $SU(3) \times SU(3)$ representations under which $\mathcal{H}'$ transforms, if accurate data for $a_0^{(I)}$ were available. Unfortunately, as we shall see, present experimental results provide us with only some consistency checks of theoretical predictions.

D. Low-energy theorem for Meson-Nucleon scattering

For practical use, the low-energy theorems for meson-nucleon scattering (elastic πN or $K^\pm N$ for example) are presently by far more important for testing theories of (broken) chiral symmetries. This is because present meson–nucleon scattering data are, compared to the meson–meson data, less controversial and ambiguous.

Consider the process

$$M_a(q) + N(p) \to M_b(q') + N(p') \tag{2.32}$$

with four momenta of the particles indicated in parentheses and a, b denoting the $SU(3)$ indices of the mesons. The kinematic invariants for this process are defined by

$$s = (q+p)^2, \quad t = (q-q')^2, \quad u = (q-p')^2 \tag{2.33}$$

or equivalently, for the sake of convenience, we shall also use

$$\nu = (p+p')\cdot(q+q')/4M_N = (s-u)/4M_N$$
$$= \omega + t/4M_N,$$
$$\nu_B = -q\cdot q'/2M_N = (t - q^2 - q'^2)/4M_N, \tag{2.34}$$

where ω denotes the total laboratory energy of the incoming meson. The amplitude for the process (2.32) can be continued off-mass shell by means of the definition

$$T_{ba}(\nu, t, q^2, q'^2)$$
$$= i(q^2 - m_a^2)(q'^2 - m_b^2)(m_a^2 f_a)^{-1}(m_b^2 f_b)^{-1}$$
$$\times \int d^4x \exp(iq'\cdot x)$$
$$\times \langle p' \mid T(\partial_\mu A_b^\mu(x)\partial_\nu A_a^\nu(0)) \mid p\rangle. \tag{2.35}$$

The basic relation for deriving low-energy theorems is the so-called generalized Ward–Takahashi identity, which can be written in the form

$$\int d^4x\, d^4y \exp(iq'\cdot x) \exp(-iq\cdot y)$$
$$\times \langle p' \mid T(\partial_\mu A_b^\mu(x)\partial_\nu A_a^\nu(y)) \mid p\rangle$$
$$= \int d^4x\, d^4y \exp(iq'\cdot x) \exp(-iq\cdot y)$$
$$\times \langle p' \mid \{q_\mu' q_\nu T(A_b^\mu(x) A_a^\nu(y))$$
$$+ iq_\mu'\delta(x_0 - y_0)[A_b^\mu(x), A_a^0(y)]$$
$$- \delta(x_0 - y_0)[A_b^0(x), \partial_\nu A_a^\nu(y)]\} \mid p\rangle \tag{2.36}$$

which can be derived by using standard techniques[1] of pulling the derivatives through the time-ordered product. The first term on the right-hand side of Eq. (2.36) contributes both to the symmetric and antisymmetric part of the amplitude with respect to the $SU(3)$ indices a, b, whereas the second term, the equal-time commutator known from current algebra (Sec. IIA), is antisymmetric in a, b and of first order in the meson momentum. The last term corresponds to the σ commutator and is *not* determined by Gell-Mann's current algebra. Inserting Eq. (2.36) into (2.35) and taking the soft-meson limit $q_\nu \to 0$, $q_\mu' \to 0$, we obtain the low-energy theorem

$$T_{ba}(0,0,0,0) = -f_a^{-1} f_b^{-1}\sigma_{NN}^{ba}, \tag{2.37}$$

where

$$\sigma_{NN}^{ba} \equiv i\int d^4x\, \delta(x_0)\langle p' \mid [A_b^0(x), \partial_\nu A_a^\nu(0)] \mid p\rangle$$
$$= \langle p' \mid [F_b^5(0), [F_a^5(0), \epsilon\mathcal{H}'(0)]] \mid p\rangle \tag{2.38}$$

is the so-called meson–nucleon sigma term, and use has been made of Eq. (2.11b). That σ_{NN}^{ba} is symmetric in the

$SU(3)$ indices a, b can easily be seen by writing a Jacobi identity for the double commutator in Eq. (2.38):

$$[F_b^5, [F_a^5, \mathcal{H}']] + [\mathcal{H}', [F_b^5, F_a^5]] + [F_a^5, [\mathcal{H}', F_b^5]] = 0$$

and using the fact that $[\mathcal{H}', [F_b^5, F_a^5]] = 0$ by isospin and hypercharge conservation. Therefore, a consistent calculation of the σ-terms should make use of the (isospin) even amplitude $T_{ba} + T_{ab}$ which we shall denote hereafter by $T^{(+)}$ or T^+ (see below). Since we are only interested in elastic processes, we always will have $f_a = f_b$ in Eq. (2.37).

Since σ terms are directly related to the symmetry-breaking Hamiltonian, Eq. (2.38), it would clearly be a good thing to know $\sigma_{NN}{}^{ba}$. Since Eq. (2.37) relates the nucleon expectation value of the sigma commutator to an off-mass-shell amplitude, objects like $\sigma_{NN}{}^{ba}$ cannot be measured directly, but can be obtained by extrapolation from on-shell scattering amplitudes. Going off the mass shell through a power series expansion in q^2 and q'^2, for example, which was first proposed by Cheng and Dashen (1971),

$$\begin{aligned} T^+(0, 2m_a^2, m_a^2, m_a^2) &= T^+(0,0,0,0) + m_a^2(\partial/\partial q^2)T^+(0,0,0,0) \\ &\quad + m_a^2(\partial/\partial q'^2)T^+(0,0,0,0) + O(\epsilon^2), \end{aligned} \tag{2.39}$$

where we assumed $a = b$, since we will be only interested in elastic meson–nucleon scattering. The Adler consistency conditions (PCAC) (Adler, 1965; Adler and Dashen, 1968)

$$T^+(0, m_a^2, m_a^2, 0) = T^+(0, m_a^2, 0, m_a^2) = 0 \tag{2.40}$$

(where we always define the point $\nu = 0$, $\nu_B = 0$ by taking the limit $\nu_B \to 0$ followed by $\nu \to 0$ which eliminates the Born pole terms) can also be expanded into a power series

$$\begin{aligned} T^+(0, m_a^2, m_a^2, 0) &= T^+(0,0,0,0) + m_a^2(\partial/\partial q^2)T^+(0,0,0,0) + O(\epsilon^2) \\ T^+(0, m_a^2, 0, m_a^2) &= T^+(0,0,0,0) + m_a^2(\partial/\partial q'^2)T^+(0,0,0,0) + O(\epsilon^2) \end{aligned} \tag{2.41}$$

which implies

$$\begin{aligned} T^+(0,0,0,0) &= -m_a^2(\partial/\partial q^2)T^+(0,0,0,0) + O(\epsilon^2) \\ &= -m_a^2(\partial/\partial q'^2)T^+(0,0,0,0) + O(\epsilon^2) \end{aligned} \tag{2.42}$$

and together with Eq. (2.39) it follows that

$$T^+(0, 2m_a^2, m_a^2, m_a^2) = -T^+(0,0,0,0) + O(\epsilon^2). \tag{2.43}$$

Equation (2.37) can then be written as

$$T^+(0, 2m_a^2, m_a^2, m_a^2) = f_a^{-2}\sigma_{NN}{}^{aa}, \tag{2.44}$$

where we have dropped higher-order terms, since $O(\epsilon^2) = O[(m_a^2/M_N^2)^2]$ with the nucleon mass M_N representing a "typical strong-interaction mass." For the case of πN scattering, such higher-order correction can be safely neglected, as actually has been shown (Brown, Pardee, and Peccei, 1971), because of the exceedingly small factor m_π^4. A result not to be expected *a priori* for kaon–nucleon scattering, where second-order corrections in Eq. (2.44) can be as large as m_K^4/M_N^4. Thus, provided these higher-order terms in chiral symmetry breaking can be neglected, Eq. (2.44) offers a unique relation between the symmetry-violating interaction $\mathcal{H}'$ and the on-mass-shell amplitude independent of any (ambiguous) model-dependent off-mass-shell extrapolation procedure. In addition, the point $\nu = 0$, $t = 2m_a^2$ is clearly outside the physical region. To reach this unphysical (but on-mass-shell) point $\nu = \nu_B = 0$, $q^2 = q'^2 = m_a^2$, one has to use, for instance, fixed-t dispersion relations.

Another equally important method to relate Eq. (2.37) to experimental observables would be, for example, a linear expansion of the isospin-even amplitude, by making full use of Weinberg's smoothness hypothesis (Weinberg, 1966b). This and similar methods will be discussed in the following two chapters.

E. K_{l3} decays

Finally, let us mention another low-energy theorem, namely the Callan–Treiman—Mathur–Okubo–Pandit relation for K_{l3} decays, which reads (Callan and Treiman, 1966; Mathur, Okubo, and Pandit, 1966)

$$f_+(m_K^2) + f_-(m_K^2) = f_K/f_\pi \tag{2.45}$$

where the semileptonic decay constants f_π and f_K are again given by Eq. (2.9) and the scalar form factors are defined by

$$\begin{aligned} &\langle \pi^0 \mid V_{4+i5}{}^\mu(0) \mid K^+ \rangle \\ &\quad = (1/\sqrt{2})[(p_K + p_\pi)^\mu f_+(t) + (p_K - p_\pi)^\mu f_-(t)], \end{aligned} \tag{2.46}$$

with $t = (p_K - p_\pi)^2$. The similarly defined form factors for the K^0 decay must be the same as those in Eq. (2.46), assuming the validity of the $\Delta I = \frac{1}{2}$ rule. Equation (2.45) is an interesting relation because it uses PCAC to relate $SU(3)$ breaking effects (see, for example, Dashen, 1971b). Since Eq. (2.45) is merely an identity, it cannot be used as a consistency check on any form of symmetry breaking, i.e., it does not discriminate between different transformation properties of $\mathcal{H}'$. This would not be the case, of course, if the vacuum were not an $SU(3)$ singlet in the chiral limit, but this would anyway imply a scalar (κ) Goldstone boson; work along this line has been done, especially in order to explain the observed value of

$$\xi(t) = f_-(t)/f_+(t), \tag{2.47}$$

but we will refer to it later. However, corrections to Eq. (2.45) can be calculated (Dashen and Weinstein, 1969c; Li and Pagels, 1971a) on a more or less model-independent basis. Although the experimental situation in K_{l3} decays is still very confused (Chounet, Gaillard, and Gaillard, 1972; Wojcicki, 1972), the real importance of testing such correction terms is not that it differentiates between models of symmetry breaking, but that it is an (essentially) model-independent test of the idea of expanding in powers of symmetry breaking.

III. CALCULATIONS OF THE πN SIGMA TERM

Before discussing various recent estimates of $\sigma_{NN}^{\pi\pi}$, let us briefly summarize some relevant properties of pion–nucleon amplitudes.

A. Pion–nucleon amplitudes and dispersion relations

Considering the reaction in Eq. (2.32), where M_a stands now for π_a, the T-matrix is conventionally (Moorhouse, 1969) decomposed into

$$T_{ba} = A_{ba} + \tfrac{1}{2}\gamma\cdot(q+q')B_{ba} \tag{3.1}$$

where the two invariant amplitudes A and B are chosen to be scalar functions of ν, t. To specify the various charge states, these amplitudes are decomposed into

$$A_{ba} = \tfrac{1}{2}\{\tau_b, \tau_a\}A^{(+)} + \tfrac{1}{2}[\tau_b, \tau_a]A^{(-)} \tag{3.2}$$

and similarly for B_{ba}, where τ_a denotes the 2×2 isospin matrices. With respect to crossing ($\nu\to-\nu$, t fixed), $A^{(+)}$ and $B^{(-)}$ are even functions, whereas $A^{(-)}$ and $B^{(+)}$ are odd. The amplitudes corresponding to definite isospin $I = 1/2$, $3/2$ are given by

$$A^{(+)} = \tfrac{1}{3}A^{1/2} + \tfrac{2}{3}A^{3/2}, \qquad A^{(-)} = \tfrac{1}{3}A^{1/2} - \tfrac{1}{3}A^{3/2} \tag{3.3}$$

and similarly for $B^{(\pm)}$, and therefore the amplitudes for $\pi^{\pm}N$ scattering are

$$A_{\pm} = A^{(+)} \mp A^{(-)}, \qquad B_{\pm} = B^{(+)} \mp B^{(-)}. \tag{3.4}$$

The T-matrix normalization is chosen such that the differential cross section in the c.m. system is given by

$$d\sigma/d\Omega = (M_N/4\pi W)^2 \sum |\bar{u}(p')Tu(p)|^2, \tag{3.5}$$

where $W = (s)^{1/2}$ and $\sum$ denotes the sum and/or average over nucleon spins. The invariant amplitudes may be decomposed into partial-wave amplitudes by

$$\frac{1}{4\pi}A(\nu,t) = \frac{W+M_N}{E+M_N}f_1 - \frac{W-M_N}{E-M_N}f_2$$
$$\frac{1}{4\pi}B(\nu,t) = \frac{1}{E+M_N}f_1 + \frac{1}{E-M_N}f_2 \tag{3.6}$$

where E is the total c.m. energy of the nucleon and

$$f_1 = \frac{1}{k}\sum_{l=0}^{\infty}[f_{l+}P_{l+1}'(x) - f_{l-}P_{l-1}'(x)]$$
$$f_2 = \frac{1}{k}\sum_{l=1}^{\infty}(f_{l-} - f_{l+})P_l'(x), \tag{3.7}$$

with $P_l' = dP_l/dx$, $x = \cos\theta$, and k and θ being the c.m. momentum and scattering angle, respectively. The partial waves and phase shifts corresponding to total angular momentum $j = l \pm \frac{1}{2}$ are denoted by $f_{l\pm}$ and $\delta_{l\pm}$, respectively, where

$$f_{l\pm} = (2i)^{-1}[\exp(2i\delta_{l\pm}) - 1]. \tag{3.8}$$

Of special interest in our discussion will be the values of scattering amplitudes at various energy points; at physical threshold, $\nu = m_\pi$, $t = 0$, we have

$$\lim_{k\to0} f_1 = a_{0+}, \qquad \lim_{k\to0} f_2 = 0, \tag{3.9}$$

where $a_{l\pm}$ is the scattering length of the lth πN partial wave $f_{l\pm}$, defined by

$$a_{l\pm} = \lim_{k\to0} f_{l\pm}/k^{2l+1}. \tag{3.10}$$

In addition we will need the value of the amplitudes at $\nu = m_\pi + m_\pi^2/4M_N$, $t = 2m_\pi^2$ (always keeping $q^2 = q'^2 = m_\pi^2$), where we have

$$\cos\theta\,|_{t=2m_\pi^2} = (m_\pi^2 + k^2)/k^2$$

and since

$$P_l'(x) \underset{x\to\infty}{\sim} \frac{l(2l)!}{2^l(l!)^2}x^{l-1} \tag{3.11}$$

we obtain, using Eq. (3.11) in (3.7),

$$\lim_{k\to0} f_1\,|_{t=2m_\pi^2} = \sum_{l=0}^{\infty} a_{l+}\frac{(l+1)[2(l+1)]!}{2^{l+1}[(l+1)!]^2}m_\pi^{2l}$$
$$\lim_{k\to0}(f_2/k^2)\,|_{t=2m_\pi^2} = \sum_{l=1}^{\infty}(a_{l-} - a_{l+})\frac{l(2l)!}{2^l(l!)^2}m_\pi^{2(l-1)}. \tag{3.12}$$

In practice, only the s- and p-wave scattering lengths will be of interest; they are the only ones that are experimentally rather well known and that are the dominant contributions to Eq. (3.12).

The fixed momentum transfer (on-mass-shell) dispersion relations may now be written in the form (Moorhouse, 1969)

$$\mathrm{Re}A^{(\pm)}(\nu,t) = \frac{1}{\pi}P\int_{m_\pi+t/4M_N}^{\infty} d\nu'\,\mathrm{Im}A^{(\pm)}(\nu',t) \times\left(\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'+\nu}\right)$$
$$\mathrm{Re}B^{(\pm)}(\nu,t) = \frac{g^2}{2M_N}\left(\frac{1}{\nu_B-\nu} \mp \frac{1}{\nu_B+\nu}\right) + \frac{1}{\pi}P\int_{m_\pi+t/4M_N}^{\infty} d\nu'\,\mathrm{Im}B^{(\pm)}(\nu',t) \times\left(\frac{1}{\nu'-\nu} \mp \frac{1}{\nu'+\nu}\right), \tag{3.13}$$

where, according to Eq. (2.34), $\nu_B = -m_\pi^2/2M_N + t/4M_N$ and the pion–nucleon coupling constant in the Born term is given by $g^2/4\pi = 14.6$. As we have emphasized in Sec. IID, the important amplitudes for calculating chiral symmetry breaking effects are the isospin-even ones, given by

$$T^{(+)}(\nu, t) = A^{(+)} + \nu B^{(+)}, \qquad (3.14)$$

where, unless stated otherwise, on-mass-shell amplitudes will be denoted by $T(\nu, t) \equiv T(\nu, t, m_\pi^2, m_\pi^2)$.

B. Estimates of the πN sigma term

Soon after the $(3, \bar{3}) + (\bar{3}, 3)$ model (Gell-Mann, Oakes, and Renner, 1968; Glashow and Weinberg, 1968) of chiral symmetry breaking was proposed, von Hippel and Kim (1969, 1970) estimated the σ terms for elastic πN scattering as well as for several elastic and inelastic $K^\pm N$ and $\pi\Sigma$ reactions, and found excellent agreement with the predictions of the $(3, \bar{3}) + (\bar{3}, 3)$ symmetry-breaking scheme. In order to reach the unphysical off-mass-shell point in Eq. (2.37), they (von Hippel and Kim 1969, 1970) employed an off-mass-shell dispersion relation (which turns out to be just the Low equation in the laboratory frame) using the Fubini–Furlan extrapolation technique (Fubini and Furlan, 1968) which relates the current algebra soft-meson point to the scattering amplitude at threshold. The result obtained in this way for πN scattering is

$$\sigma_{NN}^{\pi\pi} = 26 \text{ MeV} \quad \text{(von Hippel and Kim, 1969, 1970)}. \qquad (3.15)$$

Although these authors did not explicitly state an error estimate, they emphasized that a large statistical and systematic error should be attached to (3.15), coming mainly from rescattering corrections. (Typically such errors have been shown to be about 30%). However, these von Hippel–Kim calculations have been criticized (Chan and Meiere 1969; Brown, Pardee, and Peccei, 1971; Kleinert, Steiner, and Weisz. 1971) for several reasons: Off-mass-shell dispersion relations although not obviously pathological are certainly not well enough understood to exclude unexpected sources of error. In addition, the result of von Hippel and Kim is especially questionable (Brown, Pardee, and Peccei, 1971), for they work with amplitudes of definite isospin rather than with isospin-even [Eq. (3.3)] amplitudes, which yields inconsistent results as discussed by Brown, Pardee, and Peccei (1971) and in Sec. IID. Furthermore, since they evaluate their (definite isospin) amplitudes at threshold rather than at the point $\nu = \nu_B = 0$, such an analysis incurs (Brown, Pardee, and Peccei, 1971) errors of order m_π^2/M_N^2 and thus does not furnish a reliable evaluation of the sigma term, because the contribution of the essentially unknown continuum in the dispersion integral is of the same order.

Similar calculations have been performed (Chan and Meiere, 1969) for the σ terms of $\pi\Lambda$ and $\pi\Sigma$ scattering and the $(3, \bar{3}) + (\bar{3}, 3)$ predictions confirmed.

More recently, Cheng and Dashen (1971) carried out an analysis using the completely different method outlined in Eqs. (2.39)–(2.43), and obtained a value for $\sigma_{NN}^{\pi\pi}$ considerably larger than that in Eq. (3.15). In order to reach the on mass shell unphysical energy point $\nu = 0$, $t = 2m_\pi^2$ of $T^{(+)}$ in Eq. (2.44), where $T^{(+)}$ is defined in Eq. (3.14), they used a so-called broad-area subtracted dispersion relation (Adler, 1965) of the form

$$\begin{aligned} T^{(+)}(\nu, t) = {} & \frac{g^2}{M_N}\frac{\nu_B^2}{\nu_B^2 - \nu^2}\frac{(\nu_1^2 - \nu^2)^\beta(\nu_2^2 - \nu^2)^{1-\beta}}{(\nu_1^2 - \nu_B^2)^\beta(\nu_2^2 - \nu_B^2)^{1-\beta}} \\ & + \frac{2}{\pi}(\nu_1^2 - \nu^2)^\beta(\nu_2^2 - \nu^2)^{1-\beta} \\ & \times \left[\int_{m_\pi + t/4M_N}^{\nu_1} d\nu' \frac{\nu'}{\nu'^2 - \nu^2} \right. \\ & \times \frac{\mathrm{Im}T^{(+)}(\nu', t)}{(\nu_1^2 - \nu'^2)^\beta(\nu_2^2 - \nu'^2)^{1-\beta}} \\ & + \int_{\nu_1}^{\nu_2} d\nu' \frac{\nu'}{\nu'^2 - \nu^2} \\ & \times \frac{\sin\beta\pi\ \mathrm{Re}T^{(+)}(\nu', t) + \cos\beta\pi\ \mathrm{Im}T^{(+)}(\nu', t)}{(\nu'^2 - \nu_1^2)^\beta(\nu_2^2 - \nu'^2)^{1-\beta}} \\ & \left. - \int_{\nu_2}^{\infty} d\nu' \frac{\nu'}{\nu'^2 - \nu^2}\frac{\mathrm{Im}T^{(+)}(\nu', t)}{(\nu'^2 - \nu_1^2)^\beta(\nu'^2 - \nu_2^2)^{1-\beta}}\right] \end{aligned} \qquad (3.16)$$

which can be obtained from Eq. (3.13) using the original amplitude $T^{(+)}$ divided by $(\nu_1^2 - \nu'^2)^\beta(\nu_2^2 - \nu'^2)^{1-\beta}$. This denominator introduces a new cut on the real axis from ν_1 to ν_2 in the ν plane $m_\pi + t/4M_N < \nu_1 < \nu_2 < \infty$ and the discontinuity across this artificial cut is determined by the imaginary and real parts of $T^{(+)}$. Thus it has the advantageous effect of smearing the needed subtraction for $T^{(+)}$ over a finite segment of the real axis so that the results will not be very sensitive to errors in the phase shifts at any one point. Using broad-area subtractions has the additional advantage in the presence of the three parameters ν_1, ν_2 and β $0 < \beta < 1$ providing us built-in checks on the compatibility of the various phase-shift solutions used with respect to conventionally subtracted dispersion relations. Different values for ν_1, ν_2 have been used but the optimal choice with respect to existing experimental information, turned out to be $\nu_1 = 1.52m_\pi$, $\nu_2 = 2.85m_\pi$ and β variable. Feeding various πN phase-shift solutions into the right-hand side of Eq. (3.16), Cheng and Dashen (1971) obtained an average value of

$$T^{(+)}(0, 2m_\pi^2) = 1.7m_\pi^{-1} \qquad (3.17)$$

which together with Eq. (2.44), using $f_\pi = 96$ MeV, yields (Cheng and Dashen, 1971)

$$\sigma_{NN}^{\pi\pi} = 110 \text{ MeV} \qquad (3.18)$$

in violent disagreement with Eq. (3.15) and, as we shall see later, also in serious disagreement with the $(3, \bar{3}) + (\bar{3}, 3)$ model. This calculation has been questioned (Höhler, Jakob, and Strauss, 1971; Liu and Vermaseren, 1973) because of the claim that Cheng and Dashen (1971) may not have selected the most reliable input data in their analysis.

Very recently, Liu and Vermaseren (1973) have undertaken the arduous task of recalculating the Cheng–Dashen estimate using the most recent phase-shift analyses. Doing a very thorough evaluation of the dispersion integrals in Eq. (3.16), and choosing the artificial cut in (3.16) to go from $\nu_1 = 1.52m_\pi$ to $\nu_2 = 2.84m_\pi$, they obtain an average value for $T^{(+)}(0, 2m_\pi^2)$ of (Liu and Vermaseren, 1973)

$$T^{(+)}(0, 2m_\pi^2) = 1.1m_\pi^{-1} \quad \text{or } 1.3m_\pi^{-1} \tag{3.19}$$

depending on whether the CERN 71 (Almehed and Lovelace, 1972) or CERN 68 phase-shift analysis is used, respectively. Equation (3.19), to be compared with (3.17), then implies, using Eq. (2.44), (Liu and Vermaseren, 1973)

$$\sigma_{NN}^{\pi\pi} = 72 \text{ MeV} \quad \text{or} \quad 85 \text{ MeV} \tag{3.20}$$

in contrast to Eq. (3.18). In calculations of this kind, a reliable error estimate is extremely difficult, but the variation in the outputs (for different β's and ν_1, ν_2) gives us some idea of the uncertainties in the final result. This indicates (Liu and Vermaseren, 1973) that the error lies around some 30% (similarly for the Cheng–Dashen calculation). Because of this calculation, and estimates to be discussed subsequently which use the same off-mass-shell extrapolation but different forms of subtracted dispersion relations, it appears now rather certain that the original Cheng–Dashen result in Eq. (3.18) should be reduced by about 30 to 40 MeV.

The second group who cast doubt on the validity of the Cheng–Dashen estimate was Höhler, Jakob, and Strauss (1971). These authors, however, used "conventionally" subtracted forward and forward-derivative dispersion relations, because the broad-area subtraction technique seemed to overemphasize the low-energy data points. However, the reconstruction of the forward-derivative amplitude from partial-wave series is less convergent. Jakob (1971) later repeated the same kind of calculation by means of more recent phase-shift analyses and total-cross-section data. In brief, this calculation goes as follows. Let us start with an expansion of the isospin-even πN amplitude

$$A^{(+)}(\nu, t) = a_1^+ + a_2^+ t + a_3^+ \nu^2 + a_4^+ \nu^2 t + \cdots \tag{3.21}$$

which coincides with the $T^{(+)}$ amplitude considered by Cheng and Dashen (1971) at the kinematical point of interest ($\nu = 0$, $t = 2m_\pi^2$ or $\nu_B = 0$) up to the constant g^2/M_N, stemming from the pseudoscalar Born term. From the general results of Osypowski (1970), based on Ward identities applied to three- and four-point functions, one gets (Höhler, Jakob, and Strauss, 1971; Jakob, 1971)

$$T^{(+)}(0, 2m_\pi^2, m_\pi^2, m_\pi^2) = a_1^+ - (g^2/M_N) + 2m_\pi^2 a_2^+ = f_\pi^{-2}\sigma_{NN}^{\pi\pi}, \tag{3.22}$$

where in the last step we again used Eq. (2.44). [Although Höhler Jakob, and Strauss (1971) and Jakob (1971) started from Ward identities, there is no difference from the Cheng–Dashen calculation as far as the mass extrapolation is concerned.] In order to calculate the two expansion parameters a_1^+ and a_2^+ one can proceed as follows: Consider the fixed-t dispersion relation for the (on-mass-shell) amplitude

$$C^{(+)}(\nu, t) = A^{(+)}(\nu, t) + [4M_N^2\nu/(4M_N^2 - t)]B^{(+)}(\nu, t) \tag{3.23}$$

in the forward direction

$$\mathrm{Re}C^{(+)}(\nu, 0) = A^{(+)}(0, 0) - \frac{g^2}{M_N}\frac{\nu^2}{\nu^2 - \nu_B^2} + \frac{2\nu^2}{\pi} P \times \int_{m_\pi}^\infty \frac{d\nu'}{\nu'}\frac{k_L'\sigma^{(+)}(\nu')}{\nu'^2 - \nu^2}, \tag{3.24}$$

with $k_L'^2 = \nu'^2 - m_\pi^2$ and, according to Eq. (3.4), the $\pi^\pm p$ total cross sections are given by $\sigma^{(+)} = (\sigma_{\pi^+p} + \sigma_{\pi^-p})/2$. Equation (3.24) is obtained by making one subtraction at $\nu = 0$ in Eq. (3.13), and using the optical theorem of the form

$$\mathrm{Im}C^{(+)}(\nu, 0) = k_L\sigma^{(+)}. \tag{3.25}$$

From Eqs. (3.21) and (3.24) we then obtain

$$a_1^+ - \frac{g^2}{M_N} = \mathrm{Re}C^{(+)}(\nu, 0) + \frac{g^2}{4M_N^3}\frac{m_\pi^4}{\nu^2 - m_\pi^4/4M_N^2} - \frac{2\nu^2}{\pi} P \int_{m_\pi}^\infty \frac{d\nu'}{\nu'}\frac{k_L'\sigma^{(+)}}{\nu'^2 - \nu^2}. \tag{3.26}$$

Although Jakob (1971) used three times more data than in the previous analysis (Höhler, Jakob, and Strauss, 1971) in order to calculate the right-hand side of (3.26), the consistency of the resulting real parts and of Eq. (3.26) for different ν turned out not to be very good [see Fig. 1 of Höhler, Jakob, and Strauss (1971) and Jakob (1971)]. This is mainly due to systematic differences between the phase-shift analyses of different authors. The best average result obtained is

$$a_1^+ - (g^2/M_N) = (-1.53 \pm 0.2)m_\pi^{-1} \tag{3.27}$$

corresponding to an s-wave scattering length value of

$$a_{0+}^{(+)} = (-0.014 \pm 0.014)m_\pi^{-1} \tag{3.28}$$

where we have used Eqs. (3.6) and (3.9) to obtain

$$\mathrm{Re}C^{(+)}(m_\pi, 0) = 4\pi[(m_\pi + M_N)/M_N]a_{0+}^{(+)}. \tag{3.29}$$

The second coefficient in Eq. (3.22), a_2^+, can be determined by a study of the dispersion relation for the derivative of the amplitude $C^{(+)}(\nu, t)$ at $t = 0$. Assuming the usual parametrization for the derivative of $\mathrm{Im}C^{(+)}$

$$\partial/\partial t\, \mathrm{Im}C^{(+)}(\omega, t)\,|_{t=0} = \tfrac{1}{2}b^+(\omega)k_L\sigma^{(+)}, \tag{3.30}$$

where $b^+(\omega)$ denotes the slope of the diffraction peak, one obtains the following relation:

$$\frac{\partial}{\partial t}\,\mathrm{Re}C^{(+)}(\omega,t)\,|_{t=0}-\frac{\partial}{\partial t}C_N{}^{(+)}(\omega,t)\,|_{t=0}$$

$$-\frac{2\omega^2}{\pi}P\int_{m_\pi}^{\bar\omega}\frac{d\omega'}{\omega'}\frac{(\partial/\partial t)\ \mathrm{Im}C^{(+)}(\omega',t)\,|_{t=0}}{\omega'^2-\omega^2}$$

$$-\frac{\omega}{2\pi M_N}\int_{m_\pi}^{\infty}\frac{d\omega'}{\omega'^2}\frac{(2\omega'+\omega)k_L'\sigma^{(+)}}{(\omega'+\omega)^2}$$

$$=a_2{}^+ + \frac{\langle b^+\rangle}{2}\zeta, \tag{3.31}$$

with

$$\zeta=\frac{2\omega^2}{\pi}\int_{\bar\omega}^{\infty}\frac{d\omega'}{\omega'}\frac{k_L'\sigma^{(+)}}{\omega'^2-\omega^2}$$

and the possible energy dependence of b^+ taken into account by an average value $\langle b^+\rangle$. The subscript N denotes the nucleon Born term. The left-hand side of Eq. (3.31) can be calculated in the interval $m_\pi<\omega<\bar\omega\simeq 2$ GeV from phase shifts and total cross sections, whereas ζ is given by total-cross-section data alone. Thus, the left-hand side of Eq. (3.31) can be plotted as a function of ζ and the parameters $a_2{}^+$ and $\langle b^+\rangle$ are very accurately obtained by a straight line fit to be

$$a_2{}^+=(1.11\pm 0.02)m_\pi{}^{-3},\qquad \langle b^+\rangle=6.1\ (\mathrm{GeV}/c)^{-2}. \tag{3.32}$$

Note that the value for $\langle b^+\rangle$ is consistent with estimates using only high-energy data, which obviously would not be the case if only points in the low-energy region are considered, as has been done in the calculation of Cheng and Dashen (1971), for example. Inserting Eqs. (3.27) and (3.32) in Eq. (3.22) yields

$$T^{(+)}(0,2m_\pi{}^2)=(0.69\pm 0.24)m_\pi{}^{-1} \tag{3.33}$$

to be compared with Eq. (3.17), which implies (Jakob, 1971)

$$\sigma_{NN}{}^{\pi\pi}=(45\pm 16)\ \mathrm{MeV} \tag{3.34}$$

a value one third as large as the original Cheng–Dashen estimate, Eq. (3.18), and half as large as the result in Eq. (3.20). This discrepancy is due to the fact that the broad-area dispersion relation strongly overemphasizes the low-energy region, as has been already noted by Höhler, Jakob, and Strauss (1971) and Jakob (1971), because the (ν_1,ν_2) interval of Eq. (3.16) covers only a very small low-energy region close to threshold: $\zeta_1(\nu_1)\simeq 0.13m_\pi{}^{-1}$ to $\zeta_2(\nu_2)\simeq 0.48m_\pi{}^{-1}$ on the ζ scale. This interval corresponds exactly to the energy region which was selected by Cheng and Dashen (1971) to give the best information on the real parts for the broad-area subtraction method. However, it is just this energy interval where the real parts of amplitudes from phase-shift analyses must be considered as doubtful (Höhler, Jakob, and Strauss, 1971; Jakob, 1971). Jakob (1971) has shown explicitly how such an inconsistency occurs, if this small low-energy region is too dominating, by writing a subtracted dispersion relation for $C^{(+)}$ at $t=2m_\pi{}^2$ which is related [similarly to Eq. (3.31)] to $T^{(+)}(0,2m_\pi{}^2)+\langle e^{b^+}\rangle\zeta$ and making use of Eq. (3.30) in the high-energy part, as it was also done by Cheng and Dashen (1971). Again he obtained $\langle e^{b^+}\rangle=1.13$, whereas a straight-line extrapolation of the dispersively calculated points from the interval (ζ_1,ζ_2) to the Cheng–Dashen value of Eq. (3.17) yields a *negative* value for $\langle e^{b^+}\rangle$, which is of course unphysical [see, for example, Fig. 3 of Jakob (1971)].

Similarly to Eq. (3.34), the original estimate of Höhler, Jakob, and Strauss (1971) yielded a value of (Höhler, Jakob, and Strauss, 1971)

$$\sigma_{NN}{}^{\pi\pi}=40\ \mathrm{MeV}. \tag{3.35}$$

However, these calculations are not entirely unique, because of the rather poorly known (Pilkuhn *et al.*, 1973) s-wave scattering length $a_{0+}{}^{(+)}$. Eliminating $a_1{}^+-g^2/M_N$ from Eq. (3.22) by using Eq. (3.26) at $\nu=m_\pi$ one obtains (Höhler, Jakob, and Strauss, 1971)

$$T^{(+)}(0,2m_\pi{}^2)=4\pi[(m_\pi+M_N)/M_N]a_{0+}{}^{(+)}+2m_\pi{}^2a_2{}^+-1.31m_\pi{}^{-1} \tag{3.36}$$

where use has been made of Eq. (3.29), and the factor $1.31m_\pi{}^{-1}$ is the value of the dispersion integral in Eq. (3.26) minus the Born term calculated at $\nu=m_\pi$. Equation (3.35) corresponds to $a_{0+}{}^{(+)}=-0.025m_\pi{}^{-1}$ which, however, is about three times the present world average (Pilkuhn *et al.*, 1973). Since the various phase-shift solutions for $a_{0+}{}^{(+)}$ scatter rather widely, the error in Eq. (3.35) could be as large as about 50%. It is interesting to note that for $\sigma_{NN}{}^{\pi\pi}=0$ Eq. (3.36) implies $a_{0+}{}^{(+)}=(-0.066\pm 0.015)m_\pi{}^{-1}$ which is excluded by the low-energy data (Pilkuhn *et al.*, 1973).

A very similar method has been used recently by Scadron and Thebaud (1973), using in addition p-wave scattering lengths for determining $a_2{}^+$ in Eq. (3.22). In particular they claim that the right-hand side of Eq. (3.27) should be changed to $(-1.40\pm 0.15)m_\pi{}^{-1}$, with the final result (Scadron and Thebaud, 1973)

$$\sigma_{NN}{}^{\pi\pi}=(73\pm 21)\ \mathrm{MeV}. \tag{3.37}$$

Again, as in the above calculations, the same ambiguities are inherent in this estimate coming from the poorly known s-wave scattering lengths.

Using Ward identity techniques and linear expansions of πN amplitudes, Osypowski (1970) obtained a value of $\sigma_{NN}{}^{\pi\pi}\simeq 60$ MeV consistent with the above estimates.

Shih and Shepard (1972 obtained $\sigma_{NN}{}^{\pi\pi}=-46\pm 140$ MeV using the amplitude $A^{(+)}(\nu=0,\ t=2m_\pi{}^2)$ only. Since the σ term in this calculation happened to be the difference of two big but nearly equal numbers, the result is subject to large errors.

An elegant field theoretical method for calculating $\sigma_{NN}{}^{\pi\pi}$ has been proposed by Altarelli, Cabibbo, and Maiani

(1971a, b), by making full use of the Weinberg smoothness hypothesis (Weinberg, 1966b). The outcome of this analysis is a relation expressing the σ term in terms of s- and p-wave scattering lengths, and a well known integral over total πN cross sections. Consider the isospin-even amplitude defined by

$$F(\nu,t,q^2,q'^2) = A^{(+)} + \nu B^{(+)} - \frac{g(q^2)g(q'^2)}{M_N}\frac{\nu_B^2}{\nu_B^2-\nu^2}, \tag{3.38}$$

where the Born term has been explicitly subtracted so that F is expected to be a smooth function of all its arguments. On the mass shell we have $g(m_\pi^2) = g$, while the off-mass-shell behavior of $g(q^2)$ is completely specified by Eq. (2.9) in terms of the axial and induced pseudoscalar form factors of the nucleon in Eq. (2.16). Taking into account the symmetry properties of F we can expand F, following Weinberg's original suggestion (Weinberg, 1966b), around $\nu = t = q^2 = q'^2 = 0$ in the following way:

$$F(\nu,t,q^2,q'^2) = Am_\pi^2 + Bt + C(q^2+q'^2) + D\nu^2 + R(\nu,t,q^2,q'^2), \tag{3.39}$$

where R measures the deviations of F from linearity. These deviations can occur due to the presence of nearby singularities in each of the variables ν, t, q^2, and q'^2. In the t, q^2, and q'^2 channels, such higher-order effects are of the order m_π^4/M_N^4, compared to the first-order terms m_π^2/M_N^2, and can be safely neglected. The main contribution to R from s-channel poles comes from the $\Delta(1236)$, and its contribution is rather small (Altarelli, Cabibbo, and Maiani, 1971b). We therefore will neglect R in Eq. (3.39) for the time being, but will come back to it later. Together with the low-energy theorem in Eq. (2.37), Eq. (3.39) then yields

$$\sigma_{NN}^{\pi\pi} = -f_\pi^2 m_\pi^2 A \tag{3.40}$$

[here, A is of course a different quantity than in Eq. (2.30)]. The next step is to find four equations for the four low-energy expansion parameters A, B, C, and D. One equation is obtained from the Adler consistency condition, Eq. (2.40), which yields

$$F(0,m_\pi^2,m_\pi^2,0) = m_\pi^2(A+B+C) = 0. \tag{3.41}$$

The second condition can be obtained at physical threshold $\nu = m_\pi$, $t = 0$:

$$\begin{aligned} F(m_\pi,0,m_\pi^2,m_\pi^2) &= m_\pi^2(A+2C+D) \\ &= 4\pi\left(1+\frac{m_\pi}{M_N}\right)\tfrac{1}{3}(a_1+2a_3) - \frac{g^2}{M_N}\frac{m_\pi^2}{m_\pi^2-4M_N^2} \end{aligned} \tag{3.42}$$

where use has been made of Eq. (3.29) and the isospin decomposition Eq. (3.3). The s-wave scattering length for a definite isospin channel $I = 1/2,\ 3/2$ is denoted by a_{2I}, whereas the p-wave scattering lengths will be denoted by $a_{2I,2J}$. Thirdly, at $\nu = t = 0$ we have

$$\begin{aligned} F(0,0,m_\pi^2,m_\pi^2) &= m_\pi^2(A+2C) \\ &= F(m_\pi,0,m_\pi^2,m_\pi^2) - \frac{2m_\pi^2}{\pi}P\int_{m_\pi}^{\infty}\frac{d\nu'}{\nu'} \times \frac{k_L'\sigma^{(+)}(\nu')}{\nu'^2-m_\pi^2} \\ &= F(m_\pi,0,m_\pi^2,m_\pi^2) - (1.45\pm0.02)m_\pi^{-1}, \end{aligned} \tag{3.43}$$

where the second step follows by making a subtraction in (3.13) at $\nu = t = 0$ and then taking the resulting expression for $F(\nu,0,m_\pi^2,m_\pi^2)$ at $\nu = m_\pi$. The fourth and last equation for determining the expansion parameters in Eq. (3.39) comes from an expansion of F in powers of k^2 and $\cos\theta$ around the physical threshold and comparing the coefficients of $k^2\cos\theta$. (The coefficient in $A^{(+)} + \nu B^{(+)}$ is then related to a linear combination of s- and p-wave scattering lengths.) In this way one obtains the sum rule

$$\begin{aligned} &\frac{m_\pi^2}{4\pi}\left(2B+\frac{m_\pi}{M_N}D\right) + \frac{g^2}{4\pi}4m_\pi^2\frac{2M_N+m_\pi}{(4M_N^2-m_\pi^2)^2} \\ &= \frac{m_\pi^2}{3}\left[\frac{a_1+2a_3}{4M_N^2} + (a_{11}+2a_{31}) + 2\left(1+\frac{3m_\pi}{2M_N}\right)(a_{13}+2a_{33})\right] \end{aligned} \tag{3.44}$$

which is nearly saturated (Altarelli, Cabibbo, and Maiani, 1971a) and thus confirms the reliability of using this equation for determining the parameters in Eq. (3.39). Solving Eqs. (3.41) to (3.44), A is found to be

$$\begin{aligned} -m_\pi^2 A = 4\pi\Bigg[&\left(1+\frac{m_\pi}{2M_N}\right)^2\frac{a_1+2a_3}{3} + m_\pi^2\frac{a_{11}+2a_{31}}{3} + 2m_\pi^2\left(1+\frac{3m_\pi}{2M_N}\right)\frac{a_{13}+2a_{33}}{3}\Bigg] \\ &- m_\pi^2\left(1+\frac{m_\pi}{M_N}\right)\left[\frac{g^2}{4M_N^3} + \frac{2}{\pi}P \times \int_{m_\pi}^{\infty}d\nu'\frac{\sigma^{(+)}(\nu')}{\nu' k_L'}\right], \end{aligned} \tag{3.45}$$

where terms proportional to m_π^4 have been neglected. Inserting Eq. (3.45) into (3.40) we find the desired expression for $\sigma_{NN}^{\pi\pi}$ in terms of s- and p-wave scattering lengths and a well known integral over total πN cross sections. The only problem which remains is to estimate the nonlinear contribution R to Eq. (3.39). This has been done (Altarelli, Cabibbo, and Maiani, 1971b) on the basis of field theory, using for the $\pi N\Delta$ vertex an effective Lagrangian of the form

$$\mathfrak{L}_{\pi N\Delta} = (g^*/m_\pi)\bar{\psi}_\Delta{}^\mu\psi\partial_\mu\pi + \text{H.c.} \tag{3.46}$$

with $\psi_\Delta{}^\mu$ representing the spin-$\frac{3}{2}$ Rarita–Schwinger field, and ψ and π the nucleon and pion fields respectively. The $\pi N\Delta$ coupling g^* is found (Altarelli, Cabibbo, and Maiani, 1971b) to be $g^* \simeq 3.3$. With this, the correction term Δ_R, which has to be added to the right-hand side of Eq. (3.45), turns out to be

$$\Delta_R = (-0.47 \pm 0.05) m_\pi{}^{-1} \tag{3.47}$$

which constitutes a 20% correction to Eq. (3.45). Putting everything together one obtains (Altarelli, Cabibbo, and Maiani, 197

$$\sigma_{NN}{}^{\pi\pi} = (80 \pm 30) \text{ MeV}. \tag{3.48}$$

As in the previous estimates, this result is critically dependent on the poorly known isospin-even combination of s-wave scattering lengths $a_1 + 2a_3$, and Eq. (3.48) represents somewhat a world average. Since the unknown higher-order contributions in this calculation represent at most a 10% correction, the estimate in Eq. (3.48) stands on rather firm ground. In addition, with respect to t-channel unitarity corrections, it has been shown (Geddes and Graham, 1973) that the expansion (3.39), neglecting R, holds within one percent. [However, a recent study of Höhler *et al.* (1972) which attempts to determine the expansion coefficients in Eq. (3.39) by a slightly different method, finds $\sigma_{NN}{}^{\pi\pi} \simeq 45$ MeV preferable to Eq. (3.48). This is consistent with Eq. (3.48), but indicates that the lower limit of the Altarelli–Cabibbo–Maiani estimate should be favored.] It is interesting to note that with the present method one can also estimate the higher-order correction $O(\epsilon^2)$ in Eq. (2.43) of the Cheng–Dashen mass extrapolation, which turns out (Altarelli, Cabibbo, and Maiani, 1971b) to be of the order of $10^{-2} m_\pi{}^{-1}$, i.e., about 1%. It should be emphasized that the Altarelli–Cabibbo–Maiani method obviously can never yield better, i.e. more accurate, results than methods based entirely on dispersion relations (for instance, Höhler, Jakob, and Strauss, 1971, 1972). The reason lies simply in the fact that accurate scattering lengths are determined from the same dispersion integrals with the same input as the coefficients of the expansion (3.39). Therefore the errors are practically the same.

A more recent similar analysis, taking into account N, ρ, Δ, and σ exchanges, yielded the result (Olsson, Osypowski, and Turner, 1973)

$$\sigma_{NN}{}^{\pi\pi} = (42 \pm 10) \text{ MeV}. \tag{3.49}$$

It should be noted that these authors use the same low-energy parameter $a_2{}^+$ as given by Höhler, Jakob, and Strauss (1972), in addition to the theoretical assumption that $\sigma_{NN}{}^{\pi\pi}$ transforms as a $(\frac{1}{2}, \frac{1}{2})$ representation of $SU(2) \times SU(2)$, thus avoiding the use of the less well-determined expansion parameter $a_1{}^+$ in Eq. (3.21). Since the value (3.49) obtained in this way is practically identical with Eqs. (3.34) and (3.35), this indicates that the chiral transformation property of the symmetry breaking Hamiltonian $\mathcal{H}'$ is the simplest possible. (We shall come back to this point in Sec. V.) The other interesting part of this paper (Olsson *et al.*, 1973) is the treatment of Δ and ρ exchange.

Very recently precise new data have become available on the differential cross section of low-energy elastic πN scattering at laboratory kinetic energies from $T_{\text{lab}} = 80$ to 300 MeV (Bussey *et al.*, 1973). The accuracy and completeness of these very high-statistics measurements are so much superior to all the old low-energy πN data used so far, that they make it interesting to recalculate the expansion coefficients of the πN invariant amplitudes about the point $\nu = t = 0$, as for instance in Eq. (3.21). In this way, Carter, Bugg, and Carter (1973), and Nielsen and Oades (1974) obtained a new value for the πN sigma term, probably the most accurate and reliable one to date (Höhler, 1973). Using the method of Höhler, Jakob, and Strauss (1971) as described above, Carter, Bugg, and Carter (1973) obtained

$$\sigma_{NN}{}^{\pi\pi} = (83 \pm 12) \text{ MeV}. \tag{3.50}$$

More recently this calculation has been seriously questioned by Nielsen and Oades (1974), mainly because of an incorrect treatment of d and f waves, which cannot be determined from phase-shift analyses at low energies. Therefore the error in Eq. (3.50) is a far too optimistic estimate, and should be much larger due to these d- and f-wave uncertainties. Using the same method as Höhler, Jakob, and Strauss (1972), Nielsen and Oades (1974) calculated the low-energy expansion parameters of the (on-mass-shell) πN invariant amplitudes and found, contrary to what is usually assumed, that these amplitudes have a nonlinear t dependence, i.e., terms proportional to t^2 cannot be neglected anymore in an on-shell amplitude [as, for example, in Eq. (3.21)]. The σ-term turns out to be (Nielsen and Oades, 1974)

$$\sigma_{NN}{}^{\pi\pi} = (66 \pm 9) \text{ MeV}. \tag{3.51}$$

Although the nonlinear t dependence is a very interesting and far-reaching result as far as on-mass-shell amplitudes are concerned, its use for calculating $\sigma_{NN}{}^{\pi\pi}$ in Eq. (3.51) is obviously inconsistent: On the one hand one has to neglect q^4 (or $m_\pi{}^4$) corrections, which cannot be calculated exactly, in the off-mass-shell extrapolation given by Eq. (2.43). On the other hand, $m_\pi{}^4$ corrections stemming from the t^2 term have been taken into account in the t-channel extrapolation in order to derive Eq. (3.51). It is simple to show, for example, that the t^2 contribution will be already totally compensated for by off-shell $m_\pi{}^4$ corrections stemming just from deviations of the *exact* GT relation (Reya, 1973c). In order to obtain a correct and consistent result, one has to work in one definite order! Therefore, since the t^2 contribution to $\sigma_{NN}{}^{\pi\pi}$ in Eq. (3.51) amounts to about 9 MeV, the correct Nielsen and Oades estimate should read (Reya, 1973c)

$$\sigma_{NN}{}^{\pi\pi} = (57 \pm 9) \text{ MeV} \tag{3.52}$$

which, at the present time, probably represents the most accurate and reliable value for the πN sigma term (Höhler, 1973). Contrary to the calculation of Carter, Bugg, and Carter (1973), the treatment of Nielsen and Oades (1974) has the additional advantage of giving a reasonable consistency with the CERN phase shifts at higher energies in the dispersion relation for $\partial C^{(+)}/\partial t$ at $t = 0$. It should be kept in mind, however, that the value for $\sigma_{NN}^{\pi\pi}$ is very sensitive to small effects like d and f waves at low energies and probably also to details of electromagnetic (Coulomb) corrections.

Within a light-cone approach, using finite mass dispersion relations and commutation relations of the gluon model as input for the numerical evaluation, Ng and Vinciarelli (1971) estimated $\sigma_{NN}^{\pi\pi}$ to be approximately 220 MeV. This calculation, however, appears to be too model-dependent for its result to be considered as being "unique" and final.

Using entirely different experimental information, Ericson and Rho (1971) estimated the πN σ term from a study of π-nuclei interactions in π-mesonic atoms making use of the Fubini–Furlan off-mass-shell extrapolation (Fubini and Furlan, 1968). The main advantage of working with nuclei instead of with πN scattering lies in the fact that, in contrast to the very poorly known charge-symmetric πN amplitude (see above discussion concerning $a_{0+}^{(+)}$), the low-energy π-nuclear amplitudes are very well known. Although there exist no scattering data, they can be extracted from the energy shifts in π-mesic atoms, which have been measured with a high precision for a number of elements (Backenstoss, 1970). The calculations have been done for several nuclei ($Z = 3$ to $Z = 12$) with the result (Ericson and Rho, 1971)

$$\sigma_{NN}^{\pi\pi} = 34 \text{ MeV} \tag{3.53}$$

under the assumption that the pion–nucleus sigma term can be written as a coherent sum of πN σ terms. Again a 20% to 30% uncertainty should be attached to Eq. (3.53). Similar calculations have been done by Gensini (1971b) and Hakim (1972a) with a somewhat smaller result than in Eq. (3.53), in agreement with the original estimate of von Hippel and Kim (1969, 1970).

There appears to be a systematic difference, by about a factor of two, between estimates resulting from calculations using the Fubini–Furlan extrapolation technique (von Hippel and Kim, 1969, 1970; Ericson and Rho, 1971; Gensini, 1971a, b; Hakim, 1972a, b) and calculations employing one or the other off-mass-shell extrapolation [Eq. (2.43) for example] (Cheng and Dashen, 1971; Höhler, Jakob, and Strauss, 1971; Osypowski, 1970; Altarelli, Cabibbo, and Maiani, 1971b; Nielsen and Oades, 1974; Scadron and Thebaud, 1973). The question can thus be raised as to the reasons behind such a systematic disagreement. Hakim (1972c) has looked into this problem and found that estimates using the Fubini–Furlan extrapolation are critically dependent on the extrapolation path chosen. Without deforming this extrapolation path, a value for $\sigma_{NN}^{\pi\pi}$ twice as large as the above-mentioned has been found (Hakim, 1972c).

To summarize this section, we state in Table I the "experimental" estimates for $\sigma_{NN}^{\pi\pi}$ done so far. At the present stage of the game we are not able to deduce the *exact* value of $\sigma_{NN}^{\pi\pi}$ (this can only be achieved when more accurate future experiments are available), but, according to Table I and to what has been said in the preceding discussions, the magnitude (or range) of $\sigma_{NN}^{\pi\pi}$ can be considered to be rather reliably known: The "world average" of Table I lies around

TABLE I. Estimates of the πN sigma term. For values where no error has been stated explicitly, an error typically of some 30% should be assigned.

Authors	$\sigma_{NN}^{\pi\pi}$ (MeV)
von Hippel and Kim (1969, 1970)	26
Cheng and Dashen (1971)	110
Liu and Vermaseren (1973)	72 to 85
Höhler, Jakob, and Strauss (1971)	40
Jakob (1971)	45 ± 16
Scadron and Thebaud (1973)	73 ± 21
Osypowski (1970)	60
Shih and Shepard (1972)	-46 ± 140
Altarelli, Cabibbo, and Maiani (1971b)	80 ± 30
Olsson, Osypowski, and Turner (1973)	42 ± 10
Carter, Bugg, and Carter (1973)	83 ± 12
Nielsen and Oades (1974)	66 ± 9
Ericson and Rho (1971)	34
Gensini (1971b)	26 ± 8
Hakim (1972a)	22 ± 1
Hakim (1972c)	51 ± 9

50 MeV, and it appears improbable that $\sigma_{NN}^{\pi\pi}$ exceeds 70 MeV. With some confidence, therefore, it can be concluded that $\sigma_{NN}^{\pi\pi}$ lies within the range

$$30 \text{ MeV} \lesssim \sigma_{NN}^{\pi\pi} \lesssim 70 \text{ MeV} \tag{3.54}$$

and values as large as the Cheng–Dashen estimate of 110 MeV (or larger) can be rather convincingly excluded. Probably the most accurate and reliable estimate to date is given by the corrected Nielsen and Oades (1974) result (3.52).

The question remains open, to what extent Eq. (3.54) or the estimates in Table I are consistent, within model predictions, with other scattering processes. With respect to the experimental information presently available, the next most obvious but theoretically exceedingly more complicated reaction is kaon–nucleon scattering, to which we will now turn. (In conventional chiral symmetry-breaking models, as we shall see later, the kaon–nucleon σ term is expected to be about one order of magnitude larger than $\sigma_{NN}^{\pi\pi}$).

IV. CALCULATIONS OF THE KAON–NUCLEON SIGMA TERM

For the sake of clarity we briefly summarize the main properties of kaon–nucleon scattering amplitudes, before going into the details of various calculations of the kaon–nucleon σ term.

A. Kaon–nucleon amplitudes

For a reaction of the form (2.32) with M_a denoting $K^\pm$, the T-matrix decomposition is the same as in Eq. (3.1), whereas the isospin-even and isospin-odd amplitudes are now defined by

$$\tilde{A}(\nu, t) = \tilde{A}^+(\nu, t) + (\boldsymbol{\tau}_N \cdot \boldsymbol{\tau}_K)\tilde{A}^-(\nu, t) \tag{4.1}$$

and similarly $\tilde{B}^\pm$, where $\boldsymbol{\tau}_N$ and $\boldsymbol{\tau}_K$ are the isospin matrices for the nucleon and kaon, respectively. (A tilde always refers to the kaon–nucleon system.) The crossing property is the same as for the equivalent πN amplitudes $A^{(\pm)}$ and $B^{(\pm)}$. In the s channel, the amplitudes in Eq. (4.1) are related to the amplitudes for a definite isospin state $I = 0, 1$ by

$$\tilde{A}^{+} = \tfrac{1}{4}(\tilde{A}^{0} + 3\tilde{A}^{1}), \qquad \tilde{A}^{-} = \tfrac{1}{4}(\tilde{A}^{1} - \tilde{A}^{0}) \tag{4.2}$$

and similarly for $\tilde{B}^{\pm}$. Therefore the amplitudes for $K^{\pm}N$ scattering are given by

$$\tilde{A}_{\pm} = \tilde{A}^{+} \pm \tilde{A}^{-}, \qquad \tilde{B}_{\pm} = \tilde{B}^{+} \pm \tilde{B}^{-}. \tag{4.3}$$

The T-matrix normalization and partial-wave expansions of the invariant amplitudes remain the same as in Sec. IIIA. The amplitudes of interest will again be the combination

$$\tilde{T}_{\pm}(\nu, t) = \tilde{A}_{\pm} + \nu \tilde{B}_{\pm}. \tag{4.4}$$

Using Eqs. (3.6) and (3.9) we find at physical threshold $\nu = m_K$, $t = 0$ that

$$\tilde{T}_{\pm}(m_K, 0) = 4\pi(1 + m_K/M_N)\tilde{a}_{0+}{}^{\pm}, \tag{4.5}$$

where $\tilde{a}_{l\pm}{}^{\pm}$ is the scattering length of the lth $K^{\pm}N$ partial wave $\tilde{f}_{l\pm}{}^{\pm}$, defined as in Eq. (3.10). At $\nu = m_K + m_K{}^2/4M_N$, $t = 2m_K{}^2$, we obtain from Eqs. (3.6) and (3.12) rewritten for kaons (i.e., $a_{l\pm} \to \tilde{a}_{l\pm}$, $m_\pi \to m_K$),

$$\begin{aligned}\tilde{T}_{\pm}&(m_K + m_K{}^2/4M_N, 2m_K{}^2) \\ &= 4\pi\left(1 + \frac{m_K}{M_N} + \frac{m_K{}^2}{4M_N{}^2}\right)\sum_{l=0}^{\infty}\tilde{a}_{l+}{}^{\pm} \\ &\quad\times \frac{(l+1)[2(l+1)]!}{2^{l+1}[(l+1)!]^2}\,m_K{}^{2l} \\ &\quad + 4\pi m_K{}^2\sum_{l=1}^{\infty}(\tilde{a}_{l-}{}^{\pm} - \tilde{a}_{l+}{}^{\pm})\frac{l(2l)!}{2^l(l!)^2}\,m_K{}^{2(l-1)}.\end{aligned} \tag{4.6}$$

Finally, the subtracted fixed-t dispersion relations for $K^{\pm}N$ scattering may now be written (Martin, 1970) in the form

$$\begin{aligned}\mathrm{Re}\tilde{T}_{\pm}&(\nu, t) \\ &= \mathrm{Re}\tilde{T}_{\pm}(\nu_0, t) \mp (\nu - \nu_0) \\ &\quad\times \sum_{y=\Lambda,\Sigma}\frac{R_y}{(\nu_B + \Delta_y \pm \nu_0)(\nu_B + \Delta_y \pm \nu)} \\ &\quad\mp \frac{(\nu - \nu_0)}{\pi}P\int_{\bar{\nu}}^{\nu_0} d\nu'\,\frac{\mathrm{Im}\tilde{T}_{-}(\nu', t)}{(\nu' \pm \nu_0)(\nu' \pm \nu)} \\ &\quad\mp \frac{(\nu - \nu_0)}{\pi}P\int_{\nu_0}^{\infty} d\nu'\,\frac{\mathrm{Im}\tilde{T}_{-}(\nu', t)}{(\nu' \pm \nu_0)(\nu' \pm \nu)} \\ &\quad\pm \frac{(\nu - \nu_0)}{\pi}P\int_{\nu_0}^{\infty} d\nu'\,\frac{\mathrm{Im}\tilde{T}_{+}(\nu', t)}{(\nu' \mp \nu_0)(\nu' \mp \nu)},\end{aligned} \tag{4.7}$$

with $\nu_0 = m_K + t/4M_N$, and

$$R_y = (g_y{}^2/2M_N)(\nu_B + \Delta_y + M_N - M_y)$$

$$\bar{\nu} = \nu_B + [(M_\Lambda + m_\pi)^2 - M_N{}^2]/2M_N,$$

where $\Delta_y = (M_y{}^2 - M_N{}^2)/2M_N$ and g_y is the rationalized pseudoscalar coupling constant for the KyN vertex. For the KN Born terms they are taken to be (Pilkuhn *et al.*, 1973)

$$g_\Lambda{}^2/4\pi = 5.0 \pm 1.9, \qquad g_\Sigma{}^2/4\pi = 1.0 \pm 1.5 \tag{4.8}$$

whereas the additional couplings of interest are (Warnock and Frye, 1965; Kim and von Hippel, 1969; Martin, 1970)

$$\begin{aligned}&g_{Y_1^*}{}^2/4\pi = 1.2 \pm 0.6, \qquad g_{Y_0^*}{}^2/4\pi = 0.32 \pm 0.04, \\ &g_{\Lambda'(1520)}{}^2/4\pi = 0.55 \pm 0.10,\end{aligned} \tag{4.9}$$

where for $g_{\Lambda'}$ a partial width of $\Gamma_{\Lambda' \to \bar{K}N} = 7.2 \pm 1.1$ MeV has been assumed. It should be noted that the value of $g_{Y_1^*}$ is smaller by a factor 2-3 than the $SU(3)$ prediction ($g_{Y_1^*}{}^2/4\pi \simeq 2.4$), as suggested (Martin, 1970) by high-energy photoproduction of $Y_1^*(1385)$.

B. Estimates of the kaon–nucleon sigma term

Partly because of the complexity of the problem and the rather incomplete experimental information available, considerably fewer estimates of the kaon–nucleon sigma term $\sigma_{NN}{}^{KK}$ have been done up until now than in the case of πN scattering. Calculations using off-mass-shell dispersion relations and the Fubini–Furlan extrapolation technique yield (von Hippel and Kim, 1969, 1970; Hakim, 1972b)

$$\sigma_{NN}{}^{KK} \simeq 170 \text{ MeV} \tag{4.10}$$

and the same critiques raised in Sec. IIIB with respect to off-mass-shell dispersion relations and the Fubini–Furlan method apply, in addition to the doubtful use of definite isospin amplitudes.

Most of the recent calculations based on fixed-t dispersion relations and isospin-even amplitudes have made use of the Cheng–Dashen off-mass-shell extrapolation, Eq. (2.43), applied to kaons. However, as an additional source of uncertainty, it should be kept in mind that the off-mass-shell points $q^2 = q'^2 = 0$ and the point $t = 2m_K{}^2$ are now, in contrast to πN scattering, relatively far away from the physical region. The correction terms neglected in Eq. (2.44) are now of the order $m_K{}^4/M_N{}^4$ which may not be negligible *a priori* in contrast to terms like $m_\pi{}^4/M_N{}^4$, but we will come back to this point later. Moreover, since $\sigma_{NN}{}^{KK}$ itself is expected to be $O(m_K{}^2/M_N{}^2)$ this approximation can introduce an error as large as 30%.

To reach the unphysical (but on-mass-shell) point $\nu = \nu_B = 0$ in Eq. (2.44) one has to use a fixed-t dispersion relation. From Eq. (4.7), using (4.3), we get for $\nu = 0$,

$t = 2m_K^2$,

$$\tilde{T}^+(0, 2m_K^2) = \mathrm{Re}\tilde{T}^+(\nu_0, 2m_K^2) - \nu_0^2 \sum_{y=\Lambda,\Sigma} \frac{g_y^2}{4M_N^2} \frac{(m_y - M_N)^2}{\Delta_y(\Delta_y^2 - \nu_0^2)} - \frac{2\nu_0^2}{\pi} P \int_{\nu_0}^{\infty} d\nu' \frac{\mathrm{Im}\tilde{T}^+(\nu', 2m_K^2)}{\nu'(\nu'^2 - \nu_0^2)} - \frac{\nu_0^2}{\pi} P \int_{\bar{\nu}}^{\nu_0} d\nu' \frac{\mathrm{Im}\tilde{T}_-(\nu', 2m_K^2)}{\nu'(\nu'^2 - \nu_0^2)}. \quad (4.11)$$

The subtraction constant in Eq. (4.11) is given by Eq. (4.6) using s- and p-wave scattering lengths and similarly the physical region integral can be calculated (Reya, 1972) by feeding the various partial waves $\tilde{f}_{l\pm}$ into $\tilde{T}^+(\nu', 2m_K^2)$. The evaluation of the last term in Eq. (4.11), the integral over the unphysical region below the $\bar{K}N$ threshold representing the $\Lambda\pi$ and $\Sigma\pi$ discontinuities, requires some care. The $I = 0, 1$ s-wave parts are evaluated using the K-matrix solution of Martin and Sakitt (1969), continued below the $\bar{K}N$ threshold (by the prescription $\tilde{k} \to i|\tilde{k}|$; $\tilde{k}$ and $\tilde{\theta}$ being the c.m. momentum and scattering angle in the kaon–nucleon system), with the result

$$\mathrm{Im}\left(\frac{\tilde{f}_{0+}}{\tilde{k}}\right)^I_{\tilde{k}<0} = \frac{\tilde{c}_I}{(1 + |\tilde{k}|\tilde{b}_I)^2 + (|\tilde{k}|\tilde{c}_I)^2} \quad (4.12)$$

and the KN and $\bar{K}N$ s-wave scattering lengths for a definite isospin channel are given by $\tilde{a}_I$ and $\tilde{A}_I = \tilde{b}_I + i\tilde{c}_I$, respectively, whereas the p-wave scattering lengths are denoted by $\tilde{a}_{I,2J}$ and $\tilde{A}_{I,2J} = \tilde{b}_{I,2J} + i\tilde{c}_{I,2J}$, respectively. The p-wave unphysical region is assumed to be dominated by the $Y_1^*(1385)$ resonance, and here the narrow-width approximation was used:

$$\mathrm{Im}(\tilde{f}_{1+}/\tilde{k})_{Y_1^*} = \tfrac{1}{2}\pi\tilde{\alpha}(M_{Y_1^*}/M_N)\delta(\nu - \nu_R) \quad (4.13)$$

with

$$\tilde{\alpha} = \frac{g_{Y_1^*}^2}{4\pi} \frac{(M_N + M_{Y_1^*})^2 - m_K^2}{3M_{Y_1^*}^2} \left(\frac{\tilde{k}_R}{M_N}\right),$$

where the c.m. momentum at the Y_1^* resonance is denoted by $\tilde{k}_R$. With these input data, Eq. (4.11) yields (Reya, 1972)

$$\tilde{T}(0, 2m_K^2) = (18.2 \pm 5.5) m_K^{-1} \quad (4.14)$$

and from Eq. (2.44) we get (Reya, 1972)

$$\sigma_{NN}^{KK} = (540 \pm 160)\ \mathrm{MeV} \quad (4.15)$$

using $f_K = 118$ MeV. In spite of the extrapolation from $t = 0$ to $t = 2m_K^2$, Eq. (4.15) shows that the errors in the partial waves are still kept within tolerable limits when extrapolated to the unphysical region. Because of this rather long-range extrapolation one might argue that the above results strongly depend on the extrapolation procedure used. This is, however, not the case as one can see from the partial-wave decomposition in Eq. (3.7): At low energies the main contributions are coming from s and p waves; the $s_{1/2}$ and $p_{1/2}$ contributions are independent of the extrapolation procedure, whereas the $p_{3/2}$ term depends only linearly on $\cos\tilde{\theta}$. Another possible enhancement in the t-channel extrapolation could come from $(K\bar{K})$ poles. However, we do not have any experimental evidence (Petersen, 1971) that the $\epsilon(700)$, say, is coupled to the $K\bar{K}$ channel, giving rise to a significant contribution at $t = 2m_K^2$. The major sources of uncertainty come from somewhere else. First, the neglected second-order terms in the mass extrapolation are of order m_K^4/M_N^4 which could introduce an error as large as 30%. Secondly, it is the real part of the $p_{3/2}$ K^-p scattering length in the isospin $I = 1$ channel, which experimentally suffers a sign ambiguity. For calculating the subtraction constant in Eq. (4.11), we have used a positive value for this scattering length, which is the favored solution of most of the $\bar{K}N$ phase-shift analyses performed up to now (Martin, 1970). However, changing the sign of this scattering length only *decreases* the value of σ_{NN}^{KK} given in Eq. (4.15); the importance of this statement will become clear later.

A similar calculation has been done by Thompson (1971) with the result

$$\sigma_{NN}^{KK} = (-370 \pm 110)\ \mathrm{MeV} \quad (4.16)$$

to be compared with Eq. (4.15). As just remarked, the main reason for this discrepancy comes from the poorly determined $\bar{K}N$ p-wave scattering length in the $I = 1$ channel, which enters critically into the evaluation of subtraction constants.

Using the same Cheng–Dashen off-mass-shell extrapolation and fixed-t dispersion relations, but a different extrapolation method to reach the point $\nu = \nu_B = 0$, Nasrallah and Schilcher (1973) found

$$\sigma_{NN}^{KK} \simeq 160\ \mathrm{MeV}. \quad (4.17)$$

The extrapolation to the unphysical energy point has been done in two steps. For the extrapolation in ν an amplitude of the form

$$\bar{T}^+(\nu, 0) = \frac{\nu^2 - \nu^{*2}}{\nu^2 - \bar{\nu}^{*2}} \tilde{T}^+(\nu, 0) \quad (4.18)$$

has been used, which is supposed to minimize the uncertainties arising from the unphysical region contributions by choosing the parameter ν^* such that $\bar{T}^+$ vanishes at the $Y_{0,1}^*$ peaks. A subtracted dispersion relation has then been used to evaluate $\bar{T}^+(0, 0)$, where the calculation has been performed for various values of $\bar{\nu}^*$ all lying on the physical cut where $\mathrm{Re}\tilde{T}^+(\bar{\nu}^*, 0)$ is known. Having obtained $\tilde{T}^+(0, 0)$, the extrapolation in t to $t = 2m_K^2$ has been performed by writing an unsubtracted dispersion relation in ν for arbitrary t and saturating it with the Λ, Σ, Y_0^*, Y_1^* poles. Of course, this method suffers the same uncertainties in the q^2 and q'^2 extrapolations, but seems to be rather insensitive to ambiguities stemming from the poorly known scattering lengths.

Recently, another method has been suggested (Reya, 1973a, b) based on field theoretical techniques and making full use of Weinberg's smoothness hypothesis, which is similar to the Altarelli–Cabibbo–Maiani (1971b) calculation (see Sec. IIIB) generalized to kaon–nucleon scattering. In brief, this calculation goes as follows. Since we are interested in the (smooth) behavior of the scattering amplitude in the neighborhood of the Weinberg point $\nu = t = q^2 = q'^2 = 0$, we first have to construct a smoothly varying amplitude in this region. In order to do this we must subtract from $\tilde{T}^+$ the Λ, Σ Born terms and the various nonsmooth contributions of the $\Lambda\pi$ and $\Sigma\pi$ discontinuities below the $\bar{K}N$ threshold. Taking this into account we define

$$\tilde{F}(\nu, t, q^2, q'^2) = \text{Re}\tilde{T}^+ - \tilde{T}_B{}^+ - \tilde{T}_{Y_0*}{}^+ - \tilde{T}_{Y_1*}{}^+ \quad (4.19)$$

where we did not include possible t-channel contributions, since in this case, as mentioned above, there is no direct experimental evidence for such effects. The Born terms in Eq. (4.19) are given by

$$\tilde{T}_B{}^+ = \sum_{y=\Lambda,\Sigma} \frac{g_y{}^2}{2M_N}\left[\frac{\nu_B{}^y(\nu_B{}^y + M_N - M_y)}{(\nu_B{}^y)^2 - \nu^2} + \frac{M_N - M_y}{M_N + M_y}\right], \quad (4.20)$$

where $\nu_B{}^y = \nu_B + \Delta_y$. The unphysical regions in the $\bar{K}N$ channel are dominated by the $I = 0$ s-wave $Y_0{}^*(1405)$ and by the $I = 1$ p-wave $Y_1{}^*(1385)$. Those two contributions are assumed to be described by effective Lagrangians where, in the gradient coupling theory, we have for the $K^-Y_0{}^*N$ vertex

$$\mathcal{L}_{Y_0*} = \tilde{g}_{Y_0*}\bar{\psi}_{Y_0*}\gamma^\mu\psi\partial_\mu\phi + \text{H.c.} \quad (4.21)$$

with ψ_{Y_0*} representing the spin-$\frac{1}{2}$ $Y_0{}^*$ field, and ψ and ϕ are the nucleon and kaon fields, respectively. This Lagrangian yields for the isospin-even amplitude the following expression:

$$\tilde{T}_{Y_0*}{}^+ = \frac{g_{Y_0*}{}^2}{M_N}\left[\frac{\nu_B{}^{Y_0*}(\nu_B{}^{Y_0*} + M_N + M_{Y_0*})}{(\nu_B{}^{Y_0*})^2 - \nu^2} + \frac{M_N + M_{Y_0*}}{M_N - M_{Y_0*}}\right] \quad (4.22)$$

with $g_{Y_0*}{}^2 = \tilde{g}_{Y_0*}{}^2(M_{Y_0*} - M_N)^2$, which is taken from Eq. (4.9). The $K^-Y_1{}^*N$ vertex is described by

$$\mathcal{L}_{Y_1*} = (g_{Y_1*}/M_N)\bar{\psi}_\mu\psi\partial^\mu\phi + \text{H.c.}, \quad (4.23)$$

where ψ_μ denotes the spin-$\frac{3}{2}$ Rarita–Schwinger field. We have used the following form for the $Y_1{}^*$ ($J^P = \frac{3}{2}^+$) propagator:

$$\begin{aligned} D_{\mu\nu}{}^{3/2+}(M, P) &= (\gamma\cdot P + M)/(P^2 - M^2) \\ &\times [g_{\mu\nu} + (1/3M)(P_\mu\gamma_\nu - \gamma_\mu P_\nu) \\ &- (2/3M^2)P_\mu P_\nu - \tfrac{1}{3}\gamma_\mu\gamma_\nu] \end{aligned} \quad (4.24)$$

with $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. It is well known that the spin-$\frac{3}{2}$ propagator is not unique.[5] This nonuniqueness is immaterial as far as our analysis is concerned, in that any other possible choice of $D_{\mu\nu}{}^{3/2+}$ leads to a Born term which differs from ours by a polynomial in external momenta, which turns out to be entirely negligible and would only amount to a redefinition of $\tilde{F}$. The result for $\tilde{T}_{Y_1*}{}^+$ is

$$\begin{aligned} T_{Y_1*}{}^+ = \frac{1}{3}\frac{(g_{Y_1*}/M_N)^2}{(\nu_B{}^{Y_1*})^2 - \nu^2}\Bigg\{ &\frac{1}{4M_1}\left(1 + \frac{M_N}{M_1}\right)(q^2 + q'^2) \\ &\times\left[\left(\frac{1}{2} + \frac{M_1}{M_N}\right)\nu^2 - \nu_B\nu_B{}^{Y_1*}\right] \\ &+ \frac{q^2q'^2}{4M_NM_1}\left[\left(1 + \frac{M_N}{2M_1}\right)\nu_B{}^{Y_1*} + \frac{\nu^2}{2M_1}\right] \\ &+ \nu_B{}^{Y_1*}\Bigg[(M_N + M_1)\nu_B + \frac{M_N}{2M_1}\nu^2 \\ &+ \frac{M_N{}^2}{2M_1{}^2}(\nu^2 + \nu_B{}^2)\Bigg] \\ &+ \frac{M_N}{2M_1{}^2}\nu^2\left[\nu^2 - \nu_B{}^2 + \nu_B\left(\frac{2M_1{}^2}{M_N} - M_N - M_1\right)\right]\Bigg\} \end{aligned} \quad (4.25)$$

where M_1 denotes the $Y_1{}^*$ mass, and g_{Y_1*} is given by Eq. (4.9). Besides those two $\Lambda\pi$ and $\Sigma\pi$ discontinuities below the $\bar{K}N$ threshold, there are two additional three-particle channels open: the $\Lambda\pi\pi$ and $\Sigma\pi\pi$ just below and above threshold, respectively. However, close to threshold such final states are strongly suppressed experimentally, and the measured branching ratios for those decays are small compared to the two-particle final states (Particle Data Group, 1972). It is therefore reasonable to assume that such small three-body final state contributions (if they are important at all) are included and well accounted for in our amplitude $\tilde{F}$ defined in Eq. (4.19). The amplitudes in Eqs. (4.20), (4.22), and (4.25) vanish at the Weinberg point and fulfill Adler's PCAC condition, Eq. (2.40), as a natural consequence of the gradient coupling theory.

Following Weinberg's original suggestion for low-energy πN scattering (Weinberg, 1966b), and taking into account the symmetry properties of $\tilde{F}$, we can write:

$$\tilde{F}(\nu, t, q^2, q'^2) = \tilde{A}m_K{}^2 + \tilde{B}t + \tilde{C}(q^2 + q'^2) + \tilde{D}\nu^2 + \tilde{R}(\nu, t, q^2, q'^2), \quad (4.26)$$

where $\tilde{R}$ measures the deviations of $\tilde{F}$ from linearity, and it will turn out (Reya, 1973b) that the only nonnegligible contribution comes mostly from the $\Lambda'(1520)$ which lies closest to the physical threshold. It is difficult, however, to say anything reliable about possible enhancements in the q^2 and q'^2 channels, but compared to the rather dominant effects of the $\Lambda'(1520)$, they should not be of substantial importance apart from $O(m_K{}^4/M_N{}^4)$ corrections, which remain undetermined in all estimates to date. The next step is to find four equations for the four expansion parameters $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$. Expanding $\tilde{F}$ in powers of $\tilde{k}^2$ and $\cos\tilde{\theta}$ around the physical threshold and comparing the

[5] See, for example, Nath, Etemadi, and Kimel (1971).

coefficients of $\tilde{k}^2 \cos\tilde{\theta}$, we obtain

$$\frac{1}{4\pi}\left(2\tilde{B} + \frac{m_K}{M_N}\tilde{D}\right) + C_k + \frac{1}{4\pi}\frac{m_K}{M_N}\frac{\partial\tilde{R}}{\partial\nu^2}(m_K, 0, m_K^2, m_K^2) + \frac{1}{2\pi}\frac{\partial\tilde{R}}{\partial t}(m_K, 0, m_K^2, m_K^2) = X, \tag{4.27}$$

where

$$C_k = \frac{1}{4\pi}(\tilde{T}_{B,k}{}^+ + \tilde{T}_{Y_0{}^*,k}{}^+ + \tilde{T}_{Y_1{}^*,k}{}^+),$$

with $\tilde{T}_{y,k}{}^+ = \partial\tilde{T}_y{}^+/\partial(\tilde{k}^2\cos\tilde{\theta})\,|_{\tilde{k}^2\cos\theta=0}$, and

$$X = (16M_N{}^2)^{-1}(2\tilde{a}_1 + \tilde{b}_0 + \tilde{b}_1) + \tfrac{1}{4}(2\tilde{a}_{11} + \tilde{b}_{01} + \tilde{b}_{11}) + [\tfrac{1}{2} + \tfrac{3}{4}(m_K/M_N)](2\tilde{a}_{13} + \tilde{b}_{03} + \tilde{b}_{13}).$$

Neglecting the (small) terms proportional to $\tilde{R}$, the sum rule in Eq. (4.27) is nearly saturated (Reya, 1973a)

$$(2.89 \pm 0.18)m_K{}^{-3} = (2.76 \pm 0.56)m_K{}^{-3}, \tag{4.28}$$

confirming the reliable use of Eq. (4.27) for determining the parameters in Eq. (4.26). The second equation follows from PCAC

$$\begin{aligned}\tilde{F}(0, m_K{}^2, m_K{}^2, 0) &= m_K{}^2(\tilde{A} + \tilde{B} + \tilde{C}) \\ &\quad + \tilde{R}(0, m_K{}^2, m_K{}^2, 0) \\ &= 0.\end{aligned} \tag{4.29}$$

[Contrary to previous calculations (Martin, 1966), PCAC for kaons has been found (Reya, 1973b) at least compatible with experiment, as suggested by generalized Goldberger–Treiman relations (Nieh, 1968; Dashen and Weinstein, 1969b).] At the physical threshold we get

$$\begin{aligned}\tilde{F}(m_K, 0, m_K{}^2, m_K{}^2) &= m_K{}^2(\tilde{A} + 2\tilde{C} + \tilde{D}) \\ &\quad + \tilde{R}(m_K, 0, m_K{}^2, m_K{}^2),\end{aligned} \tag{4.30}$$

where the left-hand side is given by Eq. (4.5), using Eq. (4.3). Finally, the fourth equation is obtained at $\nu = t = 0$, where we have

$$\begin{aligned}\tilde{F}(0, 0, m_K{}^2, m_K{}^2) &= m_K{}^2(\tilde{A} + 2\tilde{C}) \\ &\quad + \tilde{R}(0, 0, m_K{}^2, m_K{}^2).\end{aligned} \tag{4.31}$$

Using Eqs. (4.20), (4.22), and (4.25), it is straightforward to calculate $\tilde{F}(0, 0)$ in (4.31), apart from $\tilde{R}$, once we know $\tilde{T}^+(0, 0)$. This quantity appears as a subtraction constant in a subtracted forward dispersion relation at threshold; from Eq. (4.7) we get, for $\nu = m_K, t = 0$,

$$\tilde{T}^+(0, 0) = \mathrm{Re}\tilde{T}^+(m_K, 0) - m_K{}^2 \times \sum_{y=\Lambda,\Sigma}\frac{g_y{}^2}{4M_N{}^2}\frac{(M_y - M_N)^2 - m_K{}^2}{\omega_y(\omega_y{}^2 - m_K{}^2)} - I^+ \tag{4.32}$$

with

$$I^+ = \frac{m_K{}^2}{\pi}P\int_{\bar{\omega}}^{m_K} d\omega'\,\frac{\mathrm{Im}\tilde{T}_-(\omega', 0)}{\omega' k_L{}'^2} + \frac{m_K{}^2}{\pi}P\int_{m_K}^{\infty} d\omega'\,\frac{\tilde{\sigma}_+ + \tilde{\sigma}_-}{\omega' k_L{}'},$$

where $\omega_y = \Delta_y - m_K{}^2/2M_N$ and

$$\bar{\omega} = [(M_\Lambda + m_\pi)^2 - M_N{}^2 - m_K{}^2]/2M_N$$

and the total $K^\pm N$ cross sections are given by the optical theorem: $\mathrm{Im}\tilde{T}_\pm(\omega, 0) = k_L\tilde{\sigma}_\pm$. Equations (4.27) and (4.29) through (4.31) now completely determine the amplitude $\tilde{F}$ in Eq. (4.26), provided we can calculate $\tilde{R}$. As discussed above the only important contributions are expected to come from $\bar{K}N$ channel resonances. In this case we found (Reya, 1973b) that the only nonnegligible contribution is due to the $\Lambda'(1520)$. Its pole term can be calculated using the effective Lagrangian

$$\mathcal{L}_{\Lambda'} = (g_{\Lambda'}/M_N)\bar{\psi}_\mu\gamma_5\psi\partial^\mu\phi + \text{H.c.} \tag{4.33}$$

where $g_{\Lambda'}$ is given by Eq. (4.9). The propagator of this $J^P = \frac{3}{2}^-$ resonance can easily be related to that of the $\frac{3}{2}^+$ exchange in Eq. (4.24):

$$\begin{aligned}D_{\mu\nu}{}^{3/2-}(M, P) &= \bar{\gamma}_5 D_{\mu\nu}{}^{3/2+}(M, P)\gamma_5 \\ &= D_{\mu\nu}{}^{3/2+}(-M, P),\end{aligned} \tag{4.34}$$

where the standard commutation relations for the γ matrices have been used. According to Eq. (4.34), the $\Lambda'(1520)$ contribution can now be obtained from Eq. (4.25) by making the substitution $M_1 \to -M'$, where M' denotes the Λ' mass, and in order to obtain $\tilde{R}$ we have to subtract from this expression its linear expansion around the Weinberg point, with the final result

$$\begin{aligned}&\tilde{R}(\nu, t, q^2, q'^2) \\ &= \frac{1}{3}\frac{(g_{\Lambda'}/M_N)^2}{(\nu_B{}^{\Lambda'})^2 - \nu^2}\left\{\frac{1}{4M'}\left(1 - \frac{M_N}{M'}\right)(q^2 + q'^2)\right. \\ &\quad\times\left[\nu_B\nu_B{}^{\Lambda'} - \left(\frac{1}{2} - \frac{M'}{M_N}\right)\nu^2\right] \\ &\quad - \frac{q^2q'^2}{4M_NM'}\left[\left(1 - \frac{M_N}{2M'}\right)\nu_B{}^{\Lambda'} - \frac{\nu^2}{2M'}\right] \\ &\quad + \nu_B{}^{\Lambda'}\left[\left(\frac{2M_N}{M_N + M'} + \frac{M_N{}^2}{2M'^2}\right)\nu_B{}^2\right. \\ &\quad \left. - \frac{M_N}{2M'}\left(1 - \frac{M_N}{M'}\right)\nu^2\right] \\ &\quad + \frac{M_N}{2M'^2}\nu^2\left[\nu^2 - \nu_B{}^2 - \frac{2M_N}{M_N + M'}[\nu^2 - (\nu_B{}^{\Lambda'})^2]\right. \\ &\quad \left.\left. + \nu_B\left(\frac{2M'^2(M' - M_N)}{M_N(M' + M_N)} - M_N + M'\right)\right]\right\}.\end{aligned} \tag{4.35}$$

TABLE II. Estimates of the kaon–nucleon σ term.

Authors	σ_{NN}^{KK} (MeV)
von Hippel and Kim (1969, 1970)	170
Hakin (1972b)	180 ± 30
Thompson (1971)	-370 ± 110
Köpp, Walsh, and Zerwas (1972)	$345 {+270 \atop -550}$
Reya (1972)	540 ± 160
Nasrallah and Schilcher (1973)	160
Reya (1973a, b)	480 ± 110
Gensini (1973)	$410 {+22 \atop -28}$

Solving Eqs. (4.27) and (4.29) through (4.31), the parameter $\tilde{A}$ in Eq. (4.26) is found to be

$$-m_K^2\tilde{A} = (1 + m_K/M_N)\tilde{F}(0,0) - (m_K/M_N) \times \tilde{F}(m_K, 0) + 4\pi m_K^2(X - C_K) + \tilde{R}_\Delta \quad (4.36)$$

with

$$\begin{aligned}\tilde{R}_\Delta = &(m_K/M_N)\tilde{R}(m_K, 0) - (1 + m_K/M_N)\tilde{R}(0,0) \\ &+ 2\tilde{R}(0, m_K^2, m_K^2, 0) \\ &- (m_K^3/M_N)(\partial\tilde{R}/\partial\nu^2)(m_K, 0, m_K^2, m_K^2) \\ &- 2m_K^2(\partial\tilde{R}/\partial t)(m_K, 0, m_K^2, m_K^2).\end{aligned}$$

Together with the low-energy theorem Eq. (2.37) and Eq. (4.26),

$$\sigma_{NN}^{KK} = -f_K^2 m_K^2 \tilde{A}, \quad (4.37)$$

Eq. (4.36) completely determines σ_{NN}^{KK}. Equation (4.37), therefore, relates the nucleon expectation value of the σ commutator to s- and p-wave scattering lengths and to a rather well known integral over total $K^\pm N$ cross sections, where full use has been made of the smoothness hypothesis for $\tilde{F}$. From Eqs. (4.36) and (4.37) we finally obtain (Reya, 1973b)

$$\sigma_{NN}^{KK} = (480 \pm 110) \text{ MeV}. \quad (4.38)$$

As in the dispersive approaches discussed above, it is very hard to make a reliable estimate of the lower limit in Eq. (4.38) due to the possibility mentioned earlier (Martin, 1970) of a negative solution for b_{13}; in this case, Eq. (4.38) would read: $\sigma_{NN}^{KK} = 480^{+110}_{-600}$ MeV. A negative result is at least not favored by our analysis, and we regard a negative σ_{NN}^{KK} as unlikely in this context. In this respect much work remains to be done, in that, especially for the p-wave scattering lengths, we are far from having universally accepted values. The nonsmooth correction $\tilde{R}_\Delta$ in Eq. (4.36), stemming from the s-channel $\Lambda'(1520)$ resonance, turns out to be about $-0.2m_K^{-1}$ constituting some 5% correction to the final value of $\tilde{A}$. Possible enhancements could come from t-channel unitarity corrections to Eq. (4.26), and a study similar to that of Geddes and Graham (1973) would certainly be of interest. Again, higher-order corrections to Eq. (2.43) turn out to be negligibly small (as far as s-channel corrections are concerned). According to Eq. (2.43), taking into account $\tilde{R}$ in Eq. (4.26), we obtain

$$-\tilde{F}(0,0,0,0) = \tilde{F}(0, 2m_K^2, m_K^2, m_K^2) + \tilde{\Delta} \quad (4.39)$$

with

$$\tilde{\Delta} = 2\tilde{R}(0, m_K^2, m_K^2, 0) - \tilde{R}(0, 2m_K^2, m_K^2, m_K^2)$$

which turns out to be small: $\tilde{\Delta} \simeq 10^{-2}m_K^{-1}$. It therefore appears that the low-energy theorem is not appreciably modified if the amplitude is approximated by Eq. (2.43). The major source of uncertainty of course remains, the unknown terms $O(m_K^4/M_N^4)$.

Using entirely different methods, Köpp, Walsh, and Zerwas (1972) derived an off-mass-shell finite-energy sum rule for the s-channel isospin-zero A amplitude and found

$$\sigma_{NN}^{KK} = 345^{+270}_{-550} \text{ MeV} \quad (4.40)$$

and, according to these authors, a negative value is at least not favored in their analysis. Since their calculation has been done very thoroughly, there appears to be no reason to doubt the result in Eq. (4.40), provided off-shell finite-energy sum rules are granted.

In Table II we summarize the results obtained so far. Within present uncertainties, these estimates are compatible with each other, except the strongly negative result of Thompson (1971). Since two very similar calculations (Reya, 1972; Nasrallah and Schilcher, 1973) using fixed-t dispersion relations yielded positive values for σ_{NN}^{KK}, in agreement with other estimates, a negative value for the kaon–nucleon σ term appears to be more than unlikely. Although, at the present stage, it is virtually impossible to state a reliable lower limit for σ_{NN}^{KK}, Table II suggests an approximate upper limit of

$$\sigma_{NN}^{KK} < 600 \text{ MeV}, \quad (4.41)$$

whereas the world average according to Table II lies around 350 MeV, but we are presently not in the position to deduce an *exact* value for σ_{NN}^{KK}. The main point, however, is that a value of σ_{NN}^{KK} as large as 1300 MeV, say, is definitely excluded. This is a magnitude which, as we shall see in the next section, would be implied by the large Cheng–Dashen result (Cheng and Dashen, 1971) for $\sigma_{NN}^{\pi\pi}$ in the framework of the $(3, \bar{3}) + (\bar{3}, 3)$ chiral symmetry-breaking model. Such large σ terms would upset (Altarelli, Cabibbo, and Maiani, 1971a; Crewther, 1971; Mathur, 1971) our "conventional" understanding of symmetry-breaking mechanisms, which we are going to discuss now.

V. THEORIES OF CHIRAL SYMMETRY BREAKING

So far, in the last two sections, we have been mainly concerned with the "experimental" aspect of chiral symmetry by studying how symmetry-breaking effects can be extracted from presently available scattering data. We are now going to discuss the purely theoretical aspect of this problem, namely how to construct models for the symmetry-breaking Hamiltonian $\mathcal{H}'$ in Eq. (2.10), that is, to find appropriate representations under which $\mathcal{H}'$ transforms. The predictions and compatibility of such symmetry-breaking schemes will be discussed and compared with experiment.

A. The $(3, \bar{3}) + (\bar{3}, 3)$ model

If, for the time being, we assume that the symmetry-breaking piece of the total Hamiltonian does not admit operators of isospin or hypercharge 2, then the only representations allowed for the components of $\mathcal{H}'$ are the $(3, \bar{3}) + (\bar{3}, 3)$ and $(1, 8) + (8, 1)$ representations of $SU(3) \times SU(3)$. Originally it was shown (Gell-Mann, Oakes, and Renner, 1968) that $(1, 8) + (8, 1)$ contributions are unlikely to be large, compared to $(3, \bar{3}) + (\bar{3}, 3)$ breaking terms, and therefore constitute at most admixtures to other dominating terms in $\mathcal{H}'$. Moreover, we know by now that a *pure* $(1, 8) + (8, 1)$ model is ruled by the data. Possible $(1, 8) + (8, 1)$ admixtures, if they are important at all, will be discussed at the end of this chapter. [Note that as long as the strong breaking of $SU(3) \times SU(3)$ transforms as a $(3, \bar{3}) + (\bar{3}, 3)$ representation, parity and strangeness are conserved at a strong level, which is not true if the explicit breaking transforms as a $(1, 8) + (8, 1)$.] We are therefore led naturally (Renner and Sudbery, 1969) to the $(3, \bar{3}) + (\bar{3}, 3)$ model of Gell-Mann, Oakes, and Renner (1968) and Glashow and Weinberg (1968), the simplest and most elegant chiral symmetry-breaking model having just one free (universal) parameter which fixes the relative scale between $SU(3)$ breaking and $SU(3)$-invariant chiral symmetry breaking. Before going into the details of this model, let us briefly review the construction of the $(3, \bar{3}) + (\bar{3}, 3)$ representation.

The 3 and $\bar{3}$ representations T_i and W_i of $SU(3)$ are defined by the commutation relations

$$[F_a, T_i] = \tfrac{1}{2} T_j (\lambda_a)_{ji}$$
$$[F_a, W_i] = -\tfrac{1}{2} (\lambda_a)_{ij} W_j, \tag{5.1}$$

where the λ_a are the eight traceless 3×3 Gell-Mann matrices (Gell-Mann, 1962; Adler and Dashen, 1968) of the three-dimensional representation of $SU(3)$. Similarly to Eq. (5.1) we can write for $(3, \bar{3})$ in $SU(3) \times SU(3)$

$$[F_a{}^+, T_{ij}] = \tfrac{1}{2} (\lambda_a{}^*)_{ik} T_{kj}$$
$$[F_a{}^-, T_{ij}] = -\tfrac{1}{2} (\lambda_a)_{jk} T_{ik} \tag{5.2}$$

and for $(\bar{3}, 3)$

$$[F_a{}^+, W_{ij}] = -\tfrac{1}{2} (\lambda_a)_{ik} W_{kj}$$
$$[F_a{}^-, W_{ij}] = \tfrac{1}{2} (\lambda_a{}^*)_{jk} W_{ik}, \tag{5.3}$$

where the $F_a{}^\pm$ are defined in Eq. (2.4). Since $T_{ij}{}^\dagger$ transforms like $(\bar{3}, 3)$, we can parity double our decomposition by requiring, as in Eq. (2.6),

$$P T_{ij} P^{-1} = T_{ji}{}^\dagger$$

so that T_{ij} is now said to transform under $(3, \bar{3}) + (\bar{3}, 3)$. In order to reduce this representation under parity, we define the two even and odd linear combinations

$$P_{ij} = T_{ij} + T_{ji}{}^\dagger$$
$$M_{ij} = i(T_{ij} - T_{ji}{}^\dagger) \tag{5.4}$$

so that

$$P P_{ij} P^{-1} = P_{ij}, \qquad P_{ij}{}^\dagger = P_{ji}$$
$$P M_{ij} P^{-1} = -M_{ij}, \qquad M_{ij}{}^\dagger = M_{ji}.$$

The $SU(3)$ content can now be made manifest by writing

$$P_{ij} = (1/\sqrt{3}) u_0 \delta_{ij} + (1/\sqrt{2}) (\lambda_a)_{ji} u_a$$
$$M_{ij} = (1/\sqrt{3}) v_0 \delta_{ij} + (1/\sqrt{2}) (\lambda_a)_{ji} v_a. \tag{5.5}$$

Inverting these relations as

$$u_0 = (1/\sqrt{3}) P_{ii}, \qquad u_a = (1/\sqrt{2}) (\lambda_a)_{ij} P_{ij}$$
$$v_0 = (1/\sqrt{3}) M_{ii}, \qquad v_a = (1/\sqrt{2}) (\lambda_a)_{ij} M_{ij}, \tag{5.6}$$

where the u's and v's are Hermitian scalar and pseudoscalar operators, respectively, we obtain the well known commutation relations (Gell-Mann, 1962, 1964; Gell-Mann, Oakes, and Renner, 1968)

$$[F_a, u_0] = 0, \qquad [F_a, v_0] = 0$$
$$[F_a, u_b] = i f_{abc} u_c, \qquad [F_a, v_b] = i f_{abc} v_c,$$
$$[F_a{}^5, u_0] = -i(\tfrac{2}{3})^{1/2} v_a, \qquad [F_a{}^5, v_0] = i(\tfrac{2}{3})^{1/2} u_a,$$
$$[F_a{}^5, u_b] = -i d_{abc} v_c - i(\tfrac{2}{3})^{1/2} \delta_{ab} v_0,$$
$$[F_a{}^5, v_b] = i d_{abc} u_c + i(\tfrac{2}{3})^{1/2} \delta_{ab} u_0. \tag{5.7}$$

The commutators involving the F_a's in (5.7) obviously identify u_0 and v_0 as $SU(3)$ singlets and $\{u_a\}$ and $\{v_a\}$ as $SU(3)$ octets. For $(3, \bar{3}) + (\bar{3}, 3)$ calculations it is customary to work directly with the u's and v's since their commutation relations are simple and the f_{abc} and d_{abc} are well tabulated (Gell-Mann, 1962; Adler and Dashen, 1968). However, as we shall see, for more complicated higher-dimensional representations, it proves simpler to work directly in terms of the analogues of the T_{ij}.

From the requirement that $\mathcal{H}'$ conserve isospin, hypercharge, and parity, the most general form for $\epsilon \mathcal{H}'$ is

$$\epsilon \mathcal{H}' = u_0 + c u_8 \tag{5.8}$$

which has been suggested by Gell-Mann, Oakes, and Renner (1968) (GMOR), and where c is the only parameter of the model to be determined. Having specified $\mathcal{H}'$, Eq. (5.8), we can calculate explicitly every quantity of interest, by using the commutation relations (5.7). From Eq. (2.23) we get to first order in ϵ (singlet vacuum)

$$m_\pi{}^2 f_\pi{}^2 = -\tfrac{2}{3}[1 + (c/\sqrt{2})]\langle 0 | u_0 | 0 \rangle$$
$$m_K{}^2 f_K{}^2 = -\tfrac{2}{3}[1 - (c/2\sqrt{2})]\langle 0 | u_0 | 0 \rangle$$
$$m_\eta{}^2 f_\eta{}^2 = -\tfrac{2}{3}[1 - (c/\sqrt{2})]\langle 0 | u_0 | 0 \rangle. \tag{5.9}$$

Since these relations contain only two unknown parameters, there is one relation among the three masses, but it is just the Gell-Mann–Okubo mass formula

$$m_K^2 = (3m_\eta^2 + m_\pi^2)/4 \qquad (5.10)$$

which is no surprise, since we built octet breaking of $SU(3)$ into our choice of $\mathcal{H}'$. (Since we are working only to lowest order $\mathcal{H}'$ we take $f_\pi = f_K = f_\eta \equiv f$.) The only really interesting thing about the meson mass formulae in Eq. (5.9) is that they determine the parameter of the model according to

$$c = -2\sqrt{2}[(m_K^2 - m_\pi^2)/(2m_K^2 + m_\pi^2)] \simeq -1.25. \qquad (5.11)$$

In the limit $m_\pi^2 \to 0$ we have $c = -\sqrt{2}$, i.e., exact $SU(2) \times SU(2)$ symmetry, since $u_0 - \sqrt{2}u_8$ commutes with F_a and F_a^5 for $a = 1, 2, 3$ in Eq. (2.11). Therefore, according to Eq. (5.11), it is clear that, using Eq. (2.11), $\partial_\mu A_a^\mu$ is very small for $a = 1$, 2, and 3, implying that $SU(2) \times SU(2)$ is a particularly good symmetry within the $(3, \bar{3}) + (\bar{3}, 3)$ breaking scheme.

We just have seen, as mentioned earlier, that the meson mass formula in Eq. (2.23) merely fixes the parameters of a model, but does not discriminate between different symmetry-breaking schemes. In order to proceed let us consider the axial-vector divergences. For $\pi\pi$ scattering, the σ commutator in Eq. (2.30) can be easily calculated, using Eqs. (5.8), (5.7), and (5.9):

$$A = -f^{-4}[(2 + \sqrt{2}c)/3]\langle 0 \mid u_0 \mid 0\rangle + O(\epsilon^2) = m_\pi^2/f^2 + O(\epsilon^2) \qquad (5.12)$$

which, of course, is the original Weinberg (1966b) result, since the $(3, \bar{3}) + (\bar{3}, 3)$ representation does not contain $I = 2$ components. Inserting Eq. (5.12) into Eq. (2.29) we get the prediction $a_0^{(0)} \simeq 0.16 m_\pi^{-1}$ which is quite consistent with the most recent experimental result from K_{e4} decays (Beier, 1973; Beier *et al.*, 1973)

$$(a_0^{(0)})_{\text{exp}} = (0.17 \pm 0.13) m_\pi^{-1}. \qquad (5.13)$$

Because of the large experimental uncertainties the conclusions are obviously not very restrictive. A similar situation holds for the $I = 2$ scattering length, obtained (Petersen, 1971) from applying the Chew–Low extrapolation to πN scattering data, where the Weinberg prediction, Eq. (5.12) into Eq. (2.29), $a_0^{(2)} \simeq -0.05 m_\pi^{-1}$ is close to the experimental value. Again, the experimental errors in such an extrapolation to the pion pole are difficult to ascertain (Petersen, 1971).

Using the commutation relations (5.7), we obtain the following expressions for the axial-vector divergences in Eq. (2.11b):

$$\partial_\mu A_a^\mu = -(1/\sqrt{3})(\sqrt{2} + c)v_a \qquad \text{for } a = 1, 2, 3$$
$$\partial_\mu A_a^\mu = -(1/\sqrt{3})(\sqrt{2} - \tfrac{1}{2}c)v_a \qquad \text{for } a = 4, 5, 6, 7$$
$$\partial_\mu A_8^\mu = -(1/\sqrt{3})(\sqrt{2} - c)v_8 - (\tfrac{2}{3})^{1/2}v_0 \qquad (5.14)$$

and therefore the πN and KN σ terms in Eq. (2.38) take the form

$$\sigma_{NN}^{\pi\pi} = \tfrac{1}{3}(\sqrt{2} + c)\langle N \mid \sqrt{2}u_0 + u_8 \mid N\rangle$$
$$\sigma_{NN}^{KK} = \tfrac{1}{3}(\sqrt{2} - c/2)\langle N \mid \sqrt{2}u_0 + (\sqrt{3}/2)u_3 - \tfrac{1}{2}u_8 \mid N\rangle. \qquad (5.15)$$

[Note that using Eq. (2.13) together with (5.14) for the meson mass spectrum, instead of Eq. (2.23), would yield the same results as Eq. (5.9), which is to be expected if one is working in a consistent approximation.] The matrix elements of u_a in Eq. (5.15) are given by the mass differences in the $SU(3)$ baryon octet. According to Eq. (2.10), the baryon masses are given, to lowest order, by

$$M_B = M_0 + \langle B \mid \epsilon\mathcal{H}' \mid B\rangle, \qquad (5.16)$$

where $M_0 = \langle B \mid \mathcal{H}_0 \mid B\rangle$ is the (degenerate) average baryon mass in the $SU(3) \times SU(3)$ limit. Using the Wigner–Eckart theorem

$$\langle B_a \mid u_b \mid B_c\rangle = i f_{abc}F + d_{abc}D$$
$$\langle B_a \mid u_0 \mid B_c\rangle = \alpha\delta_{ac} \qquad (5.17)$$

in Eq. (5.16), we obtain

$$\langle N \mid u_3 \mid N\rangle = (2\sqrt{3}c)^{-1}(2M_N + M_\Sigma - 3M_\Lambda)$$
$$\langle N \mid u_8 \mid N\rangle = (2c)^{-1}(2M_N - M_\Sigma - M_\Lambda). \qquad (5.18)$$

The matrix element $\langle N \mid u_0 \mid N\rangle$ is not known, but its magnitude is expected to be similar to that of $\langle N \mid u_8 \mid N\rangle$. The reason for this is that $SU(3)$ mass splittings are always of the same order as the masses of the pseudoscalar meson octet. This observation suggests that the strength of the two symmetry-violating terms are comparable. Since u_0 breaks $SU(3) \times SU(3)$ and u_8 breaks $SU(3)$ as well as $SU(3) \times SU(3)$, we cannot allow $\langle N \mid u_0 \mid N\rangle$ to be different by as much as an order of magnitude, say, from $\langle N \mid u_8 \mid N\rangle$ and still have the two symmetries broken by a comparable amount. [There could be a possible enhancement of $\langle N \mid u_0 \mid N\rangle$ with respect to $\langle N \mid u_8 \mid N\rangle$, if one assumes (Altarelli, Cabibbo, and Maiani, 1971a; Crewther, 1971; Mathur, 1971) u_0 to be coupled to the Goldstone boson of a further symmetry, namely scale invariance. However, the main motivation for a strongly enhanced $\langle N \mid u_0 \mid N\rangle$ appears to be obsolete by now, since, according to Table I and to what we have said in Sec. III, values for $\sigma_{NN}^{\pi\pi}$ of about 100 MeV or more are most likely to be ruled out. We shall come back to this point at the end of Sec. V.] Therefore, one obtains the following estimates for Eq. (5.15):

$$|\sigma_{NN}^{\pi\pi}| \simeq 10 \text{ to } 20 \text{ MeV} \qquad (5.19a)$$
$$|\sigma_{NN}^{KK}| \simeq 100 \text{ to } 200 \text{ MeV}. \qquad (5.19b)$$

It should be noted that the estimate of σ_{NN}^{KK} seems to be more reliable than in the case of πN scattering, Eq. (5.19a), where $\sigma_{NN}^{\pi\pi}$ is proportional to $(\sqrt{2} + c)$ which is very sensitive to slight variations of the negative number c.

Comparing Eq. (5.19a) with the results of Sec. III and Table I, we find that the $(3, \bar{3}) + (\bar{3}, 3)$ model is, within the present experimental uncertainties, consistent with the "experimental" estimates done so far. The same applies to σ_{NN}^{KK}, Eq. (5.19b), when compared with Table II. That the πN and kaon–nucleon calculations are indeed consistent with each other can be simply shown by calculating the nucleon expectation value of u_0. Taking the average value for σ_{NN}^{KK} of Table II to be about 350 MeV, Eq. (5.15), using Eq. (5.18), tells us that $\langle N \mid u_0 \mid N \rangle \simeq 400$ MeV; inserting this value into $\sigma_{NN}^{\pi\pi}$ of Eq. (5.15) gives $\sigma_{NN}^{\pi\pi} \simeq$ 40 MeV, which is roughly the world average of the estimated πN sigma terms in Table I. In addition, no significant enhancement of $\langle N \mid u_0 \mid N \rangle$ is found with respect to $\langle N \mid u_8 \mid N \rangle$. A strong enhancement comes about, as just mentioned above, if u_0 is coupled to the Goldstone boson of scale invariance: In this case one expects (Altarelli, Cabibbo, and Maiani, 1971a; Crewther, 1971; Mathur, 1971) $\langle N \mid u_0 \mid N \rangle \simeq 1500$ MeV, in clear contradiction to the above result.

Although the present results for $\sigma_{NN}^{\pi\pi}$ and σ_{NN}^{KK} favor within quoted uncertainties the $(3, \bar{3}) + (\bar{3}, 3)$ scheme for chiral symmetry breaking, most of them yield values somewhat larger than the theoretical estimates, Eq. (5.19), of the GMOR model. Since those entirely independent calculations produce slightly enhanced results (by about a factor of 2) with respect to the conventional $(3, \bar{3}) + (\bar{3}, 3)$ estimates, it appears that this could be something more than just an accidental coincidence. Taking these enhancements literally, this could mean that further admixtures in the symmetry-breaking Hamiltonian are required in addition to the $(3, \bar{3}) + (\bar{3}, 3)$ transforming part, or the transformation property of $\mathcal{H}'$ is entirely different from that suggested by the GMOR model; these possibilities will be studied later. Before discussing other alternatives, namely mechanisms producing enhanced $(3, \bar{3}) + (\bar{3}, 3)$ chiral symmetry-breaking effects, let us look at the connection of σ terms with the average mass M_0, Eq. (5.16), in a given baryon octet.

Considering the πN σ term for example, we obtain by applying the Wigner–Eckart theorem (5.17) to Eq. (5.16) and $\sigma_{NN}^{\pi\pi}$ in (5.15), the following relation:

$$M_0 = \frac{M_N}{\sqrt{2}c} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}c}\right)(M_\Lambda + M_\Sigma) - \frac{3}{2 + \sqrt{2}c}\sigma_{NN}^{\pi\pi}$$
$$\simeq 1300 - 13\sigma_{NN}^{\pi\pi} \text{ (MeV)}, \tag{5.20}$$

for $c = -1.25$. Using $\sigma_{NN}^{\pi\pi} \simeq 40$ MeV, say, we find that the (degenerate) average mass of the $\frac{1}{2}^+$ baryon octet is about $M_0 \simeq 800$ MeV. This confirms our physical "intuition" in that it implies that exact $SU(3)$ symmetry is responsible for the main portion of the baryon masses in the $\frac{1}{2}^+$ octet, say, whereas the much smaller mass splitting between the $\frac{1}{2}^+$ baryons are generated by (comparatively small) symmetry-breaking effects. Such a situation has to hold, after all, if the whole concept of symmetries and their (first-order) breaking effects is believed to be correct. From this point of view $(3, \bar{3}) + (\bar{3}, 3)$ appears to be the only acceptable choice, in contrast to higher-dimensional representations like the $(6, \bar{6}) + (\bar{6}, 6)$ or $(8, 8)$, say, as we shall see below.

Taking literally the enhanced σ terms, with respect to the conventional GMOR estimates in Eq. (5.19), in Tables I and II, we will now discuss mechanisms which generate and explain such enhancements without changing the transformation properties of $\mathcal{H}'$. Recently, Renner (1972b) estimated the meson–nucleon sigma terms using the Li–Pagels mechanism (Li and Pagels, 1971a, b; 1972) of calculating $(3, \bar{3}) + (\bar{3}, 3)$ chiral symmetry breaking. In this model the octet enhancement is achieved by the threshold dominance of Goldstone-boson-pair states. To be more explicit, the matrix elements of the scalar operators u_a, which vanish in the chiral symmetry limit, are calculated in terms of the contributions of two low-energy pseudoscalar Goldstone bosons to dispersion relations in momentum transfer. The dominance of these contributions is compelling for those matrix elements which approach the chiral symmetry limit nonanalytically (Li and Pagels, 1971a) like $\epsilon \ln\epsilon$. Considering the matrix element $\langle B \mid cu_8 \mid B \rangle \times \bar{u}_B u_B F(t)$, u_B denotes the baryon Dirac spinor, Li and Pagels (1971b, 1972) introduced dispersion relations for the form factors of the form

$$M_B - M_0 = F(0) = \frac{1}{\pi}\int_{t_0}^{\infty} (dt'/t')\, \mathrm{Im}F(t') \tag{5.21}$$

where, from unitarity, the absorptive part is given by

$$\mathrm{Im}F(t') = (2\pi)^4 \sum_n \delta(\textstyle\sum p)\langle 0 \mid cu_8 \mid n \rangle\langle n \mid T^\dagger \mid \bar{B}B \rangle \tag{5.22}$$

and they retain only the two pseudoscalar meson states ($n = PP$) over a limited t' range. Pretending to be in the neighborhood of the chiral symmetry limit, they use results of chiral symmetry in evaluating the discontinuity in Eq. (5.22): In particular, by setting $\langle 0 \mid u_8 \mid PP \rangle$ constant and taking pseudovector Born terms to estimate

$$\langle PP \mid T^\dagger \mid \bar{B}B \rangle \sim (t')^{1/2},$$

the lower integration limit in Eq. (5.21) is taken down to zero and all pseudoscalar meson masses (except for the scale implied in $\langle 0 \mid cu_8 \mid PP \rangle$) are set equal to zero. In this way, Li and Pagels (1971b, 1972) obtained

$$(3/10)(F/D) = (F/D)_{Ax}/[3(F/D)_{Ax}^2 - 1]. \tag{5.23}$$

This previously unknown relation between the baryon octet mass splitting (F/D) ratio, Eq. (5.17), and the $(F/D)_{Ax}$ ratio of the weak axial-vector current to the baryons in semileptonic hyperon decays, turns out to be in excellent agreement with experiment: With

$$F/D = \tfrac{2}{3}(M_N - M_\Xi)/(M_\Sigma - M_\Lambda),$$

which follows from Eqs. (5.16) and (5.17), one obtains from Eq. (5.28) that $(F/D)_{Ax} \simeq 0.45$ in agreement with the data (Marshak, Riazuddin, and Ryan, 1969; Pilkuhn *et al.*, 1973). Using, in addition, a similar estimate for $\langle B \mid u_0 \mid B \rangle = G(t)$, Renner (1972b) obtained $\langle N \mid u_0 \mid N \rangle =$ 400 MeV which, by Eq. (5.15), implies

$$\sigma_{NN}^{\pi\pi} \simeq 40 \text{ MeV}, \quad \sigma_{NN}^{KK} \simeq 400 \text{ MeV (Renner, 1972b)} \tag{5.24}$$

and appears to be in excellent average agreement with

Tables I and II. However, this close agreement could be accidental, since various uncertainties are contained in the above estimates: Besides the uncertainty in the ratio of the $SU(3)$ singlet and octet t-channel cut-off masses (Renner, 1972b), the strict use of chiral symmetry both in evaluating the discontinuity in Eq. (5.22) and in setting $t_0 = 0$ in (5.21) appears to be a further source of considerable uncertainty. Nonetheless, this relatively simple example shows that enhanced σ terms, especially if they are unambiguously confirmed by more accurate future experiments, do not require the $(3, \bar{3}) + (\bar{3}, 3)$ model to be discarded at all.

Evidently, Eq. (5.11) requires, inserted in the first equation of (5.14), that $SU(2) \times SU(2)$ be regarded as a much better symmetry than $SU(3) \times SU(3)$ or $SU(3)$. This has been challenged by Gaillard (1969) and by Brandt and Preparata (1970), who prefer $-c \ll \sqrt{2}$ which, by Eq. (5.15), then also implies large σ terms. This is the so-called weak version of PCAC where pion pole dominance is *a priori* assumed only for matrix elements of $\partial_\mu A_a{}^\mu$ between physical states (and not for *all* Green's functions involving $\partial_\mu A_a{}^\mu$, as in the conventional "strong" PCAC considered so far), implying $SU(3)$ to be a much better symmetry than $SU(2) \times SU(2)$. The result $-c \ll \sqrt{2}$ is mainly based on their analyses of K_{l3} decays. The main assumption is $\xi(m_K{}^2) \simeq \xi(0)$, where $\xi(t)$ is defined in Eq. (2.47); then a small value of c is required if the confused experimental situation (Chounet and Gaillard, 1970; Chounet, Gaillard, and Gaillard, 1972; Wojcicki, 1972) is supposed to favor $\xi(0) \lesssim -0.5$. [Together with Eq. (5.11), the GMOR model, however, predicts (Gerstein and Schnitzer, 1968; Deshpande, 1970) $\xi(0)$ to be close to zero, $\xi(0) \simeq -0.1$, unless one allows the slope λ_+ of $f_+(t)$ in Eq. (2.46) to be as large as $\lambda_+ \simeq 0.1$ which could account for (Weinstein, 1971a, b) $\xi(0) \simeq -0.5$. Note that a simple $K^*(890)$ pole dominance of $f_+(t)$ predicts $\lambda_+ = m_\pi{}^2/m_{K^*}{}^2 \simeq 0.024$.] However, when the collinear dispersion relations of Banerjee (1970) are examined, it is difficult to avoid the conclusion that $-c \ll \sqrt{2}$ implies $\xi(m_K{}^2) \simeq -1$ and $\xi(0) \gg 0$; only by having $c \simeq -\sqrt{2}$ can a value $\xi(0) \lesssim -0.5$ be obtained. In addition, it has been pointed out by Weinstein (1971b) that the calculation of the ξ parameter of Gaillard (1969) and Brandt and Preparata (1970) depends upon the introduction of (large) $SU(3)$ violation (by renormalization constants of the scalar densities u_K and u_π) and not upon a modification of the PCAC hypothesis. We therefore retain the usual estimate $c \simeq -1.25$, Eq. (5.11), and exclude "weak PCAC" as a possible explanation for enhanced σ terms.

As already mentioned, most of the K_{l3} experiments done so far (typically with about 10^4 $K_{\mu3}{}^0$ or $K_{\mu3}{}^+$ decays) yielded rather contradictory results (Wojcicki, 1972): Whereas polarization measurements, measuring the μ polarization in $K \to \pi\mu\nu_\mu$, yield values for ξ between -1 and -2, most of the Dalitz plot experiments give consistently less negative ξ values between -0.5 and -1. The more recent high-statistics polarization experiment (10^5 $K_L{}^0 \to \pi^-\mu^+\nu_\mu$ events) of Sandweiss *et al.* (1973), however, resulted in a much less negative ξ value than stated above, of $\xi(0) \simeq -0.5$. Most recently, Donaldson *et al.* (1973) made a very high-statistics Dalitz plot experiment (probably the most reliable one to date) by analyzing about 10^6 $K_L{}^0 \to \pi\mu\nu_\mu$ decays, and found $\xi(t)$ to be around -0.1, and to be independent of t; similarly they found $\lambda_+ \simeq 0.03$. This is in excellent agreement with current algebra and ("strong") PCAC predictions, as well as $K^*(890)$ dominance of the vector form factor $f_+(t)$, leaving the $(3, \bar{3}) + (\bar{3}, 3)$ model unrivalled; therefore, at the present time, most of the difficulties in understanding the Dalitz plot data which were raised by Chounet, Gaillard, and Gaillard (1972) can be fairly convincingly resolved. Thus, the real discrepancy which must yet be resolved (hopefully by the current very high-statistics K^0 polarization measurements) is between the Dalitz plot and polarization experiments.

In closing this section let us remark briefly on possible connections of strong symmetry violations with weak and electromagnetic effects (Gell-Mann, 1969; Cabibbo and Maiani, 1970b; Gatto, 1970). Various unsolved problems have led us to believe that isospin is also broken by some purely hadronic interaction. Among these problems are the still unexplained $\Delta I = 1$ mass differences (the p–n mass difference for instance) using pure electromagnetic interactions, or the $SU(3) \times SU(3)$ Dashen sum rule (Dashen, 1969)

$$m_{\pi^+}{}^2 - m_{\pi^0}{}^2 = m_{K^+}{}^2 - m_{K^0}{}^2 + O(\epsilon e^2) \qquad (5.25)$$

which follows if isospin breaking is of purely electromagnetic origin. Whereas Eq. (5.25) is obviously in violent disagreement with experiment, its exact analogue for baryon masses, the Coleman–Glashow sum rule (Coleman and Glashow, 1961)

$$(M_p - M_n) + M_{\Sigma^-} - M_{\Sigma^+}) - (M_{\Xi^-} - M_{\Xi^0}) = 0 + O(\epsilon e^2) \qquad (5.26)$$

is, in fact, satisfied very well. On the other hand, we are faced with the aggravating problem of the $\eta \to 3\pi$ decay, which, according to a theorem due to Bell and Sutherland (Sutherland, 1967), cannot proceed through electromagnetic interactions in the soft pion limit. Even introducing the corrections for a finite pion mass, the predicted decay rate is smaller, by at least two orders of magnitude, than the observed one (Sutherland, 1967). On a more fundamental basis there remains the problem of explaining the empirical suppression factor for strangeness-changing weak amplitudes, i.e., the Cabibbo angle θ_C in Eq. (2.14). These problems can be resolved, in a more or less satisfactory way, by adding an $I = 1$, $I_3 = 0$ "field" to the chiral symmetry-breaking Hamiltonian of the form

$$\epsilon\mathcal{H}' = u_0 + cu_8 + c_3u_3, \qquad (5.27)$$

where u_3 causes a breakdown of isospin symmetry due to strong interactions (independent of the isospin breaking induced by the electromagnetic interaction), and c_3 is clearly of the order e^2. The original "tadpole" scheme of Coleman and Glashow (1964) results in a term much like u_3. Various attempts based on Eq. (5.27) have been made to relate the strong symmetry-breaking parameters in Eq. (5.27) to the Cabibbo angle θ_C and to the fine structure constant $e^2/4\pi$; in other words, symmetry breakings in the strong Hamiltonian are determined by weak and electromagnetic forces in a self-consistent way. For example, constraints on these parameters may be obtained from the requirement that weak (Gatto, Sartori, and Tonin, 1968) or weak plus electromagnetic (Cabibbo and Maiani, 1968, 1970a) contributions to the "quark masses" are finite to first order in perturbation theory. Another scheme has been proposed by Oakes (1969a) by demanding that the strong-interaction Hamiltonian for $m_\pi \neq 0$ be obtained by rotating the $SU(2) \times SU(2)$ invariant ($m_\pi = 0$) Hamiltonian $u_0 - \sqrt{2}u_8$ by an angle $2\theta_C$ about the 7th axis in $SU(3)$ space and imposing strangeness conservation on the result. (This is a natural extension of the fact that the strangeness-nonconserving weak Cabibbo current Eq. (2.14) can be thought of as arising from such a rotation of the strangeness-conserving weak current.) Thus, θ_C is directly related (Gell-Mann, 1969; Palmer, 1973) to the symmetry-breaking parameters in Eq. (5.27) or equivalently (Oakes, 1969a) to the pion mass itself. All these approaches yield values for the "tadpole strength" c_3 between $c_3 \simeq -0.02$ to -0.06, which is of order e^2 as it should be. The resulting predictions for $\eta \to 3\pi$ decays (Oakes, 1969b) are in reasonable agreement with experiment, and so are the electromagnetic mass differences [see, for example, Cabibbo and Maiani (1970b) and Gatto (1970)]. Similar conclusions have been reached by studying "u_3" terms in higher-dimensional representations of $SU(3) \times SU(3)$ (Genz, Katz, Ram Mohan, and Tatur, 1972; Dittner, Dondi, and Eliezer, 1972). It has also been pointed out by Wilson (1969) that, in the framework of short-distance operator-product expansions, the u_3 term in Eq. (5.27) occurs quite naturally in order to cancel divergences in the effective Lagrangian describing second-order radiative corrections. Very recently, model-independent tests based on pseudoscalar meson mass sum rules have been presented (Cicogna, Strocchi, and Vergara–Caffarelli, 1972, 1973) for the presence of the u_3 term in the Hamiltonian density, and its presence has been strongly supported by the experimental data. It therefore has become conceivable that the chiral symmetry-breaking Hamiltonian might have the form (5.27), rather than having a structure as in Eq. (5.8). Although including a u_3 tadpole in the Hamiltonian still remains an *ad hoc* assumption, it would clearly provide a beautiful and desirable interrelation of the strong, weak and electromagnetic interactions. Because of the smallness of the u_3 term in Eq. (5.27), such contributions to meson–nucleon σ terms are obviously negligible.

B. The $(6, \bar{6}) + (\bar{6}, 6)$ representation

Another alternative to account for the enhanced σ terms is, provided the observed enhancements with respect to the conventional $(3, \bar{3}) + (\bar{3}, 3)$ predictions can indeed be taken literally at the present stage, to attribute these larger effects to $I = 2$ contributions. In order to have isospin-two we require at least the 27-dimensional representation of $SU(3)$. The two smallest $SU(3) \times SU(3)$ representations containing this are $(8, 8)$, having decuplets and a 27-plet, and $(6, \bar{6}) + (\bar{6}, 6)$, having 27-plets, in addition to the singlets and octets as in a $(3, \bar{3}) + (\bar{3}, 3)$ representation. The $(6, \bar{6}) + (\bar{6}, 6)$ representation has been recently suggested by Auvil (1972) and Dittner, Dondi and Eliezer (1972a) and can be constructed in the same way as the $(3, \bar{3}) + (\bar{3}, 3)$ representation in the last section, Eqs. (5.1) through (5.6). The analogues to the even and odd parity tensors in Eq. (5.4) are now denoted by T_{ab}^+ and T_{ab}^- which obey the following $(6, \bar{6}) + (\bar{6}, 6)$ commutation relations (Auvil, 1972; Dittner, Dondi, and Eliezer, 1972a):

$$[F_a, T_{bc}^\pm] = if_{abd}T_{dc}^\pm + if_{acd}T_{bd}^\pm$$
$$[F_a^5, T_{bc}^\pm] = -(d_{abd}T^\mp_{dc} + d_{acd}T_{bd}^\mp + \delta_{ab}d_{cde}T_{de}^\mp + \delta_{ac}d_{bde}T_{de}^\mp) \quad (5.28)$$

where as usual the $a, b, c, \cdots$ indices go from 1 to 8, and

$$T_{ab}^\pm = T_{ba}^\pm.$$

The $SU(3)$ decomposition of $T_{ab}^\pm$ into irreducible parts [similarly to Eq. (5.5)] is

$$T_{ab}^\pm = T_{ab}^\pm(1) + T_{ab}^\pm(8) + T_{ab}^\pm(27) \quad (5.29)$$

where 1, 8, and 27 denote the dimensions of the $SU(3)$ representations, and

$$T_{ab}^\pm(1) = \tfrac{1}{8}\delta_{ab}T_{pp}^\pm$$
$$T_{ab}^\pm(8) = \tfrac{3}{5}d_{abc}d_{cpq}T_{pq}^\pm$$
$$T_{ab}^\pm(27) = T_{ab}^\pm - \tfrac{1}{8}\delta_{ab}T_{pp}^\pm - \tfrac{3}{5}d_{abc}d_{cpq}T_{pq}^\pm. \quad (5.30)$$

We are now in the position to write down the chiral symmetry-breaking Hamiltonian which, for the time being, should transform only as a $(6, \bar{6}) + (\bar{6}, 6)$ representation. The only parts of $T_{ab}^\pm$ which conserve isospin, hypercharge, and parity are $T_{88}^+(1)$, $T_{88}^+(8)$, and $T_{88}^+(27)$. Thus

$$\epsilon\mathcal{H}' = T_{88}^+(1) + \epsilon_8 T_{88}^+(8) + \epsilon_{27}T_{88}^+(27) \quad (5.31)$$

where, according to Eq. (5.30), we have

$$T_{88}^+(1) = \tfrac{1}{8}T_{pp}^+$$
$$T_{88}^+(8) = -(\sqrt{3}/5)d_{8pq}T_{pq}^+$$
$$T_{88}^+(27) = T_{88}^+ - \tfrac{1}{8}T_{pp}^+ + (\sqrt{3}/5)d_{8pq}T_{pq}^+. \quad (5.32)$$

In order to compare this model with experiment, we proceed in the usual way. First, the meson mass formula (2.23) fixes the free parameters in Eq. (5.31): Inserting Eq. (5.31) into Eq. (2.23) and making use of Eq. (5.28), we find (Dittner, Dondi, and Eliezer, 1972a)

$$m_\pi^2 f^2 = -(1/120)(25 - 14\epsilon_8 + 9\epsilon_{27})\langle 0 | T_{pp}^+ | 0\rangle,$$
$$m_K^2 f^2 = -(1/120)(25 + 7\epsilon_8 - 27\epsilon_{27})\langle 0 | T_{pp}^+ | 0\rangle,$$
$$m_\eta^2 f^2 = -(1/120)(25 + 14\epsilon_8 + 81\epsilon_{27})\langle 0 | T_{pp}^+ | 0\rangle, \quad (5.33)$$

which gives[6]

$$\epsilon_8 = \frac{20}{7}\frac{(-3m_\pi^2 + 2m_K^2 + m_\eta^2)}{(3m_\pi^2 + 4m_K^2 + m_\eta^2)},$$
$$\epsilon_{27} = \frac{5}{9}\frac{(m_\pi^2 - 4m_K^2 + 3m_\eta^2)}{(3m_\pi^2 + 4m_K^2 + m_\eta^2)},$$
$$\langle 0 | T_{pp}^+ | 0\rangle / f^2 = -\tfrac{3}{5}(3m_\pi^2 + 4m_K^2 + m_\eta^2). \quad (5.34)$$

[6] It should be emphasized that the determination of ϵ_{27} from Eq. (5.33) depends solely on a breakdown of the Gell-Mann–Okubo mass formula (5.10), i.e., on second-order effects of chiral symmetry breaking.

As expected, only a small 27 contribution, $\epsilon_{27} \simeq -0.03$, is required to fit the pseudoscalar meson masses, while the octet part is of the same order of magnitude, $\epsilon_8 \simeq 1.57$, as the singlet part. Thus, to a good approximation, the 27-plet component of $\mathcal{H}'$ in Eq. (5.31) can be neglected. Note that even for $\epsilon_{27} = 0$ one can have large $(I = 2)$ 27-plet contributions to $\pi\pi$ scattering lengths and meson-nucleon σ terms, say, since terms involving $T_{88}^+(1)$ and $T_{88}^+(8)$ when computed with F_a^5, generate 27-plet components in addition to singlets and octets.

Having determined the parameters of the model, we now can compare the predictions with experiment. For $\pi\pi$ scattering, the σ commutator in Eq. (2.30) can be calculated using Eq. (5.28), (5.31), and (5.34), and to lowest order in ϵ one obtains

$$\begin{aligned} A &= -(1/120f^4)(85 - 62\epsilon_8 + 117\epsilon_{27})\langle 0 \mid T_{pp}^+ \mid 0\rangle \\ &= (1/35f^2)(149m_\pi^2 - 48m_K^2 + 18m_\eta^2) \\ &\simeq -5m_\pi^2/f^2 \end{aligned} \quad (5.35)$$

to be compared with the $(3, \bar{3}) + (\bar{3}, 3)$ Weinberg prediction in Eq. (5.12). Using Eq. (2.29) one finds for the $\pi\pi$ s-wave scattering lengths

$$a_0^{(0)} \simeq -0.08m_\pi^{-1}, \qquad a_0^{(2)} \simeq -0.14m_\pi^{-1}. \quad (5.36)$$

Whereas $a_0^{(2)}$ is consistent with the experimental value (Petersen, 1971), the prediction for $a_0^{(0)}$ seems to be ruled out by the most recent experiments, Eq. (5.13).

The meson–nucleon σ terms cannot be uniquely determined since, even if $\epsilon_{27} = 0$ in Eq. (5.31), we get 27-plet contributions (Auvil, 1972) by commuting $\mathcal{H}'$ with F_a^5, which remain arbitrary, i.e., we do not have enough constraints to fix $\langle N \mid T_{88}^+(27) \mid N\rangle$, say, and thus account for a rather wide range of values of σ terms. To be more specific, let us study $\sigma_{NN}^{\pi\pi}$ in order to see what in general awaits us. The Hamiltonian (5.31) with the commutation relations (5.28) gives (Dittner, Dondi, and Eliezer, 1972a)

$$\begin{aligned} &\tfrac{1}{3}\sum_{\pi=1}^{3}[F_\pi^5, [F_\pi^5, \epsilon\mathcal{H}']] \\ &\quad = (4/45)(5 - 4\epsilon_8 + 9\epsilon_{27})\sum_{a=1}^{3} T_{aa}^+ \\ &\qquad + (1/60)(5 + 2\epsilon_8 - 27\epsilon_{27})\sum_{a=4}^{7} T_{aa}^+. \end{aligned} \quad (5.37)$$

Defining now the reduced matrix elements of the $(6, \bar{6}) + (\bar{6}, 6)$ tensors T_{pq}^+ between octet baryon states by using the Wigner–Eckart theorem as in Eq. (5.17),

$$\begin{aligned} \langle B_a \mid T_{pq}^+ \mid B_b\rangle &= \alpha\delta_{ab}\delta_{pq} + \beta(\delta_{ap}\delta_{bq} + \delta_{aq}\delta_{bp}) \\ &\quad + \gamma d_{pqc}d_{cab} + i\delta d_{pqc}f_{cab} \end{aligned} \quad (5.38)$$

and inserting this expression into the nucleon expectation value of Eq. (5.37), we obtain

$$\begin{aligned} \sigma_{NN}^{\pi\pi} &= \tfrac{1}{3}\sum_{\pi=1}^{3}\langle N \mid [F_\pi^5, [F_\pi^5, \epsilon\mathcal{H}']] \mid N\rangle \\ &= \frac{\alpha}{15}(25 - 14\epsilon_8 + 9\epsilon_{27}) \\ &\quad + \frac{\beta}{30}(5 + 2\epsilon_8 - 27\epsilon_{27}) \\ &\quad + \frac{\gamma - 3\delta}{180}(-35 + 34\epsilon_8 - 99\epsilon_{27}). \end{aligned} \quad (5.39)$$

The baryon masses are given by Eq. (5.16) which yields four equations, but now for the five unknowns $M_0, \alpha, \beta, \gamma, \delta$. This explains, as we just mentioned, the arbitrariness of the magnitude of the σ terms. Nonetheless, one can derive a relation similar to that in Eq. (5.20) by using Eq. (5.16) together with Eqs. (5.31) and (5.38) in order to eliminate the reduced matrix elements α, β, γ, and δ from Eq. (5.39), with the result

$$M_0 \simeq 550 - 5\sigma_{NN}^{\pi\pi}\ (\text{MeV}) \quad (5.40)$$

where use has been made of Eq. (5.34). Taking, according to Table I, again an average value of $\sigma_{NN}^{\pi\pi} \simeq 40$ MeV, say, we find that in the symmetry limit the degenerate average mass of the baryon octet is about $M_0 \simeq 350$ MeV. This implies that within this scheme the major part of the baryon mass comes from the $SU(3) \times SU(3)$ breaking, in contrast to Eq. (5.20), which makes the usefulness of the $(6, \bar{6}) + (\bar{6}, 6)$ symmetry-breaking scheme questionable [see also the discussion following Eq. (5.20)]. For larger values of $\sigma_{NN}^{\pi\pi}$ the situation only becomes worse. Of course, the same arguments apply to the kaon–nucleon sigma term σ_{NN}^{KK}.

Because of the somewhat unacceptable consequences of Eq. (5.40) and because of the disagreement of $a_0^{(0)}$ in Eq. (5.36) with the latest experimental results, a pure $(6, \bar{6}) + (\bar{6}, 6)$ chiral symmetry-breaking scheme seems not to be favored at least.

C. The (8, 8) representation

The smallest $SU(3) \times SU(3)$ representation containing isospin-two pieces is the 64-dimensional (8, 8) representation (Barnes and Isham, 1970; Genz and Katz, 1970, 1972; Brehm, 1971; Cornwell, Genz, Katz, and Steiner, 1973). This representation is spanned by 64 operators S_{ab}, with $a, b = 1, \cdots, 8$, which satisfy the following equal time commutation relations:

$$\begin{aligned} [F_a, S_{bc}] &= i(f_{abd}S_{dc} + f_{acd}S_{bd}) \\ [F_a^5, S_{bc}] &= i(-f_{abd}S_{dc} + f_{acd}S_{bd}). \end{aligned} \quad (5.41)$$

Under parity we have

$$PS_{ab}P^{-1} = S_{ba}.$$

The even and odd parity tensors S_{ab}^+ and S_{ab}^-, respectively, are defined by

$$S_{ab}^{\pm} = \tfrac{1}{2}(S_{ab} \pm S_{ba})$$

and their $SU(3)$ decomposition into irreducible parts may be written as

$$S_{ab}^{+} = \tfrac{1}{8}\delta_{ab}S^{+}(1) + d_{abc}S_c^{+}(8) + \zeta_{ab\theta}S_\theta^{+}(27)$$
$$S_{ab}^{-} = f_{abc}S_c^{-}(8) + \hat{S}_{ab}^{-} \qquad (5.42)$$

where we have used the same obvious notation as in the last section. The singlet and even and odd octets in Eq. (5.42) are given by

$$S^{+}(1) = S_{pp}, \qquad S_c^{+}(8) = \tfrac{3}{5}d_{cpq}S_{pq} \qquad (5.43a)$$
$$S_c^{-}(8) = \tfrac{1}{3}f_{cpq}S_{pq} \qquad (5.43b)$$

and similarly the even 27-plet

$$S_\theta^{+}(27) = \zeta_{\theta pq}S_{pq},$$

where $\zeta_{\theta ab}$ is a Clebsch–Gordan coefficient (Brehm, 1971) with $\theta = 1, 2, \cdots, 27$ and symmetric in ab. [Note that the θ summation in Eq. (5.42) goes from 1 to 27.] The 10- and $\overline{10}$-plets in Eq. (5.42) are contained in $\hat{S}_{ab}^{-}$ which we need not specify any further, except that $f_{cab}\hat{S}_{ab}^{-} = 0$ for all $c = 1, \cdots, 8$. From the requirement of parity and isospin conservation, the symmetry-breaking (Barnes-Isham) Hamiltonian takes the form (Barnes and Isham, 1970)

$$\epsilon\mathcal{H}' = z_0 + dz_8 \qquad (5.44)$$

where the singlet and octet operators are defined by

$$z_0 = (1/2\sqrt{2})\,S_{aa}, \qquad z_8 = (\tfrac{3}{5})^{1/2}d_{8ab}S_{ab}. \qquad (5.45)$$

Note that these operators can be written (Barnes and Isham, 1970) in the suggestive form of products of $SU(3) \times SU(3)$ vector and axial-vector currents as

$$S_{ab} = (V_a - A_a)^\mu(V_b + A_b)_\mu.$$

Again, the meson mass formula fixes the free parameter d of the model: Inserting Eq. (5.44) into Eq. (2.23) and using the commutation relations (5.41), we find to lowest order in chiral symmetry breaking

$$m_\pi^2f^2 = -\tfrac{3}{2}[1 + (\tfrac{2}{5})^{1/2}d]\langle 0 \mid z_0 \mid 0\rangle$$
$$m_K^2f^2 = -\tfrac{3}{2}[1 - (10)^{-1/2}d]\langle 0 \mid z_0 \mid 0\rangle$$
$$m_\eta^2f^2 = -\tfrac{3}{2}(1 - (\tfrac{2}{5})^{1/2}d)\,\langle 0 \mid z_0 \mid 0\rangle \qquad (5.46)$$

which yields

$$d = -(10)^{1/2}[(m_K^2 - m_\pi^2)/(2m_K^2 + m_\pi^2)], \qquad (5.47)$$

i.e., $d \simeq -1.4$, and we see that the singlet and octet contributions are of the same order of magnitude, as was to be expected. It should be emphasized that in the (8, 8) model, as well as in *any* higher-dimensional representation of $SU(3) \times SU(3)$ like the $(6, \bar{6}) + (\bar{6}, 6)$ model, the smallness of the pion mass does not imply that $SU(2) \times SU(2)$ is a better symmetry than $SU(3) \times SU(3)$ or $SU(3)$. This can easily be seen by calculating the divergence of the axial-vector current of the pion. Inserting Eq. (5.44) into Eq. (2.11b) and using Eqs. (5.41) and (5.42), we obtain

$$\partial_\mu A_\pi^\mu = \left(\frac{3}{\sqrt{2}} + \frac{3}{5^{1/2}}\,d\right)S_\pi^{-}(8)$$
$$+ \left(\frac{1}{\sqrt{2}} - \frac{1}{5^{1/2}}\,d + \frac{3}{5^{1/2}}\,d\sum_{i=1}^{3}\delta_{ib}\right)f_{\pi bc}\hat{S}_{bc}^{-} \qquad (5.48)$$

with $\pi = 1, 2, 3$, and where use has been made of the well known (Macfarlane, Sudbery, and Weisz, 1968) properties of the structure constants f_{abc} and d_{abc}. Equation (5.48) clearly shows that the smallness of the pion mass only ensures that the octet contribution to $\partial_\mu A_\pi^\mu$ is small: For vanishing pion mass, i.e., $d = -(5/2)^{1/2}$ in Eq. (5.46), only the first term in Eq. (5.48) vanishes, whereas the coefficient of the second one (the contributions from the 10- and $\overline{10}$-plets) remains finite. Thus, in the (8, 8) model the smallness of the pion mass is just accidental, insofar as it is not related to a particularly good subsymmetry. The same applies to the $(6, \bar{6}) + (\bar{6}, 6)$ representation, where $\partial_\mu A_\pi^\mu$ has singlet- and 27-parts in addition to the octet contribution. It is *only* in the $(3, \bar{3}) + (\bar{3}, 3)$ model that the smallness of m_π implies that $SU(2) \times SU(2)$ is a much better symmetry than $SU(3) \times SU(3)$ or $SU(3)$ itself: $\partial_\mu A_\pi^\mu$ in Eq. (5.14) indeed vanishes for $c = -\sqrt{2}$, i.e., $m_\pi = 0$ in Eq. (5.9).

Together with Eqs. (2.29) and (2.30), it is now straightforward to calculate the $\pi\pi$ scattering lengths: Equation (2.30) yields (to lowest order in ϵ)

$$A = -\frac{9}{2f^4}\left(1 + \frac{(10)^{1/2}}{3}\,d\right)\langle 0 \mid z_0 \mid 0\rangle$$
$$\simeq -12.5m_\pi^2/f^2 \qquad (5.49)$$

and thus

$$a_0^{(0)} \simeq -0.36m_\pi^{-1}, \qquad a_0^{(2)} \simeq -0.25m_\pi^{-1}, \qquad (5.50)$$

in striking disagreement with experiment [Eq. (5.13) and discussion thereafter]. Note that the $(6, \bar{6}) + (\bar{6}, 6)$ predictions, Eq. (5.36), are appreciably closer to the experimental data than Eq. (5.50). The main reason for the (8, 8) model's failure to describe the observed $\pi\pi$ data is that the $I = 2$ component of the σ commutator turns out to be too large in an (8, 8) representation. This is the general result of a thorough analysis of $\pi\pi$ scattering by Brehm (1971), which strongly favors the $(3, \bar{3}) + (\bar{3}, 3)$ model over the (8, 8) or more complicated representations.

As in the $(6, \bar{6}) + (\bar{6}, 6)$ model, the meson–nucleon σ terms cannot be uniquely determined because of the unknown nucleon expectation values of the various multiplet operators. However, we again can derive a relation as in (5.40) between the degenerate baryon mass and the πN σ term, say. Calculating Eq. (2.38) from Eqs. (5.41) and (5.44) and using an $SU(3)$ decomposition for the baryon matrix element of S_{ab}^{+} in Eq. (5.42),

$$\langle B_a \mid S_{pq}^{+} \mid B_b \rangle = \alpha' \delta_{ab}\delta_{pq} + \beta' d_{abc}d_{cpq} + i\gamma' d_{pqc} f_{cab} + \delta' \zeta_{\theta ab}\zeta_{\theta pq} \qquad (5.51)$$

we obtain

$$M_0 \simeq -200 - 6\sigma_{NN}^{\pi\pi} \text{ (MeV)}, \qquad (5.52)$$

where we used the four equations coming from Eq. (5.16), together with Eq. (5.51), in order to eliminate the reduced matrix elements α', β', γ' and δ', and d as given in Eq. (5.47). Note that the θ sum of the two 27-plet Clebsch–Gordan coefficients is given by (Macfarlane, Sudbery, and Weisz, 1968; Brehm, 1971)

$$\zeta_{\theta ab}\zeta_{\theta pq} = \tfrac{1}{2}(\delta_{ap}\delta_{bq} + \delta_{aq}\delta_{bp}) - \tfrac{1}{8}\delta_{ab}\delta_{pq} - \tfrac{3}{5}d_{abc}d_{cpq}.$$

With respect to what we said before about the general concept of symmetries and symmetry breaking, Eq. (5.52) is clearly in much worse shape than its $(6,\bar{6}) + (\bar{6},6)$ analog Eq. (5.40): Using $\sigma_{NN}^{\pi\pi} \simeq 40$ MeV, Table I, Eq. (5.52) yields $M_0 \simeq -440$ MeV. Thus we get a negative (!) degenerate baryon mass in the exact symmetry limit, and hence the symmetry breaking has to compensate for this negative average mass and generate, in addition, the entire (physical) baryon mass spectrum—a situation which is obviously not acceptable. From this point of view $(3,\bar{3}) + (\bar{3},3)$ is the preferable choice, but certainly $(6,\bar{6}) + (\bar{6},6)$ is better than $(8,8)$.

Since, at the present stage, sigma terms cannot be predicted uniquely in this model, one could use Eq. (5.51) and parametrize all the meson–baryon σ in terms of reduced matrix elements, and fit these free parameters to the available "experimental" estimates of meson–baryon σ terms. This has indeed been done (Genz, Katz, and Steiner, 1972) and the agreement with estimated sigma terms was found to be good. Because of the many free parameters and relatively few estimated meson–baryon σ terms available, such a result is not very surprising and cannot be seriously used to test a model.

The $(8,8)$ model has also been studied in connection with K_{l3} decays (Genz and Katz, 1972; Ali and Razmi, 1973) but, although the experimental data are too scarce to draw definite conclusions, the $(8,8)$ breaking scheme seems not to be favored.

To summarize, we can say that present experiments together with the unphysical implications of the $(8,8)$ model rule out a chiral symmetry-breaking Hamiltonian which has only $(8,8)$ transformation properties.

D. The (1, 8) + (8, 1) model

For completeness we should also mention the $(1,8) + (8,1)$ representation of $SU(3) \times SU(3)$, which is spanned by scalar and pseudoscalar operators g_a and h_a, respectively, transforming as

$$[F_a, g_b] = if_{abc}g_c, \qquad [F_a, h_b] = if_{abc}h_c$$
$$[F_a^5, g_b] = if_{abc}h_c, \qquad [F_a^5, h_b] = if_{abc}g_c. \qquad (5.53)$$

The symmetry-breaking Hamiltonian in this scheme is given by

$$\epsilon\mathcal{H}' = g_8 \qquad (5.54)$$

which, by Eq. (5.53), implies that $SU(2) \times SU(2)$ is always conserved since $[F_\pi, g_8] = [F_\pi^5, g_8] = 0$ for $\pi = 1, 2, 3$. Originally it was argued by Gell-Mann (1964) and Gell-Mann, Oakes, and Renner (1968) that the $(1,8) + (8,1)$ representation cannot play a dominant role in chiral symmetry-breaking mechanisms, and at most could serve as a possible admixture in other symmetry-breaking schemes like the $(3,\bar{3}) + (\bar{3},3)$ model. Their arguments were based on the fact that Eq. (5.54) vanishes in the PCAC approximation if $SU(3)$ symmetry is applied to single-particle matrix elements of g_a (Gell-Mann, Oakes, and Renner, 1968). This latter assumption might be questionable since the violation of $SU(3)$ could be as large as that of $SU(3) \times SU(3)$.

However, recent studies of meson–baryon σ terms have shown (Gensini, 1971a, b, c; Kleinert, Steiner, and Weisz, 1971) that a pure $(1,8) + (8,1)$ breaking can be definitely ruled out despite the uncertainties of the data. This is not surprising when we realize that, since $SU(2) \times SU(2)$ is an exact symmetry in a pure $(1,8) + (8,1)$ breaking model, the σ terms for elastic scattering of pions on any target vanish identically, i.e., $\sigma_{NN}^{\pi\pi} = \sigma_{\Sigma\Sigma}^{\pi\pi} = \cdots = 0$. Table I clearly shows, for example, that $\sigma_{NN}^{\pi\pi} = 0$ is in violent disagreement with the data.

So far we have seen that, in addition to a pure $(1,8) + (8,1)$ breaking, higher-dimensional representations of $SU(3) \times SU(3)$ than the $(3,\bar{3}) + (\bar{3},3)$ appear to be ruled out as a dominant transformation property of the chiral symmetry-breaking Hamiltonian. Whereas the $(8,8)$ model, together with its unphysical implications, seems to be totally incompatible with experiment, the inconsistency of the $(6,\bar{6}) + (\bar{6},6)$ representation appears not to be so violent. However, as we shall see later, very recently it has been shown that any triangular representation $(X,\bar{X}) + (\bar{X},X)$ with $X \neq 3$ is inconsistent with the present meson–baryon data, if nonlinear effective Lagrangians are used with chiral symmetry-breaking components contained in a *single* representation. Furthermore we have seen that octet dominance plays the most important role in symmetry-breaking Hamiltonians, and small $I = 2$ components might be required, as suggested by enhanced σ terms, strongly negative values of the $\xi(0)$ parameter in K_{l3} decays, or $\pi\pi$ s-wave scattering lengths appreciably deviating from the Weinberg prediction. A natural (although not very elegant) way to account for such effects would be to work more closely to the $(3,\bar{3}) + (\bar{3},3)$ GMOR model but to introduce a (small) breaking of octet dominance. These are then models where the symmetry-breaking Hamiltonian has mixed transformation properties and we will turn to their discussion now.

E. The $(3, \bar{3}) + (\bar{3}, 3) + (8, 8)$ model

A symmetry-breaking scheme where the Hamiltonian consists of a dominant $(3, \bar{3}) + (\bar{3}, 3)$ representation in addition to an $(8, 8)$ admixture has been recently suggested by Brehm (1971, 1972) and Sirlin and Weinstein (1972). Such mixed symmetry-breaking models can explain virtually all presently available data, in view of the large numbers of free parameters. In order to see this, let us consider the following Hamiltonian (Sirlin and Weinstein, 1972)

$$\epsilon\mathcal{H}' = u_0 + cu_8 + d_0z_0 + d_8z_8, \tag{5.55}$$

where u_0 and u_8 are the familiar even-parity $(3, \bar{3}) + (\bar{3}, 3)$ operators, defined in Sec. VA, and the $(8, 8)$ operators z_0 and z_8 are given in Eq. (5.45). As in Eq. (5.44) we did not include a 27-plet term in Eq. (5.55), even though it occurs in the decomposition (5.42), simply because the accuracy of the $SU(3)$ mass formulae implies its coefficient to be small, as we have seen in the $(6, \bar{6}) + (\bar{6}, 6)$ model. Since we already calculated the pseudoscalar meson masses for each model separately, we simply obtain from Eqs. (5.9) and (5.46)

$$\begin{aligned} m_\pi^2f^2 &= -\frac{2}{3}\left(1+\frac{c}{\sqrt{2}}\right)\langle 0 \mid u_0 \mid 0\rangle - \frac{3}{2}\left[d_0 + \left(\frac{2}{5}\right)^{1/2} d_8\right] \\ &\quad \times \langle 0 \mid z_0 \mid 0\rangle, \\ m_K^2f^2 &= -\frac{2}{3}\left(1-\frac{c}{2\sqrt{2}}\right)\langle 0 \mid u_0 \mid 0\rangle - \frac{3}{2}\left[d_0 - \frac{1}{(10)^{1/2}} d_8\right] \\ &\quad \times \langle 0 \mid z_0 \mid 0\rangle, \\ m_\eta^2f^2 &= -\frac{2}{3}\left(1-\frac{c}{\sqrt{2}}\right)\langle 0 \mid u_0 \mid 0\rangle - \frac{3}{2}\left[d_0 - \left(\frac{2}{5}\right)^{1/2} d_8\right] \\ &\quad \times \langle 0 \mid z_0 \mid 0\rangle. \end{aligned} \tag{5.56}$$

Since m_π^2, m_K^2, and m_η^2 satisfy the Gell-Mann–Okubo sum rule (5.10), Eqs. (5.56) are really two equations in four unknowns. To find a further equation we could use Eq. (2.30) and find, according to Eq. (5.55), from (5.12) and (5.49) the following expression for A:

$$A = -f^{-4}\left[\frac{2}{3}\left(1+\frac{c}{\sqrt{2}}\right)\langle 0 \mid u_0 \mid 0\rangle + \frac{9}{2}\left(d_0 + \frac{(10)^{1/2}}{3} d_8\right) \times \langle 0 \mid z_0 \mid 0\rangle\right]. \tag{5.57}$$

Thus, Eqs. (5.56) and (5.57) still provide only three independent equations in the four unknowns c, d_8/d_0, $\langle 0 \mid u_0 \mid 0\rangle$, and $d_0\langle 0 \mid z_0 \mid 0\rangle$, and so, even with $a_0^{(0)}$ fixed, they are unconstrained. In order to constrain the theory let us consider the rather appealing model where $c = -\sqrt{2}$, i.e., the $(3, \bar{3}) + (\bar{3}, 3)$ part of $\mathcal{H}'$ in Eq. (5.55) leaves $SU(2) \times SU(2)$ exact, and so all of the $SU(2) \times SU(2)$ breaking (finite pion mass) is done by the $(8, 8)$ terms. Solving Eqs. (5.56) and (5.57) one finds (Sirlin and Weinstein, 1972)

$$D_0 \equiv d_0\frac{\langle 0 \mid z_0 \mid 0\rangle}{\langle 0 \mid u_0 \mid 0\rangle} = \frac{1}{12}\frac{(10-\rho)}{(4.6+0.19\rho)} \tag{5.58a}$$

$$\frac{d_8}{d_0} = \frac{(10)^{1/2}}{2}\frac{(6-\rho)}{(\rho-10)}, \tag{5.58b}$$

with $\rho = 2Af^2/m_\pi^2$. Note that $\rho = 2$ corresponds to the Weinberg prediction (5.12). From Eq. (5.58a) we see that in the range

$$0.16m_\pi^{-1} \leq a_0^{(0)} \leq 0.5m_\pi^{-1} \qquad \text{for } 2 \leq \rho \leq 20, \tag{5.59}$$

D_0 remains less than 0.13, which means that the $(8, 8)$ part contributes at most about 10% to the dominating $(3, \bar{3}) + (\bar{3}, 3)$ component. Hence, by admixing a small amount of $(8, 8)$ breaking into the GMOR model, one can obtain a wide range of values for the $\pi\pi$ scattering lengths, which are entirely consistent with the present experimental results (5.13), and still having $SU(2) \times SU(2)$ a much better symmetry than $SU(3) \times SU(3)$. As we have seen above, this is in contrast to the situation in which one has pure $(8, 8)$ or $(6, \bar{6}) + (\bar{6}, 6)$ breaking. However, because of the large number of free parameters, a mixed symmetry-breaking model is rather ambiguous and unconstrained and therefore can only explain (or fit) experimental results but not predict them.

Even more ambiguous is the situation for σ terms; as an example let us consider the πN sigma term. Again, in order to constrain the theory, we assume $c = -\sqrt{2}$ in (5.55) which, after all, is close to its actual value in Eq. (5.11). Thus, the $(3, \bar{3}) + (\bar{3}, 3)$ contribution to $\sigma_{NN}^{\pi\pi}$, as given by Eq. (5.15), vanishes; however, this is not true for the $(8, 8)$ contribution. The terms involving z_0 and z_8 in Eq. (5.55) generate, when commuted with F_π^5, $\pi = 1, 2, 3$, three terms transforming as z_0, z_8 and the $I = Y = 0$ component of the 27-dimensional representation of $SU(3)$. Inserting the $(8, 8)$ term $d_0z_0 + d_8z_8$ of Eq. (5.55) into Eq. (2.38) and calculating the double commutator with the help of Eq. (5.41), we obtain for the singlet and octet contributions to $\sigma_{NN}^{\pi\pi}$ the following expressions:

$$\begin{aligned} (\sigma_{NN}^{\pi\pi})_{\text{singlet}} &= \left(\frac{3}{2}d_0 + \frac{3}{(10)^{1/2}}d_8\right)\langle N \mid z_0 \mid N\rangle \\ (\sigma_{NN}^{\pi\pi})_{\text{octet}} &= \left(\frac{3}{(10)^{1/2}}d_0 + \frac{9}{5}d_8\right)\langle N \mid z_8 \mid N\rangle, \end{aligned} \tag{5.60}$$

where we used the $SU(3)$ decomposition for S_{ab}^+ in (5.42) and the well known properties (Macfarlane, Sudbery, and Weisz, 1968) of the structure constants f_{abc} and d_{abc}. Thus, even neglecting the 27-plet contributions which hopefully are small, Eq. (5.60) shows that $\sigma_{NN}^{\pi\pi}$ still cannot be unambiguously computed, since the nucleon matrix elements of z_0 and z_8 are unknown. However, an order-of-magnitude estimate may be obtained if we make the further assumption that $|\langle N \mid z_0 \mid N\rangle| \simeq |\langle N \mid z_8 \mid N\rangle| \simeq |\langle N \mid u_8 \mid N\rangle|$. Using Eq. (5.18) and $|d_0| \simeq 0.1$, we see that a value of $\sigma_{NN}^{\pi\pi}$ in the range 10 to 100 MeV can be accommodated in this theory without requiring a large value of $\langle N \mid u_0 \mid N\rangle$. [In fact, since we use $c = -\sqrt{2}$, the term $\langle N \mid u_0 \mid N\rangle$ is completely decoupled from this calculation because the $(3, \bar{3}) + (\bar{3}, 3)$ part of the Hamiltonian does not contribute to $\sigma_{NN}^{\pi\pi}$.] Hence, even for a πN σ term of about 50 MeV one can maintain the physically very pleasant situation in which only a small fraction of the average baryon octet mass is due to $SU(3) \times SU(3)$ breaking, i.e., a small $\langle N \mid u_0 \mid N\rangle$. Otherwise the familiar Goldberger–Treiman relations fails (Dashen and Weinstein, 1969b; Dashen, 1971b) to lowest order in $SU(3) \times SU(3)$, but remains good to lowest order in $SU(2) \times SU(2)$. It is even possible to set $d_8/d_0 = -(5/2)^{1/2}$, so that $m_\pi = 0$ to first order in ϵ, Eq. (5.56), and still obtain an order-of-magnitude estimate for $\sigma_{NN}{}^{\pi\pi}$ of about 50–70 MeV.

Therefore it becomes clear that a mixed symmetry-breaking model, as long as it is dominated by $(3, \bar{3}) + (\bar{3}, 3)$ transformation properties, can account for and explain all presently available data because of the large number of free parameters. Considering only elastic πN and kaon–nucleon scattering, such models are rather unconstrained and ambiguous and therefore have practically no predictive power; it remains doubtful whether more accurate future elastic scattering experiments will be able either to confirm or reject them. The same situation holds, for example, for $(6, \bar{6}) + (\bar{6}, 6)$ admixtures to the GMOR model. In fact, the possibilities for constructing models of this kind are virtually limitless, provided the $(3, \bar{3}) + (\bar{3}, 3)$ representation plays the dominant role in the chiral symmetry-breaking Hamiltonian.

In general, mixed symmetry models can be constrained by not limiting oneself to elastic scattering processes only. For example, as pointed out by Glashow and Weinberg (1968) and Dashen and Weinstein (1969a, b), if one also studies the sum rule for the deviations from the generalized GT relations, the relevant low-energy theorems for $\pi N \rightarrow \pi\pi N$ and pion-photoproduction processes, then one can, for instance, overconstrain the $(3, \bar{3}) + (\bar{3}, 3) + (8, 8)$ model. At the present experimental stage, however, such additional constraints (unfortunately) appear to be rather academic.

F. The $(3, \bar{3}) + (\bar{3}, 3) + (1, 8) + (8, 1)$ model

This model, described by a symmetry-breaking Hamiltonian of the form

$$\epsilon \mathcal{H}' = u_0 + c u_8 + \delta_8 g_8, \tag{5.61}$$

where g_8 is defined by Eq. (5.53), was originally suggested (Arnowitt, Friedman, Nath, and Suitor, 1971; Schilcher, 1971) in connection with K_{l3} decays in order to account for large negative values of $\xi(0)$ defined by Eq. (2.47). The additional $(1, 8) + (8, 1)$ term $\delta_8 g_8$ in Eq. (5.61) has the attractive feature of not contributing to $SU(2) \times SU(2)$ breaking and thus leaves unchanged the Weinberg predictions (Weinberg, 1966b) of $\pi\pi$ scattering lengths which, as we already discussed, are in agreement with recent experimental results. Because of the number of free parameters, this model is again not fully constrained by the pseudoscalar meson masses, and a further constraint can be found by fitting $\xi(0)$, say. This parameter then becomes essentially a function of (Schilcher, 1971) $\delta_8 \langle 0 \mid g_8 \mid 0 \rangle / c \langle 0 \mid u_8 \mid 0 \rangle$ and $(1 + \sqrt{2}/c)$, which are related by spectral sum rules where the spectral functions can be approximately determined using meson-pole dominance. Despite the presently confused experimental situation (Chounet, Gaillard, and Gaillard, 1972; Wojcicki, 1972) which by now appears to favor a small value (Donaldson *et al.*, 1973) for $\xi(0)$, this model is still rather controversial (Arnowitt, Friedman, Nath, and Suitor, 1971; Schilcher, 1971; Khelashvili, 1972) in explaining the K_{l3} data—for example the possible strong violation of an $SU(3)$-symmetric vacuum and renormalization constants. Furthermore, as discussed at the end of Sec. VA, it is not clear whether an additional contribution to the $(3, \bar{3}) + (\bar{3}, 3)$ transforming part is indeed required in order to explain the measured value of $\xi(0)$.

On the other hand, the methods of Arnowitt, Friedman, Nath, and Suitor (1971) were believed to be independent of the details of symmetry breaking. For example, in this treatment the meson-mass spectrum is fed in *ab initio* rather than related to the form of the symmetry breaking, and symmetry breaking in the vacuum states does not appear in the formalism. Various authors accounted for these problems either by using (Barker, 1972) the Lagrangian model of Schechter and Ueda (1971) for the scalar and pseudoscalar mesons to calculate the above effects explicitly, or by employing the generalized σ model with and without massive gauge fields (Uchida and Suzuki, 1973). Both calculations resulted in the conclusion that a $(1, 8) + (8, 1)$ symmetry breaking large enough to accommodate the K_{l3} data would lead to unacceptable consequences for the meson mass spectrum, and that the generalized σ model gives too smooth an off-shell extrapolation of the amplitudes to fully account for the present experimental values of λ_+ and $\xi(0)$.

With respect to the enhanced πN σ terms of Table I, the Hamiltonian in Eq. (5.61) is certainly not a desirable choice, simply because $(1, 8) + (8, 1)$ conserves $SU(2) \times SU(2)$ and thus the σ commutator of g_8 vanishes, leaving us with the $(3, \bar{3}) + (\bar{3}, 3)$ prediction of Eq. (5.19a). However, the term $\delta_8 g_8$ can influence the size of $\sigma_{NN}{}^{\pi\pi}$ via the baryon mass formula [Eq. (5.16)] which we used to estimate $\langle N \mid u_8 \mid N \rangle$. Since we have octet dominance of the $(3, \bar{3}) + (\bar{3}, 3)$ term in Eq. (5.61), the contribution of $\delta_8 g_8$ to nucleon matrix elements of u_a is expected to be small. To see this, one can derive relations between σ terms and δ_8. (Note that this model cannot give a definite prediction for $\sigma_{NN}{}^{\pi\pi}$, say, since δ_8 remains unconstrained.) Making, in order to constrain the theory, the assumption that chiral symmetry breaking does not strongly influence the nucleon mass (Gell-Mann, 1969), i.e.,

$$\langle N \mid u_0 + c u_8 + \delta_8 g_8 \mid N \rangle \simeq 0,$$

and using Eqs. (5.16) and (5.17) together with

$$\langle B_a \mid g_b \mid B_c \rangle = i f_{abc} F' + d_{abc} D', \tag{5.62}$$

one obtains from Eq. (5.15) (Khelashvili, 1972)

$$\sigma_{NN}{}^{\pi\pi} \simeq 25 - 0.13 \delta_8 D' \text{ (MeV)},$$

$$\sigma_{NN}{}^{KK} \simeq 180 - 2 \delta_8 D' \text{ (MeV)}.$$

This implies that for $\sigma_{NN}{}^{KK} \simeq 350$ MeV we get $\sigma_{NN}{}^{\pi\pi} \simeq 36$ MeV, which as expected is very similar to a pure $(3, \bar{3}) + (\bar{3}, 3)$ model where $\sigma_{NN}{}^{KK} \simeq 350$ MeV implies $\sigma_{NN}{}^{\pi\pi} \simeq 40$ MeV.

Although the $(3, \bar{3}) + (\bar{3}, 3) + (1, 8) + (8, 1)$ model can neither be confirmed nor ruled out by present experimental data, the mixed $(3, \bar{3}) + (\bar{3}, 3) + (8, 8)$ model appears to be in somewhat better shape, provided mixed symmetry-breaking schemes are to be used at all.

G. Implications from effective Lagrangians

The importance of effective Lagrangians and field algebras as means of treating chiral symmetry and PCAC has been widely discussed in the literature.[7] Here, we will briefly discuss a recent unified approach, proposed by Rosen and McDonald (1971) and McDonald and Rosen (1972, 1973), which is based on nonlinear Lagrangians consisting of $SU(3)$ singlet and octet symmetry-breaking components of a single representation $(X,\bar{X})+(\bar{X},X)$ of $SU(3)\times SU(3)$. These authors have shown how to construct meson–meson and meson–baryon Lagrangians from pseudoscalar meson octets transforming nonlinearly under $SU(3)\times SU(3)$, by relating these nonlinear representations to linear representations $(X,\bar{X})+(\bar{X},X)$ and where all the symmetry breaking occurs in the mass term belonging to $(X,\bar{X})+\bar{X},X)$. In this way one obtains a unified dynamical description of meson–meson and meson–baryon scattering, where the various symmetry-breaking components, i.e., various breaking schemes of the form $(X,\bar{X})+(\bar{X},X)$ as considered so far, are closely intertwined and therefore can be checked against each other on an equal footing, contrary to what we did previously. To be more explicit, under the assumption that the symmetry-breaking meson–meson Lagrangian $\mathfrak{L}_{MM}'((X,\bar{X})+(\bar{X},X))$ is dominated by singlet and octet contributions, which ensures that the Gell-Mann–Okubo mass formula will be satisfied, one obtains for the nonlinear representations of $SU(3)\times SU(3)$ up to second order in the meson fields (Rosen and McDonald, 1971)

$$\begin{aligned}\mathfrak{L}_{MM}'((X,\bar{X})+(\bar{X},X))\\ &= -\tfrac{1}{2}c_0\pi_a\pi_a + d_{8ab}\pi_a\pi_b\\ &\times\left[\frac{3x_3}{5x_2}\left(\frac{3}{2}\right)^{1/2}c_8-\frac{3(2x_2+3)}{50}\left(\frac{3}{2}\right)^{1/2}\tilde{c}_8\right],\end{aligned}\qquad(5.63)$$

where the parameter c_0 describes the strength of a singlet in an $(X,\bar{X})$ representation of $SU(3)\times SU(3)$, and c_8 and $\tilde{c}_8$ denote the relative strengths of the two octets in $(X,\bar{X})$. [Note, the normalization in Eq. (5.63) is such that in the $(3,\bar{3})+(\bar{3},3)$ model one recovers $c=c_8/c_0$ given by Eq. (5.11).] The octet of pseudoscalar mesons in Eq. (5.63) is denoted by π_a, $a=1,\cdots,8$, and x_2 and x_3 are the eigenvalues of the quadratic and cubic Casimir operators:

$$\begin{aligned}x_2 &= \tfrac{2}{3}(\mu_1^2+\mu_2^2+\mu_1\mu_2+3\mu_1+3\mu_2)\\ x_3 &= \tfrac{1}{9}(\mu_2-\mu_1)[(\mu_1+2\mu_2)(\mu_2+2\mu_1)\\ &\quad+9(\mu_1+\mu_2+1)]\end{aligned}\qquad(5.64)$$

where the representation (X) of $SU(3)$ and its conjugate $(\bar{X})$ is described by

$$(X)\equiv(\mu_1,\mu_2),\qquad(\bar{X})\equiv(\mu_2,\mu_1)\qquad(5.65)$$

and the two integers (μ_1,μ_2) represent the number of quark and antiquark indices, respectively. The dimension $\mathfrak{D}$ of (X) is

$$\mathfrak{D}(\mu_1,\mu_2)=\tfrac{1}{2}(\mu_1+1)(\mu_2+1)(\mu_1+\mu_2+2)\qquad(5.66)$$

and μ_1 and μ_2 can take all positive integer values. Therefore, in the case of triangular representations $\mu_2=0$, the two octets in $(X,\bar{X})$ become proportional (Rosen and McDonald, 1971), and so we can take $\tilde{c}_8=0$ in Eq. (5.63). For example, the lowest-dimensional triangular representations are then given by

$$\begin{aligned}&(3,\bar{3})+(\bar{3},3): &&\mu_1=1,\quad\mu_2=0\\ &(6,\bar{6})+(\bar{6},6): &&\mu_1=2,\quad\mu_2=0\\ &(10,\overline{10})+(\overline{10}+10): &&\mu_1=3,\quad\mu_2=0\end{aligned}\qquad(5.67)$$

whereas the self-adjoint $(8,8)$ representation is described by $\mu_1=\mu_2=1$. From Eq. (5.63) one can immediately read off the following relationship between the pseudoscalar meson masses and c_0, c_8 and $\tilde{c}_8$:

$$\frac{6x_3}{5x_2}\frac{c_8}{\sqrt{2}c_0}-\frac{3(2x_2+3)}{25}\frac{\tilde{c}_8}{\sqrt{2}c_0}=2\frac{m_K^2-m_\pi^2}{2m_K^2+m_\pi^2}.\qquad(5.68)$$

For triangular and self-adjoint representations the theory is fully constrained since $\tilde{c}_8=0$ and $x_3=0$ (since $\mu_1=\mu_2$), respectively. The Lagrangian in (5.63) can therefore be tested unambiguously for different chiral symmetry-breaking representations $(X,\bar{X})+(\bar{X},X)$. Using Eq. (5.63) it is straightforward to calculate the various meson–meson scattering lengths, and the main conclusions for $\pi\pi$ scattering lengths are the following: For self-adjoint representations one finds for the $(8,8)$ that $a_0^{(0)}$ and $a_0^{(2)}$ are both negative and of comparable magnitudes; for higher representations, $(27,27)$, $(64,64)$ etc., the ratio $a_0^{(2)}/a_0^{(0)}$ tends to 0.4. These predictions are inconsistent with recent data, Eq. (5.13) for example, and thus if we always want $a_0^{(0)}$ to be positive and much larger than $|a_0^{(2)}|$ we must rule out all symmetry breaking which is described by (dominant) self-adjoint representations. In the case of triangular representations, the $(3,\bar{3})+(\bar{3},3)$ predictions $a_0^{(0)}\simeq0.15m_\pi^{-1}$ and $a_0^{(2)}\simeq-0.04m_\pi^{-1}$ are in agreement with experiment, whereas $(6,\bar{6})+(\bar{6},6)$ predicts $a_0^{(2)}$ to be much larger than $a_0^{(0)}$ in magnitude, and the disagreement with the data becomes even worse for the higher-dimensional representations. Although present experimental results on $\pi\pi$ scattering include large uncertainties, one obtains fairly strong restrictions on the manner of $SU(3)\times SU(3)$ breaking and again we find it has to have (dominant) $(3,\bar{3})+(\bar{3},3)$ transformation properties.

Similarly one can construct meson–baryon Lagrangians with well defined chiral transformation properties, under the usual assumption that both the baryon kinetic and meson–baryon interaction terms are invariant under chiral transformations, and the symmetry is broken only by mass terms belonging to an $(X,\bar{X})$ representation of $SU(3)\times SU(3)$. Expanding these terms up to second order in the meson fields, the symmetry-breaking meson–baryon Lagrangian belonging to triangular representations $(\tilde{c}_8=0)$ can

[7] See, for example, Weinberg (1966a, 1968a, b) Dashen and Weinstein (1969a), Gasiorowicz and Geffen (1969), Schechter and Ueda (1971), and Dondi and Eliezer (1973).

TABLE III. Predictions for meson–nucleon σ terms from effective Lagrangians, according to McDonald and Rosen (1963)

I M_0 (MeV)	II Expt.	III 940 $(3,\bar{3})+(\bar{3},3)$	IV 940 $(6,\bar{6})+(\bar{6},6)$	V 940 $(10,\overline{10})+(\overline{10},10)$	VI 645 $(3,\bar{3})+(\bar{3},3)$	VII 645 $(6,\bar{6})+(\bar{6},6)$	VIII 300 $(3,\bar{3})+(\bar{3},3)$	IX 300 $(6,\bar{6})+(\bar{6},6)$
$\sigma_{NN}^{\pi\pi}$(MeV)	50	26	−40	−126	(50)	20	78	89
σ_{NN}^{KK} (MeV)	350	148	250	386	431	958	762	1786

be written as (McDonald and Rosen, 1972)

$$\begin{aligned}
-\mathcal{L}_{MB}'((X,\bar{X})+(\bar{X},X))
&= c_0\Bigg[\bar{\Psi}\Psi\left(1-\frac{x_2}{8f^2}\pi_a\pi_a\right) \\
&+\left(\frac{2}{3}\right)^{1/2}\frac{3x_3}{5x_2f^2}\bar{\Psi}(F_af'+D_ad')\Psi d_{abc}\pi_b\pi_c\Bigg] \\
&+\left(\frac{3}{2}\right)^{1/2}c_8\Bigg\{\frac{3x_3}{20f^2}\bar{\Psi}\Psi d_{8bc}\pi_b\pi_c \\
&+\left(\frac{2}{3}\right)^{1/2}\bar{\Psi}(F_af'+D_ad')\Psi \\
&\times\Bigg[\delta_{8a}+\frac{\delta_{8a}}{f^2}\left(\frac{11}{60}-\frac{x_2}{10}\right)\pi_c\pi_c+\frac{1}{4f^2}d_{8ab}d_{bcd}\pi_c\pi_d \\
&+\frac{1}{10f^2}\left(\tfrac{1}{3}-2x_2\right)\pi_8\pi_a\Bigg]\Bigg\}
\end{aligned} \tag{5.69}$$

where Ψ represents the octet of baryons, and f is our common meson decay constant, $f\simeq 96$ MeV. The quantities F_a are the usual $SU(3)$ generators which are now, regardless of current algebra, represented by $F_a=\frac{1}{2}\lambda_a$ satisfying, of course, Eq. (2.3a). Similarly we have $D_a=(2/3)d_{abc}F_bF_c$. [Note that the combination $F_af'+D_ad'$ in Eq. (5.69) results from constructing baryon–baryon–meson $SU(3)$ couplings.] The contributions from the Lagrangian in (5.69) to the baryon masses, along with the contribution from the kinetic energy term, which describes the average octet mass M_0, are given by

$$\begin{aligned}
c_8f' &= (1/3)^{1/2}(M_N-M_\Xi), \\
c_8d' &= (\sqrt{3}/2)(M_\Sigma-M_\Lambda), \qquad c_0+M_0=\tfrac{1}{2}(M_\Sigma+M_\Lambda).
\end{aligned} \tag{5.70}$$

Thus, apart from one free parameter M_0, say, the theory is fully constrained. An interesting application of Eq. (5.69) is to calculate meson–baryon σ terms: Using standard perturbation theory and the low-energy theorem (2.37), one obtains (McDonald and Rosen, 1973)

$$\begin{aligned}
\sigma_{NN}^{\pi\pi} &\simeq (1/50)\mu_1(\mu_1+3)(395-M_0)+70\ (\text{MeV}) \\
\sigma_{NN}^{KK} &\simeq (6/25)\mu_1(\mu_1+3)(1011-M_0)+79.5\ (\text{MeV}).
\end{aligned} \tag{5.71}$$

Since we consider only triangular representations, we have $\mu_2=0$ according to Eq. (5.67).

Following McDonald and Rosen (1973), we list the predictions of Eq. (5.71) in Table III. In column II we state the present world average of the estimated σ terms according to Tables I and II. In columns III, IV, and V of Table III we list the values of the σ terms for the three lowest triangular representations, Eq. (5.67), under the assumption that $M_0=940$ MeV. As can be seen, present analyses of πN data already exclude the $(6,\bar{6})+(\bar{6},6)$ representation as a possible alternative to the $(3,\bar{3})+(\bar{3},3)$; matters get worse as we go to higher representations. In columns VI and VII we use $\sigma_{NN}^{\pi\pi}=50$ MeV as input, which corresponds to $M_0\simeq 645$ MeV. Whereas the $(3,\bar{3})+(\bar{3},3)$ prediction for σ_{NN}^{KK} is in agreement with the data, the $(6,\bar{6})+(\bar{6},6)$ prediction for the same quantity is clearly in disagreement with the data. As in the previous case, matters are not improved in higher representations. In columns VIII and IX we arbitrarily assumed M_0 to be far from the nucleon mass which, as we have seen in Sec. VB, corresponds to the $(6,\bar{6})+(\bar{6},6)$ representation, for example. Although $\sigma_{NN}^{\pi\pi}$ is not entirely inconsistent with the data, the kaon–nucleon σ terms are far too large. In higher representations they become even larger.

The conclusion is that the present experimental information seems to rule out all triangular (and self-adjoint) representations of $SU(3)\times SU(3)$ except the $(3,\bar{3})+(\bar{3},3)$ as possibilities for describing chiral symmetry breaking. Of course, a large number of nontriangular representations have not been considered, and these as well as mixed symmetry-breaking models have enough free parameters to fit the present data. Nevertheless, the results, along with those obtained in the previous sections by studying each symmetry-breaking scheme separately, suggest that the $(3,\bar{3})+(\bar{3},3)$ representation plays a central role in breaking chiral symmetries, and higher-dimensional irreducible representations should only serve as (small) admixtures, if indeed required, for breaking octet dominance.

H. Scale invariance and chiral symmetry

Finally, let us conclude with some remarks concerning the intimate relation between broken scale invariance and broken chiral symmetry, which has been extensively discussed in the recent literature (Gell-Mann, 1969; Wilson, 1969; Fritzsch and Gell-Mann, 1971). This connection becomes clear when we observe that (Gell-Mann, 1969)

$$\frac{d\hat{D}}{dx_0}=\int d^3x\,\theta_\mu{}^\mu, \tag{5.72}$$

where $\theta_\mu{}^\mu$ is the trace of the energy-momentum tensor $\theta_{\mu\nu}$, and $\hat{D}$ is the dilation operator of the conformal group. Thus, in the limit of scale invariance corresponding to a theory free from dimensional quantities, we must have $\theta_\mu{}^\mu \to 0$ in order to get $d\hat{D}/dx_0 = 0$. The parallel that can now be drawn between the violation of scale breaking and that of $SU(3) \times SU(3)$ invariance involves the idea that the term $u \equiv \epsilon\mathcal{H}'$ in the energy density θ_{00} that violates $SU(3) \times SU(3)$ is a world scalar and so is, by assumption, the term in θ_{00} that violates $\hat{D}$ conservation. It is then most natural to suppose that the $SU(3) \times SU(3)$ violating term u is found among the scale-breaking terms (Gell-Mann, 1969; Wilson, 1969), namely

$$\theta_{00} = \mathcal{H}_0 + \delta + u, \tag{5.73}$$

where $\mathcal{H}_0$ is our usual chiral and scale-invariant Hamiltonian with dimension 4, δ is invariant under $SU(3) \times SU(3)$ but violates scale invariance and has scale dimension $d_\delta \neq 4$, and u violates both chiral and scale invariance and, hopefully, has an unique dimension d_u (Renner, 1972a). From Eq. (5.73) one obtains (Gell-Mann, 1969) the following "virial theorem" for $\theta_\mu{}^\mu$:

$$\theta_\mu{}^\mu = (4 - d_\delta)\delta + (4 - d_u)u \tag{5.74}$$

and, by Eq. (5.72), the interrelationship between broken scale invariance and chiral symmetry becomes transparent. In the simplest possible theories δ is merely a c number, which implies $d_\delta = 0$ (Gell-Mann, 1969; Ellis, Weisz, and Zumino, 1971). In this case, Eq. (5.74) gives

$$M_N = (4 - d_u)\langle N \mid u_0 + cu_8 \mid N\rangle \tag{5.75}$$

where we used $\langle N \mid \theta_\mu{}^\mu \mid N\rangle = M_N$ and $u = u_0 + cu_8$, and the matrix element always refers, of course, to the connected one. In general one expects (Wilson, 1969; Ellis, 1970a) $1 \leq d_u < 4$; this constraint is necessary (Wilson, 1969) to make PCAC work when $SU(3) \times SU(3)$ is broken (the lower bound is a consequence of the Källén–Lehmann representation). Therefore Eq. (5.75) is incompatible with $\langle N \mid u_0 \mid N\rangle \simeq M_N/2$ corresponding to $\sigma_{NN}{}^{\pi\pi} \simeq 50$ MeV, and use has been made of Eq. (5.18). The way out of this paradox (provided δ is indeed a c-number) has been shown by Ellis (1970b) in assuming $\langle 0 \mid u_{0,8} \mid \sigma\rangle \neq 0$, where σ would be the massless Goldstone boson of exact scale invariance; for broken scale invariance, i.e., spontaneously broken conformal symmetry, this scalar σ meson, most plausibly the $\epsilon(700)$, dominates the matrix elements of $\theta_\mu{}^\mu$. The main point is that this does not require (Ellis, 1970b) strongly enhanced values for $\langle N \mid u_0 \mid N\rangle$ with respect to $\langle N \mid u_8 \mid N\rangle$. Since σ terms of about $\sigma_{NN}{}^{\pi\pi} \simeq 110$ MeV are most likely to be ruled out by now, approaches (Altarelli, Cabibbo, and Maiani, 1971a; Brown, Pardee, and Peccei, 1971; Crewther, 1971; Mathur, 1971) using $\langle N \mid u_0 \mid N\rangle \simeq 1500$ MeV in order to obtain $d_u = 3$ from Eq. (5.75) are questionable. Other estimates for the dimension of the chiral symmetry-breaking energy density, based on low-energy theorems (Kleinert and Weisz, 1971) derived from Eq. (5.74) for $\langle \pi \mid \theta_\mu{}^\mu \mid \pi\rangle$, indicate that (Levin, Okubo, and Palmer, 1971; Renner and Staunton, 1972; Pennington, 1972; Haan, Nasrallah, and Schilcher, 1974)

$$1 \leq d_u < 3. \tag{5.76}$$

These estimates plausibly identify the σ meson with the $\epsilon(700)$. If, in addition, the $S^*(980)$ which strongly couples to the $K\bar{K}$ channel is included for calculating $\langle \pi \mid \theta_\mu{}^\mu \mid \pi\rangle$, then the lower limit in Eq. (5.76) is to be favored (Renner and Staunton, 1972; Pennington, 1972). Similarly, using the Fubini–Furlan mass-dispersion relations and the πN s-wave scattering length combination $a_1 + 2a_3$ as input, one obtains $d_u \leq 3$ (O'Donnell and Wong, 1972). It is also interesting to note that an entirely different determination of d_u, using the experimental information on the forward differential cross section for high-energy pseudoscalar meson photoproduction assuming PCAC and conservation of the scale dimension on the light cone, yields (Chikashige and Inagaki, 1972) $d_u = 2$ in agreement with Eq. (5.76). However, a previous estimate (Brown, 1971) resulted in $3 \leq d_u < 4$, but using $\sigma_L/\sigma_T \to 0$ in deep-inelastic electron scattering makes such a result not very surprising.

It should be emphasized that the assumption that δ is a c number has been made mainly in order to prevent uncontrolled parameters from entering the theory and, apart from being possibly consistent (Ellis, 1970b; Ellis, Weisz, and Zumino, 1971) with our present knowledge about $SU(3) \times SU(3)$ breaking, lacks any theoretical justification. If, for example, one is not willing to accept the existence of a Goldstone σ boson as considered above, then the relatively small value of $\langle N \mid u_0 \mid N\rangle$ indicates that the possibility of a c-number scale breaking, but $SU(3) \times SU(3)$ conserving, part of the hadronic energy density should be ruled out. The case in which δ is a q number, i.e., $d_\delta \neq 0$, can apparently not be disregarded (Gell-Mann, 1969; Wilson, 1969; Kleinert and Weisz, 1971). Finally, Eq. (5.74) implicitly assumes that δ and u have unique dimensions, an assumption which is far from obvious (Renner, 1972a).

VI. CONCLUSIONS

We have described in considerable detail chiral symmetries, their importance, and consequences for the study of elementary particle physics. We concentrated mainly on meson–nucleon σ terms, presently the most powerful tools for studying chiral symmetry violations, and discussed and developed various methods and techniques to relate these quantities to actual meson–baryon scattering experiments, as well as to $\pi\pi$ scattering data. Where necessary, we also included discussions concerning the consequences of broken chiral symmetries for K_{l3} decays. Apart from giving a critical and detailed review of most of the "experimental" estimates for pion–nucleon and kaon–nucleon sigma terms done so far, we also outlined in some detail the most common chiral symmetry-breaking schemes at present and compared their predictions with available experimental data. Implications from nonlinear effective Lagrangians for broken chiral symmetries were briefly discussed, and the connection between scale invariance and chiral symmetry was outlined, with emphasis on our present knowledge of the magnitudes of meson–nucleon σ terms.

By now numerous estimates of the πN sigma term $\sigma_{NN}{}^{\pi\pi}$ are available and, viewed as a whole, the results are fairly compatible with each other, keeping in mind the rather large experimental uncertainties. The present world average for $\sigma_{NN}{}^{\pi\pi}$ lies around 50 MeV, and it appears to be rather unlikely that $\sigma_{NN}{}^{\pi\pi}$ exceeds 70 MeV, say. Values as large as the Cheng–Dashen estimate of 110 MeV (or larger), which originally stimulated all the recent reestimates of σ terms, can be quite convincingly excluded. This is due to a very recent recalculation using the same broad-area subtraction method as Cheng and Dashen did, but different, more recent, phase-shift analyses. In addition, much smaller values than 110 MeV are confirmed by practically all other calculations carried out to date.

The situation with respect to the exceedingly more complicated kaon–nucleon reactions is not so clear, since considerably fewer estimates for $\sigma_{NN}{}^{KK}$ have been performed. The upper limit of $\sigma_{NN}{}^{KK}$ lies around 600 MeV, whereas the world average is about 350 MeV. A negative value for the kaon–nucleon σ term, as has been claimed in one previous calculation using fixed-t dispersion relations, appears to be more than unlikely, since two very similar recent estimates definitely favor positive values for $\sigma_{NN}{}^{KK}$, in agreement with various other calculations employing different methods. Since, compared to πN scattering, the kaon–nucleon data are rather poor and experimental results scatter widely, it would be of interest to estimate $\sigma_{NN}{}^{KK}$ using broad-area subtracted dispersion relations which provide us with built-in consistency checks on the compatibility of the various phase-shift solutions used.

At the present stage we are certainly not in a position to deduce *exact* values for σ terms. What we can deduce with some reliability, however, is the magnitude of meson–nucleon sigma terms. The main point is that values of $\sigma_{NN}{}^{\pi\pi}$ and $\sigma_{NN}{}^{KK}$ as large as 110 MeV and 1300 MeV, respectively, can be excluded; these are values which would upset our whole "conventional" understanding of symmetries and symmetry-breaking mechanisms.

Although, within experimental uncertainties, the present world average for $\sigma_{NN}{}^{\pi\pi}$ and $\sigma_{NN}{}^{KK}$ of about 50 and 350 MeV, respectively, do not entirely disagree with the $(3,\bar{3})+(\bar{3},3)$ predictions of about 20 and 200 MeV, respectively, there appears to be a persistent enhancement present. Taking these enhancements quite literally and assuming, for the time being, that the chiral symmetry-breaking Hamiltonian transforms under a *single* representation of $SU(3)\times SU(3)$, there are only two alternatives left which could account for this octet enhancement: Either the $(3,\bar{3})+(\bar{3},3)$ model is wrong and higher-dimensional representations have to be considered, or one accepts the $(3,\bar{3})+(\bar{3},3)$ model of Gell-Mann, Oakes, and Renner and attributes the octet breaking to some other enhancement mechanism. Together with $(1,8)+(8,1)$, the higher-dimensional (irreducible) triangular and self-adjoint representations yield predictions which are inconsistent with present experimental results. A pure $(8,8)$ model predicts $\pi\pi$ scattering lengths totally incompatible with the data, whereas the $(6,\bar{6})+(\bar{6},6)$ representation appears to be in better shape. Although the σ terms cannot be predicted uniquely in these models, the degenerate average mass of the baryon octet, say, turns out to be much smaller than the nucleon mass itself, taking into account the present estimates for meson-nucleon σ terms. This result is obviously not acceptable, if our general concept of symmetries and symmetry breaking is correct. From this point of view the $(3,\bar{3})+(\bar{3},3)$ representation is the preferable choice, but certainly $(6,\bar{6})+(\bar{6},6)$ is better than $(8,8)$. Similar conclusions are reached from a study of nonlinear effective Lagrangians which rule out all triangular and self-adjoint representations of $SU(3)\times SU(3)$ except the $(3,\bar{3})+(\bar{3},3)$ as possibilities for describing chiral symmetry breaking. However, not very much can be said about non-triangular or reducible representations, since these have enough free parameters to fit all the present data.

Contrary to previous claims, therefore, one is naturally led to the conclusion that the $(3,\bar{3})+(\bar{3},3)$ model plays at least a dominant role in chiral symmetry-breaking mechanisms. In this case, the Li–Pagels mechanism of calculating $(3,\bar{3})+(\bar{3},3)$ chiral symmetry breaking is a rather plausible way to explain the octet enhancement which is achieved by the threshold dominance of Goldstone-boson-pair states. Although quantitative calculations are not free of ambiguities and uncertainties, the qualitative agreement with the enhanced pion–nucleon and kaon–nucleon σ terms is good.

Another possible way to achieve a breaking of octet dominance is to include, in addition to the dominant $(3,\bar{3})+(\bar{3},3)$ component in the symmetry-breaking Hamiltonian, (small) contributions of higher-dimensional representations of $SU(3)\times SU(3)$. This case corresponds to the so-called mixed symmetry-breaking models. The $(3,\bar{3})+(\bar{3},3)+(8,8)$ model, for example, can account for and explain all presently available data on $\pi\pi$ scattering and meson–nucleon σ terms, because of the large number of free parameters available. Since models of this kind are rather unconstrained and ambiguous, and therefore have practically no predictive power, it remains doubtful whether more accurate future elastic scattering experiments will be able either to confirm or to reject them. The same situation holds, for example, for $(1,8)+(8,1)$ and $(6,\bar{6})+(\bar{6},6)$ admixtures to the GMOR model. In fact, the possibilities for constructing models of this kind are virtually limitless, provided the $(3,\bar{3})+(\bar{3},3)$ representation plays the dominant role in the chiral symmetry-breaking Hamiltonian.

The relation between broken scale invariance and broken chiral symmetry is also intimately related to the magnitude of σ terms. A pion–nucleon sigma term of about 100 MeV or more has proven to be consistent with the assumption that the scale breaking but chiral invariant term δ in the total energy density is merely a c number, implying the dimension of the $SU(3)\times SU(3)$ and scale breaking Hamiltonian to be $d=3$. Because such large values for $\sigma_{NN}{}^{\pi\pi}$ are most likely to be ruled out, a q number δ term appears to be favored, unless one assumes a massless Goldstone boson of exact scale invariance, which is most plausibly identified with the $\epsilon(700)$ in the actual broken world.

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