REVIEWS OF MODERN PHYSICS

VOLUME 44, NUMBER 4 OCTOBER 1972

Rigorous Constraints, Bounds, and Relations for Scattering Amplitudes

F. J. YNDURÁIN

Departamento de Física, Universidad Autónoma de Madrid, Cantoblanco, Madrid 34, Spain

We present a review of rigorous results on scattering amplitudes at low, intermediate, and high energies. The emphasis is on constraints that can be compared with experiment. Some new results are presented, in particular the existence of absolute bounds for inelastic processes, constraints for odd pion-pion waves, and a lower bound for scattering amplitudes at positive t. Most of the results rest only upon unitarity and what is popularly known as axiomatic analyticity, but there are a few cases where larger analyticity domains are needed.

CONTENTS

FOREWORD

This review has grown out of lectures given by the author at the G.I.F.T. meeting in Barcelona (June 1970) and at the School in Elementary Particles at Gif-sur-Yvette (September 1971). The emphasis in both cases (and this is reflected in these notes) was on results that can be proved rigorously and are directly related to phenomenology. We have preferred to give a few typical proofs in detail, rather than a long list of sketchy discussions.

These notes contain some new results as well as others that, although probably known to specialists, have never been published in detail. Among such results, let me quote the positivity bounds for odd $\pi\pi$ waves (and the ensuing justification of Weinberg's linear approximation) (Sec. 2.2), the existence of absolute bounds for inelastic processes (Sec. 3.4), and the lower bounds for scattering amplitudes at positive values of the momentum transfer (Sec. 4.1).

A bibliographical note and a fairly complete list of references is given at the end.

1. PRELIMINARIES

1.1. Introduction

Since the pioneering work of Froissart (F1), and through the many improvements and extensions of it carried on mainly by Martin and his school, it has become obvious that the analyticity that follows rigorously from, say, axiomatic field theory, can be used effectively, when combined with unitarity, to put fairly strong constraints on scattering amplitudes at low, medium, and high energies. The bounding art, described at times as applied field theory, and at other times as telling what a sensible S-matrix cannot be, has come now to a state such that meaningful comparisons with experiment can be made. To the rejoicing of the faithful, Goldberger's statement at the 1961 Solvay meeting ("The predictive power of axiomatic field theory is less than any positive ϵ, however small") has been proven wrong—even if, many times, the predictions are still an order of magnitude too coarse. What is perhaps better, the output of the bounding game has already found applications to phenomenology. To quote just an example, of the several solutions of Iliopoulos unitarized current algebra calculations of low energy $\pi\pi$(I1), all but one can be discarded on the basis of the crossing constraints of Martin and collaborators (M6; ABMM1).

Since these notes are about bounds, let me begin by bounding the notes. Among the many topics that shall *not* be discussed here are the actual applications of the limitations obtained [see the reviews of Wanders (W1) and references therein]; what may be described as "high energy theorems," although this is more and more becoming of the rigorous-bound-type (EK1; AKM1; GY1; SR2 for multiple references) and, finally, I will not discuss the very interesting work of Atkinson, Kupsch, and collaborators, who, by making the Mandelstam iteration procedure work, threaten us with explicit construction of the exact S-matrix, thereby killing the goose that lays the golden eggs. Also, there are up-to-date reviews (A1; A2) of this work.

Then comes the point of what is to be our input. We have unitarity, of course, and analyticity. With respect to the last we shall consider three different possibilities: First, we have what may be called axiomatic analyticity, namely, the analyticity that can be proved rigorously from axiomatic field theory. Perhaps it is not necessary to remark that by axiomatic field theory one need not stick to the old Wightman framework in which one needed such difficulty justifiable assumptions as temperedness; temperedness disappears altogether, locality simply means that measurements separated by spacelike intervals do not interfere, if these intervals are large enough, and the concept of interpolating fields can be dispensed with. This generalization has been completed recently [(EGM1) and references therein].

Second, we may assume quasiaxiomatic analyticity. This means that one assumes for any process the analyticity corresponding to that rigorously proven for $\pi\pi$ or πK. Quite likely, quasiaxiomatic analyticity will be, after all, also a consequence of axiomatic field theory. Let me note that quasiaxiomatic analtyicity seems to hold to all orders of Feynman perturbation theory (ELOP1). The third possibility is to assume the full Mandelstam representation. Surprisingly enough, and with only one major exception, the last two hypotheses are seldom any help.

Unless we state the contrary, we will use *only* axiomatic analyticity. A brief description of this analyticity can be found in Sec. 1.2; for more complete discussions we refer to (S3; M7).

Finally, I will mention that, as is unfortunately always the case in rigorous treatments, we shall have to neglect weak and electromagnetic interactions and restrict the word "particle" to mean "stable hadron."

1.2. Review of Kinematics, Analyticity and Unitarity

A. Kinematics

We will consider scattering processes such as that in Fig. 1. Here p and λ denote the four-momenta and helicities of the various particles; their masses will be denoted by

$$M^2=p_1^2, \qquad M'^2=P_1'^2, \qquad \mu^2=p_2^2, \qquad \mu'^2=p_2'^2.$$

The scattering amplitude for such process will be denoted by

$$\langle p_1\lambda_1;\, p_2\lambda_2 \mid \mathfrak{T} \mid p_1'\lambda_1';\, p_2'\lambda_2'\rangle = T_{\lambda_1\lambda_2;\lambda_1'\lambda_2'}{}^{AB\to A'B'}(s,\, t,\, u),$$

where

$$i\delta_4(p_1+p_2-p_1'-p_2')\mathfrak{T}\propto S-1$$

and

$$s=(p_1+p_2)^2, \qquad t=(p_2-p_2')^2, \qquad u=(p_1-p_2')^2.$$

These three variables are not independent but satisfy the relation

$$s+t+u=M^2+M'^2+\mu^2+\mu'^2,$$

which makes it possible to drop whichever of them that may be irrelevant for each particular problem considered.

Here T can be extended to complex values of the arguments. In particular, the same function will represent the scattering amplitudes for different channels; this is the celebrated crossing symmetry, and is a rigorous consequence of axiomatic field theory (BEG1). These "channels" correspond to different ways of looking at Fig. 1. They are traditionally denoted by

$$\begin{aligned} s \text{ channel:} &\quad A+B\to A'+B',\\ u \text{ channel:} &\quad A+\bar{B}\to A'+\bar{B}',\\ t \text{ channel:} &\quad A+A\to B+\bar{B}, \end{aligned}$$

where $\bar{X}$ is the antiparticle of X. All other possible reactions can be obtained from these three by use of TPC.

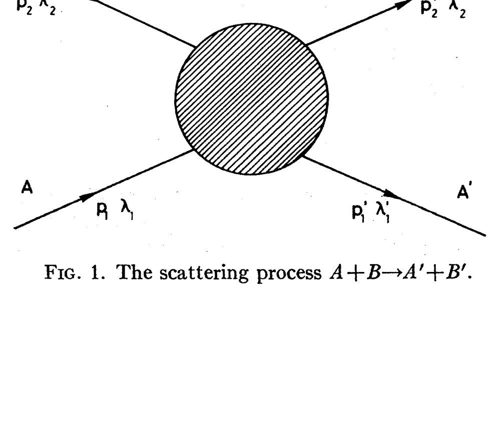


Fig. 1. The scattering process $A+B\to A'+B'$.

Next, we give the expressions for various physical quantities; in view of the applications, we shall consider the case $A=A'$, $B=B'$. Then, we have

$$W_s=s^{1/2}; \qquad q_s^2=\lambda(u, M^2, \mu^2)/4s; \qquad \cos\theta_s=1+t/2q_s^2,$$
$$W_u=u^{1/2}; \qquad q_u^2=\lambda(u, M^2, \mu^2)/4s; \qquad \cos\theta_u=1+t/2q_u^2,$$
$$W_t=t^{1/2}; \qquad q_t^2=t/4-\mu^2; \qquad Q_t=t/4-M^2;$$
$$\cos\theta_t=(s+q_t^2+Q_t^2)/2q_tQ_t.$$

Here λ is Källen's quadratic form, $\lambda(x, a^2, b^2)=[x-(a+b)^2][x-(a-b)^2]$, W_x is the c.m. energy in channel x, $\cos\theta_x$ the scattering angle, $q_{s/u}$ the c.m. momentum in channel s/u, q_t/Q_t the c.m. momentum of particle B/A (respectively) in channel t. We shall also use the laboratory energy of the projectile (particle B) in the s channel, $E=(s-M^2-\mu^2)/2\mu$.

B. Analyticity

$T_{\lambda_1\lambda_2;\lambda_1'\lambda_2'}{}^{AB\to AB}(s, t, u)$ may have kinematical or dynamical singularities. We shall show how to dispense with the first in a moment; as to the last, we remark that: (i) For fixed s (we drop u now, and forget temporarily about spin), $T(s, t)$ is analytic in what may be called the Lehmann–Mandelstam–Martin (LMM) ellipse[1] which, in the variable $\cos\theta_s$, has foci at $\cos\theta_s=\pm1$ and right extremity at $\cos\theta_s=1+t_0/q_s^2$; t_0 depends on the particular process under consideration. For $B=$octet meson, $A=$octet baryon, $t_0=4\mu^2$; for $B=$octet baryon, $A=$octet baryon, $t_0=\mu^2$ ($\mu=$mass of pion here). (ii) For t fixed inside the LMM ellipse, $T(s, t, u)$ has cuts starting at $s_0=(M_C+M_D)^2$ and running to $+\infty$, and starting at $u_0=(M_E+M_F)^2$ and running to $+\infty$. Here C, D (respectively: E, F) are the particles with lowest mass such that the process $A+B\to C+D$ (respectively: $A+\bar{B}\to E+F$) is possible, physically or virtually. Also, T has poles at every $s=M_G^2$ (respectively: $u=M_H^2$) if there exist single particle states, G(respectively: H) that couple to $A+B$ (respectively: $A+\bar{B}$). (iii) In addition there may exist a finite but eventually large region of possible nonanalyticity (the Bros–Epstein–Glaser region) referred to in the jargon as the potato, which is attached to the existing unphysical cuts. The best way to see this is pictorially as shown in Figs. 2 and 3 (p. 652). Quasiaxiomatic analyticity then means that the potato isn't really there. May I perhaps remark that the overwhelming majority of the results we will present are potato independent. (iv) Two variable analyticity. We refer to the review of Sommer (S3) and remark that enough has been proved (BEG1) to show that one can get, within the region of analyticity of T, from the physical region of any of the three channels to the physical region of any other one. If we denote by $T^{(x\to y)}$ to the continuation of T from channel x to channel y, this result implies the

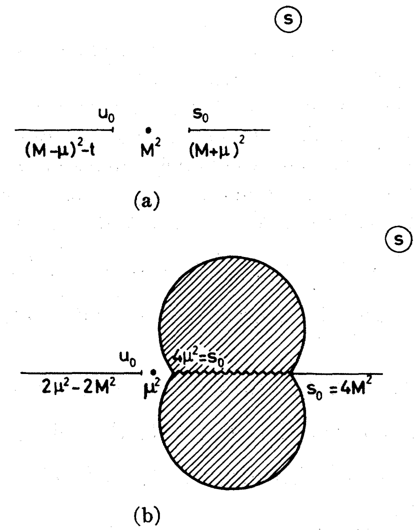


FIG. 2. (a) Singularities for πN scattering. (b) Singularities for NN scattering.

famous property of crossing symmetry, valid for any process $A+B\to A'+B'$

$$T_{\lambda_1\lambda_2;\lambda_1'\lambda_2'}{}^{(x\to y)}(s, t, u)=\sum d_{\lambda_1\bar\lambda_1}{}^{j_1}(\chi_{xy})d_{\lambda_1\bar\lambda_2}{}^{j_2}(\chi_{xy})$$
$$\times T_{\bar\lambda_1\bar\lambda_2;\bar\lambda_1'\bar\lambda_2'}{}^{(y)}(s, t, u)d_{\lambda_1'\bar\lambda_1'}{}^{j_1'}(\chi_{xy})d_{\lambda_2'\bar\lambda_2'}{}^{j_2'}(\chi_{xy}), \qquad (1.1)$$

where $T^{(y)}$ stands for the physical amplitude in channel y, $j_1\cdots j_2'$ are the spins of $A, \cdots, B'$, the d^j the well-known Wigner functions, and χ_{yx} the so-called crossing angle. Here χ_{xy} depends on the s, t, u, and the masses of the particles; its explicit form can be found in (MM1; CMN1).

Equation (1.1) does not take care of internal quantum numbers; these will give rise to other, now numerical, crossing matrices: if we denote by I_z the quantum numbers in channel z, we have

$$T_{I_x}{}^{(x\to y)}=\sum_{I_y}\alpha_{I_xI_y}T_{I_y}{}^{(y)}.$$

Now we turn to kinematical singularities. The helicity amplitudes are plagued by them. Very general ways of removing them are described in (CMN1) [see also (B1)]. Here we shall only construct one amplitude free from kinematical singularities; this will be sufficient for our purposes. Again considering the case $A=A'$, $B=B'$, it can be shown that

$$\hat{F}(s, t, u)\equiv[\lambda(s, M^2, \mu^2)]^{j_1+j_2}\xi_{j_1+j_2}(s, t)$$
$$\times\sum_{\lambda_1\lambda_2}T_{\lambda_1\lambda_2;\lambda_1\lambda_2}{}^{(s)}(s, t, u), \qquad (1.2)$$

where

$$\xi_{j_1+j_2}(s, t)=1, \qquad \text{if } j_1+j_2=\text{integer}$$
$$=[s-(M+\mu)^2]^2+st, \qquad \text{if } j_1+j_2\neq\text{integer},$$

is free from kinematical singularities in all three channels (MM2), (CMN1).

---

[1] We could throw in still another M if we recall that the extension to spinning particles involved also G. Mahoux.

C. Unitarity

We now turn to the ever-useful partial wave expansions whose main interest (for us) lies in that they diagonalize unitarity. Because we will mainly work with pions and nucleons, we shall assume that only particle A has spin, j. Also only s-channel expansions will be given now; some u- and t-channel expansions shall be presented later. In this situation one has (JW1)

$$T_{\lambda\lambda'}(s,t)=\sum(2s+1)d_{\lambda\lambda'}{}^{J}(\theta_s)f_{\lambda\lambda'}{}^{J}(s), \quad (1.3\text{a})$$

where f are the partial waves, with total angular momentum J:

$$f_{\lambda\lambda'}{}^{J}(s)=s^{1/2}/2q_s\langle B, A\lambda; s, J\mid \mathfrak{T}\mid B, A\lambda'; s, J\rangle, \quad (1.3\text{b})$$

with obvious notation. Unitarity then gives

$$\operatorname{Abs} f_{\lambda\lambda'}{}^{J}(s)=\sum_n\langle B, A\lambda; s, J\mid \mathfrak{T}\mid n; s, J\rangle \times\langle B, A\lambda'; s, J\mid \mathfrak{T}\mid n; s, J\rangle^{*}(2q_s/s^{1/2}). \quad (1.4\text{a})$$

Therefore, if we define

$$f_{\lambda\lambda'}{}^{J}(s)\equiv(s^{1/2}/2q_s)\,\hat{f}_{\lambda\lambda'}{}^{J}(s),$$

we shall have the relations

$$\operatorname{Abs}\hat{f}_{\lambda\lambda'}{}^{J}(s)=\sum_n\alpha_\lambda(n)\alpha_{\lambda'}{}^{*}(n), \quad (1.4\text{b})$$

$$0\leq\sum_{\lambda''}\mid\hat{f}_{\lambda''\lambda}{}^{J}(s)\mid^2\leq \operatorname{Im}\hat{f}_{\lambda\lambda}(s)=\operatorname{Abs}\hat{f}_{\lambda\lambda}(s)\leq 1. \quad (1.4\text{c})$$

The equality holds between $\mid\hat{f}\mid^2$ and $\operatorname{Im}\hat{f}$ in the elastic region. In writing Eqs. (1.4) we have assumed invariance under time reversal; this is not necessary for anything that follows, but simplifies the writing.

Then we introduce the important notion of *positivity*. This was first introduced by A. Martin (M5) for spinless particles, and generalized by Martin and Mahoux (MM2; M2) for arbitrary spin. We will say that the amplitude

$$F(s,t,u)=\sum_{\{\lambda\}}C_{\lambda_1\lambda_2;\lambda_1'\lambda_2'}(s,t)\,T_{\lambda_1\lambda_2;\lambda_1'\lambda_2'}(s,t,u),$$

has *positivity* (M5) if it is free from kinematical singularities and, both in the s and u channels, we can write

$$\operatorname{Im} F=\sum_n a_n\cos n\theta,$$

$a_n\geq 0$ in the physical s and u regions. It may be shown (MM2) that $\hat{F}$, as defined in (1.2), possesses such positivity. Other amplitudes with positivity have been given by Bell (B1).

One fundamental consequence of positivity is the fact that

$$\partial^n/\partial t^n \operatorname{Im} F(s,t)\geq 0 \quad \text{for} \quad 0\leq t\leq t_0, \qquad n=0,1,2,\cdots.$$

This follows by explicit differentiation, and noting that

$$(\partial/\partial t)\cos n\theta=\sum_{n'}b_{nn'}\cos n'\theta,$$

with $b_{nn'}\geq 0$, so that the derivatives of $\operatorname{Im} F(s,t)$ have the same structure as $\operatorname{Im} F(s,t)$ itself. This property ($\partial^n/\partial t^n \operatorname{Im} F\geq 0$) is also referred to as (weak) positivity It shall be used repeatedly in all that follows.

Finally two points of notation must be mentioned. For spinless particles, $\hat{F}=T$; the amplitude will be denoted by F. We will use T without helicity labels when we consider particles that may eventually have spin, but whose spin we are temporily ignoring. Second, we shall denote by $F_x(T_x)$ the absorptive (=imaginary) part of $F(T)$ in channel x.

1.3. The Froissart–Gribov Representation

One of the most useful tools for squeezing information out of unitarity and analyticity is the Froissart–Gribov representation (F2; G1). This is obtained as follows: sit in the t channel, in the unphysical region ($0\leq t<t_0$; we are again considering $A=A'$, $B=B'$). Taking spinless particles, to begin with, one can write

$$f_l{}^{(+)}(t)=\frac{1}{2}\int_{-1}^{+1}d\cos\theta_t P_l(\cos\theta_t)F^{(+)}(s,t,u), \quad (1.6\text{a})$$

where we have altered slightly the notation $f_l\equiv f_{00}{}^{J=l}$, and $F^{(+)}(s,t,u)=F(s,t,u)+F(u,t,s)$. In this t region, F satisfies fixed t dispersion relations; we shall prove below (Sec. 1.4) that these are at most twice subtracted. Hence we have

$$F^{(+)}(s,t,u)=a(t)+b(t)s+\frac{(s-s_1)(s-s_2)}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{F_s{}^{(+)}(s',t)}{(s'-s_1)(s'-s_2)(s'-s)}+\frac{(s-s_1)(s-s_2)}{\pi}\int_{(M+\mu)^2}^{\infty}du'\frac{F_u{}^{(+)}(u',t)}{(s'-s_1)(s'-s_2)(u'-u)}. \quad (1.6\text{b})$$

Now, $\cos\theta_t$ is a linear function of s: hence, if $l\geq 2$, the subtraction polynomial will give no contribution if we insert Eq. (1.6b) into Eq. (1.6a). Furthermore, as all integrations are convergent, we may exchange orders of integration so that

$$f_l{}^{(+)}(t)=\pi^{-1}\int_{(M+\mu)^2}^{\infty}ds'\left\{F_s{}^{(+)}(s',t)\,\frac{1}{2}\int_{-1}^{+1}d\cos\theta_t \times P_l(\cos\theta_t)\frac{(s-s_1)(s-s_2)}{(s'-s_1)(s'-s_2)(s'-s)}+[s\leftrightarrow u]\right\}. \quad (1.6\text{c})$$

Then, one uses Neumann's formula:

$$\frac{1}{2}\int_{-1}^{+1}dxP_l(x)\,\frac{c_0+c_1x+\cdots+c_lx^l}{c_0+c_1z+\cdots+c_lz^l}\,(z-x)^{-1}=Q_l(z),$$

to integrate the first term in Eq. (1.6c). It is *not* necessary to integrate the second ($s\leftrightarrow u$) because, as $\cos\theta_t\propto(s-u)$, the exchange $s\leftrightarrow u$ amounts to exchanging $\cos\theta_t$ with $-\cos\theta_t$: since $Q(-z)=Q(z)\times(-1)^{l+1}$, it follows that the ($s\leftrightarrow u$) term has only the effect of killing odd waves and doubling even ones. With all this, we obtain the Froissart–Gribov formula,

$$f_l^{(+)}(t)=(\pi q_tQ_t)^{-1}\int_{(M+\mu)^2}^{\infty}ds\{F_s(s,t)+F_u(s,t)\}$$
$$\times Q_l\left(\frac{s+q_t^2+Q_t^2}{2q_tQ_t}\right),\qquad l=\text{ even }\geq 2. \quad (1.7)$$

Where will this formula be valid? Clearly, until Eq. (1.6b) breaks down, i.e., up to $t=t_0$.

The incorporation of spin is trivial; all that we must do is to use $\hat{F}$ as given by Eq. (1.2), and substitute f_l by appropriate projections as, e.g.,[2]

$$k_l^{(+)}(t)=\int_{-1}^{+1}d\cos\theta_t P_l^{(\alpha\beta)}(\cos\theta_t)\hat{F}(s,t);$$

Neumann's formula is substituted by

$$\frac{1}{2}\int_{-1}^{+1}dxP_l^{(\alpha\beta)}(x)\,\frac{c_0+c_1x+\cdots+c_lx^l}{c_0+c_1z+\cdots+c_lz^l}\,(z-x)^{-1}=Q_l^{(\alpha\beta)}(z),$$

where α and β depend on the spins involved. For example, for πN scattering, we find

$$k_l^{(+)}(t)=\frac{t-2\mu^2}{2Q_tq_t}\left[\left(\frac{M^2+q_t^2+Q_t^2}{2q_tQ_t}\right)^2-1\right]$$
$$\times G_{NN\pi}{}^2Q_l^{(11)}\left(\frac{M^2+q_t^2+Q_t^2}{2q_tQ_t}\right)$$
$$+(\pi q_tQ_t)^{-1}\int_{(M+\mu)^2}^{\infty}ds[\hat{F}_s(s,t)+\hat{F}_u(s,t)]$$
$$\times\left[\left(\frac{s+q_t^2+Q_t^2}{2q_tQ_t}\right)^2-1\right]Q_l^{(11)}\left(\frac{s+q_t^2+Q_t^2}{2q_tQ_t}\right).$$

The first term in the right hand side comes from the contribution of the nucleon pole. The $k_l^{(+)}(t)$ are related to the usual helicity waves, $g_J^{(\pm)}(t)$ by

$$k_l^{(+)}(t)=\frac{2t^{1/2}}{q_t^{1/2}Q_t^{3/2}}\frac{l+1}{2l+3}\left\{g_l^{(+)}(t)-g_{l+2}^{(+)}(t)-\frac{t^{1/2}}{2M}\right.$$
$$\left.\times\left[\left(\frac{l}{l+1}\right)^{1/2}g_l^{(-)}(t)-\left(\frac{l+2}{l+3}\right)^{1/2}g_{l+2}^{(-)}(t)\right]\right\}.$$

For details, see (C1; CY2).

1.4. Polynomial Bounds on Scattering Amplitudes

We said that, from axiomatic field theory, one can prove that $|T(s,t)|<C's^{N'}$, N' independent of t, if t is physical. Furthermore, Jin and Martin (JM1; M5) have proven a theorem which shows that the same bound, except perhaps increased by one power of s, if N' is integer, also holds in the analyticity ellipse with right extremity t_0. We will now determine the value of N, starting from

$$|T(s,t)|<Cs^N,\qquad t\leq t_0. \quad (1.8)$$

As usual we begin by neglecting spin. Then, using the partial wave expansion, (1.8) can be rewritten as

$$Cs^N>|T(s,t_0)|>T_s(s,t_0)=\sum_0^\infty(2l+1)P_l(\cos\theta_s^{(0)})$$
$$\times\operatorname{Im}f_l(s)>(2l+1)P_l(\cos\theta_s^{(0)})\operatorname{Im}f_l(s);$$
$$\cos\theta_s^{(0)}\equiv\cos\theta_s(t=t_0)=s+(t_0/2q_s^2),$$

i.e.,

$$[\operatorname{Im}\hat{f}_l(s)]^{1/2}<(2q_s)^{1/2}C^{1/2}s^{N/2}/s^{1/4}[(2l+1)P_l(\cos\theta_s^{(0)})]^{1/2}.$$

On the other hand, for t physical, we have $|P_l(\cos\theta_s)|\leq 1$: therefore, for t physical now,

$$\frac{2q_s}{s^{1/2}}|T(s,t)|=|\sum_0^\infty(2l+1)P_l(\cos\theta_s)f_l(s)|$$
$$\leq\sum_0^\infty(2l+1)|f_l(s)|\leq\sum_0^\infty(2l+1)[\operatorname{Im}f_l(s)]^{1/2}$$
$$=(\sum_0^{L(s)}+\sum_{L(s)+1}^\infty)(2l+1)[\operatorname{Im}f_l(s)]^{1/2}$$
$$\leq\sum_0^{L(s)}(2l+1)+\sum_{L(s)+1}^\infty(2l+1)[\operatorname{Im}f_l(s)]^{1/2}$$
$$<[L(s)]^2+2L(s)+1+\sum_{L(s)+1}^\infty(2l+1)[\operatorname{Im}f_l(s)]^{1/2}$$

we have used the unitarity constraints, Eq. (1.4), $0\leq|\hat{f}_l|^2\leq\operatorname{Im}\hat{f}_l\leq 1$, and the above result is valid for any L. Then, one remembers (MOS1) that $P_l(\cos\theta_s^{(0)})>C'\exp[C''l/s^{1/2}]$: therefore, taking $L(s)=ns^{1/2}\log s$, and using both inequalities together, we get (t physical)

$$|T(s,t)|\lesssim n^2s\log^2s+C'''s^\nu\sum_{ns^{1/2}\log s}^\infty(2l+1)^{1/2}\times\exp[-C''l/s^{1/2}].$$

The second term can be summed explicitly; it behaves as $s^{-nC_0+C_1}$, so that it is negligible if n is large enough. Thus, we have

$$|T(s,t)|<(\text{constant})s\log^2s. \quad (1.9)$$

This is the original Froissart bound (F1); it will be elaborated further on. As we have derived it, it holds only for s positive. However, exchanging the roles of particles B and $\bar{B}$, it will, due to crossing symmetry, still hold for u positive, i.e., for s negative. Then, the Phragmén–Lindelöf theorem tells us that, in fact, (1.9) is valid along any direction in the complex s plane. Therefore, we have only to apply the Jin–Martin theorem to find that the N of (1.8) was indeed equal to

[2] For the definitions of the special functions, $P_l^{(\alpha\beta)}$, $Q_l^{(\alpha\beta)}$, etc., we follow (MOS1).

2. Actually, one can prove more: one can show that

$$\int^{\infty} ds\, s^{-3}\, |T(s,t)| < \infty,$$

or

$$|T(s,t)| < Cs^{2-\epsilon}, \qquad \epsilon>0, \quad \text{for} \quad t<t_0. \quad (1.10)$$

The inclusion of spin is fairly simple. One substitutes T by $\hat{F}$ as given by Eq. (1.2), and the bounds

$$P_l(\cos\theta_s^{(0)}) > C' \exp[C''l/(s)^{1/2}]$$

by corresponding bounds for the $d_{\lambda\lambda'}^{J}(\theta_s)$; the rest is similar. The result is that one finds (1.9) for any $T_{\lambda_1\lambda_2;\lambda_1'\lambda_2'}$, and (1.10) is replaced by

$$|\hat{F}(s,t)| < Cs^{2(j_1+j_2)} s^{\rho_{j_1+j_2}} s^{2-\epsilon}, \qquad \epsilon>0, \quad \text{for} \quad t<t_0, \quad (1.11)$$

where

$$\rho_{j_1+j_2}=0 \qquad \text{if } j_1+j_2=\text{integer}$$
$$=2 \qquad \text{if } j_1+j_2\neq\text{integer}.$$

We refer to (MM1; M1; B1) for details.

2. LOW-ENERGY BOUNDS AND CONSTRAINTS

2.1. Absolute Bounds inside the Mandelstam Triangle for $\pi\pi$ Scattering

By repeated use of analyticity, crossing, and unitarity, supplemented with refined minimization techniques, Lukaszuk and Martin (LM1) obtained a set of absolute bounds on the $\pi\pi$ scattering amplitude, $F(s,t,u)$, inside the Mandelstam triangle: $s\geq 0$, $t\geq 0$, $u\geq 0$. Although these bounds are valid only in unphysical regions, they are very interesting for two reasons: first, from a purely theoretical point of view, they show that the $\pi\pi$ forces have a saturation point, i.e., they cannot be made arbitrarily strong. This is perhaps best understood by recalling that a bound on $|F(s,t,u)|$ inside the Mandelstam region implies a bound on the four-pion coupling constant defined as[3] (CM2) $\lambda_{4\pi}\equiv |F(\frac{4}{3},\frac{4}{3},\frac{4}{3})|$. Second, it will be obvious that bounds on $|F(s,t,u)|$ imply bounds on the S-wave scattering length, which are certainly useful for phenomenological analyses of $\pi\pi$ scattering (I1).

Instead of giving the long and complicated general proofs, we shall exemplify the methods by giving lower bounds for $F(4,2,-2)$ and to the scattering length, a_0, for $\pi^0\pi^0$ scattering, following a method suggested by Martin and developed by Common and Wit (CW1).

To begin, we recall the definitions of the scattering lengths,

$$a_l \equiv \lim_{q_s\to 0} \hat{f}_l(s)/q_s^{2l}, \qquad (2.1)$$

[3] Throughout Sec. 2.1 we take units such that μ = mass of pion = 1.

and the partial wave expansions

$$F(s,t)=\sum_{l=0}^{\infty}(2l+1)P_l\left(1+\frac{2t}{s-4}\right)f_l(s),$$
$$f_l(s)=[s/(s-4)]^{1/2}\hat{f}_l(s). \qquad (2.2)$$

Due to the statistics of the pions, we have $f_l(s)=0$ if l=odd; we shall understand that all sums over l run over even values of l only. Then, we will use a result that will be proven in the following section

$$0<a_{l+2}<(1/16)[(l+1)(l+2)/(l+\tfrac{3}{2})(l+\tfrac{5}{2})]a_l. \quad (2.3)$$

Using well-known properties of the Legendre polynomials (MOS1) and (2.3), one sees at once that

$$F(4,2,-2)=a_0+\pi^{-1/2}\sum_{l=2}^{\infty}(2l+1)2^l\frac{\Gamma(l+\frac{1}{2})}{\Gamma(l+1)}a_l$$
$$\leq a_0+Na_2, \quad (2.4a)$$

where

$$N=\frac{16}{\pi^{1/2}}\sum_{l=2}^{\infty}(2l+2)2^{-l}\frac{\Gamma(l+\frac{1}{2})}{\Gamma(l+1)}\prod_{j=4}^{l}\frac{j(j-1)}{j^2-\frac{1}{4}}. \quad (2.4b)$$

The product $\prod_{j=4}^{l}$ is defined to be unity for $l=2$. N can be computed numerically quite easily. Now, numerical estimates using phenomenological estimates for a_2 ($a_2\sim 10^{-3}$) show that Na_2 is small; therefore we neglect this term (later we will indicate the results one obtains with an exact treatment, that is, without neglecting a_2). If we do so, (2.4a) becomes

$$a_0\geq F(4,2,-2);$$

Therefore, a *lower* bound on $F(4,2,-2)$ implies a lower bound on a_0.

To find the bound to F we begin by assuming that $F(4,2,-2)<0$: this is no restriction because we are looking for a *lower* bound. Then, we write a dispersion relation for $F(s,2)$:

$$F(s,2,2-s)-F(4,2,-2)=\frac{2(s+2)(s-4)}{\pi}$$
$$\times\int_4^{\infty}ds'\frac{(s'-1)F_s(s',2)}{(s'-4)(s'-s)(s'+2)(s'-2+s)}. \quad (2.5)$$

Since $F_s(s',2)\geq 0$ from positivity [cf. Eq. (2.2)] and we have assumed $F(4,2,-2)<0$, it is clear from Eq. (2.5) that $F(s,2,2-s)$ has no zero in the cut s plane and is convex between $s=1$ and $s=4$, i.e.,

$$(d/ds)F(s,2,2-s)>0, \qquad 1\leq s\leq 4.$$

Now take $s=2$; Eq. (2.5) yields the majorization,

$$|F(2,2,0)|>\frac{16}{\pi}\int_4^{\infty}ds'\frac{(s'-1)F_s(s',2)}{(s'-4)(s'-2)(s'+2)s'}.$$

The next step is to apply repeatedly Schwartz's inequality to F; letting $x=1+4(s-4)^{-1}$, we get

$$|F(s,0)|^2=\tfrac{1}{4}|(s^{1/2}/q_s)\sum(2l+1)f_l(s)|^2$$
$$\leq(s/4q_s^2)\{\sum(2l+1)|f_l(s)|^2P_l(x)\}$$
$$\times\{\sum'(2l+1)[P_l(x)]^{-1}\}$$
$$\leq(s^{1/2}/2q_s)F_s(s,2)\sum[(2l+1)/P_l(x)]$$
$$\equiv(s^{1/2}/2q_s)\Phi(s)F_s(s,2),$$
$$\Phi(s)\equiv\sum_0^\infty\frac{2l+1}{P_l[1+4/(s-4)]}.$$

Therefore,

$$|F(2,2,0)|>\frac{16}{\pi}\int_4^\infty ds'\frac{2q_{s'}(s'-1)|F(s,0)|^2}{(s'-4)(s'-2)(s'+2)s'\Phi(s')(s')^{1/2}}.$$

This can be rewritten as

$$|F(2,2,0)|^{-1}>(2\pi)^{-1}\int_{-\pi}^{\pi}d\alpha W(\alpha)|F_R(e^{i\alpha})|^2,\quad(2.6a)$$

where

$$F_R(e^{i\alpha})\equiv F(s,0)/F(2,2,0),\quad(2.6b)$$

$$W(\alpha)\equiv\frac{1+2v}{2(1+v)(2+v)\sum(2l+1)/P_l[1+2(v-1)^{-1}]},\quad(2.6c)$$

$$v\equiv[1-\tan(2\alpha/2)]^{1/2},$$

and we have made the change of variables

$$w=\frac{i-[(s-2)^2/4-1]^{1/2}}{i+[(s-2)^2/4+1]^{1/2}},\qquad r\equiv|w|,\qquad\alpha\equiv\arg w.$$

Again using the Schwartz inequality, (2.6a) gives

$$|F(2,2,0)|^{-1}\geq\left\{\exp\left[(2\pi)^{-1}\int_{-\pi}^{\pi}d\alpha W(\alpha)\right]\right\}$$
$$\times\left\{\exp\left[(2\pi)^{-1}\int_{-\pi}^{\pi}d\alpha\log|F_R(e^{i\alpha})|^2\right]\right\}.$$

Since $F(s,2,2-s)$ has no zeros, we have

$$(2\pi)^{-1}\int_{-\pi}^{\pi}d\alpha\log|F_R(e^{i\alpha})|^2=2\log F_R(w=0).$$

But $w=0$ corresponds to $s=2$, so that using (2.6b) gives $\log F_R(w=0)=0$: we have proved

$$|F(2,2,0)|^{-1}\geq\exp[(2\pi)^{-1}\int_{-\pi}^{\pi}d\alpha W(\alpha)].$$

Here $W(\alpha)$ is an explicit function; thus we can perform (numerically!) the integration in the right hand side above getting

$$|F(2,2,0)|\leq19;\qquad F(2,2,0)\geq-8.9;$$

the second if $F(2,2,0)$ was negative. Due to the convexity of $F(s,2,2-s)$, we therefore obtain

$$-8.9\leq F(2,2,0)\leq F(4,2,-2)\leq a_0:$$

these are the desired lower bounds. The analysis can be much refined, and we give a list of what has been obtained:

$$a_0\geq-19\quad[\text{exact (LM1)}],$$
$$a_0\geq-4\quad[\text{exact (BV1)}],$$
$$a_0\geq-3.5\quad[\text{exact (M8)}],$$
$$a_0\geq-1.15\;[\text{if } a_2 \text{ small (CW1)}];\quad(2.7a)$$

for definite isospins one has, e.g.,

$$a_0^{(0)}\geq-7.5,\qquad a_0^{(2)}\geq-7.1\;[\text{exact (C3)}].\quad(2.7b)$$

This is to be compared with the phenomenological estimate (MP1) $a_0\simeq0.16$. The constant improvement with time is apparent. For the whole amplitude one gets, for example,

$$|F(2,2,0)|\leq8.5\;[\text{exact (CW1)}],$$
$$|F(3,2,-1)|\leq75\;[\text{exact (LM1)}],$$
$$-50\leq F(\tfrac{4}{3},\tfrac{4}{3},\tfrac{4}{3})\leq8\;[\text{exact (BV1)}].\quad(2.8)$$

Apart from these bounds there is a variety of other results that be proven for the S wave in the unphysical region; for example, one gets $f_0(3.205)>f_0(0.2134)>f_0(2.9863)$. In particular, these inequalities (which include constraints for df_0/ds as well) imply that $f_0(s)$ has a unique minimum inside $0\leq s\leq4$; this minimum must lie somewhere between 1.12 and 1.7. For further details on these constraints, cf. (M6). Some others will be presented in Sec. 2.3 here.

2.2. Positivity Constraints

Before we can really take profit from any constraints derived on partial waves it is clear that one has to know how many waves one has to take into account (for a given accuracy). Certainly S and P waves are needed, but when should one stop?

The answer to this question is one of the important consequences of the set of constraints that will be described below.

The starting point in finding such constraints is the Froissart–Gribov representation, Eq. (1.7). We will begin with $\pi^0\pi^0$ scattering, and generalize the result afterwards.

Due to positivity, we have $F_s+F_u\geq0$ in the region of integration of Eq. (1.7). Furthermore, it can be proved[4] that

$$(\partial/\partial z)[Q_{l+n}^{(\alpha\beta)}(z)/Q_l^{(\alpha\beta)}(z)]<0,\qquad z\geq1.$$

Now, in the region of integration of Eq. (1.7), we have

$$z\equiv\frac{2(s+t/2-2\mu^2)}{4\mu^2-t}\geq z_0\equiv\frac{t+4\mu^2}{4\mu^2-t}\geq1;$$

[4] A general proof of the equation below may be found in (Y3), Appendix A.

therefore,

$$\int_{4\mu^2}^{\infty} ds\{F_s+F_u\}\{Q_{l+2}(z)/Q_{l+2}(z_0)-Q_l(z)/Q_{l+2}(z_0)\}<0,$$

so that one has the bounds

$$0<f_{l+2}^{00}(t)<\{Q_{l+2}[z_0(t)]/Q_l[z_0(t)]\}f_l^{00}(t),\qquad l=\text{even}\geq 2. \quad (2.9a)$$

If the limit $q_t\rightarrow 0$ is taken, this becomes

$$0<a_{l+2}^{00}<(1/16)[(l+1)(l+2)/(l+\tfrac{3}{2})(l+\tfrac{5}{2})]a_l^{00},\qquad l=\text{even}\geq 2. \quad (2.9b)$$

Many other relations can be obtained by relating Eq. (1.7) to the so-called Hausdorff-like moment problems (ST1); we refer to (C4; Y1; CP2). The generalization of (2.9) is now simple. With respect to isospin, all we need is that F_s+F_u be positive; hence, (2.9) holds automatically if replacing, e.g., $f^{\pi^0\pi^0}\equiv f^{00}$ and a^{00} by, respectively, $f^{(I_t=0)}$ and $a^{(I_t=0)}$, or by $f^{+0}=(1/3)f^{(I_t=0)}-(1/3)f^{(I_t=2)}$ and $a^{+0}=(1/3)a^{(I_t=0)}-(1/3)a^{(I_t=2)}$. To get results for $I_t=1$ one must elaborate a bit further. First, one notes that

$$F_s^{00}+F_u^{00}=\tfrac{1}{3}F_s^{(I_s=0)}+\tfrac{2}{3}F^{(I_s=2)},$$
$$F_s^{+0}+F_u^{+0}=\tfrac{1}{2}F_s^{(I_s=1)}+\tfrac{1}{2}F^{(I_s=2)}.$$

Then, the Froissart–Gribov representation for $f^{(I_t=1)}$ reads (note that $F^{(I_t=1)}$ is antisymmetric in $\cos\theta_t$)

$$f_l^{(I_t=1)}(t)=(\pi\mid q_t\mid^2)^{-1}\int_{4\mu^2}^{\infty} dsF_s^{(I_t=1)}(s,t)\times Q_l\left(\frac{2s+t-4\mu^2}{4\mu^2-t}\right),\qquad l=\text{odd}\geq 3.$$

Now, we have

$$F_s^{(I_t=1)}=\tfrac{1}{3}F^{(I_s=0)}+\tfrac{1}{2}F^{(I_s=1)}-\tfrac{5}{6}F^{(I_s=2)}$$

so that

$$\mid F_s^{(I_t=1)}\mid\leq\tfrac{1}{3}F_s^{(I_s=0)}+\tfrac{1}{2}F_s^{(I_s=1)}+\tfrac{5}{6}F_s^{(I_s=2)}\leq(F_s^{00}+F_u^{00})+(F_s^{+0}+F_u^{+0}):$$

therefore,

$$\mid f_l^{(I_t=1)}(t)\mid\leq(\pi\mid q_t\mid^2)^{-1}\int_{4\mu^2}^{\infty} ds[F_s^{00}+F_u^{00}]Q_l+(\pi\mid q_t\mid^2)^{-1}\int_{4\mu^2}^{\infty} ds[F_s^{+0}+F_u^{+0}]Q_l,$$

and, hence, reasoning as above, because now both brackets are positive.

$$\mid f_l^{(I_t=1)}(t)\mid<\{Q_l[z_0(t)]/Q_2[z_0(t)]\}[f_2^{+0}(t)+f_2^{00}(t)],\qquad l=\text{odd}\geq 3. \quad (2.10a)$$

In particular, for $l=3$, $q_t\rightarrow 0$, we have

$$a_3^{(1)}<(2/9)(a_2^{+0}+a_2^{00}). \quad (2.10b)$$

Before discussing the implications of (2.9), (2.10), I will extend the analysis to particles other than π's. The extension to $\pi\pi\rightarrow K\bar{K}$ is trivial, and we leave it to the reader. In respect to spin, one can generalize the results for arbitrary spin (C1): however, since the only case of interest is $\pi\pi\rightarrow N\bar{N}$, we shall concentrate on that. Also here the generalizations are trivial; all one has to do (CY2) is to substitute in Eq. (2.9a) (say), $f_l(t)$ by

$$\bar{k}_l^{(+)}(t)=k_l^{(+)}(t)-[(t-2\mu^2)/2Q_tq_t]\times\{[(M^2+q_t^2+Q_t^2)/2q_tQ_t]^2-1\}\times G_{NN\pi}^2Q_l^{(11)}[(M^2+q_t^2+Q_t^2)/2q_tQ_t],$$

(the notation is that of Sec. 1.3), and $Q_n[z_0(t)]$ by $Q_n^{(11)}[(t+4\mu M)/4q_tQ_t]$. Thus, Eq. (1.9b) becomes[5]

$$0<\bar{a}_{l+2}^{(+)}<\{1/[2\mu(M+\mu)]^2\}\bar{a}_l^{(+)},\qquad l=\text{even}\geq 2,$$

where the $\bar{a}_l^{(+)}$ are given by

$$\bar{a}_l^{(+)}=[4\pi^2M/2^l\gamma_l(M^2-\mu^2)]\bar{\alpha}_l^{(+)}-[2\pi^2\mu^2Mc_{ll}/2^l\gamma_l(M^2-\mu^2)]\bar{\alpha}_l^{(-)},$$

$$\gamma_l\equiv\pi^{1/2}l!/2^{l+1}\Gamma(l+\tfrac{3}{2}),$$

$$c_{ll}\equiv\frac{2l+1}{[l(l+1)]^{1/2}}\int_{-1}^{+1} dxP_l'(x)P_l(x)x,$$

and the $\bar{\alpha}_l^{(\pm)}$ are the standard helicity scattering lengths,

$$\bar{\alpha}_l^{(+)}=\lim_{q_t\rightarrow 0}[-t^{1/2}Q_t^{1/2}/2q_t(q_tQ_t)^l]\bar{g}_l^{(+)}(t)$$
$$\bar{\alpha}_l^{(-)}=\lim_{q_t\rightarrow 0}[-Q_t^{1/2}/q_t^{1/2}(q_tQ_t)^l]\bar{g}_l^{(-)}(t).$$

The constraints for $I_t=1$ odd waves can also be generalized (Y3).

Coming back now to $\pi\pi$ scattering, Eqs. (2.9) and (2.10) tell us that, within an error of $[Q_3(z_0)/Q_2(z_0)]\times(f_2^{+0}+f_2^{00})$, one can approximate any $\pi\pi$ scattering amplitude by only the S, P and D waves in the t channel. Since $\pi\pi$ is fully crossing symmetric, this holds true in all three channels, and hence we can write

$$F(s,t)=C_0+C_1t+C_2s+C_3t^2+C_4s^2+C_5st, \quad (2.11)$$

where the C's only depend on the isospin. From Eqs.

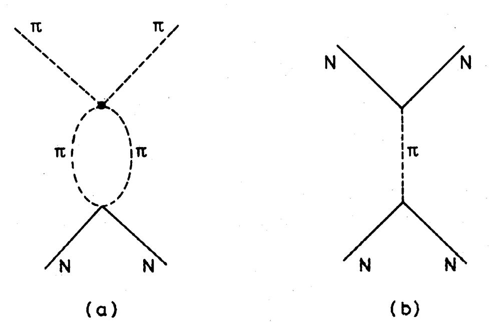


Fig. 3. Lowest possible exchanges. (a) For πN scattering. (b) For NN scattering.

[5] The bar over a quantity means that, just as for the k_l, we have subtracted the (explicitly known) contribution of the nucleon pole.

(2.9b), (2.10b) we expect the error to be typically $\sim 20\%$ $(a_2^{+0}+a_2^{00})$. With the existing values for a_2 (MP1), this is a very small quantity. Of course one has to bear in mind that these considerations hold rigorously only inside the Mandelstam triangle and in a finite but unspecified neighborhood of it. However, one will expect them to be true as long as the threshold expansion, $f_l(t)\simeq a_l q_t^{2l}$ is a good approximation, that is to say, more or less up to the resonance region.

2.3. Crossing Constraints

There are two types of constraints imposed by crossing symmetry: those that follow from crossing alone, and ones that also contain some positivity. The first are in the form of integral constraints, while the second are point-wise.

A. Pure Crossing Constraints

All the constraints of this type were contained in the work of Balachandran and Nuyts (BN1), and it is only because of the group-theoretic sauce with which this paper was dressed that it took some time for people to realize that. The rough idea behind these constraints is the following: inside the Mandelstam region partial wave expansions converge in all three channels, so it is obvious that crossing symmetry must give consistency conditions among the partial waves. I will refer to the original papers (BN1; R1; BCM1) for the (impressive) list of results, and just discuss a simple case, following a method suggested by Nussinov and developed by Roskies (R1).

Take $\pi^0\pi^0$ scattering. If Δ denotes the Mandelstam triangle, one has $(\mu=1)$

$$\iint ds\, dt(s+t+u-4)F(s,t,u)=0.$$

This formula is obvious, as $s+t+u=4$. However, since F is symmetric, one can replace $s+t+u-4$ by $3s-4$ so that

$$\int ds(3s-4)\int dtF(s,t,u)=0.$$

Writing then $dt=2q_s^2 d\cos\theta_s$, we get

$$\int_0^4 ds2q_s^2(3s-4)\int_{-1}^{+1} d\cos\theta_s F(s,t,u)=0,$$

and, since

$$F(s,t,u)=\sum(2l+1)P_l(\cos\theta_s)f_l^{\pi^0\pi^0}(s)$$

and

$$\int_{-1}^{+1} dxP_l(x)=2\delta_{l0},$$

this gives at once

$$\int_0^4 ds(4-s)(3s-4)f_0^{00}(s)=0. \qquad (2.12a)$$

With extra work, this method can be extended to a complete set of constraints giving necessary and sufficient conditions for crossing. Here one starts with $\int ds\int dtp(s,t,u)F(s,t)=0$, p any polynomial antisymmetric in s and t or t and u, and one gets, for example,

$$\int_0^4 ds\{(4-s)s[2f_0^{(0)}(s)-5f_0^{(2)}(s)] +3(4-s)f_1^{(1)}(s)\}=0, \quad \text{etc.} \qquad (2.12b)$$

Also, one can extend the results to any process. As an illustration, we show one of the simplest constraints for $\pi N\to\pi N$ (BCM1):

$$\int_{(M-\mu)^2}^{(M+\mu)^2} dsq_s^2\{[(M^2+q_s^2)^{1/2}+M][f_{l+1/2}^{(1/2)}(s)-f_{l+1/2}^{(3/2)}(s)] -[(M^2+q_s^2)^{1/2}-M][f_{l-1/2}^{(1/2)}(s)-f_{l-1/2}^{(3/2)}(s)]\}=0, \qquad (2.12c)$$

where we use the notation $f_{L,J-L}^{(I_s)}$, L=orbital angular momentum, I_s=isospin, J=total angular momentum, for the πN waves.

B. Crossing Plus Positivity Constraints

Crossing conditions that include positivity go back to the work of Martin (M6; M7), for $\pi^0\pi^0$ scattering. They were then extended to include isospin (ABMM1) and have been further improved recently (B3). The method is both astute and simple. Working with $\pi^0\pi^0$, one uses a twice-subtracted dispersion relation,

$$F(s,\cos\theta_s)=C(s)+\frac{2\cos^2\theta_s}{\pi}\int_{z_0(s)}^{\infty} dz\frac{F_s(s,z)}{z(z^2-\cos^2\theta_s)}, \qquad (2.13)$$

and the Froissart–Gribov representation for $l=$ even ≥ 2,

$$f_l(s)=\frac{2}{\pi}\int_{z_0(s)}^{\infty} dzQ_l(z)F_s(s,z); \qquad (2.14)$$

we have changed variables,

$$(s,t)\to[s, z=(s-4+2t)/(4-s)],$$

and

$$z_0(s)=(s+4)/(4-s).$$

Equations (2.13) and (2.14) will be taken for $0<s<4$. (We are setting $\mu=1$ throughout this section). Then one uses the Darboux–Christoffel formula

$$[z(z^2-x^2)]^{-1}=\sum_0^{L-2}(2l+1)Q_l(z)P_l(x) +L\frac{zP_L(x)Q_{L-1}(z)-xP_{L-1}(x)Q_L(z)}{z^2-x^2}; \qquad (2.15)$$

putting it into Eq. (2.13), one gets

$$F(s,\cos\theta_s)=\frac{2\cos^2\theta_s}{\pi}\sum_{l=2}^{L-2}(2l+1)$$
$$\times\int_{z_0(s)}^{\infty}dzQ_l(z)F_s(s,z)P_l(\cos\theta_s)$$
$$+\frac{2L}{\pi}\int_{(z_0)s}^{\infty}dzF(s,z)$$
$$\times\frac{zQ_{L-1}(z)P_L(\cos\theta_s)-(\cos\theta_s)Q_L(z)P_{L-1}(\cos\theta_s)}{z^2-\cos^2\theta_s}.$$

Note that the contribution from the term $l=0$ from Eq. (2.15) cancels the $C(s)$ in Eq. (2.13). Using then Eq. (2.14) this becomes

$$F(s,\cos\theta_s)-\sum_{l=0}^{L-2}(2l+1)P_l(\cos\theta_s)f_l(s)=R_L(s,\cos\theta_s),$$

$$R_L(s,\cos\theta_s)\equiv\frac{2L}{\pi}\int_{z_0}^{\infty}dzF_s(s,z)$$
$$\times\frac{zQ_{L-1}(z)P_L(\cos\theta_s)-(\cos\theta_s)Q_L(z)P_{L-1}(\cos\theta_s)}{z^2-\cos^2\theta_s}.\qquad(2.16)$$

Now, we have $F_s(s,z)/(z^2-\cos^2\theta_s)\geq0$ in the region of integration, so that the sign of the left hand side of Eq. (2.16) is $+/-$ wherever

$$zQ_{L-1}(z)P_L(\cos\theta_s)-(\cos\theta_s)P_{L-1}(\cos\theta_s)Q_L(z)$$

is positive/negative inside the region of integration. Actually, since $Q_{L-1}(z)/Q_L(z)$ is an increasing function of z, it is easy to see that sign [left hand side of Eq. (2.16)] = sign $P_L(\cos\theta_s)$ if

$$z_0(s)Q_{L-1}(z_0)/Q_L(z_0)\geq(\cos\theta_s)P_{L-1}(\cos\theta_s)/P_L(\cos\theta_s).$$

For example, with $L=4$, we have

$$F-\sum_{0}^{2}(2l+1)P_l(\cos\theta_s)f_l(s)$$
$$>0\quad\text{for } 0.8640<|\cos\theta_s|<1\text{(a)},$$
$$<0\quad\text{for } 0.3439<|\cos\theta_s|<0.8611\text{(b)}\qquad(2.17)$$

Since F is fully symmetric, we can, in all that has been said above, exchange s and t. Hence, we find following the above example, that

$$F-\sum_{0}^{2}(2l+1)P_l(\cos\theta_t)f_l(t)$$
$$>0\quad\text{for } 0.8640<|\cos\theta_t|<1\text{(c)}$$
$$<0\quad\text{for } 0.3439<|\cos\theta_t|<0.8611\text{(d)}\qquad(2.17')$$

F is the same in both. Taking then the intersection of the region (a) in (2.17) with the region (d) in (2.17′), and subtracting (2.17′) from (2.17) the F drops out and we have,

$$\sum_{0}^{2}(2l+1)\{P_l[\cos\theta_{t=t_1}(s=s_1)]f_l(t_1)$$
$$-P_l[\cos\theta_{s=s_1}(t=t_1)]f_l(s_1)\}>0,\qquad(2.18)$$

where (s_1,t_1) are any values of s, t such that

$$0.8640<|\cos\theta_{s=s_1}(t=t_1)|<1$$

and

$$0.3439<|\cos\theta_{t=t_1}(s=s_1)|<0.8611.$$

This method can obviously be extended to any L and then optimized (finding the particular values of s_1, t_1 that give tighter bounds). It may also be proved that the ensuing inequalities will give necessary and sufficient conditions for crossing symmetry (and some positivity of F_s, too). Furthermore, one can improve the results by taking into account isospin and by using the fact that not only $F_s\ (s,\cos\theta_s)$, but also $(\partial/\partial\cos\theta_s)F_s(s,\cos\theta_s)$ is positive. Thus, for example, one gets

$$f_0^{(0)}(s_1)-f_0^{(2)}(s_1)+(0.0843)f_1^{(1)}(s_1)$$
$$>\tfrac{1}{3}f_0^{(0)}(s_2)+\tfrac{2}{3}f_0^{(2)}(s_2)+(4.6476)f_1^{(1)}(s_2),\qquad(2.19)$$

with $s_1=0.3650$, $s_2=1.7557$.

Still other constraints can be obtained by mixing the positivity constraints of Sec. 2.2 with the purely crossing contraints of the beginning of this section. The problem of course, is very involved, and we refer to the original papers: (PW1; R2; BB1; P1; CP1).

What is the moral of all this? To me it seems to be that given any model for $\pi\pi$ scattering one has a simple way of checking its consistency. In particular, in the very low energy region, using Eq. (2.11) together with the crossing constraints tells us that once one knows one of the two scattering lengths a_0 or a_1 by some independent method [as current algebra (W2) for a_0 or ρ dominance for a_1] and the fact that a_2 is small, everything gets fixed within fairly narrow errors. For example, this explains why the Veneziano (V2) and Weinberg (W2) models are practically equal at very low energy in spite of their radical dissimilarities at the resonance region.

3. MEDIUM AND HIGH ENERGY BOUNDS

The bounds and constraints developed in Sec. 2 are useful in the very low energy region. We are now going to describe bounds and relations whose range of application extends to larger energies, in particular to the important regions of asymptotic s and the resonance region.

3.1. Bounds without Analyticity

Somewhat surprisingly it turns out that it is possible to write bounds without analyticity if one does not insist on getting pointwise bounds, but rather considers averages of the form

$$(\cos\theta_s''-\cos\theta_s')^{-1}\int_{\cos\theta_s'}^{\cos\theta_s''} d\cos\theta_s T(s,t)\equiv\tilde{T}(s,t). \tag{3.1}$$

We will discuss later what one gains and loses by so doing, and turn now to the proof, due to Glaser and Martin (unpublished). Disregarding spin, we write, with obvious notation,

$$T(s,t)=\langle p_1p_2\mid\mathfrak{T}\mid p_1'p_2'\rangle\equiv\langle\mathbf{k}_i\mid\mathfrak{T}\mid\mathbf{k}_f\rangle,$$

where $\mathbf{k}_{i/f}$ represents the c.m. relative momenta of the incoming/outgoing particles. Let me denote by θ the angle of $\mathbf{k}$ with the z axis; taking then a wave function, $\varphi(\mathbf{k})=\varphi(\theta)$, one defines the average for, e.g., Im T, as

$$\operatorname{Im}\tilde{T}(s,\cos\theta_s)\equiv\int(d\Omega_i/4\pi)(d\Omega_f/4\pi)\times\varphi^*(\mathbf{k}_i)\langle\mathbf{k}_i\mid\mathfrak{T}\mid\mathbf{k}_f\rangle\varphi(\mathbf{k}_f). \tag{3.2}$$

If one assumes φ normalized to unity, we then have

$$\int(d\Omega/4\pi)\mid\varphi(\mathbf{k})\mid^2=1.$$

Im $\tilde{T}$ is the quantity which is closer to what one really measures experimentally, viz., $|\tilde{T}|^2$. Then one has the well-known summation formula

$$P_l(\cos\theta_{\mathbf{k}_i\mathbf{k}_f})=\frac{4\pi}{2l+1}\sum_{m=-l}^{+l}Y_l^m(\Omega_i)Y_l^m(\Omega_f)^*;$$

putting this into the partial wave expansion of T, and substituting the result into Eq. (3.2) gives

$$\operatorname{Im}\tilde{T}(s,\cos\theta_s)=4\pi\sum_l\sum_m\left|\int(d\Omega/4\pi)\varphi(\mathbf{k})Y_l^m(\Omega)\right|^2\operatorname{Im}f_l(s).$$

Now, we may expand φ in a Legendre series

$$\varphi(\mathbf{k})=\varphi(\theta)=\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\varphi_l;$$

if we substitute P_l by its expression in terms of Y_l^m's, we see immediately that

$$1=\int\frac{d\Omega}{4\pi}\mid\varphi(\mathbf{k})\mid^2=\sum_0^{\infty}(2l+1)\mid\varphi_l\mid^2$$

and also

$$\operatorname{Im}\tilde{T}(s,\cos\theta_s)=\sum_0^{\infty}(2l+1)\operatorname{Im}f_l(s)\mid\varphi_l\mid^2,$$

so that the unitarity bound on the Im f_l gives us

$$|\operatorname{Im}\tilde{T}(s,\cos\theta_s)|\leq s^{1/2}/2q_s. \tag{3.3a}$$

To see more clearly what this means, let us use that $T(s,\cos\theta_s)$ is continuous (actually we know it is even analytic) in θ_s for physical θ_s. Then, one can apply the mean value theorem so that Eq. (3.2) gives

$$\operatorname{Im}\tilde{T}(s,\cos\theta_s)=\tfrac{1}{4}\operatorname{Im}T(s,\cos\tilde{\theta}_s)\left(\int_0^{\pi}d\theta\varphi(\theta)\right)^2.$$

Taking then the simplest possible choice for φ, viz.

$$\varphi(\theta)=1/2\delta,\qquad 0\leq\theta\leq\delta$$
$$=0,\qquad \text{elsewhere},$$

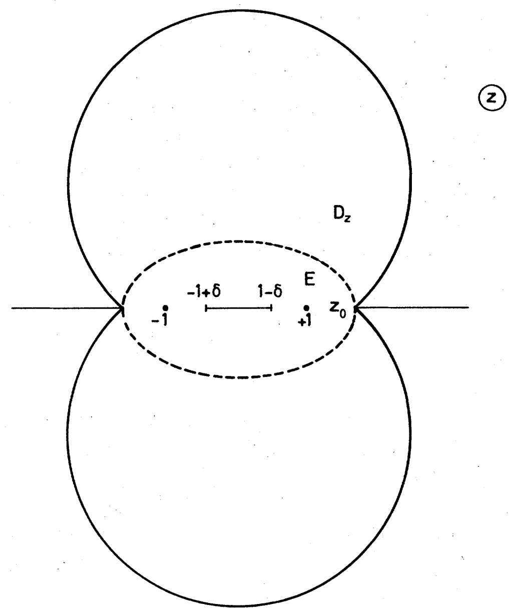


FIG. 4. The z plane.

we obtain finally

$$\operatorname{Im}T(s,\cos\tilde{\theta}_s)\leq 2s^{1/2}/\delta^2q_s,\qquad 0\leq\tilde{\theta}_s\leq\delta. \tag{3.3b}$$

It is not difficult to extend this to fixed t or to Re T; we refer to (M8; M10) for details.

It is surprising that (3.3a) and (3.3b) are in some respects better than what one (gets) using analyticity; there one has for, e.g., $\pi\pi$ scattering (cf. Sec. 3.3 here)

$$|T(s,\cos\theta_s)|<C[\log^{3/2}(s/s_0)/\sin^2\theta_s],$$

even using the Mandelstam representation. Further advantages of (3.3a) and (3.3b) are that they give absolute bounds, i.e., are valid at all energies and without unknown constants, and that they occur in quantities directly measurable. Of course, the price to pay for this is that if one takes averages over too narrow regions (i.e., with high precision) the bounds blow up as is evident from (3.3b). Therefore, for practical applications both types of bounds (these bounds and the bounds using analyticity that will be described below) should be used in conjunction.

3.2. Froissart Bounds and Improvements. (i) A Simple Version for $\pi\pi$

In Sec. 1.4 we gave a quick proof of the Froissart bound, Eq. (1.9). As stated there, the result presents some ugly features. First, the constant in front of the bound was at first unknown. This problem was solved by Łukaszuk and Martin (LM1) who showed that it actually equals $4\pi/t_0$, t_0 being, as usual, the right extremity of the Lehmann–Mandelstam–Martin ellipse ($t_0=4\mu^2$ for $\pi\pi$, πN, $KN\cdots$, $t_0=\mu^2$ for NN). This looks nice because it corresponds to the possible exchanges of particles (Fig. 4) and because $\pi/\mu^2\simeq 60$ mb which is only twice the experimental value for πN total cross sections, but there is still some work to be done before meaningful comparisons with experiment can be done; in fact, (a) there is an unknown scale factor in the log of Eq. (1.9) which should actually be written as $\log^2(s/s_0)$, s_0 unknown; (b) the bound is only valid asymptotically, i.e., as $s\rightarrow\infty$, but no indication is given about when we are in asymptopia, and (c) the bound is only valid in the mean, that is, strictly speaking, Eq. (1.9) may be violated in isolated intervals provided that averages of the type

$$(2\delta)^{-1}\int_{(1-\delta)s}^{(1+\delta)s} ds'\sigma_{\text{tot}}(s'), \qquad \delta>0 \text{ fixed},$$

satisfy it.

It turns out that it is possible to solve all these problems at the same time by using t channel information through the Froissart–Gribov representation. I will now show how this is done, following, with slight modifications, the original work (Y2) (in spite of the fact that it can be greatly improved) because it shows very clearly the mechanism at work.

To begin, one defines convenient averages

$$\bar{\sigma}_{\text{tot}}^{(n)}(s)\equiv\frac{n+1}{(s-4\mu^2)^{n+1}}\int_{4\mu^2}^{s} ds'(s'-4\mu^2)^n\sigma_{\text{tot}}(s'). \tag{3.4}$$

The advantages of using Eq. (3.4) are that as $n\rightarrow\infty$, $\sigma_{\text{tot}}^{(n)}(s)\rightarrow\sigma_{\text{tot}}(s)$ and, if σ_{tot} does not oscillate too much, also as $s\rightarrow\infty$, $\sigma_{\text{tot}}^{(n)}(s)\rightarrow\sigma_{\text{tot}}(s)$ for any n (GY1). Then, one writes the partial wave expansion[6] of σ_{tot}

$$\bar{\sigma}_{\text{tot}}^{(n)}(s)=16\pi\frac{n+1}{(s-4\mu^2)^{n+1}}\int_{4\mu^2}^{s} ds'\frac{(s'-4\mu^2)^{n-1/2}}{s'^{1/2}} \times\sum_{0}^{\infty}(2l+1)\,\text{Im}\,f_l(s').$$

We will split the sum into two pieces: one up to

$$l=L(s)\equiv N[(s-4\mu^2)/\mu^2]^{1/2}\log(s/\mu^2)-1, \tag{3.5}$$

where N will be fixed in a moment, and the other piece from $l=L(s)+1$ on. In the first half we majorize $\text{Im}\,f_l$ by its unitary bound, $[s/(s-4\mu^2)]^{1/2}$: hence,

$$\bar{\sigma}_{\text{tot}}^{(n)}(s)<16\pi\frac{n+1}{(s-4\mu^2)^{n+1}}\int_{4\mu^2}^{s} ds'(s'-4\mu^2)^{n-1} \times\sum_{0}^{L(s)}(2l+1)+16\pi\frac{n+1}{(s-4\mu^2)^{n+1}}\int_{4\mu^2}^{s} ds' \times\frac{(s'-4\mu^2)^{n-1/2}}{s'^{1/2}}\sum_{L(s)+1}^{\infty}(2l+1)\,\text{Im}\,f_l(s'). \tag{3.6a}$$

The first term in the right hand side of (3.6a) can be summed explicitly giving

$$16\pi N^2[(n+1)/n\mu^2]\log^2(s/\mu^2). \tag{3.6b}$$

Next, one recalls the Froissart–Gribov representation, Eq. (1.7) ($M=\mu$ now). Using the threshold behavior, $f_2(t)\simeq(t-4\mu^2)^3a_2^t$, where a_2^t is the t-channel D-wave scattering length, and the asymptotic behavior of Legendre functions (MOS1), $Q_2(z)\simeq(2/15)z^{-3}$ as $z\rightarrow\infty$, we find that in the limit $q_t\rightarrow0$, Eq. (1.7) becomes

$$a_2^t=\frac{16}{15\pi}\int_{4\mu^2}^{\infty} ds's'^{-3}F_s(s',t=4\mu^2). \tag{3.7}$$

Expanding the right hand side in Legendre polynomials, and noting that $P(\cos\theta^{(0)})$ increases with l and decreases with s [here $\cos\theta_s^{(0)}=\cos\theta_s(t=4\mu^2)$] one has the chain of inequalities

$$\begin{aligned}
a_2^t &\geq \frac{16}{15\pi}\int_{4\mu^2}^{\infty} ds's'^{-3}\sum_{0}^{\infty}(2l+1)P_l(\cos\theta_s^{(0)})\,\text{Im}\,f_l(s')\\
&> \frac{16}{15\pi}\int_{4\mu^2}^{s} ds's'^{-3}\sum_{L+1}^{\infty}(2l+1)P_l(\cos\theta_s^{(0)})\,\text{Im}\,f_l(s')\\
&> \frac{16P_{L+1}(\cos\theta_s^{(0)})}{15\pi}\int_{4\mu^2}^{s} ds's'^{-3}\frac{(s'-4\mu^2)^{n-1/2}}{s'^{1/2}} \times\frac{s'^{1/2}}{(s'-4\mu^2)^{n-1/2}}\sum_{L+1}^{\infty}(2l+1)\,\text{Im}\,f_l(s')\\
&> \frac{16P_{L+1}(\cos\theta_s^{(0)})}{15\pi s^{2+1/2}(s-4\mu^2)^{n-1/2}}\int_{4\mu^2}^{s} ds'\frac{(s'-4\mu^2)^{n-1/2}}{s'^{1/2}} \times\sum_{L+1}^{\infty}(2l+1)\,\text{Im}\,f_l(s').
\end{aligned}$$

Comparing this with (3.6), it gives at once that

$$\bar{\sigma}_{\text{tot}}^{(n)}(s)<16\pi N^2\frac{n+1}{n\mu^2}\log^2\frac{s}{\mu^2} +\frac{15\pi^2(n+1)s^{5/2}}{(s-4\mu^2)^{3/2}P_{L+1}[1+8\mu^2/(s-4\mu^2)]}a_2^t. \tag{3.8}$$

Since we know from the work of Łukaszuk and Martin that one cannot improve upon

$$\bar{\sigma}_{\text{tot}}\lesssim(\pi/\mu^2)\log^2(s/s_0),$$

we take $N=1/4$; this makes (3.8) an absolute bound, valid at all energies and with no unknown constants. A more explicit form of the bound, valid for $s^{1/2}>0.42$ GeV/c can easily be obtained taking $n=\log(s/\mu^2)$, and using the asymptotic form of the P_L [cf. (A1) for a simple discussion] getting

$$\bar{\sigma}_{\text{tot}}(s)<\frac{\pi}{\mu^2}\log^2\frac{s}{\mu^2}+\frac{\pi}{\mu^2}\log\frac{s}{\mu^2}+\frac{15\pi^3}{\sqrt{2}}\left(\frac{s}{s-4\mu^2}\right)^{3/2} \times\left[\left(\log\frac{s}{\mu^2}\right)^{1/2}+\left(\log\frac{s}{\mu^2}\right)^{3/2}\right]a_2^t \times\exp\frac{2\mu\log(s/\mu^2)}{(s-4\mu^2)^{1/2}}, \tag{3.9}$$

$$\bar{\sigma}_{\text{tot}}(s)\rightarrow\sigma_{\text{tot}}(s) \quad \text{as} \quad s\rightarrow\infty.$$

The first term in the right hand side of (3.9) gives the usual Froissart bound, and the rest is the correction to

[6] To avoid complications due to identity of particles, we consider $\pi^0\pi^+$ scattering.

it that allows the whole bound to be valid at finite energies.

3.3. Froissart Bounds. (ii) General Treatment

Before improving the analysis of Sec. 3.2, I will show how one can extend it to the general case. It will be noted that one can generalize the method to any scattering process; however, except in favorable situations, an unknown constant will be introduced. We will only consider here such "favorable" situations, referring to (MM2; CY1) for details about the "unfavorable" situations.

From the analysis of the previous section we see that the ingredients we need to obtain absolute bounds are the following: analyticity in t up to the first t-channel threshold; positivity of $\hat{F}$ in both s and u channels [in particular, this makes the method unsuitable for processes like Kp that possess unphysical regions without positivity (M1; M3)]; and the fact that we can write

$$\hat{F}_s(s,t)+\hat{F}_u(s,t) \geq \sum (2l+1)\Phi_l(\cos\theta_s)\ \mathrm{Im}\ \varphi_l(s), \quad (3.10)$$

where

$$0 \leq \mathrm{Im}\ \varphi_l(s) \leq s^{1/2}/2q_s,$$

$\Phi_l(\cos\theta_s)$ increases with l, and decreases with s. In the previous situation, we had $\varphi_l = f_l$, $\Phi_l = P_l$ and the equality sign held in (3.10).

As already hinted by the notation, the answer will be furnished by $\hat{F}$ as given by Eq. (1.2). Let me prove this is so; to be definite, only the case $j_2=0$, $j_1=\text{integer}\equiv j$ will be considered.

As shown by Mahoux and Martin (MM1), $\hat{F}$ is free from kinematical singularities, and has positivity in both s and u channels. Hence, if the process under consideration has no unphysical cuts, the two first requisites above will be satisfied. As to the last, continuing $\hat{F}$ to the u channel gives

$$\hat{F}^{(s\to u)}(s,t) = [\lambda(s,M^2,\mu^2)]^j \sum_{\lambda\lambda'} d_{\lambda'\lambda}{}^j(2\chi_{su}) T_{\lambda\lambda'}{}^{(u)}(u,t),$$

we have used the results described in Eqs. (1.1) and (1.2). Expanding in partial waves, we get

$$\hat{F}_s(s,t)+\hat{F}_u(s,t) = [\lambda(s,M^2,\mu^2)]^j \sum_J (2J+1) \times \{\sum_\lambda d_{\lambda\lambda}{}^J(\theta_s)\ \mathrm{Im}\ f_{\lambda\lambda}{}^J(s) + \sum_{\lambda\lambda'} [\lambda(u,M^2,\mu^2)/\lambda(s,M^2,\mu^2)]^j d_{\lambda'\lambda}{}^j \times (2\chi_{su}) d_{\lambda'\lambda}{}^J(\theta_s)\ \mathrm{Abs}\ \bar{f}_{\lambda\lambda'}{}^J(s)\}, \quad (3.11)$$

where $\bar{f}$ are the u channel waves. Now, it is not hard to check, using explicit expressions for the d's and for χ that, in the region of interest for us [$t\geq 0$, $s\geq (M+\mu)^2$] one has the relation

$$\sum_{\lambda\lambda'} d_{\lambda'\lambda}{}^j(2\chi_{su}) d_{\lambda'\lambda}{}^J(\theta_s)\ \mathrm{Abs}\ \bar{f}_{\lambda\lambda'}{}^J(s) > \exp(-2j\,|\,\chi_{su}\,|) \sum_\lambda d_{\lambda\lambda}{}^J(\theta_s)\ \mathrm{Im}\ f_{\lambda\lambda}{}^J(s).$$

Combining this with (3.11) we find that

$$\hat{F}_s(s,t)+\hat{F}_u(s,t) > \left[1+\left(\frac{\lambda(u,M^2,\mu^2)}{\exp(2j\,|\,\chi_{su}\,|)\lambda(s,M^2,\mu^2)}\right)^j\right] \times \sum_J (2J+1) \sum_\lambda d_{\lambda\lambda}{}^J(\theta_s)[\mathrm{Im}\ f_{\lambda\lambda}{}^J(s) + \mathrm{Im}\ \bar{f}_{\lambda\lambda}(s)].$$

This satisfies requisite (3.10) as the $d_{\lambda\lambda}{}^J(\theta_s)$ decrease with s and increase with J, and f, $\bar{f}$ fulfill the unitarity bounds, Eqs. (1.4). Thus, one can again use the Froissart–Gribov representation, Eq. (1.7), and everything happens as before, using that, from Eq. (1.7),

$$\bar{a}_2{}^{(+)} = M \int_{(M+\mu)^2}^\infty ds\{\hat{F}_s(s,t)+\hat{F}_u(s,t)\}\frac{1}{(s+\mu^2-M^2)^3},$$

$\bar{a}_2{}^{(+)}$ being as in Sec. 2.2.

Now we will turn to the announced improvements on the rather gross majorizations carried on up to now. We shall carry over the discussion giving pertinent references, but no detailed proofs. We shall do so because the methods are totally similar to those we will use later to put bounds on differential cross sections for elastic (Sec. 3.5) and inelastic (Sec. 3.4) processes.

First of all, in the original work of Froissart (F1) and in the subsequent discussions of Lukaszuk and Martin (LM1) one found bounds not only on $\sigma_{\text{tot}}(s)$, but also on $d\sigma/d\Omega$. Certainly, this will also be true in our case: it is not difficult to extend the analysis to find absolute bounds not only on $\mathrm{Im}\ T(s,0)$, but also on $|\,T(s,t)\,|$, t physical. This has been done by Common (C5), who has obtained the optimum bounds on this quantity. For finite energies, Common's bounds can only be found numerically; their form for large s, however, can be computed explicitly giving (for, e.g., $\pi\pi$ scattering)

$$|\,F(s,t)\,| < \frac{2}{3(\pi)^{1/2}}\frac{s[\log(s/s_0)]^{3/2}}{[-4(4\mu^2)^3 t]^{1/4}}; \qquad t<0, \quad (3.12\mathrm{a})$$

$$|\,F(s,\cos\theta_s)\,| < \frac{2s^{3/4}[\log(s/s_0)]^{3/2}}{3(4\mu^2)^{3/4}(\pi\sin\theta_s)^{1/2}}; \qquad -1<\cos\theta_s<+1. \quad (3.12\mathrm{b})$$

where

$$s_0 = (s_1{}^{3/2}/8\mu)\log\frac{8s}{3(s_1{}^3/4\mu^2)^{1/2}\log(s/s_1)};$$

$$s_1 = e/30\pi^2\mu a_2{}^t,$$

valid respectively, for the fixed t and fixed $\cos\theta_s$ amplitudes. This is one of the few cases in which a bound can be improved by using the Mandelstam representation; if we assume it, then (3.12b) is improved to (KLM1)

$$|\,F(s,\cos\theta_s)\,| < C\frac{[\log(s/s_0)]^{3/2}}{\sin^2\theta_s}.$$

The constants C, s_0 are unknown and the bound only holds asymptotically. As yet, nobody has been able to cure these diseases of the last bound.

Secondly, some people may feel unhappy that the bounds contain the quantity $a_2{}^t$. Personally, I think

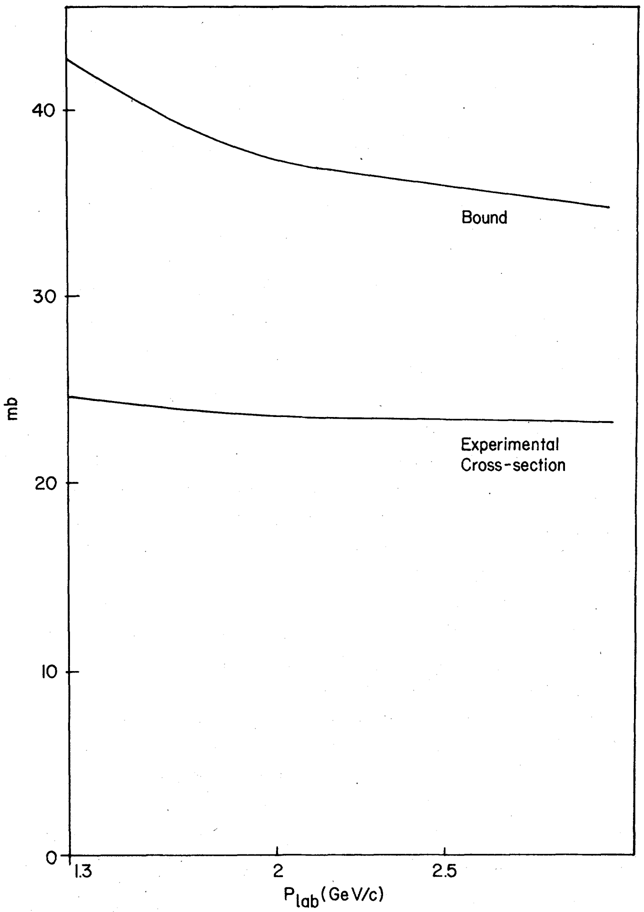


FIG. 5. Low energy $\pi^\circ\pi^+$ cross section and bound, using only the D wave scattering length. The average taken was

$$\bar{\sigma}(s)=\left(\int_4^s ds' q_{s'} s'^{-5/2}\right)^{-1}\int_4^s ds' q_{s'} s'^{-5/2}\sigma(s').$$

that this is good because physically $a_2{}^t$ is a very transparent measure of the strength of the interaction; furthermore, in all cases where $a_2{}^t$ can be estimated ($\pi\pi$, πK, πN) it turns out to be rather small so that good bounds obtain. However, just as a matter of principle, let me mention that, at least for $\pi\pi$ scattering, one can use the Froissart–Gribov representation not at $t=4\mu^2$, but somewhere between 0 and $4\mu^2$: there one has absolute bounds on $|F(s,t)|$ and hence on $f_2(t)$, which depend only on the pion mass [cf. Eqs. (2.8) here]. Therefore, one can get absolute bounds analogous to Eq. (3.9), but depending *only* on the pion mass. This was noticed by the author (seminar given at Orsay, February 1970, unpublished) and independently by Common (C5) who worked out the details.

Thirdly, it is obvious that the bounding procedure used in Sec. 3.2 is quite poor, as one is throwing away a lot that should be taken better care of. Also, one should allow freedom in the choice of the averages; Eq. (3.4) is satisfactory for large s, but one can use different averages to optimize at each value of s. A first step towards solving the first problem was taken by Common (C5) but much more remains to be done as recent investigations (S4; R4) show. In particular, one would expect improvement by using the techniques developed by Blankenbecler and Einhorn (BE1) and, indeed, the preliminary results referred to [(S4; R4); see also (R3)] give bounds only slightly above "experiment" for $\pi\pi$ in the resonance region. (Fig. 5. See also the Appendix for the best available high energy bounds.)

3.4. Bounds on Inelastic Processes

The methods just described can easily be generalized to give absolute bounds for inelastic processes. Inelastic processes can be classified into two categories: (a) "conversion" processes, of the type $A+B\rightarrow C+D$; and (b) "production" processes, of the type $A+B\rightarrow C+D+X+\cdots+Z$. One can find bounds for both, but only conversion processes will be considered here: we will take the particular case $N\bar{N}\rightarrow\pi\pi$, as it exhibits all relevant features.

Denoting by T_+, T_-, the helicity amplitudes $T_{+1/2,+1/2;0,0}$, $T_{+1/2,-1/2;0,0}$, we shall get absolute bounds on averages $|\bar{T}_\pm{}^{(n)}|$; this implies absolute bounds on the differential cross-sections

$$d\sigma_{N\bar{N}\rightarrow\pi\pi}{}^{(\pm)}/d\Omega$$

which are proportional to $|T_\pm|^2$. We shall, for definiteness, take the π's to be in isospin zero and, to be consistent with the rest of these notes, take $N\bar{N}\rightarrow\pi\pi$ to be the t channel (cf. Sec. 1.2.A).

The partial wave expansions of $T_\pm$ are (JW1)

$$T_+(s,t)=\sum(2J+1)P_J(\cos\theta_t)g_J{}^{(+)}(t),$$
$$T_-(s,t)=\sum(2J+1)/[J(J+1)]^{1/2}$$
$$\times\sin\theta_t P_J'(\cos\theta_t)g_J{}^{(-)}(t).\quad(3.14)$$

The partial waves g_J are given by [cf. Eqs. (1.3), (1.4)]

$$g_J{}^{(\pm)}(t)=(t^{1/2}/2q_t{}^{1/2}Q_t{}^{1/2})\hat{g}_J{}^{(\pm)}(t),$$
$$\hat{g}_J{}^{(\pm)}(t)=\langle\pi\pi;t,J\,|\,\mathfrak{T}\,|\,N,\tfrac{1}{2};\bar{N},(\pm1/2);t,J\rangle,$$

the notation being as in Sec. 1.2.C.

Now, if we denote by f_J the $\pi\pi$ waves with isospin zero, we have, for $t\geq 4M^2$, (S1; S2)

$$\begin{aligned}\operatorname{Im}\hat{f}_J(t)&\equiv(2q_t/t^{1/2})\operatorname{Im}f_J(t)\\&=(2i)^{-1}\langle\pi\pi;t,J\,|\,\mathfrak{T}-\mathfrak{T}^+\,|\,\pi\pi;t,J\rangle\\&=\sum_n|\langle\pi\pi;t,J\,|\,\mathfrak{T}\,|\,n;t,J\rangle|^2\\&>|\langle\pi\pi;t,J\,|\,\mathfrak{T}\,|\,N,\tfrac{1}{2};\bar{N},\tfrac{1}{2};t,J\rangle|^2\\&\qquad+|\langle\pi\pi;t,J\,|\,\mathfrak{T}\,|\,N,\tfrac{1}{2};\bar{N},-\tfrac{1}{2};t,J\rangle|^2\\&=|\hat{g}_J{}^{(+)}(t)|^2+|\hat{g}_J{}^{(-)}(t)|^2+|\hat{f}_J(t)|^2.\end{aligned}$$

This is an expression of the obvious fact that when using completeness for $\pi\pi$ we sum over all intermediate states—in particular, if $t\geq 4M^2$, over $N\bar{N}$ states. The above relation then gives the unitarity constraints, valid for $t \geq 4M^2$,

$$|g_J^{(\pm)}(t)| < t^{1/2}/4q_t^{1/2}Q_t^{1/2},$$
$$|g_J^{(\pm)}(t)| < (t^{1/4}/\sqrt{2}Q_t^{1/2})\{\mathrm{Im}\, f_J(t) - [\mathrm{Im}\, f_J(t)]^2\}^{1/2}. \tag{3.15}$$

We can obviously write for, e.g., the $(+)$ choice,

$$|T_+(s,t)| < \sum (2J+1)\,|P_J(\cos\theta_t)|\,|g_J^{(+)}(t)|,$$

so that, if we define

$$|\bar{T}_+^{(n)}(s,t)| \equiv \frac{(n+1)t^{1/2}}{(t-4\mu^2)^{n+1}q_t^{1/2}Q_t^{1/2}} \times \int_{4M^2}^{t} dt'(t'-4\mu^2)^n \frac{q_{t'}Q_{t'}}{(t')^{1/2}}\,|T_+(s,t')|,$$

we can apply to this the minimization procedure of (C4), using the relation

$$b_2^s \equiv \frac{16}{15\pi}\int_{4M^2}^{\infty} dt' t'^{-3} F_t(s=4\mu^2,t') < a_2^s, \tag{3.16}$$

where a_2^s, b_2^s, F refer to $\pi\pi$ scattering. However, and again to show clearly the mechanism at work, we shall *not* use minimization techniques but rather give a simple discussion.

Just as in Sec. 3.2, we write

$$|\bar{T}^{(n)}| \leq \frac{(n+1)t^{1/2}}{(t-4\mu^2)^{n+1}q_t^{1/2}Q_t^{1/2}} \times \int_{4M^2}^{t} dt'(t'-4\mu^2)^n \frac{q_{t'}^{1/2}Q_{t'}^{1/2}}{(t')^{1/2}}$$
$$\times\{\sum_0^L (2J+1)\,|P_J(\cos\theta_{t'})|\,|g_J^{(+)}(t')| + \sum_{L+2}^{\infty} (\text{same})\}.$$

In the first sum, we use the unitary bound, and, in both sums, since we are being crude, that, for $\cos\theta_{t'}$ physical $|P_J(\cos\theta_{t'})| \leq 1$; hence, using the Schwartz inequality for the $\sum_{L+2}^{\infty}$ term,

$$|\bar{T}_+^{(n)}(s,t)| < \frac{t^{1/2}}{4q_t^{1/2}Q_t^{1/2}}\left[1-\left(\frac{4M^2-4\mu^2}{t-4\mu^2}\right)^{n+1}\right] \times (L^2/2+L) + R_L(t),$$

$$R_L(t) < \frac{(n+1)(t)^{1/2}}{(t-4\mu^2)^{n+1}q_t^{1/2}Q_t^{1/2}} \times \left[\int_{4M^2}^{t} dt' \frac{(t'-4\mu^2)^{2n}}{t'/4-\mu^2}\right]^{1/2} \times \left[\sum_{L+2}^{\infty}\int_{4M^2}^{t} dt'(2J+1)^2\frac{q_{t'}^3Q_{t'}}{t'}\,|g_J^{(+)}(t')|^2\right]^{1/2}$$
$$< \frac{(n+1)(t)^{1/2}}{4(n)^{1/2}q_t^{2+1/2}Q_t^{1/2}}\left[1-\left(\frac{4M^2-4\mu^2}{t-4\mu^2}\right)^{2n+1}\right] \times \sum_{L+2}^{\infty}(2J+1)\left[\int_{4M^2}^{t} dt' \frac{q_{t'}^3}{t'^{1/2}}\mathrm{Im}\, f_J(t')\right]^{1/2}.$$

In the last inequality we have used again (3.15). From (3.16), on the other hand, we see that, for any $J \geq L+2$,

$$b_2^s > \frac{16 P_J(\cos\theta_t^{(0)})}{15\pi q_t^3 t^{2+1/2}}(2J+1)\int_{4M^2}^{t} dt' \frac{q_{t'}^3}{t'^{1/2}}\mathrm{Im}\, f_J(t'),$$
$$\cos\theta_t^{(0)} = 1 + 8\mu^2/(t-4\mu^2).$$

Taking now

$$L = [2q_t/(1-\varphi_0)\mu]\log(t/\mu^2),$$

and using the majorization

$$\sum_{L+2}^{\infty}[2J+1/P_J(\cos\theta_t^{(0)})]^{1/2} < 2\sqrt{2}\frac{q_tL}{\mu\varphi_0}\exp[-(1-\varphi_0)L\mu/2q_t],$$

φ_0 arbitrary between 0 and 1, it is a matter of substituting all of it into the bound for R_L to find the bound

$$|\bar{T}_+^{(n)}(s,t)| < \frac{t^{1/2}q_t^2}{2(1-\varphi_0)^2\mu^2 q_t^{1/2}Q_t^{1/2}}\log^2\frac{t}{\mu^2} + \frac{t^{1/2}q_t}{(1-\varphi_0)\mu q_t^{1/2}Q_t^{1/2}}\log\frac{t}{\mu^2} + \frac{n+1}{\varphi_0(1-\varphi_0)\mu^2}\left(\frac{15\pi b_2^s}{8n}\right)^{1/2}\frac{t^{3/4}q_t}{Q_t^{1/2}}\log\frac{t}{\mu^2}; \tag{3.17}$$

we have replaced

$$1-[(4M^2-4\mu^2)/(t-4\mu^2)]^2 \simeq 1.$$

At high energy, the first term in (3.17) dominates and we have

$$|\bar{T}_+^{(n)}(s,t)| \lesssim (t/4\mu^2)\log^2(t/\mu^2),$$

which is equivalent to a Froissart bound. This is the result one also finds by other methods (AM1; LMV1; TT1). Using optimization techniques, we expect (3.17) to be improved by a factor of 5 to 10. Although the result is quite poor at high energy, where Regge theory makes one expect

$$|\bar{T}_+^{(n)}(\cos\theta_t=0,t)| \simeq (\text{constant})\,t^{\alpha_N(0)},$$
$$\alpha_N(0) \simeq -0.3,$$

Equation (3.17) (or its improved version) should be reasonably tight up to $t \simeq 4\ \mathrm{GeV}^2$ because experimentally the cross sections $N\bar{N}\to\pi\pi$ are fairly large there. A detailed analysis with numerical results may be found in (CY3). One has, for example, $\sigma_{\bar{p}p\to\pi^+\pi^-} < 20$ mb for a laboratory momentum of 0.5 GeV.

Finally, we refer to (AM1; LMV1; TT1; R3) for asymptotic bounds on processes of the type $A+B\to C+D+X\cdots+Z$.

3.5. Relations Involving Diffraction Peaks, Elastic Cross Sections, etc.

All bounds and constraints discussed so far have one merit in common, namely, they require only the masses of the particles involved and, at times, one extra parameter, generally a scattering length. Apart from such results, it turns out that a richness of further relations can be obtained to link different measurable quantities among themselves. (The word "bounds" is also used for such results, but we believe "relations" to be a more appropriate term.) Some of them, in particular the impressive amount recently discussed by Singh and Roy, are extremely tight and, at times, even able to kill theoretical models, as the isospin triangular inequalities for πN polarization recently discussed by Dass *et al.* [CERN Preprint TH. 1367 (1971)]. We will not here aim to present a complete list of such relations [cf. the recent review of Roy (R3)]; instead, two examples will be discussed in some detail. Also we shall neglect spin throughout, noting however that all the results can be generalized to arbitrary spin with the help of standard techniques.

A. Bounds on the Diffraction Peak

The width of the diffraction peak may be conveniently defined as

$$a_s'(0)\equiv \left.\frac{(\partial/\partial t)T_s(s,t)}{T_s(s,t)}\right|_{t=0}. \qquad (3.18)$$

This amounts to writing $T_s(s,t)=\exp a_s(t)$. In a Regge model, $a_s'(0)\simeq\alpha_P'(0)\log s$, $\alpha_P'(0)$ being the slope of the Pomeranchuk trajectory.

A lower bound on (3.18) is very simple to obtain (MM1). Clearly, making a partial wave expansion of T_s, we have

$$a_s'(0)=\left(\sum_1^\infty (2l+1)(l+1)l\,\mathrm{Im}\,f_l(s)\right)$$

$$\times\left(4q_s^2\left\{\sum_1^\infty (2l+1)\ \mathrm{Im}\,f_l(s)+\ \mathrm{Im}\,f_0(s)\right\}\right)^{-1},$$

and we have used that $P_l'(1)=l(l+1)/2$. Now, we have

$$\sigma_{\mathrm{tot}}=\frac{4\pi}{q_s^2}\sum_0^\infty (2l+1)\ \mathrm{Im}\,\hat{f}_l(s).$$

This suggests that we define $L(s)$ to be the maximum integer such that

$$\sum_1^{L(s)}(2l+1)\le\frac{q_s^2}{4\pi}\sigma_{\mathrm{tot}}(s)-\mathrm{Im}\,\hat{f}_0(s). \qquad (3.19)$$

Using by now familiar arguments, we can verify that

$$\sum_1^\infty (2l+1)(l+1)l\,\mathrm{Im}\,f_l(s)\ge\frac{s^{1/2}}{2q_s}\Phi(s)$$

$$\equiv\frac{s^{1/2}}{2q_s}\sum_1^{L(s)}(2l+1)(l+1)l$$

[we have used (1.4) and (3.9)] and hence that

$$a_s'(0)\ge\pi\Phi(s)/q_s^4\sigma_{\mathrm{tot}}(s). \qquad (3.20)$$

This is an absolute bound, valid at all energies. At intermediate energies it is just a few times the experimental value for πN, NN; for medium-heavy nuclei, (3.20) is only about 50% too high. To see what happens at high energies, we assume that asymptotically $\sigma_{\mathrm{tot}}\ge$ constant; then, $4L(s)\simeq[s\sigma_{\mathrm{tot}}(s)/\pi]^{1/2}$, $\Phi(s)\simeq s^2[\sigma_{\mathrm{tot}}(s)]^2/32\pi^2$, and (3.20) becomes $\frac{1}{2}a_s'(0)\ge\sigma_{\mathrm{tot}}/8\pi$. This bound is only a factor of $\log s$ short of the Regge value if we have constant total cross sections, and is saturated if the Froissart bound was saturated or if we had a flat Pomeranchuk.

To get an upper bound on $a_s'(0)$ we proceed differently (B2; K2; K3). We define now $L(s)\equiv ns^{1/2}\log s$; recalling the crude version of the Froissart bound given in Sec. 1.4, we notice that we had

$$\mathrm{Im}\,f_l(s)<Cs^2/(2l+1)P_l[1+(2\mu^2/q_s^2)].$$

Summing explicitly, it follows from this that, if n is large enough, we have

$$\sum_{L(s)+1}^\infty (2l+1)(l+1)l\,\mathrm{Im}\,f_l(s)<C's^{-\nu_n}$$

with ν_n large at will provided that n be sufficiently large. Thus, *for asymptotic s*, we can neglect, in the sum

$$\sum_1^\infty (2l+1)(l+1)l\,\mathrm{Im}\,f_l(s),$$

all terms from $L(s)+1$ onward. Hence we have

$$a_s'(0)\simeq\sum_0^L (2l+1)(l+1)l\,\mathrm{Im}\,f_l(s)/\pi s^2\sigma_{\mathrm{tot}}(s)$$

$$<\sum_0^L (2l+1)(l+1)l/\pi s^2\sigma_{\mathrm{tot}}(s)\simeq n^2[\log^4 s/2\pi\sigma_{\mathrm{tot}}(s)]$$

This can be refined, using minimization techniques, [see, e.g., (BE1; R3)], to read

$$a_s'(0)<C\log^2 s/\sigma_{\mathrm{tot}}(s). \qquad (3.21)$$

Again we fall one $\log s$ off Regge theory, but this bound is also saturated if the Pomeranchuk theorem was violated (AKM1; EK1) or if the Froissart bound was saturated (E1).

The best bounds on diffraction peaks involve also elastic cross sections, and can be found in (SR1), (R3).

B. Relations among σ_{el}, $d\sigma/d\Omega$

We now present a bound on $d\sigma/d\Omega$, involving the elastic cross section σ_{el}. We will follow the work of Singh and Roy (SR1); this will exemplify the use of minimization techniques, with which all bounds in the preceding sections can be improved.[7]

We begin by defining (we will carry over the proofs only for $\cos\theta_s=1$)

$$|T(s,t=0)|\le\Psi(s)\equiv\frac{s^{1/2}}{2q_s}\sum_0^\infty (2l+1)|\hat{f}_l(s)|, \qquad (3.22a)$$

$$T_s(s,t=4\mu^2)\ge G(s)\equiv\frac{s^{1/2}}{2q_s}\sum_0^\infty (2l+1)P_l\left(1+\frac{2\mu^2}{q_s^2}\right)\times|\hat{f}_l(s)|^2. \qquad (3.22b)$$

[7] A very complete survey of minimization techniques, with applications, can be found in (BE1).

and

$$\sigma_{\text{el}}(s)=\frac{4\pi}{q_s^2}\sum_0^\infty(2l+1)|\hat{f}_l(s)|^2. \tag{3.22c}$$

Then, we define the functional

$$\Lambda[|f_l|]\equiv(2q_s/s^{1/2})\Psi(s)+\alpha(2q_s/s^{1/2})G(s)+\beta(q_s^2/4\pi)\sigma_{\text{el}}(s).$$

The condition of extremum is $\delta\Lambda[|\hat{f}_l|]/\delta|\hat{f}_l|=0$; one can check that this extremum is, indeed, a maximum. Substituting Ψ, G, σ_{el}, this gives

$$1+2[\alpha P_l(1+2\mu^2/q_s^2)+\beta]|\hat{f}_l|_{\max}=0,$$

i.e., with $a=-\frac{1}{2}\alpha$, $b=\beta/\alpha$,

$$|\hat{f}_l(s)|=\min\left\{1,\frac{a}{b+P_l(1+2\mu^2/q_s^2)}\right\}. \tag{3.23}$$

To get a and b we use the constraints (3.22b and c):

$$a^2\sum_0^\infty\frac{(2l+1)P_l(1+2\mu^2/q_s^2)}{[b+P_l(1+2\mu^2/q_s^2)]^2}\leq T_s(s,t=4\mu^2),$$

$$a^2\sum_0^\infty\frac{(2l+1)}{[b+P_l(1+2\mu^2/q_s^2)]^2}=\frac{q_s^2}{4\pi}\sigma_{\text{el}}(s). \tag{3.24}$$

These implicit relations give a, b and hence $|\hat{f}_l|_{\max}$; therefore, substituting Eq. (3.23) into the constraint (3.22a) we get the desired upper bound on $|T(s,t=0)|$, hence on

$$d\sigma/d\Omega|_{t=0}=1/s\,|T(s,t=0)|^2.$$

Equations (3.24) can only be solved numerically [given $T_s(s,\ t=4\mu^2)$], for finite values of s. Asymptotically, however, it is not difficult to get a and b explicitly obtaining the bound

$$d\sigma/d\Omega|_{t=0}\lesssim(\sigma_{\text{el}}/4\pi)[L(s)+1]^2,$$

$$L(s)\equiv\tfrac{1}{2}(s/4\mu^2)^{1/2}\log(s/\sigma_{\text{el}}). \tag{3.25}$$

For arbitrary (physical) angle the corresponding result is

$$d\sigma/d\Omega\lesssim(\sigma_{\text{el}}/4\pi)\{[L(s)+1]^2[P_L(\cos\theta_s)]^2+\sin^2\theta_s[P_L'(\cos\theta_s)]^2\},$$

L as before. It will be noted that, up to logarithms, these bounds are what one would find if putting, naively,

$$\sigma_{\text{el}}=\int d\Omega\frac{d\sigma}{d\Omega}\simeq\frac{4\pi}{4q_s^2}\int_{-4q_s^2}^0 dt\frac{d\sigma}{d\Omega}$$

$$\simeq\frac{4\pi}{s}\int_{-4q_s^2}^0 dt\frac{d\sigma}{d\Omega}\gtrsim\frac{4\pi}{s}\int_{-8\mu^2}^0 dt\frac{d\sigma}{d\Omega}\simeq\frac{\pi}{2s}\frac{d\sigma}{d\Omega}\bigg|_{t\simeq0};$$

we have used that $d\sigma/d\Omega$ is peaked at $t=0$ to cut the integral at $-8\mu^2$ and take the average value of $d\sigma/d\Omega$ to be at $t=0$ when t is inside $(-8\mu^2,0)$.

The field of rigorous relations between different physical quantities has swelled enormously in the last years. The reader is referred to the review of Roy (R3) for a comprehensive list and references.

C. *Lower Bounds on D Waves*

Although we shall work with $\pi\pi$ scattering, the results can be generalized to the πN system (CY2; Y3). Also, although we shall not put isospin indices explicitly, the results will hold for the $\pi^0\pi^0$, $\pi^0\pi^+$, or any positive combination thereof.

The point is, that as was noted in (CY1; CY2), a_2^t cannot be made too small or else the experimental $\pi\pi$ cross sections will violate the absolute bounds (as given in Sec. 3.2 here and references therein). Preliminary computations were performed for $\pi\pi$ and πN in (CY2); we shall present here a method[8] which is both simpler and much more powerful. We begin by defining $L(s)$, and $\eta(s)$ as follows: $L(s)$ is the maximum integer such that

$$\frac{4\pi}{q_s^2}\sum_0^{L(s)}(2l+1)\leq\sigma_{\text{tot}}(s). \tag{3.26a}$$

where, again to avoid problems with identical particles we take the $\pi^0\pi^+$ case, and $\eta(s)$ is such that

$$\frac{4\pi}{q_s^2}[2L(s)+1)\eta(s)]=\sigma_{\text{tot}}(s)-\frac{4\pi}{q_s^2}\sum_0^{L(s)}(2l+1). \tag{3.26b}$$

Equation (3.26) fix uniquely L, η given $\sigma_{\text{tot}}(s)$. Now, with $\cos\theta_s^{(0)}=1+8\mu^2/(s-4\mu^2)$, we have

$$\sum_0^\infty(2l+1)P_l(\cos\theta_s^{(0)})\,\text{Im}\,\hat{f}_l(s)$$

$$\geq\{\sum_0^{L(s)}(2l+1)P_l(\cos\theta_s^{(0)})+[2L(s)+1]\times P_{L(s)+1}(\cos\theta_s^{(0)})\eta(s)\}.$$

because $P_l(\cos\theta_s^{(0)})$ is an increasing function of l, and $\text{Im}\,\hat{f}_l(s)\leq1$. Therefore, we get the bound

$$a_2^t\geq\frac{16}{15\pi}\int_{4\mu^2}^\infty ds s^{-3}\frac{s^{1/2}}{2q_s}\{\sum_0^{L(s)}(2l+1)P_l(\cos\theta_s^{(0)})+[2L(s)+1]P_{L(s)+1}(\cos\theta_s^{(0)})\eta(s)\}, \tag{3.27}$$

this sets a minimum for a_2^t once $\sigma_{\text{tot}}(s)$ is given. Preliminary calculations (R4; S4) show that (3.27) is very tight—so tight indeed that some of the current estimates for the width and mass of the ϵ resonance, or for a_2^t (MP1) are ruled out.

4. LOWER BOUNDS

Unlike the low, medium, and high energy upper bounds we have discussed previously, lower bounds based on axiomatic analyticity are very weak. However, the situation may improve a bit for fixed t bounds if we postulate somewhat more analyticity and very interesting large t bounds are obtained if we assume the

[8] We thank S. M. Roy for discussions about this. We also thank R. J. Eden for communicating some pertinent results prior to publication.

Mandelstam representation. Both types of bounds will be described here.

4.1. Lower Bounds at Fixed t

For physical fixed t the best one can do is to prove that there should exist an integer, N, such that

$$|T(s,t)|>Cs^{-N}.$$

This result is due to Martin (unpublished).[9] One fares better when $t=0$. In this case, it is better to use the variable $E=(s-M^2-\mu^2)/2\mu$ (laboratory energy of the projectile) because, under crossing $s\to u$, E goes over $-E$. Neglecting spin, and for reactions without unphysical cuts (this last restriction *is* essential), we can write a dispersion relation for the symmetric amplitude

$$F^{(+)}(E)\equiv\tfrac{1}{2}\{T(s,t=0)+T(u,t=0)\},$$

$$F^{(+)}(E)=\pi^{-1}\int_{(M+\mu)^2}^{\infty}dE'^2\,\frac{\operatorname{Im}F^{(+)}(E')}{E'^2-E^2}. \quad (4.1)$$

We are allowed to write (4.1) without subtractions because we are looking for *lower* bounds. Let now $E=ir$, $r>0$; (4.1) can be rewritten as

$$r^2F^{(+)}(ir)=\pi^{-1}\int_{(M+\mu)^2}^{\infty}dE'^2\,\frac{\operatorname{Im}F^{(+)}(E')}{(E'/r)^2+1}.$$

Due to unitarity, we have $\operatorname{Im}F^{(+)}(E')\geq 0$, so that we may take the limit as $r\to\infty$ inside the integral to find

$$\lim_{r\to\infty}r^2F^{(+)}(ir)=\pi^{-1}\int_{(M+\mu)^2}^{\infty}dE'^2\operatorname{Im}F^{(+)}(E'). \quad (4.2)$$

The right hand side above may be finite or infinite, but at any rate is positive. Therefore, there exists $C>0$ such that it is larger than C: hence, as $r\to\infty$,

$$|F^{(+)}(ir)|\gtrsim C/r^2. \quad (4.3)$$

Suppose now that one had $|F^{(+)}(E)|<C/|E|^2$ along the real axis: then, the Phragmén–Lindelöf theorem (T1) would imply that $|F^{(+)}(z)|<C/|z|^2$ along any direction. Since this contradicts (4.3), we get the bound

$$|F^{(+)}(E)|>C/|E|^2. \quad (4.4)$$

This bound, as all previous bounds, is to be understood to hold in the mean. Disentangling (4.4), we see that the sum of $T^{AB\to AB}(s,0)$ and $T^{A\bar{B}\to A\bar{B}}(s,0)$ is bounded from below by $C's^{-2}$; therefore, at least one of them satisfies this bound. Take it to be $T^{AB\to AB}$. Then, one can still show that $T^{A\bar{B}\to A\bar{B}}>C''s^{-2}/\log s$. We shall not prove this [cf. (JM1)].

This gives lower bounds on $d\sigma/d\Omega_{t=0}$. If we recall the fixed t bounds mentioned at the beginning of this section, it is clear that $\sigma_{\mathrm{el}}(s)>C_0s^{-N}$ with some N. Therefore, using this together with (4.4) and (3.25) we see that

$$\sigma_{\mathrm{tot}}^{AB}\geq\sigma_{\mathrm{el}}^{AB}>C_1/s^6\log^2 s,$$
$$\sigma_{\mathrm{tot}}^{A\bar{B}}\geq\sigma_{\mathrm{el}}^{A\bar{B}}>C_2/s^6\log^4 s. \quad (4.5)$$

These bounds were first obtained, using complicated techniques of Herglotz functions, by Jin and Martin (JM1). It is quite possible that they can be improved; some of the logarithms have been removed recently by Cornille (C6) and it may be possible to go further. Indeed, for positive t, the bounds described here imply that

$$|T(s,t)|>\bar{C}s^{-1-\epsilon} \quad (4.6)$$

valid for any $\epsilon>0$ and where $4\mu^2-\delta\leq t\leq 4\mu^2$ (δ is a finite, but unknown number). However, assuming analyticity in t in a small neighborhood of $t=4\mu^2$ (minus the cut) continuity in t at $t=4\mu^2$ and polynominal boundedness for $T(s,t)$ in s when t is there, one can show that

$$T_s(s,t)>\bar{C}'s^{-1/2-\epsilon};\qquad 4\mu^2-\delta\leq t\leq 4\mu^2, \quad (4.7)$$

which is much stronger than (4.6).

Let me prove (4.7). Assume it did not hold, and one had $T_s(s,t)<Cs^{-1/2-\epsilon}$ for $t=4\mu^2$ and hence, due to positivity, for all t, $0\leq t\leq 4\mu^2$. First, by using the Froissart–Gribov representation, one can interpolate analytically $\hat{f}_l(t)$ when $t>4\mu^2$, Re $l>$ some N. Take, for simplicity, the case of $\pi\pi$ scattering; then $\hat{f}_l(t)$ satisfies elastic unitarity, and we can write (AR1)[10]

$$S(\lambda,q_t)=[Z(\lambda,q_t^2)-iq_t^{2\lambda}e^{2\pi i\lambda}]/[Z(\lambda,q_t^2)-iq_t^{2\lambda}],$$

where $S(\lambda,q_t)=1+i\hat{f}_{\lambda-1/2}(t)$ for $\lambda-\frac{1}{2}=$ even integer, $t\geq 4\mu^2$. Here Z is analytic in λ, q_t. Continuing back to $t<4\mu^2$, we see that, since $T_s(s,t)<Cs^{-1/2-\epsilon}$, the Froissart–Gribov representation holds for all λ with Re $\lambda\geq-\frac{1}{2}$: thus, there also, one has the above representation for S, valid now for Re $\lambda\geq-\frac{1}{2}$ and $0<t\leq 4\mu^2$. In particular, when $\lambda=$ integer, one can check (AR1) that both numerator and denominator in the expression for S vanish, and S is given by

$$S(\lambda,q_t)=1+2\pi q_t^{2\lambda}/[(\partial/\partial\lambda)Z(\lambda,q_t)-2iq_t^{2\lambda}\log q_t].$$

Elastic unitarity requires $S^*(\lambda^*,q_t^*)S(\lambda,q_t)=1$; in terms of Z, this becomes (we use that $\lambda=$ integer to set $\lambda^*=\lambda$)

$$(\partial/\partial\lambda)Z(\lambda,q_t)+(\partial/\partial\lambda)Z^*(\lambda,q_t^*)$$
$$=-2\pi q_t^{2\lambda}+2iq_t^{2\lambda}[\log q_t-(\log q_t^*)^*]$$
$$=-2\pi q_t^{2\lambda}.$$

Since, however, Z was analytic in q_t^2, one can write

$$\frac{\partial}{\partial\lambda}Z(\lambda,q_t^2)=\sum_{0}^{\infty}a_{2n}q_t^{2n}.$$

[9] It seems that it is now possible to calculate explicitly N (Cornille, private communication).

[10] Although the proofs of (AR1) are in the framework of potential scattering, they are formal and thus hold without changes in general; cf., e.g., (F3).

Comparing, we see that $(\partial/\partial\lambda)Z = i\sum_{n\neq\lambda} r_{2n} q_t^{2n} - \pi q_t^{2\lambda}$, r_{2n} = real. Thus, writing $q_t = ik$, k = real positive, we have

$$S(\lambda, q_t) = 1 + i\{2\pi k^{2\lambda} / [\sum_{n\neq\lambda} (-1)^{n-\lambda} r_{2n} k^{2n} - 2k^{2\lambda} \log k]\}.$$

This gives for the quotient $\hat{f}_{\lambda-1/2}(t)/k^{2\lambda}$ the value

$$2\pi / \{\sum_{n\neq\lambda} (-1)^{n-\lambda} r_{2n} k^{2n} - 2k^{2\lambda} \log k\}.$$

Let now $\lambda = 0$. On the one hand, letting q_t (hence k) tend to zero, we see from this result that

$$\lim_{k\to 0} \hat{f}_{\lambda-1/2}(t)/k^{2\lambda} = 0.$$

On the other hand, however, the Froissart–Gribov representation says that

$$\lim_{k\to 0} \hat{f}_{\lambda-1/2}(t)/k^{2\lambda} = \pi^{-1} \int_{4\mu^2}^{\infty} ds s^{-1/2} T_s(s, t = 4\mu^2),$$

which, due to positivity, can only vanish if $T_s \equiv 0$, and $\sigma_{\text{tot}} \equiv 0$. Thus we have reached a contradiction: QED.

The proof of this result is classical in Regge pole theory. The extension to the general situation was achieved independently by Martin and the present author (both unpublished).

4.2. Large Angle Bounds

Large angle bounds using only axiomatic analyticity have been obtained by Ciulli and by Van Hieu [see, e.g., (V1; C2)]. We refer there for the results. Here we shall describe the original lower bound of Cerulus and Martin (CM1). The assumptions are that it is possible to continue $T(s, z = \cos\theta_s)$ inside the domain D_z of Fig. 4, where $z_0 = 1 + 2\mu^2/q_s^2$ (also shown there is the axiomatic ellipse, E) and that $T(s, z)$ is bounded in s as $|T(s, z)| < Cs^N$, N independent of s and z. This is certainly weaker than the Mandelstam representation, but not much is gained or lost by assuming the latter to hold.

We begin by assuming that, inside the segment $(-1+\delta, 1-\delta)$, $\delta > 0$,

$$|T(s, z)| < e^{-\varphi(s)}, \tag{4.8}$$

and we will show that, if φ is too large, we shall get a contradiction with (4.4), i.e., at $z = 1$ ($t = 0$). To do so, we first map D_z into D_ζ (Fig. 6a) by defining

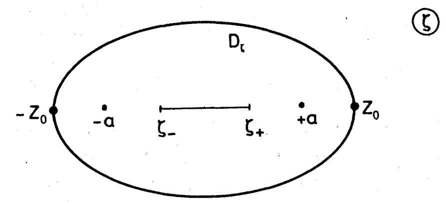
$$\zeta = (z_0/z)[z - (z_0^2 - z^2)^{1/2}];$$

$\pm a$ are the images of ± 1, $\zeta_\pm$ the images of $\pm(1-\delta)$. Then, one changes variables again writing

$$w = \zeta_+^{-1}[\zeta + (\zeta^2 - \zeta_+^2)^{1/2}].$$

This maps D_ζ over the disc D_w (Fig. 6b), the segment $[-a, +a]$ over the circle C_a, and the segment $[\zeta_-, \zeta_+]$ over the circle C_ζ. We will denote by r_D, r_a, $r_\delta = 1$ to the radii of D_w, C_a, C_δ, respectively.

The next step is to recall Hadamard's three circles theorem (T1) which states that, if $|T(s, w)|$ is bounded



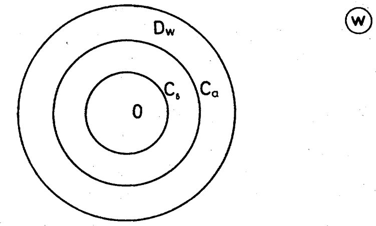


FIG. 6. (a) Mapping onto ζ plane. (b) Mapping onto w plane.

by M_D on the boundary of D_w, and by M_δ on C_δ, then, on C_a,

$$|T(s, w)| < M_\delta M_D[(\log r_D - \log r_a)(\log r_a)]/\log^2 r_D. \tag{4.9}$$

It now remains only to compute. From (4.8) we know that, for z inside $[-1+\delta, 1-\delta]$, $|T(s, z)| < \exp[-\varphi(s)]$; from our original assumptions, we have that $|T(s, z)| < Cs^N$, z on the boundary of D_z: making the change of variables $z \to \zeta \to w$ this gives M_δ, M_D: (4.9) gives an explicit bound for $|T(s, w)|$ when w is in C_a. Transforming back, $w \to \zeta \to z$, we get an explicit upper bound on $|T(s, z)|$ for $-1 \le z \le 1$. In particular, for $z = \cos\theta_s = 1$, and large s (so that, e.g., we can replace $\log r_a / \log r_D$ by $1 - C_0/q_s$), this reads

$$|T(s, \cos\theta_s = 1)| < \exp\{-C\varphi(s)(2\mu/q_s) + N(\log s)[1 - (2\mu C/q_s)]\}, \tag{4.10}$$

where C is a positive constant that depends only on δ. Now, if $\varphi(s) > C' s^{1/2+\epsilon}$ (any $\epsilon > 0$), (4.10) gives

$$|T(s, \cos\theta_s = 1)| \lesssim \exp(-C'' s^\epsilon), \qquad C'' > 0,$$

which contradicts (4.4). Therefore we have proven that $|T(s, \cos\theta_s)| > \exp\{-C' s^{1/2+\epsilon}\}$: this is the desired lower bound. A more refined analysis (K1) gives the bound

$$|T(s, \cos\theta_s)| > \exp[-C(\theta_s s)^{1/2} \log s], \qquad C > 0, \quad -1 < \cos\theta_s < 1. \tag{4.11}$$

Although not directly related to our main topic in these notes, we shall mention that a bound similar to (4.11) can be proved for form factors. In fact, using only axiomatic analyticity, Martin (M4) has proven that

$$|F(t)| > \exp[-C(t)^{1/2}], \qquad C>0. \quad (4.12)$$

Both bounds (4.11) and (4.12) have been on and off saturated by experiment. The present status seems to be that they are compatible with experimental results, although a few years back phenomenological fits (in particular Orear's fit) saturated them and left ground for believing in violation of (4.11).

5. DISCUSSION

In the next two sections we shall try to explain the physical reasons why we have all the bounds described so far for hadron interactions.

5.1. Finite Range of Strong Interactions

Here I will try to give physical (i.e., intuitive) arguments why we have bounds for scattering amplitudes. Of course the reason is the finite range of strong interactions: if we take an impact parameter model, then the distance r of the projectile to the target is related to the momentum by $q_s r \simeq l$. Now, we will have interaction if projectile and target come close enough so that they can exchange something. The lightest object a pion and a nucleon can exchange (cf. Fig. 3) is a two-pion pair, and for two nucleons a pion (for example). Hence, we will have interaction if $l<\bar{L}=q_s r_0$, with $r_0=1/t_0^{1/2}$, $t_0=4\mu^2$ for πN, $t_0=\mu^2$ for NN. Actually, we shall have interaction even farther because we must take into account the tail of, for example, a Yukawa potential. Since the interaction will then decrease exponentially, this means that we shall have interaction if $l<L=\bar{L}\log(q_s r_0)$. Because of the strength of the interactions, we expect that $2\,\mathrm{Im}\,f_l(s)\simeq s^{1/2}/q_s$ if $l<L$, but, for $l>K$ the particle will altogether miss the target and we will neglect the corresponding waves. Thus, we have

$$\sigma_{\mathrm{tot}}\simeq \frac{4\pi}{q_s^2}\sum_0^L (2l+1)\simeq \frac{4\pi}{t_0}\log^2\frac{s}{\mu^2}.$$

This is the Froissart bound. In this model one can understand the improvement of Sec. 3.2: the a_2^t gives a limit on the contribution of the neglected waves, $l>L$. All other bounds of Secs. 3.2 to 3.5 can be similarly understood by neglecting waves with $l>L$.

Harder to understand are the crossing constraints of Sec. 2.3; however, the positivity constraints of Sec. 2.2 are quite simple: dimensionally, $a_l=[\mathrm{mass}]^{-2l}$. The typical mass being that of the lightest exchanged object (a two-pion pair for both $\pi\pi\to N\bar{N}$ and $\pi\pi\to\pi\pi$), we would expect $a_l\simeq[2\mu]^{-2l}$ and hence $a_{l+2}\lesssim[2\mu]^4 a_l$. As for the absolute bounds of Sec. 2.1, they are a reflection of the fact that, if the four-pion forces were too strong, they would give rise to bound states. Actually, this is one of the reasons why one cannot extend the results to πN or NN because there bound states do exist.

5.2. Sizes of Particles

Concentrating on Froissart-like bounds, one can also say that bounds on σ_{tot} simply mean bounds on the sizes of the particles involved. To show this clearly, I will discuss an amusing application. Consider πA scattering, A a nucleus with atomic number A. Assume temporarily that one had $t_0=4\mu^2$; we shall come to this at the end. In the absolute bounds of Sec. 3.2, $\bar{a}_2^t(A)$ (with obvious notation) enters linearly, but in the improved version of Common (C5) it turns out that only $\log\bar{a}_2^t(A)$ appears. Here $a_2^t(A)$ may be computed by saturating the t-channel D wave with the contribution of the nuclear "resonance" corresponding to N_{33}^*, that we shall denote by A^*, for the left-hand cut, and the f^0 for the right-hand cut. Hence $\bar{a}_2^t(A)$ is proportional to the coupling constants $g_{A^*A\pi}^2$ and $g_{A\bar{A}f^0}$. Now, all known coupling constants of nuclear physics scale: hence, we take $g_{A^*A\pi}^2\simeq A^2 g_{N_{33}^*N\pi}^2$, $g_{A\bar{A}f^0}\simeq A g_{NNf^0}$, so that $\bar{a}_2^t(A)\simeq A^2\bar{a}_2(\pi\pi\to N\bar{N})$. Since $\bar{a}_2^t(A)$ enters only logarithmically in the bounds, the bounds to $\bar{\sigma}_{\mathrm{tot}}$ ($\pi A\to$ all) will be $\log A$ times the bounds for $\bar{\sigma}_{\mathrm{tot}}$ ($\pi N\to$ all). Experimentally, however, we have $\bar{\sigma}_{\mathrm{tot}}$ ($\pi A\to$ all)$\simeq A^{2/3}\bar{\sigma}_{\mathrm{tot}}$ ($\pi N\to$ all), so that we shall have a violation of the bound if A is large enough. In fact, a detailed calculation (CY1) gives violation for $A\gtrsim 20$.

What then is wrong here? Clearly, it is the assumption that $t_0=4\mu^2$. Now, one can prove that t_0 *always* equals $4\mu^2$ *except* if one has anomalous thresholds in the t channel. Denoting by B the binding energy of a nucleon in A, if can be shown that one has anomalous thresholds if, and only if, $B<10$ MeV: therefore, we are predicting that, for $A>20$, B is less than 10 MeV, to be compared with the experimental figure of 8 MeV. Few detailed nuclear physics calculations would give such a good estimate! Since the size of a nucleus is inversely related to the binding energy, we see that Froissart bounds measure with fairly good accuracy the sizes of the objects involved.

ACKNOWLEDGMENTS

It is with pleasure that the author acknowledges innumerable discussions with our colleagues, in particular A. Martin, A. K. Common, R. J. Eden and S. M. Roy. We are also indebted for hospitality at the Theory Division of CERN, where a good part of these notes were written.

APPENDIX—SOME RECENT DEVELOPMENTS

In this section we will present very quickly some of the results that have been obtained since this review was originally written (December 1971). The references will be found at the end of the list of references.

The problem of merging positivity and crossing constraints (Secs. 2.2, 2.3) has been developed by Roskies and Yen (RY1-A), who have been able to find conditions necessary and sufficient to have full crossing symmetry plus the requirement $\mathrm{Im}\,\hat{f}_l\geq 0$. Grassberger

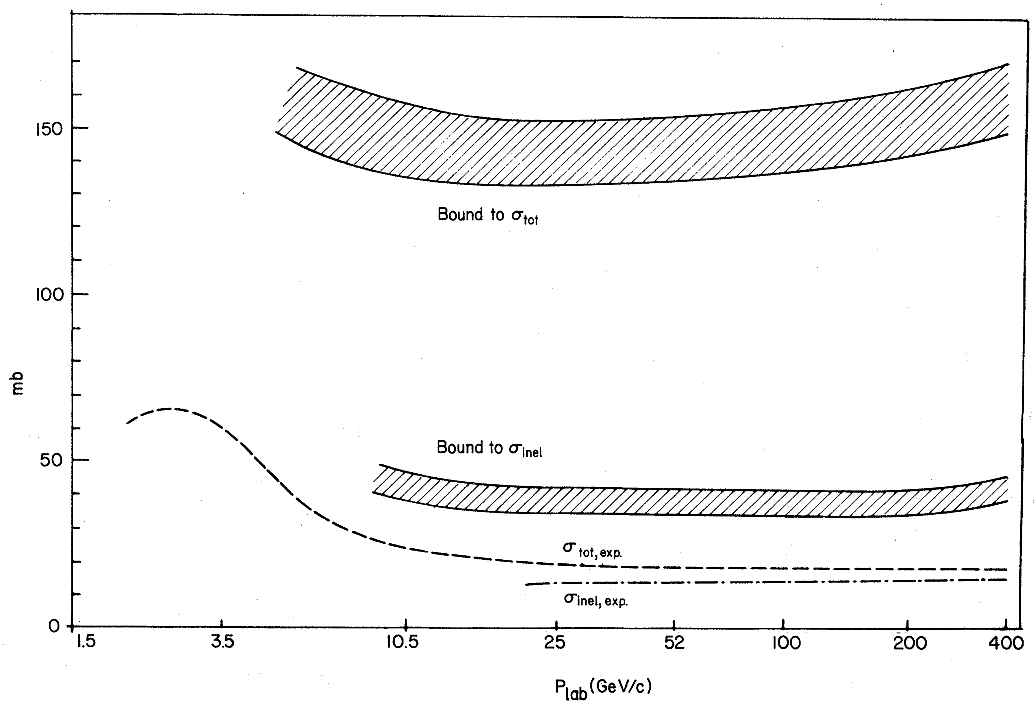


FIG. 7. High energy bounds on total and total inelastic $\pi^0\pi^+$ cross sections, together with "experimental" values (using Regge theory or the Quark Model). Solid lines, bounds. Broken line, experimental σ_{total}. Broken-plus-dotted line, $\sigma_{inelastic}$.

(G2-A) has been able to take into account the nonlinear part of positivity, $0\leq \mathrm{Im}\,\hat{f}_l\leq 1$. The constraints are so stringent that no model for $\pi\pi$ scattering seems to satisfy them (PY1-A). The meaning of this last result is not clear, however. Indeed, crossing constraints (for example) such as those in Eqs. (2.12) give relations between integrals of polynomials in s times partial waves over the s-channel unphysical region, and this means that what one is testing are derivatives of $f_l(s)$ with respect to s. It is not surprising that admittedly rough models fail to reproduce such fine details as the constraints are testing. Other crossing plus positivity constraints which involve one single wave (the D wave, for example) have been found by Common and Pidcock (CP1-A). For this, one uses a set of relations found by Roy (R1-A) which give $\mathrm{Re}\,f_l(s)$ for $4\mu^2\leq s\leq 60\mu^2$ in terms of the absorptive part of the scattering amplitude in the physical region only. In the same paper, Roy has produced crossing constraints that involve integrals over $F_s(s,t)$ for s physical. These results are very useful for consistency checks of $\pi\pi$ partial wave analyses. Finally, Grassberger (G3-A) has improved the constraints on the $\pi^0\pi^0$ S wave by showing that $df_0^{00}(s)/ds<0$ for $s<1.21895\mu^2$. This is an optimum result, as Mahoux and Martin (MM1-A) have found examples saturating it.

It is known that one cannot remove the logarithm in the Froissart bound for, e.g., $\pi\pi$,

$$\sigma_{tot}(s)<(\pi/\mu^2)\log^2 s/s_0.$$

However, what one can do is to swell the scale factor, s_0.

This is the case if to the constraints given by $a_2^t=\lim f_2(t)/q_t^4$ [recall Eq. (3.7)] one adds the requirement that both

$$\lim_{q_t\to 0}(d/dt)[f_2(t)/q_t^4],\quad \lim_{q_t\to 0}(d^2/d_t^2)[f_2(t)/q_t^4] \qquad \text{(A1)}$$

should also be finite, as follows from elastic unitarity in the t channel. Then one can show (Y1-A) that this gives $s_0=\log^7 s$; numerically, this gives practically constant bounds up to ISR energies. In Fig. 7 we have shown the best bounds at high energies, using also the constraints (A1) (Fig. 7).

On the other hand, if one is willing to make the *assumption* $\sigma_{tot}(s)<C$, then one can apply the methods of Sec. 3.2 to get bounds on C (BS1-A). These bounds are quite reasonable (of the order of a few dozen millibarn). Alternatively, one can improve the bounds of Secs. 3.2–3.3 by using minimization techniques.

Lower bounds which are better and more complete than those in Sec. 4 have been developed by Cornille and Martin (CM1-A), but it is hard to give details as the work is still unpublished.

Apart from the bounds which involve only hadrons, it is possible to apply positivity à la Glaser (G1-A) to derive relations which bound processes involving on and off mass shell photons or, more generally, electromagnetic or weak currents. These relations, which have been extensively studied by de Rafael and De Rujula, are refined generalizations of the obvious bound

$$\text{Im}\,\langle p_1p_2\,|\,\mathfrak{T}\,|\,p_1p_2\rangle \geq |\,\text{Im}\,\langle p_1p_2\,|\,\mathfrak{T}\,|\,p_1'p_2'\rangle|$$

valid directly from positivity for massless or massive particles and physical energies and scattering angles. A complete treatment together with references can be found in (DR1-A). Finally, very tight phenomenological bounds on scattering amplitudes have been discussed by Hahn and Hodgkinson [(HH1-A); cf., also the lecture notes of Eden (E1-A)], while inequalities that bound inclusive, multiparticle, etc. cross sections in terms of elastic ones have been recently discussed by many people (TT1-A), (DKS1-A), (R3).

BIBLIOGRAPHICAL NOTE

A comprehensive survey of the field of high energy collisions is presented in Eden's treatise (E1) and review (E2), supplemented by (AKM1; EK1) for high-energy theorems, and the review of Roy (R3) for high-energy bounds. Low-energy constraints are surveyed by Wanders (W1). For the proofs of the required analyticity, we recommend (M10; S3) for axiomatic results, and (ELOP1) for perturbation and "pure" S-matrix analyticity.

REFERENCES

References to Text

(A1) D. Atkinson, in "Strong Interaction Physics", edited by G. Höhler, Springer Tracts No., 57, Berlin, 1971.

(A2) ——, in "Analytical Theory of the S Matrix" G.I.F.T. Spring Seminar on Theoretical Physics, G.I.F.T., (University of Saragossa, Spain, 1971).

(ABMM1) G. Auberson, O. Brander, G. Mahoux, and A. Martin, Nuovo Cimento **65A**, 743 (1970).

(AKM1) ——, T. Kinoshita, and A. Martin, Phys. Rev. D **3**, 3185 (1971).

(AM1) R. Ascoli and A. Minguzzi, Phys. Rev. **118**, 1435 (1960).

(AR1) V. De Alfaro and T. Regge, "Mathematical Theory of Potential Scattering" (North Holland Publ. Co., Amsterdam, 1966).

(B1) J. S. Bell, Nuovo Cimento **61A**, 541 (1969).

(B2) D. J. Bessis, Nuovo Cimento **45**, 974 (1966).

(B3) O. Brander, CERN Preprint TH-1353 (1971).

(BB1) A. P. Balachandran and M. L. Blackmon, Phys. Letters **31B**, 665 (1970).

(BCM1) J. L. Basdevant, G. Cohen-Tannoudji, and A. Morel, Nuovo Cimento **64A**, 585 (1969).

(BE1) R. Blankenbecler and M. Einhorn, Ann Phys. (N.Y.), **67**, 480 (1971).

(BEG1) J. Bros, H. Epstein, and V. Glaser, Nuovo Cimento **31**, 1265 (1964); Commun. Math. Phys. **1**, 240 (1965).

(BN1) A. P. Balachandran and J. Nuyts, Phys. Rev. **172**, 1821 (1968).

(BV1) B. Bonnier and R. Vihn Mau, Phys. Letters **24B**, 477 (1967); Phys. Rev. **165**, 1323 (1967).

(C1) W. Case, Syracuse Preprint 1206–231 (1971).

(C2) S. Ciulli, Nuovo Cimento **61A**, 463 (1969).

(C3) A. K. Common, Nuovo Cimento **63A**, 451 (1969).

(C4) ——, Nuovo Cimento **63A**, 863 (1969) and **65A**, 581 (1970).

(C5) ——, Nuovo Cimento **69A**, 115 (1970).

(C6) H. Cornille, Nuovo Cimento **4A**, 549 (1971).

(CM1) F. Cerulus and A. Martin, Phys. Letters **8**, 80 (1964).

(CM2) G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

(CMN1) G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N.Y.) **46**, 111 (1968).

(CP1) A. K. Common and M. R. Pennington, Westfield College Preprint (1971).

(CP2) ——, and M. R. Pennington, Nuovo Cimento Letters, **2**, 429 (1971).

(CW1) ——, and R. Wit, Nuovo Cimento **3A**, 179 (1971).

(CY1) ——, and F. J. Yndurâin, Nucl. Phys. **B26**, 167 (1971).

(CY2) ——, and F. J. Yndurâin Nucl. Phys. **B34**, 509 (1971).

(CY3) ——, and F. J. Yndurâin, Nuclear Phys. B, **45**, 577 (1972).

(E1) R. J. Eden, "High Energy Collisions of Elementary Particles", Cambridge (U.K.) 1967.

(E2) ——, Rev. Mod. Phys. **43**, 15 (1971).

(EK1) ——, and G. Kaiser, Nucl. Phys. **B28**, 253 (1971).

(ELOP1) ——, P. V. Landshoff, D. J. Olive, and J. C. Polkinghorne, "The Analytic S Matrix", (Cambridge U. P., London, 1966).

(EGM1) H. Epstein, V. Glaser, and A. Martin, Commun. Math. Phys. **13**, 257 (1969).

(F1) R. Froissart, Phys. Rev. **123**, 1053 (1961).

(F2) ——, Proc. La Jolla Conference on Elementary Particles, (1961), unpublished.

(F3) ——, in *Theoretical Physics*, IAEA, Vienna, 1962.

(G1) V. N. Gribov, Sov. Phys. JETP **14**, 1395 (1962).

(GY1) J. L. Gervais and F. J. Yndurâin, Phys. Rev. **167**, 1289 and **169**, 1187 (1968); Phys. Rev. **1D**, 1835 (1970).

(I1) J. Iliopoulos, Nuovo Cimento **52A**, 192 (1967) and **53A**, 552 (1968).

(JM1) Y. S. Jin and A. Martin, Phys. Rev. **135**, B1369 and B1375 (1964).

(JW1) M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) **7**, 404 (1959).

(K1) T. Kinoshita, Phys. Rev. Letters **12**, 256 (1964).

(K2) ——, in *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colorado, 1964), Vol. 7B.

(KLM1) ——, J. J. Loeffel, and A. Martin, Phys. Rev. Letters **10**, 460 (1967); Phys. Rev. **135B**, 1464 (1964).

(K3) K. L. Kowalski, Phys. Rev. **137B**, 1350 (1965).

(LM1) L. Lukaszuk and A. Martin, Nuovo Cimento **52A**, 122 (1967).

(LMV1) A. A. Logunov, M. A. Mestrivishvili and N. Van Hieu, *Proceedings 1967 International Conference on Particles and Fields* (Interscience, New York, 1967).

(M1) S. W. MacDowell, Phys. Rev. **D1**, 3501 (1970).

(M2) G. Mahoux, Thèse, Université de Paris, 1969.

(M3) ——, Nuovo Cimento Letters **3**, 221 (1970).

(M4) A. Martin, Nuovo Cimento **37**, 671 (1965).

(M5) ——, Nuovo Cimento **42**, 930; **44**, 1219 (1966).

(M6) ——, Nuovo Cimento **47A**, 265 (1967) and **58A**, 303 (1968).

(M7) ——, Nuovo Cimento **63A**, 167 (1969).

(M8) ——, "Scattering Theory: Unitarity, Analyticity and Crossing", Lecture Notes in Physics, No. 3, Springer, Berlin 1969.

(M9) ——, in *Problems in Theoretical Physics*, Nauka, Moscow, 1969.

(M10) ——, in "Analytical Theory of the *S* Matrix", G.I.F.T. Spring Seminar on Theoretical Physics, G.I.F.T. (University of Saragossa, Spain, 1971).

(MM1) S. W. MacDowell and A. Martin, Phys. Rev. **135B**, 960 (1964).

(MM2) G. Mahoux and A. Martin, Phys. Rev. **174,** 2140 (1968).

(MOS1) W. Magnus, F. Oberhettinger, and R. P. Soni, "Formulas and Theorems for the Special Functions of Mathematical Physics," Springer, Berlin 1966.

(MP1) D. Morgan and J. Pisut, in *Low Energy Hadron Interactions*, edited by G. Höhler, Springer Tracts No. 55, Berlin, 1970.

(P1) N. R. Pennington, Nucl. Phys. **B24,** 317 (1970); **B25,** 621 (1971).

(PW1) O. Piguet and G. Wanders, Phys. Letters **30B,** 418 (1969).

(R1) R. Roskies, Phys. Letters **30B,** 42 (1969); Nuovo Cimento **65A,** 467 (1970).

(R2) ——, J. Math. Phys. **11,** 2913 (1970); Phys. Rev. **2D,** 247 (1970).

(R3) S. M. Roy, "High Energy Theorems for Strong Interactions and their Comparison with Experimental Data", Phys. Reports, to be published.

(R4) ——, Private communication.

(RS1) ——, and V. Singh, Phys. Letters **32B,** 50 (1970).

(S1) G. Sommer, Nuovo Cimento **48A,** 92 (1967).

(S2) ——, Nuovo Cimento **52A,** 373, 866 (1967).

(S3) ——, Fortschritte der Phys. **18,** 577 (1970).

(S4) A. Steven, Cavendish Preprints, 1972.

(SR1) V. Singh and S. M. Roy, Ann. Phys. (N.Y.) **57,** 461 (1970); Phys. Rev. **1D,** 2638 (1970).

(ST1) J. A. Shohat and J. D. Tamarkin, *The Problem of Moments,* Amer. Math. Soc. Monographs, Providence (R.I.) **1943**.

(T1) E. C. Titchmarsh, *Theory of Functions* (Oxford U. P., Oxford, England, 1939).

(TT1) G. Tiktopoulos and S. B. Treiman, Phys. Rev. **167,** 1437 (1968).

(V1) N. Van Hieu, J.I.N.R. Preprint E-2-3728, Dubna 1968.

(V2) G. Veneziano, Nuovo Cimento **57A,** 190(1968).

(W1) G. Wanders, in *Strong Interaction Physics,* edited by G. Höhler, Springer Tracts No. 57, Berlin 1971; in "Analytical Theory of the S Matrix", G.I.F.T. Spring Seminar on Theoretical Physics, G.I.F.T. (Spain) 1971.

(W2) S. Weinberg, Phys. Rev. Letters **17,** 616 (1966).

(Y1) F. J. Ynduráin, Nuovo Cimento **64A,** 225 (1969).

(Y2) ——, Phys. Letters **31B,** 368, 620 (1970).

(Y3) ——, "Stable Extrapolations of Scattering Amplitudes Using Unitarity," CERN Preprint TH-1372 (1971), and Ann. Phys. (N.Y.) to te published.

References to Appendix

(BS1-A) R. Blankenbecler and R. Savit, SLAC-PUB-1018 preprint (1972).

(CM1-A) H. Cornille and A. Martin, private communication.

(CP1-A) A. K. Common and M. K. Pidcock, Canterbury preprint (1972).

(DKS1-A) P. P. Divakaran, M. Kugler, and J. Soffer, Weizman Institute Preprint WIS 72/22 Ph (1972).

(DR1-A) A. De Rújula and E. de Rafael, IHES Preprint (1972), to be published in Ann. Phys. (N.Y.); see also A. De Rujula, in 1972 Rencontres de Moriond.

(E1-A) R. J. Eden, Lecture Notes for the Kaiserslautern Summer School in Theoretical Strong Interaction Physics, 1972.

(G1-A) V. Glaser, in "Problems in Theoretical Physics", Nauka publ., Moscow (1969).

(G2-A) P. Grassberger, Nuclear Phys. **B41,** 474 (1972).

(G3-A) ——, Bonn Preprint (1972).

(HH1-A) B. D. Hahn and D. P. Hodgkinson, Cavendish Preprint HEP 72-5, (1972) to be published in Nucl. Phys. B.; D. P. Hodgkinson, Nuovo Cimento Letters **4A,** 731 (1972).

(MM1-A) G. Mahoux and A. Martin, private communication.

(R1-A) S. M. Roy, Phys. Letters **36B,** 353 (1971).

(RY1-A) R. Roskies and H. C. Yen, Phys. Rev. **4D,** 1873 (1971).

(TT1-A) G. Tiktopoulos and S. B. Treiman, Rockefeller University Preprint COO-3505-20 (1972).

(Y1-A) F. J. Ynduráin, CERN Preprint TH-1554 (1972), to be published in Phys. Letters.