Two-Photon Processes for Particle Production at High Energies*

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The literature of the past three years on the two-photon process for particle production $e^{\pm} + e^{-} \rightarrow e^{\pm} + \gamma^{*} + e^{-} + \gamma^{*} \rightarrow e^{\pm} + e^{-} + X$ (where X is any $C = +$ state) or $p + p \rightarrow X_1 + \gamma^{*} + X_2 + \gamma^{*} \rightarrow X_1 + X_2 + \mu^{+} + \mu^{-}$ (where X_1, X_2 are hadron states) is reviewed in some detail. Both the theoretical aspects and the experimental feasibility of various processes are discussed especially for experimentalists' convenience.

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I. INTRODUCTION

This review article is addressed mainly to high-energy experimental physicists who are concerned with either particle production by electron–positron (or electron–electron[1]) colliding beams or massive lepton-pair creation in hadron–hadron collisions at high energies. We review more than one hundred papers which have appeared in the literature during the last three years on the two-photon process for particle production.[2] By the two-photon process we mean processes of the type

$$e^{\pm}+e^{-}\rightarrow e^{\pm}+\gamma^{*}+e^{-}+\gamma^{*}\rightarrow e^{\pm}+e^{-}+X \quad (1.1)$$

in which electrons of both incident beams emit virtual (spacelike) photons (γ^{*}'s), which in turn annihilate, producing a final state X, where X may be a lepton state such as $e^{+}e^{-}$, $\mu^{+}\mu^{-}$, or any possible neutral $C=+$ hadron state such as $\pi^{+}\pi^{-}$, $\pi^{+}\pi^{-}\pi^{0}$, π^{0}, η, $K^{+}K^{-}$, etc. [See Fig. 1(a).] Another process of the type

$$p+p\rightarrow X_1+\gamma^{*}+X_2+\gamma^{*}\rightarrow X_1+X_2+\mu^{+}+\mu^{-}, \quad (1.2)$$

where X_1 and X_2 are arbitrary hadron states is also called the (generalized) two-photon process. We shall discuss the former process in the following thirteen sections, leaving the latter for Sec. XV.

A few years ago, the importance of the two-photon process for lepton and hadron production in electron–electron colliding-beam experiments was emphasized by three independent groups (Arteaga-Romero, Jaccarini, and Kessler, 1969; Balakin, Budnev, and Ginzburg, 1970; Brodsky, Kinoshita, and Terazawa, 1970). The cross section for the process (1.1) is obviously of order α^4 and is completely negligible at low beam energies (up to several hundred MeV) compared to the cross section for the one-photon annihilation process[3]

$$e^{+}+e^{-}\rightarrow\gamma^{*}\rightarrow\pi^{+}+\pi^{-}, \text{ etc.}, \quad (1.3)$$

which is of order α^2.

[1] In the two-photon processes, the incident particles participate only as suppliers of virtual photons. Thus, it is irrelevant whether they are electrons or positrons. For simplicity we shall call them both electrons throughout this paper unless specified otherwise.

[2] A brief review talk on the two-photon process was given by Brodsky (1972) at the 1971 International Symposium on Electron and Photon Interactions at High Energies, Cornell, 1971. A brief review of previous work may also be found in the paper by Brodsky, Kinoshita, and Terazawa (1971b). One of the main purposes of the present paper is to bring up to date the review of the work on the two-photon process. This includes many papers which have been published since the Symposium. We basically follow and even duplicate many parts of the above reference (Brodsky, Kinoshita, and Terazawa 1971b) in writing Secs. II–VI, XVI, and XVII of this review in order to make it self-contained. Therefore, readers familiar with the contents of the reference may skip these sections and start at Sec. VII. For a comparison between the one- and two-photon processes, see also the article by Pais (1972).

[3] As for theoretical studies of the one-photon annihilation process, see especially Cabibbo and Gatto (1961) and Gatto (1965). References to recent works on this process may be found in Pais (1971) and Bjorken (1972).

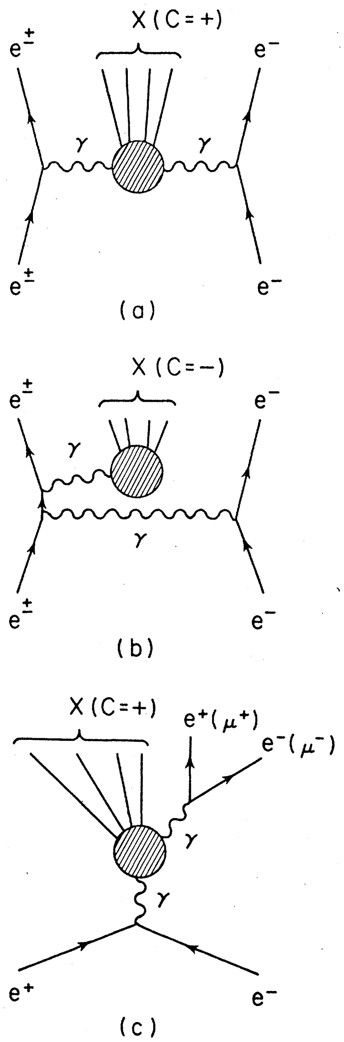

Fig. 1. The three types of diagrams which contribute to the production process $e^{\pm}+e^{-}\rightarrow e^{\pm}+e^{-}+X$ in $e^{\pm}e^{-}$ colliding-beam experiments. The product states have positive charge conjugation ($C=+$) in processes (a) and (c) and negative charge conjugation in process (b). In the case of $e^{-}e^{-}$ collisions, there is another diagram in which the two final electron lines in (a) are exchanged.

However, as the beam energy increases, two factors operate to reverse the relative importance of these two processes: (1) Whereas the cross section for the one-photon process (1.3) will eventually decrease with the beam energy E as $\sim E^{-2}$ (Bjorken, 1966; Gribov, Ioffe, and Pomeranchuk, 1967), this energy factor is replaced in the cross section for (1.1) by a constant m^{-2}, where m is usually the threshold mass of the state X. This exemplifies the observation by Cheng and Wu (1969, 1970a) that the asymptotic behavior of higher-order terms may be completely different from that of lower-order terms. (2) In the process (1.1) both incident particles are electrons which radiate photons so easily that the corresponding cross section is enhanced by two factors of ln (E/m_e) ($\simeq 7.6$ for $E=1$ GeV), in addition to other possibly logarithmic terms which can be inferred from the Cheng–Wu analysis of massive-photon quantum electrodynamics. For these reasons, the two-photon cross section for (1.1) behaves asymptotically[4] as

$$\sigma(E) \propto (\alpha^4/m^2)[\ln (E/m_e)]^2[\ln (E/m)]^n, \quad (1.4)$$

where n is a number ≥ 1 which depends on the high-energy behavior of the cross section for $\gamma+\gamma\rightarrow X$. In the case $X=\pi^+\pi^-$, this cross section becomes comparable to that of the one-photon cross section ($\sigma \propto \alpha^2 E^{-2}$) at an energy per beam $E\simeq 1.5$ GeV, even if we treat pions as pointlike (Brodsky, Kinoshita, and Terazawa, 1971b). For higher energies, the two-photon process becomes clearly the dominant one. This means, on the one hand, that the magnitude of the two-photon cross sections at high energy is large enough to open the way to a complete exploration of the photon–photon annihilation process including the production of $C=+$ hadron resonances. On the other hand, these produced hadrons would also form a serious background to other processes of interest, especially the one-photon annihilation process (1.3), and make these experiments more difficult at high energies unless experimentalists set up appropriate devices to detect at least one of the electrons scattered predominantly forward and to discriminate the two-photon process (1.1) from the one-photon annihilation process (1.3). In any case, in order to be able to extract interesting physics from the high-energy colliding-beam experiments, it is imperative to understand not only the qualitative but also the quantitative characteristics of the two-photon cross sections. If they are understood correctly, the value of the colliding-beam facilities will be much enhanced, since they may serve to investigate the two-photon process as well as the one-photon annihilation process.

The history of the two-photon process can be traced back to 1934 when Landau and Lifshitz (1934) and, independently, Williams (1934) studied the production of electron–positron pairs by two fast moving charged particles. Around 1960 when the colliding-beam facilities were about to become available (cf. Barber, Gittelman, O'Neilt, and Richter, 1966), the two-photon process caught some attention. Calogero and Zemach (1960) studied pion pair production in electron–electron collisions. They calculated the differential cross section for $e+e\rightarrow e+e+\pi^++\pi^-$ by assuming pointlike pions in a special kinematical region where the produced pair of pions have identical energies and exactly opposite directions in the center-of-mass system of the incident beams. This kinematical situation turns out to be useful, as it is related to the question of pion electromagnetic self-mass (see Sec. XI). Low (1960) cal-

[4] The term "asymptotically" means that the asymptotic condition E^2, $m^2 \gg m_e^2$ is satisfied. We shall discuss in more detail this asymptotic behavior of the cross section in connection with the validity of equivalent-photon approximation in Sec. IV.

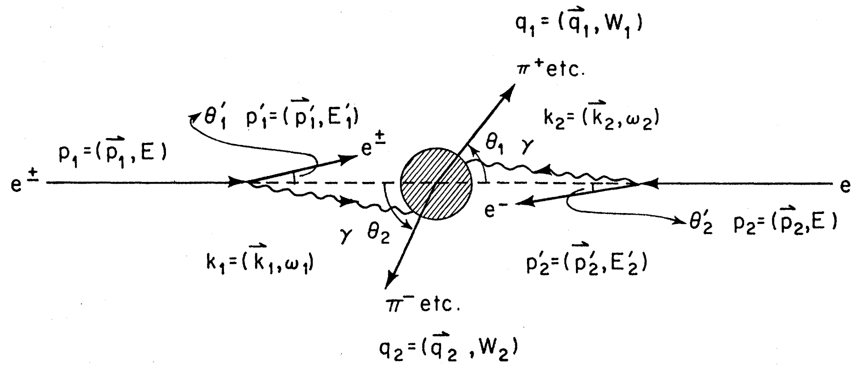

FIG. 2. Kinematics and notation for the pair production of various particles by the two-photon mechanism.

culated the cross section for π^0 production by colliding electrons, which he proposed as a means of measuring the π^0 lifetime. In his paper the $[\ln(E/m_e)]^2\ln(E/m_\pi)$ energy dependence of the two-photon cross section is clearly evident (see Sec. IV). For nearly ten subsequent years, however, the two-photon process attracted little attention. This seems to be simply because the energies available at the existing colliding-beam facilities were too low for this process to be important and attention was focused solely on the production of $C=-$ resonances by the e^+e^--annihilation process (1.3). In the late sixties, we can only find the paper by DeCelles and Goehl (1969) who studied some aspects of the process (1.1) for σ production in order to explore the feasibility of experimental determination of S-wave $\pi\pi$ phase shifts (see Sec. VIII for details).

As beam energies have steadily increased and new data have begun to be reported (Alles-Borelli *et al.*, 1970; Baldini-Celio *et al.*, 1970; Barbiellini *et al.*, 1970; Bartoli *et al.*, 1970) the two-photon process has again become a subject of intense investigation. Arteaga-Romero, Jaccarini, and Kessler (1969), and Arteaga-Romero *et al.* (1970, 1971a, b) have calculated the two-photon cross sections for $\mu^+\mu^-$, $\pi^+\pi^-$, and K^+K^- production at $E=2$ and 3 GeV and pointed out that these cross sections are rather large and increase with beam energy. Independently, two other theoretical groups, one consisting of Balakin, Budnev, and Ginzburg (1970; Budnev and Ginzburg, 1971a, b), and the other of Brodsky, Kinoshita, and Terazawa (1970, 1971b) have investigated the two-photon process. The latter group has paid particular attention to detailed features such as angular distributions, angular correlations, and mass distributions of produced hadrons which would help to distinguish the two-photon process from the one-photon process (see details in Sec. V). In all these previous works, reported at the Kiev Conference in 1970, calculations have been carried out in the equivalent-photon approximation (Fermi, 1924; Weizsäcker and Williams, 1934; Landau and Lifshitz, 1934; Curtis, 1956; Dalitz and Yennie, 1957), which is useful for an understanding of the main qualitative features. In this approximation (see Sec. III for details), the leading term for $E/m_e\to\infty$ of the cross section for the process (1.1) when the scattered electrons are not detected is given by (see Fig. 2 for notation and Secs. III and IV for the proof)

$$d\sigma_{ee\to eeX}(E)\cong\left(\frac{2\alpha}{\pi}\right)^2\left(\ln\frac{E}{m_e}\right)^2 \times\int\frac{d\omega_1\,d\omega_2}{\omega_1\omega_2}\frac{(E^2+E_1'^2)(E^2+E_2'^2)}{4E^4}d\sigma_{\gamma\gamma\to X}(\omega_1,\omega_2), \tag{1.5}$$

where $d\sigma_{\gamma\gamma\to X}$ is the differential cross section for the annihilation of two oppositely directed real unpolarized photons of energy ω_1 and ω_2 into a state X. For the total cross section, we obtain [noting that $\sigma_{\gamma\gamma\to X}$ is a function of $s=(k_1+k_2)^2\simeq4\omega_1\omega_2$ only]

$$\sigma_{ee\to eeX}(E)\cong2\left(\frac{\alpha}{\pi}\right)^2\left(\ln\frac{E}{m_e}\right)^2\int_0^{4E^2}\frac{ds}{s} \times f\left(\frac{s^{1/2}}{2E}\right)\sigma_{\gamma\gamma\to X}(s), \tag{1.6}$$

as in Low's work (1960), where $f(x)$ is given by

$$f(x)=(2+x^2)^2\ln(1/x)-(1-x^2)(3+x^2). \tag{1.7}$$

Substitution of the explicit form of the cross section, for example,

$$\sigma_{\gamma\gamma\to X}\sim(\ln s)^{n'}, \tag{1.8}$$

where n' depends on the state X, leads to a result of the form (1.4) with $n=n'+2$ if $n'\geq-1$ and $n=1$ if $n'<-1$.

Thus, during 1970 it was qualitatively established that the two-photon process becomes dominant in colliding-beam experiments as the beam energy increases above $\sim$1 GeV. Much of the work done on the two-photon process during the following year was devoted to two different aspects. Firstly, more quantitative knowledge of this process is required. From this point of view previous results are not completely satisfactory because they have all been obtained by means of the equivalent-photon approximation or variants thereof. Although this approximation will give the leading logarithmic E/m_e dependence of the cross section, the reliability of the method at laboratory energies of approximately a few GeV has not been established. Brodsky, Kinoshita, and Terazawa (1971b) have investigated how good this approximation actually is (see Secs. IV and VI). Cheng and Wu (1971a) also have looked into the same question and found that the approximation can never be good in the differential cross section for $\pi^+\pi^-$ production when the pair of pions are detected in certain kinematical regions. However, there is no disagreement between these two exact calculations (see Sec. VI).

The second question which has caught the attention of many theorists during last two years is the following: "What kinds of physics can be done by means of the two-photon process?" or, more bluntly, "Is it useful at all?" Over a hundred physicists have given various answers to this question in their theoretical papers, which we shall review in Sec. VII and the sequel.

At the International Symposium on Electron and Photon Interactions at High Energies held in 1971, exciting data on the first evidence of the two-photon process were reported by both the Novosibirsk (Balakin *et al.*, 1971) and the Frascati (Bacci *et al.*, 1971, 1972) experimental groups. The angular distribution of the produced pair of electrons observed in the Novosibirsk colliding-beam machine fits remarkably well the theoretical curve calculated by Baier and Fadin (1971a, b, c, 1972).

II. GENERAL KINEMATICS AND FORMULAS

Diagrams of the type shown in Figs. 1(a) and (b) both contribute to the process $e^\pm+e^-\rightarrow e^\pm+e^-+X$. In this paper, however, we shall concentrate on diagrams of the first kind (two-photon diagrams) because (1) the diagrams of the second kind will have fewer factors of $\ln(E/m_e)$, (2) the contributions of the first $(C=+)$ and second $(C=-)$ kinds will not interfere (assuming the C invariance of strong and electromagnetic interactions) unless the charges of the produced particles are distinguished, and (3) the contribution of the second diagrams is found (Arteaga-Romero, Jaccarini, Kessler, and Parisi, 1970, 1971a) to be negligible if electrons are detected in the forward direction. In Sec. XVI we shall give a brief discussion of diagrams of the type of Fig. 1(b) for $C=-$ states; an estimate of their contribution will be given in Sec. XIII for the case of electrons scattered at fairly large angles.

To facilitate the calculation further we shall omit the Møller interference term e^-e^- collisions. The effect of interference of electrons in the final states in e^-e^- collisions is clearly negligible since the amplitude for both incident electrons to scatter backwards is very small. In the case of e^+e^- collisions we omit the contribution of the Bhabha type diagram [Fig. 1(c)] because it is undoubtedly small compared with the others [(a) and (b)] (see XIII for the reasons).

Ideally speaking, it is desirable to detect scattered electrons in addition to produced particles. In the processes of interest most electrons are scattered into very small forward angles $[\sim(m_e/E)^{1/2}\simeq 1.3°$ for $E=1$ GeV$]$ and therefore it would seem to be hard to separate them from the unscattered beams. However, since the electrons which have lost their energies substantially (say by more than 5%) after emitting virtual photons are bent by the existing magnetic field in the storage rings, they will eventually come out of the rings and be detected by appropriate counters set up along the storage rings. This method of detecting scattered electrons has already been adopted successfully at Frascati (Bacci *et al.*, 1971, 1972). Of course, the counting rate becomes smaller as one specifies the final state more closely. We shall discuss in Secs. III–X primarily those experiments in which scattered electrons are not detected or in which scattering angles of the electrons are limited to a certain region (e.g., $0<\theta'<5°$ or $5°<\theta'<60°$). In the case of e^+e^- colliding beams, however, this arrangement would make it difficult to distinguish between the e^+e^- annihilation process and the two-photon process. To avoid this problem it would be necessary to either measure the moments of all produced particles accurately or detect at least one of the scattered electrons in coincidence with the produced particles. The latter approach may not be too unreasonable since, in order to distinguish these two processes, it suffices to detect the mere presence of the scattered electrons. Of course more quantitative information can be obtained (at the expense of diminishing cross sections) if the energy and/or the angle of the scattered electron is measured. For the case where the energy (but not the angle) of the scattered electron is measured, the cross sections can easily be derived from the cross sections in this and following sections by undoing the integration with respect to the energy of the scattered electron. Necessary modifications of the formulas for the case in which electrons are detected scattering either into small forward angles or between two small angles are discussed at the end of Sec. III. The case in which at least one of the electrons is scattered into large angles is treated in Secs. XI–XIV.

We are now ready to write down the two-photon cross section for the process (1.1) for the production of a $C=+$ state X integrated over the scattered-electron phase space (see Fig. 2 for the kinematics and notation):

$$\begin{aligned} d\sigma = {} & \left(\frac{\alpha}{2\pi^2}\right)^2 \frac{1}{E^2}\int \frac{d^3p_1' d^3p_2'}{E_1'E_2'}\left(\frac{1}{k_1^2k_2^2}\right)^2 \\ & \times(p_1^\mu p_1'^\nu + p_1'^\mu p_1^\nu + \tfrac{1}{2}k_1^2 g^{\mu\nu}) \\ & \times(p_2^\alpha p_2'^\beta + p_2'^\alpha p_2^\beta + \tfrac{1}{2}k_2^2 g^{\alpha\beta}) \\ & \times \tfrac{1}{8} M_{\mu\alpha}{}^\dagger M_{\nu\beta}\, d\tilde{\Gamma}, \\ M_{\mu\nu} = {} & i\int d^4x \exp(-ik_1x)\langle X \mid T^*(J_\mu(x), J_\nu(0)) \mid 0\rangle, \\ d\tilde{\Gamma} = {} & (2\pi)^4\delta^4(k_1+k_2-P)\,d\Gamma, \end{aligned} \tag{2.1}$$

where J_μ is the electromagnetic current,

$$d\Gamma=\Pi_i[\rho_i d^3q_i(2\pi)^{-3}]$$

is the invariant phase space of the state X [$\rho_i=1/(2W_i)$ or m_i/W_i according to whether the particle i is a boson or a fermion], and P is the energy–momentum four vector of the state X. The symbol T^* means that all Schwinger terms are subtracted in $M_{\mu\nu}$ so that both Lorentz covariance and gauge invariance are guaranteed. Throughout this paper we ignore, whenever this is safe, the electron mass m_e in comparison with the electron-beam energy E. In the following, we shall refer to (2.1) as an exact formula (in contrast to the equivalent-photon-approximation formulas) even though some approximations have been made.

In the limit in which the photons of momenta k_1 and k_2 become real, the integrand of (2.1) satisfies the relation

$$\lim_{k_1^2\to 0,k_2^2\to 0} \tfrac{1}{8}M_{\mu\alpha}{}^\dagger M^{\mu\alpha}d\bar{\Gamma}=(2\omega_1)(2\omega_2)d\sigma_{\gamma\gamma\to X}, \tag{2.2}$$

where $d\sigma_{\gamma\gamma\to X}$ is the corresponding cross section for the production of the state by two (oppositely directed) unpolarized photons of energy ω_1 and ω_2, respectively. In the applications, we shall be interested in $d\sigma/ds$ where s is the invariant mass squared of the produced system:

$$s=P^2=(k_1+k_2)^2=m_X^2. \tag{2.3}$$

To exhibit the s dependence of the cross section, it is convenient to rewrite (2.1) by introducing the factor $\delta[(k_1+k_2)^2-s]$ in its integrand and integrating the result over the variable s.

We shall introduce the variables ω, q, and $\bar{s}$ by

$$\begin{aligned}\omega_1&=E-E_1'=\tfrac{1}{2}(\omega+q)\\ \omega_2&=E-E_2'=\tfrac{1}{2}(\omega-q)\\ \bar{s}&=\omega^2-q^2=4\omega_1\omega_2.\end{aligned} \tag{2.4}$$

Then we can write (for $m_e^2\ll E^2$, E'^2)

$$\delta[(p_1+p_2-p_1'-p_2')^2-s] =\delta[\bar{s}-2(1+\cos\theta')(E^2+\tfrac{1}{4}\bar{s}-E\omega)-s], \tag{2.5}$$

where

$$\cos\theta'=\hat{p}_1'\cdot\hat{p}_2'=-\cos\theta_1'\cos\theta_2'+\sin\theta_1'\sin\theta_2'\cos\varphi'. \tag{2.6}$$

In the region where $\cos\theta_i'\sim 1$ ($i=1,2$), which gives the dominant contribution to the two-photon process, we have $\bar{s}\sim s$ and ω and q can be identified as the energy and momentum of the produced system X.

In terms of the new variables ω and $\bar{s}$, the $\bar{s}$ integration can be carried out immediately and the required invariant lepton phase space becomes

$$\int\frac{d^3p_1'\,d^3p_2'}{E_1'E_2'}\delta[(p_1+p_2-p_1'-p_2')^2-s]=\tfrac{1}{2}\pi\int_{-1}^{1}dx_1\,dx_2 \times\int_0^{2\pi}d\varphi'\int_{s^{1/2}}^{E+s/4E}\frac{d\omega}{|q|}\sum_{q=\pm|q|}\frac{E_1'E_2'\theta(\omega^2-\bar{s}_0)}{1-\frac{1}{2}(1+\cos\theta')}, \tag{2.7}$$

with

$$x_i=\cos\theta_i', \qquad i=1,2$$
$$|q|=(\omega^2-\bar{s}_0)^{1/2}, \tag{2.8}$$

and

$$\bar{s}_0=[s+2(1+\cos\theta')(E^2-E\omega)]/[1-\tfrac{1}{2}(1+\cos\theta')]. \tag{2.9}$$

The upper limit on ω is determined by the condition that E_1' and E_2' are positive ($\geq m_e$). Notice also that $\bar{s}_0$ attains its minimum value $\bar{s}_{\min}=s$ for $\cos\theta'=-1$ or $\omega=E+s/4E$.

Apart from the omission of the Møller interference term (or the Bhabha term) and the approximation $m_e^2\ll E^2$, E'^2, the cross section for the production of $C=+$ states by electron–electron collision is therefore given by[5]

$$\begin{aligned}\frac{d\sigma}{dsd\Gamma}=\frac{\alpha^2}{8\pi^3}\int_{-1}^{1}dx_1\,dx_2\int_0^{2\pi}d\varphi'\int_{s^{1/2}}^{E+s/4E}\frac{d\omega}{|q|}\left(\frac{1}{k_1^2k_2^2}\right)^2\\ \times(2\pi)^4\delta^4(k_1+k_2-P)\frac{E_1'E_2'\theta(\omega^2-\bar{s}_0)}{E^2[1-\frac{1}{2}(1+\cos\theta')]}\\ \times(p_1^\mu p_1'^\nu+p_1'^\mu p_1^\nu+\tfrac{1}{2}k_1^2g^{\mu\nu})\\ \times(p_2^\alpha p_2'^\beta+p_2'^\alpha p_2^\beta+\tfrac{1}{2}k_2^2g^{\alpha\beta})\tfrac{1}{8}M_{\mu\alpha}{}^\dagger M_{\nu\beta}.\end{aligned} \tag{2.10}$$

In practice, the four-dimensional integration can be handled in a straightforward manner by numerical integration.

So far, we have obtained the exact formula (2.10) for the differential cross section $d\sigma/dsd\Gamma$ expressed by the integration with the directly measurable variables x_1, x_2, φ' and Low's variable ω, starting with the manifestly covariant expression (2.1). There is another way to derive many useful formulas for the two-photon cross section. Many authors [Brown and Muzinich (1971), Carlson and Tung (1971), Budnev, Chernyak, and Ginzburg (1971), Terent'ev (1971c), and Starke (1972)] have expressed the same cross section in terms of helicity amplitudes. Their way of doing this is the following: (1) Choose the "brick wall" frame where $k_1=[0,0,0,(-k_1^2)^{1/2}]$ and express the first leptonic tensor $(p_1^\mu p_1'^\nu+p_1'^\mu p_1^\nu+\frac{1}{2}k_1^2g^{\mu\nu})$ in terms of the $O(2,1)$ variables. (2) Proceed likewise for the second leptonic tensor. (3) Decompose the central part $M_{\mu\alpha}{}^+M_{\nu\beta}$ into independent helicity amplitudes defined in the center-of-mass system of the two virtual photons. (4) Finally, combine these three pieces into one by

[5] $q=|q|$ or $-|q|$ depending on $d\Gamma$.

making use of the $O(2,1)$ boosts. Their expressions for the two-photon cross sections have two advantages. The first is that it is easier to derive the equivalent-photon approximation formula from them. The second is that, in some cases, one can more easily perform additional integrations over the $O(2,1)$ variables analytically. In fact, this turns out to be the case in calculating the differential cross section for massive muon-pair production in hadron–hadron collisions, which we shall discuss in Sec. XV. Until then, we will continue to use the more straightforward formulas given in this section.

III. EQUIVALENT-PHOTON APPROXIMATION

The equivalent-photon method is a useful technique for obtaining the leading high-energy behavior of electroproduction cross sections in which the scattered electron is either undetected or detected only if it is scattered into small forward angles. This technique, which can be traced to early work by Fermi (1924), Weizsäcker and Williams (1934), and Landau and Lifshitz (1934), gives the general connection between electroproduction and photoproduction cross sections. A corresponding treatment in terms of Feynman diagrams has been given by Curtis (1956) and by Dalitz and Yennie (1957). We shall briefly review the formulas required for our present application.

Let us take, as an example, the production of the state X in e–p collisions. Then the electroproduction cross section integrated over the final-electron phase space can be written as

$$d\sigma_{ep\to eX}=\frac{\alpha}{2\pi^2}\int\frac{d^3p'}{EE'}\frac{g_{\mu\nu}}{k^2}\frac{g_{\alpha\beta}}{k^2}$$

$$\times(p^\mu p'^\alpha+p'^\mu p^\alpha+\tfrac{1}{2}k^2g^{\mu\alpha})\tfrac{1}{4}M^{\nu\dagger}M^\beta\,d\tilde{\Gamma},$$

$$d\tilde{\Gamma}=(2\pi)^4\delta^4(k+p-P)\,d\Gamma, \qquad (3.1)$$

where p, p' are the initial and final electron momenta, k is the photon momentum, and P the total momentum of the state X. In this and the following formulas we ignore the electron mass m_e whenever it is safe to do so. For $k^2=(p-p')^2\to0$ we can identify

$$\lim_{k^2\to0}(-\tfrac{1}{4}M_\mu{}^\dagger M^\mu d\tilde{\Gamma})=2\omega d\sigma_{\gamma p\to X}, \qquad (3.2)$$

where $d\sigma_{\gamma p\to X}$ is the corresponding photoproduction cross section for real unpolarized photons of energy ω directed along the electron beam direction.

It is convenient to perform the photon polarization sums in the radiation (Coulomb) gauge. Thus we shall make the following substitution in (3.1):

$$g_{\mu\nu}/k^2\to(-g_{0\mu}g_{0\nu}/\mathbf{k}^2)-\sum_{i=1,2}(g_{i\mu}g_{i\nu}/k^2). \qquad (3.3)$$

The polarization directions i are orthogonal to $\hat{k}$. The contribution of the transverse-current terms to (3.1) is

$$(\alpha/2\pi^2)\int(d^3p'/EE')(1/k^2)^2\tfrac{1}{4}\sum_{i,j=1,2}M_i{}^\dagger M_j\,d\tilde{\Gamma}$$

$$\times(-\tfrac{1}{2}k^2\delta_{ij}+2p_ip_j), \qquad (3.4)$$

which becomes

$$\frac{\alpha}{2\pi^2}\int\frac{d^3p'}{EE'}\left(\frac{1}{k^2}\right)^2\tfrac{1}{4}\sum_{j=1,2}|M_j|^2\,d\tilde{\Gamma}$$

$$\times[-\tfrac{1}{2}k^2+(E^2E'^2/\mathbf{k}^2)\sin^2\theta'], \qquad (3.5)$$

when average over the azimuthal angle φ' of p'; $\cos\theta'=\hat{p}\cdot\hat{p}'$. If we approximate $\frac{1}{4}\sum_j|M_j|^2\,d\tilde{\Gamma}$ by its value (3.2) on the photon mass shell ($k^2=0$) and ignore the longitudinal contribution, then we obtain the equivalent-photon-approximation result (Curtis, 1956; Dalitz and Yennie, 1957)

$$d\sigma_{ep\to eX}{}^{(0)}=\int_0^E(d\omega/\omega)N(\omega)\,d\sigma_{\gamma p\to X}(\omega), \qquad (3.6)$$

where $\omega=E-E'$ and

$$N(\omega)=\frac{\alpha}{2\pi^2}(2\pi)\int_{-1}^{1}d\cos\theta'\,\frac{2\omega^2E'}{E}\left(\frac{1}{k^2}\right)^2$$

$$\times\left(-\tfrac{1}{2}k^2+\frac{E^2E'^2}{\mathbf{k}^2}\sin^2\theta'\right)$$

$$=\frac{\alpha}{\pi}\left[\frac{E^2+E'^2}{E^2}\left(\ln\frac{E}{m_e}-\tfrac{1}{2}\right)+\frac{(E-E')^2}{2E^2}\right.$$

$$\left.\times\left(\ln\frac{2E'}{E-E'}+1\right)+\frac{(E+E')^2}{2E^2}\ln\frac{2E'}{E+E'}\right] \qquad (3.7)$$

in the limit $m_e\ll E$. The origin of the leading $\ln(E/m_e)$ contribution is the logarithmic dependence of the θ' integration near $\theta'\sim0$, where

$$-k^2\simeq2EE'(1-\cos\theta')+[m_e^2(E-E')^2/EE']. \qquad (3.8)$$

Note that the leading $\ln(E/m_e)$ contribution arises only from (3.6) and not from the Coulomb excitation (scalar photon) term or the remainder terms of the transverse cross section which are not singular at $k^2=0$. Therefore, unless the Coulomb excitation current is anomalously large or the transverse cross section has an anomalous dependence on k^2, the equivalent-photon contribution dominates for $\ln(E/m_e)\gg1$.

Although we have found the Coulomb gauge the most convenient for our purpose, the same result can of course be obtained in any gauge.

We shall now apply this equivalent-photon method to the two-photon cross section $d\sigma$ [see (2.1)]. According to the method reviewed above, the leading contribution to $d\sigma$ can be obtained by (1) performing the photon polarization sums in the radiation (Coulomb) gauge, (2) retaining only the transverse-current contribution, and (3) approximating the transverse current and the phase space $d\tilde{\Gamma}$ by their values at $k_1{}^2=k_2{}^2=0$,

$\theta_1'=\theta_2'=0$:

$$\sum_{i,j,l,m=1,2}(2p_1^ip_1^j-\tfrac{1}{2}k_1^2g^{ij})(2p_2^lp_2^m-\tfrac{1}{2}k_2^2g^{lm})$$
$$\times\tfrac{1}{8}M_{il}{}^{\dagger}M_{jm}d\bar{\Gamma}\Rightarrow\left(\frac{\mathbf{p}^2\mathbf{p}_1'^2\sin^2\theta_1'}{\mathbf{k}_1^2}-\tfrac{1}{2}k_1^2\right)$$
$$\times\left(\frac{\mathbf{p}^2\mathbf{p}_2'^2\sin^2\theta_2'}{\mathbf{k}_2^2}-\tfrac{1}{2}k_2^2\right)(2\omega_1)(2\omega_2)d\sigma_{\gamma\gamma\to X},\quad(3.9)$$

where we have averaged over the azimuthal angles φ_1', φ_2', and $d\sigma_{\gamma\gamma\to X}$ is defined by (2.2). Under these approximations the phase space $d\bar{\Gamma}$ loses all terms correlating the variables θ_1', φ_1' and θ_2', φ_2'. [The suppression of this correlation is an unavoidable feature in the simultaneous application of the equivalent-photon method to both electrons. It introduces an additional complication in this method whose effect is hard to evaluate. (See Sec. VI for a more detailed discussion on this possible defect of the method.)] The integrand of (2.1) also reduces to a product of two factors, one a function of θ_1' and the other a function of θ_2'. The integrations over the angles (θ_1', φ_1') and (θ_2', φ_2') of the scattered electrons can thus be carried out independently, giving

$$d\sigma^{(0)}=\int_0^E\frac{d\omega_1\,d\omega_2}{\omega_1\omega_2}N(\omega_1)N(\omega_2)\,d\sigma_{\gamma\gamma\to X}(s)\quad(3.10)$$

as the leading approximation to the $C=+$ cross section, where $s=4\omega_1\omega_2$ and $N(\omega)$ is given by (3.7).

If we introduce the variables ω, q [see (2.4)], we obtain from (3.10) (putting $\bar{s}=s$)[5]

$$\frac{d\sigma^{(0)}}{ds\,d\Gamma}=s^{-1}\int_{s^{1/2}}^{E+s/4E}\frac{d\omega}{|q|}N(\omega_1)N(\omega_2)\frac{d\sigma_{\gamma\gamma\to X}(s)}{d\Gamma}$$
$$=2\left(\frac{\alpha}{\pi}\right)^2\left[\left(\ln\frac{E}{m_e}-\tfrac{1}{2}\right)^2f(\gamma)+\left(\ln\frac{E}{m_e}-\tfrac{1}{2}\right)g(\gamma)+h(\gamma)\right]s^{-1}\frac{d\sigma_{\gamma\gamma\to X}(s)}{d\Gamma},\quad(3.11)$$

where

$$f(\gamma)=\int_{s^{1/2}}^{E+s/4E}\frac{d\omega}{|q|}\left(\frac{E^2+E_1'^2}{E^2}\right)\left(\frac{E^2+E_2'^2}{E^2}\right)$$
$$=(2+\gamma^2)^2\ln(1/\gamma)-(1-\gamma^2)(3+\gamma^2),$$
$$g(\gamma)=\int_{s^{1/2}}^{E+s/4E}\frac{d\omega}{|q|}\frac{E^2+E_1'^2}{E^2}n(E_2')+\text{term with }1\leftrightarrow2,$$
$$h(\gamma)=\int_{s^{1/2}}^{E+s/4E}\frac{d\omega}{|q|}n(E_1')n(E_2'),\quad(3.12)$$

and $\gamma=s^{1/2}/2E$. Here $n(E')$ denotes the last two terms in the square brackets of (3.7) and vanishes for $E'\to E$.

The result (3.10) gives the relationship between the electron–electron collision and two-photon collision cross sections for the production of $C=+$ states in the equivalent-photon approximation. The remainder of the cross section [from the difference of (2.1) and (3.10)],[5]

$$d\sigma^{(1)}=\frac{\alpha^2}{8\pi^3}\int_{-1}^1dx_1\,dx_2\int_0^{2\pi}d\varphi'\int_{s^{1/2}}^{E+s/4E}\frac{d\omega}{|q|}\frac{E_1'E_2'}{E^2}\left(\frac{1}{k_1^2k_2^2}\right)^2[(p_1^{\mu}p_1'^{\nu}+p_1'^{\mu}p_1^{\nu}+\tfrac{1}{2}k_1^2g^{\mu\nu})(p_2^{\alpha}p_2'^{\beta}+p_2'^{\alpha}p_2^{\beta}+\tfrac{1}{2}k_2^2g^{\alpha\beta})$$
$$\times\tfrac{1}{8}M_{\mu\alpha}{}^{\dagger}M_{\nu\beta}\,d\bar{\Gamma}\frac{\theta(\omega^2-\bar{s}_0)}{1-\frac{1}{2}(1+\cos\theta')}-\left(\frac{\mathbf{p}^2\mathbf{p}_1'^2\sin^2\theta_1'}{\mathbf{k}_1^2}-\tfrac{1}{2}k_1^2\right)\left(\frac{\mathbf{p}^2\mathbf{p}_2'^2\sin\theta_2'}{\mathbf{k}_2^2}-\tfrac{1}{2}k_2^2\right)(\tfrac{1}{8}M_{\mu\alpha}{}^{\dagger}M^{\mu\alpha}\,d\bar{\Gamma})_{k_1^2=k_2^2=0,\theta'=\pi}],\quad(3.13)$$

can yield only a single power of $\ln(E/m_e)$ since the surviving contribution [from Coulomb excitation (scalar-longitudinal current contribution) and deviation of the transverse cross section from its value at $k_1^2=k_2^2=0$, $\theta_1'=\theta_2'=0$] is not singular when both k_1^2 and $k_2^2\to0$. Thus the equivalent-photon term $d\sigma^{(0)}$ dominates for large E/m_e provided there are no anomalous enhancements from the scalar current or variations of cross section with photon mass. The approximation

$$\sigma_{ee\to eeX}\simeq2\left(\frac{\alpha}{\pi}\right)^2\left(\ln\frac{E}{m_e}\right)^2\int_{s_{\rm th}}^{4E^2}\frac{ds}{s}$$
$$\times\sigma_{\gamma\gamma\to X}(s)f(s^{1/2}/2E)\quad(3.14)$$

is thus justified to order $[\ln(E/m_e)]^{-1}$, where $s_{\rm th}$ is the threshold value of s, and f is defined in (3.12).

We shall now discuss the angular spread of emitted (virtual) photons and scattered electrons. For this purpose we have to go back to the stage prior to (3.10) in which the integration over the angles θ_1' and θ_2' of the scattered electrons has not yet been carried out. It is easy to see that, for the small angular region, the differential cross section depends on the photon emission angle θ_γ or the electron scattering angle θ' as

$$\frac{d\theta_\gamma^2}{\theta_\gamma^2+(m_e/E)^2},\qquad\frac{d\theta'^2}{\theta'^2+[m_e(E-E')/EE']^2},\quad(3.15)$$

respectively. [Although the second formula seems to lead to a $\ln(E-E')$ singularity, it is actually suppressed in the complete expression. See Eq. (3.7).] Thus, roughly one half of the cross section comes from the angles

$$\theta_\gamma,\ \theta' < (m_e/E)^{1/2}, \tag{3.16}$$

and roughly three quarters comes from $\theta_\gamma, \theta' < (m_e/E)^{1/4}$, etc. This spreading of the virtual photon beam is expected to introduce errors of order $(m_e/E)^{1/2}$ in the angular distribution of produced particles. Thus the dominant contribution to the process $e+e \to e+e+X$ can be represented as photon–photon collisions of two oppositely directed bremsstrahlung beams each of virtual radiator strength $N \sim (2\alpha/\pi)\ln(E/m_e)$ and beam angular divergence $\sim (m_e/E)^{1/2}$.

It is easy to extend our considerations to the case in which either or both electrons are detected scattering into the small forward angles

$$\theta_1',\ \theta_2' < \theta_{\max}, \qquad (m_e/E)^2 \ll \theta_{\max}^2 \ll 1. \tag{3.17}$$

In the equivalent-photon approximation, the corresponding cross section is given by the formula (3.10) if we replace $N(\omega)$ by

$$N(\omega, \theta_{\max}) = \frac{\alpha}{\pi}\left[\frac{E^2+E'^2}{E^2}\left(\ln\frac{E\theta_{\max}}{2m_e} - \tfrac{1}{2}\right) + \frac{(E-E')^2}{2E^2}\left(\ln\frac{2E'}{E-E'}+1\right) + \frac{(E+E')^2}{2E^2} \times \ln\frac{2E'}{[(E-E')^2+EE'\theta_{\max}^2]^{1/2}}\right], \tag{3.18}$$

which is obtained by restricting the domain of integration to $0 \le \theta' \le \theta_{\max}$ in (3.7) (Brodsky, Kinoshita, and Terazawa, 1971b). In the case $\theta_{\min} \le \theta' \le \theta_{\max}$ [$(m_e/E)^2 \ll \theta_{\min}^2$], $N(\omega)$ should be replaced by $N(\omega, \theta_{\max}) - N(\omega, \theta_{\min})$. Note that the correction term $d\sigma^{(1)}$ given in (3.13) vanishes as $O(\theta_{\max}^2)$. Thus, *the equivalent-photon result approaches the exact cross section in the case where both scattered electrons are detected within small forward angles* [$\lesssim (m_e/E)^{1/2} \simeq 1.3°$ for $E = 1$ GeV]. Of course we have assumed here that the scalar-longitudinal current contribution is not abnormally large:

$$\theta_{\max}^2\,|M_{\text{long}}|^2 \ll |M_{\text{trans}}|^2.$$

The effects of the kinematical cutoff $\theta_{\max}$ on effective cross sections in experiments have been investigated in detail by Arteaga-Romero, Jaccarini, Kessler, and Parisi (1970, 1971b).

IV. TEST OF VALIDITY OF EQUIVALENT-PHOTON APPROXIMATION IN NARROW-RESONANCE PRODUCTION

The simplest example of hadron production by electron–electron collision is the narrow-resonance meson production of a single π^0, η, η', etc. We shall first consider the case of π^0 production in some detail in order to see how well the equivalent-photon approximations are justified in practice. We take as the effective Lagrangian for the $\pi^0\gamma\gamma$ coupling

$$\mathcal{L}_{\pi^0\gamma\gamma} = -(e^2 g_{\pi^0\gamma\gamma}/2!)\,\phi_{\pi^0}\epsilon^{\mu\nu\kappa\lambda}F_{\mu\nu}F_{\kappa\lambda}, \tag{4.1}$$

where ϕ_{π^0} is the neutral-pion field and $F_{\mu\nu}$ is the electromagnetic field strength. The interesting problem of how to determine the coupling constant g theoretically will be reviewed in Sec. VII. However, g can easily be determined experimentally by

$$(e^2 g_{\pi^0\gamma\gamma})^2 = 4\pi\Gamma_{\pi^0\to\gamma\gamma}/m_\pi^3, \tag{4.2}$$

where m_π is the mass of the π^0, $\Gamma_{\pi^0\to\gamma\gamma} = \tau_{\pi^0}^{-1}$ is the $\pi^0 \to \gamma\gamma$ decay width, and

$$\sigma_{\gamma\gamma\to\pi^0} = 8\pi^2(\Gamma_{\pi^0\to\gamma\gamma}/m_\pi)\,\delta(s - m_\pi^2) \tag{4.3}$$

is the narrow-width production cross section of π^0 production at $k_1^2 = k_2^2 = 0$. [In the case of spin-J production there is an additional factor of $2J+1$ in Eq. (4.3)]. Thus the leading term of the total cross section for $e+e \to e+e+\pi^0$ in the equivalent-photon approximation is (Low, 1960; Parisi, 1970),

$$\sigma_{ee\to ee\pi^0}{}^{(0)} = \frac{16\alpha^2\Gamma_{\pi^0\to\gamma\gamma}}{m_\pi^3}\left[\left(\ln\frac{E}{m_e} - \tfrac{1}{2}\right)^2 f(\gamma) + \left(\ln\frac{E}{m_e} - \tfrac{1}{2}\right) g(\gamma) + h(\gamma)\right] = \frac{64\alpha^2\Gamma_{\pi^0\to\gamma\gamma}}{m_\pi^3}\left(\ln\frac{E}{m_e}\right)^2 \ln\frac{2E}{m_\pi} + \cdots, \tag{4.4}$$

from (3.11) and (4.3) where $\gamma = m_\pi/2E$. The $g(\gamma)$ and $h(\gamma)$ terms contribute less than 0.5% (Grammer, 1972) at $E = 2$ GeV and can be neglected. The result using $\Gamma_{\pi^0\to\gamma\gamma} = 7.8$ eV is plotted in Fig. 3, see also Table I. It should be noted that the present experimental error in $\Gamma_{\pi^0\to\gamma\gamma}$ is ± 0.9 eV (Particle Data Group, 1972).

This equivalent-photon approximation result can be compared to the exact fourth-order calculation (Brodsky, Kinoshita, and Terazawa, 1971b). The quantity

$$(p_1^\mu p_1'^\nu + p_1'^\mu p_1^\nu + \tfrac{1}{2}k_1^2 g^{\mu\nu})(p_2^\alpha p_2'^\beta + p_2'^\alpha p_2^\beta + \tfrac{1}{2}k_2^2 g^{\alpha\beta}) \times M_{\mu\alpha}{}^\dagger M_{\nu\beta}\, d\tilde{\Gamma}$$

in the integrand of (2.1) can be written in this case as

$$2(e^2 g_{\pi^0\gamma\gamma})^2\,|F|^2\,B \times 2\pi\delta[(k_1+k_2)^2 - m_\pi^2], \tag{4.5}$$

where

$$\begin{aligned} B &= \tfrac{1}{4}k_1^2k_2^2B_1 - 4B_2^2 + m_e^2B_3,\\ B_1 &= (4p_1\cdot p_2 - 2p_1\cdot k_2 - 2p_2\cdot k_1 + k_1\cdot k_2)^2 + (k_1\cdot k_2)^2 - k_1^2k_2^2 - 16m_e^4,\\ B_2 &= (p_1\cdot p_2)(k_1\cdot k_2) - (p_1\cdot k_2)(p_2\cdot k_1),\\ B_3 &= k_1^2(2p_1\cdot k_2 - k_1\cdot k_2)^2 + k_2^2(2p_2\cdot k_1 - k_1\cdot k_2)^2 + 4m_e^2(k_1\cdot k_2)^2. \end{aligned} \tag{4.6}$$

TABLE I. The total cross sections for the colliding-beam production of π^0, η, $\mu^+\mu^-$, e^+e^-, $\pi^+\pi^-$. Exact, f.f., and e.p. refer to the cross sections calculated without form factor, with the form factor (4.9), and in the equivalent-photon approximation, respectively. All the values have been taken from Brodsky, Kinoshita, and Terazawa (1971b) except for the π^0 values which are slightly changed by the new experimental data for $\Gamma_{\pi^0\to\gamma\gamma}$. The values in ( ) are the recently improved results of Grammer and Kinoshita (1972) and those in [ ] and { } are the exact results of Brown and Lyth (1973) and of Bonneau and Martin (1973), respectively.

	E(GeV)					
Process	0.5	1.0	1.5	2.0	2.5	3.0
			$\sigma_{\text{total}}(10^{-33}\text{ cm}^2)$			
$ee\to ee\pi^0$ (exact)	0.30	0.57	0.80	0.96	1.12	1.25
		(0.54)		(0.77)		(0.91)
$ee\to ee\pi^0$ (f.f.)	0.28	0.54	0.71	0.86	1.00	1.07
$ee\to ee\pi^0$ (e.p.)	0.25	0.48	0.63	0.77	0.87	0.97
$ee\to ee\eta$ (exact)	0.067	0.32	0.56	0.77	0.98	1.16
	(0.068)	(0.29)		(0.62)		(0.95)
$ee\to ee\eta$ (f.f.)	0.062	0.27	0.46	0.63	0.79	0.92
$ee\to ee\eta$ (e.p.)	0.086	0.32	0.54	0.70	0.89	1.00
$ee\to ee\mu^+\mu^-$ (e.p.)	8.1	18.7	29	38	45	50
	{5.3}	{15.4}		{32}		
$ee\to ee\pi^+\pi^-$ (e.p.)	0.46	1.37	2.2	2.9	3.5	4.1
		[0.97]		[2.1]		
$ee\to eee^+e^-$ (e.p.)	5.5×10^6	7.3×10^6	8.4×10^6	9.5×10^6	1.02×10^7	1.09×10^7
$e^+e^-\to\mu^+\mu^-$	87	22	9.7	5.4	3.5	2.4
$e^+e^-\to\pi^+\pi^-$ (pointlike)	22	5.4	2.4	1.36	0.87	0.60

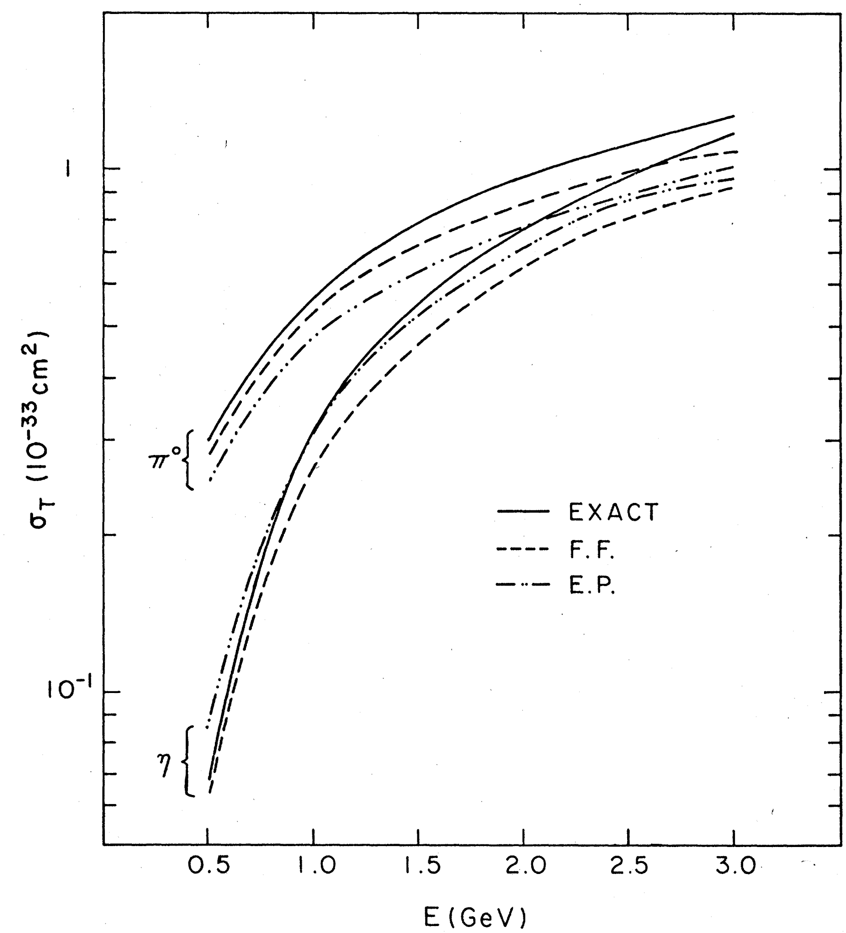

FIG. 3. The total cross sections for the colliding-beam production of π^0 and η. EXACT., F.F., and E.P. refer to the cross sections calculated without the form factor, with the form factor (4.9), and in the equivalent-photon approximation, respectively. All the curves have been taken from Brodsky, Kinoshita, and Terazawa (1971b) except for the π^0 curves which are slightly changed by the new experimental data for $\Gamma_{\pi^0\to\gamma\gamma}$.

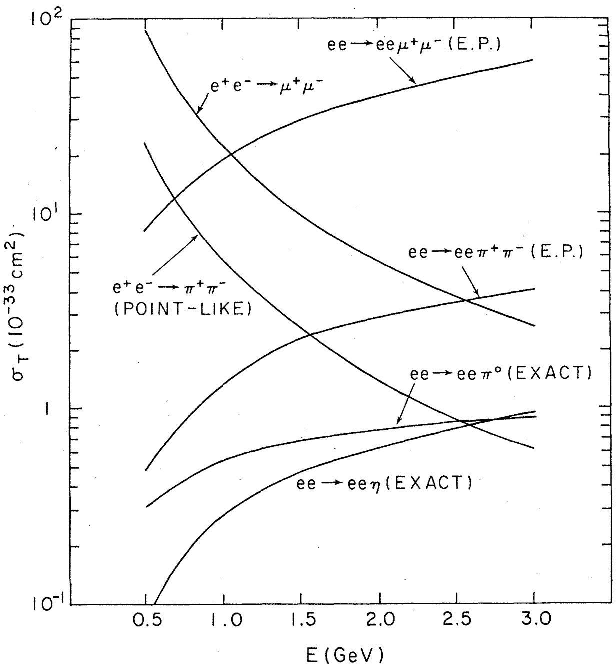

Fig. 4. The total cross sections for the colliding-beam production of π^0, η, $\pi^+\pi^-$, and $\mu^+\mu^-$. The cross sections for π^0 and η are exact and without form factors. The two-photon cross sections for $\pi^+\pi^-$ and $\mu^+\mu^-$ are calculated in the equivalent-photon approximation. All the curves have been taken from Brodsky, Kinoshita, and Terazawa (1971b) except for the π^0 and η curves which are taken from Grammer and Kinoshita (1972) and slightly changed by the new experimental data for $\Gamma_{\pi^0\to\gamma\gamma}$.

The factor F is included in (4.5) to represent a possible form-factor dependence of the cross section on the photon masses:

$$F=F(k_1^2,k_2^2) \quad \text{with} \quad F(0,0)=1. \tag{4.7}$$

The theoretical interest of this form factor will be discussed in detail in Sec. XIV.

In Fig. 3 (see also Table I) we have plotted the result for $\sigma_{ee\to ee\pi^0}$ obtained by numerical integration of (2.10) with (4.5), assuming the cases

$$\text{(a)} \qquad F=1 \tag{4.8}$$

and

$$\text{(b)} \qquad F=(1-k_1^2/m_\rho^2)^{-1}(1-k_2^2/m_\rho^2)^{-1}, \tag{4.9}$$

as suggested by ρ dominance. It is seen that the equivalent-photon approximation underestimates the total cross section by 20–30% for $1\lesssim E\lesssim 3$ GeV. However, the exact result is reduced considerably when the effect of the form factor (4.9) is taken into account.

The discrepancy between the equivalent-photon and exact calculations can be traced to the contribution to the total cross section of relatively large electron scattering angles. Note that for $\theta_\gamma>(m_e/E)^{1/4}$ ($\simeq 12°$ for $E=1$ GeV), which still contains approximately 25% of the total cross section (see Sec. III), $|k^2|$ is larger than m_π^2. The equivalent-photon approximation is not expected to work well in this region since, roughly speaking, it can be regarded as an expansion in k^2/m_π^2 as well as in k^2/E^2. For the same reason this approximation is more reliable for the production of more massive states such as η and η'. The exact (with and without form factors) and approximate total cross sections for η production are shown in Fig. 3, see also Table I. We have used $m_\eta=0.549$ GeV and $\Gamma_{\eta\to\gamma\gamma}=1.0$ keV (Particle Data Group, 1972).

Thus, as far as present predictions of $C=+$ hadron production by electron–electron collisions are concerned, the equivalent-photon approximation is adequate since errors due to lack of knowledge of coupling constants, form factors, etc., are much more serious. Of course, the detailed fitting of the decay width and the possible determination of form factors and longitudinal current contributions (from large-angle electron scattering) to resonance production will require the complete result. For the latter purpose, however, it is strongly recommended that both scattered electrons be detected and that k_1^2 and k_2^2 be measured for each event (see Sec. XIV).

Another possible explanation of the discrepancy between the equivalent-photon and exact results, which cannot be dismissed at present, is that, whereas the leading $[\ln(E/m_e)]^2$ term is given correctly by the equivalent-photon method, the exact cross section contains a contribution linear in $\ln(E/m_e)$ with a large coefficient, e.g., $[\ln(E/m_\pi)]^2$, and/or a contribution cubic in $\ln(E/m_\pi)$, which cannot be evaluated by the equivalent-photon method. Bonneau, Gourdin, and Martin (1973), in their recent paper, have shown that a term containing $[\ln(E/m_X)]^3$ always appears even in the transverse–transverse (T–T) contribution to the cross section for fixed m_X $(=s^{1/2})$. Their result for $d\sigma_{ee\to eeX}{}^{\mathrm{TT}}/ds$ is proportional to

$$\frac{2}{3}\left(\ln\frac{4E^2}{s}\right)^3+2\left(\ln\frac{4E^2}{s}\right)^2\left(\ln\frac{s}{m_e^2}\right)+\left(\ln\frac{4E^2}{s}\right)\left(\ln\frac{s}{m_e^2}\right)^2, \tag{4.10}$$

in which they find that the leading $[\ln(4E^2/s)]^3$ term for $4E^2/s\gg s/m_e^2\gg 1$ has a coefficient $\frac{2}{3}$ instead of the 1 which the leading $[\ln(4E^2/m_e^2)]^2\ln(4E^2/s)$ term has for $4E^2/m_e^2\gg 4E^2/s\gg 1$. However, *this does not mean any contradiction with the equivalent-photon approximation* in which the latter condition $4E^2/m_e^2\gg 4E^2/s\gg 1$ is always assumed. In fact, after rearranging their result (4.10) into[6]

$$[\ln(4E^2/m_e^2)][\ln(4E^2/s)]-\tfrac{1}{3}[\ln(4E^2/s)]^3, \tag{4.11}$$

we can clearly see that it is consistent with the equiva-

[6] The author is indebted to Dr. G. Grammer and Professor T. Kinoshita for discussion on this point.

lent-photon result when the condition $4E^2/m_e^2 \gg 4E^2/s \gg 1$ is satisfied. Furthermore, the equivalent-photon condition $4E^2/m_e^2 \gg 4E^2/s \gg 1$ is more practical at present accelerator energies than is their condition $4E^2/s \gg s/m_e^2 \gg 1$. The second term in (4.11) is less than 4% of the first term for $E=1$ GeV in the case of π^0 production ($s=m_\pi^2$).

Recently Grammer and Kinoshita (1972) have improved the numerical calculation of the total cross section for π^0 and η productions and found that agreement between the exact and equivalent-photon approximation results is much better ($\lesssim 10\%$). We quote their results in Table I and Fig. 4.

See also a latest work by Subbarao (1973) who refined the approximate formula (3.11) up to the single logarithm in E/m_e, including the longitudinal–transverse components.

Very lately Bonneau and Martin (1973) have greatly improved the equivalent photon approximation, including both the longitudinal–transverse and longitudinal–longitudinal contributions. Their almost exact results for the total cross sections for π^0 and η productions agree with the previous exact results by Brodsky, Kinoshita, and Terazawa (1971b) remarkably well.

A further discussion on narrow-resonance production by the two-photon process, especially concerned with its theoretical aspects will be given in Secs. VII and XIV.

V. TWO-PHOTON CROSS SECTIONS FOR $\pi^+\pi^-$ AND $\mu^+\mu^-$ PRODUCTION IN THE EQUIVALENT-PHOTON APPROXIMATION

In this section we shall discuss the two-photon production of a $\pi^+\pi^-$ or $\mu^+\mu^-$ pair

$$e+e\rightarrow e+e+\pi^+ + \pi^- \tag{5.1}$$

or

$$e+e\rightarrow e+e+\mu^+ + \mu^- \tag{5.2}$$

by colliding electron beams. The analysis of π^0 production described in Sec. IV shows that the important features of the $\pi^+\pi^-$ (or $\mu^+\mu^-$) production cross section, except for the coplanarity of pion (muon) pairs to be discussed in Sec. VI, can be investigated with reasonable accuracy by means of the equivalent-photon method. We shall therefore restrict ourselves here to this approximation, leaving the exact calculation to the following section.

A. Total Cross Sections

Calculation of the two-photon total cross section for muon pair production is straightforward in the equivalent-photon approximation. We simply have to substitute the total cross section for the $\mu^+\mu^-$ pair creation by two γ's,[7]

$$\sigma_{\gamma\gamma\rightarrow\mu^+\mu^-}(s) = \frac{4\pi\alpha^2}{s}\left\{\left(2+\frac{8m_\mu^2}{s}-\frac{16m_\mu^4}{s^2}\right)\times\ln\left[\frac{s^{1/2}}{2m_\mu}+\left(\frac{s}{4m_\mu^2}-1\right)^{1/2}\right] -\left(1-\frac{4m_\mu^2}{s}\right)^{1/2}\left(1+\frac{4m_\mu^2}{s}\right)\right\}, \tag{5.3}$$

into the formula (3.14). The result of the numerical calculation (Brodsky, Kinoshita, and Terazawa, 1971b) is plotted in Fig. 4 (see also Table I). This cross section exceeds the one-photon cross section $\sigma_{e^+e^-\rightarrow\mu^+\mu^-}$, which is equal to

$$(\pi\alpha^2/3E^2)[1-(m_\mu^2/E^2)]^{1/2}[1+(m_\mu^2/2E^2)],$$

for $E \gtrsim 1$ GeV. For very large E/m_e we have[8]

$$\sigma_{ee\rightarrow ee\mu^+\mu^-}(E) \simeq (112\alpha^4/9\pi)(1/m_\mu^2)\times[\ln(E/m_e)]^2\ln(E/m_\mu). \tag{5.4}$$

Note also that in the energy range shown in Fig. 4, the $\mu^+\mu^-$-production cross section (for both one-photon and two-photon processes) is an order of magnitude larger than any process of hadron production. Since the muon is pointlike as far as has been tested experimentally, this cross section is undoubtedly the most reliably known of the cross sections shown in Fig. 4.

The evaluation of the total cross section for pion-pair production is not as simple as is that of the muon case because of the strong interaction in the final state. In fact, the reaction (5.1) is the ideal process for studying the interaction in $C=+$ states such as the σ and f resonances. However, we shall postpone the consideration of hadron physics until Sec. VII and, for illustration, first treat π^+ and π^- as pointlike charged particles without strong interaction. There are two reasons for doing this: (1) Such a calculation serves as a reference point of hadron physics because the effect of strong interaction in the final state can be determined as the deviation of the cross section from the one calculated here. (2) For E around 1 GeV, our calculation will in fact give a reasonable estimate of the actual cross section for $\pi^+\pi^-$ production. This is because the small-s region near the threshold is not too far removed from the threshold region of the elastic photon-pion scattering, where the exact value of the cross section is determined by the low-energy theorem for Compton scattering, and because the contribution of $\sigma_{\gamma\gamma\rightarrow\pi^+\pi^-}(s)$ to the integral (3.14) is heavily weighted towards the low-s end.

If we accept this picture as a first approximation, then $\sigma_{\gamma\gamma\rightarrow\pi^+\pi^-}$ can be calculated from the Born (in-

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[7] See Akhiezer and Berestetskii (1965), p. 450.

[8] This result is larger by a factor of 3/2 than the result of Akhiezer and Berestetskii (1965), pp. 454 and 484.

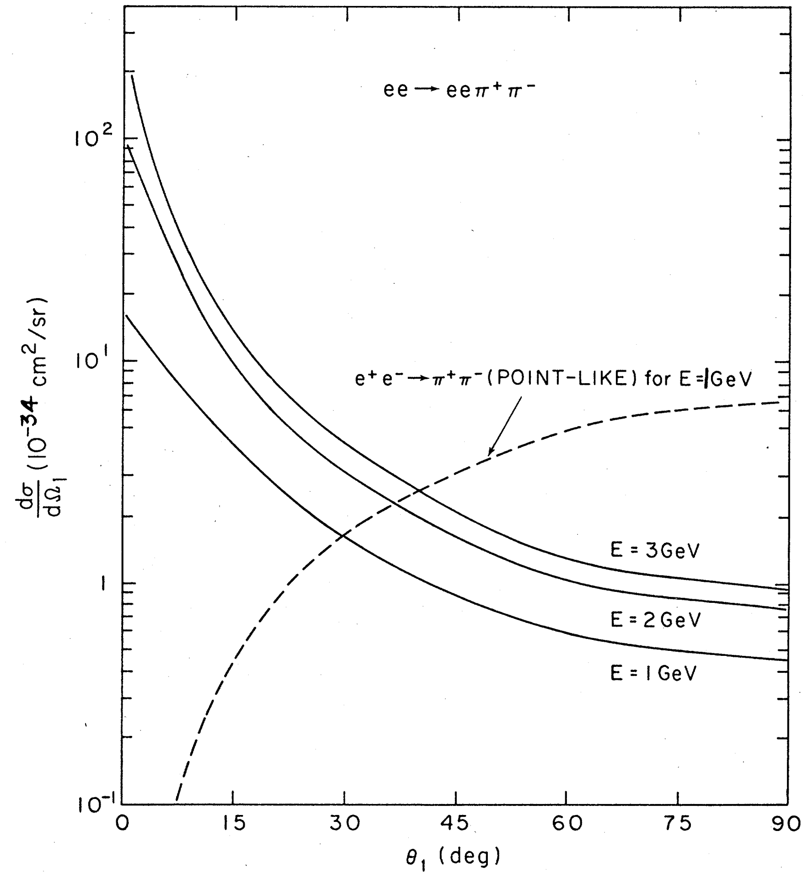

FIG. 5. The cross section $d\sigma/d\Omega_1$ for the process $e+e\rightarrow e+e+\pi^+ + \pi^-$ calculated in the equivalent-photon approximation for $E=1$, 2, and 3 GeV (Brodsky, Kinoshita, and Terazawa, 1971b). For comparison, the one-photon cross section $d\sigma_{e^+e^-\rightarrow\pi^+\pi^-\text{pointlike}}$ is also shown for $E=1$ GeV.

cluding seagull) diagrams and is given by[9]

$$\sigma_{\gamma\gamma\rightarrow\pi^+\pi^-}(s)=\frac{2\pi\alpha^2}{s}\left\{\left(1-\frac{4m_\pi^2}{s}\right)^{1/2}\left(1+\frac{4m_\pi^2}{s}\right)\right.$$

$$\left.-\frac{8m_\pi^2}{s}\left(1-\frac{2m_\pi^2}{s}\right)\ln\left[\frac{s^{1/2}}{2m_\pi}+\left(\frac{s}{4m_\pi^2}-1\right)^{1/2}\right]\right\}. \quad (5.5)$$

In Fig. 4 (see also Table I) we show the energy dependence of the total cross section for the colliding-beam production of the $\pi^+\pi^-$ pair calculated from (3.14) and (5.5) in the equivalent-photon approximation. For the pointlike pions and for $E>1.5$ GeV, this cross section exceeds the usual one-photon cross section $\sigma_{e^+e^-\rightarrow\pi^+\pi^-}$, which equals

$$(\pi\alpha^2/12E^2)[1-(m_\pi^2/E^2)]^{3/2}.$$

For very large E/m_e we have

$$\sigma_{ee\rightarrow ee\pi^+\pi^-}(E)\simeq(16\alpha^4/9\pi)(1/m_\pi^2)$$
$$\times[\ln(E/m_e)]^2\ln(E/m_\pi). \quad (5.6)$$

B. Angular Distributions

We shall now discuss the angular distribution of the produced pair (Brodsky, Kinoshita, and Terazawa, 1971b). For this purpose we need the differential cross sections for $\gamma+\gamma\rightarrow\mu^++\mu^-$ [10] and $\gamma+\gamma\rightarrow\pi^++\pi^-$ [11]:

$$d\sigma_{\gamma\gamma\rightarrow\mu^+\mu^-}/d\Omega_1=(\alpha^2/2s)[1-(4m_\mu^2/s)]^{1/2}G_\mu(W_1,\theta_1),$$

$$G_\mu(W_1,\theta_1)=2+4\left(1-\frac{m_\mu^2}{W_1^2}\right)$$
$$\times\frac{(1-m_\mu^2/W_1^2)\sin^2\theta_1\cos^2\theta_1+m_\mu^2/W_1^2}{[1-(1-m_\mu^2/W_1^2)\cos^2\theta_1]^2}$$
$$+\frac{s/W_1^2-4}{1-(1-m_\mu^2/W_1^2)\cos^2\theta_1}, \quad (5.7)$$

and

$$d\sigma_{\gamma\gamma\rightarrow\pi^+\pi^-}/d\Omega_1=(\alpha^2/2s)[1-(4m_\pi^2/s)]^{1/2}G_\pi(W_1,\theta_1),$$

$$G_\pi(W_1,\theta_1)=1-\frac{2m_\pi^2}{W_1^2}\left(1-\frac{m_\pi^2}{W_1^2}\right)$$
$$\times\frac{\sin^2\theta_1}{[1-(1-m_\pi^2/W_1^2)\cos^2\theta_1]^2}, \quad (5.8)$$

where W_1 is the energy of μ^+ (or π^+), and θ_1 is the angle between one of the incident photons and an outgoing μ^+

[9] This expression is smaller by a factor of 2 than that in Akhiezer and Berestetskii (1965), p. 844 [Eq. (60.7)].

[10] See Akheizer and Berestetskii (1965), p. 843. The last term in G_μ which disappeared in the center-of-mass system of two photons should be kept here.

[11] See Akhiezer and Berestetskii (1965), p. 450.

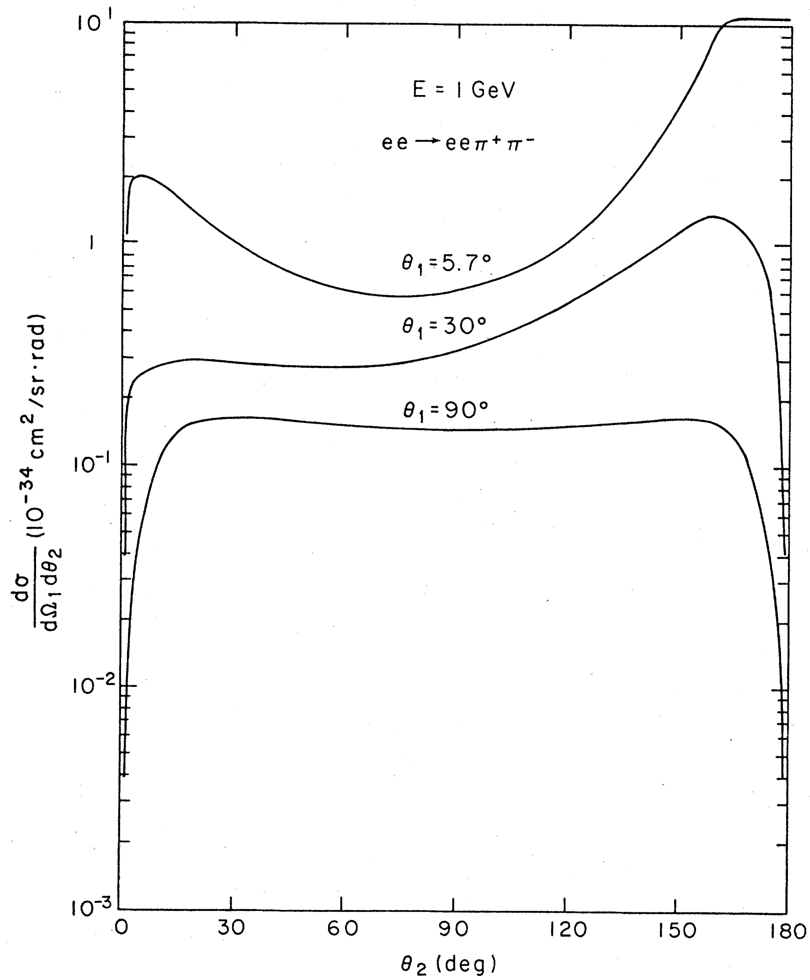

FIG. 6. The cross section $d\sigma/d\Omega_1 d\theta_2$ for the process $e+e\rightarrow e+e+\pi^++\pi^-$ calculated in the equivalent-photon approximation for $E=1$ GeV and $\theta_1=5.7°$, 30°, and 90° (Brodsky, Kinoshita, and Terazawa, 1971b).

(or π^+) in the photon–photon center-of-mass system. Before we substitute these cross sections into (3.11), we have to transform them from the photon–photon center-of-mass system to the electron–electron center-of-mass system. For this purpose it is useful to note that $G_\mu(W_1, \theta_1)$ and $G_\pi(W_1, \theta_1)$ are *form*-invariant under the Lorentz transformation along the beam direction. Thus we have only to reinterpret W_1 and θ_1 in $G_\mu(W_1, \theta_1)$ and $G_\pi(W_1, \theta_1)$ as the energy and angle of μ^+ (or π^+) in the laboratory frame (i.e., the electron–electron center-of-mass system). In this way we can derive various differential cross sections from (3.11). Of particular interest are [the superscript (0) refers to the equivalent-photon approximation as in (3.10)]

$$\frac{d\sigma_{ee\rightarrow ee\pi^+\pi^-}{}^{(0)}}{d\Omega_1}=\frac{8\alpha^4}{\pi^2}\left(\ln\frac{E}{m_e}\right)^2\int_{4m_\pi^2}^{4E^2}\frac{ds}{s^2}\int_{-q_m}^{q_m}\frac{dq}{\omega}G_\pi(W_1,\theta_1)\frac{(E^2+E_1'^2)(E^2+E_2'^2)}{4E^4}\frac{|\mathbf{q}_1|^2}{[s^2-4m_\pi^2(\omega^2-q^2\cos^2\theta_1)]^{1/2}}, \quad (5.9)$$

with

$$q_m=E-s/4E, \qquad \omega=(q^2+s)^{1/2},$$

$$|\mathbf{q}_1|=\frac{sq\cos\theta_1+\omega[s^2-4m_\pi^2(\omega^2-q^2\cos^2\theta_1)]^{1/2}}{2(\omega^2-q^2\cos^2\theta_1)}, \qquad W_1=(|\mathbf{q}_1|^2+m_\pi^2)^{1/2}; \quad (5.10)$$

and

$$\frac{d\sigma_{ee\rightarrow ee\pi^+\pi^-}{}^{(0)}}{d\Omega_1 d\theta_2}=\frac{4\alpha^4}{\pi^2}\left(\ln\frac{E}{m_e}\right)^2\int_{4m_\pi^2}^{4E^2}\frac{ds}{s^2}G_\pi(W_1,\theta_1)\frac{(E^2+E_1'^2)(E^2+E_2'^2)}{4E^4}\frac{|\mathbf{q}_1|^2|\mathbf{q}_2|\sin(\theta_2-\theta_1)}{q[\omega W_1\sin^2\theta_1+\omega W_2\sin^2\theta_2-W_1W_2\sin^2(\theta_2-\theta_1)]}, \quad (5.11)$$

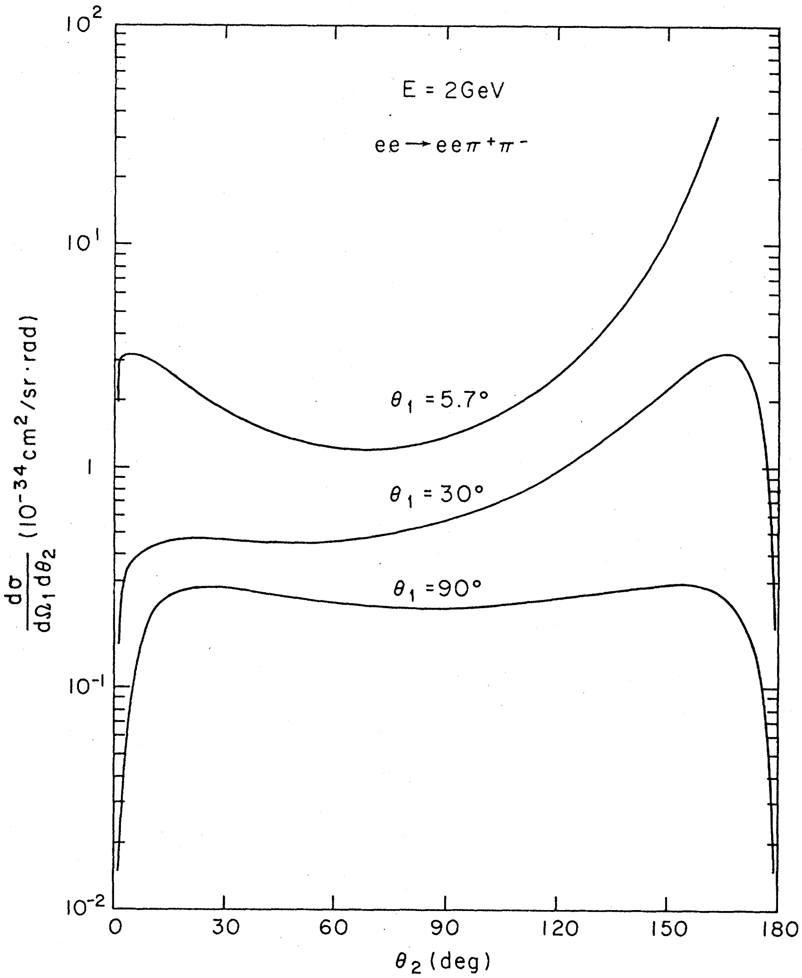

FIG. 7. The cross section $d\sigma/d\Omega_1 d\theta_2$ for the process $e+e\rightarrow e+e+\pi^+ +\pi^-$ calculated in the equivalent-photon approximation for $E=2$ GeV and $\theta_1=5.7°$, 30°, and 90° (Brodsky, Kinoshita, and Terazawa, 1971b).

for $\theta_1<\theta_2$ with,

$$\begin{aligned} &a_1=[\sin\theta_1/\sin(\theta_2-\theta_1)]^2, \qquad a_2=[\sin\theta_2/\sin(\theta_2-\theta_1)]^2, \\ &q^2=\frac{2\{a_1a_2s^2-m_\pi^2[a_1+a_2-(a_1-a_2)^2]s+m_\pi^4\}^{1/2}-(a_1+a_2-1)s-2m_\pi^2}{4a_1a_2-(a_1+a_2-1)^2}, \\ &|\mathbf{q}_1|=q\sin\theta_2/\sin(\theta_2-\theta_1), \qquad |\mathbf{q}_2|=q\sin\theta_1/\sin(\theta_2-\theta_1), \\ &W_2=(|\mathbf{q}_2|^2+m_\pi^2)^{1/2}, \qquad q^2\leq q_m^2. \end{aligned} \tag{5.12}$$

θ_1 and θ_2 are the angles of π^+ and π^- with respect to the electron beam direction. Other notations are defined in Fig. 2. We can use the symmetry property $d\sigma(\pi-\theta_1, \pi-\theta_2)=d\sigma(\theta_1, \theta_2)$ in order to obtain the cross section for $\theta_1>\theta_2$ from (5.11). Cross sections for muon pair production (5.2) are obtained by replacing $G_\pi(W_1, \theta_1)$ with $G_\mu(W_1, \theta_1)$ in (5.9) and (5.11).

In Fig. 5 (see also Table II) we show the cross section $d\sigma_{ee\rightarrow ee\pi^+\pi^-}{}^{(0)}/d\Omega_1$ calculated from (5.9) for $E=1$, 2, and 3 GeV. It is clearly seen that the pions are produced predominantly in the beam direction. For comparison, the one-photon cross section $d\sigma_{e^+e^-\rightarrow\pi^+\pi^-}/d\Omega_1$ is also shown for $E=1$ GeV. What should be emphasized here is the following: Suppose we detect produced pions at $E=1$ GeV and $\theta_1=30°$ without observing any coincidence. Then we can expect to find as many pions coming from the two-photon process as from pion-pair creation through the one-photon annihilation *even if pions are assumed to be pointlike.* In practice, we can ignore pions pair-created by the one-photon process at this energy in this type of experiment and, instead, have to worry about pions from multiple-particle production. However, one can set up appropriate counters to detect the mere presence of the scattered electrons without losing the large cross section ($d\sigma_{ee\rightarrow ee\pi^+\pi^-}/d\Omega_1\sim 10^{-34}$ cm²/sr). This is one of the simplest experiments to find ev-

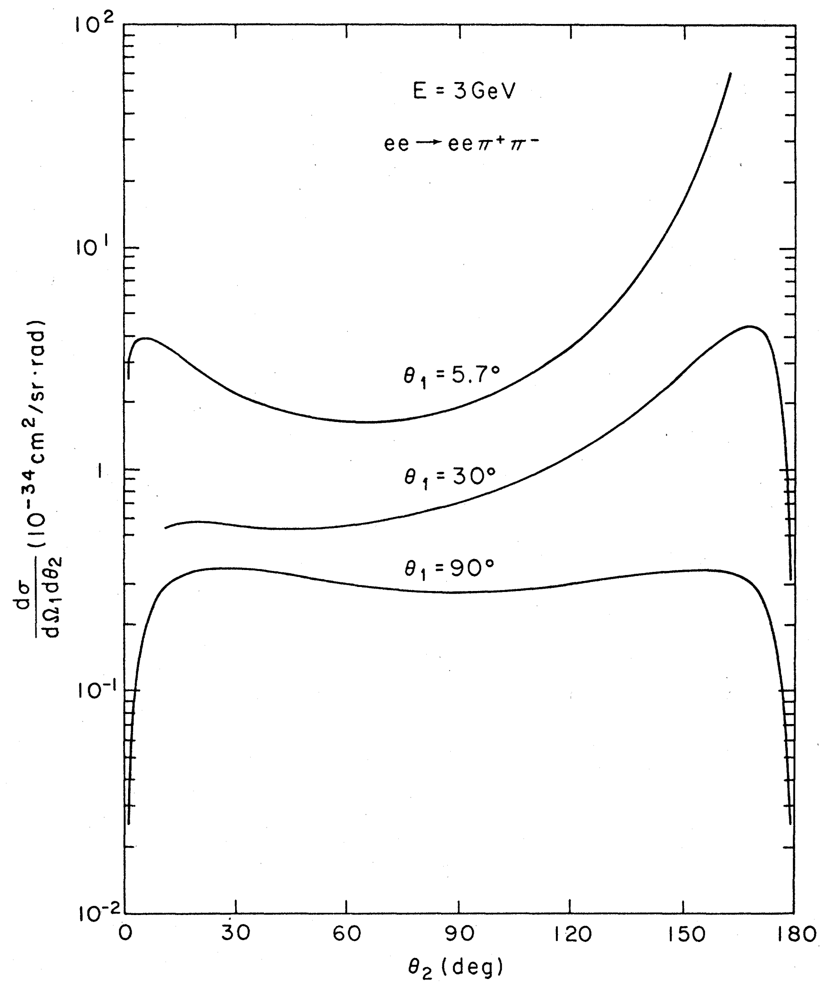

FIG. 8. The cross section $d\sigma/d\Omega_1 d\theta_2$ for the process $e+e\to e+e+\pi^++\pi^-$ calculated in the equivalent-photon approximation for $E=3$ GeV and $\theta_1=5.7°$, 30°, and 90° (Brodsky, Kinoshita, and Terazawa, 1971b).

idence of hadron production by the two-photon process.

In Figs. 6–8 (see also Table III) we show the cross section $d\sigma_{ee\to ee\pi^+\pi^-}{}^{(0)}/d\Omega_1 d\theta_2$ calculated from (5.11) for combinations of $\theta_1=5.7°$, 30°, and 90° and $E=1$, 2, and 3 GeV. One observes a strong tendency for the pion pairs to be produced with a rather narrow opening angle $(\theta_1+\pi-\theta_2)$. That is, the pion pairs tend to emerge in a strongly noncollinear fashion. This is in contrast to the pion pairs produced in the one-photon process, which must be exactly collinear.

For comparison we also show in Fig. 9 $d\sigma_{ee\to ee\mu^+\mu^-}{}^{(0)}/d\Omega_1$ and $d\sigma_{ee\to ee\mu^+\mu^-}{}^{(0)}/d\Omega_1 d\theta_2$ in Fig. 10 for $E=1$ GeV which

TABLE II. The cross section $d\sigma/d\Omega_1$ for the process $e+e\to e+e+\pi^+\pi^-$ calculated in the equivalent-photon approximation for $E=1$, 2, and 3 GeV (Brodsky, Kinoshita, and Terazawa, 1971b). The one-photon cross section $d\sigma_{e^+e^-\to\pi^+\pi^-}{}^{\text{pointlike}}/d\Omega_1$ is also given for $E=1$ GeV.

E (GeV)	θ_1 (deg) 1.0	5.7	11.5	17.2	22.9	30	60	90
	$d\sigma_{ee\to ee\pi^+\pi^-}/d\Omega_1$ (10^{-34}cm^2/sr)							
1.0	15.2	10.3	5.6	3.6	2.3	1.62	0.58	0.44
2.0	85	37	13.6	7.6	4.6	3.1	1.02	0.75
3.0	191	49	21	10.6	6.4	4.2	1.28	0.93
	$d\sigma_{e^+e^-\to\pi^+\pi^-}$(pointlike) $/d\Omega_1$ (10^{-34}cm^2/sr)							
1.0	2.0×10^{-3}	0.065	0.26	0.57	0.99	1.62	4.9	6.5

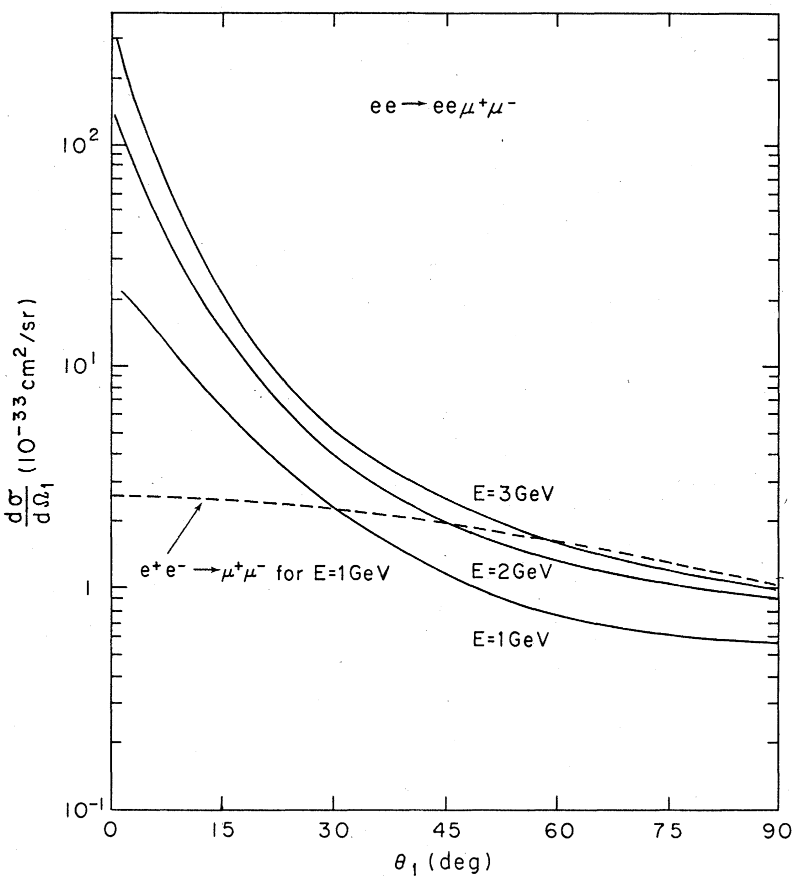

FIG. 9. The cross sections $d\sigma/d\Omega_1$ for the process $e+e+\mu^++\mu^-$ calculated in the equivalent-photon approximation for $E=1$, 2, and 3 GeV (Brodsky, Kinoshita, and Terazawa, 1971b). For comparison, the one-photon cross section $d\sigma_{e^+e^-\rightarrow\mu^+\mu^-}/d\Omega_1$ is also shown for $E=1$ GeV.

show a similar peaking in the beam direction. For details see also Tables IV and V.[12]

C. Bias Factor

The figures for the various cross sections shown above are somewhat misleading because they do not take explicit account of the diminishing phase space as the pions approach the beam direction. In order to evaluate this effect and also to exhibit some of the features of these cross sections more clearly, let us examine the cross section (Brodsky, Kinoshita, and Terazawa, 1971b)

$$\frac{d\sigma_{ee\rightarrow ee\pi^+\pi^-}{}^{(0)}}{ds\,d\Omega_1}=\left(\frac{2\alpha}{\pi}\right)^2\left(\ln\frac{E}{m_e}\right)^2\int_{-q_m}^{q_m}\frac{dq}{\omega}\,\theta(s^2-4m_\pi^2(\omega^2-q^2\cos^2\theta))\,\frac{(E^2+E_1'^2)(E^2+E_2'^2)}{4E^4}$$

$$\times(1-4m_\pi^2/s)^{-1/2}\,\frac{d\sigma_{\gamma\gamma\rightarrow\pi^+\pi^-}}{d\Omega_1}\,\frac{\{sq\cos\theta+\omega[s^2-4m_\pi^2(\omega^2-q^2\cos^2\theta)]^{1/2}\}^2}{s(\omega^2-q^2\cos^2\theta)^2[s^2-4m_\pi^2(\omega^2-q^2\cos^2\theta)]^{1/2}}. \quad (5.13)$$

This expression becomes considerably simpler in the region

$$\omega^2\ll E^2, \quad (5.14)$$

where we can simplify the virtual-photon distribution, and

$$\mathbf{q}_i^2\gg m_\pi^2, \qquad i=1,2 \quad (5.15)$$

where we can ignore the complications from the pion's velocity. Then $d\sigma_{\gamma\gamma\rightarrow\pi^+\pi^-}/d\Omega_1$ becomes isotropic in the photon–photon center-of-mass system and hence u-independent for fixed s, where $u=q/\omega$ is the velocity of the photon–photon center-of-mass system as seen in the laboratory frame (namely the center-of-mass

[12] We wish to thank Dr. Ronald Madaras for calling our attention to errors in Tables IV and V and Figs. 15–18 in the preliminary version of Brodsky, Kinoshita, and Terazawa (1971b).

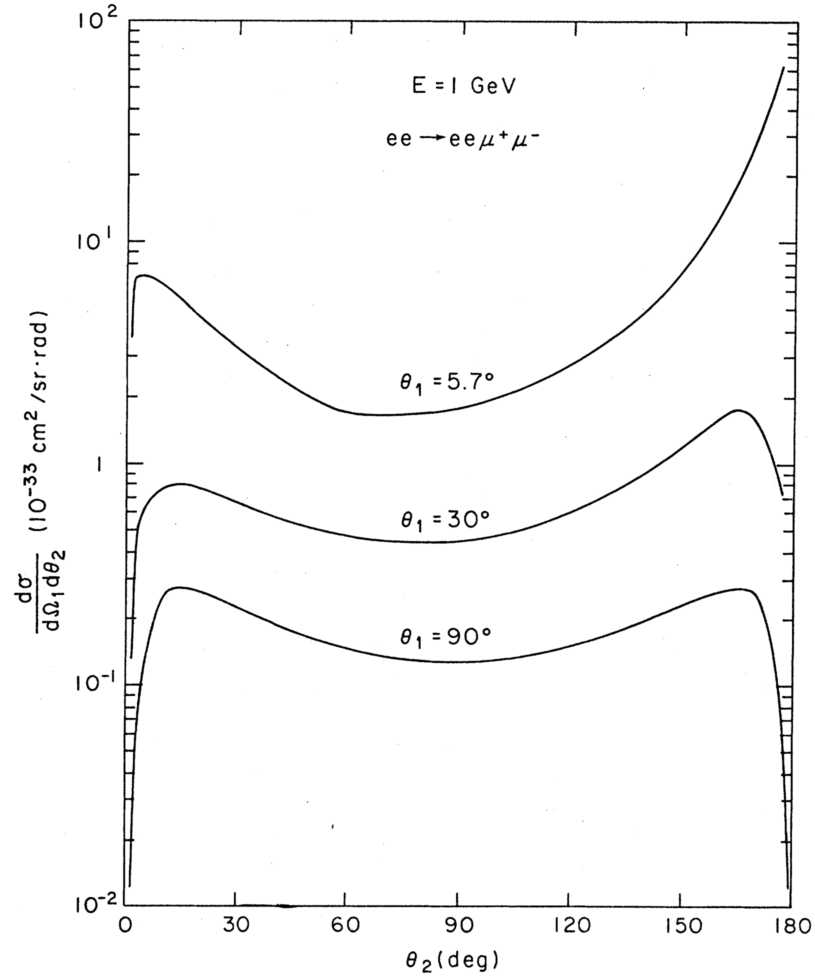

FIG. 10. The cross section $d\sigma/d\Omega_1 d\theta_2$ for the process $e+e\to e+e+\mu^+ +\mu^-$ calculated in the equivalent-photon approximation for $E=1$ GeV and $\theta_1=5.7°$, 30°, and 90° (Brodsky, Kinoshita, and Terazawa, 1971b).

system of incident beams). Thus Eq. (5.13) reduces to

$$\frac{d\sigma_{ee\to ee\pi^+\pi^-}{}^{(0)}}{dsd\Omega_1}\simeq\left(\frac{2\alpha}{\pi}\right)^2\left(\ln\frac{E}{m_e}\right)^2 s^{-1}$$

$$\times\int_{-u_m}^{u_m}\frac{du}{(1-u\cos\theta)^2}\frac{d\sigma}{d\Omega_1}$$

$$=\left(\frac{2\alpha}{\pi}\right)^2\left(\ln\frac{E}{m_e}\right)^2\frac{2}{s}$$

$$\times\frac{u_m}{1-u_m{}^2\cos^2\theta}\frac{d\sigma}{d\Omega_1},\qquad 4m_\pi{}^2\ll s\ll 4E^2,\quad(5.16)$$

where $u_m=(1-s/4E^2)/(1+s/4E^2)$ is the maximum velocity allowed kinematically. It is seen from (5.16) that the angular distribution is peaked along the beam direction, with the peaking becoming more pronounced as the mass s of the system decreases.

In order to understand how this peaking affects experiments which are usually only sensitive to particles produced at large angles, we can define a bias factor

$$G_{\Delta\Omega}(s)=\int_{\Delta\Omega}\frac{d\sigma_{ee\to eeX}}{ds\,d\Omega}d\Omega\bigg/\frac{d\sigma_{ee\to eeX}}{ds}\frac{\Delta\Omega}{4\pi},\quad(5.17)$$

which gives the ratio of the efficiency of detecting events in a given solid angle $\Delta\Omega$ compared to the efficiency of detecting events in $\Delta\Omega$ if the events were isotropic in the laboratory. For the usual experimental arrangement with symmetry in the φ coordinate, and for

$$-x_0<\cos\theta<x_0,\qquad 0<x_0\leq 1,\qquad(5.18)$$

the efficiency ratio for the cross section (5.16) will be

$$G_{\Delta\Omega}(s)=\int_{-x_0}^{x_0}\frac{dx}{1-u_m{}^2x^2}\bigg/x_0\int_{-1}^{1}\frac{dx}{1-u_m{}^2x^2}$$

$$=x_0{}^{-1}\frac{\ln[(1+u_mx_0)/(1-u_mx_0)]}{\ln[(1+u_m)/(1-u_m)]}.\quad(5.19)$$

We note that for s large and comparable to $4E^2$ ($u_m\ll 1$)

$$G_{\Delta\Omega}(s)\simeq 1-\tfrac{1}{3}u_m{}^2(1-x_0{}^2),\qquad(5.20)$$

TABLE III. The cross section $d\sigma/d\Omega_1 d\theta_2$ for the process $e+e\rightarrow e+e+\pi^++\pi^-$ calculated in the equivalent-photon approximation for combinations of $\theta_1=5.7°, 30°, 90°$, and $E=1, 2,$ and 3 GeV (Brodsky, Kinoshita, and Terazawa, 1971b).

E (GeV)	θ_1 (deg)	θ_2 (deg): 1.15	2.87	11.5	17.2	35.7	60	90	92.9	120	144.3	151.4	157.1	162.8	168.5	174.3	177.1	178.8
		$d\sigma_{ee\rightarrow ee\pi^+\pi^-}/d\Omega_1 d\theta_2$ (10^{-34}cm^2/sr rad)																
1.0	5.7	1.15	1.95	1.83	1.49	0.88	0.61	0.61	0.62	1.02	2.8	4.7	6.5	10.1	10.5	10.5	10.5	10.4
	30	0.038	0.23	0.28	0.30	0.29	0.28	0.33	0.35	0.53	0.95	1.14	1.34	1.34	1.39	0.76	0.27	0.041
	90	0.0040	0.023	0.115	0.147	0.167	0.152	0.143	0.143	0.152	0.167	0.167	0.162	0.147	0.115	0.060	0.023	0.0040
2.0	5.7	2.0	3.1	2.9	2.4	1.54	1.21	1.34	1.42	2.6	7.7	13.2	17.4	38				
	30	0.156	0.33	0.44	0.47	0.45	0.45	0.57	0.59	0.95	1.83	2.3	2.7	3.1	3.1	2.0	0.83	0.185
	90	0.0149	0.061	0.23	0.27	0.28	0.25	0.23	0.23	0.25	0.28	0.29	0.28	0.27	0.23	0.140	0.061	0.0149
3.0	5.7	2.6	3.8	3.5	2.9	1.94	1.59	1.88	2.0	3.5	10.4	20	25	62				
	30			0.54	0.57	0.54	0.56	0.71	0.74	1.20	2.3	2.9	3.6	4.1	4.4	3.2	1.47	0.32
	90	0.025	0.103	0.30	0.34	0.35	0.30	0.28	0.28	0.30	0.35	0.35	0.35	0.34	0.30	0.189	0.103	0.025

which is a negligible bias. For $s/4E^2\ll 1$, $u_m\simeq 1-2(s/4E^2)$ we have

$$G_{\Delta\Omega}(s)\simeq\frac{\ln[(1+x_0)/(1-x_0)]}{x_0\ln(4E^2/s)}. \qquad (5.21)$$

For example, we find

$$G_{\Delta\Omega}(s)\simeq 0.53 \quad\text{for}\quad x_0=(\tfrac{1}{2})^{1/2}(|\theta|<45°),$$
$$s/4E^2=1/100. \qquad (5.22)$$

Thus, if $d\sigma_{\gamma\gamma\rightarrow\pi^+\pi^-}$ were isotropic in the photon–photon center-of-mass system, the bias factor is only a logarithmic effect and does not give worse than 50% loss in the counting rate. Actually, the nonisotropic part of the cross section for $\pi^+\pi^-$ production will be relatively small, as is seen from (5.8), and the above estimate will not be affected too much. For the production of the $\mu^+\mu^-$ pair, however, the cross section is strongly nonisotropic and the bias factor would be much smaller than the above estimate. In both the $\pi^+\pi^-$ and $\mu^+\mu^-$ cases, the exact value of the bias factor can be obtained easily by numerical integration.

VI. DEGREE OF NONCOPLANARITY OF THE $\pi^+\pi^-$ PAIR

Brodsky, Kinoshita, and Terazawa (1970) have pointed out that the $\pi^+\pi^-$ (or $\mu^+\mu^-$) pair produced by the two-photon process is approximately coplanar with the incident beam. This is a simple consequence of the circumstance that the virtual photons are emitted predominately in the beam direction and hence the production plane defined by the momenta of the photons and pions (or muons) also contains the original electron momenta. However, this kinematical restriction is not very strong because of the angular spread of order $(m_e/E)^{1/2}$ of the photon beam [see (3.16)]. As we shall see, the statement that the two-particle production process (e.g., $e+e\rightarrow e+e+\pi^++\pi^-$) produces events "coplanar" with the electron-beam direction is only approximate. This approximate coplanarity has been closely examined by the authors mentioned above (1971b).

In order to examine this problem let us introduce the total momentum $\mathbf{q}$ of the produced system X:

$$\mathbf{q}=\mathbf{q}_1+\mathbf{q}_2=\mathbf{k}_1+\mathbf{k}_2=-\mathbf{p}_1'-\mathbf{p}_2'. \qquad (6.1)$$

The angle θ of $\mathbf{q}$ measured from the initial beam direction is given by

$$\cos\theta=||\mathbf{p}_1'|\cos\theta_1'-|\mathbf{p}_2'|\cos\theta_2'|/|\mathbf{q}|, \qquad 0\leq\theta\leq\tfrac{1}{2}\pi, \qquad (6.2)$$

where as before $\cos\theta_1'=\hat{p}_1\cdot\hat{p}_1'$ and $\cos\theta_2'=\hat{p}_2\cdot\hat{p}_2'$. We shall call θ the "photon–photon axis angle" in the following. We shall also define the "coplanarity angle ψ" between the two planes, one determined by $\mathbf{q}_1$ and $\mathbf{p}_1$

TABLE IV. The cross section $d\sigma/d\Omega_1$ for the process $e+e\to e+e+\mu^++\mu^-$ calculated in the equivalent-photon approximation for $E=1,2$, and 3 GeV (Brodsky, Kinoshita, and Terazawa, 1971b). The one-photon cross section $d\sigma_{e^+e^-\to\mu^+\mu^-}/d\Omega_1$ is also given for $E=1$ GeV.

E (GeV)	θ_1 (deg) 1.0	5.7	11.5	17.2	22.9	30	60	90
	$d\sigma_{ee\to ee\mu^+\mu^-}/d\Omega_1\,(10^{-33}\text{cm}^2/\text{sr})$							
1.0	22	15.7	8.6	5.5	3.5	2.3	0.75	0.55
2.0	121	52	20	10.8	6.4	3.8	1.28	0.89
3.0	290	82	28	13.8	8.3	4.9	1.56	1.13
	$d\sigma_{e^+e^-\to\mu^+\mu^-}/d\Omega_1\,(10^{-33}\text{cm}^2/\text{sr})$							
1.0	2.6	2.6	2.5	2.5	2.4	2.3	1.62	1.30

and the other by $\mathbf{q}_2$ and $\mathbf{p}_2$, by

$$\cos\psi=\frac{(\hat{q}_1\times\hat{p}_1)}{|\hat{q}_1\times\hat{p}_1|}\cdot\frac{(\hat{q}_2\times\hat{p}_2)}{|\hat{q}_2\times\hat{p}_2|}. \tag{6.3}$$

Although the coplanarity angle ψ can be defined only for two-particle productions, the photon–photon axis angle θ is general to *all* of the $C=+$ production cross sections, including production of a single hadron such as π^0 or η. It is, therefore, instructive to see how the cross section for π^0 and η production depends on θ (Brodsky, Kinoshita, and Terazawa, 1971b). In Table VI we give the ratios of these cross sections evaluated using the exact formula under the restriction $\theta>\theta_{\min}$ for various cutoff angles $\theta_{\min}$ to the total cross sections. It is seen that approximately $\frac{1}{2}$ of the total cross section still comes from θ larger than $(m_e/E)^{1/4}$ even though $\frac{1}{2}$ of the emitted photons fall in the much narrower angular region $\theta_\gamma<(m_e/E)^{1/2}$, as is seen from (3.16). It is not difficult to understand this result qualitatively: In (6.1) the longitudinal components (parallel to the electron beam) of $\mathbf{k}_1$ and $\mathbf{k}_2$ tend to cancel each other whereas the transverse components, which are of order $E(m_e/E)^{1/2}$, may add up, leading to the result $\theta\gg\theta_\gamma$. Furthermore, since this θ dependence of the cross section arises from the spreading of the total momentum $\mathbf{k}_1+\mathbf{k}_2$ of the two-photon system, it will be rather insensitive to the nature of the produced state X. Thus we may expect to find a similar situation in the case of $\pi^+\pi^-$ or $\mu^+\mu^-$ production.

We shall now examine the degree of noncoplanarity of the $\pi^+\pi^-$ pair. For this purpose we have to calculate a cross section such as $d\sigma/d\psi$. Clearly we have to carry out such a calculation without using the equivalent-photon approximation since the information on the ψ dependence is completely lost in this approximation [see (3.9) and the remarks which follow]. To facilitate the computational problem, Brodsky, Kinoshita, and Terazawa (1971b) have actually calculated the cross section $d\sigma/d\cos\theta_1 d\cos\theta_2 d\psi$, where θ_1 and θ_2 are angles which the π^+ and π^- make with the electron beams.

The cross section $d\sigma/d\cos\theta_1 d\cos\theta_2 d\psi$ is obtained by evaluating the matrix elements $M_{\mu\alpha}$ of (2.10) in perturbation theory. This leads to

$$\frac{d\sigma}{d\cos\theta_1 d\cos\theta_2 d\psi}=\frac{\pi}{2E^2}\left(\frac{\alpha}{\pi}\right)^4\int_{m_\pi}^{E}dW_1\,dW_2\int_{-1}^{1}d\cos\theta_1'\int_{E_-'}^{E_+'}dE_1'\frac{|\mathbf{q}_1||\mathbf{q}_2|}{|\mathbf{q}_1+\mathbf{q}_2|\sin\theta_{1+2}\sin\theta_1'\sin\varphi_{1+2}}\sum_{b=\pm1}\frac{\mathfrak{D}}{(k_1^2k_2^2)^2}, \tag{6.4}$$

where

$$\begin{aligned}\mathfrak{D}=&\left[2p_1\cdot p_2+\frac{p_1\cdot(k_1-2q_1)p_2\cdot(k_1-q_1+q_2)}{2k_1\cdot q_1-k_1^2}+\frac{p_1\cdot(k_1-2q_2)p_2\cdot(k_1+q_1-q_2)}{2k_1\cdot q_2-k_1^2}\right]^2\\&+\tfrac{1}{4}k_1^2\left[2p_2+\frac{(k_1-2q_1)p_2\cdot(k_1-q_1+q_2)}{2k_1\cdot q_1-k_1^2}+\frac{(k_1-2q_1)p_2\cdot(k_1+q_1-q_2)}{2k_1\cdot q_2-k_1^2}\right]^2\\&+\tfrac{1}{4}k_2^2\left[2p_1+\frac{(k_1-q_1+q_2)p_1\cdot(k_1-2q_1)}{2k_1\cdot q_1-k_1^2}+\frac{(k_1+q_1-q_2)p_1\cdot(k_1-2q_2)}{2k_1\cdot q_2-k_1^2}\right]^2\\&+\tfrac{1}{16}k_1^2k_2^2\left[16+\frac{4(k_1-2q_1)\cdot(k_1-q_1+q_2)}{2k_1\cdot q_1-k_1^2}+\frac{(k_1-2q_1)^2(k_1-q_1+q_2)^2}{(2k_1\cdot q_1-k_1^2)^2}\right.\\&\left.+\frac{4(k_1+q_1-q_2)\cdot(k_1-2q_2)}{2k_1\cdot q_2-k_1^2}+\frac{(k_1+q_1-q_2)^2(k_1-2q_2)^2}{(2k_1\cdot q_2-k_1^2)^2}+\frac{2(k_1-2q_1)\cdot(k_1-2q_2)(k_1-q_1+q_2)\cdot(k_1+q_1-q_2)}{(2k_1\cdot q_1-k_1^2)(2k_1\cdot q_2-k_1^2)}\right],\end{aligned} \tag{6.5}$$

Table V. The cross section $d\sigma/d\Omega_1 d\theta_2$ for the process $e+e\rightarrow e+e+\mu^+ +\mu^-$ calculated in the equivalent-photon approximation for $\theta_1=5.7°$, 30°, 90°, and $E=1$ GeV (Brodsky, Kinoshita, and Terazawa, 1971b).

θ_1 (deg)	θ_2 (deg) 1.15	2.9	11.5	17.2	35.7	60	92.9	120	144.3	151.4	157.1	162.8	177.1
	$d\sigma_{ee\rightarrow ee\mu^+\mu^-}/d\Omega_1 d\theta_2$ (10^{-33}cm²/sr rad)												
5.7	4.0	6.9	6.1	4.9	2.9	1.71	1.78	2.7	5.7	6.8	10.0	15.4	61
30	0.133	0.49	0.79	0.79	0.60	0.46	0.45	0.59	0.98	1.20	1.41	1.71	0.70
90	0.0123	0.066	0.27	0.27	0.21	0.144	0.124	0.144	0.21	0.23	0.25	0.27	0.066

W_1, W_2 are pion energies defined by (5.10), (5.12), and

$$\cos\theta_{1+2}=\frac{\mathbf{p}_1\cdot(\mathbf{q}_1+\mathbf{q}_2)}{|\mathbf{p}_1||\mathbf{q}_1+\mathbf{q}_2|},$$

$$\cos\beta=\frac{(2E-W_1-W_2)^2-|\mathbf{q}_1+\mathbf{q}_2|^2-2E_1'(2E-W_1-W_2)}{2|\mathbf{p}_1'||\mathbf{q}_1+\mathbf{q}_2|},$$

$$\cos\varphi_{1+2}=\frac{\cos\beta-\cos\theta_1'\cos\theta_{1+2}}{\sin\theta_1'\sin\theta_{1+2}},$$

$$E_\pm'=\frac{(2E-W_1-W_2)^2-|\mathbf{q}_1+\mathbf{q}_2|^2}{2[2E-W_1-W_2+|\mathbf{q}_1+\mathbf{q}_2|\cos(\theta_1'\pm\theta_{1+2})]}. \tag{6.6}$$

The summation in (6.4) is over the possibilities $b=+1$ and -1 in

$$\varphi_1=\varphi_{1+2}-b\cos^{-1}\left(\frac{|\mathbf{q}_1|^2\sin^2\theta_1+|\mathbf{q}_1+\mathbf{q}_2|^2\sin^2\theta_{1+2}-|\mathbf{q}_2|^2\sin^2\theta_2}{2|\mathbf{q}_1|\sin\theta_1|\mathbf{q}_1+\mathbf{q}_2|\sin\theta_{1+2}}\right),$$

$$\varphi_2=\varphi_{1+2}+b\cos^{-1}\left(\frac{|\mathbf{q}_2|^2\sin^2\theta_2+|\mathbf{q}_1+\mathbf{q}_2|^2\sin^2\theta_{1+2}-|\mathbf{q}_1|^2\sin^2\theta_1}{2|\mathbf{q}_2|\sin\theta_2|\mathbf{q}_1+\mathbf{q}_2|\sin\theta_{1+2}}\right), \tag{6.7}$$

where φ_1 and φ_2 are the azimuthal angles of $\mathbf{q}_1$ and $\mathbf{q}_2$, respectively, with respect to the plane containing $\mathbf{p}_1'$ and the incident beams.

In Fig. 11 and Table VII we show the ψ dependence of the cross section $d\sigma/d\cos\theta_1 d\cos\theta_2 d\psi$ for typical angles $\theta_1=90°$, $\theta_2=60°$; $\theta_1=30°$, $\theta_2=120°$; $\theta_1=5.7°$, $\theta_2=157°$; $\theta_1=5.7°$, $\theta_2=174.3°$ all at $E=1$ GeV. It is seen that all these curves behave approximately as ψ^{-1} for large ψ [$>2°\simeq(m_e/E)^{1/2}$]. Similar behavior is likely to be observed for other combinations of θ_1 and θ_2. If we integrate this cross section $d\sigma/d\cos\theta_1 d\cos\theta_2 d\psi$ over ψ, the result can be compared with $d\sigma^{(0)}/d\Omega_1 d\theta_2$ of (5.11) obtained in the equivalent-photon approximation where the superscript (0) refers to the equivalent-photon approximation as in (3.10). Note that the ψ^{-1} behavior for $(m_e/E)^{1/2}\lesssim\psi\lesssim1$ leads to a logarithmic factor $\ln(E/m_e)$ in $\int(d\sigma/d\cos\theta_1 d\cos\theta_2\, d\psi)\,d\psi$. Note also that the relation between this integral and $d\sigma^{(0)}/d\Omega_1 d\theta_2$ is not straightforward because a certain averaging over the azimuthal angles has to be made in deriving the latter [as is seen from (3.5)]. Nevertheless this integral and $(2\pi/\sin\theta_2)d\sigma^{(0)}/d\Omega_1 d\theta_2$ are expected to be of the same order of magnitude. The results are shown in Table VIII. The exact results are in fact in rough agreement with (although somewhat smaller than) the equivalent-photon results.

The most significant feature seen from Fig. 11 and Tables VII and VIII is that pion pairs produced by the two-photon process are much more noncoplanar than what is implied by the quantity $(m_e/E)^{1/2}$. Although results for only four sets of θ_1, θ_2 are given, the strong deviation from coplanarity seems to be a general feature judging from the almost identical shape of curves of Fig. 11 for different values of θ_1 and θ_2. It is not difficult to understand this if we recall that the largeness of the coplanarity angle is closely related to (and in fact more or less determined by) the largeness of the photon–photon axis angle θ described earlier.

From Table VIII it is seen that 40–50% of all pion pairs are emitted with a coplanarity angle greater than

Table VI. The ratios of the π^0 and η production cross sections calculated with the cutoff $\theta>\theta_{\min}$, where θ is the proton–photon axis angle defined by (6.2), to the corresponding total cross sections (Brodsky, Kinoshita, and Terazawa, 1971b). $E=1$ GeV.

X	$\theta_{\min}$ (rad) 0	$m_e/E=0.511\times10^{-3}$	$(m_e/E)^{1/2}=0.0226$	$(m_e/E)^{1/4}=0.150$	$(m_e/E)^{1/8}=0.388$
	$\sigma_{ee\rightarrow eeX}{}^{\text{exact}}(\theta>\theta_{\min})/\sigma_{ee\rightarrow eeX}{}^{\text{exact}}$				
π^0	1.00	0.98	0.78	0.48	0.30
η	1.00	1.00	0.83	0.52	0.33

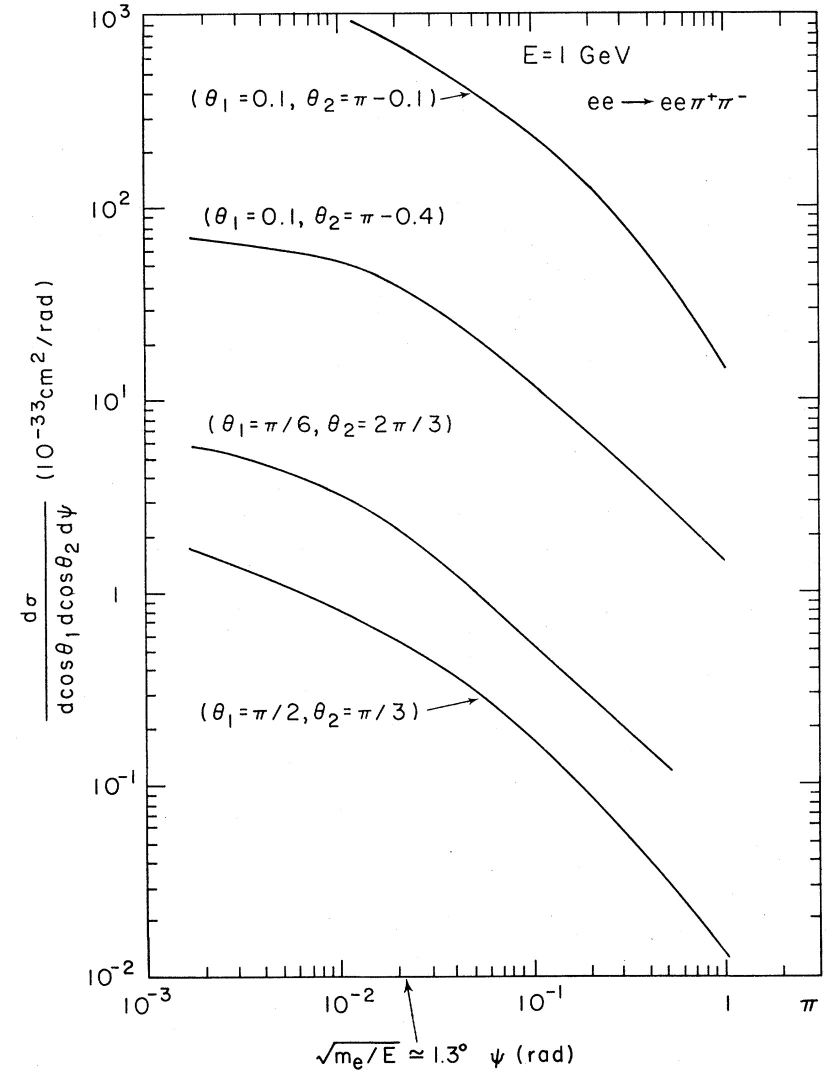

FIG. 11. Dependence of the cross section $d\sigma/d\cos\theta_1 d\cos\theta_2 d\psi$ on the coplanarity angle ψ for typical angles $\theta_1=90°$, $\theta_2=60°$; $\theta_1=30°$, $\theta_2=120°$; $\theta_1=5.7°$, $\theta_2=157°$; $\theta_1=5.7°$, $\theta_2=174.3°$ (Grammer and Kinoshita, 1972). $E=1$ GeV.

12°. If we assume that this result holds for other pairs of θ_1 and θ_2 as well, then a sizable fraction of the noncoplanar two-charged-particle events which tend to be classified as multiple production events from the one-photon annihilation process may actually have to be regarded as pion (or muon) pairs produced by the two-photon process. Experimentalists should be increasingly cautious not to mistake some two-photon events for one-photon annihilation events as the beam energy increases beyond 1 GeV.

Cheng and Wu (1971a) have also calculated numerically the differential cross section for $\pi^+\pi^-$-pair production

TABLE VII. The cross section $d\sigma/d\cos\theta_1 d\cos\theta_2 d\psi$ for the process $e+e\rightarrow e+e+\pi^++\pi^-$, where ψ is the coplanarity angle defined by (6.3) (Grammer and Kinoshita, 1972). $E=1$ GeV.

(θ_1, θ_2) (deg)	ψ (deg) 0.1	0.5	1.0	5.0	12	30	60
	$d\sigma_{ee\rightarrow ee\pi^+\pi^-}{}^{\text{exact}}/d\cos\theta_1\, d\cos\theta_2\, d\psi$ $(10^{-33}\text{cm}^2/\text{rad})$						
(5.7, 174.3)	1000.	990.	790.	270.	130.	39.	15.
(5.7, 157.1)	69.	53.	43.	16.	7.0	2.8	1.4
(30, 120)	5.6	3.4	2.3	0.61	0.30	0.12	
(90, 60)	1.7	0.86	0.64	0.20	0.086	0.032	0.013

Table VIII. The cross section $\int (d\sigma/d\cos\theta_1 d\cos\theta_2 d\psi)d\psi$ for the process $e+e\rightarrow e+e+\pi^++\pi^-$ obtained by graphical integration of the curves of Fig. 11 where ψ is the coplanarity angle defined by (6.3). $E=1$ GeV. Integrations are carried out for $\psi>0°$ and $\psi>12°$. For comparison the cross sections $d\sigma/d\cos\theta_1 d\cos\theta_2$ calculated in the equivalent-photon approximation are also shown. All values have been taken from Grammer and Kinoshita (1972).

	(θ_1, θ_2) (deg)			
	(5.7, 174.3)	(5.7, 157.1)	(30, 120)	(90, 60)
	$d\sigma_{ee}\sigma_{ee\pi^+\pi^-}/d\cos\theta_1 d\cos\theta_2 (10^{-33}\text{cm}^2)$			
Exact	130.	8.5	0.32	0.100
Exact ($\psi>12°$)	52.	4.5	0.18	0.041
e.p. approx.	170.	8.6	0.28	0.094
Exact ($\psi>12°$)/Exact	0.40	0.53	0.57	0.40
Exact/e.p. approx.	0.76	0.99	1.14	1.06

tion by the two-photon process and found that the equivalent-photon approximation is never valid in a certain kinematical region. Their differential cross section $d\sigma/d\mathbf{q}_1 d\mathbf{q}_2$ contains all the kinematical variables with respect to the produced pion-pair, certainly including the coplanarity angle ψ. As we have seen in the exact calculation of $d\sigma/d\cos\theta_1 d\cos\theta_2 d\psi$, the relation $\psi\simeq 0$ which is implied by the equivalent-photon picture is fairly strongly violated in the sense that the average ψ is roughly $(m_e/E)^{1/4}$ instead of $(m_e/E)^{1/2}$. It is therefore not surprising that their criterion for the validity of the equivalent-photon approximation, which is somewhat too restrictive, is never satisfied except in special instances. Thus we find some consistency between these two calculations (Brodsky, Kinoshita, and Terazawa, 1971b; Cheng and Wu, 1971a) rather than contradiction. We believe that the equivalent-photon results given in Sec. V, which do not involve the coplanarity angle ψ, would not be changed much by exact calculations. See the recent paper by Budnev, Ginzburg, Meledin, and Serbo (1972b) for more detailed discussion on this point.

Recently Grammer and Kinoshita (1972) have improved the numerical calculation of $d\sigma/d\cos\theta_1 d\cos\theta_2 d\psi$. We quote their results in Tables VII and VIII and in Fig. 11. They have also calculated the differential cross section for $e+e\rightarrow e+e+\mu^++\mu^-$. The details are found in their paper.

Brown and Lyth (1973) have also made an extensive numerical calculation of the cross sections for $e+e\rightarrow e+e+\pi^++\pi^-$. Their conclusion is that the equivalent-photon approximation is very good if both electron scattering angles are less than 0.1 rad, but is 20%–40% too big if either angle is integrated over. See the details in their paper.

Bonneau and Martin (1973) calculated both the differential cross section $d\sigma/ds$ and the total cross section for $e+e\rightarrow e+e+\mu^++\mu^-$ almost exactly in their analytical way. Their result is between the approximated results of Brodsky, Kinoshita, and Terazawa (1971b) and of Baier and Fadin (1971c). They concluded that the approximation of Baier and Fadin is not very good at low energy although it becomes better and better as the energy grows. On the other hand, the equivalent-photon result of Brodsky, Kinoshita, and Terazawa is about 20% larger than theirs in a range of energy up to 100 GeV.

VII. MORE ON SINGLE-MESON PRODUCTIONS

In the previous sections we have reviewed the various (rather kinematical) features which are common to every two-photon process. In this and the following seven sections we shall discuss several theoretical aspects of dynamical features which are characteristic for an individual two-photon process.

The simplest of all hadron productions by the two-photon process is the single-meson (or meson resonance) production

$$e+e\rightarrow e+e+M, \tag{7.1}$$

where M is any $C=+$ meson such as π^0, η, η' (or X^0), ϵ (or σ), etc.

A. Measurement of the $M\rightarrow\gamma\gamma$ Decay Widths

Over ten years ago, Low (1960) proposed the process

$$e+e\rightarrow e+e+\pi^0 \tag{7.2}$$

as a means of measuring the π^0 lifetime. Since, as we have seen in Sec. IV [see (4.4)], the cross section for this process is proportional to $\pi^0\rightarrow 2\gamma$ decay width in the equivalent-photon approximation, a measurement of the cross section gives the π^0 lifetime directly but approximately. The present experimental error for the $\pi^0\rightarrow\gamma\gamma$ decay width is 12% (Particle Data Group, 1972) while the theoretical error lying in the equivalent-photon approximation is greater than 10% (see Table I). Therefore, in order to improve the present accuracy of the π^0 lifetime via the experiment (7.2), the original proposal by Low should be modified in the following way: Detect the scattered electrons within a small forward angle $\theta_{\max}[\lesssim(m_e/E)^{1/2}\ll 1]$ and compare the observed cross section with the corresponding cross section calculated from (3.10), (3.18), and (4.3). Since the smallness of $\theta_{\max}$ guarantees that $k_{1,2}^2\simeq 0$ and

$d\sigma^{(0)} \simeq d\sigma$, we can then suppress the theoretical error in this way as much as we wish. Experiments of the missing mass type

$$e+e \to e+e+ \text{(missing mass)} \tag{7.3}$$

may be useful for identifying the produced state as π^0, η, η', ϵ (or σ), etc. It does not seem to be too difficult to obtain better experimental values (or upper bounds) for the $\eta \to \gamma\gamma$, η' (or X^0)$\to\gamma\gamma$, ϵ (or σ)$\to\gamma\gamma$ decay widths from this type of experiment. It will be instructive (Brodsky, Kinoshita, and Terazawa, 1971b) to derive an upper limit of the partial decay width $\Gamma_{\eta' \to \gamma\gamma}$ from the multiple-particle production data reported by the Frascati groups (Bacci *et al.*, 1971, 1972). In analogy with (4.4) we obtain

$$\begin{aligned}\sigma_{ee \to ee\eta'} &\simeq 16\alpha^2 \Gamma_{\eta' \to \gamma\gamma} [\ln (E/m_e)]^2 f(m_{\eta'}/2E) m_{\eta'}^{-3} \\ &= (2.5 \times 10^{-35}\ \text{cm}^2) \times \Gamma_{\eta' \to \gamma\gamma},\end{aligned} \tag{7.4}$$

for $E=1$ GeV, and with $\Gamma_{\eta' \to \gamma\gamma}$ in keV. On the other hand, we have the inequalities

$$\sigma_{ee \to ee2\pi^+2\pi^-\pi^0} \geq \sigma_{ee \to ee\eta'} R_{\eta' \to \eta\pi^+\pi^-} R_{\eta \to \pi^+\pi^-\pi^0} \tag{7.5}$$

and

$$\begin{aligned}\sigma_{ee \to ee\pi^+\pi^- + \text{neutrals}} \geq \sigma_{ee \to ee\eta'} (&R_{\eta' \to \eta\pi^+\pi^-} R_{\eta \to \text{neutrals}} \\ &+ R_{\eta' \to \eta\pi^0\pi^0} R_{\eta \to \pi^+\pi^-\pi^0} + R_{\eta' \to \rho\gamma} R_{\rho \to \pi^+\pi^-}),\end{aligned} \tag{7.6}$$

R being various branching ratios of the η' and η decays. Using the data from Frascati,[13] $\sigma_{e^+e^- \to 2\pi^\pm + \text{any neutrals}} = (8.5 \pm 4) \times 10^{-33}$ cm^2 and $\sigma_{e^+e^- \to 4\pi^\pm + \text{any neutrals}} = (8.5 \pm 3) \times 10^{-33}$ cm^2 for $E=1$ GeV, the present particle data (Particle Data Group, 1972), and Eqs. (7.4)–(7.6), we obtain the upper limit

$$\Gamma_{\eta' \to \gamma\gamma} \leq 500 \pm 200\ \text{keV}. \tag{7.7}$$

This value is much larger than the upper limit derived from the present particle data, $\Gamma_{\eta' \to \gamma\gamma} < (72 \pm 12)$ keV. This is of course not surprising since the Frascati experiment has not been performed to pick up any information on $\Gamma_{\eta' \to 2\gamma}$. A prediction of the decay rate from broken SU(3) is (Baracca and Bramon, 1967; Chan, Clavelli, and Torgerson, 1969; Suura, Walsh, and Young, 1972)[14]

$$\Gamma_{\eta' \to \gamma\gamma} = \begin{cases} \text{6 keV for the fractionally charged quark model} \\ \text{25.6 keV for the Han–Nambu (integrally charged three triplet) model.} \end{cases} \tag{7.8}$$

Even the smaller value leads to a prediction for the cross section of the process

$$e+e \to e+e+\eta' \searrow \pi^+ + \pi^- + \pi^+ + \pi^- + \pi^0,\ \text{etc,.} \tag{7.9}$$

of approximately the same magnitude at $E > 1.2$ GeV as that of the η-meson production shown in Fig. 4. An anomalously large coupling of the η' to two photons compared with (7.8) could yield a substantial number of events with four charged-particle tracks even at the present Frascati energy.

B. Low-Energy Theorem on the $\pi^0\gamma\gamma$ Vertex

An absolute determination of the coupling constant of the pseudoscalar mesons to two photons has been of much theoretical interest during the last several years. Let us define the $\pi^0\gamma\gamma$ vertex function $F(q^2, k^2)$ by[15,16]

$$\begin{aligned}M_{\mu\nu}(q, k) &= i \int dx \exp(-iqx) \langle P \mid T^*(J_\mu(x), J_\nu(0)) \mid 0 \rangle \\ &= \epsilon_{\mu\nu\alpha\beta} q^\alpha k^\beta F(q^2, k^2).\end{aligned} \tag{7.10}$$

In this section and in Sec. XIV we shall call the photon momenta q and k, for convenience, instead of k_1 and k_2 as defined in Sec. II. P is the pion momentum ($P = q+k$). From (4.2) and $g_{\pi^0\gamma\gamma} = F(0, 0)/4$, the $\pi^0 \to \gamma\gamma$ decay width is given by

$$\Gamma_{\pi^0 \to \gamma\gamma} = [\,| e^2 F(0, 0)|^2 / 64\pi] m_\pi^3. \tag{7.11}$$

In 1967 Sutherland (1967) and Veltman (1967) showed that the ordinary PCAC hypothesis (Nambu, 1960; Gell-Mann and Levy, 1960) and the algebra of currents (Gell-Mann, 1964) lead to the low-energy theorem

$$F(0, 0)|_{P^2=0} = 0. \tag{7.12}$$

This means that the $\pi^0 \to \gamma\gamma$ decay constant $g_{\pi^0\gamma\gamma} = F(0, 0)/4$ [see (4.1)] vanishes in the limit $m_\pi^2 \to 0$. Since $F(0, 0)|_{P^2=m_\pi^2}$ is finite experimentally and m_π^2 is small, this theorem was a puzzle until Bell and Jackiw (1969), and Adler (1969) found the PCAC anomaly[17] in the triangle diagram with a charged-fermion loop for the axial-vector A_μ vertex function. They have shown that the ordinary PCAC should be modified by adding the anomalous term which is of order e^2:

$$\partial_\mu A_\pi{}^\mu = f_\pi m_\pi^2 \phi_\pi + (e^2 S / 16\pi^2) \epsilon_{\alpha\beta\gamma\delta} : F^{\alpha\beta} F^{\gamma\delta} : \tag{7.13}$$

where f_π is the pion-leptonic-decay constant ($f_\pi \simeq 95$ MeV) and S is the anomalous constant. In the quark

[13] We are considering the possibility that some of two-photon events might be mistaken for one-photon events without detecting the scattered electrons in the experiments.

[14] These two predictions which differ much from each other should be subjected to experimental checks.

[15] This vertex function differs from the form factor defined in Sec. IV only by its normalization [see (4.7)].

[16] The electromagnetic current J_μ is defined here by the usual J_μ divided by e (electric charge).

[17] See also Schwinger (1951).

models S is predicted to be[18]

$$S=\begin{cases} 1/6 \text{ for the original Gell-Mann–Zweig (fractionally charged single triplet) quark model} \\ 1/2 \text{ for } \begin{cases}\text{the original Sakata (integrally charged triplet) model, the Han–Nambu (integrally charged three triplet) model (1965), and the fractionally charged three triplet quark model.}\end{cases}\end{cases} \qquad (7.14)$$

The low-energy theorem (7.12) must be replaced by (Bell and Jackiw, 1969; Alder, 1969)

$$F(0,0)|_{P^2=0} = -S/(2\pi^2 f_\pi). \qquad (7.15)$$

Thus, if we assume the existence of the Bell–Jackiw–Adler anomaly, the $\pi^0 \to \gamma\gamma$ decay-puzzle is solved completely. The present experimental data (Particle Data Group, 1972) $\Gamma_{\pi^0\to\gamma\gamma} = 7.8 \pm 0.9$ eV give $S \simeq 0.5$, which favors the Han–Nambu and fractionally charged three triplet quark models. Recently, Crewther (1972) has shown that the anomalous constant S is determined by a product of high energy electroproduction and electron–neutrino annihilation cross sections, by assuming Wilson's theory of broken scale invariance (1969).

The anomaly also affects the low-energy theorem on the $\eta \to \gamma\gamma$ decay. However, the low-energy theorem is less practical in this decay than in the $\pi^0 \to \gamma\gamma$ decay because the extrapolation needed to reach the physical constant for $P^2 = m_\eta^2$ is much more demanding.

More precise measurements of the decay width of pseudoscalar mesons are extremely interesting and may possibly be performed by the two-photon process $e+e \to e+e+\pi^0$, η, η' etc. (see Sec. VIIA above).

C. Predictions for the $\sigma\gamma\gamma$ Coupling Constant

The coupling constant of σ or ϵ (700) and two photons is theoretically as interesting as the $\pi^0 \to \gamma\gamma$ decay constant. In order to measure the former, it is desirable to do either the experiment of (7.3) type or the one of the type

$$e+e \to e+e+\pi^+ + \pi^- \qquad (7.16)$$

in which both pion momenta are measured in order to find the broad σ (or ϵ) enhancement in the mass squared of the pion pair. It should be noticed here that the $\sigma\gamma\gamma$ coupling constant can be directly measured in the experiment (7.3) while what can be measured in the experiment (7.16) is the product of the $\sigma\gamma\gamma$ and $\sigma\pi^+\pi^-$ coupling constants but not the $\sigma\gamma\gamma$ coupling constant alone. Theoretical predictions for the product of these coupling constants and their consequences will be discussed in Sec. VIII.

Predictions for the $\sigma\gamma\gamma$ coupling constant have been made by reasoning in parallel with the $\pi^0 \to \gamma\gamma$ decay constant argument. We can play the same game as in Sec. VIIB above by replacing PCAC by PCDC (Partially Conserved Dilation Current) (Mack, 1968; Carruthers, 1971). Kleinert, Staunton, and Weisz (1972) showed that if the $\sigma(700)$ meson dominates the trace of energy momentum tensor $\theta_\mu{}^\mu$, then the $\sigma\gamma\gamma$ coupling constant $g_{\sigma\gamma\gamma}$ defined by

$$\mathcal{L}_{\sigma\gamma\gamma} = -(e^2 g_{\sigma\gamma\gamma}/2!)\phi_\sigma F_{\mu\nu}F^{\mu\nu} \qquad (7.17)$$

should vanish in the soft-meson limit. However, Crewther (1972) and, independently, Chanowitz and Ellis (1972) have recently pointed out that the PCDC anomaly (Callan, 1970; Symanzik, 1970) affects the low-energy theorem on the coupling and that $g_{\sigma\gamma\gamma}$ does not vanish. Furthermore they have predicted the coupling constant to be

$$g_{\sigma\gamma\gamma} \simeq R/(12\pi^2 f_\sigma), \qquad (7.18)$$

where R is the asymptotic ratio of $\sigma(e^+e^- \to \text{hadrons})$ to $\sigma(e^+e^- \to \mu^+\mu^-)$ at high energies and f_σ is defined by

$$\langle 0 | \theta_\mu{}^\mu(0) | \sigma \rangle = m_\sigma^2 f_\sigma. \qquad (7.19)$$

From this result Chanowitz and Ellis (1972; 1973) estimated the $\sigma \to \gamma\gamma$ decay width to be

$$\Gamma_{\sigma\to\gamma\gamma} \simeq 0.2\, R^2 \text{ keV},$$

for

$$m_\sigma \simeq 700 \text{ MeV} \quad \text{and} \quad f_\sigma \simeq 150 \text{ MeV}. \qquad (7.20)$$

This can be checked by the two-photon process (7.3) whose cross section is given in the equivalent-photon approximation by [see (4.4)]

$$\sigma_{ee\to ee\sigma} \simeq \frac{16\alpha^2 \Gamma_{\sigma\to\gamma\gamma}}{m_\sigma^3}\left[\left(\ln\frac{E}{m_e} - \frac{1}{2}\right)^2 f\left(\frac{m_\sigma}{2E}\right) + \left(\ln\frac{E}{m_e} - \frac{1}{2}\right) g\left(\frac{m_\sigma}{2E}\right) + h\left(\frac{m_\sigma}{2E}\right)\right]. \qquad (7.21)$$

D. Other $C=+$ Mesons

Any other mesons with positive charge conjugation can also be produced in the two-photon process (7.1). They include the $A_1(1070)$, $f(1260)$, $A_2(1320)$, and so on. The decay widths of these mesons into two photons are of great theoretical interest and will be measured in the near future. In the production of these mesons we can also use formula (4.4) as an approximation, or, more precisely, the combination of (3.10), (3.18), and (4.3) in order to obtain the decay widths from the measured cross sections. Notice that we should multiply the right-hand side of (4.3) by $2J+1$ in the case of spin-J-meson production. The $f \to \gamma\gamma$ decay width has been predicted by Renner (1971) in his model for vector–vector–tensor meson

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[18] A detailed comparison of the effective values for S predicted in various models can be found in the papers by Okubo (1969), by Suura, Walsh, and Young (1972), and by Bardeen, Fritzsch, and Gell-Mann (1972).

vertices. The two-photon cross sections for production of various $C=+$mesons other than π^0 and η have been calculated by Bramon and Greco (1971) and in the equivalent-photon approximation.

E. Effects of the Form Factor

After the $M\to\gamma\gamma$ decay widths have been measured precisely by the two-photon process $e+e\to e+e+M$ in which the scattered electrons are detected within a small forward angle, interest will turn to the effects of the $M\gamma\gamma$ form factors. The best way to detect this effect is to detect both scattered electrons and also measure their momenta at a considerable cost of the large cross section in order to find the virtual-photon masses in each meson production. This experiment will be discussed in Sec. XIV. There is, however, an indirect way of doing this without losing the largeness of the cross sections. Since the $M\to\gamma\gamma$ decay width is supposed to be known very accurately by the previous experiment (say within 1%), we can calculate exactly the cross section within a few percent accuracy by assuming that the form factor is constant, i.e., $F(k_1^2, k_2^2)=1$ [see (4.8)]. Any deviation of the observed cross section from the calculated cross section implies an effect due to the form factor, i.e., $F(k_1^2, k_2^2)\neq 1$ for $k_1^2\neq 0$ or $k_2^2\neq 0$. Since, as we have seen in Sec. IV, the cross section calculated with the form factor suggested by ρ dominance [see (4.9)] is smaller roughly by 10% than the cross section without the form factor, this way of detecting the form-factor effect will be practical, though not the most desirable. Young (1970) has proposed this method together with the other method of direct measurement of the form factor in the process $e^++e^-\to\pi^0+\gamma$. A detailed numerical calculation of the cross sections for π^0 and η production has been made by Parisi and Kessler (1971, 1972) for various form factors with different k_1^2, k_2^2 dependence.

VIII. STRONG-INTERACTION MODIFICATIONS OF $\gamma+\gamma\to\pi^++\pi^-$

A. General Features and Measurement of the $\pi\pi$ Scattering Phase Shift

One of the most basic processes which can be studied by electron–electron collisions is $\gamma(k_1)+\gamma(k_2)\to\pi^+(q_1)+\pi^-(q_2)$. In general, the full Compton amplitude for k_1^2, k_2^2 spacelike and

$$s=(k_1+k_2)^2=(q_1+q_2)^2>4m_\pi^2 \tag{8.1}$$

can be studied. In this section we shall concentrate on the case of (almost) real photons, postponing the case of highly virtual photons to section XI.

In the Born approximation (no strong interactions) the amplitude for $\gamma+\gamma\to\pi^++\pi^-$ is $e^2\epsilon_\mu(k_1)M_B^{\mu\nu}\epsilon_\nu(k_2)$, where

$$M_B^{\mu\nu}=-2g^{\mu\nu}+\frac{(2q_1^\mu-k_1^\mu)(2q_2^\nu-k_2^\nu)}{2q_1\cdot k_1}+\frac{(2q_1^\nu-k_2^\nu)(2q_2^\mu-k_1^\mu)}{2q_2\cdot k_1}. \tag{8.2}$$

In the general case (including strong interactions), gauge invariance, parity conservation, and time-reversal invariance limit the complete structure to two independent amplitudes. A convenient parametrization is (Brodsky, Kinoshita, and Terazawa, 1971b)

$$(1/e^2)M^{\mu\nu}=M_B^{\mu\nu}B(s,t,u)+4(g^{\mu\nu}k_1\cdot k_2-k_1^\nu k_2^\mu)A(s,t,u), \tag{8.3}$$

where $t=(k_1-q_1)^2=(k_2-q_2)^2$ and $u=(k_1-q_2)^2=(k_2-q_1)^2$. This form has only the explicit poles dictated by the Born contribution. The Thomson limit for forward Compton scattering demands $B(0, m_\pi^2, m_\pi^2)=1$, of course. In addition to the usual conditions following from crossing symmetry (A and B are even under $t\leftrightarrow u$), unitarity to order e^2 requires

$$\operatorname{Im} F_{\gamma\gamma\to\pi\pi}{}^J\propto F_{\gamma\gamma\to\pi\pi}{}^{J*}F_{\pi\pi\to\pi\pi}{}^J. \tag{8.4}$$

This relation holds for each amplitude F^J of given angular momentum J (in the photon–photon center-of-mass system) and isotopic spin if s is in the region of $\pi\pi$ elastic scattering ($4m_\pi^2<s<16m_\pi^2$). Accordingly, the S-wave parts of A and B, both of which contribute to the $J=0$ amplitude, each contain the factor

$$\exp\left(\frac{s}{\pi}\int_{4m_\pi^2}^\infty\frac{\delta_0(s')ds'}{s'(s'-s)}\right), \tag{8.5}$$

where δ_0 can be identified with the S-wave $\pi\pi$ phase shift in the elastic region. In general, however, the A and B amplitudes can be further multiplied by entire functions and still satisfy the unitarity condition (8.4).

We also note that the general amplitude will contain additional contributions from all even-l resonances in the $\pi^+\pi^-$ system as well as t and u channel exchange contributions. Since $\gamma+\gamma\to\pi^++\pi^-$ is now readily measurable in $e+e\to e+e+\pi^++\pi^-$, this process promises to be an ideal new testing ground for various models of hadronic interactions and the search for structure in the $\pi^+\pi^-$ system.

In terms of the A and B functions we have

$$\frac{d\sigma_{\gamma\gamma\to\pi^+\pi^-}}{dt}=\frac{4\pi}{s(1-4m_\pi^2/s)^{1/2}}\frac{d\sigma_{\gamma\gamma\to\pi^+\pi^-}}{d\Omega_{\text{c.m.}}}=(2\pi\alpha^2/s^2)[(1-2r+2r^2)|B|^2+s^2|A|^2-2rs\operatorname{Re}(A^*B)], \tag{8.6}$$

where

$$r=m_\pi^2(k_1\cdot k_2)/2(k_1\cdot q_1)(k_1\cdot q_2)=m_\pi^2 s/(t-m_\pi^2)(u-m_\pi^2). \tag{8.7}$$

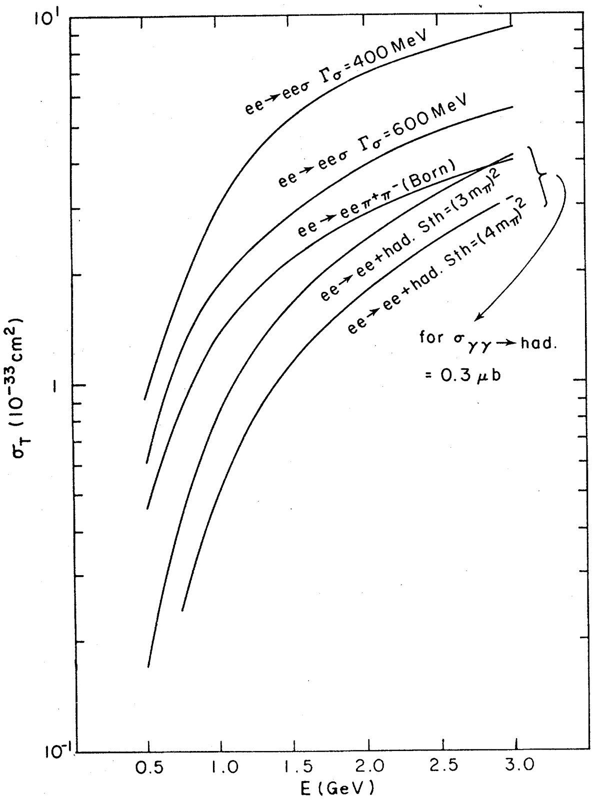

FIG. 12. Effects of hadronic interaction on the total two-photon cross sections. These curves [taken from Brodsky, Kinoshita, and Terazawa (1971b)] are given for (a) the Born cross section, (b) a completely isotropic ($J=0$) contribution from the σ-pole term plus an isotropic part of the Born amplitude (i.e., $B=1$) with $m_\sigma=700$ MeV, $\Gamma_\sigma=400$ and 600 MeV, and (c) estimated cross sections for multihadron production with assumed threshold $s_{\mathrm{th}}=(3m_\pi)^2$ and $(4m_\pi)^2$.

Notice that at threshold ($s=4m_\pi^2$, $r=1$) the cross section (8.6) is proportional to $|B-sA|^2$. This reflects the fact that at threshold the B term yields contributions only to the equal-helicity c.m. amplitude $M_{\lambda_1=\lambda_2}{}^{\mathrm{c.m.}}$, which is the entire contribution of the A term. This amplitude corresponds to an oppositely directed angular momentum and is the helicity amplitude which can contribute to the $J=0$ state. At high energies where $s\gg 4m_\pi^2/\sin^2\theta_{\mathrm{c.m.}}$, i.e., $r\ll 1$, the interference of the A and B terms disappears since then the B term contributes only to the unequal-helicity c.m. amplitude $M_{\lambda_1=-\lambda_2}{}^{\mathrm{c.m.}}$.

By inserting the formula (8.6) into Eq. (3.11) or the formula (8.3) into Eq. (2.10), we can express the differential cross section $d\sigma_{ee\to ee\pi^+\pi^-}/dsdt$ in terms of the A and B amplitudes. Thus we can measure these amplitudes and, consequently, the $\pi-\pi$ scattering phase shift by the two-photon process $e+e\to e+e+\pi^++\pi^-$. This interesting method of measuring the phase shift, which was originally proposed by DeCelles and Goehl (1969), has been more closely investigated by Brown and Lyth (Lyth 1971; Brown and Lyth, 1972; Cheng and Wu (1971b); Goble and Rosner, 1972; Khleskov, 1972; Schierholz and Sundermeyer 1972; Carlson and Tung, 1972; and Yndurain, 1972. All these authors have presented elaborate numerical results, which readers will find in the original papers, for the cross sections.

B. The σ-Contribution

There is a particular interest in understanding the $J=0$ partial-wave contribution to the $\gamma+\gamma\to\pi^++\pi^-$ process since this can yield information on $\pi-\pi$ scattering lengths and s-wave resonances, especially the broad σ or ϵ enhancement near 700 MeV. We shall define the coupling constant $g_{\sigma\pi\pi}$ in terms of the effective Lagrangian

$$\mathcal{L}_{\sigma\pi\pi}=-g_{\sigma\pi\pi}\phi_\pi{}^\dagger\phi_\pi\phi_\sigma. \tag{8.8}$$

Obviously, only the A term in (8.3) receives the σ contribution. What is relevant to the two-photon process $e+e\to e+e+\pi^++\pi^-$ is not the coupling constant $g_{\sigma\gamma\gamma}$ alone but the product $g_{\sigma\gamma\gamma}g_{\sigma\pi\pi}$ of coupling constants, as has been noted in Sec. VIIC. This product was first estimated by Sarker (1970) using a superconvergent (or finite-energy) sum rule for the helicity-flip amplitude of pion Compton scattering (Abarbanel and Goldberger, 1968). Taking account of contributions from σ and higher resonances, Sarker obtained a result $2g_{\sigma\gamma\gamma}g_{\sigma\pi\pi}\lesssim -2.1\pm 0.6$. (His definition of $g_{\sigma\gamma\gamma}$ differs from ours by a factor of 2.) Parisi and Testa (1971), and Schrempp-Otto, Schrempp, and Walsh (1971) have predicted different values of $g_{\sigma\gamma\gamma}g_{\sigma\pi\pi}$, starting with the same superconvergent sum rule. The reason for this is that the estimated values of $g_{\sigma\gamma\gamma}g_{\sigma\pi\pi}$ strongly depend on the way in which the sum rule is saturated. This can best be illustrated by the following example (Brodsky, Kinoshita, and Terazawa, 1971b). If one takes only the σ resonance into account in the superconvergent sum rule, one finds $2g_{\sigma\gamma\gamma}g_{\sigma\pi\pi}=-4$, which is by a factor of 2 different from Sarker's result. This result can also be obtained simply by requiring that the forward differential cross section fall faster at high energy than either the Born contribution or the σ contribution (A term) alone. Since $r=1$ for $\theta=0$, this means that we require

$$\begin{aligned}\lim_{s\to\infty}(B-sA)&=\lim_{s\to\infty}\{1-\tfrac{1}{2}(-g_{\sigma\gamma\gamma}g_{\sigma\pi\pi})\\&\qquad\times[s/(s-m_\sigma^2+im_\sigma\Gamma_\sigma)]\}\\&=0,\end{aligned} \tag{8.9}$$

which leads to $g_{\sigma\gamma\gamma}g_{\sigma\pi\pi}=-2$. Very recently Lyth (1972) has estimated the same product $g_{\sigma\gamma\gamma}g_{\sigma\pi\pi}$ to be five times smaller than this value from S-wave unitarization and claimed that the σ contribution is so small that the cross section for $\gamma+\gamma\to\pi^++\pi^-$ may well be approximated by the Born diagrams for small values of s.

Because of the ambiguities in the predicted values for $g_{\sigma\gamma\gamma}g_{\sigma\pi\pi}$ and because of the difficulty in estimating higher-resonance contributions, we would rather take a naive model and give representative examples of hadronic interactions on the total $ee \rightarrow ee\pi^+\pi^-$ cross section in Fig. 12. Curves are given for (a) the Born cross section using Eqs. (3.14) and (5.5), and (b) a completely isotropic ($J=0$) contribution from the σ-pole term as defined above plus an isotropic part of the Born amplitude (i.e., $B=1$). The results are shown for the simple form

$$\sigma_{\gamma\gamma\rightarrow\pi^+\pi^-} = \frac{2\pi\alpha^2}{s}\left(1-\frac{4m_\pi^2}{s}\right)^{1/2} \frac{m_\sigma^4}{(s-m_\sigma^2)^2+m_\sigma^2\Gamma_\sigma^2}, \tag{8.10}$$

assuming $m_\sigma = 700$ MeV, $\Gamma_\sigma = 400$ MeV and 600 MeV. As is seen from Fig. 12, the cross section including the σ-resonance effect may be larger than the Born cross section by a factor of about 2. A similar conclusion on the strong-interaction effect (the t-channel exchange effect instead of the s-channel resonance effect mentioned above) has been obtained by Manassah and Matsuda (1971) using a harmonic-oscillator model for hadrons (Nambu, 1970; Susskind, 1970).

More definitive information on the $\gamma+\gamma \rightarrow \pi^+ + \pi^-$ process, however, will require measurements of the s dependence (by pion-energy measurements or tagging the scattered electrons) and the angular distributions of the pion pair (see Sec. VIIIA). Some features of these distributions as seen in the laboratory have been discussed in Sec. V. Goble and Rosner (1972) conjectured that the maximum of the σ enhancement is shifted to about 450 MeV due to the various constraints on the s-wave amplitude in the low s region.

The phase-shift analysis of $\gamma+\gamma \rightarrow \pi^0+\pi^0$ was also proposed by Lyth (1971) and others although the experiment $e+e \rightarrow e+e+\pi^0+\pi^0$ is more difficult to carry out than is the $e+e \rightarrow e+e+\pi^+ + \pi^-$ one.

The effect of electromagnetic interactions in the $\pi^+\pi^-$ final state has been considered by Nandy (1972), especially from the interesting view point of pionium (the $\pi^+ - \pi^-$ electromagnetic bound state as a resonance in photon–photon scatterings).

Isaev and Khleskov (1972b) have investigated the process $\gamma+\gamma \rightarrow K+\bar{K}$ and estimated the cross section $\sigma_{ee\rightarrow eeK\bar{K}}$ to be 3.5×10^{-37} cm^2 for $E=1$ GeV.

IX. PRODUCTION OF AN ODD NUMBER OF SOFT PIONS

In this section we shall discuss the production of an odd number of pions by two real photons

$$\gamma+\gamma \rightarrow \pi^+ + \pi^- + \pi^0,\ 3\pi^0 \text{ etc.}, \tag{9.1}$$

in which all the pions are soft (their momenta are small). There is a great difference between the dynamics involved in this process and that in the two-photon production of an even number of charged soft pions

$$\gamma+\gamma \rightarrow n\pi^+ + n\pi^-, \qquad n=1, 2, 3, \cdots, \tag{9.2}$$

which will be discussed in Sec. X. In the process (9.2) the ordinary PCAC relation and the algebra of currents lead to meaningful results with nonvanishing amplitudes while in the process (9.1) the PCAC anomaly (see Sec. VIIB) affects the results seriously and, therefore, must be taken into account.

Aviv, Hari Dass, and Sawyer (1971) first found that the amplitude for the process $\gamma+\gamma \rightarrow 3\pi^0$ vanishes in the soft-pion limit ($p_{\pi^0} \rightarrow 0$) and that the amplitude for $\gamma+\gamma \rightarrow \pi^+ + \pi^- + \pi^0$ in the same limit is related to the PCAC anomaly (Alder 1969; Bell and Jackiw, 1969) and, therefore, can be written solely in terms of the $\pi^0 \rightarrow \gamma\gamma$ amplitude. Their first result: the vanishing $\gamma\gamma \rightarrow 3\pi^0$ amplitude was confirmed by Abers and Fels (1971) and by many others. However, their second result was controversial for some time. For example, Yao (1971) obtained a different result for the $\gamma\gamma \rightarrow \pi^+\pi^-\pi^0$ amplitude. This problem was finally solved by Terent'ev (1971b), Wong (1971), Adler, Lee, Treiman, and Zee (1971), Bacry and Muyts (1972), and Hari Dass (1972), who pointed out that the amplitudes for $\gamma+\gamma \rightarrow \pi^+ + \pi^- + \pi^0$ written in the previous papers are not gauge-invariant. The conclusion of these various authors is the following: (1) Both of the amplitudes for $\gamma+\gamma \rightarrow \pi^+ + \pi^- + \pi^0$ and $3\pi^0$ vanish when that π^0 momentum vanishes, while the charged-pion momenta are on the mass shell, (2) The amplitude for $\gamma+\gamma \rightarrow \pi^+ + \pi^- + \pi^0$ can be expressed in terms of those for $\pi^0 \rightarrow \gamma+\gamma$ and $\gamma \rightarrow \pi^+ + \pi^- + \pi^0$, and (3) the latter two amplitudes are simply related by $F_{\pi^0\rightarrow\gamma\gamma} = f_\pi^2 F_{\gamma\rightarrow\pi^+\pi^-\pi^0}$ [where $F_{\pi^0\rightarrow\gamma\gamma} = F(0,0)|_{P^2=0}$ defined in (7.10) and $F_{\gamma\rightarrow\pi^+\pi^-\pi^0}$ is defined by the soft-pion production amplitude $\mathfrak{M}(\gamma(k) \rightarrow \pi^0 + \pi^+(p) + \pi^-(q)) = iek^\alpha\epsilon^\beta p^\gamma q^\delta \epsilon_{\alpha\beta\gamma\delta} F_{\gamma\rightarrow\pi^+\pi^-\pi^0}$], due to the gauge invariance of the former amplitude. The proof of (3) in the presence of the PCAC anomaly has recently been given by Terent'ev (1971b; 1972b) and Adler, Lee, Treiman, and Zee (1971) although the relation between $\pi^0 \rightarrow \gamma\gamma$ and $\gamma \rightarrow \pi^+ + \pi^- + \pi^0$ was approximately derived by Kawarabayashi and Suzuki (1966) several years ago, neglecting the anomaly. A closely related discussion can be found in the paper by Wess and Zumino (1971). With the corrected version of the amplitudes for $\gamma+\gamma \rightarrow \pi^+ + \pi^- + \pi^0$ and $3\pi^0$, actual calculations of the cross sections for $e+e \rightarrow e+e+\pi^+ + \pi^- + \pi^0$ and $e+e \rightarrow e+e+3\pi^0$ have been done by Pratap, Smith, and Uy (1972), by Köberle (1972) who included hard-pion terms, and by Zee (1972). Unfortunately, these predicted cross sections are small ($\sim10^{-36}$ cm^2 for $E=1$ GeV) in the soft-pion region [$s \simeq (3m_\pi)^2$] where these soft-pion results should be tested. The result of Köberle (1972) shows, however, that the hard-pion cross section for these processes calculated from vector meson dominance may be large enough ($\sim10^{-34}$ cm^2 for $e+e \rightarrow e+e+\pi^+ + \pi^- + \pi^0$ at $E=1$ GeV and $\sim10^{-35}$ cm^2 for $e+e \rightarrow e+e+3\pi^0$ at $E=2$ GeV) to be measured in the near future. Terent'ev (1972a), Zee (1972), and Smith and Stanko (1972) have also calculated the cross section for $\pi^\pm$ production of $\pi^\pm\pi^0$ in the Coulomb field $\pi^\pm + Z \rightarrow Z + \pi^\pm + \pi^0$ to test the soft-pion results.

The author thanks Professor A. Zee for his comments on this section of the manuscript.

X. PRODUCTION OF EVEN NUMBER OF CHARGED SOFT PIONS

In the production of an even number of charged soft pions by two photons

$$\gamma+\gamma\to n\pi^{+}+n\pi^{-} \tag{10.1}$$

which is more practical experimentally than the process discussed in the previous section, we need not worry about the PCAC anomaly. Instead, there are a few ambiguities (Alder and Weisberger, 1968) in taking the soft-pion limit and in extrapolating the off-pion-mass-shell amplitude to the physical one. However, Terazawa (1971) has taken gauge invariance and the Thomson limit for forward Compton scattering as guiding principles to obtain a consistent result.

The successive application of the PCAC hypothesis (Nambu, 1960; Gell-Mann and Levy, 1960), the soft-pion technique, and the algebra of currents (Gell-Mann, 1964) makes it possible to reduce a pion pair (of momenta p and q) in the final state of the amplitude for (10.1) in the soft-pion limit $p, q\to 0$. Repeating the same procedure, we can express the amplitude in terms of the propagators of the vector and axial-vector currents, V_μ and A_μ, in the limit $p_i, q_i\to 0$ [16]:

$$\begin{aligned}\langle n\pi^{+}(p_i), n\pi^{-}(q_i) \mid T(J^\mu(x)J^\nu(0))\mid 0\rangle \underset{\text{soft pion}}{\longrightarrow} &\{-2(-3)^{n-1}[\prod_{i=1}^{n} 2\omega_{p_i}(2\pi)^3 2\omega_{q_i}(2\pi)^3]^{-1/2} f_\pi^{-2n}\}\\ &\times[\langle 0\mid T(V_3^\mu(x)V_3^\nu(0))\mid 0\rangle\\ &-\langle 0\mid T(A_3^\mu(x)A_3^\nu(0))\mid 0\rangle],\end{aligned} \tag{10.2}$$

where f_π ($\simeq$95 MeV) has been defined in (7.13). Using the spectral representations of the propagators given by Das, Gurlnik, Mathur, Low, and Young (1967), we obtain the following expression for the matrix element $M^{\mu\nu}$ for $\gamma+\gamma\to n\pi^{+}+n\pi^{-}$:

$$\begin{aligned}M^{\mu\nu}=i\int d^4x e^{-ik_1x}&\langle n\pi^{+}(p_i), n\pi^{-}(q_i);\\ &\sum_{i=1}^{n}(p_i+q_i)=k_1+k_2\mid T(J^\mu(x)J^\nu(0))\mid 0\rangle\\ \underset{\text{soft pion}}{\longrightarrow}\; &2(-3)^{n-1}[\prod_{i=1}^{n} 2\omega_{p_i}(2\pi)^3 2\omega_{q_i}(2\pi)^3]^{-1/2} f_\pi^{-2n}\\ &\times\left[\int dm^2\frac{\rho_V(m^2)-\rho_A(m^2)}{k_1\cdot k_2+m^2-i\epsilon}\left(g^{\mu\nu}+\frac{k_2^\mu k_1^\nu}{m^2}\right)\right.\\ &-g^{\mu0}g^{\nu0}\int dm^2\frac{\rho_v(m^2)-\rho_A(m^2)}{m^2}\\ &\left.-f_\pi^2\frac{k_2^\mu k_1^\nu}{k_1\cdot k_2-i\epsilon}+g^{\mu0}g^{k0}f_\pi^2\right],\end{aligned} \tag{10.3}$$

where k_1 and k_2 are the momenta of the photons,

$$[k_1+k_2=\sum_{i=1}^{n}(p_i+q_i)\to 0$$

in the soft-pion limit], and ρ_V and ρ_A are the spectral functions of the vector and axial-vector currents, respectively. It is clearly seen in expression (10.3) that not only Lorentz covariance but also gauge invariance is maintained in the soft-pion limit if, and only if, Weinberg's first sum rule (Weinberg, 1967),

$$\int dm^2[\rho_V(m^2)-\rho_A(m^2)]/m^2=f_\pi^2 \tag{10.4}$$

is valid. Therefore we shall assume the validity of the sum rule hereafter. Then we have the soft-pion theorem:

$$\begin{aligned}M^{\mu\nu}\underset{\text{soft pion}}{\longrightarrow}\; &2(-3/f_\pi^2)^{n-1}[\prod_{i=1}^{n} 2\omega_{p_i}(2\pi)^3 2\omega_{q_i}(2\pi)^3]^{-1/2}\\ &\times[g^{\mu\nu}-(k_2^\mu k_1^\nu/k_1\cdot k_2)]F(k_1\cdot k_2)\end{aligned} \tag{10.5}$$

and

$$F(Q^2)=(1/f_\pi^2)\int dm^2[\rho_V(m^2)-\rho_A(m^2)]/(Q^2+m^2-i\epsilon)$$
$$\text{with } F(0)=1[\text{see}(10.4)], \tag{10.6}$$

which give a correct Thomson limit for $n=1$ when k_1 and k_2 vanish. This relation (10.5) has also been obtained by Terent'ev (1971a) for $n=1$ [19] and has been confirmed by Goble and Rosner (1972) for $n=2$ and in the limit k_1 and $k_2\to 0$.

We can now apply the relation (10.5) to various processes $\gamma+\gamma\to n\pi^{+}+n\pi^{-}$, $e+e\to e+e+n\pi^{+}+n\pi^{-}$, and $e^{+}+e^{-}\to n\pi^{+}+n\pi^{-}+\gamma$.

A. $\gamma+\gamma\to n\pi^{+}+n\pi^{-}$

The differential cross section for $\gamma+\gamma\to n\pi^{+}+n\pi^{-}$ is given by

$$\begin{aligned}d\sigma_{\gamma\gamma\to n\pi^+n\pi^-}=&[(4\pi\alpha)^2/s(n!)^2](3/f_\pi^2)^{2(n-1)}[F(s/2)]^2\\ &\times\left[\prod_{i=1}^{n}\frac{d^3p_i d^3q_i}{2\omega_{p_i}(2\pi)^3 2\omega_{q_i}(2\pi)^3}\right](2\pi)^4\delta[P-\sum_{i=1}^{n}(p_i+q_i)],\end{aligned} \tag{10.7}$$

where $P=(s^{1/2},0,0,0)$ and s is the total energy squared in the center-of-mass system of the two photons. For $n=1$, the integration of phase space can be easily carried out. The total cross section for $\gamma+\gamma\to\pi^{+}+\pi^{-}$ is

$$\sigma_{\gamma\gamma\to\pi^+\pi^-}=(2\pi\alpha^2/s)[1-(4m_\pi^2/s)]^{1/2}[F(s/2)]^2. \tag{10.8}$$

Obviously this result is valid only for values of s close to the threshold value $(2m_\pi)^2$. If we take $s=0.1$ GeV^2= $(2.25m_\pi)^2$, for example, then $\sigma_{\gamma\gamma\to\pi^+\pi^-}=4.8\times10^{-31}$ cm^2.[20]

For n larger than 1, it is difficult to perform the phase-space integration exactly. There is, however, a

[19] The relation for $n=1$ is contained essentially in the work of Das, Gurlnik, Mathur, Low, and Young (1967).

[20] For numerical calculations we have taken the pole dominance [i.e., $\rho_V(m^2)=g_\rho^2\delta(m^2-m_\rho^2)$ and $\rho_A(m^2)=g_A^2\delta(m^2-m_A^2)$, where $m_\rho^2=0.59$ GeV^2 and $m_A^2=1.14$ GeV^2], and the result of Weinberg's second sum rule [i.e., $g_\rho^2=g_A^2=2m_\rho^2 f_\pi^2$]. See Weinberg (1967).

method developed by Bjorken and Brodsky (1970)[21] to approximate the integration. Following their method, we obtain

$$\sigma_{\gamma\gamma\to n\pi^+n\pi^-}\simeq\frac{2\pi\alpha^2}{s}\frac{(2n-1)(3s/16\pi^2f_\pi^2)^{2(n-1)}}{[n!(2n-1)!]^2}\times[F(\tfrac{1}{2}s)]^2,\qquad n\geq 2. \quad (10.9)$$

This simple result must be taken as a rough estimate of the cross section because the last approximation taken may not be reliable in the region of interest ($|\mathbf{p}_i|$ and $|\mathbf{q}_i|\ll m_\pi$). Equation (10.9) gives a prediction of the cross sections for $\gamma+\gamma\to 2\pi^++2\pi^-$ and $\gamma+\gamma\to 3\pi^++3\pi^-$, e.g.,

$$\sigma_{\gamma\gamma\to 2\pi^+2\pi^-}\simeq 2.1\times 10^{-33}\text{ cm}^2 \quad\text{at}\quad s=(6m_\pi)^2$$

and

$$\sigma_{\gamma\gamma\to 3\pi^+3\pi^-}\simeq 0.54\times 10^{-35}\text{ cm}^2 \quad\text{at}\quad s=(8m_\pi)^2. \quad (10.10)$$

Notice that the cross section for two-pion-pair production is two orders of magnitude larger than that for $\pi^+\pi^-\pi^0$ production [$\sim 10^{-35}$ cm^2 at $s=(4m_\pi)^2$] (see Sec. IX). More detailed and precise numerical results for $n=2$ can be found in the paper by Goble and Rosner (1972).

B. $e+e\to e+e+n\pi^++n\pi^-$

Within the equivalent-photon approximation we can estimate the cross sections for $e+e\to e+e+n\pi^++n\pi^-$, using Eqs. (3.14), (10.8), and (10.9). Since the low-s region dominates the total cross sections (see Sec. VA), the soft-pion results give a reasonable estimate for these processes. As a few examples, we find

$$\sigma_{ee\to ee\pi^+\pi^-}{}^{\text{soft pion}}=1.35\times 10^{-33}\text{ cm}^2$$

and

$$\sigma_{ee\to ee2\pi^+2\pi^-}{}^{\text{soft pion}}=3.5\times 10^{-36}\text{ cm}^2 \quad (10.11)$$

at the beam energy $E=1$ GeV. These numbers should be compared with either

$$\sigma_{ee\to ee\pi^+\pi^-}{}^{\text{Born}}=1.37\times 10^{-33}\text{ cm}^2$$

or

$$\sigma_{e^+e^-\to\pi^+\pi^-}{}^{\text{pointlike}}=5.4\times 10^{-33}\text{ cm}^2 \quad (10.12)$$

at the same energy. It is remarkable that the soft-pion cross section for $n=1$ is very close to the cross section for $e+e\to e+e+\pi^++\pi^-$ obtained by using the Born diagrams for $\gamma+\gamma\to\pi^++\pi^-$. The reason is that the factor $F(s/2)$ which depends upon the spectral functions in (10.8) simulates the effect of the interference between the isotropic and anisotropic parts of the Born cross section (5.5) for small values of s.

Other applications of the relation (10.5) will be found in Sec. XI, the paper by Terazawa (1971) ($e^++e^-\to n\pi^++n\pi^-+\gamma$), and the papers by Buchl and Nigam (1972a, b) ($e^++e^-\to n\pi^++n\pi^-$ and $e+\gamma\to e+n\pi^++n\pi^-$).

[21] Since we take our results seriously only in the region where all the pion momenta $|\mathbf{p}_i|$ and $|\mathbf{q}_i|$ are small compared with the mass m_π, we do not introduce *a priori* a factor which suppresses large-momentum components.

XI. PION ELECTROMAGNETIC MASS DIFFERENCE

In this section we shall discuss the possibility of determining the pion electromagnetic mass difference, a quantity of great theoretical interest, by the two-photon process

$$e(p_1)+e(p_2)\to e(p_1')+e(p_2')+\gamma^*(k_1)+\gamma^*(k_2), \quad (11.1)$$

$$\gamma^*(k_1)+\gamma^*(k_2)\to\pi^+(q_1)+\pi^-(q_2), \quad (11.2)$$

where the momenta of the various particles are designated in parentheses following the particle symbols. This has been proposed by Yan (1971) whose paper we shall follow here. As we saw in Sec. X, the PCAC hypothesis and the algebra of currents determine the amplitude for (11.2) at the unphysical point

$$k_1+k_2=0,\qquad q_1=q_2=0. \quad (11.3)$$

From this unphysical point one must extrapolate to the nearest physical point, namely, the production threshold

$$q_1=q_2=(m_\pi,0,0,0), \quad (11.4)$$

and, therefore,

$$k_1+k_2=(2m_\pi,0,0,0)$$

or

$$\omega_1=\omega_2=m_\pi \quad\text{and}\quad \mathbf{k}_1=-\mathbf{k}_2. \quad (11.5)$$

In addition to k_1^2 and k_2^2, there are two invariant variables available in the process (11.2), namely, $s=(k_1+k_2)^2=(q_1+q_2)^2$ and $k_1\cdot q_1$. These two quantities vary from zero to $4m_\pi^2$ and m_π^2, respectively, in the minimum extrapolation. As a consequence of the spacelike nature of the two virtual photons, the range of extrapolation for s and $k_1\cdot q_1$ from the unphysical point (11.3) to the threshold (11.4) is only of order m_π^2 and is *independent* of the virtual-photon squared masses $k_1^2=k_2^2$. Therefore, the usual smootheness assumption of PCAC suggests that the amplitude at the unphysical point (11.3) should be a good approximation to the amplitude at the physical threshold (11.4).

Thus, Yan (1971) has found that the soft-pion amplitude (10.5) derived by Terazawa (1971) for real photons ($k_1^2=k_2^2=0$) can be extrapolated into the threshold amplitude for any highly virtual photons ($k_1^2=k_2^2\neq 0$) without losing the validity of soft-pion approximation. He has also found that a gauge-invariant generalization of (10.5) to $k_1+k_2\neq 0$ is unique if terms quadratic in the pion momenta are neglected. The threshold amplitude for (11.2) is finally given by

$$M^{\mu\nu}=[2/(2\pi)^3(2W_12W_2)^{1/2}][g^{\mu\nu}-(k_2{}^\mu k_1{}^\nu/k_1\cdot k_2)]F(Q^2), \quad (11.6)$$

where $Q^2=-k_1^2=-k_2^2\geq 0$ and the function $F(Q^2)$ has been defined in (10.6). The differential cross section for $e+e\to e+e+\pi^++\pi^-$ with the special kinematics of Calogero and Zemach (1960)

$$\mathbf{p}_1+\mathbf{p}_2=\mathbf{k}_1+\mathbf{k}_2=\mathbf{p}_1'+\mathbf{p}_2'=0, \qquad k_1^2=k_2^2 \tag{11.7}$$

can now be given in terms of $F(Q^2)$ (Yan, 1971)

$$\frac{d\sigma}{dE_1'd\Omega_1dE_2'd\Omega_2}=(8\pi)^{-1}\left(\frac{\alpha^2}{\pi}\right)^2\left(\frac{4E^2}{Q^2}\right)^2\left(\frac{x}{Q^2+\frac{1}{2}s}\right)^2 \times\left(1-\frac{4m_\pi^2}{s}\right)^{1/2}\left[\tfrac{1}{4}(1+x^2)^2+x^2\cos^4\tfrac{1}{2}\theta\right]\left[F(Q^2)\right]^2, \tag{11.8}$$

where $4E^2=(p_1+p_2)^2$, $E_1'=E_2'=E'$ are the energies of the two final electrons, Ω_1 and Ω_2 are the solid angles of the two final electrons, θ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_1'$, and $x=E'/E$. By comparing experimentally observed cross sections for $e+e\to e+e+\pi^++\pi^-$ with the formula (11.8), we can readily measure $F(Q^2)$ as a function of Q^2, which is by itself of great threoetical interest.

On the other hand, Das, Guralnik, Mathur, Low, and Young (1967) have derived an expression for the electromagnetic mass difference of the pions based on the same assumptions that lead to (11.6). It is given by

$$m_{\pi^\pm}^2-m_{\pi^0}^2=\frac{3\alpha}{4\pi}\int_0^\infty dQ^2F(Q^2). \tag{11.9}$$

Combining (11.8) and (11.9), Yan (1971) has arrived at a sum rule:

$$m_{\pi^\pm}^2-m_{\pi^0}^2=\frac{3\alpha}{4\pi}\int_0^\infty dQ^2\lim_{s\to 4m_\pi^2}\left\{\frac{(d\sigma/dE_1'\,d\Omega_1\,dE_2'\,d\Omega_2)\,(ee\to ee\pi^+\pi^-)\,|_{\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_1'+\mathbf{p}_2'=0}}{(8\pi)^{-1}(\alpha^2/\pi)^2(4E^2/Q^2)^2[x/(Q^2+\frac{1}{2}s)]^2[1-(4m_\pi^2/s)]^{1/2}[\frac{1}{4}(1+x^2)^2+x^2\cos^4\frac{1}{2}\theta]}\right\}^{1/2}. \tag{11.10}$$

This sum rule is reminiscent of Cottingham's formula (1963), except that the pion mass difference is expressed in terms of the virtual Compton scattering amplitude in the crossed channel; its validity depends on the reliability of PCAC and Weinberg's first spectral sum rule (10.4). It should be noted that the algebraic sign of the pion mass difference is not determined theoretically by the sum rule (11.10).

To give a rough idea of the order of magnitude involved in this experiment, Yan (1971) has presented the following numbers. If $s=(2.5m_\pi)^2$, $Q^2=0.5$ GeV^2, and $E=3$ GeV, then (11.8) gives $d\sigma/dE_1'd\Omega_1dE_2'd\Omega_2=0.7\times10^{-34}[F(Q^2)]^2$ cm^2/GeV^2 and $F(Q^2)=0.4$ assuming ρ and A_1 dominance, Weinberg's second spectral sum rule, and the KSRF relation (Kawarabayashi and Suzuki, 1966; Riazuddin and Fayyazudin, 1966). Suppose $dE_1'=dE_2'=50$ MeV, $d\Omega_1=d\Omega_2=0.1$, then we have $d\sigma=2\times10^{-38}$ cm^2, where a factor of 2π has been included by integrating over the common azimuth of both final electrons. This is too small to be measurable at presently available colliding-beam facilities. If $Q^2=0.1$ GeV^2 with other conditions unchanged, then $F(Q^2)=0.8$ and $d\sigma=3\times10^{-35}$ cm^2, which should be within experimental reach in the near future.

This experiment is extremely interesting, although it seems to be rather difficult at present, for the following two reasons: (1) a measurement of the function $F(Q^2)$ is of great theoretical interest because it may answer such questions as the convergence of Weinberg's second sum rule, the behavior of the spectral function of the axial-vector current, and so on. (2) The origin of the pion mass difference is not known. Most people may believe that it is electromagnetic. We think, however, that it is still an open and interesting question whether the pion mass difference is really and entirely electromagnetic or not. We hope that the future colliding-beam experiments will give an answer to this question.

More detailed calculations of the cross section for this experiment can be found in the paper by Isaev and Khleskov (1972a).

XII. DEEP INELASTIC $e\gamma$ SCATTERING

In the previous sections we have discussed the two-photon processes $e+e\to e+e+X$ in which X is identified with a particular particle state with positive charge conjugation such as π^0, η, η', $\mu^+\mu^-$, $\pi^+\pi^-$, $\pi^+\pi^-\pi^0$, $n\pi^+n\pi^-$, etc. These two-photon processes are classified as "exclusive." In this and the next sections, we shall consider the "inclusive" processes $e+e\to e+e+X$ in which the hadronic state X is not specified but both of the scattered electrons are detected, their momenta being measured so that we can specify the following three invariants: the squared masses of the virtual photons, k_1^2 and k_2^2, and the squared mass of the final state X, $(k_1+k_2)^2$. In the typical arrangement of colliding-beam experiments in which all charged hadrons emitted at large angles are detected, there would be no particular complication in detecting one or both of the elctrons scattered into large angles in addition to the produced hadrons. Although the cross section $d\sigma_{ee\to eeX}$ is small for large $k_i^2=(p_i-p_i')^2$ for any specific hadron state (see Sec. XI), it might be large enough to be observable if it is summed over all possible final states X. Furthermore, from the cross sections observed in the inclusive reactions $e+e\to e+e+X$ we shall be able to obtain completely different and even more interesting information on the hadronic structure of particles.

From such points of view, Brodsky, Kinoshita, and Terazawa (1971a), Carlson and Tung (1971), Kunszt and Ter-Antonyan (1971) and Walsh (1971), have independently pursued the study of deep-inelastic electron–photon scattering, the analogy of the inelastic electroproduction in which the target is a *photon* rather than a nucleon:

$$e(p)+\gamma(P)\rightarrow e(p')+\text{any hadrons}. \qquad (12.1)$$

Let us first consider the inelastic scattering of an electron of momentum p on a boson target B of mass M and momentum P. In parallel with the electron–proton case, we shall define the inelastic form factors W_1^B and W_2^B by

$$2P_0\sum_n \langle P\,|\,J_\alpha(0)\,|\,n\rangle\langle n\,|\,J_\beta(0)\,|\,P\rangle(2\pi)^3\delta^4(q+P-P_n)$$
$$=-[g_{\alpha\beta}-(q_\alpha q_\beta/q^2)]W_1^B(Q^2,\nu)$$
$$+[P_\alpha-(\nu q_\alpha/q^2)][P_\beta-(\nu q_\beta/q^2)]W^{2B}(Q^2,\nu), \quad (12.2)$$

where J_α is the electromagnetic current of hadrons, $q(=p-p')$ is the momentum transfer of the electron, $Q^2=-q^2$, $\nu=q\cdot P$, and averaging over the target spin is assumed. Then the cross section for the process $e+B\rightarrow e+$any hadrons can be written as

$$\frac{d\sigma_{e+B\rightarrow e+\text{any hadrons}}}{dQ^2d\nu}=-\frac{2\pi\alpha^2}{(Q^2)^2}$$
$$\times\left[W_2{}^B(Q^2,\nu)\left(1-y-\frac{y^2Q^2M^2}{4\nu^2}\right)+W_1(Q^2,\nu)\,\frac{y^2Q^2}{2\nu^2}\right], \quad (12.3)$$

where $P^2=M^2$ and $y=\nu/p\cdot P$. We have neglected the electron mass compared with the incident energy. We have defined W_1^B and W_2^B in such a way that we do not encounter any difficulty in passing to the limit $M=0$. From now on we shall regard B as a real photon. Of course in this case the rest system of the target no longer exists.

We now consider large-Q^2 and large-ν regions and assume that the Bjorken scaling limit (Bjorken, 1969) exists for the hadronic structure functions of the *photon*:

$$\lim_{\nu\rightarrow\infty,\,\omega\text{ fixed}} W_1{}^\gamma(Q^2,\nu)=F_1{}^\gamma(\omega),$$
$$\lim_{\nu\rightarrow\infty,\,\omega\text{ fixed}} \nu W_2{}^\gamma(Q^2,\nu)=F_2{}^\gamma(\omega), \quad (12.4)$$

where $\omega=2\nu/Q^2(\geq 1+m_\pi{}^2/Q^2)$, and $F_1{}^\gamma$ and $F_2{}^\gamma$ are dimensionless functions of ω and are implicitly of order α. Then for large ν and fixed ω we have the simple formula

$$\frac{d\sigma_{e+\gamma\rightarrow e+\text{any hadrons}}}{dQ^2d\nu}=-\frac{2\pi\alpha^2}{\nu(Q^2)^2}$$
$$\times\left[F_2{}^\gamma(\omega)\,(1-y)+F_1{}^\gamma(\omega)\,\frac{y^2}{\omega}\right]. \quad (12.5)$$

This shows clearly that electron–photon scattering experiments in the deep inelastic kinematical region will give us information on the hadronic structure functions $F_1{}^\gamma$ and $F_2{}^\gamma$ of the photon.

In practice it is hard to find a suitable free-photon target. However, we have already seen that e^-e^- or e^+e^- colliding beams are wonderful suppliers of virtual photons. We shall therefore examine the feasibility of using the two-photon process $e+e\rightarrow e+e+$any hadrons in order to measure the hadronic structure functions of the photon.

If we consider the situation in which the incident electron 1 of energy E is scattered into an angle θ with energy E' while the incident electron 2 is scattered into a small angle ($<\theta_{\max}$), emitting an almost-real photon of energy E_γ, the corresponding cross section in the laboratory frame of colliding beams can be written in general (i.e., before the Bjorken limit is taken) as

$$\frac{d\sigma}{dE'd\cos\theta dE_\gamma}=\frac{8\pi\alpha^2EE'}{(Q^2)^2}N(E_\gamma,\theta_{\max})$$
$$\times\left[W_2(Q^2,\nu)\,(1-y)+W_1(Q^2,\nu)\,\frac{Q^2y^2}{2\nu^2}\right], \quad (12.6)$$

where the equivalent-photon method is applied to the electron 2 and

$$N(E_\gamma,\theta_{\max})=\frac{\alpha}{\pi}\left\{\frac{E^2+(E-E_\gamma)^2}{E^2}\left[\ln\left(\frac{E\theta_{\max}}{2m_e}\right)-\frac{1}{2}\right]\right.$$
$$+\frac{E_\gamma{}^2}{2E^2}\left[\ln\left(\frac{2(E-E_\gamma)}{E_\gamma}\right)+1\right]$$
$$\left.+\frac{(2E-E_\gamma)^2}{2E^2}\ln\frac{2(E-E_\gamma)}{[E_\gamma{}^2+E(E-E_\gamma)\theta_{\max}{}^2]^{1/2}}\right\}, \quad (12.7)$$

for $(m_e/E)^2\ll\theta_{\max}{}^2\ll1$ [see (3.18)]. In the case $\theta_{\min}<\theta<\theta_{\max}[(m_e/E)^2\ll\theta_{\min}{}^2]$, $N(E_\gamma,\theta_{\max})$ in (12.6) should be replaced by $N(E_\gamma,\ \theta_{\max})-N(E_\gamma,\ \theta_{\min})$. Notice that the equivalent-photon method applied to the electron 2 can be taken as an extremely good approximation if the condition $(m_e/E)^2\ll\theta_{\max}{}^2\ll1$ is actually satisfied.

In the deep inelastic region, which may be defined by

$$\nu=2E_\gamma[E-E'\cos^2(\theta/2)]>\nu_{\min},$$
$$Q^2=4EE'\sin^2(\theta/2)>Q_{\min}{}^2, \quad (12.8)$$

for an appropriate choice of $\nu_{\min}$ and $Q_{\min}{}^2$, the cross section (12.6) can be reduced to the form

$$\frac{d\sigma}{dE'd\cos\theta dE_\gamma}=\frac{4\pi\alpha^2EE'N(E_\gamma,\theta_{\max})}{(Q^2)^2[E-E'\cos^2(\theta/2)]E_\gamma}$$
$$\times[F_2{}^\gamma(\omega)\,(1-y)+F_1{}^\gamma(\omega)\,(y^2/\omega)] \quad (12.9)$$

and

$$y=1-(E'/E)\cos^2(\theta/2). \quad (12.10)$$

If we do not measure the energy of the electron 2 scattered into forward angles but only specify the upper bound of its energy to be $E-E_{\min}$ so that the deep inelastic kinematics is guaranteed, it is more useful to integrate (12.10) over E_γ:

$$\frac{d\sigma}{dE'd\cos\theta}=\frac{4\pi\alpha^2EE'}{(Q^2)^2[E-E'\cos^2(\theta/2)]}\times\int_{E_{\min}}^{E}\frac{dE_\gamma}{E_\gamma}N(E_\gamma,\theta_{\max})[F_2{}^\gamma(\omega)(1-y)+F_1{}^\gamma(\omega)(y^2/\omega)], \quad (12.11)$$

where $E_{\min}=\nu_{\min}/[2(E-E')\cos^2(\theta/2)]$.

In the ep case the scaling region seems to start at $\nu\sim$several GeV^2 and $Q^2\sim 0.5$ GeV^2. Thus $Q_{\min}{}^2$ for ep scattering is less than $\frac{1}{2}$ of the threshold value for the πp continuum. If a similar situation prevails in the ee scattering as well, $Q_{\min}{}^2$ will be close to the threshold of a few-pion system and will probably be of order 0.1 GeV^2. We may choose $\nu_{\min}\sim 1$ GeV^2 in the same analogy. Of course we do not know whether such an argument is correct: After all these are quantities to be determined by experiment.

In order to estimate the magnitude of the cross sections (12.9) and (12.11), let us assume the parton model with spin $\frac{1}{2}$ constituents. Then we have the relation (Callan and Gross, 1969)

$$F_1{}^\gamma(\omega)=\tfrac{1}{2}\omega F_2{}^\gamma(\omega). \quad (12.12)$$

We shall further make use of the very rough estimate

$$F_2{}^\gamma(\omega)\simeq(\sigma_{\gamma p}/\sigma_{pp})F_2{}^p(\omega)\simeq(1/300)\times 0.3\simeq 1\times 10^{-3} \quad (12.13)$$

based on factorization, where $\sigma_{\gamma p}$ and σ_{pp} are the total γp and pp cross sections. Then, for typical values $E=2.5$ GeV, $E'=1.0$ GeV, $\theta=15°$, $E_\gamma=0.5$ GeV, $\theta_{\max}=5.7°$, the cross section (12.9) becomes

$$d\sigma/dE'd\cos\theta dE_\gamma\simeq 4.2\times 10^{-34}\ \mathrm{cm}^2/\mathrm{GeV}^2. \quad (12.14)$$

(The values of other quantities are $\nu=1.5$ GeV^2, $Q^2=0.17$ GeV^2, $\omega=18$, $y=0.61$.) For the same values of $E, E', \theta, \theta_{\max}$, and $E_{\min}=0.5$ GeV (hence $\nu\geq 1.5$ GeV^2), we obtain from (12.11)

$$d\sigma/dE'd\cos\theta\simeq 3.3\times 10^{-34}\ \mathrm{cm}^2/\mathrm{GeV}. \quad (12.15)$$

Integrating (12.11) over E' and $\cos\theta$ in the deep inelastic region we obtain

$$\sigma\simeq\frac{4\alpha^3}{Q_{\min}{}^2}\left(\ln\frac{E\theta_{\max}}{m_e}\right)\left(\ln\frac{E}{E_{\min}}\right)\left(\ln\frac{2E^2}{\nu_{\min}}\right)\times F_2{}^\gamma\simeq 5.4\times 10^{-35}\ \mathrm{cm}^2, \quad (12.16)$$

for $E=2.5$ GeV, $\nu_{\min}=1.5$ GeV, $E_{\min}=0.5$ GeV, and $Q_{\min}{}^2=0.17$ GeV. From these examples we see that the deep inelastic scattering on a photon target will provide a practical and exciting opportunity for the new high-energy, high-luminosity colliding-beam facilities at SLAC and DESY.

We can give an additional argument to support the assumption (12.13) (Brodsky, Kinoshita, and Tarazawa, 1971a). If we follow the Bjorken–Paschos (1969) interpretation of the structure function, we anticipate the ω dependence of $F_2{}^\gamma(\omega)$ to be similar to that of $F_2{}^p(\omega)$. However, the magnitude of $F_2{}^\gamma(\omega)$ will depend on two factors: (1) the probability of finding the hadronic state in the target photon which is of the order of e^2/g^2 if we adopt the vector-dominance model with the universal coupling constant $g(g^2/4\pi\simeq 2.0)$; (2) the factor depending upon the structure of the hadronic state of the photon which is different from that of the nucleon. If we take the fractionally-charged-quark model in (2), we find the sum rule of the Bjorken–Paschos–Drell–Levy–Yan type (Bjorken and Paschos, 1969; Drell, Levy, and Yan, 1969, 1970)

$$\int_0^1 F_2{}^\gamma(\omega)d(\omega^{-1})\simeq(2/9)(e^2/g^2)\simeq 8\times 10^{-4}, \quad (12.17)$$

which is consistent with (12.13).

At $E=2.5$ GeV and $\theta=15°$, the Bhabha scattering cross section $d\sigma(e^+e^-\rightarrow e^+e^-)/d\cos\theta$ is about 20 μb, which is roughly 10^5 times larger than the cross section obtained from (12.15) by integration over E'. However, since the produced hadrons are to be detected simultaneously, no serious background problem arises from Bhabha scattering and its radiative corrections. The background due to hadrons produced in $C=-$states by the bremsstrahlung of virtual photons in ee collisions [see Fig. 1 (b)] appears to be strongly suppressed in the deep inelastic region. Fujikawa (1971a) has estimated such a background as well as the structure functions of the photons by assuming that in the inclusive hadron production by two photons, one real and the other highly virtual, $\gamma^*+\gamma\rightarrow$any hadrons, can be approximated by the muon-pair production $\gamma^*+\gamma\rightarrow\mu^++\mu^-$. One of his conclusions is that the $C=-$background process may contribute substantially (say a few times 10%) to the deep inelastic $e\gamma$ scattering in some kinematical region. See also the discussion on the same background given in Sec. XIII.

If we detect both the scattered electrons and measure all the kinematical variables, including the coplanarity angle of the electrons, we shall be able to determine six combinations of the eight independent structure functions involved in the imaginary part of forward virtual photon–photon scattering (Brown and Muzinich, 1971; Carlson and Tung, 1971). Although such a determination of more than two structure functions requires more difficult experiments, the questions of what scales and how it scales in the limit of Bjorken's type are interesting by themselves. Cheng and Zee (1972) have given an answer to these questions in the quark model for the light-cone algebra of currents (see Sec. XIII).

A more specific inclusive experiment was proposed by Roy (1971) who has shown that the reaction

$$e+e\rightarrow e+e+\pi(\mathrm{soft})+\text{anything} \quad (12.18)$$

is useful for picking up information on possible scaling of the matrix element of the axial-vector current commutator between real photon states. He has used the theoretical technique developed by Pais and Treiman (1970) for the inclusive one-photon annihilation process $e^+ + e^- \to \pi(\text{soft}) + \text{anything}$.

Recently Walsh and Zerwas (1973), in an argument based mainly on the parton picture, have conjectured that the structure functions of the target photon do not scale (or do but only approximately) in the Bjorken limit in contrast with the structure functions of hadrons.

See also Fujikawa (1971a) and Buchl (1973).

XIII. TEST OF THE ALGEBRA OF BILOCAL CURRENTS

In the previous section we discussed the inclusive two-photon process $e+e \to e+e+$any hadrons in which $-k_1^2$ is large, $k_2^2 \simeq 0$, and $(k_1+k_2)^2$ is also large with the fixed ratio $\omega = 1 + (k_1+k_2)^2/(-k_1^2)$. This is Bjorken's scaling region. In this section we shall investigate the same inclusive process in the different kinematical region where all three invariants, k_1^2, k_2^2, and $(k_1+k_2)^2$, are finite and large.

A. Various Scaling Limits

Since we have more than two invariant variables, we have several choices for the ratio to be fixed in the scaling limit. In fact, many authors have already invented different scaling limits.

Efremov and Ginzburg (1971), Shabel'skii (1971), Kingsley (1972), and Perlovskii and Kheifets (1972) have, independently, proposed "the super-scaling," which means that the amplitude for

$$\gamma^*(k_1) + \gamma^*(k_2) \to X, \qquad s = (k_1+k_2)^2 \tag{13.1}$$

is a function of only the one variable $ss_0/k_1^2k_2^2$ when both $-k_1^2$ and $-k_2^2$ are large and comparable to $(ss_0)^{1/2}$. The arbitrary dimensional constant s_0 (~ 1 GeV^2) appears in the first two papers because the authors have assumed Regge-type behavior for the $p\,p$ total cross section [i.e., $s\sigma^{pp}(s) \sim (s/s_0)^\alpha$] in addition to the factorization [i.e. $\sigma^{\gamma\gamma}(s, k_1^2, k_2^2) = \sigma^{\gamma p}(s, k_1^2) \times \sigma^{\gamma p}(s, k_2^2)/\sigma^{pp}(s)$] and the scaling law [i.e., $s\sigma^{\gamma p}(s, k^2) \sim f(s/k^2)$]. Kingsley has conjectured the same scaling from his analysis of the imaginary part of the Feynman diagram with a fermion loop drawn for the forward photon–photon scattering. Matveev, Muradyan, and Tavkhelidze (1970) have proposed a different scaling for the same amplitude in an analysis similar to Kingsley's Their conjecture is that the amplitude for (13.1) scales as a function of the two ratios k_1^2/k_2^2 and s/k_1^2 when all the three invariants, k_1^2, k_2^2, and $s \to \infty$, have the ratios fixed. Skobelev (1972) has also investigated the $\gamma\gamma$ forward scattering amplitude off the mass shell in the limit of s, $-k_1^2$, $-k_2^2 \to \infty$ with $s/-k_1^2$, $s/-k_2^2$ fixed, but large in the multiperipheral Amati–Bertocchi–Fubini–Stanghellini–Tonin model.

However, these conjectures of the scalings may not be strongly convincing since neither the reliability of the factorization assumption for such a large value of k_1^2 and k_2^2 nor the relevance of the Feynman diagram in such a limit as k_1^2, k_2^2, and $s \to \infty$ has been made clear as yet. In contrast, another scaling for the amplitude (13.1) in the limit, $-k_1^2$ and $-k_2^2 \to \infty$ with k_1^2/k_2^2 and s fixed, proposed by Gross and Treiman (1971b), is based on a more formal and mathematical basis, the algebra of bilocal currents. We shall briefly review this currently popular mathematical tool and one of its consequences, the Gross-Treiman scaling, in Sec. XIIIB and discuss the experiment proposed by many authors as a test of the algebra of bilocal currents in Sec. XIIIC.

A general consideration of kinematics for large $-k_1^2$, $-k_2^2$, and s can be found in the paper by Choban and Shekhter (1971).

B. The Algebra of Bilocal Currents

Since the SLAC–MIT experiments revealed Bjorken's scaling phenomena (1969) in deep-inelastic electroproduction, two ideas have attracted theoretical attention. One is the parton model (Feynman, 1969) and the other is the light-cone dominance of current commutators (Ioffe, 1969; Brandt, 1969; Jackiw, Van Royan, and West, 1970; Lewtwyler and Stern, 1970; Frishman, 1970; Brandt and Preparata, 1971). It has been assumed that the algebra of current commutators on the light cone possesses the structure abstracted from the free-quark model (Fritzsch and Gell-Mann, 1971) or the gluon-quark model (Cornwall and Jackiw, 1971; Gross and Treiman, 1971a). For the electromagnetic currents in the free-quark model, for example, we have

$$\begin{aligned}[J_\mu(x), J_\nu(y)] \simeq{}& \partial^\alpha D(x-y) \\ &\times \{s_{\mu\nu\alpha\beta}[V^\beta(x,y) - V^\beta(y,x)] \\ &-i\epsilon_{\mu\nu\alpha\beta}[A^\beta(x,y) + A^\beta(y,x)]\}, \qquad \text{for } (x-y)^2 \simeq 0, \\ s_{\mu\nu\alpha\beta} ={}& g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta},\end{aligned} \tag{13.2}$$

and

$$D(x) = \epsilon(x_0)\delta(x^2)/2\pi.$$

The bilocal currents V_μ and A_μ are defined by

$$V_\mu(x, y) = \bar\psi(x)\gamma_\mu Q^2\psi(y)$$

and

$$A_\mu(x, y) = \bar\psi(x)\gamma_5\gamma_\mu Q^2\psi(y), \tag{13.3}$$

where $\psi(x)$ is the quark field and Q is the quark charge (matrix). Thus the study of bilocal currents with lightlike separation has become an interesting subject in particle physics. Moreover, Fritzsch and Gell-Mann (1971) and also Gross and Treiman (1971a) have suggested the physical significance of the commutator of the light-cone commutators in which the separations of all four space–time points are lightlike. For the vector currents in the free quark model, we have

$$[V_\mu(u,x),V_\nu(y,z)]\simeq\bar\psi(u)\gamma_\mu\gamma_\alpha\gamma_\nu Q^4\psi(z)\partial^\alpha D(x-y)-\bar\psi(y)\gamma_\nu\gamma_\alpha\gamma_\mu Q^4\psi(x)\partial^\alpha D(z-u),$$

for

$$(u-x)^2\simeq(u-y)^2\simeq(u-z)^2\simeq(x-y)^2\simeq(x-z)^2\simeq(y-z)^2\simeq 0. \quad (13.4)$$

It is important to propose as many tests of this algebra of bilocal currents as possible because it is quite unfamiliar and therefore should be subjected to experimental checks. To do this, Gross and Treiman (1971b) have considered the process [see Fig. 1(c)]

$$e^+ + e^- \to \mu^+ + \mu^- + \text{any hadrons}. \quad (13.5)$$

They stress the importance of the ordered limits. First they introduce the scaling limit

$$q^2\to\infty,\quad \omega=q^2/P\cdot q,\quad \text{and } P^2 \text{ fixed}, \quad (13.6)$$

where $q=(l+k)/2$, $P=l-k$, and l and k are the total momenta of the incident electron pair and the final muon pair, respectively. After this limit, they take the second limit $P^2\to\infty$ with ω fixed and find the striking result that one can determine the complete ω dependence of the cross section for this process by using the algebra of bilocal currents (13.4). Unfortunately, their cross section is too small to be measurable in the near future.[22] More practical tests of the algebra of bilocal currents would be highly welcome.

C. Tests by Means of the Two-Photon Process

Many authors [Terazawa (1972a), Walsh and Zerwas (1972), Lee, Yand, and Yu (1972), Kunszt (1972), Kunszt and Ter-Antonyan (1972), and Chernyak (1972)] have proposed the inclusive two-photon process

$$e(p_1)+e(p_2)\to e(p_1')+e(p_2')+X(P) \quad (13.7)$$

as a test of the algebra of bilocal currents. The momenta of the leptons are indicated in the associated parentheses and X is an arbitrary hadron state with positive charge conjugation and with momentum P. Detection of at least one produced hadron as well as measurement of the momenta of both scattered electrons is required experimentally. Let E, E, E_1', and E_2' be the energies of leptons 1, 2, 1′, and 2′, respectively. We shall concentrate on the kinematical region where the two virtual-photon mass squared, $-k_1^2$ and $-k_2^2$, are very large compared to, say, the pion mass squared but still much smaller than $4E^2$, $4EE_1'$, $4EE_2'$, $4E_1'E_2'$, i.e.,

$$m_\pi^2\ll -k_1^2,\quad -k_2^2\ll 4E^2,\quad 4EE_1',\quad 4EE_2',\quad \text{and}\quad 4E_1'E_2'. \quad (13.8)$$

In the e^+e^- colliding beams we have two graphs, Figs. 1(a) and (c), which contribute to the production of hadron states with positive conjugation. The graph of Fig. 1(c), however, can be ignored completely under the conditions (13.8) because the ratio of the amplitude Fig. 1(c) to that of Fig. 1(a) has roughly the magnitude of $(k_1^2k_2^2)/[(p_1+p_2)^2(p_1'+p_2')^2]$. On the other hand, in the case of e^-e^- colliding beams we can ignore the interference between the graph of Fig. 1(a) and the graph in which the two final electron lines are exchanged, due to the fact that the backward scattering would correspond to a much higher momentum transfer than the momentum transfer involved in the forward scattering.

Therefore, we may restrict ourselves to the graph of Fig. 1(a). For this graph, all the physics is contained in the tensor

$$M_{\mu\nu}=i\int d^4x\,\exp(-iqx)\times\langle X\mid T^*(J_\mu(\tfrac12x),J_\nu(-\tfrac12x))\mid 0\rangle, \quad (13.9)$$

where J_μ is the electromagnetic current (divided by e) and q is redefined by

$$q=\tfrac12(k_1-k_2)\quad\text{and}\quad P=k_1+k_2. \quad (13.10)$$

There are two different sets of three independent and invariant quantities $(q^2, P\cdot q, P^2)$ and (k_1^2, k_2^2, P^2) whose relations are

$$q^2=\tfrac12(k_1^2+k_2^2-\tfrac12P^2)<0\quad\text{and}\quad P\cdot q=\tfrac12(k_1^2-k_2^2). \quad (13.11)$$

In the physical region of this process the scaling variable $\omega=q^2/P\cdot q$ is constrained by

$$|\,\omega\,|>(1+P^2/4q^2)^{-1}>1. \quad (13.12)$$

After summing all the final hadron states with the total momentum P fixed, we find the vacuum expectation value of the commutator of the current commutators:

$$\begin{aligned}M_{\alpha\beta;\mu\nu}&=\sum_X(2\pi)^4\delta(P-P_X)M_{\alpha\beta}{}^{\dagger}M_{\mu\nu}\\&=\int d^4x\,d^4y\,d^4z\,\exp(-iqx+iqy-iPz)\theta(x_0)\theta(y_0)\\&\quad\times\langle 0\mid[[J_\alpha(\tfrac12(y+z)),J_\beta(\tfrac12(-y+z))]^\dagger,[J_\mu(\tfrac12(x+z)),J_\nu(\tfrac12(-x+z))]]\mid 0\rangle.\end{aligned} \quad (13.13)$$

[22] For examples, $d\sigma\sim10^{-42}$ cm^2 for $E=3$ GeV, $q^2=1.5$ GeV^2, $\omega=1.8$, and $P^2=1$ GeV^2. Their final results for the structure functions g_s and g_P seem to have a wrong normalization by $2(2\pi)^4$. The author thanks Professor D. J. Gross and Professor S. B. Treiman for correspondence on this point.

Following Gross and Treiman (1971b), in the first limit of $-q^2 \rightarrow \infty$ with ω and P^2 fixed, we see that $M_{\mu\nu}$ is determined by the singularity of the product of two currents on the light cone ($x^2=0$) and that the tensor structure of $M_{\alpha\beta;\mu\nu}$ is identical to that of $M_{\alpha\beta;\mu\nu}$ appearing in the single scalar and pseudoscalar productions by the two virtual photons. Therefore, $M_{\alpha\beta;\mu\nu}$ can be expressed in terms of two structure functions g_S and g_P as follows:

$$\lim_{q^2\to-\infty,\omega \text{ and } P^2 \text{ fixed}} (P\cdot q)^2 M_{\alpha\beta;\mu\nu} = g_P(\omega, P^2)\epsilon_{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\kappa\lambda}P^\gamma P^\kappa q^\delta q^\lambda$$
$$+g_S(\omega, P^2)[P_\alpha P_\beta \omega - (P_\alpha q_\beta + q_\alpha P_\beta) + (P\cdot q) g_{\alpha\beta}][P_\mu P_\nu \omega - (P_\mu q_\nu + q_\mu P_\nu) + (P\cdot q) g_{\mu\nu}]. \quad (13.14)$$

One of the most important points here is that the functions g_S and g_P thus introduced scale as a function of ω when $q^2 \rightarrow -\infty$ with ω and P^2 fixed. Furthermore, by assuming the algebra of bilocal currents (13.4), Gross and Treiman (1971b) have shown that in the ordered limits: $q^2 \rightarrow -\infty$ with ω and P^2 fixed followed by $P^2 \rightarrow \infty$ with ω fixed; $M_{\alpha\beta;\mu\nu}$ becomes identical to what one can obtain in calculating the pair creation of massless quarks by the two virtual photons. Thus we can determine the ω dependence of the structure functions in the asymptotic limit of $P^2 \rightarrow \infty$.

The results are

$$\lim_{P^2\to\infty,\omega \text{ fixed}} g_S(\omega, P^2) = g_S(\omega) = \langle Q^4\rangle \int_{-1}^{1} \frac{dz}{2\pi} \frac{z^2(1-z^2)}{(\omega^2-z^2)^2}$$

and

$$\lim_{P^2\to\infty,\omega \text{ fixed}} g_P(\omega, P^2) = g_P(\omega) = \langle Q^4\rangle \int_{-1}^{1} \frac{dz}{2\pi} \frac{\omega^2(1-z^2)}{(\omega^2-z^2)^2}, \quad (13.15)$$

where $\langle Q^4\rangle$ is an effective value for the sum of Q^4 over quarks. This constant $\langle Q^4\rangle$ is predicted to be

$$\langle Q^4\rangle = \begin{cases} 2/9 \text{ for the original Gell-Mann–Zweig (fractionally charged single triplet) quark model} \\ 2/3 \text{ for the fractionally charged three triplet quark model} \\ 1 \text{ for the original Sakata (integrally charged triplet) model} \\ 2 \text{ for the Maki–Hara (integrally charged triplet) model (Hara, 1964; Maki, 1964)} \\ 4 \text{ for the Han–Nambu (integrally charged three triplet) model (Han and Nambu, 1965).} \end{cases} \quad (13.16)$$

In any event, it is a single parameter to be determined by experiment and to discriminate between various models. A detailed comparison of the effective values for Q determined in various processes can be found in the papers by Suura, Walsh, and Young (1972) and by Bardeen, Fritzsch, and Gell-Mann (1972).

For large $-q^2$ with fixed ω and P^2, the differential cross section for the process (13.7) is given by the following simple formula (Terazawa, 1972a):

$$\frac{d\sigma}{d\epsilon_1 d\cos\theta_1 d\epsilon_2 d\cos\theta_2 d\phi} = \frac{256\alpha^4}{\pi} \times \frac{E^4 E_1' E_2' (E-E_1')^2 (E-E_2')^2}{[k_1^2 k_2^2 (k_1^2+k_2^2)]^2} F(\omega, P^2),$$
$$\text{for } P^2 \ll -q^2 \ll 4E^2, \quad (13.17)$$

where $\epsilon_i = E_i'/E$ ($i=1, 2$), θ_i is the scattering angle of the electron, ϕ is the angle between the plane containing the first scattered electron and the incident beam and the other plane containing the second scattered electron and the beam,[23]

$$-k_i^2 = 4EE_i' \sin^2(\theta_i/2), \quad (13.18)$$

and

$$F(\omega, P^2) = \omega^2[\omega^2 g_S(\omega, P^2) + g_P(\omega, P^2)]. \quad (13.19)$$

From Eqs. (13.15) and (13.19), we obtain the asymptotic form of the reduced structure function (Terazawa, 1972a):

$$\lim_{P^2\to\infty,\omega \text{ fixed}} F(\omega, P^2) = F(\omega) = \langle Q^4\rangle \int_{-1}^{1} \frac{dz}{2\pi} \frac{\omega^4(1-z^4)}{(\omega^2-z^2)^2}$$
$$= \langle Q^4\rangle \frac{\omega^4}{\pi} \left[\frac{3\omega^4+1}{4|\omega|^3} \ln\frac{|\omega|+1}{|\omega|-1} - \frac{3\omega^2+1}{2\omega^2}\right]. \quad (13.20)$$

Since the maximum energy of the colliding-beam machines which will be available within a few years is 3 or 3.5 GeV per beam, it is hard to test this result of the algebra of bilocal currents (13.20) under the extremely idealistic condition

$$m_\pi^2 \ll P^2 \ll -q^2 \ll 4E^2. \quad (13.21)$$

However, if we assume that $F(\omega, P^2)$ will already be close to the $F(\omega)$ given in (13.20) at relatively small

[23] To make the definition of ϕ clear, note that

$$P^2 = 4(E-E_1')(E-E_2') - 2E_1'E_2'(1-\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi).$$

values for $-q^2$ and P^2,[24] then we see that this test may be feasible in rather early colliding-beam experiments. Taking, for example, $E=3$ GeV, $E_1'=E_2'=2.5$ GeV, $q^2=-0.7$ GeV^2, $\omega=1.75$, and $P^2=0.4$ GeV^2, we find[25]

$$\frac{d\sigma}{d\epsilon_1 d\cos\theta_1 d\epsilon_2 d\cos\theta_2 d\phi} \begin{aligned} &=2.9\times10^{-33}\ \mathrm{cm}^2 \quad \text{for } \langle Q^4\rangle=\tfrac{2}{9} \\ &=1.3\times10^{-32}\ \mathrm{cm}^2 \quad \text{for } \langle Q^4\rangle=1. \end{aligned} \tag{13.22}$$

Under the condition (13.8) or (13.21), the experimental phase-space volume $d\epsilon_1 d\cos\theta_1 d\epsilon_2 d\cos\theta_2\, d\phi$ is roughly 10^{-4}. Therefore, the effective cross section will be of the order of 10^{-36} cm^2 for $\langle Q^4\rangle=1$ and of 10^{-37} cm^2 for $\langle Q^4\rangle=\frac{2}{9}$. A cross section of this magnitude can be measured with future colliding-beam facilities if their luminosities reach $\sim10^{33}$–10^{34} cm^{-2}/sec.

In order to make this test of the algebra of bilocal currents more precise, Terazawa (1972a) has also investigated the background contribution in detail. His conclusion is that the $C=-$ background contribution is less than 17% of total events if $\langle Q^4\rangle=\frac{2}{9}$ and that it is at most 4.3% if $\langle Q^4\rangle=1$ at the same combination of kinematical values used previously in (13.22). He has also found that the background contribution disappears at high energies where the condition (13.8) can be better satisfied. For more details see Terazawa (1972a). See also the recent paper by Walsh (1972) for a comparison between the Regge and parton (or light-cone algebra) pictures.

In conclusion, we should emphasize that the processes (13.5) and (13.7) provide the only simple tests of the algebra of bilocal currents that do not depend on any further assumptions such as the distribution of partons. It is to be hoped that future colliding-beam facilities will have sufficiently high luminosities to make these tests possible.

Recently Ferrara, Grillo, and Parisi (1973) have applied the Crewther's short-distance analysis to $\gamma^*+\gamma^*\rightarrow X\rightarrow\gamma^*+\gamma^*$ and obtained the relation $\langle Q^4\rangle=16S^2/3R$, where S is the anomalous constant defined in (7.13) and R is the high energy limit of $\sigma(e^++e^-\rightarrow\gamma^*\rightarrow\text{hadrons})/\sigma(e^++e^-\rightarrow\mu^++\mu^-)$. See also the paper by Fritzsch (1972).

XIV. ASYMPTOTIC BEHAVIOR OF THE VERTEX FUNCTION

In this section we shall return to the exclusive process

$$e+e\rightarrow e+e+X \tag{14.1}$$

in which X is specified to be a single $C=+$ meson such as π^0, η, η', σ (or ϵ), $f(1260)$, etc. Some general features of this process have been discussed in Secs. IV and VII. Here we shall concentrate on the asymptotic behavior of the two-photon vertex function where at least one of the (mass)^2 of the virtual photons becomes large. Detection of the final electron scattered at large angles as well as measurement of the momentum is required experimentally in order to find the photon mass squared. Naturally, the cross section will become very small in this case. It is, however, one of the ultimate physical interests to measure the vertex as a function of the photon mass squared once the process (14.1) has been observed.

To be more specific, let us take $X=\pi^0$ as an example. The differential cross section for the two-photon process $e+e\rightarrow e+e+\pi^0$ at the c.m. energy of E is given by (Terazawa, 1972b)

$$\frac{d\sigma}{dE_1'd\cos\theta_1 dE_2'd\cos\theta_2 d\phi} =128\alpha^4\frac{E^2E_1'E_2'(E-E_1')^2(E-E_2')^2}{(q^2k^2)^2} \times\delta(P^2-m_\pi^2)\,|\,F(q^2,k^2)\,|^2, \quad \text{for } m_\pi^2\ll-q^2,\ -k^2\ll4E^2, \tag{14.2}$$

where we have used the same notations as in Sec. XIII [see (13.7)] except for q^2 and k^2 which here denote the mass squared of the virtual photons ($q^2=k_1^2$ and $k^2=k_2^2$) for convenience. The $\pi^0\gamma\gamma$ vertex function has been defined in Sec. VII by

$$\begin{aligned} M_{\mu\nu}(q,k) &= i\int d^4x\,\exp(-iqx) \\ &\quad\times\langle P\,|\,T^*(J_\mu(x),J_\nu(0))\,|\,0\rangle \\ &=\epsilon_{\mu\nu\alpha\beta}q^\alpha k^\beta F(q^2,k^2). \end{aligned} \tag{14.3}$$

Its normalization is given either by [see (7.11)]

$$\begin{aligned} |\,F(0,0)\,| &= (64\pi\Gamma_{\pi^0\rightarrow\gamma\gamma}/e^4m_\pi^3)^{1/2} \\ &=(2.7\pm0.2)\times10^{-4}\ \mathrm{MeV}^{-1} \end{aligned} \tag{14.4}$$

experimentally, or by [see (7.14) and (7.15)]

$$\begin{aligned} F(0,0)\,|_{P^2=0} &= -S/(2\pi^2 f_\pi) \\ &=2.7\times10^{-4}\ \mathrm{MeV}^{-1} \quad \text{for } S=1/2 \\ &=0.89\times10^{-4}\ \mathrm{MeV}^{-1} \quad \text{for } S=1/6 \end{aligned} \tag{14.5}$$

theoretically.

Several years ago, Cornwall (1966) showed that $F(q^2,k^2)$ decreases as fast as $(k^2)^{-1}$ when $k^2\rightarrow+\infty$ and $k^2/q^2\rightarrow+1$ if the q-number Schwinger term does not exist, the Bjorken–Johnson–Low theorem (Bjorken, 1966; Johnson and Low, 1966) is valid, and the space–space component of equal-time current commutators is defined as

$$[J_i(\tfrac{1}{2}x),J_j(-\tfrac{1}{2}x)]_{x_0=0}=-2i\epsilon_{0ijk}A_{Q^2}{}^k(0)\delta(\mathbf{x}), \tag{14.6}$$

[24] This conjecture has a good chance to become true because the SLAC–MIT experiments revealed Bjorken's predicted scaling at surprisingly low momentum transfers and because the threshold values of P^2 for continuum states are much smaller than those in deep-inelastic electroproduction.

[25] The angles θ_1, θ_2, and ϕ are 21°, 9.4°, and 111°, respectively.

where $A_{Q^2}{}^{\mu}$ is an axial-vector current

$$A_{Q^2}{}^{\mu}(x)=\bar{\psi}(x)\gamma_5\gamma^{\mu}Q^2\psi(x). \qquad (14.7)$$

In the paper which was discussed in Sec. XIII, Gross and Treiman (1971b) have shown, in the gluon-quark model for the light-cone algebra of currents (Cornwall and Jackiw, 1971; Gross and Treiman, 1971a), that $q^2F(q^2, k^2)$ scales as a function of k^2/q^2 in the limit $q^2\rightarrow\infty$ with k^2/q^2 fixed. It will be extremely interesting to check these predictions for the asymptotic behavior of the vertex function experimentally because this could rule out (or reveal) a few possibilities to ruin their results: i.e., the existence of the q-number Schwinger term, the failure of the Bjorken–Johnson–Low theorem, and a divergent coefficient for the space-space component of equal-time (or light-cone) current commutators.

Since these results are not only model-dependent but limited to the special kinematical region where both q^2 and k^2 are large with the ratio k^2/q^2 fixed, it is required to find a less model-dependent and more widely applicable prediction for the asymptotic behavior of the vertex function. Terazawa (1972b) has discussed this question by using the Schwartz inequality and unitarity only, as we shall review in the following.

If we take q and k below and above the 2π threshold, respectively, (i.e., $q^2<4m_\pi{}^2$ and $k^2>4m_\pi{}^2$) in Eq. (14.3), the imaginary part of the $M_{\mu\nu}(-q, k)$ simplifies and takes the form:

$$\mathrm{Im}\, M_{\mu\nu}(-q, k)=\tfrac{1}{2}\sum_n (2\pi)^4\delta(P+q-P_n) \times\langle P\mid J_\mu(0)\mid n\rangle\langle n\mid J_\nu(0)\mid 0\rangle. \qquad (14.8)$$

Let us define the spectral function of the photon propagator by

$$\sum_n (2\pi)^3\delta(k-P_n)\langle 0\mid J_\mu(0)\mid n\rangle\langle n\mid J_\nu(0)\mid 0\rangle =(-k^2g_{\mu\nu}+k_\mu k_\nu)\Pi(k^2), \qquad (14.9)$$

and the inelastic electroproduction form factors of π^0 by

$$\begin{aligned}\sum_n (2\pi)^3\delta(P+q-P_n)\langle P\mid J_\mu(0)\mid n\rangle\langle n\mid J_\nu(0)\mid P\rangle &=-[g_{\mu\nu}-(q_\mu q_\nu/q^2)]W_1(q^2, P\cdot q)\\ &+[P_\mu-(P\cdot q/q^2)q_\mu][P_\nu-(P\cdot q/q^2)q_\nu]\\ &\times W_2(q^2, P\cdot q), \qquad (14.10)\end{aligned}$$

where *we restrict n to states with $J=1$*. From Eqs. (14.8)–(14.10), we obtain the Schwatz inequality for the imaginary part of the vertex function:

$$\mid \mathrm{Im}\, F(q^2, k^2)\mid \leq 2\pi\left\{\frac{k^2\Pi(k^2)W_1[q^2, \frac{1}{2}(k^2-q^2-m_\pi{}^2)]}{(k^2-q^2)^2-2m_\pi{}^2(k^2+q^2)+m_\pi{}^4}\right\}^{1/2}, \quad \text{for } q^2<4m_\pi{}^2. \qquad (14.11)$$

It is well known that $\Pi(k^2)$ is related to the total cross section of the one-photon annihilation process:

$$\Pi(k^2)=k^2\sigma(e^+e^-\rightarrow\text{any hadrons})(k^2)/(16\pi^3\alpha^2). \qquad (14.12)$$

The total cross section is predicted to decrease as fast as $(k^2)^{-1}$ (Bjorken, 1966; Gribov, Ioffe, and Pomeranchuk, 1967) or faster. In other words, we have

$$\Pi(k^2)\rightarrow\text{constant or zero} \quad \text{as} \quad k^2\rightarrow\infty. \qquad (14.13)$$

The following results will not be changed unless the total cross section stays constant or increases as k^2 goes up. On the other hand, $W_1(q^2, \nu)$ is related to the total cross section for the inelastic scattering of the virtual transverse photon on the π^0 target:

$$W_1(q^2, \nu)\simeq\nu\sigma_T(q^2, \nu)/(2\pi^2\alpha). \qquad (14.14)$$

Since the final state is restricted to have $J=1$, unitarity tells us that $\sigma_T(q^2, \nu)$ decreases as fast as ν^{-1} or faster, namely

$$W_1(q^2, \nu)\rightarrow\text{constant or zero} \quad \text{as} \quad \nu\rightarrow\infty \text{ with } q^2 \text{ fixed}. \qquad (14.15)$$

From the inequality (14.11) together with (14.13) and (14.15), we conclude that $\mid \mathrm{Im}\, F(q^2, k^2)\mid$ decreases not slower than $(k^2)^{-1/2}$ as k^2 increases while q^2 is fixed. Unfortunately, we do not know anything about the asymptotic behavior of $\mathrm{Re}\, F(q^2, k^2)$. We shall assume that $F(q^2, k^2)$ does not contain any polynomial in k^2 or, in other words, $F(q^2, k^2)$ decreases at all.[26] Then we can use the unsubtracted dispersion relation for $F(q^2, k^2)$ with fixed q^2 to obtain the following inequality sum rule:

$$\mid F(q^2, k^2)\mid\leq 2\int_{4m_\pi{}^2}^{\infty}\frac{d\kappa^2}{\kappa^2-k^2} \times\left\{\frac{\kappa^2\Pi(\kappa^2)W_1[q^2, \frac{1}{2}(\kappa^2-q^2-m_\pi{}^2)]}{(\kappa^2-q^2)^2-2m_\pi{}^2(\kappa^2+q^2)+m_\pi{}^4}\right\}^{1/2}, \quad \text{for } q^2, k^2<4m_\pi{}^2. \qquad (14.16)$$

Now we can investigate the behavior of $F(q^2, k^2)$ for large $-k^2$ (or $-q^2$) with q^2 (or k^2) fixed at an arbitrary value, whether it is spacelike, real or timelike ($q^2, k^2<4m_\pi{}^2$). Notice the purely mathematical fact that[27]

$$\left|\int_0^\infty dt'\frac{f(t')}{t'-t}\right|\leq c(-t)^{-a}, \qquad \text{for } t<-T<0, \qquad (14.17)$$

if $\mid f(t)\mid\leq c't^{-a}$ for $0<T'<t<\infty$, where c, c', T, T', and a are some constants (c, $c'>0$ and $0<a<1$). In our case, $a=1/2$ and, therefore, the integral in the right-hand side of (14.16) is bounded by constant$\times(-k^2)^{-1/2}$ for large $-k^2$ and fixed q^2. Consequently, we conclude that the vertex function decreases not more slowly than $(-k^2)^{-1/2}$ *for any fixed values of* q^2 ($<4m_\pi{}^2$) as $-k^2$

[26] The author thanks Professor N. N. Khuri for pointing out the necessity of this assumption.

[27] The author thanks Professor V. Singh for suggesting this mathematical point.

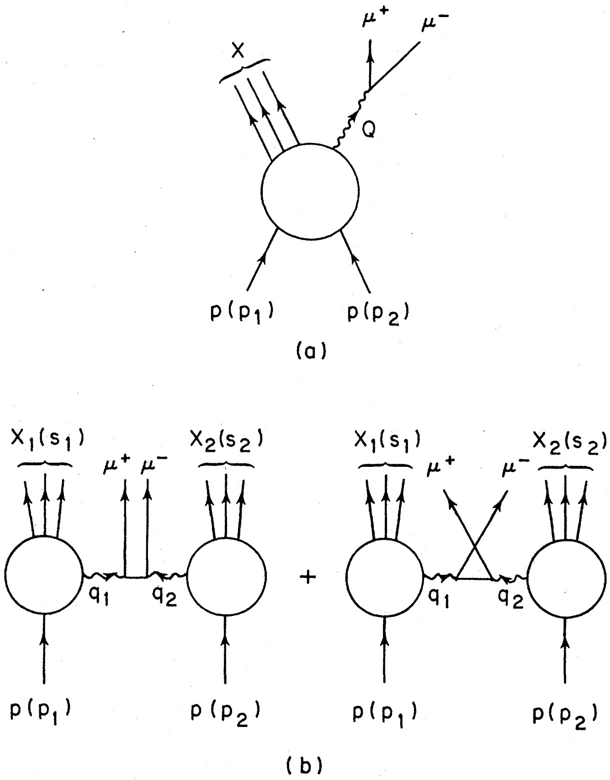

FIG. 13. The two different types of diagrams which contribute to the process $p+p\rightarrow\mu^++\mu^-+$ anything. X, X_1, and X_2 are arbitrary hadron states.

increases. The same conclusion can be reached for large $-q^2$ with k^2 fixed because $F(q^2, k^2)$ is symmetric with the two variables. Since $W_1[q^2, \frac{1}{2}(\kappa^2-q^2-m_\pi^2)]$ for fixed κ^2 vanishes quickly as $-q^2$ goes up, the main contribution to the integral in (14.16) comes from the deep inelastic region where both κ^2 and $-q^2$ are large and $\omega=(\kappa^2-q^2)/(-q^2)>1$. Introducing the Bjorken scaling function (Bjorken, 1969) for W_1 [28]

$$W_1[q^2, \tfrac{1}{2}(\kappa^2-q^2-m_\pi^2)]\rightarrow F_1(\omega)$$

as

$$-q^2\rightarrow\infty \text{ with } \omega \text{ fixed,} \qquad (14.18)$$

we transform the inequality (14.16) into

$$|F(q^2, k^2)|\leq\frac{2}{(-q^2)^{1/2}}\int_1^\infty d\omega\frac{[(\omega-1)\Pi(\infty)F_1(\omega)]^{1/2}}{\omega[\omega-1+(k^2/q^2)]}, \qquad \text{for large } -q^2. \quad (14.19)$$

This clearly shows that $F(q^2, k^2)$ decreases not more slowly than $(-q^2)^{-1/2}$ for any fixed values of k^2 ($<4m_\pi^2$) as $-q^2$ increases and that $(-q^2)^{1/2}F(q^2, k^2)$ scales as a function of k^2/q^2 or vanishes for large $-k^2$ and $-q^2$ with k^2/q^2 fixed. The latter prediction is definitely weaker but less model-dependent than that of Gross and Treiman (1971b).

An inequality of the type (14.16) holds for arbitrary single particle states X such as η, η', σ (or ϵ), etc. The experimental determination and further theoretical investigation of these vertex functions is of interest since they can potentially be better probes to the structure of hadrons than the ordinary form factors of hadrons. Some other consequences of the inequality (14.16) can be found in the recent paper by West (1973).

How the PCAC anomaly affects the asymptotic behavior of the $\pi^0\gamma\gamma$ and axial-vector-vector-vector vertex functions has been discussed by Terazawa (1973a).

XV. MASSIVE MUON-PAIR PRODUCTION IN HADRON–HADRON COLLISIONS

One of the most intriguing experiments reported recently in high-energy physics is the massive muon-pair production in hadron–hadron collisions, $p+p$ (or n)$\rightarrow\mu^++\mu^-+$anything, which was first performed by Christenson, Hicks, Lederman, Limon, and Pope (1970) with a uranium target. Drell and Yan (1970) predicted a scaling behavior for the α^2 contribution [Fig. 13(a)] to the process

$$p(p_1)+p(p_2)\rightarrow\mu^++\mu^-+\text{anything} \qquad (15.1)$$

at large and timelike values of Q^2 (the di-muon mass squared or the virtual photon mass squared) with the ratio $\rho=Q^2/s$ fixed where $s=(p_1+p_2)^2$ is redefined in this section by the total energy squared of the incident beams in the center-of-mass system. The differential cross section per di-muon mass is predicted to be of the form

$$d\sigma/d(Q^2)^{1/2}=[\alpha^2/(Q^2)^{3/2}]F(\rho). \qquad (15.2)$$

However, the magnitude and detailed behavior of the scaling function $F(\rho)$ can not be unambiguously calculated at this time. A different approach to the same process is an analysis by Altarelli, Brandt, and Preparata of the light cone singularity of products of currents (1970; Brandt and Preparata, 1972). Such an analysis predicts a nonscaling behavior with exponential decrease of the differential cross section at large Q^2.

Since the magnitude of the one-photon cross section has never been predicted, there is no reason why we should believe that the process (15.1) is dominated by the one-photon process [Fig. 13(a)] for any values of Q^2 and s. In addition to the one-photon process, one should also consider the two-photon contribution [Fig. 13(b)] to the differential cross section for (15.1). The relevance of the two-photon process in $p+p\rightarrow\mu^++\mu^-+$anything has been emphasized by Budnev, Ginzburg, Meledin, and Serbo (1970, 1972a), who estimated the elastic contribution $X_1=X_2=p$ for the

[28] Since W_1 is the contribution of $J=1$ states to the usual inelastic form factors, $F_1(\omega)$ is a part of the usual structure function. Therefore, $F_1(\omega)$ may vanish, in which case $F(q^2, k^2)$ decreases faster than $(-q^2)^{1/2}$. Notice also that $F_1(\omega)\rightarrow$constant or zero as $\omega\rightarrow\infty$, which guarantees strong convergence of the integral in (14.19).

diagrams shown in Fig. 13(b) by means of the equivalent photon approximation. As subsequently pointed out by Brodsky, Kinoshita, and Terazawa (1971b), the effects of proton form factors at the vertex $p \to p+\gamma^*$ are important and suppress the equivalent-photon result very severely at high Q^2, where the minimum momentum transfers, q_1^2 and q_2^2, to the protons are not negligible. This point was confirmed by a calculation of the same elastic contribution to $d\sigma/d(Q^2)^{1/2}$ by Fujikawa (1971b), who still neglected some of the q^2 dependence of the purely electromagnetic two virtual photons to muon pair amplitude.

It is important to know the full contribution to the differential cross section given by the diagrams in Fig. 13(b), where X includes both elastic and inelastic states. Knowledge of this contribution is necessary for an assessment of the background for the extraction of the α^2 one photon process. Since amplitudes for the one- and two-photon processes do not interfere in the differential cross section, the two-photon contribution will also provide a lower bound to the expected cross section for (15.1). Furthermore, if the one-photon process is small at large Q^2, which is a possibility with the nonscaling predictions such as that of Brandt and Preparata (1972), or if the scattering of a hadron is coherent off a nuclear target, the two-photon contribution will be more important and reveal its own physical significance. From quantum number consideration, it can be shown that the muon pair produced directly from one timelike photon and two spacelike photons have different angular distributions (Terazawa, 1973b). More detailed measurements can thus separate these two processes and use the muon pair production as a signature to measure the electromagnetic structure of other hadrons, for example, the pion form factor, in the spacelike q^2 region in hadron–hadron collisions (Geshkenbein and Terentyev, 1971).

Recently, Chen, Cheng, Muzinich, and Terazawa (1973) [see also Chen, Muzinich, and Terazawa (1972)] calculated the full two-photon contribution exactly. An outline of the details of their calculation is the following: After squaring the amplitude and integrating over the appropriate phase space variables, the relevant differential cross section has the general form

$$\frac{d\sigma}{d(Q^2)^{1/2}} = \frac{2m_p^2}{[s(s-2m_p^2)]^{1/2}}\left(\frac{\alpha}{2\pi^2}\right)^2 \int \frac{d^4q_1\, d^4q_2}{(q_1^2q_2^2)^2}$$

$$\times 2(Q^2)^{1/2}\delta[Q^2-(q_1+q_2)^2]$$

$$\times W_{\mu\nu}(-q_1, p_1)A^{\nu\beta\mu\alpha}(q_1, q_2)W_{\alpha\beta}(-q_2, p_2), \quad (15.3)$$

where m_p is the proton mass, $W_{\mu\nu}$ is the gauge invariant tensor for the absorptive part of the off-shell Compton amplitude, which is given by the usual inelastic form factors of the proton W_1 and W_2 as follows:

$$W_{\mu\nu}(q, p) = \sum_n (2\pi)^3\delta(p+q-p_n)\langle p \mid J_\mu(0) \mid n\rangle \times\langle n \mid J_\nu(0) \mid p\rangle$$

$$= -[g_{\mu\nu}-(q_\mu q_\nu/q^2)]W_1(q^2, p\cdot q)$$

$$+[p_\mu-(p\cdot q/q^2)q_\mu][p_\nu-(p\cdot q/q^2)q_\nu] \times W_2(q^2, p\cdot q), \quad (15.4)$$

and $A_{\nu\beta\mu\alpha}$ is the gauge invariant absorptive part of the α^2 term in the purely electromagnetic amplitude $\gamma(q_1)+\gamma(q_2)\to\gamma(q_1)+\gamma(q_2)$ with the muon loop and can be calculated by Feynman rules.

The region of the integration over q_1 and q_2 in (15.3) is always confined in such a way that $q_i^2 \leq 0$ and $s_i=(p_i-q_i)^2 \geq m_p^2$, which is the same as that for the electroproduction process $e+p \to e+$anything for the determination of $W_{\mu\nu}$. Therefore, we can use the knowledge of the experimentally measured proton inelastic form factors W_1 and W_2. For the elastic contribution, we can express W_1 and W_2 in terms of the known elastic form factors G_E and G_M with the approximate dipole parametrization $(1-q^2/0.71\ \mathrm{GeV}^2)^{-2}$. The inelastic contribution to W_1 and W_2 is parametrized by the fit of Bloom and Gilman (1971) to the scaling region $-q_i^2 \gtrsim 1\ \mathrm{GeV}^2$, $s_i \gtrsim 2\ \mathrm{GeV}^2$. Since the integration in (15.3) also includes the small $-q_1^2$ region, Chen, Cheng, Muzinich, and Terazawa have modified the formula of Bloom and Gilman by a factor $-q_i^2/(-q_i^2+0.15\ \mathrm{GeV}^2)$, which correctly matches with the total real photoabosprtion cross section at $q_i^2=0$. Low-energy resonance electroproduction and background contributions are taken into account in an average sense by the fit of Bloom and Gilman. After elimination of the δ-function and a trivial azimuthal integration, the expression in (15.3) can be written in terms of a six-dimensional integral over the invariants q_1^2, q_2^2, s_1, s_2, $s_{12}=(p_1+q_2)^2$, and $s_{21}=(q_2+q_1)^2$. After a parametrization in terms of the $O(2, 1)$ variables of the type in the papers by Brown and Muzinich (1971) and by Carlson and Tung (1971), subsequent integration over three of these variables can easily be carried out analytically. This is one of the advantages of using the $O(2, 1)$ variables mentioned in Sec. II. Thus the integral in (15.3) is reduced to a three-dimensional integral which is performed by a Monte Carlo program to an accuracy of less than 1%.

These results (Chen, M., and T., 1972; Chen, C., M., and T., 1973) are shown for both elastic and total contributions in Fig. 14 and Table IX as functions of $(Q^2)^{1/2}$. The total incident energies are chosen to agree with the Brookhaven–Columbia experiments at $s=56.3\ \mathrm{GeV}^2$ and the future experiments at NAL, ISR, and ISABELLE with $s=1000$, 2500, and $10^5\ \mathrm{GeV}^2$, respectively.

TABLE IX. The two-photon cross section $d\sigma/d(Q^2)^{1/2}$ for the process $p+p\rightarrow\mu^++\mu^-+X_1+X_2$ calculated for combinations of $s=56.3$, 1000, 2500, 10^5 GeV^2, and $(Q^2)^{1/2}=1-60$ GeV. All values have been taken from Chen, Cheng, Muzinich, and Terazawa (1972).

		$(Q^2)^{1/2}$(GeV)														
s(GeV^2)		1	2	3	4	5	6	9	12	15	18	20	30	40	50	60
		$d\sigma/d(Q^2)^{1/2}$ (cm^2/GeV)														
56.3	ELASTIC	1.74×10^{-35}	2.94×10^{-37}	7.6×10^{-39}	1.13×10^{-40}	2.25×10^{-43}										
56.3	TOTAL	2.05×10^{-35}	3.7×10^{-37}	8.4×10^{-39}	1.20×10^{-40}	2.26×10^{-43}										
1000	ELASTIC	4.2×10^{-34}	2.96×10^{-35}	4.9×10^{-36}			1.24×10^{-37}	8.1×10^{-39}	7.1×10^{-40}	6.2×10^{-41}	4.2×10^{-42}					
1000	TOTAL	5.3×10^{-34}	4.5×10^{-35}	8.5×10^{-36}			2.53×10^{-37}	1.70×10^{-38}	1.41×10^{-39}	1.11×10^{-40}	6.7×10^{-42}					
2500	ELASTIC			1.21×10^{-35}			4.7×10^{-37}	5.0×10^{-38}	8.0×10^{-39}	1.55×10^{-39}		1.24×10^{-40}	4.9×10^{-43}			
2500	TOTAL			2.03×10^{-35}			9.8×10^{-37}	1.32×10^{-37}	2.64×10^{-38}	6.5×10^{-39}		8.1×10^{-40}	1.63×10^{-41}			
10^5	ELASTIC			1.17×10^{-34}			9.5×10^{-36}	1.97×10^{-36}	6.1×10^{-37}	2.37×10^{-37}		6.6×10^{-38}	9.7×10^{-39}	2.21×10^{-39}	6.3×10^{-40}	2.14×10^{-40}
10^5	TOTAL			1.48×10^{-34}			1.42×10^{-35}	3.4×10^{-36}	1.17×10^{-36}	5.0×10^{-37}		1.60×10^{-37}	2.94×10^{-38}	8.2×10^{-39}	2.84×10^{-39}	1.15×10^{-39}

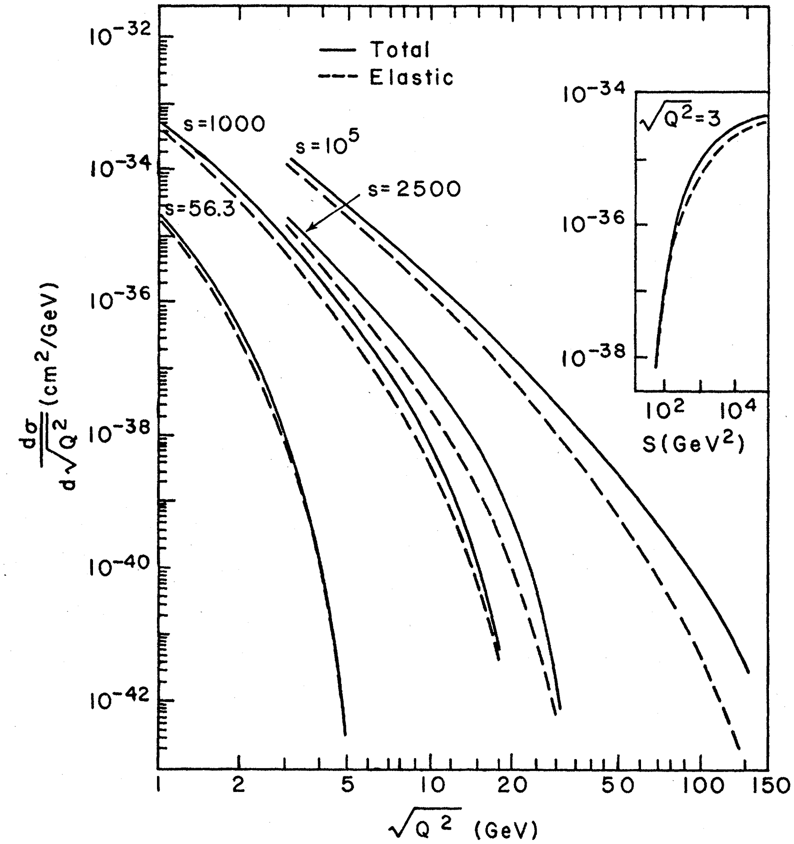

FIG. 14. The two-photon cross section $d\sigma/d(Q^2)^{1/2}$ for the process $p+p\rightarrow\mu^+ + \mu^- + X_1 + X_2$ as a function of $(Q^2)^{1/2}$ at $s=56.3$, 1000, 2500, and 10^5 GeV^2. The solid line represents the total contribution and the broken line represents the elastic contribution with $X_1=X_2=p$. The upper right-hand corner exhibits the s-dependence of the cross section at $(Q^2)^{1/2}=3$ GeV. All the curves have been taken from Chen, Muzinich, and Terazawa (1972).

The results can be summarized by the following comments:

(1) At $s=56.3$ GeV^2, the two-photon contribution is negligible compared to the experimental data of Christenson *et al.*, (1970), which indicates that at this energy there is very little background due to the two-photon process.

(2) The cross section increases very rapidly with s at fixed values of $(Q^2)^{1/2}$ between $s=56.3$ and 1000 GeV^2 and gives observable values at s greater than 1000 GeV^2 even at $(Q^2)^{1/2}=10$ GeV. The cross section asymptotically increases as $(\ln s)^3$, which is expected by the equivalent-photon method. Apart from this logarithmic factor, it approximately scales in $\rho=Q^2/s$ for large s. If one assumes that the α^2 one-photon process has already reached the scaling limit in ρ, then the two-photon process becomes comparable with the one-photon process extrapolated to $s=10^5$ GeV^2. Therefore the two-photon process has a good chance to be one of the main contributions at high s and very important.

(3) At high s values the inelastic and elastic contributions become comparable to each other and are both observable. The inelastic contribution has a substantial part coming from the Bjorken's scaling region with large s_i and $-q_i^2$. Since in this region the muon pair will have a sizable total transverse momentum, this part can even be separated out to provide a test of the scaling in W_1 and νW_2 $(\nu=p\cdot q)$ in the average sense at very high energies.

(4) At $s=1000$ GeV^2 and $(Q^2)^{1/2}\lesssim 3$ GeV, the elastic contribution is large enough so that one may have a good chance to do π, K coherent scattering on heavy nucleus targets to measure the electromagnetic form factors, as suggested by Geshkenbein and Terentyev (1971).

One might worry that the angular distribution of the lepton pair from the two-photon process might be forward peaked in the beam direction to the extent that the pair may not be well enough separated from the beam to be observable. Although the angular distribution can be straightforwardly calculated, it is sufficient to have a qualitative estimation at the present stage. Since each of the muons are preferentially emitted in the direction of the muon pair total momentum $\mathbf{Q}=(\mathbf{q}_1+\mathbf{q}_2)$, it is sufficient to estimate the portion of the cross section in which $\mathbf{Q}$ has a substantial opening angle with respect to the beam. From the previous studies of the two-photon process for π^0 and η production in Sec. VI, where the π^0 (or η) angle ("the photon–photon axis angle") corresponds to the opening angle for $\mathbf{Q}$, only about half of the cross section is lost by making an angular out of $(m/E)^{1/4}$. This cut off is of the order of 15° even for $E=200$ GeV.

It is easy to extract the one-photon contribution alone if we can perform both of the experiments $p+p\to\mu^++\mu^-+$anything and $\bar{p}+p\to\mu^++\mu^-+$anything at the same values of s. Since not only the two-photon contribution but also the contribution of parton–antiparton clouds around the proton (or antiproton) in the one-photon process to the $\bar{p}p$ collision [in the picture of Drell and Yan (1970)] is identical to those to the pp collision, the difference between these two cross sections can be interpreted as a contribution of "valence" partons inside the proton (or antiproton) in the one-photon process. In the picture of Altarelli, Brandt, and Preparata (1970), we can interpret the difference as a one-photon cross section from which the Pomeranchukon contribution is subtracted. Another way of extracting the two-photon contribution has been recently pointed out by Soni (1973). For large s, the two-photon cross section has an approximately logarithmic dependence on the lepton mass, i.e., $d\sigma/d(Q^2)^{1/2}\propto\ln(Q^2/m_l^2)$, while the one-photon cross section is independent of the lepton mass up to order m_l^2/Q^2. A characteristic difference between the electron and muon pair production cross sections is, therefore, solely controlled by the two-photon process. The ratio of the electron pair production to the muon pair production in the two-photon cross sections is approximately

$$\frac{d\sigma(e^+e^-)/d(Q^2)^{1/2}}{d\sigma(\mu^+\mu^-)/d(Q^2)^{1/2}}\simeq\frac{\ln(Q^2/m_e^2)}{\ln(Q^2/m_\mu^2)}\simeq 2.6 \quad \text{for } (Q^2)^{1/2}=3\text{ GeV}. \tag{15.5}$$

XVI. MISCELLANEOUS TOPICS

In this section we shall discuss various topics and processes which are related to the two-photon processes.

A. Other Higher-Order Contributions

In addition to the diagrams for the two-photon process which produce $C=+$ final states, the diagram shown in Fig. 1(b) for $C=-$ states will contribute logarithmically increasing total cross sections in $e^\pm-e^-$ collisions. The equivalent-photon method applied to one electron leg gives the leading contribution for $E/m_e\gg1$:

$$d\sigma_{ee\to eeX(C=-)}=\int\frac{d\omega}{\omega}N(\omega)\,d\sigma_{\gamma e\to eX(C=-)}, \tag{16.1}$$

where $d\sigma_{\gamma e\to eX(C=-)}$ is the differential cross section for a real photon in collision with an electron to produce the $C=-$ state X. This cross section is finite for $m_e\to0$ (provided X is not the state e^+e^-) and hence the cross section (16.1) is only singly logarithmic in E/m_e. Of course, the actual magnitude of this cross section must be determined by an explicit calculation, a problem to be settled before long. We note that, in principle, this $C=-$ production cross section can be completely calculated from the knowledge of the one-photon process (1.3).

Arteaga-Romero, Jaccarini, Kessler, and Parisi (1970, 1971a) have estimated the $C=-$ total cross sections for $\mu^+\mu^-$, $\pi^+\pi^-$, and K^+K^- production in the equivalent-photon approximation (16.1) and found that they are very small and negligible compared with the corresponding $C=+$ total cross sections. Altukhov (1971) has given general formulas for the $C=-$ process. An exact calculation of the $C=-$ contribution to the inclusive process $e+e\to e+e+$any hadrons can be seen in the paper by Terazawa (1972a).

Contributions of other higher-order diagrams for hadron production in which the incident e^+ and e^- beams annihilate decrease with energy as in the one-photon annihilation process (1.3). Examples are the order-α radiative correction to $e^++e^-\to\gamma^*\to X(C=-)$ including the emission of hard photons (Litke, 1970; Kunszt, Muradyan, and Ter-Antonyan, 1970; Bonneau and Martin, 1971; Berends, Gaemers, and Gastmass, 1972) by one of the leptons, namely

$$e^++e^-\to\gamma^*+\gamma,\quad \gamma^*\searrow X(C=-), \tag{16.2}$$

the two-photon annihilation (or two-photon exchange) process (Gatto, 1965; Sakurai, 1970; Lepetre and Renard, 1972)[29]

$$e^++e^-\to\gamma^*+\gamma^*\to X(C=+), \tag{16.3}$$

the e^+e^- annihilation into two virtual photons which in turn decay into hadrons separately (Cheng and Wu, 1972)

$$e^++e^-\to\gamma^*+\gamma^*,\quad \gamma^*\searrow X_1(C=-)\quad \gamma^*\searrow X_2(C=-), \tag{16.4}$$

and the one-photon annihilation into hadrons plus a single photon (Cabibbo and Gatto, 1961; Gatto, 1965; Creutz and Einhorn, 1970a,b; Kunszt, Muradyan, and Ter-Antonyan, 1970; Gakh, 1971)

$$e^++e^-\to\gamma^*\to\gamma+X(C=+). \tag{16.5}$$

The last process (16.5) recently discussed by Creutz and Einhorn (1970a,b) and by others is very closely related to the two-photon process $e^\pm+e^-\to e^\pm+e^-+X(C=+)$. It is because what is measurable in the process (16.5) is the $\gamma^*(k_1)\to X(C=+)+\gamma(k_2)$ amplitude for $k_1^2>0$ and $k_2^2=0$ while that in the two-photon process is the $\gamma^*(k_1)+\gamma^*(k_2)\to X(C=+)$ for $k_1^2\leq0$ and $k_2^2\leq0$. The same amplitude, but for the most special case of $k_1^2=k_2^2=0$, can also be observed in the generalized Primakoff effect proposed by Stodolsky (1971; Juristic and Stodolsky, 1971; see also Brown,

[29] The terminology for this process has not been established yet and is somewhat confusing. We should clearly distinguish between this process and the two-photon process $e^\pm+e^-\to e^\pm+e^-+X(C=+)$.

Muzinich, Roe, and Yao, 1972). Amazingly, another special case of the amplitude for $X=\pi^+\pi^-\pi^0$ was a current issue as its evaluation was desperately needed in solving the $K_L{}^0\to\mu^+\mu^-$ puzzle (Christ and Lee, 1971; Aviv and Sawyer, 1971; and many others).

B. Purely Leptonic Two-Photon Processes

Purely leptonic two-photon processes $e+e\to e+e+e^++e^-$ and $e+e\to e+e+\mu^++\mu^-$ can serve as checks on the fourth-order quantum-electrodynamics calculation (analogous to the trident experiments) or as normalization checks on hadron production by the two-photon processes. They have to be understood especially well because the large magnitude of their total cross sections can cause serious background problems to other colliding-beam processes if the scattered electrons in the final state are not detected.

There are three main factors that influence the experimental counting rates of these lepton-pair-production processes:

(1) The experiment will set a threshold on the minimum invariant mass $s_{\min}^{1/2}$ of the lepton pair that can be observed. Accordingly, the measured cross sections are reduced by the substitution of $s_{\min}$ for the threshold value of the s integration in (3.14), for example. Thus, the measured cross sections for the case where the produced electron–positron pairs are detected are of order

$$(\alpha^4/s_{\min})[\ln (E/m_e)]^2 \ln (4E^2/s_{\min})$$

rather than the theoretical total cross section

$$\sim(\alpha^4/m_e^2)[\ln (E/m_e)]^3.$$

(2) Some experiments will set a limit on the minimum angle of the detected particles of the produced system. As we have seen in Sec. V, this is not a particularly severe effect when $d\sigma_{\gamma\gamma\to X}$ is nearly isotropic in the photon–photon center-of-mass system as is the case $X=\pi^+\pi^-$. However, in the case of $\mu^+\mu^-$ (or e^+e^-) production, the effect of the angular cutoff will be considerable. In general, enhancement factors of $\ln (4E^2/s_{\min})$ are missing in the theoretical cross section integrated over wide-angle phase space. In addition, for the case of the electron pair-production processes, $e+e\to e+e+e^++e^-$, $e+e\to e+e+e^++e^-+e^++e^-$, etc., the requirement that at least one final-state electron be detected at a wide angle $\theta>\theta_{\min}\gg m_e/E$ eliminates the inverse dependence of the total rate on m_e^2. We should mention here that the sixth-order process $e+e\to e+e+e^++e^-+e^++e^-$ has closely been investigated by many authors [Servo (1970), Cheng and Wu (1970b), Greco (1971), Parisi and Zirilli (1971), Lipatov and Frolov (1971), and Kuraev and Lipatov (1972)].

(3) As we have discussed in Sec. V, the two-particle production cross sections are dominated by events in which the produced particle pair is noncollinear but roughly coplanar with the beam direction. In general, a considerable fraction of the events are noncoplanar (see Sec. VI) and thus this criterion is not sufficient to distinguish the two-particle production through the two-photon process and multihadron ($n>2$) production through the one-photon annihilation process. The large event rates for $e+e\to e+e+e^++e^-$ and $e+e\to e+e+\mu^++\mu^-$ can make these processes an especially serious background for multihadron production without complete particle identification. In general, the necessity for experimental arrangements which have provision for detecting and possibly tagging the scattered electrons seems to be unavoidable.

Recently Pesic (1973) has calculated the total cross section for $e^++e^-\to e^++e^-+W^++W^-$, the intermediate-vector-boson pair production by the two-photon process.

C. Total Cross Section for Hadron Production

In Secs. XII and XIII we considered the inclusive processes $e+e\to e+e+$any hadrons in which at least one of the scattered electrons is detected at a large angle. What has not been considered yet is the totally inclusive hadron production by the two-photon process $e+e\to e+e+$any hadrons in which neither of the scattered electrons is detected or in which the mere presence of them is detected in order to discriminate the two-photon process from the one-photon annihilation process. Of course, no known theoretical model is powerful enough to give a precise prediction for the cross section for this process. It is, however, desirable to have at least a rough estimate of the total hadron-production cross section via the two-photon process. Several authors (Budnev and Ginzburg, 1971a; Arteaga-Romero, Jaccarini, Kessler, and Parisi, 1971a; Brodsky, Kinoshita, and Terazawa, 1971b) have presented a simple argument based on the equivalent-photon method, local-duality, and the factorization of $\sigma_{\gamma\gamma\to\text{any hadrons}}$.

The general components of the total cross section $\sigma_{\gamma\gamma\to\text{any hadrons}}$ consist of

(a) the contribution of narrow $C=+$ resonances (π^0, η, η', etc.) described in Secs. IV and VII,

(b) two-pion production starting at the threshold $s_{\text{th}}=(2m_\pi)^2$ modulated by the even l resonances and enhancements in the π–π system described in Sec. VIII,

(c) the contribution of $C=+$ resonances which decay into other hadronic systems than the π–π system,

(d) and, finally, a nearly flat asymptotic component which may be estimated by factorization of the cross section at high energy (universal Pomeranchukon coupling) to be

$$\sigma_{\gamma\gamma\to\text{any hadrons}}{}^{\text{asympt}}\cong(\sigma_{\gamma p}{}^{\text{asympt}})^2/\sigma_{pp}{}^{\text{asympt}}$$
$$\cong 0.3\ \mu\text{b for large } s. \qquad (16.6)$$

From a duality point of view we can consider

$$\sigma_{\gamma\gamma\to\text{any hadron}}$$

as being essentially equal to

$$\sigma_{\gamma\gamma\to\text{any hadrons}}{}^{\text{asympt}}$$

with the $C=+$ resonances modulating the asymptotic value. Thus, very roughly, the multihadron cross section might be expected to average out to 0.3 μb starting at a threshold s_{th} for s of order $(3m_\pi)^2$ or $(4m_\pi)^2$. We thus estimate (Brodsky, Kinoshita, and Terazawa, 1971b)

$$\begin{aligned}\sigma_{ee\to ee+\text{any hadrons but }2\pi} &\\ \cong 2\left(\frac{\alpha}{\pi}\right)^2\left(\ln\frac{E}{m_e}\right)^2 &\int_{s_{\text{th}}}^{4E^2}\frac{ds}{s}f\left(\frac{s^{1/2}}{2E}\right)\sigma_{\gamma\gamma\to\text{any hadron}}{}^{\text{asympt}},\\ \cong 2(\alpha/\pi)^2[\ln(E/m_e)]^2&(0.3\ \mu\text{b})\\ &\times[(\ln y_0)^2+(\ln y_0)(3+2y_0+\tfrac{1}{4}y_0^2)\\ &\qquad+(1-y_0)(\tfrac{37}{8}+\tfrac{5}{8}y_0)],\end{aligned}\tag{16.7}$$

with $y_0=s_{\text{th}}/4E^2$. The results for the total cross section are plotted in Fig. 12. The cross section for producing three or more hadrons is seen to be comparable in magnitude to the two-pion production cross section discussed in Sec. VIII.

We may also obtain a simple estimate of the cross section for the process

$$e+e\to e+e+\rho^0+\rho^0\to e+e+\pi^++\pi^-+\pi^++\pi^-\tag{16.8}$$

via ρ dominance. For $s>(2m_\rho)^2$ we expect

$$\sigma_{\gamma\gamma\to\rho\rho}\cong(e/g)^4\sigma_{\rho\rho\to\rho\rho}\sim(1/300)^2(10\text{ mb})\sim 0.1\ \mu\text{b}.\tag{16.9}$$

This gives

$$\sigma_{ee\to ee\rho\rho}\sim 2\times 10^{-34}\text{ cm}^2\quad\text{at}\quad E=2\text{ GeV}.\tag{16.10}$$

Comparing this estimated cross section (16.9) ($\sim 10^{-31}$ cm^2) for $\gamma+\gamma\to\rho^0+\rho^0\to 2\pi^++2\pi^-$ with the soft-pion cross section given in (10.10) [$\sim 2\times 10^{-33}$ cm^2 at $s=(6m_\pi)^2$] for $\gamma+\gamma\to 2\pi^++2\pi^-$, we can anticipate that the cross section for $\gamma+\gamma\to 2\pi^++2\pi^-$ will increase very rapidly as s starts at the threshold $s=(4m_\pi)^2$ and goes beyond $s=(2m_\rho)^2$.

Gatto and Preparata (1973) have recently estimated the total cross section for $e+e\to e+e+$hadrons as well as the inclusive cross section for $e+e\to e+e+\pi+$anything, considering the resonance production and diffraction regions separately. For the details see their paper which includes an interesting comparison between one photon annihilation and the two-photon process for higher energies up to $E=15$ GeV.

D. Multiplicity of the Produced Particles

There has been very little work on the multiplicity of hadrons produced in the two photon process $e^\pm+e^-\to e^\pm+e^-+$hadrons. As far as the multiplicity as a function of the energy E is concerned, we know at present only a simply kinematical bound $n_A(E)\leq 2E/m_A$, where m_A is the mass of the particle A. We should, however, pay attention to the interesting work of Llewellyn Smith and Pais (1972) on the absolute bound of the multiplicity of neutral pions, n_0, given by the multiplicity of charged pions, n_{ch}, provided that the produced-hadronic state consists solely of pions and that the total number of the pions, N, is odd. Their bounds based on the isospin conservation in strong interactions for $e+e\to e+e+N\pi$ are

$$\tfrac{1}{2}\leq n_0/n_{\text{ch}}\leq 11/4,\qquad\text{for}\quad N=3$$

and

$$\tfrac{1}{4}\leq n_0/n_{\text{ch}}\leq(3N+2)/(2N-2),\quad\text{for } N\geq 5\text{ and odd.}\tag{16.11}$$

Therefore, for all $N\geq 5$ and odd we have

$$\tfrac{1}{4}\leq n_0/n_{\text{ch}}\leq 17/8.\tag{16.12}$$

These bounds will be useful experimentally if the absence of kaons and mesons other than pions in the final state of an event is confirmed and if an even number of produced pions are excluded in some way.

E. Other Application to Nonleptonc Collisions

The equivalent-photon formalism discussed in Sec. III can also be used to obtain an estimate of the magnitude of two-photon processes in high-energy electron–hadron and hadron–hadron collisions. As viewed from the center-of-mass frame, the dominant high-energy contribution again is obtained from (3.10) using the approximate equivalent-photon spectrum for each incident charged particle. For the case of proton–proton collisions, the dominant contribution to the cross section for the process $p+p\to p+\gamma^*+p+\gamma^*\to p+p+X$ suffers at least from a factor $[\ln(E_p/m_p)]^2/[\ln(E_e/m_e)]^2$ compared to the corresponding electron–electron-induced process. For the $E_p=28$ GeV available at the CERN intersecting storage rings, this ratio is about 1/6 compared to the electron–electron collision at $E_e=2$ GeV. Aside from the production of low-invariant-mass electron pairs (which in fact contributes $\sim$1.5 mb to the pp total cross section),[30] the two-photon processes are, in general, of negligible importance in hadron–hadron collisions. However, it can never be too much emphasized that *the two-photon process or, more generally, the electromagnetic process may play an important role in some hadron–hadron collisions.* One good example of this has already been given in Sec. XV for massive muon-pair production in hadron–hadron collisions. Another example can be

[30] As pointed out by Brodsky, Kinoshita, and Terazawa (1971b), this cross section is not included in the present measurements of the pp total cross section based on the transmission technique which requires deflection of incident beam into angles much larger than the typical angles involved in electron–positron pair production by the two-photon process.

seen in the papers by Berman, Bjorken, and Kogut (1971) and by Low and Treiman (1972), who have pointed out the possibility that the electromagnetic effect may become significant and competitive with the purely strong interaction contributions in hadron reactions for production of particles with sufficiently high transverse momentum.

In the case of inelastic $e-p$ collisions, the two-photon process producing lepton pairs corresponds to the usual trident process

$$e+p\rightarrow e+e^{+}+e^{-}+p. \tag{16.13}$$

Although *total* cross sections for electron tridents are large and of order $(Z^2\alpha^4/m_e^2)[\ln(E/m_e)]^2$, the contribution to the differential cross section at normal wide electron angles $(-k^2\gg m_e^2)$ has only a logarithmic dependence on the electron mass m_e and is a small standard component of the radiative-correction analysis in deep-inelastic $e-p$ scattering experiments. A comparison of the trident contribution with $d\sigma_{ep\rightarrow\text{anything}}/dE'd\Omega$ with the SLAC experimental results has been given in the paper by Brodsky, Kinoshita, and Terazawa (1971b). The trident cross section was obtained by numerical integration over the complete differential cross section computed by Brodsky and Ting (1966). The contribution is negligible $(<0.1\%)$ beyond $\theta_e=1.5°$.

XVII. CONCLUDING REMARKS

In this review we have discussed the production of hadrons and leptons by the two-photon process in high-energy colliding-beam experiments. This process will provide efficient means for the study of $C=+$ hadronic states and will play a role equal in importance and complementary to that of the one-photon annihilation process in the case of the $C=-$states. In order to extract information on $C=+$ and $C=-$ states from high energy colliding-beam experiments, however, we must be able to separate and identify these states experimentally. We shall examine this problem briefly in this final section (see Brodsky, Kinoshita, and Terazawa, 1971b).

Let us first discuss the $\pi^+\pi^-$ production in an e^+e^- collision. In this case pions may be produced by both e^+e^--annihilation and two-photon processes. Pions produced by the first process carry the energy E of the incident beam and are strongly constrained to be collinear and coplanar. On the other hand, pions produced by the second process have lower energies and only few of them come out in a collinear fashion as was shown in Sec. V. In the energy range $0.9\leq E\leq 1.2$ GeV covered by the colliding-beam experiments at Frascati (Alles-Borelli *et al.*, 1972; Bacci *et al.*, 1971, 1972) the latter pions, if any, therefore constitute a small background which can be easily distinguished by accurate energy measurements. Detection of scattered electrons, although desirable, is not absolutely necessary to separate the two processes. At higher beam energies, which have already become available at the CEA and SLAC e^+e^- colliding-beams or will be available at the DESY e^+e^- colliding-beams, however, the two-photon process will inevitably become the dominant process. Collinearity of a pion pair will no longer be a sufficient criterion. For positive identification of the two processes, besides accurate measurement of energy and momentum of the produced pion, detection of at least one of the scattered electrons is highly desirable.

As far as the study of the $C=+$ state of $\pi^+\pi^-$ (or $\pi^0\pi^0$) is concerned, the e^-e^- collision has an advantage over the e^+e^- collision in that it has no e^+e^- annihilation channel. In this case the $\pi^+\pi^-$ pair is produced mostly by the two-photon process and the production of $\pi^+\pi^-$ in the $C=-$state by the bremsstrahlung process [Fig. 1(b)] is expected to be a minor background. In this connection, we should recall that those events in which both electrons are detected at small angles $[\theta<(m_e/E^{1/2}]$ allow a particularly clean interpretation in terms of $C=+$production via the two photon process since the contribution of the $C=-$bremsstrahlung and contributions from nonzero photon mass become negligible in this region. Of curse, the last remark applies to the e^+e^- collision too.

The situation is enormously more complicated in the case of multihadron production $(n>3)$. Even in the energy range $0.9\leq E\leq 1.2$ GeV in which the Frascati experiments observed a surprisingly large number of multiparticle production events, its interpretation was by no means simple because forward-scattered electrons in the final state had been undetected and both identification and energy measurement had not been accurate enough to eliminate ambiguities. They apparently have made a large effort in making many cross checks to be convinced that what they observed is really multiple-hadron production by the one-photon annihilation process (Alles-Borelli *et al.*, 1972; Bartoli *et al.*, 1972).

In addition to complications from the copious lepton pair production discussed in Secs. V and XVIB, the predicted rate for hadron production by the two-photon process would exceed 3×10^{-33} cm^2 at $E=1$ GeV. As was shown in Sec. VI, a substantial fraction of these events can simulate multihadron production by the one-photon process under the experimental condition without detecting the scattered electrons. Also, additional background processes involving $C=-$production [see Fig. 1(b) and Sec. XVIA] and the hard-photon production process (see Sec. XVIA) have to be taken into account.

We believe that in most future e^+e^- colliding-beam experiments it will not be sufficient to have accurate measurement of only the energies and momenta of produced hadrons. Detection of either or both of the scattered electrons, in addition to the produced particles, will be required in order to identify and separate the one-photon and two-photon processes. There is no doubt that development of techniques for detection and possibly energy tagging of forward-scattered electrons is crucial for the success of e^+e^- colliding-beam physics. In conclusion, it should be emphasized that the $e^\pm e^-$ storage rings such as the DESY machine can separate the one-photon process and the two-photon process very easily by performing both e^+e^- and e^-e^- experiments because the former process is absent in e^-e^- collisions while the latter is common to both e^+e^- and e^-e^- collisions.

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