Multiple scattering expansions in several particle dynamics

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The problem of quantum collisions involving several particle systems is reviewed within the framework of multiple scattering theory. The basic apparatus of collision theory for nonrelativistic potential problems is first developed, and the Born and eikonal series are introduced. A general analysis is then given of multiple scattering expansions for several particle problems. We discuss in particular the Born developments, the Faddeev–Watson expansions, the Glauber method, and various multiple scattering approaches to the determination of the optical potential. Applications to atomic collision problems and to high-energy hadron–deuteron scattering are discussed at length.

CONTENTS

I. INTRODUCTION

Several particle dynamics is a problem of long-standing interest in physics. While the nonrelativistic motion of *two* particles interacting through a given force is well understood and powerful methods have been developed to deal with situations where a *very large* number of particles are present, systems containing a *few particles* have remained difficult to analyze. This is not surprising since in general these systems exhibit all the complexity of the many-body problem. We shall examine in this review some quantum systems of this type from the point of view of collision theory. Thus bound state ("spectroscopic") properties will only be discussed insofar as they influence scattering phenomena.

The theoretical methods which we shall describe to analyze these problems all share a common feature: they may be considered as *multiple scattering expansions*. Such methods have been very useful in the analysis of atomic, nuclear, and "elementary particle" collision processes. It is the purpose of this article to present this approach from a general point of view and to illustrate it on a few selected examples.

In order to introduce some of the concepts involved in multiple scattering expansions within a simple framework, we begin in Sec. II with a study of the Born and eikonal series in nonrelativistic potential scattering.

Section III is devoted to a general analysis of multiple scattering series for several particle problems. We first discuss the Born and distorted-wave Born developments, then the Faddeev–Watson expansions, and finally the Glauber "many-body" extensions of the eikonal method. We also give a brief survey of multiple scattering approaches to the determination of the optical potential.

Applications of multiple scattering expansions to atomic collision problems are the subject of Sec. IV. We first analyze electron–hydrogen collisions, a classic three-body problem. We then discuss several electron–helium scattering processes at intermediate and high (atomic) energies, for which absolute measurements of differential cross sections recently have become available.

In Sec. V we consider high-energy hadron–deuteron collisions. These processes lie at the borderline between elementary particle physics and nuclear physics and have been a locus of fruitful interaction between the two fields. After recalling a few general properties of hadron–nucleus scattering at high energies, we review the applications of Glauber's high-energy diffraction theory to hadron–deuteron collisions. Particular emphasis is given to elastic scattering, for which a comprehensive comparison of theoretical and experimental work is made. We also discuss hadron–deuteron scattering from the point of view of Regge theory. We study the connection between diffraction scattering and Regge poles and then investigate the Regge cut contained in the Glauber eclipse term. We also give a brief survey of phenomenological applications.

Finally, we summarize in Sec. VI the main results

* Supported in part by the NATO Scientific Affairs Division, Research Grant No. 586.

† Permanent address. Work at Stony Brook supported in part under National Science Foundation Grant No. GP 32998X..

discussed in this review and indicate several open problems.

II. POTENTIAL SCATTERING

A. Basic formulas

Let us consider the nonrelativistic scattering of a spinless particle of mass m by a local potential $V(\mathbf{r})$ of a range a. We denote by $\mathbf{k}_i$ and $\mathbf{k}_f$ the initial and final wave vectors of the particle while θ is the scattering angle between $\mathbf{k}_i$ and $\mathbf{k}_f$. It is also convenient to introduce the "reduced potential" $U(\mathbf{r}) = 2mV(\mathbf{r})/\hbar^2$. The particle's energy is $E = \hbar^2 k^2/2m$, where $k = |\mathbf{k}_i| = |\mathbf{k}_f|$ is its wave number. The Hamiltonian describing the system is therefore

$$H = -(\hbar^2/2m)\nabla_r^2 + V(\mathbf{r}). \qquad (2.1)$$

We shall call $\psi_{\mathbf{k}_i}^{(+)}$ the stationary scattering eigenstate of H which corresponds to an incident plane wave of momentum $\hbar\mathbf{k}_i$ and exhibits the behavior of an outgoing spherical wave. This wave function satisfies the Lippmann–Schwinger equation

$$\psi_{\mathbf{k}_i}^{(+)}(\mathbf{r}) = \Phi_{\mathbf{k}_i}(\mathbf{r}) + \int G_0^{(+)}(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\psi_{\mathbf{k}_i}^{(+)}(\mathbf{r}')\,d\mathbf{r}', \qquad (2.2)$$

where the incident plane wave is given by

$$\Phi_{\mathbf{k}_i}(\mathbf{r}) = \langle \mathbf{r}|\mathbf{k}_i\rangle = (2\pi)^{-3/2}\exp(i\,\mathbf{k}_i\cdot\mathbf{r}). \qquad (2.3)$$

The "normalization" convention which we adopt is such that for plane wave states $|\mathbf{k}\rangle$ and $|\mathbf{k}'\rangle$ the orthogonality relation reads

$$\langle \mathbf{k}'|\mathbf{k}\rangle = \delta(\mathbf{k}-\mathbf{k}'). \qquad (2.4)$$

The Green's function $G_0^{(+)}(\mathbf{r},\mathbf{r}')$ which appears in the Lippmann–Schwinger equation (2.2) is given by

$$G_0^{(+)}(\mathbf{r},\mathbf{r}') = -(2\pi)^{-3}\int \frac{\exp[i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')]}{K^2-k^2-i\epsilon}\,d\mathbf{K}, \qquad (2.5)$$

where the limiting processes $\epsilon \to 0^+$ is always implied. Explicitly, we have

$$G_0^{(+)}(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} \qquad (2.6)$$

so that the wave function $\psi_{\mathbf{k}_i}^{(+)}$ behaves asymptotically as

$$\psi_{\mathbf{k}_i}^{(+)}(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} (2\pi)^{-3/2}[\exp(i\,\mathbf{k}_i\cdot\mathbf{r}) + f\exp(ikr)/r] \qquad (2.7)$$

and the elastic scattering amplitude f is given by

$$f = -2\pi^2\langle \Phi_{\mathbf{k}_f}|U|\psi_{\mathbf{k}_i}^{(+)}\rangle. \qquad (2.8)$$

Here

$$\Phi_{\mathbf{k}_f}(\mathbf{r}) = \langle \mathbf{r}|\mathbf{k}_f\rangle = (2\pi)^{-3/2}\exp(i\,\mathbf{k}_f\cdot\mathbf{r}) \qquad (2.9)$$

is a plane wave corresponding to the final wave vector $\mathbf{k}_f$ and "normalized" according to the convention (2.4). If the potential is central, we recall that the scattering amplitude (2.8) may also be decomposed in partial waves as

$$f(k,\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)[S_l(k)-1]P_l(\cos\theta), \qquad (2.10)$$

where the coefficients $S_l(k)$ are the S-matrix elements in the angular momentum representation; they are given in terms of the phase shifts δ_l by

$$S_l(k) = \exp[2i\delta_l(k)]. \qquad (2.11)$$

B. The Born series

If we elect to solve the Lippmann–Schwinger equation (2.2) by perturbation theory, starting from the "unperturbed" incident plane wave $\Phi_{\mathbf{k}_i}(\mathbf{r})$, we generate the sequence of functions

$$\begin{aligned}
\psi_0(\mathbf{r}) &= \Phi_{\mathbf{k}_i}(\mathbf{r}),\\
\psi_1(\mathbf{r}) &= \Phi_{\mathbf{k}_i}(\mathbf{r}) + \int G_0^{(+)}(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\psi_0(\mathbf{r}')\,d\mathbf{r}'\\
&\vdots\\
\psi_n(\mathbf{r}) &= \Phi_{\mathbf{k}_i}(\mathbf{r}) + \int G_0^{(+)}(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\psi_{n-1}(\mathbf{r}')\,d\mathbf{r}'.
\end{aligned} \qquad (2.12)$$

Let us assume for the moment that this sequence converges towards $\psi_{\mathbf{k}_i}^{(+)}$. We may then write the *Born series* for the scattering wave function, namely

$$\psi_{\mathbf{k}_i}^{(+)} = \sum_{n=0}^{\infty}\varphi_n(\mathbf{r}), \qquad (2.13)$$

where $\varphi_0 = \psi_0 = \Phi_{\mathbf{k}_i}$ and

$$\varphi_n(\mathbf{r}) = \int K_n(\mathbf{r},\mathbf{r}')\psi_0(\mathbf{r}')\,d\mathbf{r}', \qquad n \geq 1 \qquad (2.14)$$

with

$$K_1(\mathbf{r},\mathbf{r}') = G_0^{(+)}(\mathbf{r},\mathbf{r}')U(\mathbf{r}') \qquad (2.15)$$

and

$$K_n(\mathbf{r},\mathbf{r}') = \int K_1(\mathbf{r},\mathbf{r}'')K_{n-1}(\mathbf{r}'',\mathbf{r}')\,d\mathbf{r}'', \quad n \geq 2. \qquad (2.16)$$

It is apparent from Eqs (2.13)–(2.16), that the Born series is a perturbation series in powers of the interaction potential. Substituting the series (2.13) into the integral representation (2.8), we obtain the corresponding *Born series for the scattering amplitude*, namely

$$f = \sum_{n=1}^{\infty}\bar{f}_{Bn}, \qquad (2.17)$$

where

$$\bar{f}_{Bn} = -2\pi^2\langle \Phi_{\mathbf{k}_f}|UG_0^{(+)}U\cdots G_0^{(+)}U|\Phi_{\mathbf{k}_i}\rangle \qquad (2.18)$$

is an expression in which the interaction appears n times and the free Green's function $(n-1)$ times. It is worth pointing out that the relation (2.18) gives the term of order n of the Born series in general circumstances, for example, when the interaction is complex and even nonlocal. It is also convenient to define the jth order Born approximation to the scattering amplitude as

$$f_{Bj} = \sum_{n=1}^{j}\bar{f}_{Bn}. \qquad (2.19)$$

In order to gain further insight into the physical content of $\bar{f}_{Bn}$, let us analyze Eq. (2.18) in momentum space. Defining (for a local potential)

$$\langle \mathbf{k}'|U|\mathbf{k}\rangle = \langle \Phi_{\mathbf{k}'}|U|\Phi_{\mathbf{k}}\rangle$$
$$= (2\pi)^{-3}\int \exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]U(\mathbf{r})\,d\mathbf{r} \quad (2.20)$$

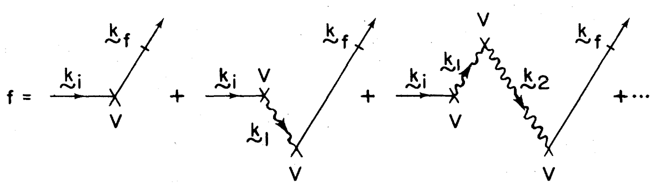
and using the integral representation (2.5) of $G_0^{(+)}$, we find that

$$\bar{f}_{B1} = f_{B1} = -2\pi^2\langle \mathbf{k}_f|U|\mathbf{k}_i\rangle \quad (2.21)$$

and

$$\bar{f}_{Bn} = -2\pi^2\int d\mathbf{k}_1\,d\mathbf{k}_2\cdots d\mathbf{k}_{n-1}\langle \mathbf{k}_f|U|\mathbf{k}_{n-1}\rangle\frac{1}{k^2-k_{n-1}^2+i\epsilon}$$
$$\cdot\langle \mathbf{k}_{n-1}|U|\mathbf{k}_{n-2}\rangle\cdots\langle \mathbf{k}_2|U|\mathbf{k}_1\rangle\frac{1}{k^2-k_1^2+i\epsilon}\langle \mathbf{k}_1|U|\mathbf{k}_i\rangle. \quad (2.22)$$

The Green's function therefore appears as a propagator, while the quantities $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_{n-1}$ are "intermediate momenta." We can thus visualize the Born series by picturing the scattering amplitude as



$$= f_{B1} + \bar{f}_{B2} + \bar{f}_{B3} + \cdots \quad (2.23)$$

namely as a *multiple scattering series* in which the projectile interacts repeatedly with the potential V and propagates freely between two such interactions. On the basis of this multiple scattering interpretation we expect that the Born series will converge if the incident particle is sufficiently fast that it cannot interact many times with the potential and (or) if the potential is weak enough. Detailed studies of the Born series (Jost and Pais, 1951; Kohn, 1954; Zemach and Klein, 1958; Aaron and Klein, 1960; Davies, 1960; Manning, 1965) confirm these intuitive considerations. In particular:

(i) For a central potential $V(r)$ less singular than r^{-2} at the origin and decreasing faster than r^{-3} as $r\to\infty$, the Born series always converges at sufficiently high energies.

(ii) For a central potential $V(r)$ the Born series converges for all energies if the potential $-|V(r)|$ cannot support any bound state.

We emphasize that the results quoted above apply only to nonrelativistic potential scattering; they may not necessarily be valid for many-body problems and (or) relativistic collisions.

C. The eikonal approximation and eikonal multiple scattering series

Let us return to the Lippmann–Schwinger equation (2.2). We assume that

$$ka \gg 1 \quad (2.24)$$

and that

$$V_0/E = U_0/k^2 \ll 1, \quad (2.25)$$

where V_0 is a typical strength of the interaction $V(\mathbf{r})$ and $U_0 = 2mV_0/\hbar^2$. Since the first, "high wave number" condition (2.24) states that the reduced de Broglie wave length $\lambda\!\!\!^- = k^{-1}$ of the particle is small with respect to the range of the potential, we expect semiclassical methods to be useful in this case. The second condition (2.25) will be referred to as the "high-energy" condition. If these two conditions are satisfied, the *eikonal approximation* (Moliere, 1947; Glauber, 1953, 1955, 1959; Watson, 1953; Schwinger, 1954; Malenka, 1954; Schiff, 1956; Saxon and Schiff, 1957) may be used to obtain for the scattering wave function $\psi_{\mathbf{k}_i}^{(+)}$ the approximate expression

$$\psi_E(\mathbf{r}) = (2\pi)^{-3/2}\exp\left[i\mathbf{k}_i\cdot\mathbf{r} - \frac{i}{2k}\int_{-\infty}^{z} U(\mathbf{b},z')\,dz'\right], \quad (2.26)$$

where we have adopted a cylindrical coordinate system such that $\mathbf{r} = \mathbf{b} + z\hat{\mathbf{k}}_i$, so that the integral is evaluated along a straight line parallel to the incident momentum $\hbar\mathbf{k}_i$. In terms of the potential $V(\mathbf{r})$, we have

$$\psi_E(\mathbf{r}) = (2\pi)^{-3/2}\exp\left[i\mathbf{k}_i\cdot\mathbf{r} - \frac{i}{\hbar v_i}\int_{-\infty}^{z} V(\mathbf{b},z')\,dz'\right], \quad (2.27)$$

where $\mathbf{v}_i = \hbar\mathbf{k}_i/m$ is the incident velocity. We shall not discuss in detail the numerous derivations of the result (2.26). We simply mention that it may be obtained from stationary-phase arguments (Schiff, 1956) or from the fact that the incoming plane wave is modulated by a function which varies slowly over the de Broglie wavelength of the incident particle (Glauber, 1959). Another interesting way of deriving the eikonal wave function (2.26) is to examine the free propagator $G_0^{(+)}$ appearing in the Lippmann–Schwinger equation (2.2) (Malenka, 1954; Schiff, 1956; Byron, Joachain, and Mund, 1973). Using its momentum space representation (2.5) and introducing the new variable $\mathbf{Q} = \mathbf{K} - \mathbf{k}_i$, one has

$$G_0^{(+)}(\mathbf{r},\mathbf{r}') = -(2\pi)^{-3}\exp[i\mathbf{k}_i\cdot(\mathbf{r}-\mathbf{r}')]$$
$$\times\int\frac{\exp[i\mathbf{Q}\cdot(\mathbf{r}-\mathbf{r}')]}{2\mathbf{k}_i\cdot\mathbf{Q}+Q^2-i\epsilon}d\mathbf{Q}. \quad (2.28)$$

We now return to the Lippmann–Schwinger equation (2.2), and provided that the two conditions (2.24) and (2.25) are satisfied, we find it is legitimate to "linearize" the denominator of the integrand (i.e., neglect the Q^2 term) and write

$$G_0^{(+)}(\mathbf{r},\mathbf{r}') \simeq -(2\pi)^{-3}\exp[i\mathbf{k}_i\cdot(\mathbf{r}-\mathbf{r}')]$$
$$\times\int\frac{\exp[i\mathbf{Q}\cdot(\mathbf{r}-\mathbf{r}')]}{2\mathbf{k}_i\cdot\mathbf{Q}-i\epsilon}d\mathbf{Q}. \quad (2.29)$$

The integral on the right of Eq. (2.29) is then readily performed, with the result

$$G_0^{(+)}(\mathbf{r},\mathbf{r}') \simeq (-i/2k)e^{ik(z-z')}\delta^2(\mathbf{b}-\mathbf{b}')\Theta(z-z'), \quad (2.30)$$

where $\mathbf{r} = \mathbf{b} + z\hat{\mathbf{k}}_i$, $\mathbf{r}' = \mathbf{b}' + z'\hat{\mathbf{k}}_i$, and Θ is the step function such that

$$\Theta(x) = 1 \quad x > 0$$
$$= 0 \quad x < 0. \qquad (2.31)$$

The *linearized propagator* (2.29) and (2.30), which clearly exhibits *forward propagation* between successive interactions with the potential, leads directly to the eikonal wave function (2.26). Incidentally, let us remark that the importance of the four-dimensional relativistic version of the linearized propagator in treating field theoretical problems was recognized by Schwinger (1954) and used recently by several authors (Chang and Ma, 1969; Abarbanel and Itzykson, 1969; Lévy and Sucher, 1970, Englert *et al.*, 1969) to sum the series of Feynman amplitudes corresponding to large classes of ladder diagrams.

With the eikonal wave function given by Eq. (2.26), we may now return to the integral representation (2.8) and write the scattering amplitude as

$$f_E(\mathbf{\Delta}) = -\frac{1}{4\pi}\int \exp(i\mathbf{\Delta}\cdot\mathbf{r}) \times U(\mathbf{r})\exp\left[-\frac{i}{2k}\int_{-\infty}^{z} U(\mathbf{b},z')\,dz'\right]d\mathbf{r}, \qquad (2.32)$$

where

$$\mathbf{\Delta} = \mathbf{k}_i - \mathbf{k}_f \qquad (2.33)$$

is the wave vector transfer of length $\Delta = 2k\sin(\theta/2)$.

In obtaining the eikonal wave function (2.26), we pointed out that the integration in its phase should be carried out along a straight line parallel to $\mathbf{k}_i$. In fact, since the actual phase of the corresponding semiclassical scattering wave function is evaluated along a curved trajectory, it is reasonable to expect that an improvement on Eq. (2.32) may be achieved by performing the z integration in the phase along a direction parallel to the bisector of the scattering angle (i.e , perpendicular to $\mathbf{\Delta}$). This suggestion, first made by Glauber (1959), leads directly to the *eikonal scattering amplitude*

$$f_E = \frac{k}{2\pi i}\int \exp(i\mathbf{\Delta}\cdot\mathbf{b})\{\exp[i\chi(\mathbf{b})] - 1\}\,d^2\mathbf{b}, \qquad (2.34)$$

where we work in a cylindrical coordinate system such that

$$\mathbf{r} = \mathbf{b} + z\,\hat{\mathbf{n}} \qquad (2.35)$$

and $\hat{\mathbf{n}}$ is perpendicular to $\mathbf{\Delta}$. The *eikonal phase shift function* $\chi(\mathbf{b})$ appearing in Eq. (2.34) is given in terms of the interaction by the simple linear relationship

$$\chi(\mathbf{b}) = -\frac{1}{2k}\int_{-\infty}^{+\infty} U(\mathbf{b},z)\,dz. \qquad (2.36)$$

Defining the quantity

$$\Gamma(\mathbf{b}) = 1 - \exp[i\chi(\mathbf{b})] \qquad (2.37)$$

we may also rewrite Eq. (2.34) as

$$f_E = \frac{ik}{2\pi}\int \exp(i\mathbf{\Delta}\cdot\mathbf{b})\Gamma(\mathbf{b})d^2\mathbf{b}. \qquad (2.38)$$

For potentials which possess azimuthal symmetry, Eq. (2.34) simplifies to

$$f_E = \frac{k}{i}\int_0^{\infty} J_0(\Delta b)\{\exp[i\chi(b)] - 1\}\,b\,db, \qquad (2.39)$$

where $\chi(b)$ is still given by Eq. (2.36). We may also look at this relation from a somewhat different point of view. Indeed, the right-hand side of Eq. (2.39) provides the Fourier–Bessel representation of the *exact* scattering amplitude, provided that the phase $\chi(b)$ is redefined accordingly. This representation is exact for all energies and angles (Adachi and Kotani, 1965, 1966; Predazzi, 1966; Chadan, 1968). For high-energy, small angle scattering, the phase $\chi(b)$ may be related to the phase shifts δ_l appearing in the partial wave series (2.10). The result is

$$\chi(b) = 2\delta_l, \qquad (2.40)$$

where b and l are related by $l \simeq kb$.

Two important remarks should be made about the eikonal approximation. Firstly, it is equally valid for real and complex potentials. In the latter case the phase shift function $\chi(\mathbf{b})$ becomes complex [see Eq. (2.36)]. Secondly, within its range of validity, the eikonal amplitude satisfies the optical theorem (Glauber, 1959), in contradistinction to the first Born approximation.

By analogy with the Born series, we may define an *eikonal multiple scattering series* by expanding the quantity $\Gamma(\mathbf{b})$ [see Eq. (2.37)] in powers of the interaction potential. Thus we write

$$f_E = \sum_{n=1}^{\infty} \bar{f}_{En}, \qquad (2.41)$$

where

$$\bar{f}_{En} = -\frac{ik}{2\pi}\frac{i^n}{n!}\int \exp(i\mathbf{\Delta}\cdot\mathbf{b})[\chi(\mathbf{b})]^n d^2\mathbf{b}. \qquad (2.42)$$

In particular, for potentials which possess azimuthal symmetry, Eq. (2.42) reduces to

$$\bar{f}_{En} = -ik\frac{i^n}{n!}\int_0^{\infty} J_0(\Delta b)[\chi(b)]^n\,b\,db. \qquad (2.43)$$

We note that in the case of a real potential the objects $\bar{f}_{En}$ given by Eq. (2.43) are alternately real and imaginary. As in the case of the Born series [see Eq. (2.19)] we also introduce the quantities

$$f_{Ej} = \sum_{n=1}^{j} \bar{f}_{En}. \qquad (2.44)$$

We now investigate the relationship between the Born and eikonal series. First of all, it is a simple matter to show that

$$f_{E1} = f_{B1} \qquad (2.45)$$

for all energies and all momentum transfers (Glauber, 1959). We emphasize that the result (2.45) is valid for all angles only when the z axis used in evaluating the eikonal phase shift function [Eq. (2.36)] is chosen along a direction perpendicular to $\mathbf{\Delta}$. If, for example, the z axis were chosen along $\mathbf{k}_i$, then we would only have approximately $\mathbf{\Delta}\cdot\mathbf{b} \simeq \mathbf{\Delta}\cdot\mathbf{r}$ for small Δ's and Eq. (2.45) would hold only for small scattering angles. In what follows we shall consistently choose $\hat{\mathbf{z}}$ perpendicular to $\mathbf{\Delta}$.

Remarkable relationships between the higher terms of the eikonal and Born series have also been noticed recently (Moore, 1970; Byron and Joachain, 1973a;

Byron, Joachain, and Mund, 1973; Swift, 1974). We shall concentrate on real central potentials and follow the treatment of Byron, Joachain, and Mund (1973), who have made a detailed analysis of this problem for a variety of interaction potentials.

First of all, we note that $\text{Re}\,\bar{f}_{E2} = 0$ [see Eq. (2.43)] while in general $\text{Re}\,\bar{f}_{B2} \neq 0$; hence there is no analog of Eq. (2.45) for $\text{Re}\,\bar{f}_{E2}$ and $\text{Re}\,\bar{f}_{B2}$. We shall return shortly to this point while discussing the relative merits of the second Born and eikonal approximations. For the moment, we focus our attention on $\text{Im}\,\bar{f}_{B2}$ and $\text{Im}\,\bar{f}_{E2}$ and consider the particular case of interactions having the form of a superposition of Yukawa potentials, namely

$$U(r) = U_0 \int_{\alpha_0>0}^{\infty} \rho(\alpha) \frac{e^{-\alpha r}}{r} d\alpha. \tag{2.46}$$

A simple calculation shows that in this case the *leading term* of the second Born expression $\bar{f}_{B2}(k, \Delta)$ for *large k* is such that

$$\bar{f}_{B2}(k, \Delta) = A_{B2}(\Delta)/k + \cdots, \tag{2.47}$$

where the quantity $A_{B2}(\Delta)$ depends only on Δ and is purely imaginary. In writing Eq. (2.47) we have neglected terms of higher order in k^{-1}. On the other hand, we may use Eq. (2.43) and the fact that the eikonal phase shift function χ is proportional to k^{-1} [see Eq. (2.36)] to write

$$\bar{f}_{E2}(k, \Delta) = A_{E2}(\Delta)/k, \tag{2.48}$$

where $A_{E2}(\Delta)$ is also purely imaginary and depends only on Δ.

What is the relationship between the quantities $A_{E2}(\Delta)$ and $A_{B2}(\Delta)$? For Yukawa-type potentials having the form (2.46) it turns out that

$$A_{E2}(\Delta) = A_{B2}(\Delta) \tag{2.49}$$

for *all momentum transfers* Δ. Thus, when k is large enough so that Eq. (2.47) holds, we have (remembering that A_{E2} and A_{B2} are purely imaginary)

$$\text{Im}\,\bar{f}_{E2}(k, \Delta) = \text{Im}\,\bar{f}_{B2}(k, \Delta) \tag{2.50}$$

for *all values of* Δ. For other interactions such as a square-well potential, a Gaussian potential $U(r) = U_0 \exp(-\alpha r^2)$ or a "polarization" potential of the form $U(r) = U_0(r^2 + d^2)^{-2}$, the relation (2.50) holds only for small scattering angles.

The comparison of the terms $\bar{f}_{Bn}$ and $\bar{f}_{En}$ for $n \geq 3$ is a difficult problem which we shall not treat in detail. On the basis of a careful analysis of the first few terms of the Born and eikonal series (up to $n = 4$) and for Yukawa-type potentials of the form (2.46), Byron, Joachain, and Mund (1973) have suggested that the following relations hold: If one writes

$$\bar{f}_{Bn}(k, \Delta) \xrightarrow[k\to\infty]{} A_{Bn}(\Delta)/k^{n-1} + O(k^{-n}) \tag{2.51}$$

and

$$\bar{f}_{En}(k, \Delta) = A_{En}(\Delta)/k^{n-1}, \tag{2.52}$$

then one has

$$A_{En}(\Delta) = A_{Bn}(\Delta) \tag{2.53}$$

for all values of Δ. We remark in this connection that Moore (1970) has shown for a Yukawa potential (and hence by a direct generalization for a superposition of Yukawa potentials) that for large Δ the leading term in $A_{Bn}(\Delta)$, which is proportional to $\log^{n-1}\Delta/\Delta^2$, is equal to the first term in $A_{En}(\Delta)$. However, for Yukawa-type potentials the quantities $A_{Bn}(\Delta)$ and $A_{En}(\Delta)$ are linear combinations of terms of the form $\log^m \Delta/\Delta^2$, with $0 \leq m \leq n - 1$, so that a general proof of Eq. (2.53) is much more difficult to construct. Such a proof has been given recently by Swift (1974).

Remembering that the quantities $\bar{f}_{En}$ are alternately real and purely imaginary, we see that for *large enough k* (and Yukawa-type potentials) the relations (2.51)–(2.53) imply that in addition to Eq. (2.50) we have the higher-order relations

$$\bar{f}_{E3}(k, \Delta) = \text{Re}\,\bar{f}_{B3}(k, \Delta), \tag{2.54a}$$

$$\text{Im}\,\bar{f}_{E4}(k, \Delta) = \text{Im}\,\bar{f}_{B4}(k, \Delta), \tag{2.54b}$$

$$\vdots$$

$$\bar{f}_{En}(k, \Delta) = \text{Re}\,\bar{f}_{Bn}(k, \Delta), \qquad n \text{ odd} \tag{2.54c}$$

$$\text{Im}\,\bar{f}_{En}(k, \Delta) = \text{Im}\,\bar{f}_{B}(k, \Delta), \qquad n \text{ even}, \tag{2.54d}$$

which imply that the eikonal amplitude may be obtained, for fixed momentum transfer, by summing the leading term in each order of the Born series, in the large k limit.

The relationships (2.54) have some important consequences. Let us first consider the weak coupling situation such that the condition

$$\frac{|V_0|a}{\hbar v_i} = \frac{|U_0|a}{2k} \ll 1 \tag{2.55}$$

is added to the inequalities (2.24) and (2.25). In this case the Born series converges rapidly and the relations (2.54) imply that the eikonal amplitude gives a consistently *poorer* approximation to the exact amplitude than does the second Born approximation f_{B2}. This is due to the fact that for $ka \gg 1$ the *exact* amplitude may be written for Yukawa-type potentials as

$$f(k, \Delta) = f_{B1}(\Delta) + \underbrace{\left[\frac{A(\Delta)}{k^2} + i\frac{B(\Delta)}{k}\right]}_{\bar{f}_{B2}} + \underbrace{\left[\frac{C(\Delta)}{k^2} + i\frac{D(\Delta)}{k^3}\right]}_{\bar{f}_{B3}} + \cdots \tag{2.56}$$

where we have introduced the quantities A, B, C, and D which depend only on Δ. In terms of the objects A_{Bn} introduced above, we note that $A_{B2}(\Delta) = iB(\Delta)$, $A_{B3}(\Delta) = C(\Delta)$, etc. On the other hand, the eikonal amplitude has the structure

$$f_E(k, \Delta) = f_{B1}(\Delta) + \underbrace{i\frac{B(\Delta)}{k}}_{\bar{f}_{E2}} + \underbrace{\frac{C(\Delta)}{k^2}}_{\bar{f}_{E3}} + \cdots. \tag{2.57}$$

Therefore neither f_{B2} nor f_E are correct to order k^{-2}. However, since the coefficient A is proportional to U_0^2 while C is proportional to U_0^3, it is clear that for small values of $|U_0|$ the second Born amplitude should be more precise than the eikonal amplitude.

As the coupling increases in such a way that $|V_0|a/\hbar v_i \simeq 1$ but $|V_0|/E < 1$, we expect from the foregoing discussion that the eikonal method should improve steadily. That this is indeed the case may be seen from Fig. 1, which displays the real part of the exact, eikonal, and second Born amplitudes for a superposition of two Yukawa potentials of the form

$$U(r) = U_0(e^{-r/a} - \rho e^{-2r/a})/r \qquad (2.58)$$

with $U_0 = -20$, $a = 1$, $\rho = 1.125$, and $ka = 5$. The excellent agreement between the eikonal and exact results, even at large angles, is particularly striking. Similar conclusions may be drawn from Fig. 2, where the corresponding imaginary parts are shown.

Let us now comment briefly on the *strong coupling situation*, for which $|V_0|a/\hbar v_i > 1$ and $|V_0|/E > 1$. In this case the Born series is useless. On the other hand, and despite the fact that the condition (2.25) is violated, the eikonal approximation is still quite accurate at *small angles* if the high wave number condition (2.24) is satisfied. This feature is illustrated in Figs. 3 and 4 where the real and imaginary parts of the exact and eikonal amplitudes are displayed for an interaction of the type (2.58) with $U_0 = -20$, $a = 1$, $\rho = 1.125$, and $ka = 2$.

We shall not attempt to discuss here various other forms of the eikonal approximation (Saxon and Schiff,

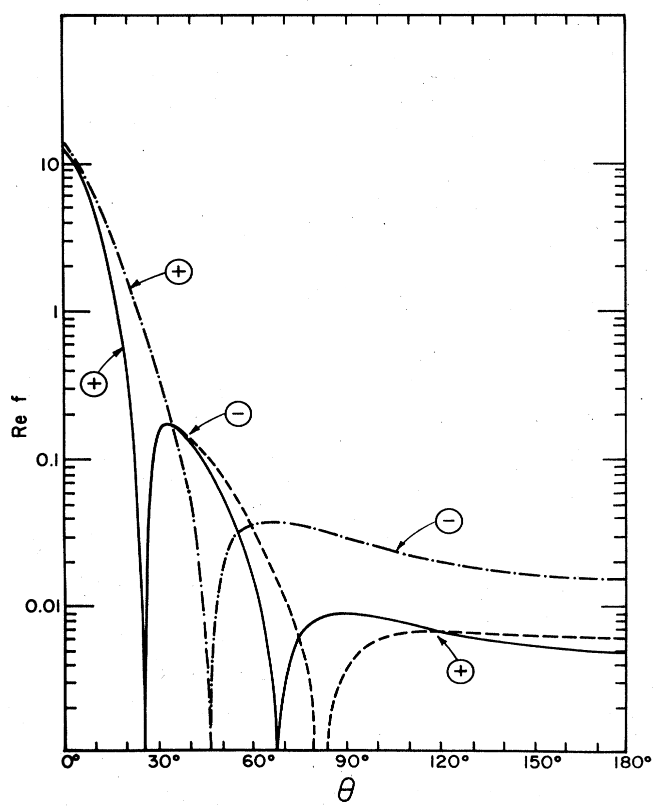


FIG. 1. The real part of the scattering amplitude for a superposition of two Yukawa potentials of the form given in Eq. (2.58), with $U_0 = -20$, $a = 1$, $\rho = 1.125$, and $ka = 5$. The solid curve shows the exact result, the dashed curve gives the eikonal result, and the dashed-dotted curve is the second Born approximation. (From Byron, Joachain, and Mund, 1973.)

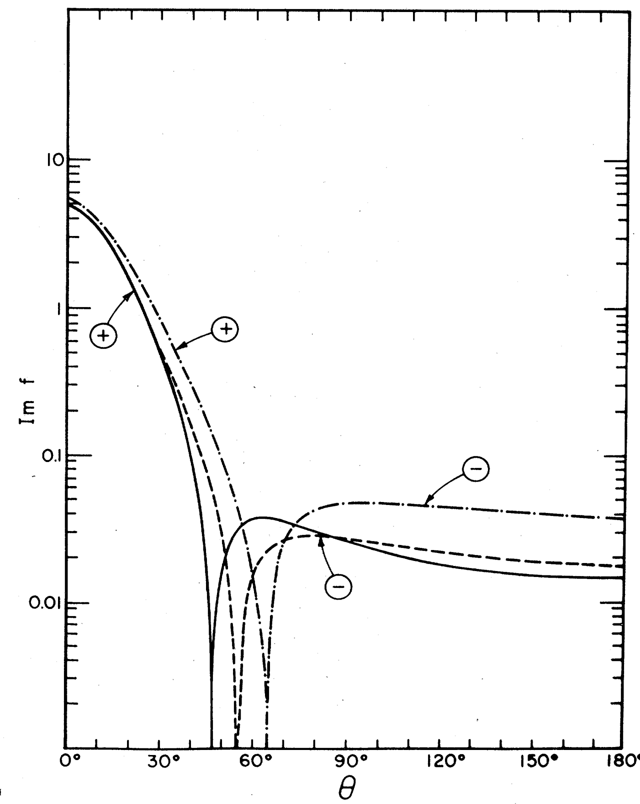


FIG. 2. Same as Fig. 1 except that the imaginary part of the amplitude is shown. (From Byron, Joachain, and Mund, 1973.)

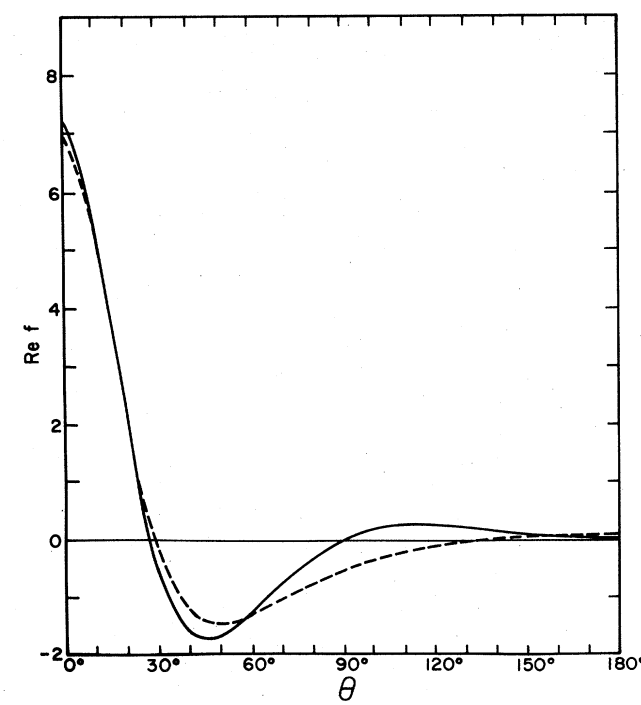


FIG. 3. The real part of the scattering amplitude for a superposition of two Yukawa potentials of the form given in Eq. (2.58), with $U_0 = -20$, $a = 1$, $\rho = 1.125$, and $ka = 2$. The solid curve shows the exact result and the dashed curve gives the eikonal result. (From Byron, Joachain, and Mund, 1973.)

1957; Wu, 1957; Blankenbecler and Goldberger, 1962; Feshbach, 1967; Schiff, 1968; Wilets and Wallace, 1968; Sugar and Blankenbecler, 1969; Hahn, 1969, 1970; Lévy and Sucher, 1969, 1970; Abarbanel and Itzykson, 1969; Moore, 1970; Kujawski, 1971; Wallace, 1971, 1973a, b, c; Baker, 1972, 1973; Swift, 1974). We note, however, that the Glauber form which we have discussed above is probably the simplest eikonal approximation, a feature which is very important when one wants to generalize the method to many-body collisions.

Finally, we note that the derivation of the eikonal scattering amplitude (2.34) may be generalized to relativistic collisions and does not require the existence of a potential to describe the collision process, although an optical potential can always be found to describe the scattering in the eikonal approximation (Glauber, 1959; Omnès, 1965; see also Sec. III.D). Moreover, for high-energy small angle scattering, the basic formula (2.34) is valid in the laboratory system as well as in the center of mass system (Franco and Glauber, 1966). The only modifications are that the center of mass wave vectors $\mathbf{k}_i$ and $\mathbf{k}_f$ must now be replaced by the corresponding laboratory quantities $\mathbf{k}$ and $\mathbf{k}'$, while $\boldsymbol{\Delta} = \mathbf{k}_i - \mathbf{k}_f$ is replaced by $\mathbf{q} = \mathbf{k} - \mathbf{k}'$. Of course the magnitude of $\mathbf{k}'$ is now smaller than that of $\mathbf{k}$ because of recoil effects, but these effects are small for scattering near the forward direction and can be minimized by interpreting the quantity $(-q^2)$ as the Mandelstam variable t, namely the square of the *four*-momentum transfer of the collision.

III. SEVERAL PARTICLE PROBLEMS

A. The Born series and the distorted-wave Born series

Let us consider a general quantum collision process $a \to b$ for which we denote the S-matrix element by $\langle b|S|a\rangle$. The theoretical analysis is conveniently carried out in terms of the $\mathfrak{T}$-matrix elements such that (see, for example, Goldberger and Watson, 1964)

$$\langle b|S|a\rangle = \delta_{ba} - 2\pi i \delta(E_b - E_a)\langle b|\mathfrak{T}|a\rangle. \tag{3.1}$$

It will sometimes prove convenient to use a somewhat more explicit notation and write $a \equiv (i, \alpha)$ and $b \equiv (f, \beta)$ where i and f are "arrangement channel" indices and α and β denote, respectively, the state of the system in the initial and final channel. Thus in the initial channel the total Hamiltonian of the system may be decomposed as

$$H = H_i + V_i, \tag{3.2}$$

where V_i is the interaction between the two colliding particles and the channel Hamiltonian H_i describes these particles when they are far apart and do not interact. We then have

$$H_i \Phi_a = E_a \Phi_a, \tag{3.3}$$

where Φ_a is the corresponding free state vector. Similarly, in the final channel,

$$H = H_f + V_f \tag{3.4}$$

with

$$H_f \Phi_b = E_b \Phi_b. \tag{3.5}$$

We also introduce the Green's operators

$$G^{(\pm)}(E) = (E - H \pm i\epsilon)^{-1}, \tag{3.6a}$$

$$G_i^{(\pm)}(E) = (E - H_i \pm i\epsilon)^{-1}, \tag{3.6b}$$

and

$$G_f^{(\pm)}(E) = (E - H_f \pm i\epsilon)^{-1}. \tag{3.6c}$$

More generally, if c is any arrangement channel index such that $H = H_c + V_c$, we have

$$G_c^{(\pm)} = (E - H_c \pm i\epsilon)^{-1}. \tag{3.6d}$$

Direct collisions are characterized by the fact that the channel Hamiltonians are the same in the initial and final states. Writing $H_i = H_f = H_d$ and $V_i = V_f = V_d$ in this case, we also define

$$G_d^{(\pm)} = (E - H_d \pm i\epsilon)^{-1}. \tag{3.6e}$$

Finally, we shall denote by H_0 the kinetic energy operator of the entire system (i.e., the Hamiltonian H from which the total interaction V has been removed). The corresponding free Green's operator is then

$$G_0^{(\pm)} = (E - H_0 \pm i\epsilon)^{-1}. \tag{3.6f}$$

Let us now examine the "on the energy shell" transition matrix elements $\langle b|\mathfrak{T}|a\rangle$ appearing in Eq. (3.1). It is convenient to factor out a momentum-conserving delta function and to introduce the reduced T-matrix elements T_{ba} such that $\langle b|\mathfrak{T}|a\rangle = \delta(\mathbf{P}_b - \mathbf{P}_a)T_{ba}$. Then (Goldberger and Watson, 1964)

$$T_{ba} = \langle \Phi_b|V_f|\Psi_a^{(+)}\rangle \tag{3.7a}$$

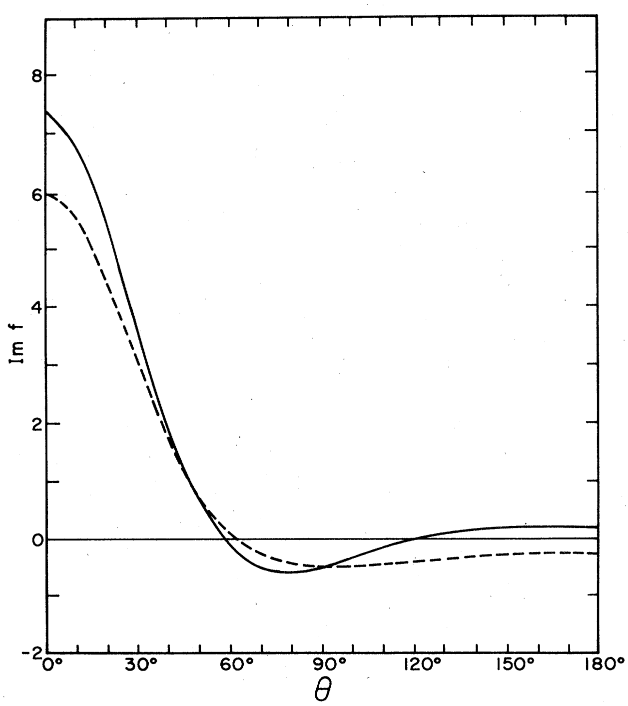


FIG. 4. Same as Fig. 3 except that the imaginary part of the amplitude is shown. (From Byron, Joachain, and Mund, 1973).

or

$$T_{ba} = \langle \Psi_b^{(-)} | V_i | \Phi_a \rangle, \tag{3.7b}$$

where the state vectors $\Psi_a^{(+)}$ and $\Psi_b^{(-)}$ are such that

$$\Psi_a^{(+)} = \Phi_a + G^{(+)} V_i \Phi_a, \tag{3.8a}$$

$$\Psi_b^{(-)} = \Phi_b + G^{(-)} V_f \Phi_b \tag{3.8b}$$

and satisfy the Lippmann–Schwinger equations

$$\Psi_a^{(+)} = \Phi_a + G_i^{(+)} V_i \Psi_a^{(+)}, \tag{3.9a}$$

$$\Psi_b^{(-)} = \Phi_b + G_f^{(-)} V_f \Psi_b^{(-)}. \tag{3.9b}$$

We note that for any arrangement channel c the Green's operators $G^{(\pm)}$ satisfy the relations

$$G^{(\pm)} = G_c^{(\pm)} + G_c^{(\pm)} V_c G^{(\pm)} \tag{3.10a}$$

or

$$G^{(\pm)} = G_c^{(\pm)} + G^{(\pm)} V_c G_c^{(\pm)} \tag{3.10b}$$

which are the Lippmann–Schwinger equations for the full Green's operator. We also define the (Lovelace) transition operators

$$\overline{U}_{fi} = V_f + V_f G^{(+)} V_i, \tag{3.11a}$$

$$U_{fi} = V_i + V_f G^{(+)} V_i, \tag{3.11b}$$

and the full many-body transition operator

$$T = V + V G^{(+)} V. \tag{3.11c}$$

These operators satisfy the Lippman–Schwinger equations

$$\overline{U}_{fi} = V_f + \overline{U}_{fi} G_i^{(+)} V_i, \tag{3.12a}$$

$$U_{fi} = V_i + V_f G_f^{(+)} U_{fi}, \tag{3.12b}$$

and

$$T = V + V G_0^{(+)} T \tag{3.12c}$$

$$= V + T G_0^{(+)} V. \tag{3.12d}$$

On the energy shell $E_a = E_b$ we have $\langle \Phi_b | V_i | \Phi_a \rangle = \langle \Phi_b | V_f | \Phi_a \rangle$ and, therefore, from Eq. (3.7)

$$T_{ba} = \langle \Phi_b | U_{fi} | \Phi_a \rangle = \langle \Phi_b | \overline{U}_{fi} | \Phi_a \rangle. \tag{3.13}$$

A variety of Born series expansions for the transition matrix element T_{ba} may be obtained by solving the various Lippmann–Schwinger equations written above by successive iterations. For example, we may first solve for the full Green's operator $G^{(\pm)}$ from Eqs. (3.10) and write

$$G^{(\pm)} = G_c^{(\pm)} + G_c^{(\pm)} V_c G_c^{(\pm)} + G_c^{(\pm)} V_c G_c^{(\pm)} V_c G_c^{(\pm)} + \cdots. \tag{3.14}$$

Then, upon substitution in Eqs. (3.8) and then in Eqs. (3.7), we find the Born development

$$T_{ba} = \langle \Phi_b | V_i (\text{or } V_f) + V_f G_c^{(+)} V_i + V_f G_c^{(+)} V_c G_c^{(+)} V_i + \cdots | \Phi_a \rangle. \tag{3.15}$$

The first Born approximation consists in retaining only the first term of this expansion, namely

$$T_{ba}^{B1} = \langle \Phi_b | V_i | \Phi_a \rangle = \langle \Phi_b | V_f | \Phi_a \rangle. \tag{3.16}$$

For direct collisions it is natural to choose the propagator $G_c^{(+)}$ so that it coincides with the Green's operators $G_d^{(+)}$ defined by Eq. (3.6e). The corresponding Born series then reads

$$T_{ba} = \langle \Phi_b | V_d + V_d G_d^{(+)} V_d + V_d G_d^{(+)} V_d G_d^{(+)} V_d + \cdots | \Phi_a \rangle. \tag{3.17}$$

Little is known about the mathematical properties of the Born series (3.15). For *direct collisions* the conditions of convergence of the series (3.17) are probably similar to those discussed in Sec. II.A for potential scattering. For example, the Born series (3.17) may well be convergent for nonrelativistic direct processes at sufficiently high colliding energies; this will be illustrated in Sec. IV. In particular, we shall see in the case of the nonrelativistic *elastic* scattering of charged particles by atoms that the *first Born* approximation eventually governs the scattering at sufficiently high energies. For *inelastic* (direct) collisions of charged particles with atoms the *two first* terms of the Born series dominate the scattering at high (nonrelativistic) energies.

On the other hand, when *rearrangement collisions* occur some particles are transferred between the colliding systems during the reaction, so that $V_i \neq V_f$. The question of the convergence of the Born series (3.15) in this case has been investigated by several authors (see, for example, Aaron, Amado, and Lee, 1961; Weinberg, 1963a, b; 1964a, b, c; Bransden, 1965, 1969; Rubin, Sugar, and Tiktopoulos, 1966, 1967a, b; Dettmann and Leibfried, 1968, 1969; Shakeshaft and Spruch, 1973). At low energies the series diverges, and even at high energies its convergence is doubtful. However, Dettmann and Leibfried (1969) have pointed out that for nonrelativistic rearrangement processes occuring in three-body systems, and for a wide class of potentials, the energy variation of the T-matrix element is given correctly at high energies by the first two terms of the Born series. It is interesting to note in this context that variational methods of the Schwinger type (Lippmann and Schwinger, 1950; Lippmann, 1956; Joachain, 1965) also involve in lowest order the first- and second-order terms of the Born series. More recently, Shakeshaft and Spruch (1973) have shown that for a particular (nonrelativistic) three-body rearrangement process (i.e., a pick-up reaction involving two heavy particles of mass M and a light one of mass m, with $m/M \to 0$) the *second* Born term dominates in the forward direction at high energies.

Distorted-wave Born series are obtained by a simple application of the two-potential scattering formalism (Gell-Mann and Goldberger, 1953). Let us assume that the interaction potentials V_i and V_f may be split as

$$V_i = U_i + W_i, \tag{3.18a}$$

$$V_f = U_f + W_f \tag{3.18b}$$

and more generally, in any arrangement channel c,

$$V_c = U_c + W_c. \tag{3.18c}$$

We define the new Hamiltonians $\overline{H}_c = H_c + U_c$, together with the Green's operators $\overline{G}_c^{(\pm)} = (E - \overline{H}_c \pm i\epsilon)^{-1}$, and assume that the distorted waves

$$\chi_a^{(+)} = \Phi_a + \overline{G}_i^{(+)} U_i \Phi_a \tag{3.19a}$$

and

$$\chi_b^{(-)} = \Phi_b + \overline{G}_f^{(-)} U_f \Phi_b \tag{3.19b}$$

are known. The T-matrix elements (3.7) are then given by (Gell-Mann and Goldberger, 1953; Gerjuoy, 1958)

$$T_{ba} = \langle \chi_b^{(-)} | (V_i - W_f) | \Phi_a \rangle + \langle \chi_b^{(-)} | W_f | \Psi_a^{(+)} \rangle \tag{3.20a}$$

or

$$T_{ba} = \langle \Phi_b | (V_f - W_i) | \chi_a^{(+)} \rangle + \langle \Psi_b^{(-)} | W_i | \chi_a^{(+)} \rangle \tag{3.20b}$$

with

$$\Psi_a^{(+)} = \chi_a^{(+)} + G^{(+)} W_i \chi_a^{(+)} \tag{3.21a}$$

and

$$\Psi_b^{(-)} = \chi_b^{(-)} + G^{(-)} W_f \chi_b^{(-)}. \tag{3.21b}$$

The two-potential formulas (3.20) simplify when the distorting potentials U_i and U_f cannot induce the transition $a \to b$ considered. This may happen, for example, if the interactions U_i and U_f only generate elastic scattering and the transition $a \to b$ is an inelastic process or a rearrangement collision. In this case the first term on the right of Eqs. (3.20) vanishes, so that

$$T_{ba} = \langle \chi_b^{(-)} | W_f | \Psi_a^{(+)} \rangle \tag{3.22a}$$

or

$$T_{ba} = \langle \Psi_b^{(-)} | W_i | \chi_a^{(+)} \rangle. \tag{3.22b}$$

If we wish to treat exactly the interactions U_i and U_f but elect to use perturbation theory to handle the interactions W_i and W_f we generate the distorted-wave Born series. For example, using the fact that

$$G^{(\pm)} = \overline{G}_c^{(\pm)} + \overline{G}_c^{(\pm)} W_c \overline{G}_c^{(\pm)} + \overline{G}_c^{(\pm)} W_c \overline{G}_c^{(\pm)} W_c \overline{G}_c^{(\pm)} + \cdots \tag{3.23}$$

we see that Eqs. (3.22) yield, with the help of (3.21),

$$T_{ba} = \langle \chi_b^{(-)} | W_i (\text{or } W_f) + W_f \overline{G}_c^{(+)} W_i + W_f \overline{G}_c^{(+)} W_c \overline{G}_c^{(+)} W_i + \cdots | \chi_a^{(+)} \rangle. \tag{3.24}$$

The first term of this expansion gives the *distorted-wave Born approximation* (DWBA), namely

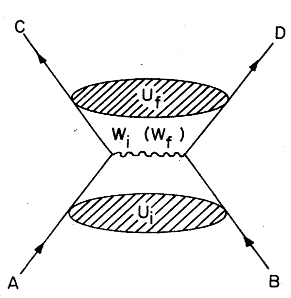


FIG. 5. Illustration of the distorted-wave Born approximation for a process $A + B \to C + D$.

$$T_{ba}^{\text{DWBA}} = \langle \chi_b^{(-)} | W_i | \chi_a^{(+)} \rangle = \langle \chi_b^{(-)} | W_f | \chi_a^{(+)} \rangle. \tag{3.25}$$

With a suitable choice of distorting potentials U_i and U_f this formula may improve significantly over the first Born approximation (3.16), at least for direct collisions. For rearrangement processes the situation is considerably more involved. As for the Born series (3.15), the convergence of the distorted-wave Born series (3.24) is again doubtful in this case (Greider and Dodd, 1966; Dodd and Greider, 1966).

A simple but physically reasonable interpretation may be given of Eq. (3.25). Let us imagine for example that the transition $a \to b$ is a process of the type $A + B \to C + D$ (see Fig. 5). We see that the two particles A and B first feel the *initial state interaction* U_i (embodied in $\chi_a^{(+)}$), then interact *once* through W_i (or W_f), and finally experience the *final state interaction* U_f while emerging from the collision. Since U_i and U_f are treated exactly, we note that the particles are allowed to interact repeatedly through the distorting potentials.

The DWBA formula (3.25) has been used extensively in atomic and nuclear physics (see for example Mott and Massey, 1965; Tobocman, 1961). It also provided an intuitive starting point for the various high-energy absorption models (Sopkovich, 1962; Gottfried and Jackson, 1964; Durand and Chiu, 1964, 1965a,b; Jackson, 1965). We shall discuss in Secs. IV and V a few applications of the *eikonal DWBA approximations*, in which the distorted waves $\chi_a^{(+)}$ and $\chi_b^{(-)}$ appearing in Eq. (3.25) are obtained with the help of the eikonal approximation.

B. The Faddeev–Lovelace–Watson expansions

In this section we shall study a nonrelativistic three-body system such that the particles 1, 2, 3 interact by means of two-body interactions. We shall denote by $V^1 \equiv V_{23}$ the potential acting between the particles 2 and 3, while $V^2 \equiv V_{13}$ acts between 1 and 3, and $V^3 \equiv V_{12}$ between 1 and 2. The total Hamiltonian of the system is then

$$H = H_0 + V, \tag{3.26}$$

where H_0 is the kinetic energy operator and

$$V = \sum_{i=1}^{3} V^i. \tag{3.27}$$

We shall also need the Hamiltonian describing two particles interacting while the third one is free, namely

$$H_i = H_0 + V^i \tag{3.28}$$

and we define the operators

$$V_i = V - V^i \tag{3.29}$$

corresponding to the interactions in which particle i participates. (For example: $V_1 = V - V^1 = V_{12} + V_{13}$.) The Green's operators corresponding to H, H_i, and H_0 are defined, respectively, by Eqs. (3.6a), (3.6b), and (3.6f).

The *two-body* T-matrices are given by

$$T_i = V^i + V^i G_i^{(+)} V^i \tag{3.30a}$$

$$= V^i + T_i G_0^{(+)} V^i \tag{3.30b}$$

$$= V^i + V^i G_0^{(+)} T_i. \tag{3.30c}$$

We also note that

$$G_i^{(\pm)} = G_0^{(\pm)} + G_0^{(\pm)} V^i G_i^{(\pm)} \quad (3.31a)$$

$$= G_0^{(\pm)} + G_i^{(\pm)} V^i G_0^{(\pm)} \quad (3.31b)$$

$$= G_0^{(\pm)} + G_0^{(\pm)} T_i G_0^{(\pm)} \quad (3.31c)$$

and

$$G_i^{(+)} V_i = G_0^{(+)} T_i. \quad (3.31d)$$

We shall describe the various possible modes of fragmentation of the three-body system by indices i, f which take on the values 0, 1, 2, 3. Thus $i = 0$ corresponds to three free particles in the initial state, $i = 1$ means that initially the particle 1 is free and the pair (2, 3) is bound, etc. A collision process $a \to b$ is then described by the reduced transition matrix T_{ba}, given by Eqs. (3.7) and (3.13). More explicitly, we shall write (on the energy shell)

$$T_{ba} \equiv T_{f\beta,i\alpha} = \langle \Phi_{f\beta} | U_{fi} | \Phi_{i\alpha} \rangle = \langle \Phi_{f\beta} | \overline{U}_{fi} | \Phi_{i\alpha} \rangle \quad (3.32)$$

where the indices α and β contain additional information on the momenta, spin, bound states, etc., of the initial and final states considered. Moreover, the operators $\overline{U}_{fi}$ and U_{fi} are given, respectively, by Eqs. (3.11a) and (3.11b) with $V_i = V - V^i$ and $V_f = V - V^f$.

Before we turn to the problem of obtaining multiple scattering expansions for the operators U_{fi} and $\overline{U}_{fi}$, we recall briefly the work of Faddeev (1960, 1961, 1962), who writes the full three-body operator $T = V + VG^{(+)}V$ as

$$T = T^{(1)} + T^{(2)} + T^{(3)}, \quad (3.33)$$

where $T^{(1)}$ represents the sum of all contributions to T in which the particles 2 and 3 interact last. The objects $T^{(i)}$ then satisfy the Faddeev equations

$$\begin{pmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} + \begin{pmatrix} 0 & T_1 & T_1 \\ T_2 & 0 & T_2 \\ T_3 & T_3 & 0 \end{pmatrix} G_0^{(+)} \begin{pmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{pmatrix} \quad (3.34)$$

which exhibit much better mathematical properties than the Lippmann–Schwinger equations (3.12). The Faddeev approach to the three-body problem immediately attracted a great deal of attention (see, for example, Lovelace, 1964a,b; Weinberg, 1964a; Omnès, 1964; Rosenberg, 1964) and possible applications to a number of nuclear and atomic problems have been investigated (a list of references may be found in Watson and Nuttall, 1967, and Chen and Joachain, 1971). Extensions of the Faddeev equations to relativistic three-body problems have also been proposed (Alessandrini and Omnès, 1965; Freedman, Lovelace, and Namysłowski, 1966; Blankenbecler and Sugar, 1966).

A slightly different version of the Faddeev equations, derived by Lovelace (1964a), involves the operators U_{fi} and $\overline{U}_{fi}$ which lead directly to the transition matrix elements (3.32) for a process $i\alpha \to f\beta$. The result is

$$\overline{U}_{fi} = V_f + \sum_{k \neq i} \overline{U}_{fk} G_0^{(+)} T_k \quad (3.35a)$$

and

$$U_{fi} = V_i + \sum_{k \neq f} T_k G_0^{(+)} U_{ki}, \quad (3.35b)$$

where $i, f, k = 1, 2, 3$. The case $i = f = 0$ may also be included by defining $T_0 = 0$. For example, if $i = 1$, i.e., the particle 1 is incident on the bound pair (2,3), we find that Eq. (3.35b) becomes

$$\begin{aligned} U_{11} &= V_1 + T_2 G_0^{(+)} U_{21} + T_3 G_0^{(+)} U_{31}, \\ U_{21} &= V_1 + T_1 G_0^{(+)} U_{11} + T_3 G_0^{(+)} U_{31}, \\ U_{31} &= V_1 + T_1 G_0^{(+)} U_{11} + T_2 G_0^{(+)} U_{21}. \end{aligned} \quad (3.36)$$

We note that the matrix kernel of the Lovelace equations (3.36) is just the transpose of the Faddeev kernel appearing in (3.34), so that all the mathematical properties of the Faddeev kernel apply equally well to the Lovelace kernel. In constrast with the Faddeev equations, however, the Lovelace equations involve interaction *potentials*. Nevertheless, a simple modification of the Lovelace formalism also yields equations which do not include any direct reference to potentials (Alt, Grassberger, and Sandhas, 1967). A comparison between the Faddeev and the Lovelace–Alt approaches to the three-body problem has been made recently by Osborn and Kowalski (1971). It is also worth pointing out that the Faddeev or Lovelace equations are closely related to Watson's multiple scattering equations (Watson, 1953, 1956, 1957; see also Goldberger and Watson, 1964; Watson and Nuttall, 1967). We shall return to this point later.

Let us now investigate how to obtain multiple scattering expansions for various three-body processes (Ekstein, 1956; Rosenberg, 1964; Queen, 1964, 1966; Bransden, 1965; Sloan, 1967, 1968; Chen and Joachain, 1971). In what follows we shall concentrate on the *intermediate and high-energy* regions such that the relative kinetic energy of the incident particle 1 with respect to the target (2,3) is large compared to the binding energy of that target ("weak binding" condition).

We start with the case $f = 1$ (elastic and inelastic direct processes) and return to the Lovelace equations (3.36). A simple iteration of these equations gives

$$\begin{aligned} U_{11} = {} & V_1 + T_2 G_0^{(+)} V_1 + T_3 G_0^{(+)} V_1 \\ & + T_2 G_0^{(+)} T_1 G_0^{(+)} V_1 + T_2 G_0^{(+)} T_3 G_0^{(+)} V_1 \\ & + T_3 G_0^{(+)} T_1 G_0^{(+)} V_1 + T_3 G_0^{(+)} T_3 G_0^{(+)} V_1 + \cdots. \end{aligned} \quad (3.37)$$

Then, using the fact that $V - V^1 = V^2 + V^3$ and eliminating the potentials in favor of the two-body T-matrices by repeated use of Eq. (3.30b), we find that

$$\begin{aligned} U_{11} = {} & T_2 + T_3 + T_2 G_0^{(+)} T_3 + T_3 G_0^{(+)} T_2 \\ & + T_2 G_0^{(+)} T_1 G_0^{(+)} T_2 + T_2 G_0^{(+)} T_3 G_0^{(+)} T_2 \\ & + T_2 G_0^{(+)} T_1 G_0^{(+)} T_3 + T_3 G_0^{(+)} T_1 G_0^{(+)} T_2 \\ & + T_3 G_0^{(+)} T_1 G_0^{(+)} T_3 + T_3 G_0^{(+)} T_2 G_0^{(+)} T_3 + \cdots. \end{aligned} \quad (3.38)$$

Similar expansions may be found for the case of rearrangement collisions. For example, when $f = 3$, i.e., for a process $1 + (2,3) \to (1,2) + 3$ we obtain from Eqs. (3.36)

$$U_{31} = V^3 + T_2 + T_1 G_0^{(+)} T_2 + T_1 G_0^{(+)} T_3 + T_2 G_0^{(+)} T_3 + \cdots. \quad (3.39)$$

For three-body breakup collisions $1 + (2,3) \rightarrow 1 + 2 + 3$, we must first include the channels $i = f = 0$ in the Lovelace equations (with $T_0 = 0$). Then

$$U_{01} = T_2 + T_3 + T_1 G_0^{(+)} T_2 + T_1 G_0^{(+)} T_3 + T_2 G_0^{(+)} T_3 + T_3 G_0^{(+)} T_2 + \cdots. \quad (3.40)$$

The rules for obtaining higher-order multiple scattering terms in the expansions (3.38)–(3.40) are easily derived:

(i) Start from the right with a two-body T-matrix for any of the pairs which participates in the initial interaction $V_i = V - V^i$

(ii) Write $G_0^{(+)}$ and T_i alternatively, avoiding the repetition of adjacent indices.

(iii) Terminate to the desired order with a two-body T-matrix for any of the pairs which participate in the final interaction $V_f = V - V^f$.

The multiple scattering expansions (3.38)–(3.40) have been obtained by using the operators U_{fi}. Similar expansions may of course be written down by making use of the operators $\overline{U}_{fi}$. For direct collisions one finds again (3.38), while the new rearrangement and breakup series are, respectively,

$$\overline{U}_{31} = V^1 + T_2 + T_1 G_0^{(+)} T_2 + T_1 G_0^{(+)} T_3 + T_2 G_0^{(+)} T_3 + \cdots \quad (3.41)$$

and

$$\overline{U}_{01} = V^1 + T_2 + T_3 + T_1 G_0^{(+)} T_2 + T_1 G_0^{(+)} T_3 + T_2 G_0^{(+)} T_3 + T_3 G_0^{(+)} T_2 + \cdots. \quad (3.42)$$

By comparing Eqs. (3.40) and (3.42) we expect that the interaction V^1 should not contribute to the breakup transition matrix element. This is easily verified since $\langle \Phi_{0\beta} | V^1 | \Phi_{1\alpha} \rangle = \langle \Phi_{0\beta} | H_1 - H_0 | \Phi_{1\alpha} \rangle = 0$.

Let us comment briefly on the multiple scattering expansions which we have generated. First of all, it is a simple matter to verify that these expansions may also be obtained from the Watson multiple scattering equations which (in the weak binding limit) read in this case

$$\Psi_{1\alpha}^{(+)} = \Phi_{1\alpha} + \sum_{j=2}^{3} G_1^{(+)} T_j \varphi_j. \quad (3.43a)$$

Here the effective waves φ_j are given by

$$\varphi_j = \Phi_{1\alpha} + \sum_{k \neq j} G_1^{(+)} T_k \varphi_k \qquad (k = 2,3) \quad (3.43b)$$

and may be readily expressed in terms of the free Green's operator $G_0^{(+)}$ by using Eqs. (3.31). We also note that the Faddeev–Lovelace–Watson expansions (3.38)–(3.42) are rearrangements of the Born series (3.15). However, in contrast with the Born development, and except for bare-potential first terms, there are *no disconnected terms* (i.e., contributions such that two particles interact while the third one remains undisturbed) in the Faddeev–Lovelace Watson expansions. Hence these expansions may exhibit better convergence properties than the Born series, especially for problems involving short-range strong interactions where the use of two-body T-matrices (instead of the corresponding potentials) is desirable.

If we write approximately (for weak coupling situations) $T_i \simeq V^i$ and limit our expansions (3.38)–(3.42) to the first order in the *interaction potentials*, we recover the first Born approximation (3.16). It is very important to note that this procedure is valid only for *weak couplings*. For any problem in which the potentials can bind particles, the validity of the first Born expression must be examined carefully.

Let us return to the multiple scattering expansion (3.38) for direct (elastic or inelastic) scattering. At sufficiently high energies a useful approximation consists in keeping only the first-order terms of this series, so that the corresponding transition matrix element reads

$$T_{ba} = T_{1\beta 1\alpha} \simeq \langle \Phi_{1\beta} | T_2 + T_3 | \Phi_{1\alpha} \rangle \quad (3.44)$$

and we recover the *impulse approximation* (Fermi, 1936; Chew, 1950; Chew and Wick, 1952; Ashkin and Wick, 1952; Chew and Goldberger, 1952) for the process considered. We note that the two-body T-matrices T_2 and T_3 describe the scattering of the incident particle 1 by the two target particles 2 and 3 as if those particles were free. The effect of the interaction $V^1 = V_{23}$ between the two target particles appears only in higher-order terms of the series (3.38).

As a final remark, we note that the Faddeev–Lovelace–Watson expansions presented in this section may be generalized to systems with more than three particles. The three-body system considered here was only selected as the obvious prototype of many-body scattering.

C. The eikonal approximation for many-body collisions

The extension of the eikonal approximation to many-body scattering problems was first proposed by Glauber (1953, 1955, 1959, 1960, 1967, 1969) in connection with high-energy, small angle hadron–nucleus collisions. The resulting *high-energy diffraction theory* is in fact a generalization of the classical Fraunhofer diffraction theory (see, for example, Born and Wolf, 1964).

Consider a fast point particle A incident on a composite target B (such as a nucleus or an atom) which contains N scatterers. We assume that the internal motion of the target particles is slow compared with the relative motion of A and B. Moreover, we suppose that the incident particle interacts with the target scatterers via two-body spin-independent interactions. The Glauber scattering amplitude for a small angle direct collision leading from an initial target state $|0\rangle$ to a final state $|m\rangle$ is given in the center of mass system by

$$F_{m0}^{G} = \frac{k_i}{2\pi i} \int d^2\mathbf{b} \exp[\, i\mathbf{\Delta} \cdot \mathbf{b}\,] \times \langle m | \{\exp[i\chi_{\text{tot}}^{G}(\mathbf{b}, \mathbf{b}_1, \ldots, \mathbf{b}_N)] - 1\} | 0 \rangle, \quad (3.45a)$$

the corresponding differential cross section being

$$d\sigma_{m0}/d\Omega = (k_f/k_i) |F_{m0}^{G}|^2. \quad (3.45b)$$

Here $\mathbf{\Delta} = \mathbf{k}_i - \mathbf{k}_f$ is the center of mass wave vector transfer, while

$$\mathbf{r} = \mathbf{b} + z\hat{\mathbf{z}} \tag{3.46}$$

is the initial relative coordinate and

$$\mathbf{r}_j = \mathbf{b}_j + z_j\hat{\mathbf{z}} \tag{3.47}$$

are the coordinates of the target particles (relative to the target center of mass). The z axis may be chosen along $\mathbf{k}_i$ for small angle collisions, but we shall also consider other choices below [see the discussion preceeding Eq. (2.34)]. The total Glauber phase shift function

$$\chi_{\text{tot}}^G(\mathbf{b}, \mathbf{b}_1, \ldots, \mathbf{b}_N) = \sum_{j=1}^{N} \chi_j(\mathbf{b} - \mathbf{b}_j) \tag{3.48}$$

is just the sum of the phase shifts χ_j contributed by each of the target scatterers as the wave representing the incident particle progresses through the target system. We note that if the elementary interactions between the incident particle and the target particles are genuine two-body problems (such as in nonrelativistic electron–atom collisions) the phase shift functions χ_j are purely real. On the contrary, if these elementary interactions lead to several final channels (such as $\pi + N \rightarrow A_1 + N$, where N is a nucleon in a target nucleus) the phase shift functions χ_j are complex.

The crucial property of phase shift additivity, expressed by Eq. (3.48), is clearly a direct consequence of the one-dimensional nature of the relative motion, together with the neglect of three-body forces, target scatterer motions, and longitudinal momentum transfer.

Another important remark concerning Eq. (3.45a) is that it applies only to collisions for which the energy transfer ΔE is small compared with the incident particle energy E_i. This is true for elastic collisions and for "mildly" inelastic ones in which the target is excited or perhaps breaks up. It is not true for "deeply" inelastic collisions in which the nature of the incident or target particles is modified or the number of particles is altered during the collision. We shall leave aside such processes in what follows and comment briefly on them in Sec. V. It is also worth noting that if we neglect recoil effects, which are small near the forward direction, we may write the Glauber scattering amplitude in the laboratory system as

$$F_{m0}^G = \frac{k}{2\pi i}\int d^2\mathbf{b}\, \exp(i\mathbf{q}\cdot\mathbf{b}) \times \langle m|\{\exp[i\chi_{\text{tot}}^G(\mathbf{b}, \mathbf{b}_1, \ldots, \mathbf{b}_N)] - 1\}|0\rangle, \tag{3.49}$$

where $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ is now the laboratory wave vector transfer, and we have denoted the initial and final laboratory wave numbers by $\mathbf{k}$, and $\mathbf{k}'$, respectively. This last expression is more convenient for analyzing high-energy hadron–nucleus collisions since we want the nuclei to remain nonrelativistic and we also wish to compare directly hadron–nucleus cross sections with those on free nucleons (see Sec. V). Defining the quantity (Glauber, 1959)

$$\Gamma_{\text{tot}}(\mathbf{b}, \mathbf{b}_1, \ldots, \mathbf{b}_N) = 1 - \exp[i\chi_{\text{tot}}^G(\mathbf{b}, \mathbf{b}_1, \ldots, \mathbf{b}_N)] \tag{3.50}$$

we see that Eq. (3.49) becomes

$$F_{m0}^G = \frac{ik}{2\pi}\int d^2\mathbf{b}\, \exp(i\mathbf{q}\cdot\mathbf{b}) \times \langle m|\Gamma_{\text{tot}}(\mathbf{b}, \mathbf{b}_1, \ldots, \mathbf{b}_N)|0\rangle. \tag{3.51}$$

Introducing the quantities

$$\Gamma_j(\mathbf{b} - \mathbf{b}_j) = 1 - \exp[i\chi_j(\mathbf{b} - \mathbf{b}_j)], \tag{3.52}$$

Glauber now writes

$$\Gamma_{\text{tot}}(\mathbf{b}, \mathbf{b}_1, \ldots, \mathbf{b}_N) = 1 - \prod_{j=1}^{N}[1 - \Gamma_j(\mathbf{b} - \mathbf{b}_j)] \tag{3.53}$$

or

$$\Gamma_{\text{tot}} = \sum_{j=1}^{N}\Gamma_j - \sum_{j\neq l}\Gamma_j\Gamma_l + \cdots + (-)^{N-1}\prod_{j=1}^{N}\Gamma_j. \tag{3.54}$$

This last equation, when substituted in Eq. (3.51), leads directly to an interpretation of the collision in terms of a multiple scattering expansion involving the incident particle and the various target scatterers. The term linear in Γ_j on the right-hand side of Eq. (3.54) accounts for the "single scattering" (impulse) contribution to the scattering amplitude, whereas the next terms provide double, triple, $\cdots$ scattering corrections. We note that the order of the multiple scattering can at most be N, reflecting the fact that the scattering is focused in the forward direction.

It is important to realize that the above generalization of the eikonal method makes no reference to interaction potentials; only the two-body phase shift functions χ_j (or the functions Γ_j) must be known in order to calculate Γ_{tot}. This fact makes the Glauber formula (3.49) particularly useful for the analysis of high-energy hadron–nucleus scattering, as we shall illustrate in Sec. V.

If the basic two-body interactions are known, as in atomic physics, we can actually gain further insight by obtaining the eikonal scattering amplitude in terms of these interaction potentials. For example, if we consider the nonrelativistic scattering of a charged "elementary" particle (i.e., a particle which does not exhibit any internal structure in the collision considered) by an atom, and if we work in the center of mass system, we may write the full eikonal wave function as a direct generalization of the expression (2.27), namely

$$\Psi_E(\mathbf{r}, X) = (2\pi)^{-3/2} \times \exp\left(i\,\mathbf{k}_i\cdot\mathbf{r} - \frac{i}{\hbar v_i}\int_{-\infty}^{z} V_i(\mathbf{b}, z', X)\,dz'\right)\psi_0(X). \tag{3.55}$$

Here $\mathbf{r}$ is the initial relative coordinate, $v_i = \hbar k_i/M_i$ is the initial relative velocity (with M_i the reduced mass in the initial channel), X denotes collectively the target coordinates, and $\psi_0(X)$ is the initial bound-state wave function of the target. The potential V_i is the full initial channel interaction between the incident particle and all the particles in the target. The corresponding transition matrix element is then given by Eq. (3.7a) in which the exact state vector $\Psi_a^{(+)}$ is replaced by Ψ_E. A similar expression may also be obtained from Eq. (3.7b). For a direct collision process ($V_i = V_f = V_d$) leading to a final target state $|m\rangle$, we may write more explicitly the many-body eikonal scattering amplitude as

$$F_{m0} = -\frac{M_i}{2\pi\hbar^2}\int d^2\mathbf{b}\,dz\,\exp(i\mathbf{\Delta}\cdot\mathbf{r}) \times \langle m|V_d(\mathbf{b},z,X)\exp\left(-\frac{i}{\hbar v_i}\int_{-\infty}^{z} V_d(\mathbf{b},z',X)\,dz'\right)|0\rangle. \tag{3.56}$$

For *elastic* scattering processes such that $|\mathbf{k}_i| = |\mathbf{k}_f| = k$, and if we choose the z axis to be perpendicular to the momentum transfer, we may perform the z integral in Eq. (3.56) to obtain the Glauber result [see Eq. (3.45) with $m = 0$]

$$F_{\text{el}}^{G} = \frac{k}{2\pi i}\int d^2\mathbf{b}\,\exp(i\mathbf{\Delta}\cdot\mathbf{b}) \times \langle 0|\{\exp[i\chi_{\text{tot}}^{G}(\mathbf{b},\mathbf{b}_1,\ldots,\mathbf{b}_N)] - 1\}|0\rangle \tag{3.57}$$

with

$$\chi_{\text{tot}}^{G}(\mathbf{b},\mathbf{b}_1,\ldots,\mathbf{b}_N) = -\frac{1}{\hbar v_i}\int_{-\infty}^{+\infty} V_d(\mathbf{b},z,X)\,dz. \tag{3.58}$$

However, for *inelastic* (direct) processes the Glauber scattering amplitude (3.45) can only be obtained from Eq. (3.56) by neglecting the longitudinal momentum transfer since $\mathbf{\Delta}$ now lies along $\mathbf{k}_i$ in the case of forward scattering (choosing the z axis perpendicular to $\mathbf{\Delta}$ would therefore be rather unnatural in this case). This neglect of the longitudinal momentum transfer is not too serious for mildly inelastic hadron–nucleus collisions at high energies, but it leads to undesirable features in atomic collisions. We shall return to this point in Sec. IV.

Instead of generating a multiple scattering expansion in terms of the quantities Γ_j (which in turn, as we shall see in Sec. V, may be obtained from the two-body scattering amplitudes describing the scattering of the incident particle by the jth scatterer), we may also write from Eq. (3.45) another multiple scattering series which is more closely related to the one we have analyzed in connection with potential scattering (see Sec. II.C). Limiting ourselves to elastic scattering, we write the Glauber scattering amplitude (3.57) as

$$F_{\text{el}}^{G} = \sum_{n=1}^{\infty} \bar{F}_{Gn}, \tag{3.59}$$

where

$$\bar{F}_{Gn} = \frac{k}{2\pi i}\frac{i^n}{n!}\int d^2\mathbf{b}\,\exp(i\mathbf{\Delta}\cdot\mathbf{b}) \times \langle 0|[\chi_{\text{tot}}^{G}(\mathbf{b},\mathbf{b}_1,\ldots,\mathbf{b}_N)]^n|0\rangle. \tag{3.60}$$

We shall also denote by F_{Gn} the sum of the first n terms of the series (3.59). Thus

$$F_{Gn} = \sum_{j=1}^{n} \bar{F}_{Gj}. \tag{3.61}$$

With the choice of z axis which we have adopted ($\hat{\mathbf{z}}$ perpendicular to $\mathbf{\Delta}$), it is a simple matter to see that for all scattering angles

$$F_{G1} = F_{B1}, \tag{3.62}$$

where F_{B1} is the corresponding first Born scattering amplitude. Higher terms of the Glauber series (3.59) and of the Born series will be examined in Sec. IV for electron–atom collisions.

We have so far studied the many-body generalization of the eikonal method proposed by Glauber. Various attempts at deriving or improving Glauber's method by starting from the multiple scattering formalism (Goldberger and Watson, 1964; Kerman, McManus, and Thaler, 1959) have been made by several authors (Czyz and Maximon, 1968, 1969; Remler, 1968, 1971; Feshbach and Hüfner, 1970; Tarasov and Tseren, 1970; Kelly, 1971, Eisenberg, 1972; Manning, 1972; Karlsson and Namysłowski, 1972; Namysłowski, 1972a; Tobocman and Pauli, 1972; Kujawski, 1972; Kujawski and Lambert, 1973). The Glauber result (3.45) may also be viewed as an eikonal approximation to a model proposed by Chase (1956), in which the target particles are frozen in a given configuration (Mittleman, 1970). Osborn (1970) has used the Faddeev equations to suggest a way of unitarizing the impulse approximation and obtaining Glauber-type results without the eikonal approximation, while Janev and Salin (1972) have studied the Faddeev–Lovelace equations in the eikonal approximation for a three-body problem with two heavy particles and a light one. We shall see in Sec. IV that the Glauber approximation is seriously deficient for the treatment of atomic collision problems. However, we shall also show that the combined use of the eikonal series and the Born series (such that higher Born terms are calculated by means of the eikonal approximation) yields very encouraging results for electron–atom collisions at intermediate energies (Byron and Joachain, 1973c, d). This *eikonal–Born series* (EBS) method will be illustrated in Sec. IV.B and IV.C for elastic scattering of fast electrons by atomic hydrogen and helium.

Many-body collisions may also be studied by using the eikonal approximation together with the *optical model formalism*. For *elastic collisions* one first tries to obtain an *optical potential* which is subsequently "eikonalized." The optical model concept may also be used within the framework of the eikonal DWBA approximation to study *inelastic collisions*. The basic problem in this approach is the determination of optical potentials, a question which we now briefly review from the point of view of multiple scattering theory.

D. Multiple scattering approach to the optical potential

The earlier applications of the optical model method were made to the analysis of the propagation of light through a refractive medium. In this case the use of a complex refractive index is in fact equivalent to the introduction of an optical potential (see, for example, Lax, 1951). A generalization of the optical model idea was made by Ostrofsky, Breit, and Johnson (1936) to the study of α decay of nuclei, while Bethe (1940) introduced the concept of an optical potential model for low-energy nuclear collisions. The description of high-energy nuclear collisions within the optical model formalism was initiated by Serber *et al.* (Serber, 1947; Fernbach, Serber, and Taylor, 1949), who first described nucleon–nucleus collisions in terms of nucleon–nucleon scattering. Their multiple scattering analysis led to the conclusion that particles should move more or less freely through nuclear matter at high energies. This fact was verified qualitatively by experiment and led to a reassessment of the optical model for low-energy nuclear scattering (see, for example, Le Levier and Saxon, 1952; Feshbach, Porter, and Weisskopf, 1954).

After the work of Serber *et al.*, several attempts were made to derive the optical model from first principles (Francis and Watson, 1953; Riesenfeld and Watson, 1956; Feshbach, 1958, 1962; Kerman, McManus, and Thaler, 1959; Glauber, 1959). We shall summarize here the multiple scattering derivations of Watson *et al.* (Goldberger and Watson, 1964; Fetter and Watson, 1965) and of Glauber (1959).

Let us assume first that the incident particle is distinct from each of the N scatterers in the target. We write the total Hamiltonian of the system as $H = H_d + V_d$, where the direct arrangement channel Hamiltonian H_d includes the kinetic energy of the colliding particles and the internal target Hamiltonian, while V_d is the interaction between the incident particle and the target system. Thus

$$V_d = \sum_{j=1}^{n} v_j,$$

where v_j is the interaction between the beam particle and the jth target scatterer. Assuming that the target is initially in the state $|0\rangle$, we call $\Psi_{c,a}^{(+)}$ that part of the complete state vector $\Psi_a^{(+)}$ corresponding to coherent (elastic) scattering. That is,

$$\Psi_{c,a}^{(+)} = \Pi_0 \Psi_a^{(+)}, \tag{3.63}$$

where Π_0 is a projection operator onto the state $|0\rangle$. We may therefore introduce formally an optical potential operator $V_{\rm opt}$ such that

$$\Psi_{c,a}^{(+)} = \Phi_a + G_d^{(+)} V_{\rm opt} \Psi_{c,a}^{(+)} \tag{3.64}$$

with $G_d^{(+)} = (E - H_d + i\epsilon)^{-1}$. Thus the optical potential is defined as an operator which, through the Lippmann–Schwinger equation (3.64) (or the corresponding one for the elastic T-matrix) leads to the exact transition amplitude for elastic scattering of the incident particle by the target.

Following the method of Watson *et al.* (Goldberger and Watson, 1964; Fetter and Watson, 1965) one may introduce an operator F defined by

$$\Psi_a^{(+)} = F\Psi_{c,a}^{(+)} \tag{3.65}$$

in terms of which the optical potential, which does not depend on the internal coordinates of the target, is given by

$$V_{\rm opt} = \langle 0|V_d F|0\rangle. \tag{3.66}$$

The operator F satisfies the Lippmann–Schwinger equation

$$F = 1 + G_d^{(+)}(1 - \Pi_0)V_d F \tag{3.67}$$

which can be solved by successive iterations. In this way one generates for F a Born series in powers of the interaction V_d, namely

$$F = 1 + G_d^{(+)}(1 - \Pi_0)V_d + \cdots \tag{3.68}$$

which, substituted into Eq. (3.66) yields

$$V_{\rm opt} = \langle 0|V_d|0\rangle + \langle 0|V_d G_d^{(+)}(1 - \Pi_0)V_d|0\rangle + \cdots \tag{3.69}$$

As an illustration of the use of Eq. (3.69), let us consider the nonrelativistic elastic scattering of an "elementary" particle A of charge Q by a neutral atom B having Z electrons (Mittleman and Watson, 1959, 1960; Mittleman, 1961, 1965). We treat the collision in the center of mass system, using the relative coordinate $\mathbf{r}$ which joins the position of the atomic nucleus (which we assume to coincide with the center of mass of the atom) to that of the particle A. We also denote by $\mathbf{r}_j(j = 1, 2, \ldots, Z)$ the vectors which determine the positions of the atomic electrons. The relative kinetic energy operator is $K = -\hbar^2\nabla_r^2/2M$, where M is the reduced mass of the two colliding particles A and B. We assume that the internal target Hamiltonian h of the atom is such that $h \mid n\rangle = w_n \mid n\rangle$, the atom being in the state $|0\rangle$ before and after the collision. The interaction V_d is the sum of the individual interactions of the incident particle A with the $(Z + 1)$ particles of the target. Neglecting all but Coulomb interactions, we have

$$V_d = \frac{ZeQ}{r} + \sum_{j=1}^{Z} \frac{-eQ}{|\mathbf{r} - \mathbf{r}_j|}. \tag{3.70}$$

We also ignore for the moment the possible effects of the Pauli principle between the incident and target particles.

The first term on the right of Eq. (3.69) is simply the *static* potential $\langle 0|V_d|0\rangle$. With the help of Eq. (3.70), we see that in the case considered here the first approximation to the optical potential is given by

$$V^{(1)}(\mathbf{r}) = \langle 0|V_d|0\rangle = \frac{ZeQ}{r} - Qe\sum_{j=1}^{Z}\left\langle 0\left|\frac{1}{|\mathbf{r} - \mathbf{r}_j|}\right|0\right\rangle. \tag{3.71}$$

This expression may be readily evaluated for simple atoms or when an independent particle model (such as the Hartree–Fock method) is used to describe the state $|0\rangle$ of the target. The static potential (3.71) has been used frequently to describe the elastic scattering of charged particles by atoms. We note, however, that this potential does not include several important features of the collision. For example, it does not take into account the *polarization* of the atom due to the presence of the incident particle having the charge Q. Moreover, at energies above the excitation threshold of the target the static potential (3.71), which is *real*, does not account for the removal of incident particles from the initial channel. Furthermore, if the incident particle is identical to one of the target scatterers (i.e., in this case an electron or an ion identical to the nucleus of the target atom), exchange effects between the incident and target particles must be considered; these effects are not present in the static potential.

It is worth noting, however, that for small values of the relative distance r, the static potential (3.71) reduces correctly to the bare Coulomb interaction ZeQ/r acting between the incident particle A and the nucleus of the target atom B. We therefore expect that the static interaction (3.71) will govern the elastic (direct) collisions involving small relative distances. Hence, if exchange effects can be neglected, and when semiclassical conditions apply, the static interaction (3.71) will give an adequate description of *large angle* elastic scattering, which precisely involves small impact parameters.

The second term on the right of Eq. (3.69) may be written as

$$V^{(2)} = \sum_{n\neq 0} \frac{\langle 0|V_d|n\rangle\langle n|V_d|0\rangle}{E - K - (w_n - w_0) + i\epsilon}, \quad (3.72)$$

where the summation runs over all the intermediate states of the target and $E = \hbar^2 k^2/2M$ is the incident relative kinetic energy. A detailed study of the expression (3.72) has been made by Mittleman and Watson (1959) (see also Goldberger and Watson, 1964). In particular, Mittleman and Watson analyzed the *adiabatic approximation*, which consists of neglecting the kinetic energy variation in the expression (3.72). Then $V^{(2)} \simeq V_{\text{ad}}^{(2)}$, where the (local and real) adiabatic potential $V_{\text{ad}}^{(2)}$ may be shown to behave at large distances as

$$V_{\text{ad}}^{(2)} \xrightarrow[r\to\infty]{} -\bar{\alpha}Q^2/2r^4, \quad (3.73)$$

with $\bar{\alpha}$ being the atomic polarizability. A convenient phenomenological parameterization of $V_{\text{ad}}^{(2)}$ is then given by the Buckingham polarization potential (Buckingham, 1937)

$$V_p(r) = -\bar{\alpha}Q^2/2(r^2 + d^2), \quad (3.74)$$

where d is a cutoff parameter. The adiabatic approximation has been shown by Mittleman and Watson (1959) to improve with decreasing incident energies and increasing values of Z.

Another approximate expression for the second-order term $V^{(2)}$, which has proved to be useful for intermediate and high incident energies, may be obtained by replacing in Eq. (3.72) the energy differences $(w_n - w_0)$ by an average excitation energy $\bar{w}$. The summation on n may then be performed by closure, so that

$$\langle \mathbf{r}|V^{(2)}|\mathbf{r}'\rangle = \frac{2M}{\hbar^2} G_0^{(+)}(k', \mathbf{r}, \mathbf{r}')A(\mathbf{r}, \mathbf{r}'). \quad (3.75)$$

Here $G_0^{(+)}(k', \mathbf{r}, \mathbf{r}')$ is the free Green's function (2.6) corresponding to a wave number $k' = (k^2 - 2\bar{w})^{1/2}$ and

$$A(\mathbf{r}, \mathbf{r}') = \langle 0|V_d(\mathbf{r}, X)V_d(\mathbf{r}', X)|0\rangle - \langle 0|V_d(\mathbf{r}, X)|0\rangle\langle 0|V_d(\mathbf{r}', X)|0\rangle, \quad (3.76)$$

where the symbol X denotes collectively the target coordinates. We note that the nonlocal second-order expression (3.75) contains explicitly an imaginary part, so that "absorption" corrections due to the nonelastic processes are now taken into account. Detailed studies of the scattering by the optical potential

$$\langle \mathbf{r}|V_{\text{opt}}|\mathbf{r}'\rangle = V^{(1)}(\mathbf{r}) + \langle \mathbf{r}|V^{(2)}|\mathbf{r}'\rangle, \quad (3.77)$$

where $V^{(1)}$ is given by Eq. (3.71) and $\langle \mathbf{r}|V^{(2)}|\mathbf{r}'\rangle$ by Eq. (3.75), have been made recently in the eikonal approximation for elastic electron–atom scattering at intermediate energies (Joachain and Mittleman, 1971a,b; Byron and Joachain, 1974). We shall return to this question in Sec. IV.

Let us now return to the Lippmann–Schwinger equation (3.67) for the operator F. An alternative way of solving this equation is to express F in terms of two-body scattering matrices. To this end we define the objects

$$t_j = v_j + v_j G_d^{(+)}(1 - \Pi_0)t_j, \quad (3.78)$$

where we recall that v_j is the two-body interaction between the incident particle and the jth target scatterer. The operator F is then given by the Watson equations (Goldberger and Watson, 1964)

$$F = 1 + \sum_{j=1}^{N} G_d^{(+)}(1 - \Pi_0)t_j F_j \quad (3.79\text{a})$$

with

$$F_j = 1 + G_d^{(+)} \sum_{k(\neq j)=1}^{N} (1 - \Pi_0)t_k F_k \quad (3.79\text{b})$$

and the optical potential is given by

$$V_{\text{opt}} = \langle 0|\sum_{j=1}^{N} t_j F_j|0\rangle. \quad (3.80)$$

This expression is still exact, but the coupled Watson equations (3.79) are in general very difficult to solve since the operators t_j include the effect of the internal target Hamiltonian. However, in the weak binding limit (i.e., when the incident particle has high energy compared to the binding energy of a target particle) one can use the impulse approximation to write $t_j \simeq T_j$, where T_j is a genuine two-body scattering matrix for the collision of the incident particle with a *free* target scatterer j. In this case the Watson equations (3.79) read

$$F = 1 + G_d^{(+)}(1 - \Pi_0) \sum_{j=1}^{N} T_j F_j \quad (3.81\text{a})$$

with

$$F_j = 1 + G_d^{(+)}(1 - \Pi_0) \sum_{j(\neq k)=1}^{N} T_k F_k \quad (3.81\text{b})$$

and the optical potential is given by

$$V_{\text{opt}} = \langle 0|\sum_{j=1}^{N} T_j F_j|0\rangle. \quad (3.82)$$

Solving the Watson equations (3.81) by iteration, we then obtain for V_{opt} the multiple scattering series

$$V_{\text{opt}} = \langle 0|\sum_{j=1}^{N} T_j|0\rangle + \langle 0|\sum_{j(\neq k)=1}^{N} T_j G_d^{(+)}(1 - \Pi_0)T_k|0\rangle + \cdots. \quad (3.83)$$

A detailed analysis of these single scattering and double scattering contributions to V_{opt} may be found in Goldberger and Watson (1964) for hadron–nucleus scattering in the weak binding limit. For a "large" nucleus of mass number A such that the concept of nuclear density is meaningful, the first term on the right of Eq. (3.83) yields the optical potential

$$V_{\text{opt}}(\mathbf{r}) = -\frac{2\pi c^2}{E} A f_0 \rho(\mathbf{r}), \quad (3.84)$$

where E is the (laboratory) energy of the incident particle, f_0 is the (laboratory) forward hadron–nucleon scattering amplitude averaged over the spins and isospins of the target nucleons, and $\rho(\mathbf{r})$ is the nuclear density normalized to one. The double scattering term in Eq.

(3.83) involves correlations between the target nucleons and has been studied by several authors (Lax, 1951; Francis and Watson, 1953; Glauber, 1959; Beg, 1960; Johnston and Watson, 1961; Goldhaber and Joachain, 1968).

Until now we have assumed that the incident particle is distinct from each of the target particles. The scattering of a particle identical with target scatterers has been considered by Takeda and Watson (1955), Bell and Squires (1959), Lippmann, Mittleman, and Watson (1959), and Feshbach (1962). The Feshbach method is particularly useful for low-energy scattering, a case which we shall not consider here.

The multiple scattering approach to the determination of the optical potential may also be formulated within the framework of the Glauber approximation (Glauber, 1959). In this case we write the eikonal elastic scattering amplitude as

$$F_{\rm el} = \frac{k}{2\pi i}\int d^2\mathbf{b}\, \exp(i\boldsymbol{\Delta}\cdot\mathbf{b})\{\exp[i\chi_{\rm opt}(\mathbf{b})]-1\}, \quad (3.85)$$

where $\chi_{\rm opt}(\mathbf{b})$ is the optical phase shift function. If we identify this amplitude with the Glauber (many-body) elastic amplitude (3.57), we define the Glauber optical phase shift function $\chi^G_{\rm opt}$ such that

$$\exp[i\chi^G_{\rm opt}(\mathbf{b})] = \langle 0|\exp[i\chi^G_{\rm tot}(\mathbf{b},\mathbf{b}_1,\ldots,\mathbf{b}_N)]|0\rangle. \quad (3.86)$$

From this relation one readily deduces that $\chi^G_{\rm opt}$ is in general complex and has a positive imaginary part as soon as nonelastic scattering can occur. We also note that, within the eikonal approximation, we may define an optical potential which corresponds to the phase shift function $\chi_{\rm opt}$. It is a local operator $V_{\rm opt}(\mathbf{r})$ such that

$$\chi_{\rm opt}(\mathbf{b}) = -\frac{1}{\hbar v_i}\int_{-\infty}^{+\infty} V_{\rm opt}(\mathbf{b},z)\,dz. \quad (3.87)$$

Glauber (1959) has given a detailed discussion of Eq. (3.86) for high-energy hadron–nucleus scattering. For a large nucleus with uncorrelated nucleons, he finds that

$$\chi^G_{\rm opt}(\mathbf{b}) = \lambda f_0\int_{-\infty}^{+\infty}\rho(\mathbf{b},z)\,dz, \quad (3.88)$$

where $\lambda = 2\pi k^{-1}$ is the de Broglie wavelength of the incident particle. We note that this result agrees with that obtained by computing first $V_{\rm opt}$ in the "single scattering" approximation of Watson's multiple scattering theory [Eq. (3.84)] and then "eikonalizing" the resulting potential by means of Eq. (3.87).

If the interaction V_d between the incident particle and the target system is known, as in atomic collision problems, we may use Eqs. (3.58) and (3.86) to expand the Glauber optical phase shift $\chi^G_{\rm opt}$ in powers of V_d (and inverse powers of v_i). Finally, we note that $\chi^G_{\rm opt}$ may also be expressed in terms of the quantity $\Gamma_{\rm tot}$ defined by Eq. (3.50). That is,

$$\exp[i\chi^G_{\rm opt}(\mathbf{b})]-1 = -\langle 0|\Gamma_{\rm tot}(\mathbf{b},\mathbf{b}_1,\ldots,\mathbf{b}_N)|0\rangle \quad (3.89)$$

and $\Gamma_{\rm tot}$ may in turn be expanded as the multiple scattering series (3.54). The first term of this series is easily shown to yield the familiar impulse approximation for $F_{\rm el}$ as we shall illustrate in Sec. V.

IV. ATOMIC COLLISIONS

A. The scattering of fast charged particles by atoms

Because the basic Coulomb interaction is well known, it should be possible to investigate systematically the validity of some of the theoretical methods discussed above, for "simple" atomic collisions. We shall give here a brief survey of recent work concerning the nonrelativistic scattering of a fast, charged, "elementary" particle by an atom.

The simplest high-energy approximation used in atomic collisions is certainly the first *Born approximation* (3.16) together with some modifications of it such as the *unitarized Born approximation* (Seaton, 1961) and the *Ochkur approximation* (Ochkur, 1963). The unitarized Born approximation is just the first Born approximation for the corresponding K-matrix element, while the Ochkur approximation is a simplified version of the Born approximation in which only the leading term of the T-matrix element in powers of k_i^{-1} (the inverse of the incident wave number) is retained. The computation of these first-order approximations is generally rather straightforward, at least for collisions involving two fragments in the final state and when simple uncorrelated wave functions (for example, of the Hartree–Fock type) are used to describe the bound atomic systems involved in the collision.

Second Born calculations imply a summation over the intermediate states of the system and are therefore much harder to perform, even approximately (see, for example, Holt and Moiseiwitsch, 1968; Holt, Hunt, and Moiseiwitsch, 1971a, 1971b; Woollings and McDowell, 1973; Byron and Joachain, 1973c). As an illustration of these difficulties, let us consider the elastic scattering of an electron by an atom of atomic number Z. We shall analyze only the direct amplitude, thus neglecting exchange effects between the incident and target electrons. The initial and final momenta of the projectile electron are denoted, respectively, by $\mathbf{k}_i$ and $\mathbf{k}_f$, with $|\mathbf{k}_i| = |\mathbf{k}_f| = k$. Neglecting recoil effects, we choose the nucleus of the target atom as the origin of our coordinate system and denote the coordinate of the projectile electron by $\mathbf{r}$, while the positions of the atomic electrons are given by $\mathbf{r}_j (j = 1, 2, \ldots, Z)$. We use atomic units (a.u.) such that the unit of length is the "first Bohr radius" a_0 while the unit of energy is e^2/a_0 (i.e., *twice* the Rydberg). The free motion of the two colliding particles is then described by the Hamiltonian

$$H_d = -\tfrac{1}{2}\nabla_r^2 + h, \quad (4.1)$$

where h is the internal target Hamiltonian, with eigenstates $|n\rangle$ and internal energies w_n. We assume that the target is in the state $|0\rangle$ before and after the collision.

The full Hamiltonian of the system is such that

$$H = H_d + V_d, \quad (4.2)$$

where the (direct) interaction potential between the incident electron and the target atom is given in a.u. by

$$V_d = -\frac{Z}{r} + \sum_{j=1}^{Z}\frac{1}{|\mathbf{r}_j-\mathbf{r}|}. \quad (4.3)$$

The second Born scattering amplitude for elastic scattering (neglecting exchange) is then given by

$$F_{el}^{B2} = F_{B1} + \bar{F}_{B2}, \tag{4.4}$$

where F_{B1} is the corresponding first Born amplitude and

$$\bar{F}_{B2} = 8\pi^2 \int d\boldsymbol{\kappa} \sum_n \frac{\langle \mathbf{k}_f, 0|V_d|\boldsymbol{\kappa}, n\rangle\langle \boldsymbol{\kappa}, n|V_d|\mathbf{k}_i, 0\rangle}{\kappa^2 - k^2 + 2(w_n - w_0) - i\epsilon}. \tag{4.5}$$

Here we have written the asymptotic initial and final free states (which are eigenstates of H_d), respectively, as $|\mathbf{k}_i, 0\rangle$ and $|\mathbf{k}_f, 0\rangle$, while a general eigenstate of H_d is denoted by $|\boldsymbol{\kappa}, n\rangle$. The normalization adopted is such that

$$\langle \boldsymbol{\kappa}', n'|\boldsymbol{\kappa}, n\rangle = \delta_{nn'}\delta(\boldsymbol{\kappa} - \boldsymbol{\kappa}'). \tag{4.6}$$

The summation over the index n appearing in Eq. (4.5) evidently implies an integration when states belonging to the continuum are concerned. As in the case of the evaluation of the second-order contribution to the optical potential [see Eq. (3.72)], we may obtain a useful approximation for the quantity $\bar{F}_{B2}$ by replacing the energy differences $(w_n - w_0)$ by an average excitation energy $\bar{w}$. The sum on intermediate states can then be done by closure, and after performing the integration on the plane wave part of the matrix elements one obtains

$$\bar{F}_{B2} = \frac{2}{\pi^2}\int d\boldsymbol{\kappa} \frac{1}{\kappa^2 - k_i'^2 - i\epsilon}\frac{1}{K_i^2 K_f^2} \times \langle 0|\{\sum_{j=1}^{Z}[\exp(-i\mathbf{K}_f\cdot\mathbf{r}_j) - 1]\} \times \{\sum_{k=1}^{Z}[\exp(i\mathbf{K}_i\cdot\mathbf{r}_k) - 1]\}|0\rangle, \tag{4.7}$$

where $\mathbf{K}_i = \mathbf{k}_i - \boldsymbol{\kappa}$, $\mathbf{K}_f = \mathbf{k}_f - \boldsymbol{\kappa}$, and $k_i' = (k_i^2 - 2\bar{w})^{1/2}$. If the state $|0\rangle$ is written as an antisymmetrized product of orbitals [whose radial part is assumed to be the sum of terms of the form $r^l \exp(-\alpha r)$] the matrix elements in Eq. (4.7) may be readily evaluated and the remaining integration on $\boldsymbol{\kappa}$ can be reduced to a single integral by using the Feynman parameterization technique (Feynman, 1949). Such calculations will be discussed below for electron–hydrogen and electron–helium scattering. In particular, we shall see that the quantity F_{B2} is an important ingredient (but not the only one) necessary to obtain a consistent expansion of the elastic differential cross section through order k^{-2}.

Let us now consider the application of the Faddeev–Watson multiple scattering (FWMS) expansions to intermediate and high-energy atomic collisions. Since a recent discussion of this method for three-body atomic problems has been given by Chen (1972), we shall only emphasize a few important points. First of all, we recall that the FWMS expansions are expressed in terms of off-shell two-body T-matrices. For the Coulomb interaction, several representations of the two-body T-matrix are available (see, for example, J. Chen and A. Chen, 1972). However, at incident energies larger than the three-body breakup threshold, particular care must be exercised in handling the cuts of the Coulomb T-matrix (Nuttall and Stagat, 1971; Chen, Chen, and Kramer, 1971; Chen and Kramer, 1971, 1972).

The application of the FWMS expansion (3.38), limited to first-order terms, has been studied for several elastic scattering processes by Chen, Chen, Sinfailam, and Hambro (1971), and Sinfailam and Chen (1972). Significant differences between the first-order FWMS expansion and the first Born approximation were found at high energies and angles as large as 0.5 rad for the case of electron and positron elastic scattering by hydrogen atoms. This effect does not appear in calculations using the Born series (Byron and Joachain, 1973b) and is in fact spurious. It is essentially removed when the second-order FWMS terms (obtained with the help of the eikonal approximation) are taken into account (Chen *et al.*, 1973).

Three-body rearrangement collisions have also been analyzed by means of first order FWMS expansions, obtained by using the multiple scattering series (3.39) or (3.41) and keeping only the two first terms on the right. (Shastry, Kumar, and Callaway, 1970; Chen and Hambro, 1971; Chen and Kramer, 1972). Of particular interest is the electron transfer or pickup reaction $p + \mathrm{H} \to \mathrm{H} + p$, in which p is a proton and H an hydrogen atom. The role of the proton–proton interaction in this reaction (at high energies) had already been the subject of numerous investigations (Oppenheimer, 1928; Brinkman and Kramers, 1930; Bates and Dalgarno, 1952; Jackson and Schiff, 1953; Drisko, 1955; Bassel and Gerjuoy, 1960; Bates, 1962; Mapleton, 1967; McCarroll and Salin, 1967; Coleman, 1968). The first-order FWMS results of Chen and Kramer (1971, 1972) indicate that at very high laboratory energies ($E > 2$MeV) the total cross sections tend towards the first Born results of Jackson and Schiff (1953), thus exhibiting a high-energy dependence of the form E^{-6}. However, since the second-order Born terms yield an $E^{-5.5}$ energy dependence of the total cross section in the high-energy limit (Drisko, 1955; Mapleton, 1967), it is clearly necessary to examine higher-order terms of the FWMS expansions (Carpenter and Tuan, 1970; Chen, Chen, and Kramer, 1971). This fact—together with the spurious effect mentioned above in connection with the elastic scattering case—illustrates some of the difficulties involved in trying to apply the Faddeev–Watson multiple scattering expansions to atomic collision problems.

We now turn to the application of *eikonal approximations* to intermediate and high-energy collisions of a charged particle by an atom. We only outline here the various methods which have been proposed. A more detailed analysis of electron–hydrogen and electron–helium collisions is given, respectively, in Secs. IV.B and IV.C.

The *many-body Glauber* amplitude, given by Eq. (3.45), has been evaluated for elastic electron–hydrogen collisions (Franco, 1968; Birman and Rosendorff, 1969; Tai, Teubner, and Bassel, 1969) and for the excitation of the lowest levels of hydrogen by electron impact (Ghosh and Sil, 1969; Ghosh, Sinha, and Sil, 1970; Tai, Bassel, Gerjuoy, and Franco, 1970; Bhadra and Ghosh, 1971; Sheorey, Gerjuoy, and Thomas, 1971; Gerjuoy, Thomas, and Sheorey, 1972). Since exchange scattering is ignored in these calculations, proton–hydrogen collisions may be treated in a formally identical manner, except for a change in the scale of the momentum transfer. Such computations have been performed by Franco and Thomas (1971), Bhadra and Ghosh (1971), and Ghosh and Sil (1971). All these calculations on atomic hydrogen, using the Glauber formula (3.45) may be reduced to the evaluation of a single dimensional integral or even, as shown by Thomas and Gerjuoy (1971), to a finite sum of hypergeometric functions (see also Gerjuoy, 1972). More recently, the Glauber amplitude has also been evaluated for the ionization of atomic hydrogen (Hidalgo, McGuire, and Doolen, 1972; McGuire *et al.*, 1973).

For target atoms more complex than atomic hydrogen, the reduction of the Glauber amplitude (3.45) to a tractable form is more difficult to achieve. Such Glauber calculations have been performed for elastic scattering from helium targets by Franco (1970) [using a Hartree–Fock wave function for the helium ground state], Johnson and Brolley (1970), and more recently by Thomas and Chan (1973). The Glauber expression (3.45) has also been used to study the excitation of the 2^1S state of helium by electron impact (Yates and Tenney, 1972). Glauber-type calculations taking into account the target degrees of freedom in a simplified way have also been carried out for alkali atoms (Mathur, Tripathi, and Joshi, 1971, 1972; Walters, 1973). A general reduction procedure of the Glauber amplitude (3.45) for many-electron atoms has been proposed by Franco (1971).

Two important reservations should be made about the above-mentioned calculations using the Glauber expression (3.45). Firstly, for elastic (atomic) scattering processes a detailed comparison of the Glauber series (3.59) and of the Born series, recently made by Byron and Joachain (1973c), shows that the Glauber approximation suffers from serious deficiencies; we shall analyze these below for elastic scattering of electrons by atomic hydrogen and helium. It is probable that such difficulties also afflict Glauber calculations dealing with *inelastic* (direct) processes. Secondly, we have already pointed out in Sec. III.C that for inelastic collisions the Glauber scattering amplitude (3.45) can only be derived from the more general expression (3.56) by neglecting the longitudinal momentum transfer. The importance of treating correctly the kinematics for inelastic atomic collisions has been stressed by Byron (1971) and by Chen, Joachain, and Watson (1972). Byron (1971) has also obtained the general Glauber expression (3.56) by treating in the eikonal approximation the complete set of close-coupling equations (with exchange neglected). He then used the Monte Carlo technique to perform the multidimensional integrals appearing in Eq. (3.56) for the excitation of various states of atomic hydrogen and helium by electron impact. Similar calculations using the Monte Carlo method have also been made by Byron and Joachain (1972) for the excitation of the 2^3S state of helium by electron impact, which is a pure rearrangement (knock-out) process when spin-dependent interactions are neglected.

We now come to eikonal calculations involving the *optical model formalism* (Joachain and Mittleman, 1971a,b; Chen, Joachain, and Watson, 1972; Joachain and Vanderpoorten, 1973, 1974; Byron and Joachain, 1974). Starting from the optical potential (3.77) and using the eikonal approximation, Joachain and Mittleman have shown that the direct elastic scattering amplitude for the collision of a charged particle by an atom is given by the *optical eikonal* expression

$$F_{\text{opt}}^{\text{eik}} = \frac{k}{2\pi i}\int d^2\mathbf{b}\exp(i\mathbf{\Delta}\cdot\mathbf{b})\{\exp[i\chi_{\text{opt}}^{(2)}(\mathbf{b})] - 1\}, \quad (4.8)$$

where the (second-order) optical phase shift function $\chi_{\text{opt}}^{(2)}(\mathbf{b})$ is obtained from

$$\chi_{\text{opt}}^{(2)}(\mathbf{b}) = -\frac{1}{v_i}\int_{-\infty}^{+\infty} V^{(1)}(\mathbf{b},z)\,dz + \frac{i}{v_i v_i'}\int_{-\infty}^{+\infty} dz \int_{-\infty}^{z} dz' \times \exp[-i(k_i - k_i')(z - z')]A(\mathbf{b},z;\mathbf{b},z') \quad (4.9)$$

Here $V^{(1)}$ is the static (first-order) optical potential, as given by Eq. (3.71), the quantity $A(\mathbf{r};\mathbf{r}')$ is defined by Eq. (3.76), and $v_i = k_i/M$ is the initial relative velocity of the two colliding particles (M being their reduced mass). Moreover, an average excitation energy $\bar{w}$ of the target states has been introduced, such that $Mv_i'^2/2 = k_i'^2/(2M) = k_i^2/(2M) - \bar{w}$. We note that within the framework of the eikonal approximation we may use Eq. (3.87) to extract from Eq. (4.9) the equivalent local (second-order) optical potential

$$V_{\text{opt}}^{(2)}(\mathbf{r}) = V^{(1)}(\mathbf{r}) - \frac{i}{v_i'}\int_{-\infty}^{z} dz' \times \exp[-i(k_i - k_i')(z - z')]A(\mathbf{b},z;\mathbf{b},z'). \quad (4.10)$$

Let us return to Eq. (4.9). We shall write it in the form

$$\chi_{\text{opt}}^{(2)} = \chi_{\text{st}} + \chi_{\text{abs}} + \chi_{\text{pol}}, \quad (4.11\text{a})$$

where

$$\chi_{\text{st}}(\mathbf{b}) = -\frac{1}{v_i}\int_{-\infty}^{+\infty} V^{(1)}(\mathbf{b},z)\,dz, \quad (4.11\text{b})$$

$$\chi_{\text{abs}}(\mathbf{b}) = \frac{i}{v_i v_i'}\int_{-\infty}^{+\infty} dz \int_{-\infty}^{z} dz' \times \cos[(k_i - k_i')(z - z')]A(\mathbf{b},z;\mathbf{b},z') \quad (4.11\text{c})$$

and

$$\chi_{\text{pol}}(\mathbf{b}) = \frac{1}{v_i v_i'}\int_{-\infty}^{+\infty} dz \int_{-\infty}^{z} dz' \times \sin[(k_i - k_i')(z - z')]A(\mathbf{b},z,\mathbf{b},z'). \quad (4.11\text{d})$$

We note that the quantity χ_{st} is simply the eikonal phase shift function corresponding to the *static* potential $V^{(1)}$. The term χ_{abs}, which is purely imaginary, accounts for *absorption effects* induced by unitarity from the open channels. Such effects, which are most important at *small angles*, have been studied by Joachain and Mittleman (1971a,b) for the case of the elastic scattering of fast electrons by atoms. The remaining term χ_{pol} has been analyzed recently by Byron and Joachain (1974), who showed that it contains the *polarization effects* induced by the long range part of the second-order optical potential. This term, whose contribution is also most important at *small angles*, corresponds to an effective local polarization potential having the form $V_{\text{pol}}(r) \sim -\bar{\alpha}Q^2/(2r^4)$ at large r [see Eq. (3.73)]. At *large momentum transfers* the *static* potential $V^{(1)}$ dominates the scattering, in accordance with the discussion following Eq. (3.71). We shall also see in Sec. IV.C how the leading *exchange effects* can be included in the eikonal optical model. Finally, we shall indicate there how the deficiencies associated with the use of the eikonal method can be remedied by using second-order perturbation theory.

It is interesting to compare the second-order optical phase shift $\chi_{\text{opt}}^{(2)}$ given by Eq. (4.9) with the Glauber optical phase shift χ_{opt}^{G} obtained from Eq. (3.86). Thus, we first write

$$\begin{aligned}\chi_{\text{opt}}^{G}(\mathbf{b}) &= -i \log\langle 0|\exp(i\chi_{\text{tot}}^{G})|0\rangle \\ &= \langle 0|\chi_{\text{tot}}^{G}|0\rangle - i \log\langle 0|\exp i[\chi_{\text{tot}}^{G} - \langle 0|\chi_{\text{tot}}^{G}|0\rangle]|0\rangle.\end{aligned} \tag{4.12}$$

Then, using Eq. (3.58) and expanding the right-hand side of Eq. (4.12) in powers of v_i^{-1}, we find that

$$\begin{aligned}\chi_{\text{opt}}^{G}(\mathbf{b}) &= -\frac{1}{v_i}\int_{-\infty}^{+\infty} V^{(1)}(\mathbf{b}, z)dz \\ &+ \frac{i}{2v_i^2}\int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} dz' A(\mathbf{b}, z; \mathbf{b}, z') + \cdots.\end{aligned} \tag{4.13}$$

Hence, by comparing this result with Eq. (4.9), we see that the Glauber optical phase shift, or the corresponding optical potential V_{opt}^{G} such that

$$\chi_{\text{opt}}^{G}(\mathbf{b}) = -\frac{1}{v_i}\int_{-\infty}^{+\infty} V_{\text{opt}}^{G}(\mathbf{b}, z)dz \tag{4.14}$$

contains no real second-order terms and corresponds to the choice $\overline{w} = 0$ for the average excitation energy of the target. The fact that $\overline{w} = 0$ in the Glauber approximation has important consequences, since it implies that the Glauber many-body elastic scattering amplitude F_{el}^{G} diverges logarithmically at zero momentum transfer (Franco, 1968a). This undesirable feature is removed in the *eikonal optical model* discussed above.

The optical model formalism may also be used together with the eikonal DWBA method to analyze inelastic or rearrangement atomic processes (Chen, Joachain, and Watson, 1972; Joachain and Vanderpoorten, 1973, 1974). For example, in the case of a direct transition such that the target, initially in the state $|0\rangle$, is left in the state $|n\rangle$, the eikonal DWBA transition matrix element obtained from Eq. (3.25), is simply

$$\begin{aligned}T_{ba}^{\text{eik}} &= (2\pi)^{-3}\int d\mathbf{r}\, \exp(i\mathbf{\Delta}\cdot\mathbf{r}) \\ &\exp\{i[\Lambda_i(\mathbf{b}, z) + \Lambda_f(\mathbf{b}, z)]\}V_{n0}(\mathbf{b}, z)\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}\Lambda_i(\mathbf{b}, z) &= -\frac{1}{v_i}\int_{-\infty}^{z} U_i(\mathbf{b}, z')dz', \\ \Lambda_f(\mathbf{b}, z) &= -\frac{1}{v_f}\int_{z}^{\infty} U_f(\mathbf{b}, z')dz',\end{aligned} \tag{4.16}$$

and

$$V_{n0}(\mathbf{b}, z) = \langle n|V|0\rangle.$$

Here U_i and U_f are, respectively, the initial and final distorting potentials, while v_i and v_f are the relative velocities in the initial and final channel. At the expense of treating to first order that part of the interaction which is responsible for the inelastic transition, this method leads to reasonably simple expressions. These take into account explicitly the longitudinal momentum transfer, allow the evaluation of exchange effects, and may be applied to complex target atoms. Applications of this method to electron–atom inelastic collisions have been made by using static distorting potentials (Chen, Joachain, and Watson, 1972) and Glauber (complex) distorting potentials (Joachain and Vanderpoorten, 1973, 1974).

To conclude this section, we would like to mention the very interesting approach recently developed by Bransden *et al.* (Bransden and Coleman, 1972; Bransden, Coleman, and Sullivan, 1972; Sullivan, Coleman, and Bransden, 1972; Berrington, Bransden, and Coleman, 1973) to analyze the scattering of charged particles by atoms. Starting from the set of close coupling equations, these authors retain explicitly a group of states in a truncated expansion of the full wave function. The remaining states are accounted for by the introduction of suitable second-order potentials, similar to those discussed above. This method has already been applied successfully to the scattering of electrons and protons by atomic hydrogen and helium.

B. Electron scattering by atomic hydrogen

We shall now analyze in more detail the scattering of electrons by atomic hydrogen at intermediate and high energies. We begin by considering elastic collisions and follow the treatment of Byron and Joachain (1973c) who have carried out a detailed comparison of the Born and the Glauber eikonal series. We write the Born series for the direct elastic scattering amplitude as

$$F_{\text{el}} = \sum_{n=1}^{\infty} \overline{F}_{Bn}, \tag{4.17}$$

where (in a.u.)

$$\overline{F}_{Bn} = -(2\pi)^2\langle \mathbf{k}_f, 0|V_d G_d^{(+)} V_d \cdots G_d^{(+)} V_d|\mathbf{k}_i, 0\rangle. \tag{4.18}$$

In this expression the potential V_d [given by Eq. (4.3) with $Z = 1$] appears n times while the propagator $G_d^{(+)} = (E - H_d + i\epsilon)^{-1}$ [with H_d given by Eq. (4.1)] is counted (n-1) times. We also define the objects

$$F_{\text{el}}^{Bn} = \sum_{j=1}^{n} \overline{F}_{Bj}. \tag{4.19}$$

We now consider the Glauber elastic scattering amplitude (3.57), together with the associated multiple scattering series defined by Eqs. (3.59)–(3.61). We recall that if the Glauber phase shift function (3.58) is evaluated with the z axis perpendicular to the momentum transfer $\mathbf{\Delta}$, we have exactly $F_{G1} = F_{B1}$ for all scattering angles [see Eq. (3.62)].

Let us now compare the higher terms of the Born series (4.17) and the Glauber series (3.59). We first remark that for the interaction potential (4.3) the Glauber phase shift function is given by

$$\begin{aligned}&\chi_{\text{tot}}^{G}(\mathbf{b}, \mathbf{b}_1, \ldots, \mathbf{b}_Z) \\ &= \frac{1}{k}\sum_{j=1}^{Z} \log\left[1 - 2\frac{b_j}{b}\cos(\phi_j - \phi) + \frac{b_j^2}{b^2}\right],\end{aligned} \tag{4.20}$$

where ϕ_j is the azimuthal angle of $\mathbf{b}_j$ in the (xy) plane.

Since χ^G_{tot} only depends on the differences $(\phi_j - \phi)$ we may choose the (xy) axes so that no ϕ dependence appears in χ^G_{tot}. Hence Eq. (3.57) yields

$$F^G_{el} = \frac{k}{i}\int_0^\infty dbb\, J_0(\Delta b)\langle 0|[\exp(i\chi^G_{tot}) - 1]|0\rangle \quad (4.21)$$

and similarly we find from Eq. (3.60) that

$$\bar{F}_{Gn} = \frac{k}{i}\frac{i^n}{n!}\int_0^\infty dbb\, J_0(\Delta b)\langle 0|[\chi^G_{tot}]^n|0\rangle. \quad (4.22)$$

It is apparent from Eq. (4.22) that, as in the case of potential scattering [cf. the discussion following Eq. (2.43)], the terms of the Glauber multiple scattering series (3.59) are alternately purely real and purely imaginary. This, again, is in contrast with the Born series (4.17), where already the term $\bar{F}_{B2}$ contains a real as well as an imaginary part.

We have already pointed out in Sec. IV.A that by using an average excitation energy $\bar{w}$ it is possible to reduce the quantity $\bar{F}_{B2}$ to the expression (4.7) which then can be evaluated in a straightforward manner. In fact, for simple target atoms like hydrogen and helium one may even include exactly a few states in the summation on n appearing in Eq. (4.5), and then evaluate the sum on the remaining states by closure methods (see, for example, Holt, Hunt, and Moiseiwitsch, 1971b; Woollings and McDowell, 1972; Byron and Joachain, 1973c). Of particular interest is the limit of large values of k for which, at small scattering angles ($\theta < 2\bar{w}/k^2$), Byron and Joachain find that Re $\bar{F}_{B2}$ varies like k^{-1}, while Im $\bar{F}_{B2}$ behaves like $k^{-1}\log k$. We note that this behavior of $\bar{F}_{B2}$ is different from that found in Sec. II.B [see Eqs. (2.47) and (2.56)] for the case of potential scattering. In particular, we emphasize that Re $\bar{F}_{B2}$ now gives the *dominant correction* to the first Born differential cross section at small angles. At larger angles $\theta > 2\bar{w}/k^2$ one retrieves the "potential scattering" behavior such that Re $\bar{F}_{B2}$ varies like k^{-2} and Im $\bar{F}_{B2}$ like k^{-1} for larger values of k. [See Eq. (2.56).] This is not surprising since the static potential dominates the large angle scattering.

Let us now examine the Glauber multiple scattering series (3.59). The second-order term $\bar{F}_{G2}$, which is purely imaginary, may easily be shown to diverge logarithmically at $\Delta = 0$. Indeed, the corresponding quantity Im $\bar{F}_{B2}$ also diverges logarithmically as $\bar{w}$, the average excitation energy, is set equal to zero. As shown explicitly by Byron (1971), the many-body Glauber result (3.57) precisely assumes that $\bar{w} = 0$. Although the quantities Im $\bar{F}_{B2}$ and Im $\bar{F}_{G2}$ differ substantially at *very small momentum transfers* because of the divergence of Im $\bar{F}_{G2}$, a detailed study of these two quantities shows that otherwise they agree very well, even in the backward direction and for rather low values of k. This is reminiscent of the relationship (2.50) proved in potential scattering for Yukawa-type potentials.

For $n \geq 3$, the terms $\bar{F}_{GN}$ of the Glauber multiple scattering series (3.59) are finite, even at $\Delta = 0$. It is therefore very likely that these terms will agree with the corresponding terms of the Born series (i.e., $\bar{F}_{G3}$ with Re $\bar{F}_{B3}$, $\bar{F}_{G4}$ with i Im $\bar{F}_{B4}$, etc.) for large enough k (Byron and Joachain, 1974b). Since the direct evaluation of the quantity Re $\bar{F}_{B3}$ (which yields contributions of order k^{-2} to the differential cross section) is an extremely difficult task, it therefore seems reasonable to use $\bar{F}_{G3}$ in place of Re $\bar{F}_{B3}$. Thus we write the *direct* elastic scattering amplitude (through terms of order k^{-2}) as

$$F_{el} = \bar{F}_{B1} + \text{Re}\,\bar{F}_{B2} + \bar{F}_{G3} + i\,\text{Im}\,\bar{F}_{B2} + \cdots \quad (4.23)$$

and shall refer to this treatment as the *eikonal–Born series* (EBS) method (Byron and Joachain, 1973c).

Before we compute the elastic differential cross section we recall that the leading (Ochkur) term of the first order *exchange* amplitude is of order k^{-2} for large k and fixed Δ. A consistent calculation of the small angle elastic differential cross section through order k^{-2} therefore requires the inclusion of this term, which we call G_1. The small angle elastic differential cross section (for unpolarized beam and target, and if no attempt is made to distinguish between the various final spin states) is then given by

$$\frac{d\sigma_{el}}{d\Omega} = \tfrac{1}{4}|F_{el} + G_1|^2 + \tfrac{3}{4}|F_{el} - G_1|^2. \quad (4.24)$$

The situation at large momentum transfers is different. For a given energy (i.e., a given value of k), the convergence of the Born series at large angles is slower than in the small angle region. This means that terms of higher order (in k^{-1}) than those included above may now play a significant role. It is also worth noting that the Ochkur term G_1 drops off like k^{-6} for large k and $\Delta \geq k$.

Further insight into large angle elastic scattering may be obtained by noting that the second Born terms Re $\bar{F}_{B2}$ and Im $\bar{F}_{B2}$ are dominated at large momentum transfers by the contribution arising from the ground state $|0\rangle$ acting as an intermediate state. A detailed study of higher terms of the Born and Glauber series (Byron and Joachain, 1973d, 1974b) yields similar conclusions. The central role of the ground state in large angle multiple scattering confirms the expectation that high-energy, large angle direct elastic scattering is mainly governed by the *static* potential $V^{(1)} = \langle 0|V_d|0\rangle$.

As an example, we display in Fig. 6 the results obtained from the EBS method for the elastic scattering of electrons by atomic hydrogen at an energy of 50 eV. Also shown on this figure are the first Born values, the Glauber differential cross section $d\sigma^G_{el}/d\Omega = |F^G_{el}|^2$ and the results obtained by solving numerically the partial wave radial equations corresponding to the static potential

$$V^{(1)}(r) = -(1 + 1/r)\exp(-2r).$$

We note from this figure that the EBS results agree reasonably well with the experimental data of Teubner *et al.* (1973) (which are normalized with respect to similar absolute measurements made in molecular hydrogen), in spite of the fact that an incident electron energy of 50 eV is rather low for the applicability of the EBS method. As we expect, the static results are good at large angles, but fail to account for the scattering at small angles, where absorption and polarization effects are important. The first Born results are quite poor, especially at small angles. Finally, the Glauber results are seen to be inaccurate over the whole angular range. We recall in this connection that the Glauber differential cross section diverges at $\theta = 0°$ (because of the term Im $\bar{F}_{G2}$) and lacks the exchange term G_1 together with the important term Re $\bar{F}_{B2}$.

A similar comparison is made in Fig. 7 for the elastic scattering of electrons from atomic hydrogen at an energy of 100 eV. Here we have shown the EBS results together with the first Born and the Glauber values. The remaining curve on Fig. 7 corresponds to positron–hydrogen scattering, calculated from the EBS amplitude (4.23). Whereas the eikonal–Born series method predicts significant differences between small angle electron and positron scattering, the Born and Glauber approximations do not distinguish between the two cases. The relative measurements of Teubner, Williams, and Carver (quoted in Tai, Teubner, and Bassel, 1969) have been normalized to the EBS curve at $\theta = 30°$. Since these data are not absolute and correspond to angles $\theta \geq 25°$ (where the shapes of the various theoretical curves are similar), they do not provide a severe test of the different theories. This is in contrast with the more recent data shown in Fig. 6 and with the situation for helium, which we shall discuss below.

We now consider briefly some inelastic transitions induced in atomic hydrogen by the impact of fast electrons. Calculations using the Glauber approximation (3.45) have been performed by several authors (Ghosh and Sil, 1969; Ghosh, Sinha, and Sil, 1970; Tai, Bassel, Gerjuoy, and Franco, 1970; Bhadra and Gosh, 1971; Sheorey, Gerjuoy, and Thomas, 1971; Gerjuoy, Thomas, and Sheorey, 1972) and reviewed by Gerjuoy (1972).

As an example, we show in Fig. 8 the total cross section for excitation of the $2p$ state of hydrogen by electron impact. Here, in addition to the first Born approximation and the Glauber results of Tai *et al.* (1970), we have also displayed the eikonal calculations of Byron (1971), the four-channel approximation results of Sullivan, Coleman and Bransden (1972), and the eikonal DWBA calculations of Joachain and Vanderpoorten (1973). Also shown for comparison are the close-coupling results of Burke, Schey, and Smith (1963). The experimental data are those of Long, Cox, and Smith (1968). They are normalized at high energies to the first Born values.

Another interesting quantity is the polarization P of the radiation emitted from the final state of the excitation process $e^- + H(1s) \to e^- + H(2p)$. This polarization results from the relative population of the magnetic sublevels of the $2p$ states. The corresponding $2p \to 1s$ line occurs at 1216 Å and has been studied experimentally by Ott, Kauppila, and Fite (1967). Using the Glauber expression (3.45) which neglects the longitudinal momentum transfer, Tai *et al.*(1970) found a selection rule $\Delta m = \pm 1$ for $s \to p$ transitions which leads to a constant polarization $P = -3/11$. This result is in disagreement with the experimental data of Ott, Kauppila, and Fite, who find that the polarization P is positive from threshold to about 250 eV. By using the more general and kinematically correct expression (3.56), Byron (1971) obtained theoretical values of P in much better agreement with the experimental data. Gerjuoy, Thomas, and Sheorey (1972) have also obtained good agreement with experiment by using the Glauber expression (3.45), with the axis of quantization chosen perpendicular to the momentum transfer, and then transforming the calculated cross sections to refer them to a quantization axis in the direction of $\mathbf{k}_i$. The results of the four-channel approximation of Sullivan, Colemen, and Bransden (1972) reproduce the experimental shape of P as a function of the energy but lie somewhat below the experimental values. It is worth noting that in this case the first Born approximation agrees surprisingly well with

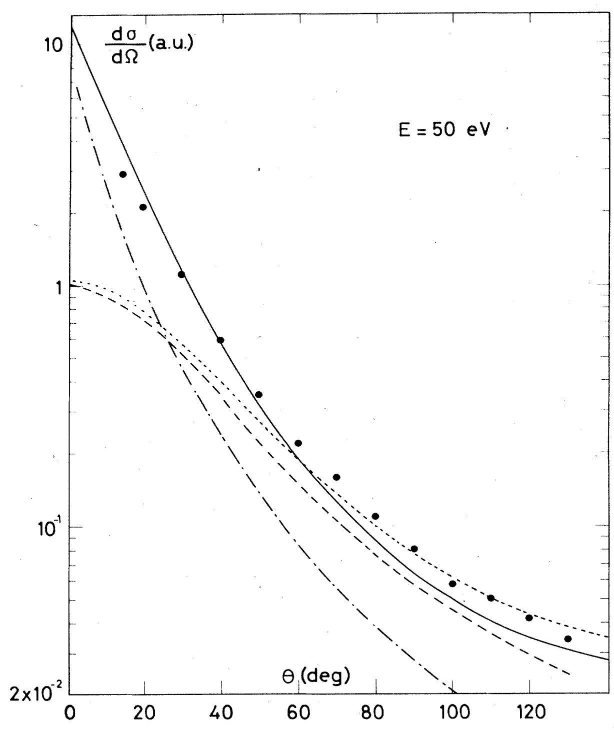


FIG. 6. Differential cross section for elastic scattering of electrons by atomic hydrogen at an energy of 50 eV. The solid curve is obtained by using the eikonal–Born series (EBS) method of Byron and Joachain (1973c). The dashed curve represents the first Born approximation and the dash–dot curve corresponds to the Glauber approximation. The dotted curve represents the results obtained from a partial wave analysis of the static potential $V^{(1)} = \langle 0|V_d|0\rangle$ corresponding to the hydrogen ground state. The experimental points refer to the work of Teubner, Lloyd, and Wiegold (1973). (From Byron and Joachain, 1974b.)

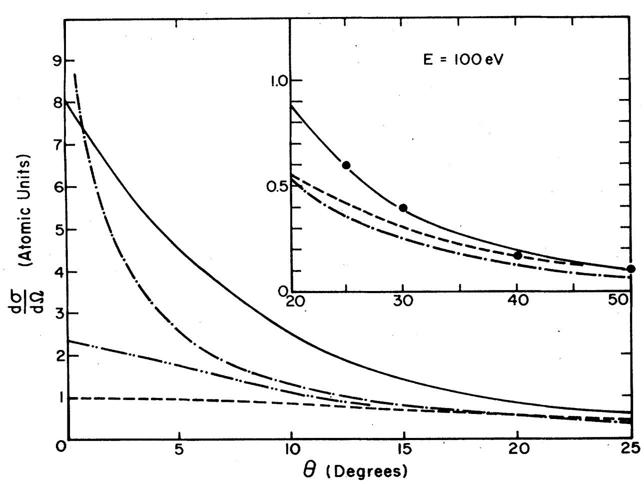


FIG. 7. Differential cross section for elastic scattering of electrons and positrons by atomic hydrogen at an energy of 100 eV. The solid curve is obtained for electrons by using the eikonal–Born series (EBS) method of Byron and Joachain (1973c). The dash–double–dot curve is the corresponding EBS curve for incident positrons. The dashed curve represents the first Born approximation, and the dash–dot curve corresponds to the Glauber approximation. The experimental points refer to the work of Teubner, Williams, and Carver, quoted in Tai, Teubner, and Bassel (1969).

measurements.

Returning to the evaluation of excitation cross sections, we note that, as in the case of elastic scattering, the Glauber expression (3.45) predicts identical results for the excitation by electron or positron impact. Using the more general Eq. (3.56), which accounts for the longitudinal momentum transfer, Byron (1971) has found significant differences between electron and positron excitation of the $2s$ states of hydrogen. Since positron scattering is presently not feasible, experimental data on proton scattering by hydrogen (in the energy range 50–150 keV, i.e., at velocities corresponding to the electron case) would be very useful to settle this question. It is worth noting that the eikonal DWBA method of Chen, Joachain, and Watson (1972) and the approach of Bransden and Coleman (1972) also predict differences between electron and positron (proton) scattering.

Before leaving the subject of electron scattering by atomic hydrogen we remark that for *elastic* collisions the first Born approximation eventually governs *all* the scattering at sufficiently high (nonrelativistic) energies. This is not the case for *inelastic* (direct) collisions where the second Born term $\bar{F}_{B2}$ dominates at high energies and *large momentum transfers*. This dominance of the second Born term over the first one also occurs in *elastic exchange* scattering (at large momentum transfers) and for *inelastic exchange* amplitudes.

C. Electron–helium collisions

We now turn to the scattering of electrons by helium at intermediate and high (atomic) energies. In this case the theoretical calculations are obviously harder to perform than for atomic hydrogen, but on the other hand, accurate, absolute experimental data for various processes have recently become available. It is therefore with helium targets that the various theories examined above can presently be tested in the most reliable way.

We begin by analyzing elastic electron–helium scattering, following the eikonal–Born series method of Byron and Joachain (1973c). By using an analytical fit (see, for example, Byron and Joachain, 1966) to the Hartree–Fock ground state helium wave function (Roothaan, Sachs, and Weiss, 1960), the reduction of the second Born expression (4.7) proceeds as in the case of hydrogen. Similarly, the Glauber expressions (4.21) and (4.22) can also be evaluated in this case. The eikonal–Born series (EBS) *direct* elastic scattering amplitude is still given by Eq. (4.23), and the elastic differential cross section now reads

$$\frac{d\sigma_{el}}{d\Omega} = |F_{el} - G_1|^2, \tag{4.25}$$

where G_1 again refers to the leading (Ochkur) term of the exchange amplitude.

As in the case of electron–hydrogen elastic scattering, we expect that the EBS method should be particularly useful for large k and small momentum transfers, in which case the Born series for F_{el} is converging rapidly and G_1 provides the leading exchange correction (of order k^{-2}). This is illustrated in Fig. 9, where we show the EBS results (Byron and Joachain, 1973c) together with small angle absolute experimental data (Bromberg, 1969) at an incident electron energy of 500 eV. Also shown on Fig. 9 are the results given by the first Born approximation and those corresponding to the Glauber approxima-

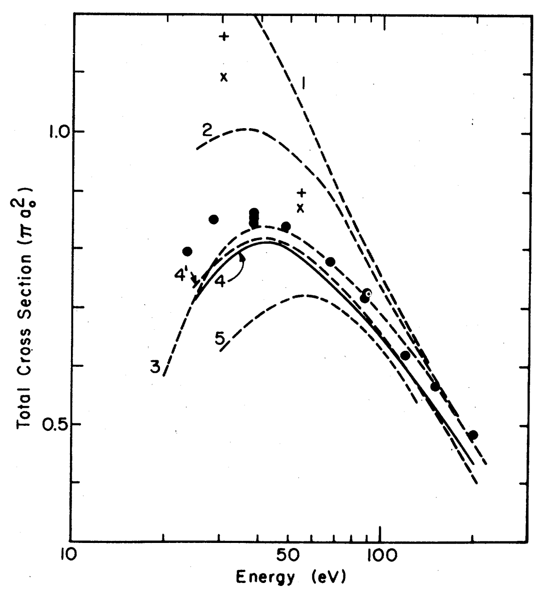


FIG. 8. Total cross section for the excitation of the $2p$ state of atomic hydrogen by electron impact as a function of the incident energy. Curve 1: first Born approximation; Curve 2: four-channel approximation of Sullivan, Coleman, and Bransden (1972); Curve 3: Glauber approximation (Tai, Bassel, Gerjuoy, and Franco, 1970); Curve 4: Eikonal DWBA method with Glauber distorting potentials (Joachain and Vanderpoorten, 1973); Curve 4′: same as curve 4, except that the quantity $Q_\perp$ defined by Long, Cox and Smith (1968) is shown; Curve 5: Eikonal calculation of Byron (1971), using Eq. (3.56), (x) = four-state close-coupling calculation for σ_{2p} (Burke, Schey, and Smith, 1963); (+): four-state close-coupling calculation for $Q_\perp$ (Burke *et al.*, 1963). The dots are the experimental data of Long, Cox, and Smith (1968). (From Joachain and Vanderpoorten, 1973.)

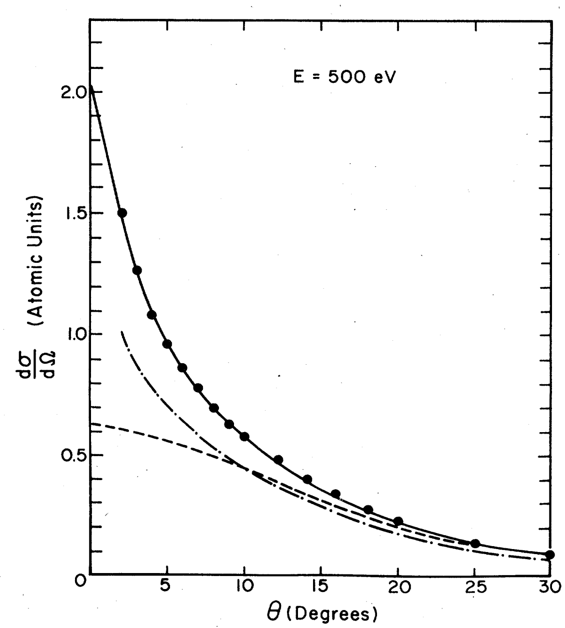


FIG. 9. Differential cross section for elastic scattering of electrons by helium at an incident electron energy of 500 eV. The solid curve is obtained from the eikonal–Born series method of Byron and Joachain (1973c). The dash–dot curve represents the Glauber approximation; the dashed curve is the first Born approximation. The dots correspond to the absolute measurements of Bromberg (1969). (From Byron and Joachain, 1973c.)

tion (Franco, 1970). It is clear from Fig. 9 that the EBS results are in excellent agreement with experiment. The Glauber differential cross section which diverges at $\theta = 0$ and lacks the terms Re $\bar{F}_{B2}$ and G_1 is seen to be deficient. We also note that even at 500 eV the first Born results are still very inaccurate at small angles, although at sufficiently high energies the first Born approximation will eventually control the scattering at all angles.

For large k and large values of Δ (the magnitude of the momentum transfer) we expect that the static potential $V^{(1)} = \langle 0|V_d|0\rangle$ will govern the scattering. This is confirmed by a detailed analysis of higher terms of the Born and Glauber series (Byron and Joachain, 1973d). To illustrate this point, we display in Figs. 10a and 10b the results obtained by solving numerically the partial wave radial equations corresponding to the static potential $V^{(1)}$, together with the experimental data of Crooks and Rudd (1972) at 200 eV and of Bromberg (1969) at 500 eV. We note from Fig. 10 that the static and EBS results agree well with each other (and with the experimental data) outside the small angle region. At small angles the static results are inaccurate since the potential $V^{(1)}$ does not include polarization, absorption, and exchange effects. As in the case of electron–hydrogen scattering, it is worth stressing that the success of the EBS method at large momentum transfers depends on cancellations between higher terms of the Born series. Such cancellations, although present in the case of electron–helium scattering, may not be present in other situations.

As we are still dealing with elastic scattering, it is interesting to discuss the results obtained from the *eikonal optical model* involving second-order optical potentials [See Eqs. (4.8) – (4.11)]. The evaluation of the phase shift functions χ_{abs} and χ_{pol} (which are dominant in correcting the first Born approximation at small angles) has been

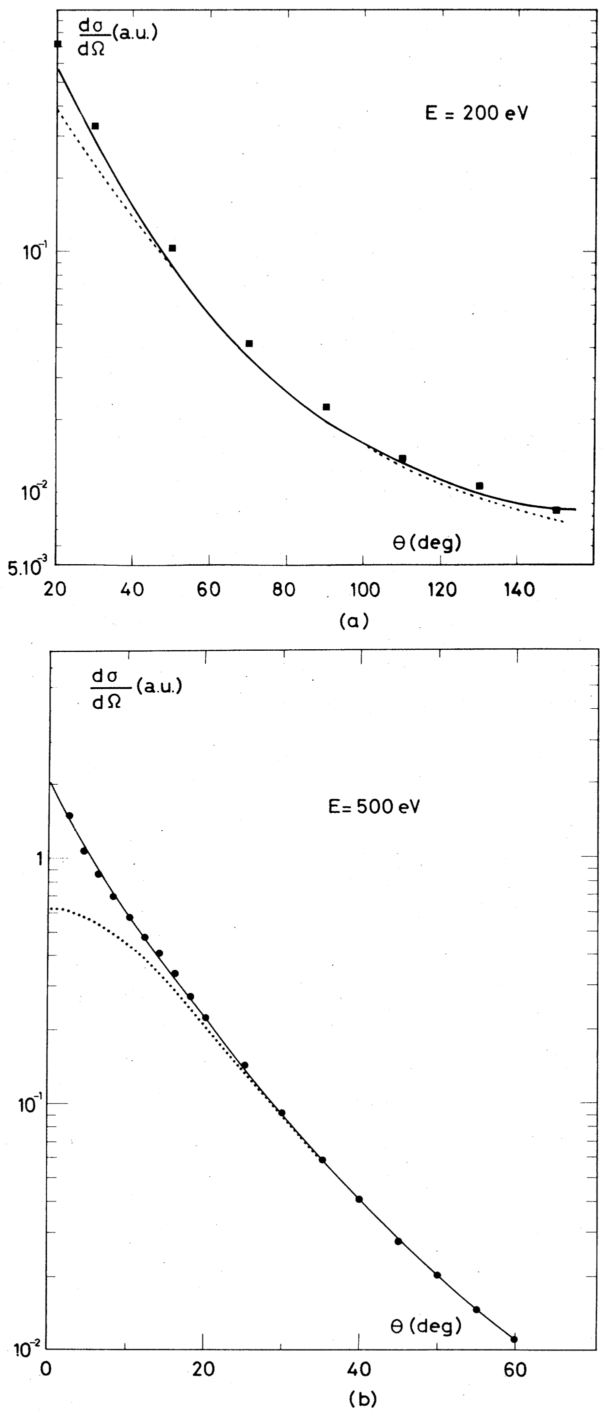


FIG. 10. Differential cross section for elastic scattering of electrons by helium (a) at 200 eV and (b) at 500 eV. The solid curve represents the eikonal–Born series results of Byron and Joachain (1973d) while the dotted curve corresponds to an "exact" (partial wave) treatment of the static potential $V^{(1)} = \langle 0|V_d|0\rangle$ corresponding to the helium ground state. The squares show the experimental results of Crooks and Rudd (1972); the circles are the experimental data of Bromberg (1969).

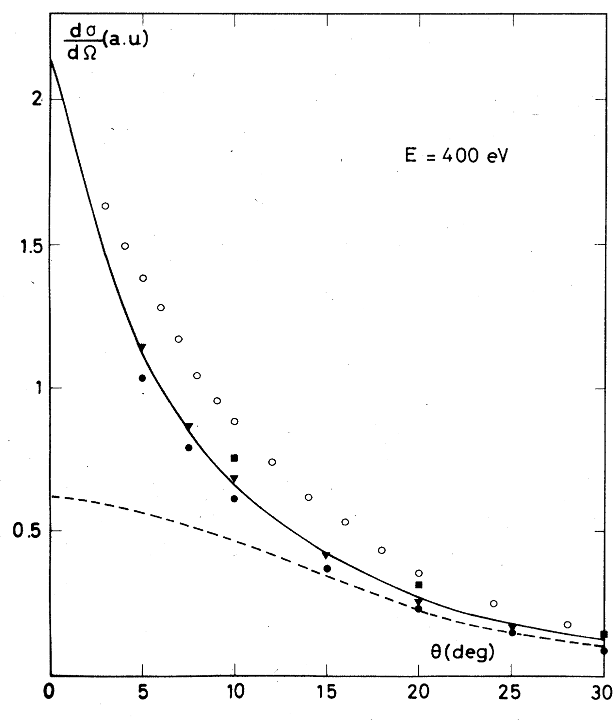


FIG. 11. The differential cross section (in atomic units) for elastic electron scattering by helium at an incident electron energy of 400 eV. The solid curve represents the results obtained by using the optical eikonal method. The dashed curve corresponds to the first Born approximation. Triangles are the experimental results of Vriens, Kuyatt, and Mielczarek (1968); solid circles are the same results as renormalized by Chamberlain, Mielczarek, and Kuyatt (1970); squares show the data of Crooks and Rudd (1972); open circles are the data of Jost, Fink, and Herrmann (1974). (From Byron and Joachain, 1974a.)

discussed by Joachain and Mittleman (1971a,b) and Byron and Joachain (1974). The static phase χ_{st} plays the dominant role at large momentum transfers.

The knowledge of the three phases χ_{st}, χ_{pol}, and χ_{abs} enables one to evaluate the *optical eikonal* amplitude F_{opt}^{eik} given by Eq. (4.8). This quantity, however, does not contain important terms of order k^{-1} (arising from Re $\bar{F}_{B2}$) which a full wave treatment of the optical potential $V^{(2)}$ would give by the iteration of the static potential $V^{(1)}$ to second order. Fortunately (at least for Hartree–Fock ground state wave functions) it is straightforward to obtain the second Born term Re $\bar{F}_{B2}^{st}$ corresponding to $V^{(1)}$. Adding this term to the optical eikonal amplitude then yields the (direct) optical elastic amplitude

$$F_{el}^{opt} = F_{opt}^{eik} + \text{Re } \bar{F}_{B2}^{st}. \tag{4.26}$$

We note that for large k an expansion of F_{el}^{opt} in powers of the full interaction V_d will duplicate the Born series through second-order. It will also give approximations to higher terms of the Born series. The third-order contribution has been discussed in detail by Byron and Joachain (1974).

As in the case of the EBS method discussed above, *exchange* effects may also be included (through leading order in k^{-1}) by using the first-order (Ochkur) expression G_1. The full optical elastic amplitude is then given by $F_{el}^{opt} - G_1$ and the corresponding differential cross section reads

$$\frac{d\sigma_{el}^{opt}}{d\Omega} = |F_{el}^{opt} - G_1|^2. \tag{4.27}$$

As an illustration of the results obtained from Eq. (4.27) we show in Fig. 11 the differential cross section for small angle electron–helium elastic scattering at an incident electron energy of 400 eV. The optical model curve, which includes the static, absorption, and polarization effects, together with the leading exchange corrections, is seen to improve considerably over the first Born results.

Finally, it is worth noting that both the EBS method and the optical model calculation yield forward scattering amplitudes which are in good agreement with the analysis of Bransden and McDowell (1970) based on dispersion relations.

We now describe briefly a few inelastic electron–helium processes. We show in Fig. 12 the differential cross section for the process $e^- + \text{He}(1^1S) \to e^- + \text{He}(2^1S)$ at an incident electron energy of 200 eV. The experimental data are those of Vriens, Simpson, and Mielczarek (1968), as renormalized by Chamberlain, Mielczarek, and Kuyatt (1970). The various theoretical predictions shown are those of the first Born approximation, of the second Born calculation performed by Woollings and McDowell (1972), of the eikonal DWBA method (Joachain and Vanderpoorten, 1974) and of the four-channel calculations of Berrington, Bransden, and Coleman (1973). In particular, Berrington *et al.* show that the 2^1S–2^1P coupling, which they take into account explicitly, strongly influences the angular distribution in the forward direction and brings it into agreement with the experimental data. The first Born and eikonal DWBA results, on the contrary, are too low at small scattering angles.

Let us now consider the excitation process $e^- + \text{He}(1^1S) \to e^- + \text{He}(2^1P)$. In this case the strong $1^1S - 2^1P$ coupling completely dominates. We should therefore expect in this case good results from the eikonal DWBA method (Joachain and Vanderpoorten, 1974).

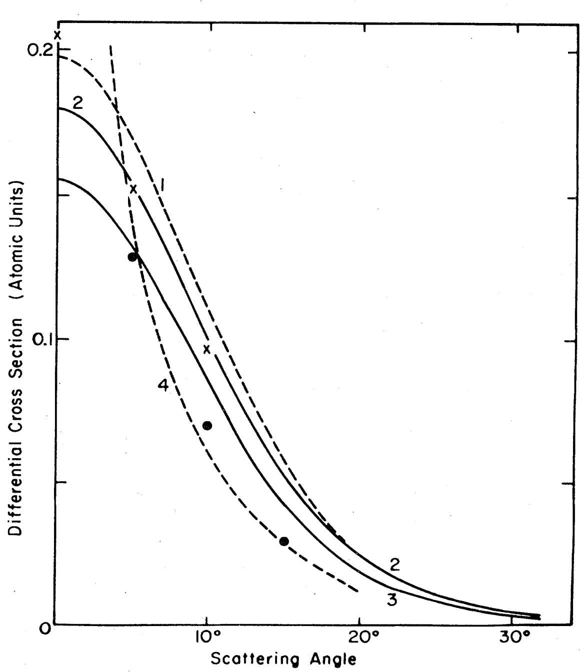


FIG. 12. Differential cross section for the process $e^- + \text{He}(1\,^1S) \to \to e^- + \text{He}(2\,^1S)$ at an incident energy of 200 eV. Curve 1: first Born approximation; Curve 2: eikonal DWBA method with static distorting potentials (Joachain and Vanderpoorten, 1974); Curve 3: eikonal DWBA method with Glauber distorting potentials (Joachain and Vanderpoorten, 1974); Curve 4: four-channel calculations of Berrington, Bransden and Coleman (1973); (x): second Born results of Woollings and McDowell (1972). The dots refer to the measurements of Vriens, Simpson and Mielczarek (1968) renormalized by Chamberlain, Mielczarek, and Kuyatt (1970). (From Joachain and Vanderpoorten, 1974.)

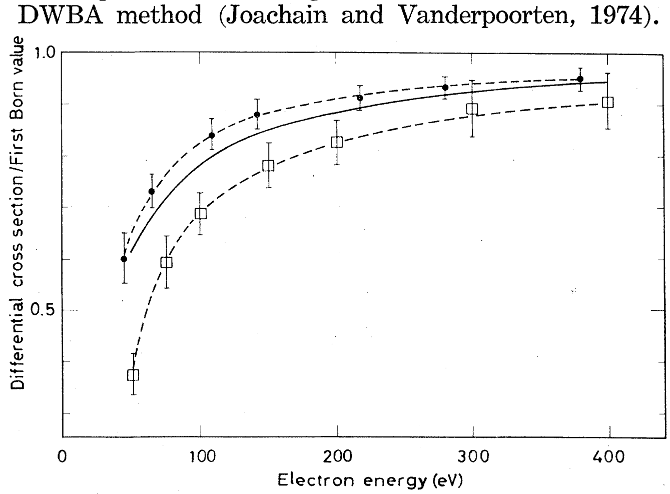


FIG. 13. The ratio of the differential cross section to the corresponding first Born approximation value, at an angle of 5°, for electron excitation of the $2\,^1P$ state of helium. The solid circles represent the results obtained by Byron (1971) from the eikonal amplitude (3.56). (The "theoretical error bars" result from the use of the Monte Carlo method to evaluate the required integrals.) The solid curve corresponds to the eikonal DWBA calculations of Joachain and Vanderpoorten (1974), using Glauber distorting potential. The experimental data (open squares) are those of Chamberlain, Mielczarek, and Kuyatt (1970). (From Joachain and Vanderpoorten, 1974.)

This is confirmed by the examination of Figs. 13 and 14, which show, respectively, the differential cross sections (at $\theta = 5°$) and the total cross sections at intermediate energies. We also note that the eikonal calculations of Byron (1971), who used the expression (3.56), are in good agreement with the experimental results.

To conclude this section, let us examine the excitation of triplet states of helium by electron impact, taking as a particular example the reaction

$$e^- + \mathrm{He}(1^1S) \to e^- + \mathrm{He}(2^3S). \qquad (4.28)$$

As we already mentioned in Sec. IV.A, this process is a pure rearrangement ("knock-out" or exchange) collision provided that very small spin-dependent interactions are neglected. Although the reaction (4.28) received a large amount of attention (Joachain and Mittleman, 1965; Ochkur and Brattsev, 1965; Bell, Eissa, and Moiseiwitsch, 1966; Miller and Krauss, 1968; Kang and Choi, 1968; Joachain and Van den Eynde, 1970), no satisfactory explanation was found for the forward peaking observed (Vriens, Simpson, and Mielczarek, 1968; Chamberlain, Mielczarek, and Kuyatt, 1970) in the differential cross section at small angles and for incident electron energies ranging from 100 to 225 eV. In particular, the first Born and the Ochkur approximations fail badly in this case, as can be seen from the examination of Fig. 15. The reasons for this failure have been given by Byron and Joachain (1972), who have also performed many-body eikonal calculations (using the Monte Carlo integration method) for the reaction (4.28). Their results, shown in Fig. 15, are seen to be in fair agreement with experiment. Given the interest concerning the understanding of rearrangement collisions, more experimental and theoretical work on the reaction (4.28) would be very desirable, particularly at high energies.

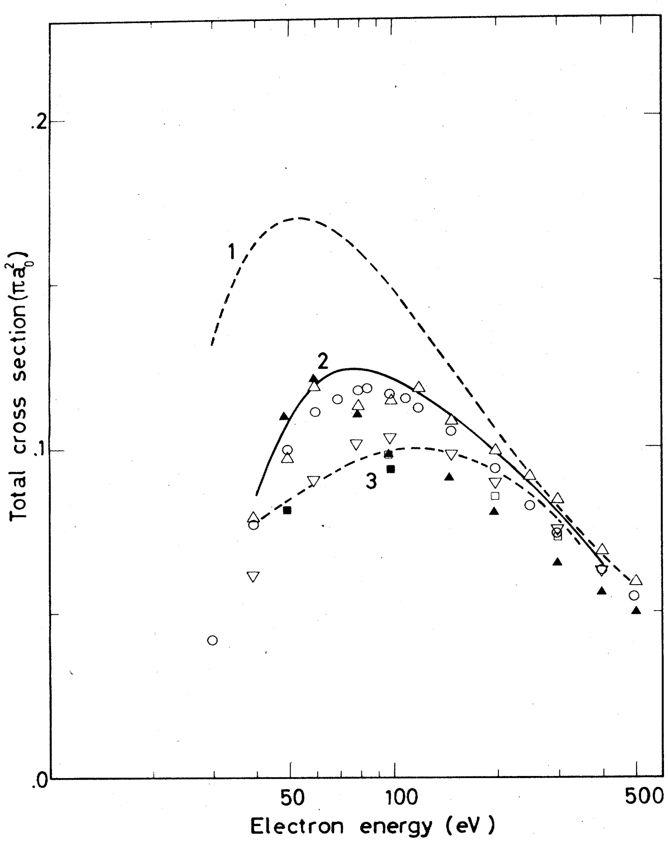


FIG. 14. Total cross section for the process $e^- + \mathrm{He}(1\,^1S) \to \to e^- + \mathrm{He}(2\,^1P)$ as a function of the incident electron energy. Curve 1: first Born approximation; Curve 2: eikonal DWBA method with Glauber distorting potentials (Joachain and Vanderpoorten, 1974); Curve 3: eikonal calculations of Byron (1971). Experimental data: ∇, Jobe and St-John (1967); □, Vriens, *et al.* (1968), renormalized by Chamberlain *et al.* (1970); ▲, Moustafa Moussa *et al.* (1969); △, de Jongh and van Eck (1971); ○, Donaldson *et al.* (1972); ■, Crooks and Rudd (1972). (From Joachain and Vanderpoorten, 1974.)

V. HIGH-ENERGY HADRON–DEUTERON COLLISIONS

The topic at hand is a vast one which we shall discuss not in general terms but with the intent of illustrating the applicability of multiple scattering expansions to a practical problem. We must be selective in our coverage, so while we will describe some of the complexities of high-energy scattering in detail, we shall have to ignore others. For the reader whose primary concern is hadron–deuteron scattering we therefore list a few of the issues we have not treated, together with one or two modern references which provide access to the literature:

(i) scattering in the resonance region (Landau, 1971),

(ii) neutron cross sections (Musgrave, 1971; Julius, 1972);

(iii) high momentum spectators (Musgrave, 1971);

(iv) Fermi motion (Atwood and West, 1972; West, 1972);

(v) presence of isobars in the deuteron wave function (Kerman and Kisslinger, 1969; Nath, Weber, and Kabir, 1971).

A. High-energy hadron-nucleus scattering

We consider a hadron X of initial laboratory energy E and momentum $\mathbf{k}$ incident on a nucleus of mass number

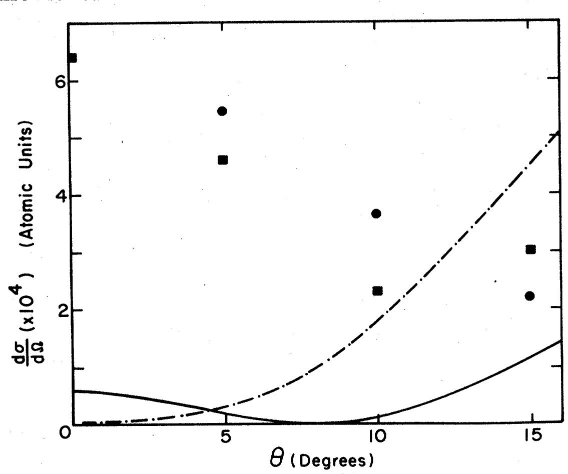


FIG. 15. The differential cross section for the process $e^- + \mathrm{He}(1\,^1S) \to e^- + \mathrm{He}(2\,^3S)$ at an incident electron energy of 225 eV. The solid curve refers to the first Born approximation. The dash-dot curve corresponds to the Ochkur approximation (Ochkur and Brattsev, 1965a). The squares are the results of the many-body eikonal approximation (Byron and Joachain, 1972). The dots refer to the measurements of Vriens, Simpson, and Mielczarek (1968), renormalized by Chamberlain, Mielczarek, and Kuyatt (1970). (From Byron and Joachain, 1972.)

A. We use units such that $\hbar = c = 1$. We assume that the incident particle travels much faster than the characteristic nuclear velocities, and that it interacts with the target nucleons via two-body spin-independent interactions. (The generalization to spin-dependent interactions will be discussed briefly below.) Furthermore, we shall only consider for the moment small angle elastic or "mildly" inelastic collisions. The transition amplitude from an initial nuclear state $|0\rangle$ to a final nuclear state $|m\rangle$ is then given (in the laboratory system) by the Glauber expression (3.51), namely (Glauber, 1959)

$$F_{m0}^{G} = \frac{ik}{2\pi}\int d^2\mathbf{b}\, \exp(i\mathbf{q}\cdot\mathbf{b})\langle m|\Gamma_{\text{tot}}(\mathbf{b},\mathbf{b}_1,\ldots,\mathbf{b}_A)|0\rangle, \quad (5.1)$$

where $\mathbf{q}$ is the laboratory momentum transfer and Γ_{tot} may be written as [see Eq. (3.54)]

$$\Gamma_{\text{tot}} = \sum_{j=1}^{A}\Gamma_j - \sum_{j\neq l}\Gamma_j\Gamma_l + \cdots (-1)^{A-1}\prod_{j=1}^{A}\Gamma_j. \quad (5.2)$$

The multiple scattering series (5.2), which contains A terms, has been particularly useful for analyzing the scattering of high-energy hadrons by light nuclei. We shall return shortly to this point in connection with hadron–deuteron scattering. We note here that according to Eq. (2.38), generalized to a high-energy two-body collision, the quantity

$$f_j(\mathbf{q}) = \frac{ik}{2\pi}\int d^2\mathbf{b}\, \exp(i\mathbf{q}\cdot\mathbf{b})\Gamma_j(\mathbf{b}) \quad (5.3)$$

is just the eikonal (laboratory) two-body scattering amplitude of the incident particle X by the jth nucleon. Hence, using Eqs. (5.1) and (5.2), we immediately deduce that the "single scattering" or "impulse" approximation, obtained by retaining only the terms linear in Γ_j on the right of Eq. (5.2), leads to the hadron–nucleus scattering amplitude

$$F_{m0} \simeq \sum_{j=1}^{A} f_j(\mathbf{q})\langle m|\exp(i\mathbf{q}\cdot\mathbf{b}_j)|0\rangle. \quad (5.4)$$

In particular, for elastic scattering, and assuming that all the f_j ' s are identical ($f_1 = f_2 = \cdots f$), we recover the familiar result of the "impulse" approximation, namely

$$\frac{d\sigma_{\text{el}}}{d\Omega} \simeq \left(\frac{d\sigma}{d\Omega}\right)_f |S(\mathbf{q})|^2, \quad (5.5)$$

where

$$\left(\frac{d\sigma}{d\Omega}\right)_f = |f|^2 \quad (5.6)$$

is the elastic differential cross section for the scattering of the incident particle by a *free* nucleon, and

$$S(\mathbf{q}) = \sum_{j=1}^{A}\langle 0|\exp(i\mathbf{q}\cdot\mathbf{r}_j)|0\rangle \quad (5.7)$$

is the elastic form factor of the target bound state. Since $S(0) = A$, Eq. (5.5) predicts that in the impulse approximation the coherent (elastic) differential cross section for hadron–nucleus scattering is enhanced by a factor A^2 in the forward direction with respect to the corresponding hadron–nucleon cross section. In fact, because hadrons interact strongly with nucleons, multiple collision effects are important in hadron–nucleus collisions. They lead to an A dependence of the forward differential cross section which increases less rapidly than A^2, although the angular distribution still remains heavily concentrated in the forward direction. This strong forward peaking is the major characteristic of high-energy coherent hadron–nucleus scattering.

The elastic scattering of hadrons by "large" nuclei is conveniently studied by means of the eikonal optical model summarized at the end of Sec. III. For example, using Eqs. (3.85) and (3.88), together with additional corrections for Coulomb and target correlation effects, Goldhaber and Joachain (1968) have analyzed the experimental data of Bellettini *et al.* (1966) on high-energy proton scattering by a variety of nuclei. Their analysis includes a study of inelastic collisions which dominate at larger angles. Goldhaber and Joachain have also proposed a simple eikonal DWBA method to deal with coherent production reactions such as

$$\pi + \text{nucleus} \rightarrow A_1 + \text{nucleus} \quad (5.8)$$

or

$$p + \text{nucleus} \rightarrow N^* + \text{nucleus}\text{ , etc.} \quad (5.9)$$

This formalism has been applied to extract the A_1–nucleon cross section from the analysis of coherent A_1 production in Freon (Goldhaber, Joachain, Lubatti, and Veillet, 1969). Such coherent production reactions in nuclei, which involve the propagation of "excited hadrons" in nuclear matter, have attracted a great deal of interest of late (Bemporad *et al.*, 1971, 1972; Rogers and Wilkin, 1972; Van Hove, 1972, 1973; Gottfried, 1972; Czyz and Maximon, 1972; Goldhaber, 1973; Kamal and Chavda, 1973; Bell, 1973).

We shall not pursue further hadron scattering by nuclei other than deuterium. The interested reader will find additional information and references in recent work (Stodolsky, 1966; Drell and Trefil, 1966; Formanek and Trefil, 1967; Bassel and Wilkin, 1967, 1968; Czyz and Lesniak, 1967; Goldhaber and Joachain, 1968; Ross, 1968; Margolis, 1968; Kolbig and Margolis, 1968; Trefil, 1969; Kofoed-Hansen, 1969; Feshbach and Hüfner, 1970; Feshbach, Gal, and Hüfner, 1971; Moniz and Nixon, 1971; Bassichis, Feshbach, and Reading, 1971; Kofoed-Hansen and Wilkin, 1971; Lambert and Feshbach, 1972, 1973; Kujawski, 1973) as well as in the review articles of Glauber (1967, 1968), Wilkin (1968), Czyz (1971); Silbar and Sternheim (1974); and Saudinos and Wilkin (1974).

B. Hadron–deuteron scattering in the Glauber formalism

Let us now concentrate on hadron–deuteron collisions, which have been studied extensively by using the Glauber generalization of the eikonal approximation. We follow here the analysis of Franco and Glauber (1966). The basic formula for elastic and mildly inelastic collisions is still Eq. (5.1), where

$$\Gamma_{\text{tot}} = 1 - \exp\{i[\chi_n(\mathbf{b} - \tfrac{1}{2}\mathbf{s}) + \chi_p(\mathbf{b} + \tfrac{1}{2}\mathbf{s})]\}. \quad (5.10)$$

The quantities χ_n and χ_p are phase shift functions contributed, respectively, by the neutron and the proton, while the vector $\mathbf{s}$ is the projection of the internal relative vector

$\mathbf{r}_d$ of the deuteron in the plane of impact parameters. If we define the quantities

$$\Gamma_n(\mathbf{b}) = 1 - \exp[i\chi_n(\mathbf{b})] \tag{5.11}$$

and

$$\Gamma_p(\mathbf{b}) = 1 - \exp[i\chi_p(\mathbf{b})], \tag{5.12}$$

we may write Eq. (5.10) as

$$\Gamma_{\text{tot}} = \Gamma_n(\mathbf{b} - \tfrac{1}{2}\mathbf{s}) + \Gamma_p(\mathbf{b} + \tfrac{1}{2}\mathbf{s}) - \Gamma_n(\mathbf{b} - \tfrac{1}{2}\mathbf{s})\Gamma_p(\mathbf{b} + \tfrac{1}{2}\mathbf{s}) \tag{5.13}$$

leading to the physical interpretation in terms of single and double scattering, as we expect from the discussion following Eq. (3.54). To analyze this situation in more detail, we note that the functions Γ_n and Γ_p can be expressed in terms of the hadron–neutron and hadron–proton scattering amplitudes f_n and f_p by an approximate two-dimensional Fourier inversion. [See Eq. (5.3).] Thus

$$\Gamma_n(\mathbf{b}) \simeq \frac{1}{2\pi ik}\int d^2\mathbf{q}\, \exp(-i\mathbf{q}\cdot\mathbf{b})f_n(\mathbf{q}). \tag{5.14}$$

A similar formula holds for Γ_p. Returning to Eq. (5.1), we now have

$$F_{m0}^G = \langle m|\{\exp(i\tfrac{1}{2}\mathbf{q}\cdot\mathbf{s})f_n(\mathbf{q}) + \exp(-i\tfrac{1}{2}\mathbf{q}\cdot\mathbf{s})f_p(\mathbf{q}) + \frac{i}{2\pi k}\int d^2\mathbf{q}'\,\exp(i\mathbf{q}'\cdot\mathbf{s})f_n(\mathbf{q}'+\tfrac{1}{2}\mathbf{q})f_p(-\mathbf{q}'+\tfrac{1}{2}\mathbf{q})\}|0\rangle \tag{5.15}$$

and for elastic scattering

$$F_{\text{el}}^G = f_n(\mathbf{q})S(\tfrac{1}{2}\mathbf{q}) + f_p(\mathbf{q})S(-\tfrac{1}{2}\mathbf{q}) + \frac{i}{2\pi k}\int d^2\mathbf{q}'S(\mathbf{q}')f_n(\mathbf{q}'+\tfrac{1}{2}\mathbf{q})f_p(-\mathbf{q}'+\tfrac{1}{2}\mathbf{q}) \tag{5.16}$$

where $S(\mathbf{q})$ is the form factor of the deuteron ground state, namely

$$S(\mathbf{q}) = \int \exp(i\mathbf{q}\cdot\mathbf{r}_d)|\psi_0(\mathbf{r}_d)|^2 d\mathbf{r}_d. \tag{5.17}$$

Here $\psi_0(\mathbf{r}_d)$ is the ground state deuteron wave function. The formulas (5.15) and (5.16) clearly justify the interpretation of the collision in terms of single and double scattering processes. The two types of diagrams which contribute to the scattering are shown in Fig. 16. Evidently, these diagrams do not, at this point, have any more content than the formulas (5.15) or (5.16). We shall return to the analysis of diagrams in Sec. V.C when dealing with analytic properties of scattering amplitudes.

We may immediately obtain the total hadron–deuteron cross section from Eq. (5.16) by using the optical theorem. Thus, writing $\sigma_{\text{tot}}^d = 4\pi\,\text{Im}\,F_{\text{el}}^G/k$, one finds that (Franco and Glauber, 1966)

$$\sigma_{\text{tot}}^d = \sigma_{\text{tot}}^n + \sigma_{\text{tot}}^p - \delta\sigma, \tag{5.18}$$

where σ_{tot}^n and σ_{tot}^p are, respectively, the total hadron–neutron and hadron–proton total cross sections, and $\delta\,\sigma$, the "cross section defect," is given by

$$\delta\sigma = -\frac{2}{k^2}\int S(\mathbf{q})\text{Re}[f_n(\mathbf{q})f_p(-\mathbf{q})]d^2\mathbf{q}. \tag{5.19}$$

If the average neutron–proton interaction has much larger range than the hadron–nucleon interaction, one can readily derive from Eq. (5.19) the approximate formula

$$\delta\sigma \simeq -\frac{4\pi}{k^2}\,\text{Re}[f_n(0)f_p(0)]\langle r_d^{-2}\rangle, \tag{5.20}$$

where $\langle r_d^{-2}\rangle$ is the inverse square of the neutron–proton distance averaged over the deuteron ground state. Further, if the amplitudes $f_n(0)$ and $f_p(0)$ are purely imaginary ("black nucleons"), one obtains the very simple result (Glauber, 1959)

$$\delta\sigma \simeq \frac{1}{4\pi}\sigma_{\text{tot}}^n\sigma_{\text{tot}}^p\langle r_d^{-2}\rangle. \tag{5.21}$$

A variety of angular distributions can be derived from Eqs. (5.15) and (5.16). The elastic differential cross section is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}} = |F_{\text{el}}^G|^2. \tag{5.22}$$

The total scattered intensity is obtained from

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{sc}} = \sum_m |F_{m0}^G|^2 \tag{5.23}$$

and can be evaluated by using the closure relation on the deuteron final states $|m\rangle$. Inelastic processes in which the deuteron is dissociated into two free nucleons are calculated from

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{in}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{sc}} - \left(\frac{d\sigma}{d\Omega}\right)_{\text{el}}. \tag{5.24}$$

The corresponding total cross sections σ_{el}, σ_{sc}, and $\sigma_{\text{in}} = \sigma_{\text{sc}} - \sigma_{\text{el}}$ are directly obtained by integrating Eqs.(5.22)–(5.24) over the angles, while the "absorption" cross section

$$\sigma_{\text{abs}} = \sigma_{\text{tot}}^d - \sigma_{\text{sc}} \tag{5.25}$$

corresponds to all processes where the incident hadron disappears during the collision or reappears with one or several produced particles.

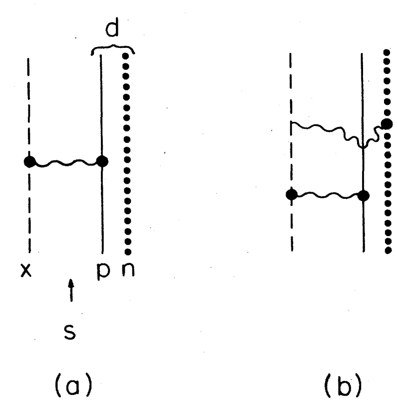


FIG. 16. The two types of diagrams which contribute to elastic hadron–deuteron scattering in the high-energy diffraction theory. [See Eq. (5.16).] (a) Single scattering diagram; (b) double scattering diagram. Another single scattering diagram with proton and neutron interchanged also contributes to the scattering.

The generalization of these considerations to include the spin and isospin degrees of freedom of the incident particle and the target nucleons has been carried out by several authors (Franco and Glauber, 1966; Wilkin, 1966; Glauber and Franco, 1967; Alberi and Bertocchi, 1968, 1969b). For example, collision processes contributing to charge-exchange scattering by the deuteron in the case of an incident hadron of isotopic spin 1/2 are represented in Fig. 17, whereas in Fig. 18 the double charge-exchange process leading to no net transfer of charge is shown. This last effect, first pointed out by Wilkin (1966), is small relative to the other cross section corrections. Indeed, if $f_c(\mathbf{q})$ is the charge-exchange amplitude, one obtains now for the cross section defect, instead of Eq. (5.19)(Glauber and Franco, 1967),

$$\delta\sigma = -\frac{2}{k^2}\mathrm{Re}\{\int S(\mathbf{q}) \times \tfrac{1}{2}[f_p(\mathbf{q})f_n(-\mathbf{q}) + f_n(\mathbf{q})f_p(-\mathbf{q}) - f_c(\mathbf{q})f_c(-\mathbf{q})]\,d^2\mathbf{q}\}, \tag{5.26}$$

or

$$\delta\sigma = -\frac{2}{k^2}\mathrm{Re}\{\int S(\mathbf{q})[2f_n(\mathbf{q})f_p(\mathbf{q}) - \tfrac{1}{2}f_p^2(\mathbf{q}) - \tfrac{1}{2}f_n^2(\mathbf{q})]\,d^2\mathbf{q}\}. \tag{5.27}$$

If the hadron–nucleon force range is small compared with the average neutron–proton interaction, one may again approximate

$$\delta\sigma \simeq -\frac{4\pi}{k^2}\mathrm{Re}[f_n(0)f_p(0) - \tfrac{1}{2}[f_n(0) - f_p(0)]^2]\langle r_d^{-2}\rangle \tag{5.28}$$

which under the assumption of purely imaginary amplitudes $f_n(0)$ and $f_p(0)$ reduces to [compare with Eq. (5.21)]

$$\delta\sigma \simeq \frac{1}{4\pi}[\sigma_{\mathrm{tot}}^n\sigma_{\mathrm{tot}}^p - \tfrac{1}{2}(\sigma_{\mathrm{tot}}^n - \sigma_{\mathrm{tot}}^p)^2]\langle r_d^{-2}\rangle. \tag{5.29}$$

Franco and Glauber (1966) have applied the theory outlined above to a detailed investigation of antiproton–deuteron collisions in the (laboratory) energy range 0.13–17.1 GeV, using various ground-state deuteron wave functions. They assume that at high energies the antiproton–nucleon amplitudes are such that

$$f_{\bar{p}n}(\mathbf{q}) = f_{\bar{p}p}(\mathbf{q}) \equiv f_{\bar{p}N}(\mathbf{q}) \tag{5.30}$$

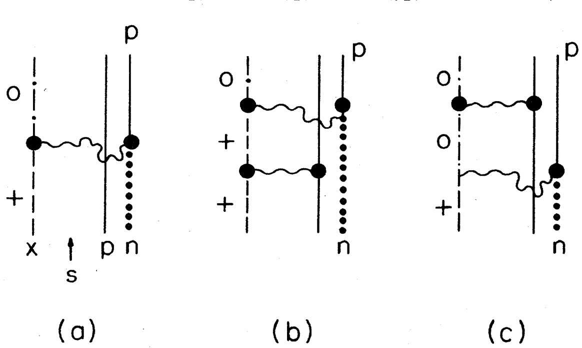


FIG. 17. The various processes which contribute to charge-exchange scattering by the deuteron in the case of a positively charged incident hadron of isotopic spin 1/2.

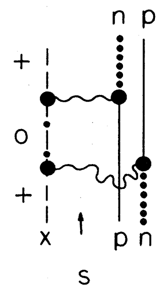


FIG. 18. The double charge-exchange process.

and can be parameterized as

$$f_{\bar{p}N} = i(k_i\sigma_{\bar{p}N}/4\pi)\exp(-\tfrac{1}{2}\alpha^2\mathbf{q}^2). \tag{5.31}$$

Using as input the measured experimental data (Elioff *et al.*, 1962; Galbraith *et al.*, 1965; Czyzewski *et al.*, 1965; Coombes *et al.*, 1958; Armenteros *et al.*, 1960; Foley *et al.*, 1963b; Ferbel *et al.*, 1965) on antiproton–proton collisions, they obtained total and absorption antiproton–deuteron cross sections in good agreement with experiment (Elioff *et al.*, 1962; Galbraith *et al.*, 1965; Chamberlain *et al.*, 1957) and showing an appreciable double scattering effect [see Fig. (19)]. They also investigated spin-dependent effects and concluded that their influence on the cross section defect should be small. Franco (1966) has also analyzed the antiproton–deuteron elastic angular distribution for small momentum transfers in the region of incident momenta between 2.78 and 10.9 GeV/c. In a subsequent work, Glauber and Franco (1967) studied the reaction

$$K^+ + d \to K^0 + p + p, \tag{5.32}$$

which, together with K^+p collisions, is used to extract information about the K^+n charge-exchange reaction (Butterworth *et al.*, 1965)

$$K^+ + n \to K^0 + p. \tag{5.33}$$

They showed that the effect of the charge-exchange correction on the values of the (pn), $(\bar{p}n)$, and (K^+n) total cross sections which are obtained indirectly through deuteron measurements is very small for incident hadron momenta above 2 GeV/c.

We now turn to a more detailed analysis of the *angular distribution* of *elastic* hadron–deuteron scattering. We start with proton–deuteron elastic scattering, which has been studied in the GeV range by various authors (Harrington, 1964, 1968a,b; Franco, 1966, 1968b; Franco and Coleman, 1966; Kujawski, Sachs, and Trefil, 1968; Franco and Glauber, 1969). To understand qualitatively the main features of the angular distribution, let us return to Eq. (5.16). We first note from the alternation of sign in Eq. (5.13) that the double scattering term has opposite sign to the single scattering term. In fact, if the amplitudes f_n and f_p were purely imaginary, the double scattering term would completely cancel the contribution of the single scattering amplitude at $-t \simeq 0.5(\mathrm{GeV}/c)^2$. The contribution of the single and double scattering terms for such a parametrization of the amplitudes is displayed in Fig. 20, which also shows that the single scattering term dominates near the forward direction. At larger momentum transfers the double scattering term, which decreases much more slowly with increasing q, becomes the domi-

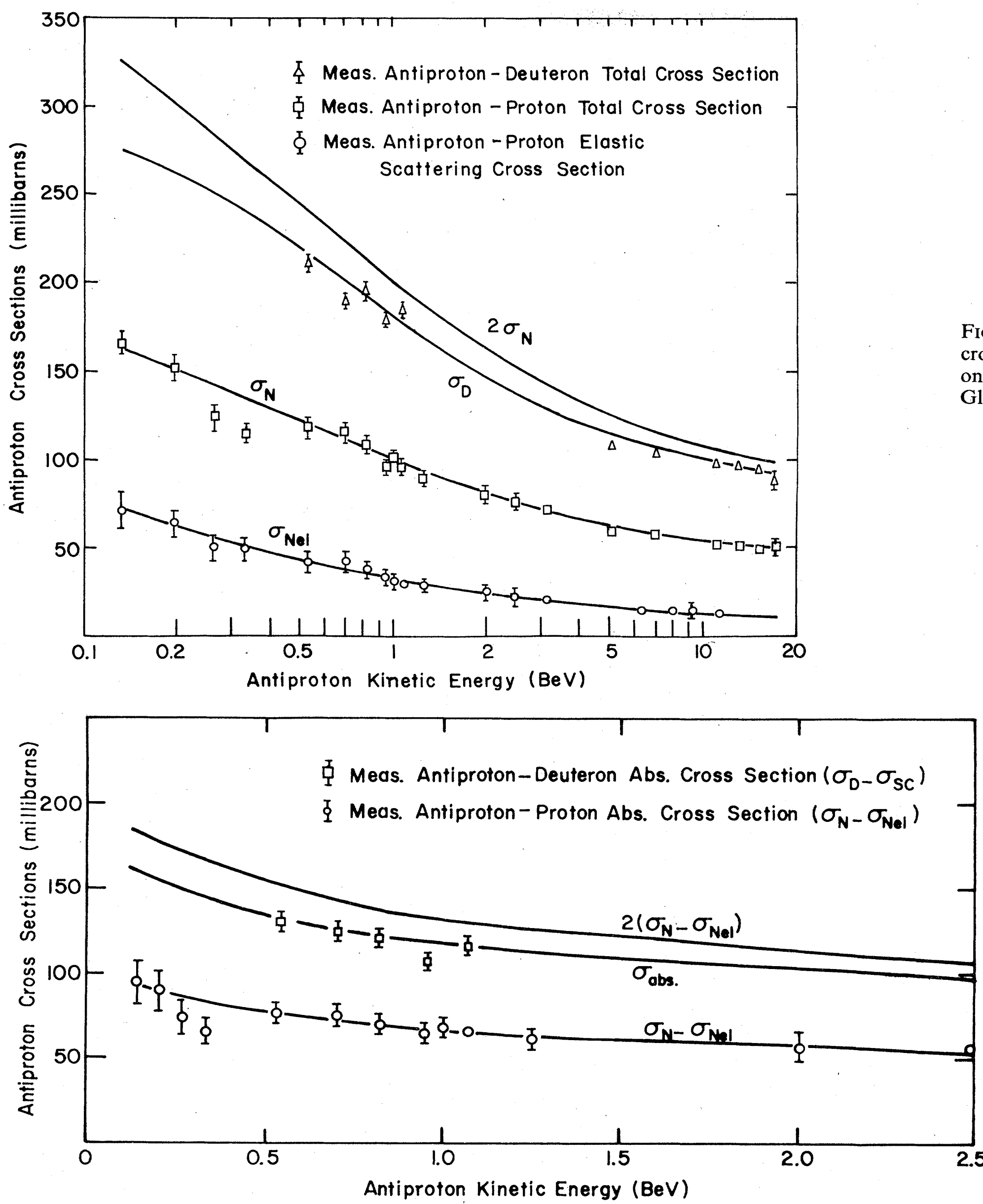


FIG. 19. The total and absorption cross sections for antiproton–deuteron scattering. From Franco and Glauber (1966).

nant contribution to the scattering amplitude.

Let us now analyze more closely the intermediate region of momentum transfers where the single and double scattering terms interfere destructively. Since the proton–neutron and proton–proton scattering amplitudes both have small real parts we do not expect the differential cross section to exhibit a zero, but instead to show a sharp dip in the interference region. This region is therefore of special interest since it depends delicately upon the phases of the hadron–nucleon amplitudes.

The first experimental data on *pd* elastic scattering (Kirillova *et al.*, 1964; Belletini *et al.*, 1965; Zolin *et al.*, 1966; Coleman *et al.*, 1966, 1967) gave encouraging agreement with Glauber's theory. For example, the large-angle measurements at 2.0 GeV (Coleman *et al.*, 1966) confirmed the importance of the double scattering term in the region of four-momentum transfers

$$0.5(\mathrm{GeV}/c)^2 \leq -t \leq 1.5(\mathrm{GeV}/c)^2 \tag{5.34}$$

and were in good agreement with the theoretical calculations of Franco and Colemen (1966). However, these larger-angle data did not fully cover the important intermediate region. It remained for Bennett *et al.* (1967) to perform a crucial *pd* experiment at 1 GeV, which showed agreement with the theory in the small and larger momentum transfer ranges, but displayed only a shoulder (no dip) in the interference region (see Fig. 21). This result was confirmed by measurements at 582 MeV (Boschitz, quoted in Glauber, 1969). A similar feature was observed in πd elastic scattering experiments (Bradamante *et al.*, 1968).

Several suggestions were proposed to understand this apparent paradox: momentum-transfer dependence of the phases of the proton–neutron and proton–proton amplitudes (Bennett *et al.*, 1967), spin effects (Kujawski, Sachs, and Trefil, 1968; Franco, 1968), influence of three-body forces (Harrington, 1968a) or of inelastic intermediate states (Pumplin and Ross, 1968; Alberi and Bertocchi, 1969a; Harrington, 1970). There is one crucial fact, though, which leads to the resolution of the puzzle, namely that interference minima *are* observed in the elastic scattering of protons by the spin zero nuclei He^4, C^{12}, and O^{16} (Palevsky *et al.*, 1967; Boschitz *et al.*, 1968). It is therefore tempting to associate the absence of the dip with the quadrupole deformation of the spin one deuteron (Harrington, 1968a).

The wave function for a deuteron of spin projection M can be written as

$$\begin{aligned}\Phi_M(r) = {} & \frac{u(r)}{r} Y_{00}(\hat{r})\langle \tfrac{1}{2}\tfrac{1}{2} m_1 m_2 | 1M\rangle \chi^p_{m_1}\chi^n_{m_2} \\ & + \frac{w(r)}{r} Y_{2,M-m_1-m_2}(\hat{r}) \\ & \times \langle 21M - m_1 - m_2\, m_1 + m_2 | 1M\rangle \\ & \times \langle \tfrac{1}{2}\tfrac{1}{2} m_1 m_2 | 1 m_1 + m_2\rangle \chi^p_{m_1}\chi^n_{m_2},\end{aligned} \tag{5.35}$$

where $\chi^p_{m_1}$ and $\chi^n_{m_2}$ are proton and neutron Pauli spinors of projection m_1 and m_2, and a summation over m_1 and m_2 is implicit. The S-wave and D-wave radial wavefunctions are chosen real and normalized by

$$\int_0^\infty dr(u^2 + w^2) = 1. \tag{5.36}$$

In the case of πd scattering, four scattering amplitudes must be considered, when spin is not ignored. In them, one may recognize (Michael and Wilkin, 1968; Sidhu and Quigg, 1973), the contributions from spherical, quadrupole, and magnetic form factors

$$\begin{aligned}\phi_S(\tfrac{1}{2}q) &= \int_0^\infty dr\, j_0(\tfrac{1}{2}qr)[|u(r)|^2 + |w(r)|^2], \\ \phi_Q(\tfrac{1}{2}q) &= \int_0^\infty dr\, j_2(\tfrac{1}{2}qr) \\ &\qquad \times [2u(r)w(r) - |w(r)|^2/\sqrt{2}\,], \\ \phi_M(\tfrac{1}{2}q) &= \int_0^\infty dr \{ j_0(\tfrac{1}{2}qr) \\ &\qquad \times [|u(r)|^2 - \tfrac{1}{2}|w(r)|^2] \\ &\qquad + j_2(\tfrac{1}{2}qr)[u(r)w(r)/\sqrt{2} + \tfrac{1}{2}|w(r)|^2]\},\end{aligned} \tag{5.37}$$

the squares of which are plotted in Fig. 22 for the hard-core model of Reid (1968). In a simple model (Michael and Wilkin, 1968) in which the πN spin-flip amplitude is neglected and the nonflip amplitude is positive imaginary, the contribution of the quadrupole form factor to the differential cross section for πd scattering remains finite at the position of the diffraction zero in the contribution of the spherical form factor and fills in the dip. This cooperation is displayed in Fig. 23.

Because the pion–nucleon scattering amplitudes are so well known, more detailed calculations have been possible. Alberi and Bertocchi (1969b) reanalyzed the data of Bradamante *et al.* (1968) by taking into account the deuteron D- state and using πN amplitudes given by phase shift analyses. Some of their results are shown in Fig. 24 which exhibits impressive agreement between theory and experiment. (For a detailed account of this work, see Bertocchi, 1969.) At higher energies the Regge pole fits of Barger and Phillips (1968, 1969) have been exploited by Alberi and Bertocchi (1969b), Michael and Wilkin (1969), and Sidhu and Quigg (1973). These calculations agree very well with the $\pi^- d$ elastic differential cross sections of Fellinger *et al.* (1969) and Bradamante *et al.* (1968, 1969, 1970a) for incident pion momenta between 2 and 15.2 GeV/c, in the single-scattering regime

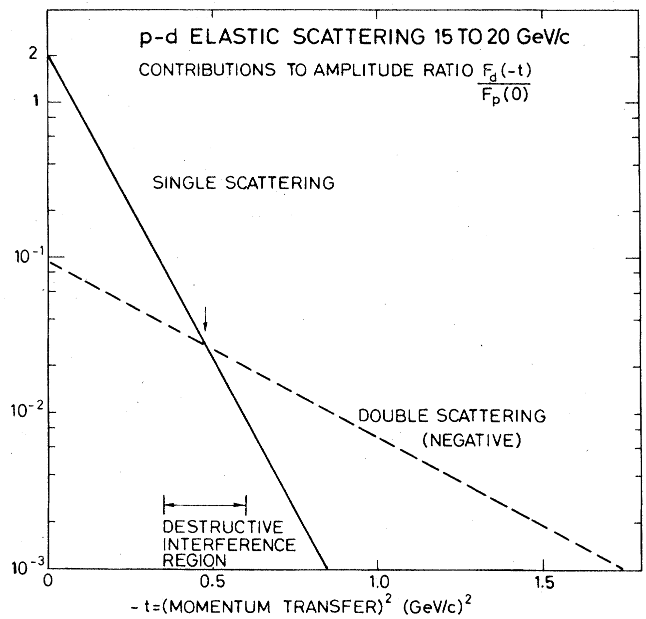


FIG. 20. The contributions to proton–deuteron elastic scattering from the single and double scattering terms in the region 15–20 GeV/c. From Glauber (1969).

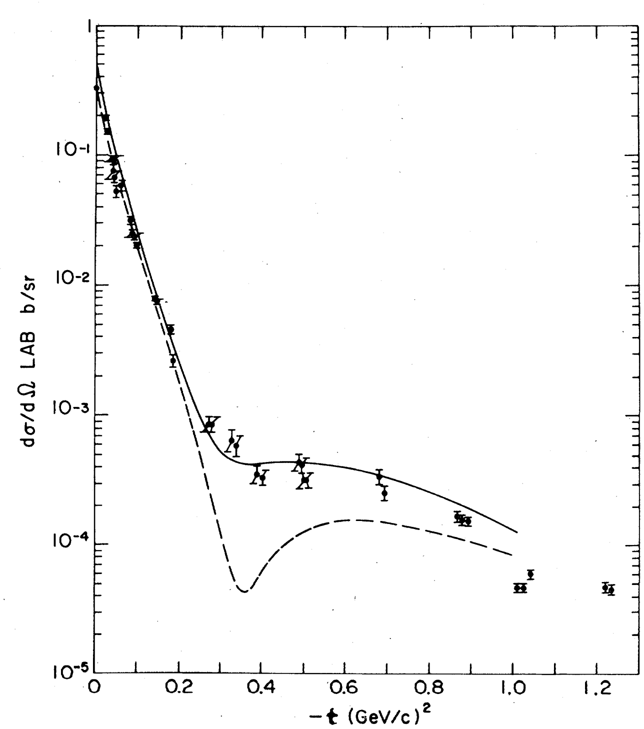


FIG. 21. The proton–deuteron elastic scattering data of Bennett *et al.* (1967), showing the absence of a dip in the "intermediate" region of momentum transfers.

and in the region of the break. Typical calculations are shown in Figs. 25 - 27. At larger angles (in the double-scattering regime), the theoretical curves lie systematically above the data. The number of detailed computations which exhibit these features supports the inference of the CERN–Trieste group (Bradamante *et al.* 1971) that the disagreement in the double-scattering regime (which is also observed in *pd* scattering) cannot be ascribed to the uncertainty in our present knowledge of the hadron–nucleon scattering amplitudes.

Similar considerations apply to proton–deuteron elastic scattering. The calculations of Franco and Glauber (1969) are compared with the experimental data at 1 and 2 GeV in Fig. 28. Recent measurements of *pd* elastic scattering at 9.7, 12.8, and 15.8 GeV/*c* (Bradamante *et al.*, 1970b) and at higher energies (Allaby *et al.*, 1969a, b; Amaldi *et al.*, 1972) are also in excellent agreement with the theory, except in the double-scattering regime.

A number of authors (Fäldt, 1971; Gunion and Blankenbecler, 1971; Cheng and Wu, 1972; Namyslowski, 1972b) have suggested that the Glauber theory without consideration of deuteron recoil overestimates the overlap integral and hence the cross section in the double-scattering regime. While it is appealing to think that the relative motion of initial- and final-state deuterons should diminish the wavefunction overlap, the magnitude of the correction (as estimated, for example, in a covariant formalism by Namyslowski, 1972b) is approximately 20%, whereas the data appear to demand a factor of 2 suppression. Additional precision experiments seem required before it is worthwhile to take the leap of formulating the entire problem covariantly, with the complica-

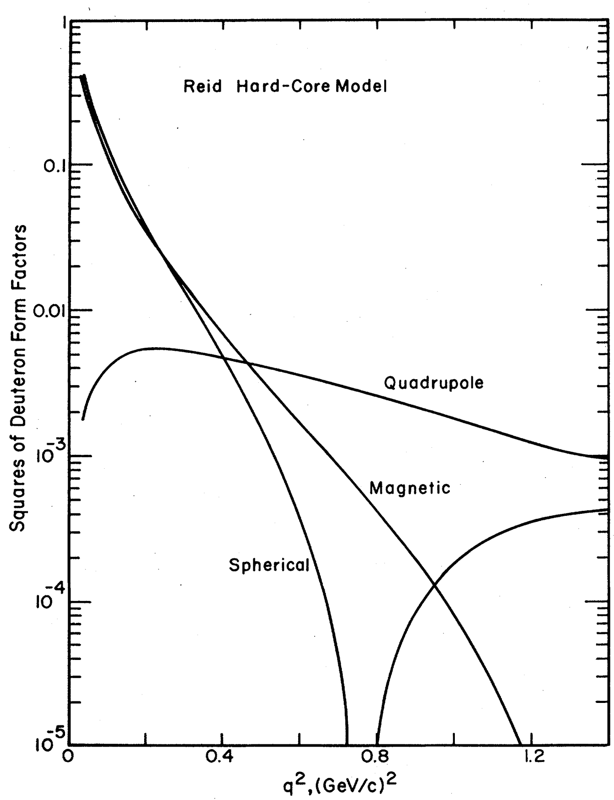


FIG. 22. Squares of the deuteron form factors given by the hard core model of Reid (1968).

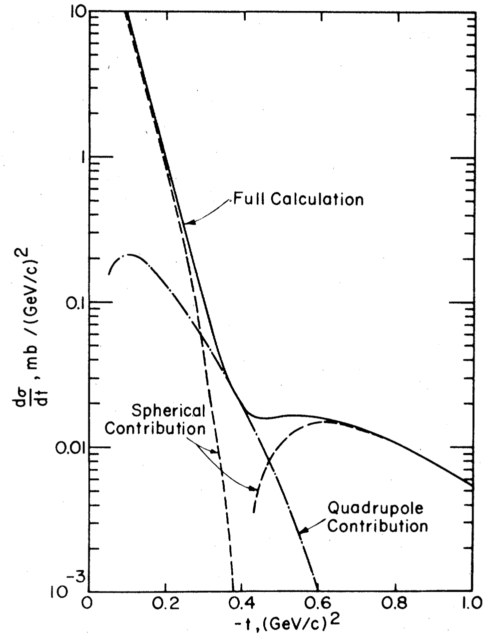


FIG. 23. A simple model calculation (Michael and Wilkin, 1968) showing how the contribution from quadrupole transitions fills in the expected dip in the πd differential cross section.

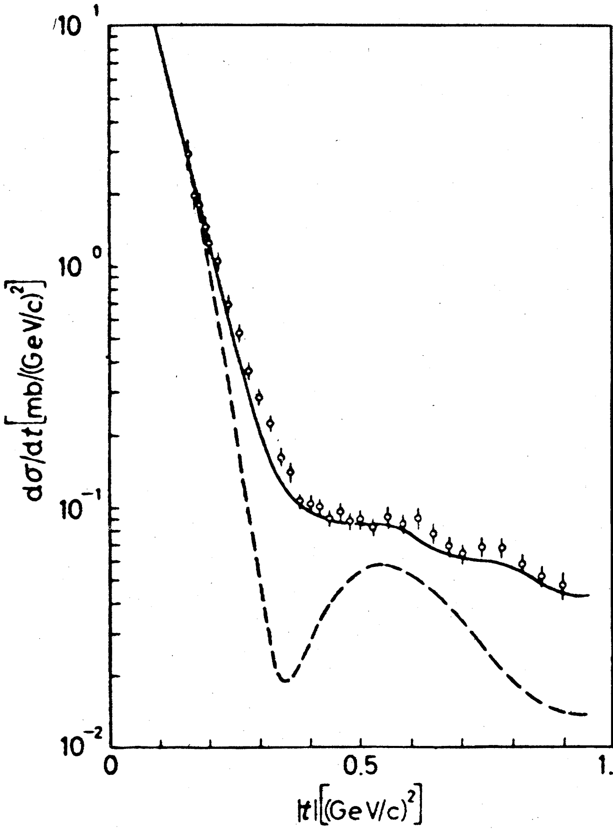


FIG. 24. Comparison of the theoretical calculations by Alberi and Bertocchi (1969b) with the πd elastic scattering data of Bradamante *et al.* (1968) at 895 MeV/*c*. The dashed curve corresponds to a pure *S*-wave deuteron wave function. The solid curve includes the effect of the *D*-wave.

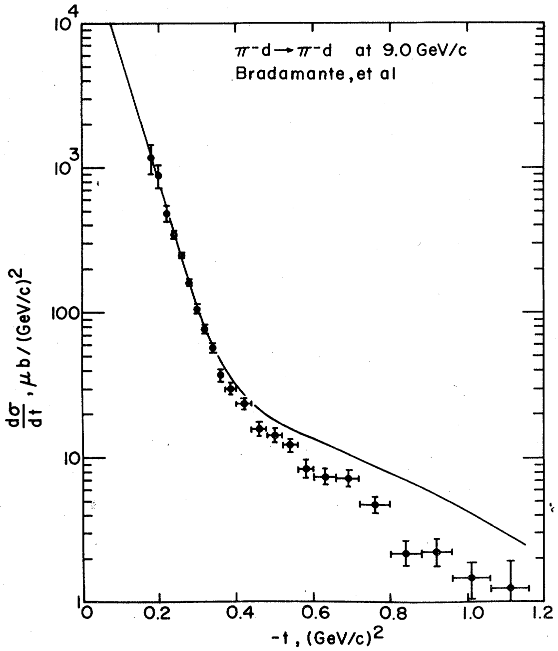


FIG. 25. The differential cross section for $\pi^- d$ elastic scattering at 9 GeV/c calculated by Sidhu and Quigg (1973) is compared with the data of the CERN–Trieste Group (Bradamante *et al.*, 1971). In addition to the statistical errors shown, the data carry an absolute normalization error of 20%.

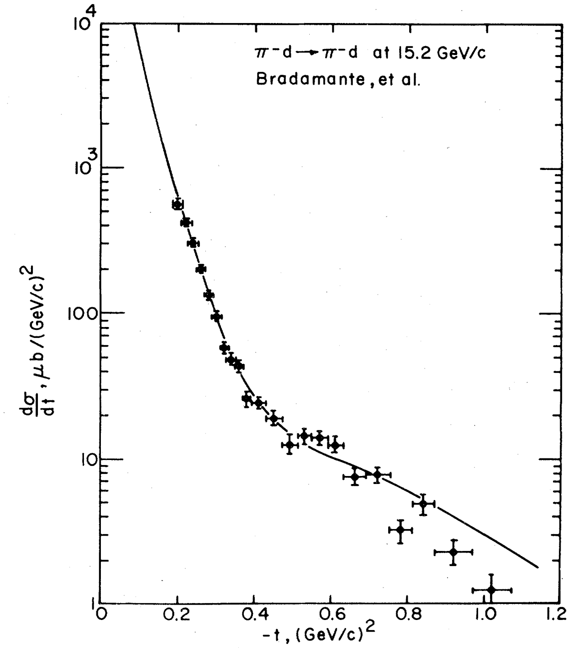


FIG. 27. Same as Fig. 25 at 15.2 GeV/c.

tions of spin fully included.

Since the scattering amplitude for elastic hadron–deuteron scattering in the intermediate momentum transfer region is dominated by quadrupole transitions between the deuteron S and D states, it is strongly dependent on the relative orientations of the momentum transfer and the deuteron spin. Thus, as Franco and Glauber (1969) remarked and Alberi and Bertocchi (1969b) demonstrated by explicit calculations, interesting effects could appear in experiments involving polarized deuteron targets. Indeed, with such a target, the interference dip can appear or not depending on the particular experimental arrangement, namely on the orientation of the polarization axis. Another interesting experiment using the spin dependence arising from the D-wave component of the deuteron to produce high-energy aligned deuterons has been proposed by Harrington (1969a). He pointed out that this spin dependence could be studied in a double-scattering experiment in which a high-energy deuteron beam is scattered from two hydrogen targets in succession. The experiment was carried out by Bunce *et*

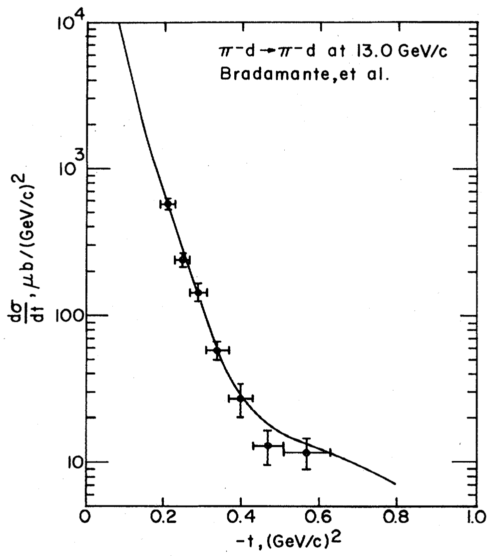


FIG. 26. Same as Fig. 25 at 13.0 GeB/c.

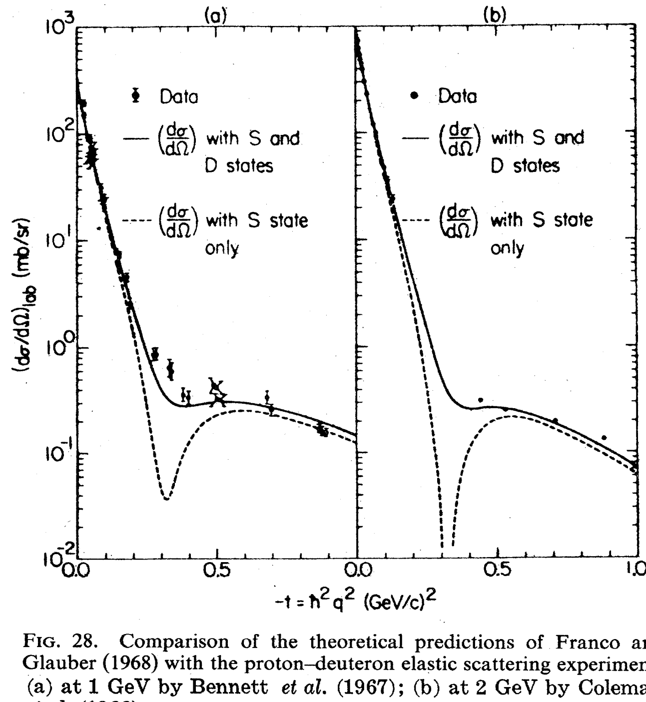


FIG. 28. Comparison of the theoretical predictions of Franco and Glauber (1968) with the proton–deuteron elastic scattering experiments (a) at 1 GeV by Bennett *et al.* (1967); (b) at 2 GeV by Coleman, *et al.* (1966).

al. (1972) using the external deuteron beam of the Princeton–Pennsylvania Accelerator at 3.6 GeV/c, a momentum corresponding to *pd* scattering at 1.0 GeV/c where the differential cross section has been measured by Bennett *et al.* (1967). In a double-scattering experiment in which a deuteron beam is polarized by the first scattering and analyzed by the second scattering, the differential cross section of the second scattering has the azimuthal dependence

$$N(\phi) = N_0(1 + A\cos 2\phi + B\cos\phi). \qquad (5.38)$$

The momentum transfer of the second scattering was fixed at $-t_b = (0.23 \pm 0.016)(\text{GeV}/c)^2$, and the azimuthal asymmetry $N(\phi)$ was measured over a range of momentum transfers of the first scattering. The measured values of the parameters A and B are shown in Fig. 29, together with fits based on Glauber theory, in which the D-state probability and real part of the NN scattering amplitude enter as parameters. Thus the experiment allows a Glauber model-dependent method for measuring the real parts of NN amplitudes at high energies.

We also mention the experiments of Carter *et al.* (1968), who measured πd cross sections, and of Chase *et al.* (1969) on *inelastic* pion–deuteron scattering at 5.53 GeV/c, leading to an outgoing pion plus anything in the final state (missing-mass experiment). The inelastic intensity, calculated from Eq.(5.24), was found to be in good agreement with the data. (See also Hsiung *et al.*, 1968.)

We now turn to a comparison of the Glauber method with the Faddeev–Watson multiple scattering equation. Bhasin (1967) has studied the first four terms of the expansion (3.38) for elastic hadron–deuteron scattering,

$$U_{11} = T_2 + T_3 + T_2 G_0^{(+)} T_3 + T_3 G_0^{(+)} T_2 + \cdots. \qquad (5.39)$$

As expected, the two first terms on the right reduce to Glauber's single scattering terms if one ignores the dependence of T_2 or T_3 on the energy of the third particle and also assumes the two-body off-the-energy-shell amplitudes to be functions only of the momentum transfer. With these assumptions and the additional requirement that $\mathbf{k}_i \simeq \mathbf{k}_f$, the double scattering terms $T_2 G_0 T_3$ and $T_3 G_0 T_2$ also reduce to the Glauber "eclipse" correction. Pumplin (1968) and Bhasin and Varma (1969) have investigated the importance of the off-shell corrections on the double scattering terms. They find that the corresponding effect for proton–deuteron scattering is largest in the interference region between single and double scattering. However, Harrington (1969b) has recently shown that in a potential model the off-energy-shell effects in the double scattering term must cancel the contribution of the remaining part of the multiple scattering series in the high-energy limit. (See also Sec. V. C.) It should be noted here that only in high-energy diffraction theory does the multiple scattering series terminate after A terms. In the deuteron case considered here the triple, quadruple, $\cdots$ terms are small, since they contain at least one (unlikely) backward scattering. Their sum could well annihilate the off-energy-shell contribution to the double scattering term, if the mechanism described by Harrington also works for interactions which cannot be described by potentials.

While we are still discussing the multiple scattering series, it is worth mentioning an analysis by Kofoed-

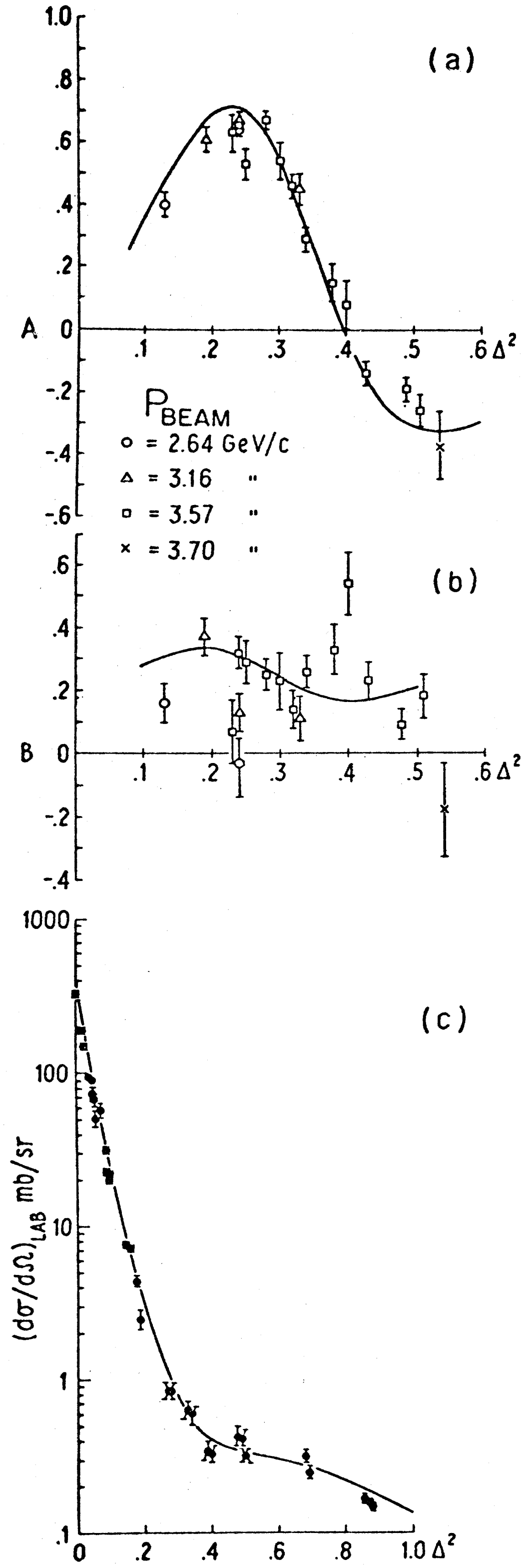


FIG. 29. Fits to the data of Bunce *et al.* (1972) using the Glauber model. (a) The coefficient A in $N(\varphi) = N_0(1 + A\cos 2\varphi + B\cos\varphi)$; (b) the coefficient B; (c) differential cross section from Bennett *et al.* (1967).

Hansen (1969), who has pointed out that truncated versions of the Glauber series (5.2) could produce misleading results since the series is slowly converging in terms of multiplicity. This remark evidently does not apply to the deuteron case—where the multiplicity is two—but it is relevant in cases such as nucleus–nucleus collisions (Franco, 1967, 1970) as well as in quark model or multiple scattering theory of hadron–hadron scattering (Harrington and Pagnamenta, 1967, 1968, 1969; Deloff, 1967; Barnhill, 1967; Schrauner, Benofy, and Cho, 1967; Chou and Yang, 1968; Frautschi and Margolis, 1968; Durand and Lipes, 1968).

We now consider briefly the effect of three-body forces in hadron–deuteron collisions. Harrington (1968b) has studied corrections to the Glauber expression due to the scattering of the incident hadron from a pion being exchanged by the two target nucleons. Numerical estimates indicate that such an effect on the total cross section is quite small ($<1\%$ at very high energies), but could possibly influence the differential cross section at large momentum transfers.

As we have emphasized above, the Glauber method is at its best for collisions in which the inelasticity is small, in particular for elastic scattering, for which the results in the high-energy small angle limits are in excellent agreement with the data. Even in that case, however, one should keep in mind that several correction terms, typified by the contribution of inelastic intermediate states (see Fig. 30) should be included in the scattering amplitude. There is no simple way to take into account the contribution of such inelastic intermediate states within the framework of Glauber's method. Fortunately, because of the mass difference, the reaction

$$\mathrm{p} + \mathrm{p} \to \mathrm{p} + \mathrm{N}^* \tag{5.40}$$

has a minimum momentum transfer greater than zero, so that two relatively violent scatterings of this type, leaving the deuteron in its bound state, are not likely to occur with high probability compared with the single and double scattering terms discussed before. Such "truly inelastic" corrections have been considered for proton–nucleus scattering by Pumplin and Ross (1968) and for pion–deuteron scattering by Alberi and Bertocchi (1969a) and Harrington (1970). We discuss them further in connection with Gribov's Reggeon calculus approach in Sec. V.C. The excellent agreement between conventional Glauber theory and the 19.1 GeV/c pd data of Allaby *et al.* (1969) indicates that inelastic corrections are negligible at that momentum.

More serious problems arise when one wants to study coherent production reactions such as

$$\pi + \text{nucleus} \to \mathrm{A}_1 + \text{nucleus}, \tag{5.41a}$$

$$\pi + \mathrm{d} \to \mathrm{A}_1 + \mathrm{d}, \tag{5.41b}$$

or

$$\mathrm{p} + \mathrm{He}^4 \to \mathrm{N}^* + \mathrm{He}^4, \text{ etc.} \tag{5.41c}$$

We shall return to this question in Sec. V.C. We note, however, that existing discussions of unstable hadron–nucleon cross sections ignore the issue of whether an unstable hadron has time to materialize as such before rescattering. Suppose the A_1 to be a normal resonance, and consider reaction (5.41a). Of particular interest is the term which describes the pion interacting with one nucleon and being excited into an A_1 which subsequently scatters from a second nucleon. Does enough time elapse between the excitation and rescattering for the excited pion to pull itself together as an A_1? The simplest estimates (Goldhaber, 1972), stimulated by the recent experiments of Bemporad *et al.* (1971, 1972) on pion + nucleus → (three or five pions) + nucleus which indicates (after a Goldhaber–Joachain analysis) rather small cross sections for nonresonant three pion and five pion systems on nucleons, suggest that the answer is no.

Coherent production of vector mesons would seem to be a special, and favorable, case since according to the ideas of vector dominance the incoming photon actually exists part of the time as an off-mass-shell vector meson. Some experimental results on the reaction $\gamma d \to \rho^0 d$ are discussed in Sec. V.C.

C. Hadron–deuteron scattering and Regge theory

How to calculate Regge cuts (branch cuts in the angular momentum plane) is one of the challenging theoretical problems of the present day for which no solution seems close at hand. We therefore choose a historical approach to the relation between the Glauber formalism and Regge theory. In this way we shall encounter some of the false steps which have been taken in the past and try to convey the theoretical atmosphere of the present. Some insight is gained into the connection between diffraction and Regge poles if, following Udgaonkar and Gell-Mann (1962), we understand the shrinkage of the diffraction peak by an optical analog.

At high energies hadron–hadron scattering is apparently dominated by Pomeranchuk exchange. The X–Y elastic scattering invariant amplitude, which we represent in Fig. 31, has the form for small angles

$$A_{XY}(s,t) = \{i - \cot[\tfrac{1}{2}\pi\alpha_P(t)]\}\gamma_X(t)\gamma_Y(t)s_0(s/s_0)^{\alpha_P(t)}, \tag{5.42}$$

where $s = -(p_X + p_Y)^2$ is the square of the total c.m.

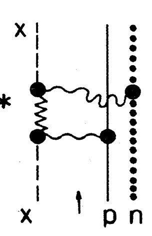


FIG. 30. Diagram corresponding to the contribution of an inelastic intermediate state for elastic scattering.

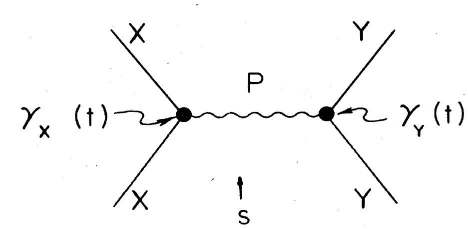


FIG. 31. Reggeon exchange diagram for X–Y elastic scattering, which is governed by Pomeranchuk (P) exchange.

energy, $t = -(p_X - p'_X)^2$ is the square of the four-momentum transfer, s_0 is the Regge scale energy-squared, and $\alpha_P(t)$ is the Pomeranchuk trajectory function: $\alpha_P(0) = 1$. The total cross section is given in terms of this amplitude by the optical theorem,

$$\sigma_{\text{total}}(s) \approx (1/s)\,\text{Im}[A_{XY}(s,0)] = \gamma_X(0)\gamma_Y(0) \quad (5.43)$$

Here we have explicitly exposed the factorization property of the pole residues. Let us rewrite (5.42) as

$$A(s,t) = \{i - \cot[\tfrac{1}{2}\pi\alpha_P(t)]\}\, s_0\left(\frac{s}{s_0}\right)^{\alpha_P(t)} \times \sigma_{\text{total}}(s)[\gamma_X(t)\gamma_Y(t)/\gamma_X(0)\gamma_Y(0)]. \quad (5.44)$$

Now assume that the Pomeranchuk trajectory is linear, $\alpha_P(t) = 1 + \epsilon t$, and that the residue functions are slowly varying, so we may set the factor in square brackets equal to 1. Then for small t, we have

$$A(s,t) \approx is\sigma_{\text{total}}(s)\exp\left[\epsilon t \log\left(\frac{s}{s_0}\right)\right], \quad (5.45)$$

which exhibits, for $\epsilon > 0$, the shrinkage of the diffraction peak.

We write the partial-wave series for $A(s,t)$

$$A(s,t) = 8\pi i \sum_l (2l+1) P_l(\cos\theta_s)(1 - e^{2i\delta_l}). \quad (5.46)$$

We turn the sum over l into an integral, introduce the impact parameter $b = 2ls^{-1/2}$, and use

$$P_l(\cos\theta_s) \approx P_l(1 + 2t/s) \approx J_0[b(-t)^{1/2}].$$

We then calculate $f(s,t)$, an amplitude such that $d\sigma/dt = |f(s,t)|^2$, which for NN scattering at high energies is $f(s,t) \approx [4s(\pi)^{1/2}]^{-1}A(s,t)$, so that

$$\begin{aligned} f(s,t) &= \frac{i}{2(\pi)^{1/2}}\int_0^\infty 2\pi b\, db[1 - S(b,s)]J_0[b(-t)^{1/2}] \\ &= \frac{i}{2(\pi)^{1/2}}\int d^2\mathbf{b}[1 - S(\mathbf{b},s)]\exp(i\mathbf{b}\cdot\mathbf{q}), \end{aligned} \quad (5.47)$$

where the transmission coefficient $S(\mathbf{b},s) = e^{2i\delta_l}$ and $q^2 = -t$. In the exponential approximation (5.45) Fourier inversion gives the absorption coefficient [now $\sigma \equiv \sigma_{\text{total}}(s)$]

$$1 - S(\mathbf{b},s) \approx \frac{\sigma}{8\pi}\left[\epsilon \log\left(\frac{s}{s_0}\right)\right]^{-1} \times \exp\left(\frac{-b^2}{4\epsilon\log(s/s_0)}\right). \quad (5.48)$$

Evidently the effective radius-squared (the value of b^2 for which the absorption coefficient is $1/e$ times its value at $b = 0$) is $4\epsilon\log(s/s_0)$, which increases logarithmically with s. Likewise the transparency, which we define as $[1 - S(b = 0,s)]^{-1}$, is logarithmically increasing with s because of the factor $\epsilon\log(s/s_0)$. Finally we find that the elastic cross section

$$\sigma_{\text{el}} = \int d^2\mathbf{b}|1 - S(\mathbf{b},s)|^2 \approx \frac{\sigma^2}{32\pi\epsilon\log(s/s_0)} \quad (5.49)$$

tends to zero as $s \to \infty$.

Let us describe a nucleus approximately as a composite system specified by a wave function referring to the individual coordinates of the constituent nucleons. Assuming that high-energy NN scattering is controlled by Regge poles, we compute the amplitude for high-energy Nd scattering. The probability distribution $|\psi|^2$ of the nucleon positions is integrated over the beam direction (z coordinates) to give a probability distribution $P(\mathbf{b}_1,\mathbf{b}_2)$ of two-dimensional vectors $\mathbf{b}_i$. Then the transmission coefficient for the deuteron, $S_d(\mathbf{b},s)$, is just the averaged product of the transmission coefficients for the constituent nucleons:

$$S_d(\mathbf{b},s) = \int d^2\mathbf{b}_1 \int d^2\mathbf{b}_2 P(\mathbf{b}_1,\mathbf{b}_2) \times S(\mathbf{b}-\mathbf{b}_1,s)S(\mathbf{b}-\mathbf{b}_2,s). \quad (5.50)$$

Now we take the deuteron c.m. as the origin so that $\mathbf{b}_1 = \frac{1}{2}\boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is the two-dimensional relative coordinate. Let the wave function—ignoring spin–be $\psi(\boldsymbol{\rho},z)$ and define

$$G(p^2) = \int_{-\infty}^{\infty} dz \int d^2\boldsymbol{\rho}|\psi(\boldsymbol{\rho},z)|^2\exp(i\mathbf{p}\cdot\boldsymbol{\rho}). \quad (5.51)$$

Then we get for the scattering amplitude and total cross section

$$\begin{aligned} f_{Xd}(s,t) = \frac{i}{4(\pi)^{1/2}}\Bigg\{ & 2\sigma G(-\tfrac{1}{4}t)B(t)\left(\frac{s}{s_0}\right)^{\alpha(t)-1} \\ & - \frac{\sigma^2}{8\pi^2}\int d^2\mathbf{p}G(p^2)B[-(\tfrac{1}{2}\mathbf{q}-\mathbf{p})^2]B[-(\tfrac{1}{2}\mathbf{q}+\mathbf{p})^2] \\ & \times\left(\frac{s}{s_0}\right)^{\alpha[-(\frac{1}{2}\mathbf{q}+\mathbf{p})^2]+\alpha[-(\frac{1}{2}\mathbf{q}-\mathbf{p})^2]-2}\Bigg\} \end{aligned} \quad (5.52)$$

and

$$\sigma_{Xd}^{\text{tot}} = 2\sigma - \frac{\sigma^2}{8\pi^2}\,\text{Re}\int d^2\mathbf{p}G(p^2)[B(-p^2)]^2\left(\frac{s}{s_0}\right)^{2\alpha(-p^2)-2}, \quad (5.53)$$

where

$$B(t) = \{1 + i\cot[\tfrac{1}{2}\pi\alpha(t)]\}s_0\gamma_X(t)\gamma_Y(t)/\gamma_X(0)\gamma_Y(0)$$

is the Regge residue function.

In addition to the Pomeranchuk pole term, with a coefficient twice as large in the forward direction as in the NN case, there is an eclipse term which corresponds to a continuous "smear" of Regge poles, i.e., to a Regge cut with branch point at

$$\alpha_c = 2\alpha(\tfrac{1}{4}t) - 1. \quad (5.54)$$

This is the result of Udgaonkar and Gell-Mann (1962). At very high energies, the eclipse term at $t = 0$ vanishes like $1/\log(s/s_0)$ and $\sigma_{Xd}^{\text{tot}} \to 2\sigma$. (See also Gribov, Ioffe, Pomeranchuk, and Rudik, 1962.) This is sensible because, as we saw above, the nucleons become very transparent at high energies. For intermediate energies, the eclipse term can be identified with Glauber's.

Abers *et al.* (1966) observed that from the point of view of Feynman graphs the double scattering term contains no Regge cut, so the validity of the result of Udgaonkar and Gell-Mann and, by extension, of Glauber theory at high energies is questionable. To compress this discussion somewhat we draw from a lecture by Wilkin (1969). We may represent the Glauber terms graphically as the impulse (or single scattering) terms of Figs. 32(a) (and 32(b)) and the eclipse (or double scattering) term of Fig. 32(c). Regarded as a Feynman diagram, the double scattering graph has no Regge cut, because the off-mass-shell part of the loop integral cancels the Regge cut from the on-mass-shell contribution which is obtained by replacing the propagator by a delta function. Thus it contributes asymptotically only as s^{-3}, not as $s/\log s$, which is given by the Glauber formula. A general Feynman diagram as in Fig. 33 has a j-plane branch cut on the physical sheet only if both the left-hand and the right-hand blobs have nonzero third double spectral functions $[\rho_{su}(s,t)]$ in the t-channel sense. In other words, crossed lines are required on both sides of the graph; the simplest diagram with a Regge cut appears in Fig. 34 [cf. Mandelstam, 1963; Wilkin, 1964]. Such a result must be a source of embarrassment either for the Glauber theory as embodied in the calculation of Udgaonkar and Gell-Mann or for Feynman graphs, if not for both. On the one hand Feynman diagrams are "fundamental" and therefore to be believed. On the other, Glauber theory has been checked experimentally for energies up to a few GeV.

One may try to circumvent the difficulty by imputing to the projectile hadron an internal structure which includes a cross, e.g., Fig. 34(b), and claiming that the compositeness of hadrons restores the Regge cut. Such a calculation was performed by Abers *et al.* (1966), who thereby proposed to replace the Glauber eclipse with a complicated expression dependent upon the internal structure of the projectile. Assigning a particular internal structure to the projectile seems artificial, especially when the imputed structure may be absent. As Quigg (1970) emphasizes, the statement $\rho_{su} \neq 0$ is equivalent to the statement that the projectile has definite (s-channel) signature. To the extent that exchange degeneracy is exact, hadrons do not have definite signature and the cross, artificial or not, is unrealistic. Under the assumption that duality diagrams are meaningful for Reggeon–hadron scattering, Finkelstein (1971) derived a selection rule for Regge cuts which makes more precise the conflict between arbitrary inputed structure and exchange degeneracy. This phenomenological argument provides strong circumstantial evidence against the imputed structure Feynman graph approach.

Landshoff (1969) has estimated the energy at which the Glauber theory result (the Regge cut of Udgaonkar and Gell-Mann) ceases to be valid numerically under the assumption that the relevant amplitude *is* given by the Feynman graph of Fig. 32(c), without assigning any structure to the projectile. Since the deuteron is very lightly bound, the critical laboratory energy at which the Glauber theory should break down is very large. On the basis of heuristic arguments about the off-mass-shell behavior of scattering amplitudes, Landshoff estimates

$$E_{\text{critical}} \approx m_{\text{projectile}}(M_{\text{nucleon}}/\text{deuteron binding energy})^{1/2}. \quad (5.55)$$

For incident nucleons this is about 20 GeV. Thus while the Feynman diagram considered has no cut in the j plane its numerical properties are quite similar over a wide range of energy to those of the Glauber eclipse term.

Further doubt has been cast upon the simple diagram approach by a potential theory calculation of Harrington (1969b). In Glauber theory the amplitude for scattering from a potential V is given by (see Sec. II. C)

$$f(\mathbf{q}) = \frac{k}{2\pi i}\int d^2\mathbf{b}\, \exp(i\mathbf{q}\cdot\mathbf{b})\{\exp[i\chi(\mathbf{b})] - 1\}, \quad (5.56)$$

where

$$\chi(\mathbf{b}) = \frac{1}{v_i}\int_{-\infty}^{\infty} dz V(\mathbf{b}, z). \quad (5.57)$$

We let $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}'$ and invert the Fourier integral (5.56). Thus

$$\tilde{f}(\mathbf{b}) = \frac{1}{2\pi}\int d^2\mathbf{q}\, \exp(-i\mathbf{q}\cdot\mathbf{b}) f(\mathbf{q}). \quad (5.58)$$

In momentum space we have

$$\tilde{V}(\mathbf{p}) = \frac{1}{(2\pi)^3}\int d^3\mathbf{X}\, \exp(i\mathbf{p}\cdot\mathbf{X}) V(\mathbf{X}), \quad (5.59)$$

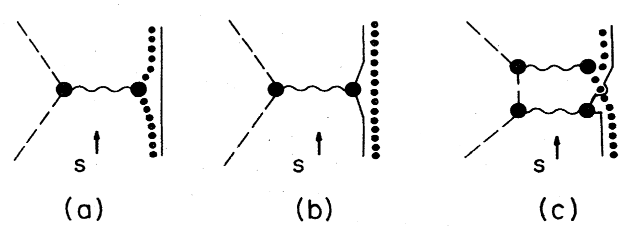


FIG. 32. Graphical representation of the Glauber series for hadron (dashed line)–deuteron scattering: (a) and (b) impulse terms; (c) eclipse term. The wavy lines are Regge poles, the solid line the proton and the dotted line the neutron.

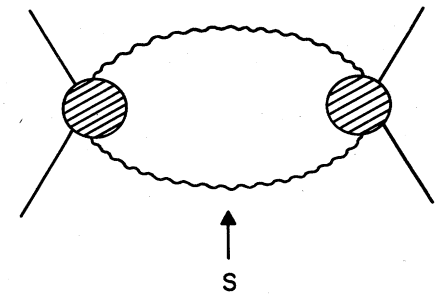


FIG. 33. General Feynman graph for two-Reggeon exchange in (quasi) two-body scattering. The blobs may have complicated structure.

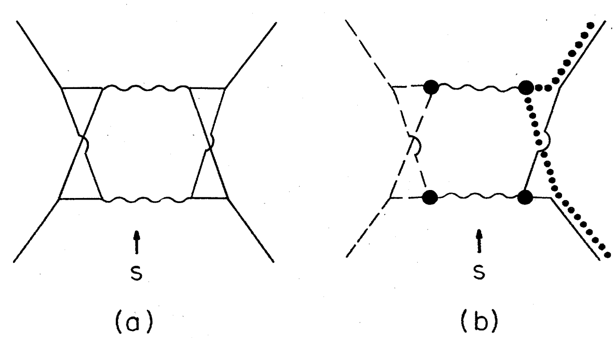


FIG. 34. (a) The simplest Feynman graph which has a Regge cut; (b) redrawn for hadron–deuteron scattering.

and the phase shift expressed in terms of $\tilde{V}$ is

$$\chi(\mathbf{b}) = -\frac{2\pi}{v_i}\int d^2\mathbf{q}\, \exp(-i\mathbf{q}\cdot\mathbf{b})\tilde{V}(\mathbf{q}). \quad (5.60)$$

We expand the integrand of (5.56) in powers of $i\chi(\mathbf{b})$,

$$f = \frac{k}{2\pi i}\int d^2\mathbf{b}\, \exp(i\mathbf{q}\cdot\mathbf{b})\left(\sum_{n=1}^{\infty}\frac{[i\chi(\mathbf{b})]^n}{n!}\right), \quad (5.61)$$

and substitute (5.60) into (5.61) to obtain

$$f = -2\pi ik\sum_{n=1}^{\infty}\frac{1}{n!}\int d^2\mathbf{q}_1\cdots\int d^2\mathbf{q}_{n-1}\left(\frac{-2\pi i}{v_i}\right)^n \times \tilde{V}(\mathbf{q}_1)\tilde{V}(\mathbf{q}_2)\cdots\tilde{V}(\mathbf{q}_{n-1})\tilde{V}\left(\mathbf{q}-\sum_{i=1}^{n-1}\mathbf{q}_i\right). \quad (5.62)$$

This represents an infinite sum of ladder graphs in which the Feynman loop integrals are integrated only over transverse momentum components. We can reexpress (5.56) in terms of the Born amplitude

$$f_B(\mathbf{q}) = -2\pi^2\tilde{V}(\mathbf{q}), \quad (5.63)$$

$$f = -2\pi ik\sum_{n=1}^{\infty}\frac{1}{n!}\int d^2\mathbf{q}_1\cdots\int d^2\mathbf{q}_{n-1} \times\left(\frac{if_B(\mathbf{q}_1)}{v_i\pi}\right)\cdots\left(\frac{if_B(\mathbf{q}-\sum_{i=1}^{n-1}\mathbf{q}_i)}{v_i\pi}\right) \quad (5.64)$$

Thus we have a prescription for calculating the "absorption corrections" to any Born term f_B. Wilkin next applied these rules to πd scattering to give some intuitive background to Harrington's result. First notice that the vertex $d\to np$ is merely a deuteron wave function which we write in momentum space as $\phi(\mathbf{p})$. If the πp amplitude of Fig. 32(b) is the Born term $f_B^p(\mathbf{q})$ we get

$$\int d^3\mathbf{q}'\phi(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')\phi(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_B^p(\mathbf{q}), \quad (5.65)$$

which is the expected result. It is straightforward to verify that the right answer is obtained for Fig. 32(c).

Now consider the graphs in Fig. 35. Remarkably, both of these give the same answer,

$$\int d^3\mathbf{p}\int d^2\mathbf{q}_1\phi(\mathbf{p})\phi(\mathbf{p}+\mathbf{q}_1)f_B^n(\mathbf{q}_1-\tfrac{1}{2}\mathbf{q}) \times\int d^2\mathbf{q}_2 f_B^p(\mathbf{q}_2)f_B^p(\tfrac{1}{2}\mathbf{q}+\mathbf{q}_1-\mathbf{q}_2), \quad (5.66)$$

which is recognizable as part of the Glauber multiple scattering term expanded in a Born series. Thus the Glauber theory includes *triple* scattering terms such as those in Fig. 35. Notice that the ordering of the πp and πn potential interactions does not affect the contribution of the graph. This is true for any complicated graph, as can be proved from the rules obtained above. It is then a basic property of Glauber theory that the order in which the interactions take place does not matter. A picturesque explanation of this fact (Wilkin, 1969) is that in deriving Glauber theory it is always assumed that the incident energy is large and any changes are very small. Complementary to this certainty in energy is an uncertainty in time: it is impossible to tell which interaction takes place first and hence there is a commutativity among the several scatterings. Glauber theory exploits this independence of time order by lumping all the πp interactions together at one end of the (Glauber, not Feynman!) diagram and pushing all the πn interactions to the other end.

Harrington's calculation goes further. Employing the Faddeev multiple scattering series (cf. Sec. III.B of this review) he proves that in the high-energy limit and in the Glauber approximation the off-shell contribution to the double scattering term is canceled by the higher-order terms in the series. The proof consists in observing that in the high-energy limit the scattering is given by the Glauber approximation

$$T \sim T_{\text{Glauber}} = \sum_n T_{\text{Glauber}}^{(n)}, \quad (5.67)$$

where $T_{\text{Glauber}}^{(n)}$ is $T^{(n)}$ after the Glauber approximations have been made. If we break the linearized propagator into its δ function $[\delta]$ and principal value $[P]$ (off-mass-shell) parts and correspondingly separate $T_{\text{Glauber}}^{(2)}$ as

$$T_{\text{Glauber}}^{(2)} = T_{\text{Glauber},\delta}^{(2)} + T_{\text{Glauber},P}^{(2)}, \quad (5.68)$$

then

$$T_{\text{Glauber}} = T_{\text{Glauber}}^{(1)} + T_{\text{Glauber},\delta}^{(2)}. \quad (5.69)$$

Thereby it follows that in the high-energy limit the off-shell contribution to $T^{(2)}$ must be canceled by the higher-order terms in the multiple scattering series

$$T_P^{(2)} + \sum_{n=3}^{\infty} T^{(n)} \sim T_{\text{Glauber},P}^{(2)} + \sum_{n=3}^{\infty} T_{\text{Glauber}}^{(n)} = 0. \quad (5.70)$$

It is not known whether this exact cancellation carries over to the relativistic domain, but the likelihood that more complicated diagrams will continue to be important means that the use of a few Feynman graphs to debunk (or derive!) Glauber is a very dubious procedure. There is a lesson here for Regge cut calculations in non-nuclear hadron–hadron scattering as well. (We do not pursue nondeuteron scattering any further here, but for the connection between multiple scattering and Regge cuts see the discussion by Jackson, 1970.)

We now turn to the question of *singularities in the Mandelstam variables*. We shall not dwell on the analytic structure of the hadron–deuteron scattering amplitude in the momentum variables, for we are able to refer the reader to the elegant review by Ericson and Locher (1969) on hadron–nucleus forward dispersion relations. In the language of *S*-matrix theory, the lightly bound structure of the deuteron is evidenced through the existence of anomalous threshold singularities (so called because they cannot be discerned in straightforward fashion from unitarity) in $d\to ab$ Regge residue functions

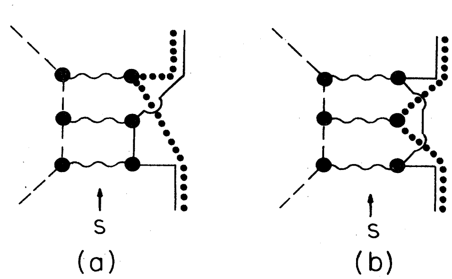


FIG. 35. Triple scattering Feynman graphs which appear in the Born series for the Glauber eclipse term.

(Karplus *et al.*, 1958). A rather complete discussion of the singularities of the *dpn* Regge residue function has been given by Lee (1968). Here we content ourselves with recalling for the reader what anomalous singularities are, by giving an intuitive discussion due to Bohr (1960).

Consider the virtual process $d \rightleftarrows np$. The deuteron is stable in the usual sense because $M_d < M_p + M_n$. For states below threshold, with energies $|\omega_i| < M_i$, a virtual decay can take place if all the particles have positive imaginary momenta $(+i\kappa)$ in the z direction, say. The four-momentum vector of a particle with imaginary three-momentum is Euclidean: $M^2 = \omega^2 + \kappa^2$. The energy momentum conservation equation can be represented geometrically by a triangle in the ω–κ plane as in Fig. 36. For the virtual decay to occur all the energies ω and pseudomomenta κ must be positive, which means the triangle will close if $M_d^2 > M_p^2 + M_n^2$. Hence an anomalous singularity will occur for the deuteron because the deuteron mass satisfies

$$(M_p + M_n)^2 > M_d^2 > M_p^2 + M_n^2. \qquad (5.71)$$

For the deuteron this anomalous threshold lies very near the physical region, at

$$t_0 = 4M_p^2 - \frac{(M_d^2 - M_p^2 - M_n^2)^2}{M_d^2} \approx 0.03\ (\mathrm{GeV}/c)^2. \qquad (5.72)$$

In most phenomenological studies the full complications of kinematics (in particular, of the anomalous threshold) have been ignored. As an example we cite the analysis of coherent K^* (890) production $Kd \to K^*d$ at 4.5 GeV/c of Eisner *et al.* (1968), in which the deuteron is treated as a structureless spin one object. Typically, statistics have been so low that more sophisticated analysis would be unwarranted. For example, see Buchner *et al.* (1969) for coherent K^* production at 3 GeV/c. Alberi and Bertocchi (1969a) estimated the contribution of inelastic intermediate meson states in $\pi d \to \pi d$. Again the subtleties of kinematics were ignored as the Regge pole parametrization was used to give the Phragmén–Lindelöff theorem connection between asymptotic energy dependence and the phase of an amplitude. Given the success of theories for $\pi d \to \pi d$ which take proper account of spin (cf. Sec. V.B), the corrections due to inelastic intermediate states are likely to be small at intermediate energies. An exception to the general rule is the paper by Barger and Michael (1969) in which the full apparatus of Lee's kinematics is applied to $pp \to \pi^+ d$, despite the relative absence of data.

Having analyzed Glauber theory in the J plane and the singularities in the Mandelstam variables, we now consider a modification of the theory recently proposed by Gribov (1969b). This author has argued that for incident momenta $\gtrsim 10$ GeV/c the screening effect changes markedly as inelastic rescattering becomes competitive with the elastic rescattering responsible for the conventional Glauber screening. Similar proposals have been advanced on intuitive or phenomenological grounds by Pumplin and Ross (1968), by Alberi and Bertocchi (1969a), and by Harrington (1970). Gribov's proposed modification of Glauber theory has its roots in his earlier work on a Reggeon calculus (Gribov, 1967) and on the question of the vanishing of the eclipse term at very high energies (Gribov, 1969a). The essence of the suggestion is that inelastic scattering leading not only to discrete resonances, but also to continuum excitation, be taken into account in the computation of rescattering corrections. These inelastic intermediate states are indicated in Fig. 37; their contributions are to be evaluated as usual, by putting the intermediate states on the mass shell.

It is straightforward to apply these ideas to an evaluation of the πd total cross section defect at high energies. Neglecting spin and assuming all production amplitudes to be purely positive imaginary, Gribov (1969b) writes the inelastic screening correction as

$$\delta\sigma_{\mathrm{inelastic}} \approx 2 \sum_l \int dt\, \rho(4t)\, d\sigma_l(t)/dt, \qquad (5.73)$$

where $\rho(t)$ is the deuteron form factor and $d\sigma_l/dt$ is the differential cross section for production of the lth intermediate state. As data become available on the inclusive reaction

$$\pi p \to p + \text{anything} \qquad (5.74)$$

(See, for example, Antipov *et al.*, 1972) it may prove useful to recast (5.73) as

$$\delta\sigma_{\mathrm{inelastic}} \approx 2 \int d(\mathfrak{M}^2/s) \int_{-\infty}^{t_{\min}(\mathfrak{M}^2)} dt\, \rho(4t) \frac{d\sigma}{d(\mathfrak{M}^2/s)dt}, \qquad (5.75)$$

where $d\sigma/d(\mathfrak{M}^2/s)dt$ is the inclusive cross section to produce a proton recoiling against missing mass $\mathfrak{M}$.

Several attempts have been made to estimate inelastic screening effects on the basis of (5.73). Gurvits and Marinov (1970) predicted that inelastic effects should diminish above 20 GeV/c. However, their conclusion was based on the identification of a decreasing experimental cross section as "diffractive" and hence with a purely imaginary amplitude, in conflict with analyticity, and should be disregarded. Kancheli and Matinyan (1970), employing the triple-Regge techniques introduced by Kancheli (1970), traced qualitatively the energy dependence of the eclipse term. They found that, with the onset of inelastic rescattering, the eclipse term increases until the inelastic screening reaches its asymptotic limit, then decreases as the conventional Glauber term diminishes á la Udgaonkar, Gell-Mann, Gribov, Ioffe, Pomeranchuk, and Rudik and approaches a constant limit given solely by inelastic screening. This is in accord with the expectations of Gribov (1969b).

More recently, Sidhu and Quigg (1973) have given a

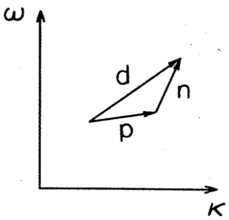


FIG. 36. The virtual dissociation $d \leftrightarrows np$ for imaginary momenta of the three particles. The length of a vector is proportional to the mass of the corresponding particle.

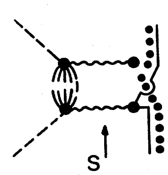


FIG. 37. Gribov's (1969b) proposed double scattering diagram which contains the full spectrum of physical states into which the projectile may be excited. In conventional Glauber theory, only the projectile itself is retained in the intermediate state.

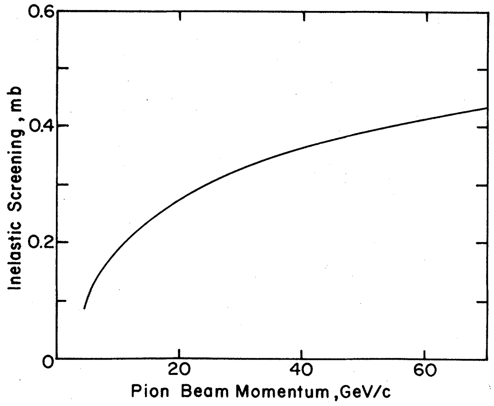


FIG. 38. Inelastic screening corrections to the pion–deuteron total cross section calculated in a nova model by Sidhu and Quigg (1973). Compare the experimental results shown in Fig. 41.

quantitative estimate of the inelastic screening to be expected at high energies. They included as intermediate states all those multipion states which may be reached from the incident pion by diffractive excitation. For simplicity the differential cross sections are parameterized as exponentials

$$d\sigma_l/dt = A_l\sigma_l \exp\{A_l\sigma_l[t - t_l(p_{\text{lab}})]\}, \qquad (5.76)$$

where t_l is the minimum squared momentum transfer required to produce the state l from an incident beam of momentum p_{lab}, and A_l is the slope of the differential cross section. If the deuteron form factor is approximated by an exponential as well, $\rho(t) = e^{(1/4)at}$, one may simplify (5.73) to

$$\delta\sigma_{\text{inelastic}} \approx 2 \sum_l A_l\sigma_l(a + A_l)^{-1}\exp[at_l(p_{\text{lab}})]. \quad (5.77)$$

They took as intermediate states all channels containing an odd number of pions ($>$ 1) and assigned them the cross section suggested by the Nova model for inclusive distributions (Jacob and Slansky, 1972). Choosing $A_l = 2.5\ (\text{GeV}/c)^{-2}$ for every l, they computed the inelastic screening contributions shown in Fig. 38, which they estimate reliable within a factor of 2 in magnitude. The energy dependence is in agreement with the qualitative description given by Kancheli and Matinyan (1970).

Recent measurement of the $\pi^{\pm}p$ total cross sections reveal several interesting features. Unlike the $\pi^{\pm}p$ total cross sections [Fig. 39], which remain constant above 30 GeV/c, the πd cross sections [Fig. 40] continue to fall. To the extent that the πp cross sections are constant, the decrease of the πd cross sections must be laid to inelastic

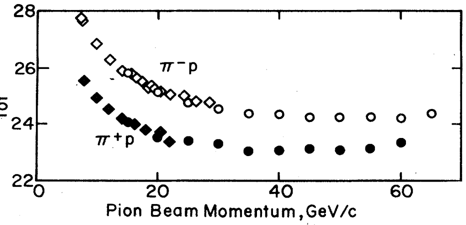


FIG. 39. Pion–nucleon total cross sections at high energies. The π^-p points are from Foley *et al.* (1967) [◇] and from Gorin *et al.* (1971) [○]; the π^+p points are from Foley *et al.* (1967) [◆] and from Denisov, *et al.* (1971) [●].

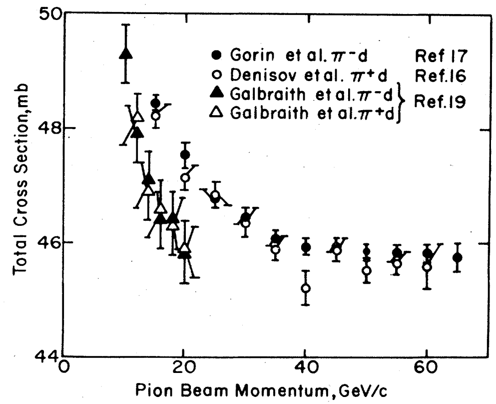


FIG. 40. Pion–deuteron total cross sections at high energies. Notice that whereas the pion–nucleon total cross sections shown in Fig. 39 are essentially constant between 30 and 60 GeV/c, the pion–deuteron cross sections continue to decrease.

screening corrections. Gorin *et al.* (1972) determined the amount of screening directly from the high-energy data as

$$\delta\sigma = \sigma_t(\pi^+p) + \sigma_t(\pi^-p) - \tfrac{1}{2}[\sigma_t(\pi^+d) + \sigma_t(\pi^-d)]. \quad (5.78)$$

Their results are shown in Fig. 41 together with the corresponding results of Galbraith *et al.* (1965). The newly measured screening corrections are clearly increasing with energy; the Serpukhov points are well fitted by the form $\delta\sigma = (1.39 + 0.004[p_{\text{lab}}/(1\text{GeV}/c)]^{1.05})$ mb. Taken literally, they seem to indicate the presence of additional screening corrections, above and beyond those predicted by Glauber theory, of roughly the magnitude that Sidhu and Quigg (1973) found plausible in Gribov's (1969b) formulation. Less convincing evidence for a similar effect in pd scattering is shown in Fig. 42, compiled by Kreisler *et al.* (1968).

Following the lead of Kancheli and Matinyan (1970),

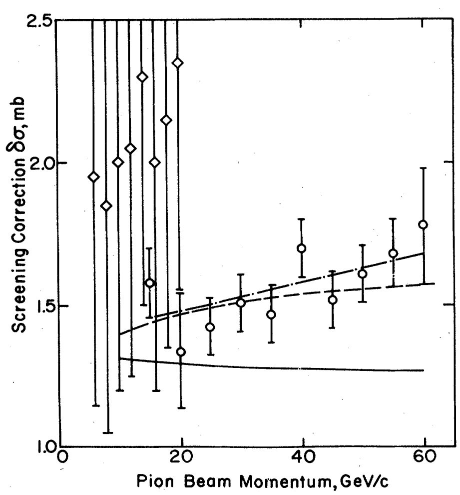


FIG. 41. Experimental results for the screening correction $\delta\sigma$ are shown together with the expectations of Glauber theory (solid curve). Data are from Galbraith *et al.* (1965) [◇] and from Gorin *et al.* (1971) [○]. The dotted line is a best fit of the form $A + Bp^C$ to the Serpukhov data. The dashed line is the calculation of Quigg and Wang (1973) which combines Gribov's scheme with a triple-Reggeon fit to inclusive spectra.

Quigg and Wang (1973) have calculated the πd total cross section defect using as input a triple-Regge analysis of the reaction $\pi^- p \to p +$ anything published by Paige and Wang (1972). In this way the phases of amplitudes are *prescribed* by the Regge pole signature factors and need not be assumed. The results of their calculation (cf. Fig. 41) are in remarkable agreement with the trend of the Serpukhov data and differ markedly from the conventional Glauber theory prediction at high energies. Indeed it does not seem too much to hope that deuteron corrections can provide an important consistency check on the Reggeon calculus program in which Gribov vertices are extracted from data on inclusive reactions. Similar results have been presented by Anisovich *et al.* (1972), and by Kwieciński *et al.* (1974).

To conclude this section, we shall now present a few remarks concerning the experimental situation. As we have indicated in the introduction to this review of high-energy hadron–deuteron scattering, Glauber theory has been tested and refined extensively for *elastic* hadron–deuteron collisions. Such detailed comparison of theory with experiment has not yet been made in inelastic reactions, and we therefore wish to close by making some simple remarks about inelastic scattering. Little is known about the catastrophic case in which the deuteron is broken up and one of the constituent nucleons is transformed into a nucleon resonance or a hyperon. A purely experimental investigation of great value is the comparison of N^* production cross sections off deuterons with the corresponding cross sections off protons. For example, examination of

$$K^+ d \to K^0 \Delta^{++} n_s \text{ vs } K^+ p \to K^0 \Delta^{++} \qquad (5.79)$$

will reveal whether the neutron is truly a spectator or not. This kind of information is needed for one to critically assess the evidence for "exotic" $I = 2$ exchange reported in a comparison of $\gamma p \to \pi^\pm \Delta$ with $\gamma d \to \pi^\pm \Delta N_s$. (See the discussion by Diebold, 1969.) One such comparison has been published by Buchner *et al.* (1971), who claim that at 2.97 GeV/c the differential cross section for $K^+ d \to K^{*0} \Delta^{++} n_s$ is not distinguishible from the differential cross section for $K^+ p \to K^{*0} \Delta^{++}$. In their data the impulse approximation seems completely adequate.

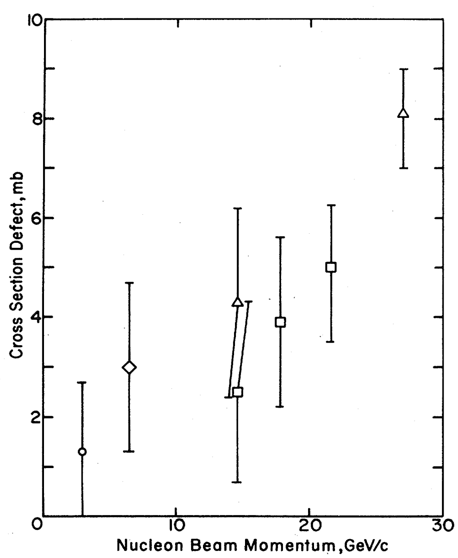


FIG. 42. Compilation of data on the pd total cross section defect (from Kreisler, *et al.*, 1968).

Backward hadron–deuteron scattering is a case in which the Glauber approximation would presumably break down. The most straightforward reaction is $pd \to dp$, for which Bertocchi and Capella (1967) proposed a double scattering mechanism with nucleon exchange which was in satisfactory agreement with the data of Coleman, *et al.* (1966). No single (known) particle exchange is allowed in $\pi d \to d\pi$, so any explanation of this reaction will suffer all the ambiguities of exotic Regge cut box graphs for hadron–hadron scattering.

Coherent excitation of the projectile seems a more tractable problem theoretically, and several experiments have been proposed (Bertocchi and Caneschi, 1967; Formanek and Trefil, 1967) as means to unstable hadron–nucleon cross sections. Of these we mention in particular

$$\begin{array}{ll} \text{(i)} & \pi d \to A_1 d, \\ \text{(ii)} & pd \to N^*(1688)d, \\ \text{(iii)} & Kd \to Qd, \\ \text{(iv)} & \gamma d \to \rho^0 d. \end{array} \qquad (5.80)$$

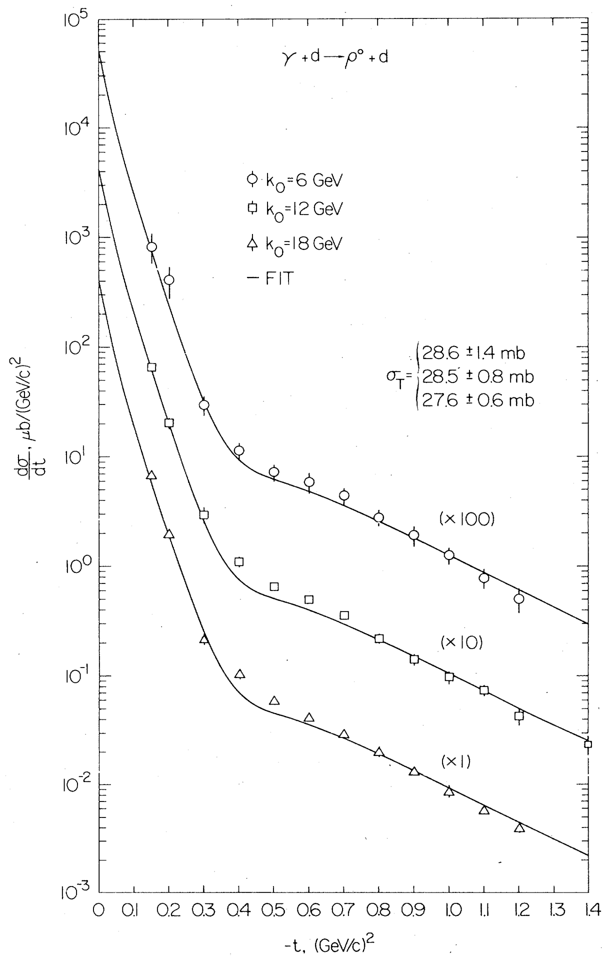


FIG. 43. Differential cross sections for the reaction $\gamma d \to \rho^0 d$ at 6, 12, and 18 GeV/c from the experiment of Anderson *et al.* (1971). The solid lines are Glauber theory fits made to extract information on ρ^0-nucleon scattering.

All of these final states may be obtained by vacuum exchange from the initial states. Using the multiple scattering formalism, we can formulate the problem to show explicitly what is to be learned from this class of experiments.

For a general coherent production

$$X + d \to X^* + d \tag{5.81}$$

we generalize the multiple scattering expansion (3.38) in an obvious way to write the transition operator as

$$T_{X^*X} = T_p + T_n + E_{X^*n}G_{X^*}T_p + E_{X^*p}G_{X^*}T_n + T_nG_XE_{Xp} + T_pG_XE_{Xn} + \cdots, \tag{5.82}$$

where E_{ij} describes the elastic scattering of particles i and j, and T_k is the amplitude corresponding to the process $Xk \to X^*k$. For applications one assumes in the spirit of Harrington (1969b) that the infinite series implied by (5.82) can be replaced by the on-shell contributions to the terms we have displayed explicitly. Then for the reactions (5.80) above everything is known (or otherwise measurable) except the X^*–nucleon elastic scattering amplitude. Thus diffractive excitation of hadron resonances off deuterons becomes a technique for studying unstable hadron–nucleon scattering. Here we have committed the usual sin, criticized in Sec. V.B, of assuming that the excited object indeed corresponds to X^* when it interacts with the second nucleon.

We have already remarked that this implicit assumption is more plausible for reaction (iv) than for the others. In a recent experiment Anderson *et al.* (1971) have studied $\gamma d \to \rho^0 d$ at 6, 12, and 18 GeV over a wide range of momentum transfer. Their data, which are shown in Fig. 43, greatly extend the older results of Hilpert *et al.*

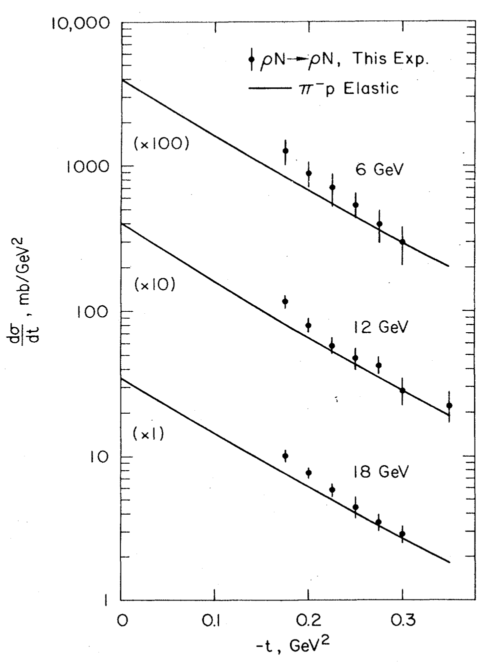


FIG. 44. Differential ρ^0–nucleon scattering cross sections as derived from the experiment of Anderson *et al.* (1971). Only experimental errors are indicated; the authors estimate a theoretical uncertainty of about 10%. For comparison, the solid lines represent π^-p elastic scattering results of Foley *et al.* (1963a) at 7, 13, and 17 GeV/c.

(1970). The shape of the differential cross section is the one characteristic of elastic hadron–deuteron scattering that we have seen already in Figs. 21, 24–29. Using the spin formalism of Michael and Wilkin (1969) and assuming equality of the ρ^0p and ρ^0n elastic scattering amplitudes, Anderson *et al.* extracted from their data the differential cross section for ρ^0–nucleon scattering over a limited range of momentum transfer. Their results are compared with the differential cross sections for $\pi^-p \to \pi^-p$ in Fig. 44. The rough agreement exhibited there is in accord with simple quark model ideas. Because of the ambiguities in the details of the theory in the region of the break in $d\sigma/dt$ for $\gamma d \to \rho^0 d$, the analysis cannot reliably be extended to larger values of $-t$.

VI. SUMMARY

Collision phenomena involving several particles have been analyzed in this review within the framework of multiple scattering theory. We have discussed various multiple scattering expansions, namely the Born series and distorted wave Born series, the Faddeev–Watson expansions, and the eikonal multiple scattering series. We then reviewed the application of these methods to some specific atomic and high-energy collision processes.

Let us recapitulate some important points and indicate various open problems. First of all, we recall that the Born series are perturbation expansions of exact scattering amplitudes in terms of interaction potentials, whereas the Faddeev–Lovelace–Watson expansions are rearrangements of the Born series in which two-body scattering amplitudes (or T-matrices) appear explicitly. Although the low-order terms of the Born and the Faddeev–Lovelace–Watson expansions have been analyzed in simple cases, the elucidation of the general properties of these series for realistic multiparticle problems still remains to be done.

One possible line of approach toward the solution of these problems is to use an approximate but simpler form of the many-body scattering amplitude, provided by the eikonal method. Eikonal multiple scattering expansions may then be generated in terms of either interaction potentials or two-body amplitudes. These eikonal multiple scattering expansions have a much simpler structure than the corresponding Born or Faddeev–Lovelace–Watson series. Moreover, remarkable relationships exist between the terms of the eikonal and the Born (or Faddeev–Lovelace–Watson) expansions. This was illustrated in Sec. II for the simple case of potential scattering, where we analyzed in some detail the correspondence between the terms of the eikonal and Born series. The results obtained in this way allow one to gain considerable insight into the limits of validity of the eikonal approximation. They also lead directly to the eikonal–Born series method, wherein terms of the Born and the eikonal series are combined to yield a consistent expansion of the differential cross section through a given order in k^{-1}. As we have shown in Sec. IV, this approach has been generalized successfully to the analysis of the elastic scattering of charged particles by simple atoms. The generalization of these ideas to inelastic and rearrangement collisions would be most desirable.

The deuteron problem at high energies is further complicated by the strength of the interaction and by the need at some point to incorporate relativistic recoil effects. No satisfactory theoretical proof of the validity of the Glauber approximation in the relativistic, hadronic regime has been given, so even in the presence of impressive experimental successes, a basic and probably deep question remains. Hints in total cross section data that inelastic screening corrections begin to appear at energies above 20 GeV add to the theoretical interest in a "fundamental" understanding of the approximation. It seems likely to us that deuteron targets may provide a controlled situation in which to probe the properties of Regge cuts. A less exotic question, but one of increasing practical importance, deals with the extraction of inelastic and inclusive cross sections from deuterium. This is a problem whose time has come.

ACKNOWLEDGMENTS

This work was initiated when both of us were at the University of California at Berkeley and the Lawrence Berkeley Laboratory. One of us (C.J.J.) is grateful to Professor G. H. Trilling and Professor K. M. Watson for their hospitality in the Physics Department of the University of California during the 1969–1970 academic year. We also wish to thank Professor G. F. Chew for his support at the Lawrence Berkeley Laboratory.

We are indebted to Professor R. J. Glauber for a number of helpful suggestions, and to Professor F. W. Byron, Jr. for many fruitful discussions. Among the many colleagues with whom we have had the opportunity to discuss the material contained in this article, particular thanks are due to Professor B. H. Bransden, Professor J. C. Y. Chen, Professor A. S. Goldhaber, Professor V. N. Gribov, Professor M. H Mittleman, and Professor K. M. Watson. We thank Professor L. Bertocchi for sending us a copy of the lectures he delivered at the Ecole Internationale de la Physique des Particules Elémentaires, Herceg-Novi, in 1969.

We are happy to thank our editors, Professor J. D. Jackson, Professor A. K. Kerman, and Professor S. B. Treiman, for their encouragement and for numerous helpful remarks.

One of us (C.Q.) is pleased to acknowledge the kind hospitality extended him by Professor E. L. Goldwasser and Professor J. D. Jackson at the National Accelerator Laboratory during the Fall Semester, 1972, and the generous grant of a leave of absence for that period by Professor O. Ames and Professor C. N. Yang.

REFERENCES

Aaron, R., R. D. Amado, and B. W. Lee, 1961, Phys Rev. **121**, 319.
Aaron, R., and A. Klein, 1960, J. Math. Phys. **1**, 131.
Abarbanel, H. D. I., and C. Itzykson, 1969, Phys. Rev. Lett. **23**, 53.
Abers, E., H. Burkhardt, V. L. Teplitz, and C. Wilkin, 1966, Phys. Lett. **21**, 339; Nuovo Cimento A **42**, 365.
Adachi, T., 1966, Prog. Theor. Phys. (Kyoto) **35**, 463, 483.
Adachi, T., and T. Kotani, 1965, Prog. Theor. Phys. Suppl., Extra No. 316.
Alberi, G., and L. Bertocchi, 1968, Phys. Lett. B **28**, 186.
Alberi, G., and L. Bertocchi, 1969a, Nuovo Cimento A **61**, 201.
Alberi, G., and L. Bertocchi, 1969b, Nuovo Cimento A **63**, 285.
Alessandrini, V. A., and R. L. Omnès, 1965, Phys. Rev. **139**, B167.
Allaby, J. V., Yu B. Bushnin, S. P. Denisov, A. N. Diddens, R. W. Dobinson, S. V. Donskov, G. Giacomelli, Yu. P. Gorin, A. Klovning, A. I. Petrukhin, Yu D. Prokoshkin, R. S. Shuvalov, C. A. Stahlbrandt, and D. A. Stoyanova, 1969a, Phys. Lett. B **30**, 500.
Allaby, J. V., A. N. Diddens, R. J. Galuber, A. Klovning, Kofoed-Hansen, E. J. Sacharidis, K. Schlüpmann, A. M. Thorndike, and A. M. Wetherall, 1969b, Phys. Lett. B **30**, 549.
Alt, E. O., P. Grassberger, and W. Sandhas, 1967, Nucl. Phys. B **2**, 167.
Amaldi, U., R. Biancastelli, C. Bosio, G. Matthiae, J. V. Allaby, A. N. Diddens, R. W. Dobinson, A. Klovning, J. Litt, L. S. Rochester, K. Schlüpmann, and A. M. Wetherell, 1972, Nucl. Phys. B **39**, 39.
Anderson, R. L., D. Gustavson, J. Johnson, L. Overman, D. M. Ritson, B. H. Wiik, R. Talman, and D. Worcester, 1971. Phys Rev. D **4**, 3245.
Anisovich, V. V., P. E. Volkovitsky, and L. G. Dakhno, 1972, Phys. Lett. B **42**, 224.
Antipov, Y. M., R. Baud, R. Busnello, G. Damgaard, M. N. Kienzle-Focacci, W. Kienzle, R. Klanner, L. G. Landsberg, A. A. Lebedev, C. Lechanoine, P. Lecomte, M. Martin, V. Roinishvili, R. D. Sard, F. A. Yotch, and A. Weitsch, 1972, Phys. Lett. B **40**, 147.
Armenteros, R., C. A. Coombes, B. Cork, G. R. Lambertson, and W. A. Wenzel, 1960, Phys. Rev. **119**, 2068.
Ashkin, J., and G.C. Wick, 1952, Phys. Rev. **85**, 686.
Atwood, W. B., and G. B. West, 1973, Phys. Rev. D **7**, 773.
Baker, A., 1972, Phys. Rev. D **6**, 3462.
Baker, A., 1973, Phys. Rev. D **8**, 1937.
Barger, V., and C. Michael, 1969, Phys. Rev. Lett. **22**, 1330; **23**, 208(E).
Barger, V., and R. J. N. Phillips, 1968, Phys. Lett. B **26**, 730.
Barger, V., and R. J. N. Phillips, 1969, Phys. Rev. **187**, 2210.
Barnhill, M. V., 1967, Phys. Rev. **163**, 1735.
Bassel, R. H. and E. Gerjuoy, 1960, Phys. Rev. **117**, 749.
Bassel, R. H., and C. Wilkin, 1967, Phys. Rev. Lett. **18**, 871.
Bassel, R. H. and C. Wilkin, 1968, Phys. Rev. **174**, 1179.
Bassichis, W. H., H. Feshbach, and J. F. Reading, 1971, Ann. Phys. (N. Y.) **68**, 462.
Bates, D. R., 1962, in *Atomic and Molecular Processes*, edited by D. R. Bates (Academic, New York) p. 596.
Bates, D. R., and A. Dalgarno, 1952, Proc. Phys. Soc. Lond. A **65**, 919.
Beg, M. A. B., 1960, Phys. Rev. **120**, 1867.
Bell, J. S., 1973, Nucl. Phys. B **66**, 293.
Bell, J. S., and E. J. Squires, 1959, Phys. Rev. Lett. **3**, 96.
Bell, K. L., H. Eissa and B. L. Moiseiwitsch, 1966, Proc. Phys. Soc. Lond. **88**, 57.
Bell, K. L., D. J. Kennedy and A. E. Kingston, 1969, J. Phys. B **2**, 26.
Bellettini, G., 1966, Nucl. Phys. **79**, 609.
Bellettini, G., G. Cocconi, A. N. Diddens, E. Lillethun, G. Matthiae, J. P. Scanlon, and A. M. Wetherell, 1965, Phys. Lett. **19**, 341.
Bemporad, C., W. Beusch, A. C. Melissinos, E. Polgar, D. Websdale, J. D. Wilson, J. P. Dufey, K. Freudenreich, R. Frosch, F. X. Gentit, P. Mhlemann, J. Codling, J. G. Lee, M. Letherpn, G. Bellini, M. di Corato, and G. Vegni, 1971, Nucl. Phys. B **33**, 397.
Bemporad, C., W. Beusch, A. C. Melissinos, E. Polgar, D. Websdale, J. P. Dufey, K. Freudenreich, R. Frosch, F. X. Gentit, P. Mühlemann, J. Codling, J. G. Lee, M. Letheren, G. Bellini, M. di Corato, and G. Vegni, 1972, Nucl. Phys. B **42**, 627.
Bennett, G. W., J. L. Friedes, H. Palevsky, R. J. Sutter, G. J. Igo, W. D. Simpson, G. C. Phillips, R. L. Stearns, and D. M. Corley, 1967, Phys. Rev. Lett. **19**, 387.
Berrington, K. A., B. H. Bransden, and J. P. Coleman, 1973, J. Phys. B **6**, 436.
Bertocchi, L., 1969, in Lecture Notes of the Ecole Internationale de la Physique des Particules Elémentaires, Herceg-Novi.
Bertocchi, L., and L. Caneschi, 1967, Nuovo Cimento A **52**, 295.
Bertocchi, L., and A. Capella, 1967, Nuovo Cimento A **51**, 369.
Bethe, H. A., 1940, Phys. Rev. **57**, 1125.
Bhadra, K., and A. S. Ghosh, 1971, Phys. Rev. Lett. **26**, 737.
Bhasin, V. S., 1967, Nuovo Cimento A **49**, 736.
Bhasin, V. S., and V. S. Varma, 1969, Phys. Rev. **184**, 1338.
Birman, A., and S. Rosendorff, 1969, Nuovo Cimento B **63**, 89.
Blankenbecler, R., and M. L. Goldberger, 1962, Phys. Rev. **126**, 766.
Blankenbecler, R., and R. Sugar, 1966, Phys. Rev. **142**, 1051.
Bohr, A., 1960, in *Lectures in Theoretical Physics*, edited by W. Brittin *et al.* (Interscience, New York), Vol. III, p. 1.
Born, M., and E. Wolf, 1964, *Principles of Optics*, (New York: Pergamon Press, MacMillan), 2nd edition, Chap. 8 .
Boschitz, E. T., W. K. Roberts, J. S. Vincent, K. Gotow, P. C. Gugelot, C. F. Perdrisat, and L. W. Swenson, 1968, Phys. Rev. Lett. **20**, 1116.
Bradamante, F., S. Conetti, G. Fidecaro, M. Fidecaro, M. Giorgi, A. Penzo, L. Piemontese, F. Sauli, and P. Schiavon, 1968, Phys. Lett. B **28**, 191.
Bradamante, F., S. Conetti, G. Fidecaro, M. Fidecaro, M. Giorgi, P. Palazzi, A. Penzo, L. Piemontese, F. Sauli, P. Schiavon, and A. Vascotto, 1969, Nuovo Cimento Lett. **1**, 894.
Bradamante, F., G. Fidecaro, M. Fidecaro, M. Giorgi, P. Palazzi, A. Penzo, L. Piemontese, F. Sauli, P. Schiavon, and A. Vascotto, 1970a, Phys. Lett. B **31**, 87.
Bradamante, F., G. Fidecaro, M. Fidecaro, M. Giorgi, P. Palazzi, A. Penzo, L. Piemontese, F. Sauli, P. Schiavon, and A. Vascotto, 1970b, Phys. Lett. B **32**, 303.

Bradamante, F., S. Conetti, G. Fidecaro, M. Fidecaro, M. Giorgi, A. Penzo, L. Piemontese, and P. Schiavon, 1971, Nucl. Phys. B **33**, 165.
Bransden, B H , 1965, in *Advances in Atomic and Molecular Physics*, edited by D. R. Bates and I. Esterman (Academic, New York) Vol. I, p. 85.
Bransden, B. H., 1969, in *Lectures in Theoretical Physics*, edited by S. Geltman, K. T. Mahantappa, and W. E. Brittin (New York, Gordon and Breach), Vol. XI-C, p. 139.
Bransden, B. H. and J. P. Coleman, 1972, J. Phys. B **5**, 537.
Bransden, B. H., J. P. Coleman and J. Sullivan, 1972, J. Phys B **5**, 546.
Bransden, B. H. and M. R. C. McDowell, 1970, J. Phys. B **3**, 29.
Brinkman, H. C. and H. A. Kramers, 1930, Proc. Acad. Sci. Amsterdam **33**, 1973.
Bromberg, J. P., 1969, J. Chem. Phys. **50**, 3906.
Buchner, K., G. Dehm, G. Goebel, H. Hupe, T. Joldersma, I. S. Mittra, W. Wittek, J. M. Crispeels, J. Debaisieux, M. Delabaye, P. Dufour, F. Grard, J.Heughebaert, J. Naisse, G. Thill, A. Grant, V. P. Henri, B. Jongejans, U. Kundt, F. Muller, R. L. Sekulin, and G. Wolf, 1969, Nucl. Phys. B **9**, 286.
Buchner, K., G. Dehm, G. Goebel, H. Hupe, W. Wittek, G. Wolf, Y. Goldschmidt-Clermont, A. Grant, V. P. Henri, B. Jongejans, F. Muller, R. L. Sekulin, J. Debaisieux, G. De Jongh, P. Dufour, F. Grard, J. Heughebaert, S. Tavernier, G. Thill, and R. Windmolders, 1971, Nucl. Phys. B **29**, 381.
Buckingham, R. A., 1937, Proc. R. Soc. Lond. A **160**, 94.
Bunce, G., O. E. Overseth, J. Walker, H. Halpern, R. Handler, L. Pondrom, and S. Olsen, 1972, Phys. Rev. Lett. **28**, 120.
Burke, P. G., H. M. Schey, and K. Smith, 1963, Phys. Rev. **129**, 1258.
Butterworth, I., J. L. Brown, G. Goldhaber, S. Goldhaber, A. A. Hirata, J. A. Kadyk, B. M. Schwartzschild, and G. H. Trilling, 1965, Phys. Rev. Lett. **15**, 734.
Byron, F. W., 1971, Phys. Rev. A **4**, 1907.
Byron, F. W., and C. J. Joachain, 1966, Phys. Rev. **146**, 1.
Byron, F. W., and C. J. Joachain, 1972, Phys. Lett. A **38**, 185.
Byron, F. W., and C. J. Joachain, 1973a, Physica **66**, 33.
Byron, F. W., and C. J. Joachain, 1973b, J. Phys. B (Letters), L169.
Byron, F. W., and C. J. Joachain, 1973c, Phys. Rev. A **8**, 1267.
Byron, F. W., and C. J. Joachain, 1973d, Phys. Rev. A **8**, 3266.
Byron, F. W., and C. J. Joachain, 1974a, Phys. Rev. (to be published).
Byron, F. W., and C. J. Joachain, 1974b, J. Phys. B (Letters) (to be published).
Byron, F. W., C. J. Joachain, and E. Mund, 1973, Phys. Rev. D **8**, 2622.
Carpenter, C. P., and T. F. Tuan, 1970, Phys. Rev. A **2**, 1811.
Carter, A. A., K. F. Riley, R. J. Tapper, D. V. Bugg, R. S. Gilmore, K. M. Knight, D. C. Salter, G. H. Stafford, E. J. N. Wilson, J. D. Davies, J. D. Dowell, P. M. Hattersley, R. J. Homer, and A. W. O'Dell, 1968, Phys. Rev. **168**, 1457.
Chadan, K., 1968, Nuovo Cimento B **53**, 12.
Chamberlain, G. E., S. R. Mielczarek and C. E. Kuyatt, 1970, Phys. Rev. A **2**, 1905.
Chamberlain, O., D. V. Keller, R. Mermod, E. Segrè, H. M. Steiner, and T. Ypsilantis, 1957, Phys. Rev. **108**, 1553.
Chang, S. -J., and S. -K Ma, 1969, Phys. Rev. Lett. **22**, 1334.
Chase, D. M., 1956, Phys. Rev. **104**, 838.
Chase, R. C. , E. Coleman, and H. W. J. Courant, 1969, Phys. Rev. Lett. **23**, 811.
Chen, A. C., J. C. Y. Chen, A. L. Sinfailam, and L. Hambro, 1971, Phys. Rev. A **4**, 1988.
Chen, J. C. Y., 1972, in *The Physics of Electronic and Atomic Collisions*, edited by T. R. Govers and F. J. de Heer (North-Holland Publ. Co., Amsterdam), p. 232.
Chen, J. C. Y., and A. C. Chen, 1972, in *Advances in Atomic and Molecular Physics*, edited by D. R. Bates and I. Estermann (Academic, New York), Vol. 8.
Chen, J. C. Y., A. C. Chen, and P. J. Kramer, 1971, Phys. Rev. A **4**, 1982.
Chen, J. C. Y., and L. Hambro, 1971, J. Phys. B **4**, 191.
Chen, J. C. Y., L. Hambro, A. L. Sinfailam, and K. T. Chung, 1973, Phys. Rev. A **7**, 2003.
Chen, J. C. Y., and C. J. Joachain, 1971, Physica **53**, 333.
Chen, J. C. Y., C. J. Jóachain, and K. M. Watson, 1972, Phys. Rev. A **5**, 2460.
Chen, J. C. Y., and P. J. Kramer, 1971, Phys. Rev. Lett. **27**, 899.
Chen, J. C. Y., and P. J. Kramer, 1972, Phys. Rev. A **5**, 1207.
Cheng, H., and T. T. Wu, 1972, Harvard–MIT preprint.
Chew, G. F., 1950, Phys. Rev. **80**, 196.
Chew, G. F., and M. L. Goldberger, 1952, Phys. Rev. **87**, 778.
Chew, G. F., and G. C. Wick, 1952, Phys. Rev. **85**, 636.
Chou, T. T , and C. N. Yang, 1968, Phys. Rev. Lett. **20**, 1213.
Coleman, E., R. M. Heinz, O. E. Overseth, and D. E. Pellett, 1966, Phys. Rev. Lett. **16**, 761.
Coleman, E., 1967, Phys. Rev. **164**, 1655.
Coleman, E., R. M. Heinz, O. E. Overseth, and D. E. Pellett, 1967, Phys. Rev. **164**, 1655.
Coombes, C. A., B. Cork, W. Galbraith, G. R. Lambertson, and W. A. Wenzel, 1958, Phys. Rev. **112**, 1303.
Crooks, G. B., 1972, University of Nebraska Thesis (unpublished).
Crooks, G. B., and M. E. Rudd, 1972, Bull. Am. Phys. Soc. **17**, 131.
Czyz, W., 1971, in *Advances in Nuclear Physics*, edited by M. Baranger and E. Vogt (Plenum, New York), Vol. 4, p. 61.
Czyz, W., and L. Lešniak, 1967, Phys. Lett. B **24**, 227, 319.
Czyz, W., and L. C. Maximon, 1968, Phys. Lett. B **27**, 354.
Czyz, W., and L. C Maximon, 1972, Acta Phys. Polon. B **3**, 897.
Czyzewski, O., *et al.*, 1965, Phys. Lett. **15**, 188.
Davies, H., 1960, Nucl. Phys. **14**, 465.
de Jongh, J. P., and J. van Eck, 1971, in *Abstracts of Papers in the VIIth International Conference on the Physics of Electronic and Atomic Collisions*, edited by L. M. Branscomb *et al.* (North Holland Publ. Co., Amsterdam), p. 701.
Deloff, A., 1967, Nucl. Phys. B **2**, 597.
Denisov, S. P., S. V. Donskov, Yu. P. Gorin, A. I. Petrukhin, Yu. D. Prokoshkin, D. A. Stoyanova, J. V. Allaby, and G. Giacomelli, 1971, Phys. Lett. B **36**, 415.
Dettman, K., and G. Leibfried, 1968, Z. Phys. **210**, 43.
Dettman, K., and G. Leibfried, 1969, Z. Phys. **218**, 1.
Diebold, R., 1969, in *Proceedings of the Boulder Conference on High Energy Physics*, edited by K. Mahantappa *et al.* (Associated Universities Press, Boulder, Colorado), p. 3.
Dodd, L. R., and K. R. Greider, 1966, Phys. Rev. **146**, 675.
Donaldson, F. G., M. A. Hender and J. W. McConkey, 1972, J. Phys. B **5**, 1192.
Drell, S. D., and J. S. Trefil, 1966, Phys. Rev. Lett. **16**, 552.
Drisko, D. R., 1955, thesis, Carnegie Institute of Technology (unpublished).
Durand, L., III, and Y. T. Chiu, 1964, Phys. Rev. Lett. **12**, 399; **13**, 45(E).
Durand, L., III, and Y. T. Chiu, 1965a, Phys. Rev. **137**, B646.
Durand, L., III, and Y. T. Chiu, 1965b, Phys. Rev. **137**, B1530.
Durand, L., III, and R. Lipes, 1968, Phys. Rev. Lett. **20**, 637.
Eisenberg, J. M., 1972, Ann. Phys. (N.Y.) **71**, 542.
Eisner, R. L., A. C. Ammann, D. D. Carmony, A. F. Garfinkel, L. J. Gutay, R. V. Lakshmi, D. H. Miller, G. W. Tautfest, and W. L. Yen, 1968, Phys. Lett. B **28**, 356.
Ekstein, H., 1956, Phys. Rev. **101**, 880.
Elioff, T., L. Agnew, O. Chamberlain, H. M. Steiner, C. Wiegand, and T. Ypsilantis, 1962, Phys. Rev. **128**, 869.
Englert, F., P. Nicoletopoulos, R. Brout, and C. Truffin, 1969, Nuovo Cimento A **64**, 561.
Ericson, T. E. O., and M. P. Locher, 1969, CERN Report 69-30.
Faddeev, L. D., 1960, Zh. Eksp.Teor. Fiz. **39**, 1459 [Sov. Phys. -JETP **12**, 1014 (1961)].
Faddeev, L. D.,1961, Dokl. Akad. Nauk SSSR **138**, 565 [Sov. Phys. -Dokl. **6**, 384].
Faddeev, L. D., 1962, Dokl. Akad. Nauk SSSR **145**, 301 [Sov. Phys. -Dokl. **7**, 600].
Fäldt, G., 1971, Nucl. Phys. B **29**, 16.
Fellinger, M., *et al.*, 1969, Phys. Rev. Lett. **22**, 1265.
Ferbel, T., *et al.*, 1965, Phys. Rev. **131**, B1250.
Fermi, E., 1936, Ric. Sci. **7–2**, 13.
Fernbach, S., R. Serber, and T. B. Taylor, 1949, Phys. Rev. **75**, 1352.
Feshbach, H., 1958, Ann. Phys. (N.Y.) **5**, 537.
Feshbach, H., 1962, Ann. Phys. (N.Y.) **19**, 287.
Feshbach, H., 1967, in *Interactions of High-Energy Particles with Nuclei*, (Course XXXVIII of the International School of Physics "Enrico Fermi") edited by T. E. O. Ericson (Academic, New York), p. 183.
Feshbach, H., A. Gal, and J. Hüfner, 1971, Ann. Phys. (N.Y.) **66**, 20.
Feshbach, H., and J. Hüfner, 1970, Ann. Phys. (N.Y) **56**, 268.
Feshbach, H., C. E. Porter, and V. F. Weisskopf, 1954, Phys. Rev. **96**, 448.
Fetter, A. L., and K. M. Watson, 1965, in *Advances in Theoretical Physics*, edited by K. A. Brueckner (Academic, New York), Vol. 1, p. 115.
Feynman, R. P., 1949, Phys. Rev. **76**, 769.
Finkelstein, J., 1971, Nuovo Cimento A **5**, 413.
Foley, K. J., R. S. Jones, S. J. Lindenbaum, W. A. Love, S. Ozaki, E.

D. Platner, C. A. Quarles and E. H. Willen, 1967, Phys. Rev. Lett. **19**, 193.
Foley, K. J., S. J. Lindenbaum, W. A. Love, S. Ozaki, J. J. Russell, and L. C. L. Yuan, 1963a, Phys. Rev. Lett. **11**, 425.
Foley, K. J., S. J., Lindenbaum, W. A. Love, S. Ozaki, J. J. Russell, and L. C. L. Yuan, 1963b, Phys. Rev. Lett. **11**, 503.
Formànek, J., and J. S. Trefil, 1967, Nucl. Phys. B **3**, 155; **4**, 165.
Francis, N. C., and K. M. Watson, 1953, Phys. Rev. **92**, 291.
Franco, V., 1966, Phys. Rev. Lett. **16**, 944.
Franco, V., 1967, Phys. Rev. Lett. **18**, 1159.
Franco, V., 1968a, Phys. Rev. Lett. **20**, 709.
Franco, V., 1968b, Phys. Rev. Lett. **21**, 1360.
Franco, V., 1970, Phys. Rev. A **1**, 1705.
Franco, V., 1971, Phys. Rev. Lett. **26**, 1088.
Franco, V., and E. Coleman, 1966, Phys. Rev. Lett. **17**, 827.
Franco, V., and R. J. Glauber, 1966, Phys. Rev. **142**, 1195.
Franco, V., and R. J. Glauber, 1969, Phys. Rev. Lett. **22**, 370.
Franco, V., and B. K. Thomas, 1971, Phys. Rev. A **4**, 1945.
Frautschi, S., and B. Margolis, 1968, Nuovo Cimento, A **56**, 1155.
Freedman, D., C. Lovelace, and J. Namyslowski, 1966, Nuovo Cimento A **43**, 258.
Galbraith, W., E. W. Jenkins, T. F. Kycia, B. A. Leontic, R. H. Phillips, A. L. Read, and R. Rubinstein, 1965, Phys. Rev. **138**, B913.
Gell-Mann, M., and M. L. Goldberger, 1953, Phys. Rev. **91**, 398.
Gerjuoy, E., 1958, Ann. Phys. (N.Y.) **5**, 58.
Gerjuoy, E., 1972, in *The Physics of Electronic and Atomic Collisions*, edited by T. R. Govers and F. J. de Heer (North-Holland Publ. Co., Amsterdam) p. 247.
Gerjuoy, E., B. K. Thomas and V. B. Sheorey, 1972, J. Phys. B **5**, 321.
Ghosh, A. S., and N. C. Sil, 1969, Indian J. Phys. **43**, 490.
Ghosh, A. S., and N. C. Sil, 1970, Indian J. Phys. **44**, 153.
Ghosh, A. S., and N. C. Sil, 1971, J. Phys. B **4**, 836.
Ghosh, A. S., P. Sinha, and N. C. Sil, 1970, J. Phys. B **3**, L58.
Glauber, R. J., 1953, Phys. Rev. **91**, 459.
Glauber, R. J., 1955, Phys. Rev. **100**, 242.
Glauber, R. J., 1959, in *Lectures in Theoretical Physics*, Vol. I, edited by W. E. Brittin (Interscience, New York), p. 315.
Glauber, R. J., 1960, in *Proceedings of the Conference on Nuclear Forces and the Few Nucleon Problem*, Vol. I, edited by T.C. Griffith and E. A. Power (Pergamon Press, New York), p. 233.
Glauber, R. J., 1967, in *High Energy Physics and Nuclear Structure*, edited by G. Alexander (North-Holland Publ. Co., Amsterdam), p. 311.
Glauber, R. J., 1969, in *High Energy Physics and Nuclear Structure*, edited by S. Devons (Plenum Press, New York, 1970), p. 207.
Glauber, R. J., and V. Franco, 1967, Phys. Rev. **156**, 1685.
Goldberger, M. L., and K. M. Watson, 1964, *Collision Theory* (Wiley, New York), Chap. 11.
Goldhaber, A. S., 1972, in *Experimental Meson Spectroscopy- 72*, edited by A. H. Rosenfeld and Kwan-Wu Lai (American Institute of Physics, New York), p. 333.
Goldhaber, A. S., 1973, Phys. Rev. D **7**, 765.
Goldhaber, A. S., and C. J. Joachain, 1968, Phys. Rev. **171**, 1566.
Goldhaber, A. S., C. J. Joachain, H. J. Lubatti, and J. J. Veillet, 1969, Phys. Rev. Lett. **22**, 802.
Gorin, Yu. P., S. P. Denisov, S. V. Donskov, A. I. Petrukhin, Yu. D. Prokoshkin, and D. A. Stoyanova, 1972, Yad. Fiz **15**, 953; [Sov. J. Nucl. Phys. **15**, 530].
Gottfried, K., 1971, Ann. Phys. (N.Y.) **66**, 868.
Gottfried, K., 1972, Acta Phys. Polon. B **3**, 769.
Gottfried, K., and J. D. Jackson, 1964, Nuovo Cimento **34**, 735.
Greider, K. R., and L. R. Dodd, 1966, Phys. Rev. **146**, 671.
Gribov, V. N., 1967, Zh. Eksp. Teor. Fiz. **53**, 654 [Sov Phys. -JETP **26**, 414 (1968)].
Gribov, V. N., 1969a, Yad. Fiz. **9**, 640 [Sov. J. Nucl. Phys. **9**, 369].
Gribov, V. N., 1969b, Zh. Eksp. Teor. Fiz. **56**, 892 [Sov. Phys. -JETP **29**, 483].
Gribov, V. N., B. L. Ioffe, I. Ya. Pomeranchuk, and A. P. Rudik, 1962, Zh. Eksp. Teor. Fiz. **42**, 1419 [Sov. Phys. -JETP **15**, 984].
Gunion, J. F., and R. Blankenbecler, 1971, Phys. Rev. D **3**, 2125.
Gurvits, S. A., and M. S. Marinov, 1970, Phys. Lett. B **32**, 55.
Hahn, Y., 1969, Phys. Rev. **184**, 1022.
Hahn, Y., 1970, Phys. Rev. C **2**, 775.
Harrington, D. R., 1964, Phys. Rev. **135**, B358; **137**, AB3(E).
Harrington, D. R., 1968a, Phys. Rev. Lett. **21**, 1496.
Harrington, D. R., 1968b, Phys. Rev. **176**, 1982.
Harrington, D. R., 1969a, Phys. Lett. B **29**, 188.
Harrington, D. R., 1969b, Phys. Rev. **184**, 1745; see also "Cut Diagrams in Potential Theory," in *Proceedings of the Regge Cut Conference*, Madison.
Harrington, D. R., 1970, Phys. Rev. D **1**, 260.
Harrington, D. R., and A. Pagnamenta, 1967, Phys. Rev. Lett. **18**, 1147.
Harrington, D. R., and A. Pagnamenta, 1968, Phys. Rev. **173**, 1599.
Harrington, D. R., and A. Pagnamenta, 1969, Phys. Rev. **184**, 1908.
Hidalgo, M. B., J. H. McGuire, and D. G. Doolen, 1972, J. Phys. B **5**, L70.
Hilpert, H. G., P. Lauscher, M. Matziolis, U. Idschok, K. Müller, A. Kolb, D. Lüke, P. Raulefs, H. Spitzer, F. Storim, G. Knies, S. Brandt, O. Braun, J. Stiewe, P. Schlamp, and J. Weigl, 1970, Nucl. Phys. B **23**, 45.
Holt, A. R., J. L. Hunt, and B. L. Moiseiwitsch, 1971a, J. Phys. B **4**, L241.
Holt, A. R., J. L. Hunt, and B. L. Moiseiwitsch, 1971b, J. Phys. B **4**, 1318.
Holt, A. R., and B. L. Moiseiwitsch, 1968, Proc. Phys. Soc. Lond. B **1**, 36.
Hsiung, H. C., E. Coleman, B. Roe, D. Sinclair, and J. Vander Velde, 1968, Phys. Rev. Lett. **21**, 187.
Jackson, J. D., 1965, Rev. Mod. Phys. **37**, 484.
Jackson, J. D., 1970, Rev. Mod. Phys. **42**, 12.
Jackson, J. D., and H. Schiff, 1953, Phys. Rev. **89**, 359.
Jacob, M., and R. Slansky, 1972, Phys. Rev. D **5**, 1847.
Janev, R. K., and A. Salin, 1972, Ann. Phys. (N. Y.) **73**, 136.
Joachain, C. J., 1965, Nucl. Phys. **64**, 548.
Joachain, C. J., and M. H. Mittleman, 1965, Phys. Rev. **140**, A432.
Joachain, C. J., and M. H. Mittleman, 1971a, Phys. Lett. A **36**, 209.
Joachain, C. J., and M. H. Mittleman, 1971b, Phys. Rev. A **4**, 1492.
Joachain, C. J., and C. Quigg, 1970, Lawrence Radiation Lab. Report UCRL-19851.
Joachain, C. J., and R. K. Van den Eynde, 1970, Physica, **46**, 8.
Joachain, C. J., and R. Vanderpoorten, 1973, J. Phys. B **6**, 622.
Joachain, C. J., and R. Vanderpoorten, 1974, J. Phys. **B** (to be published)
Jobe, J. D., and R. M. St. John, 1967, Phys. Rev. **164**, 117.
Johnson, J. D., and J. E. Brolley, 1970, Prog. Theor. Phys. (Kyoto) **44**, 1700.
Johnston, R. R. and K. M. Watson, 1961, Nucl. Phys. **28**, 583.
Jost, K., M. Fink, and P. Herrmann, 1974, Phys. Rev. (to be published).
Jost, R., and A. Pais, 1951, Phys. Rev. **82**, 840.
Julius, D. I., 1972 "The Extraction of Neutron Target Cross Sections from Deuterium," Daresbury Lecture Notes DNPL/R20.
Kamal, A. N., and L. K. Chavda, 1973, Nucl. Phys. B **57**, 617.
Kancheli, O. V., 1970, Zh. Eksp. Teor. Fiz. Pis'ma Red. **11**, 397 [JETP Lett. **11**, 267].
Kancheli, O. V., and S. G. Matinyan, 1970, Zh. Eksp. Teor. Fiz. Pis'ma Red. **12**, 41 [JETP Lett. **12**, 30].
Kang, Ik-Ju, and Chan Kyoo Choi, 1968, Phys. Rev. **171**, 123.
Karlsson, B. R., and J. M. Namysłowski, 1972, Nuovo Cimento A **8**, 520.
Karplus, R., C. M. Sommerfield, and E. H. Wichmann, 1958, Phys. Rev. **111**, 1187.
Kelly, R. L., 1971, Phys. Rev. D **4**, 1755.
Kerman, A. K., and L. S. Kisslinger, 1969, Phys. Rev. **180**, 1483.
Kerman, A. K., H. McManus, and R. M. Thaler, 1959, Ann. Phys. (N.Y.) **8**, 551.
Kirillova, L. F., *et al.*, 1964, *Proceedings of the XII International Conference on High Energy Physics*, Dubna.
Kölbig, K., and B. Margolis, 1968, Nucl. Phys. B **6**, 85.
Kofoed-Hansen, O., 1969, Nuovo Cimento A **60**, 621.
Kofoed-Hansen, O., and C. Wilkin, 1971, Ann. Phys. (N. Y.) **63**, 309.
Kohn, W., 1954, Rev. Mod. Phys. **26**, 293.
Kreisler, M. N., L. W. Jones, M. J. Longo, and J. R. O'Fallon, 1968, Phys. Rev. Lett. **20**, 468.
Kujawski, E., 1971, Phys. Rev. D **4**, 2573.
Kujawski, E., 1972, Ann. Phys. (N. Y.) **74**, 567.
Kujawski, E., 1973, Phys. Rev. C **7**, 18.
Kujawski, E., and E. Lambert, 1973, Ann. Phys. (N. Y.) **81**, 591.
Kujawaski, E., D. Sachs, and J. Trefil, 1968, Phys. Rev. Lett **21**, 581.
Kwieciński, J., L. Leśniak, and K. Zalewski, 1974, Cracow preprint TPJU-4/74 (unpublished).
Lambert, E., and H. Feshbach, 1972, Phys. Lett, B **38**, 487.
Lambert, E., and H. Feshbach, 1973, Ann. Phys. (N. Y.) **76**, 80.
Landau, R. H., 1971, Phys. Rev. D **3**, 81.
Landshoff, P. V., 1969, Phys. Rev. **177**, 253.
Lax, M., 1951, Rev. Mod. Phys. **23**, 287.

Le Levier, R. E., and D. S. Saxon, 1952, Phys. Rev. **87**, 40.
Lee, H., 1968, Phys. Rev. **174**, 2130.
Lévy, M., and J. Sucher, 1969, Phys. Rev. **186**, 1656.
Lévy, M., and J. Sucher, 1970, Phys. Rev. D **2**, 1716.
Lippmann, B., 1956, Phys. Rev. **102**, 264.
Lippmann, B., M. H. Mittleman, and K. M. Watson, 1959, Phys. Rev. **116**, 920.
Lippmann, B., and J. Schwinger, 1950, Phys. Rev. **79**, 469.
Long, R. L., D. M. Cox and S. S. Smith, 1968, J. Res. Nat. Bur. Stand. (U.S.) A **72**, 521.
Lovelace, C., 1964a, Phys. Rev. **135**, B1225.
Lovelace, C., 1964b, in *Strong Interactions and High Energy Physics*, edited by R. G. Moorhouse (Oliver and Boyd, London).
Malenka, B. J., 1954, Phys. Rev. **95**, 522.
Mandelstam, S., 1963, Nuovo Cimento **30**, 1127, 1148.
Manning, I., 1965, Phys. Rev. **139**, B495.
Manning, I., 1972, Phys. Rev. D **5**, 1472.
Mapleton, R. A., 1967, Proc. Phys. Soc. (Lond.) **91**, 868.
Margolis, B., 1968a, Phys. Lett. **26B**, 524.
Margolis, B., 1968b, Nucl. Phys. B **4**, 433.
Mathur, K. C., A. N. Tripathi, and S. K. Joshi, 1971, J. Phys. B **4**, L39.
Mathur, K. C., A. N. Tripathi, and S. K. Joshi, 1972, Phys. Rev. A **5**, 746.
McCarroll, R., and A. Salin, 1967, Proc. R. Soc. Lond. A 300, 202.
McGuire, J. H., M. B. Hidalgo, G. D. Doolen, and J. Nuttall, 1973, Phys Rev. A **7**, 973.
Michael, C., and C. Wilkin, 1969, Nucl. Phys. B **11**, 99.
Miller, K. J., and M. Krauss, 1968, J. Chem. Phys. **48**, 2611.
Mittleman, M. H., 1961, Ann. Phys. (N.Y.) **14**, 94.
Mittleman, M. H., 1965, in *Advances in Theoretical Physics*, Vol. 1, edited by K. A. Brueckner (Academic, New York), p. 283.
Mittleman, M. H., 1970, Phys. Rev. A **2**, 1846.
Mittleman, M. H., and K. M. Watson, 1959, Phys. Rev. **113**, 198.
Mittleman, M. H., and K. M. Watson, 1960, Ann. Phys. (N.Y.) **10**, 268.
Molière, G., 1947, Z. Naturforsch. A **2**, 133.
Moniz, E. J., and G. D. Nixon, 1971, Ann. Phys. (N.Y.) **67**, 58.
Moore, R. J., 1970, Phys. Rev. D **2**, 313.
Mott, N. F., and H. S. W. Massey, 1965, *Theory of Atomic Collisions*, (Clarendon Press, Oxford), 3rd edition.
Moustafa Moussa, H. R., F. J. de Heer, and J. Schutten, 1969, Physica **40**, 517.
Musgrave, B., 1971, in *Phenomenology in Particle Physics, 1971*, edited by C. B. Chiu, G. C. Fox, and A.J. G. Hey (Caltech, Pasadena), p. 467.
Namysłowski, J. M., 1972a, Nuovo Cimento A **12**, 321.
Namysłowski, J. M., 1972b, Nuovo Cimento B **12**, 331; Nuovo Cimento Lett. **4**, 517; **5**, 991.
Nath, N. R., H. J. Weber, and P. K. Kabir, 1971, Phys. Rev. Lett. **26**, 1404.
Nuttall, J., and R. W. Stagat, 1971, Phys. Rev. A **3**, 1355.
Ochkur, V. I., 1963, Zh. Eksp. Teor. Fiz. **45**, 734. [Sov. Phys.-JETP **18**, 503 (1964)].
Ochkur, V. I., and V. F. Brattsev, 1965a, Opt. Spektrosk. **19**, 490 [Opt. Spectrosc. USSR **19**, 274 (1965)].
Ochkur, V. I., and V. F. Brattsev, 1965b, Astron. Zh. **42**, 1035 [Sov. Astron. -AJ **9**, 797 (1966)].
Omnès, R., 1964, Phys. Rev. **134**, B1358.
Omnès, R., 1965, Phys. Rev. **137**, B649.
Oppenheimer, J. R., 1928, Phys. Rev. **31**, 349.
Osborn, T. A., 1970, Ann. Phys. (N.Y.) **58**, 417.
Osborn, T. A., and K. L. Kowalski, 1971, Ann. Phys. (N.Y.) **68**, 361.
Ostrofsky, M., G. Breit, and D. P. Johnson, 1936, Phys. Rev. **49**, 22.
Ott, W. R., W. E. Kauppila, and W. L. Fite, 1967, Phys. Rev. Lett. **19**, 1361.
Paige, F. E., and L. -L. Wang, 1972, Nucl. Phys. B **46**, 477.
Palevsky, H., J. L. Friedes, R. J. Sutter, G. W. Bennett, G. J. Igo, W. D. Simpson, G. C. Phillips, D. M. Corley, N. S. Wall, R. L. Stearns, and B. Gottschalk, 1967, Phys. Rev. Lett. **18**, 1200.
Predazzi, E., 1966, Ann. Phys. (N.Y.) **36**, 228, 250.
Pumplin, J., 1968, Phys. Rev. **173**, 1651.
Pumplin, J., and M. Ross, 1968, Phys. Rev. Lett. **21**, 1778.
Queen, N. M., 1964, Nucl. Phys. **55**, 177.
Queen, N. M., 1966, Nucl. Phys. **80**, 593.
Quigg, C., 1970, University of California-Berkeley thesis, UCRL-20032 (unpublished).
Quigg, C., and L. -L. Wang, 1973, Phys. Lett. B **43**, 314.
Reid, R. V., 1968, Ann. Phys. (N Y.) **50**, 411.
Remler, E. A., 1968, Phys. Rev. **176**, 2108.
Remler, E. A., 1971, Ann. Phys. (N.Y.) **67**, 114.
Riesenfeld, W. B., and K. M. Watson, 1956, Phys. Rev. **104**, 492.
Rogers, C., and C. Wilkin, 1972, Nucl. Phys. B **45**, 47.
Roothann, C. C. J., L. M. Sachs, and A. W. Weiss, 1960, Rev. Mod. Phys. **32**, 186.
Rosenberg, L., 1964, Phys. Rev. B **135**, 715.
Ross, D., 1968, Phys. Rev. **173**, 1695.
Rubin, M. R., R. L. Sugar, and G. Tiktopoulos, 1966, Phys. Rev. **146**, 1130.
Rubin, M. R., R. L. Sugar, and G. Tiktopoulos, 1967a, Phys. Rev. **159**, 1348.
Rubin, M. H., R. L. Sugar, and G. Tiktopoulos, 1967b, Phys. Rev. **162**, 1555.
Saudinos, J., and C. Wilkin, 1974, Ann. Rev. Nucl. Sci. **24** (to be published).
Saxon, D. S., and L. I. Schiff, 1957, Nuovo Cimento **6**, 614.
Schiff, L. I., 1956, Phys Rev. **103**, 443.
Schiff, L. I., 1968, Phys. Rev. **176**, 1390.
Schrauner, E., L. Benofy, and D. W. Cho, 1969, Phys. Rev. **177**, 2590.
Schwinger, J., 1954, Phys. Rev. **94**, 1382.
Seaton, M. J., 1961, Proc. Phys. Soc. Lond. **77**, 174.
Serber, R., 1947, Phys. Rev. **72**, 1114.
Shakeshaft, R., and L. Spruch, 1973, Phys. Rev. A **8**, 206.
Shastry, C. S., L. Kumar and J. Callaway, 1970, Phys. Rev. A **1**, 1137.
Sheorey, V. B., E. Gerjuoy and B. K. Thomas, 1971, J. Phys. B **4**, 657.
Sidhu, D. P., and C. Quigg, 1973, Phys. Rev. D **7**, 755.
Silbar, R., and M. M. Sternheim, 1974, Ann. Rev. Nucl. Sci. **24** (to be published).
Sinfailam, A. L., and J. C. Y. Chen, 1972, Phys. Rev. A **5**, 1218.
Sloan, I. H., 1967, Phys. Rev. **162**, 855.
Sloan, I. H., 1968, Phys. Rev. **165**, 1587.
Sopkovich, N. J., 1962, Nuovo Cimento **26**, 186.
Stodolsky, L., 1966, Phys. Rev. **144**, 1145.
Sugar, R. L., and R. Blankenbecler, 1969, Phys. Rev. **183**, 1387.
Sullivan, J., J. P. Coleman and B. H. Bransden, 1972, J. Phys. B **5**, 2061.
Swift, A. R., 1974, Phys. Rev. D (to be published).
Tai, H., R. H. Bassel, E. Gerjuoy, and V. Franco, 1970, Phys. Rev. A **1**, 1819.
Tai, H., P. J. Teubner, and R. H. Bassel, 1969, Phys. Rev. Lett. **22**, 1415.
Takeda, G , and K. M. Watson, 1955, Phys. Rev. **97**, 1336.
Tarasov, A. V., and C. Tsérén, 1970, Yad. Fiz. **12**, 978. [Sov. J. Nucl. Phys. **12**, 533 (1971)].
Teubner, P. J. O., C. R. Lloyd, and E. Wiegold, 1973, J. Phys. B (Letters) **6**, L134.
Thomas, B. K., and F. T. Chan, 1973, Phys. Rev. A **8**, 252.
Thomas, B. K., and E. Gerjuoy, 1971, J. Math. Phys. **12**, 1567.
Tobocman, W., 1961, *Theory of Direct Nuclear Reactions* Clarendon Press, Oxford.
Tobocman, W., and M. Pauli, 1972, Phys. Rev. D **5**, 2088.
Trefil, J., 1969, Phys. Rev. **180**, 1366, 1379.
Tsérén, C. 1968, Yad. Fiz. **8**, 1227 [Sov. J. Nucl. Phys. **8**, 710 (1969)].
Udgaonkar, B. M., and M. Gell-Mann, 1962, Phy. Rev. Lett. **8**, 346.
Van Hove, L., 1972, Nucl. Phys. B **46**, 75.
Van Hove, L., 1973, CERN preprint TH-1685.
Vriens, L., C. E. Kuyatt, and S. R. Mielczarek, 1968, Phys. Rev. **170**, 163.
Vriens, L., J. A. Simpson, and S. R. Mielczarek, 1968, Phys. Rev. **165**, 7.
Wallace, S. J., 1971, Phys. Rev. Lett. **27**, 622.
Wallace, S. J., 1973a, Ann. Phys. (N.Y.) **78**, 190.
Wallace, S. J., 1973b, Phys. Rev. D **8**, 1846.
Wallace, S. J., 1973c, Phys. Rev. D **8**, 1934.
Walters, H. R. J., 1973, J. Phys. B **6**, 1003.
Watson, K. M., 1953, Phys. Rev. **89**, 575.
Watson, K. M., 1956, Phys. Rev. **103**, 489.
Watson, K. M., 1957, Phys. Rev. **105**, 1388.
Watson, K. M., and J. Nuttall, 1967, *Topics in Several Particle Dynamics*, (Holden-Day. San Francisco), Chap. 4.
Weinberg, S., 1963a, Phys. Rev. **130**, 776.
Weinberg, S., 1963b, Phys. Rev. **131**, 140.
Weinberg, S., 1964a, Phys. Rev. B **133**, 232.
Weinberg, S., 1964b, Phys. Rev. B **133**, 1589.
Weinberg, S., 1964c, Phys. Rev. B **135**, 202.
West, G. B., 1972, Ann. Phys. (N.Y.) **74**, 464.
Wilets, L., and S. J. Wallace, 1968, Phys. Rev. **169**, 84.
Wilkin, C., 1964, Nuovo Cimento **31**, 377.
Wilkin, C., 1966, Phys. Rev. Lett **17**, 561.
Wilkin, C., 1968, in *Nuclear and Particle Physics*, edited by B. Margolis and C. Lam (Benjamin, New York), p. 439.

Wilkin, C., 1969, "Graphs and Glauber," Lecture given at the Summer School on Diffractive Processes, McGill University, Montreal.
Woollings, M. J., and M. R. C. McDowell, 1972, J. Phys. B **5**, 1320.
Wu, T. T., 1957, Phys. Rev. **108**, 466.
Yates, A. C., and A. Tenney, 1972, Phys. Rev. A **6**, 1451.
Zemach, C., and A. Klein, 1958, Nuovo Cimento **10**, 1018.
Zolin, L. S., *et al.*, 1966, Zh Eksp. Teor. Fiz. Pis'ma Red. **3**, 15 [JETP Lett. **3**, 8].