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Application of Orthogonal and Unitary Group Methods to the N-Body Problem*†

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Methods for constructing states of good total orbital angular momenta of N identical, free, structureless particles through the use of the orthogonal and unitary groups are developed. The first part of the paper reviews the existing literature, particularly for the three-particle problem. New results include the discrete symmetry properties of the SU(3) states vectors of the three-particle problem. The general N-particle problem is approached through the use of the subgroup property $O(n)\subset U(n)$. An imbedding of $O(n)$ in $U(n)$ is given which greatly simplifies the study of the $O(n)$ subgroup of $U(n)$. Particular applications of this imbedding are: (1) an explicit constructive procedure for obtaining all the single-valued irreducible representations of $O(n)$, and (2) an explicit constructive procedure for obtaining all N-particle states of good angular momenta up through the degree four solid harmonics.

CONTENTS

I. INTRODUCTION

The method of K harmonics was introduced by Zickendraht (Zi65) and Simonov (Si66) as a viable technique for determining the wave functions and binding energies of the three-nucleon system. Since then the method has undergone extensive applications (Si67, Ba67, Br70) and developments (Ba66, Si68, Zi69, Ri69, Ba70) toward the goal of developing a calculational scheme applicable to arbitrary nuclei.

The basic formulas of the K harmonic method are derived from an expansion of the A nucleon nonrelativistic wave function in terms of the spherical harmonics on the $3A-3$ sphere. The construction of such spherical harmonics is basic to the technique.

The spherical harmonics which occur in the K harmonic method are, of course, just the angular functions which occur in the solution of Schrödinger's equation for the relative motion of N identical noninteracting particles, i.e., they are solutions to Laplace's equation on the $(3N-3)$-sphere ($N=A$ for the nuclear problem). In place of the spherical harmonics, we can choose the *solid harmonics* which are then characterized as being homogeneous polynomials of degree p ($=K$) in the $3N-3$ relative coordinates which one introduces to describe the motion of N identical noninteracting particles relative to the center of mass.

In the present work, we consider the problem of obtaining polynomial solutions to Laplace's equation in $(3N-3)$-space in considerable detail. The principal physical motivation for this study is the basic role of the solid harmonics in the K harmonic technique for investigating the properties of actual N-particle systems. From a mathematical viewpoint, this seemingly simple problem presents a natural and physical framework, rich in structure, to which the more recent techniques in orthogonal and unitary group theory are applicable. We would like to view this aspect of the problem not so much as one more application of abstract group theoretical results, but rather as an opportunity to give such results a concrete realization in terms of a physically meaningful problem.

It is simple enough to obtain a basis for the solid harmonics in arbitrary n-space. The difficulties begin when one requires that this basis [in $(3N-3)$-space] contain *explicitly* each state of sharp total (relative) orbital angular momentum of the system of N particles. The properties of the basis solid harmonics under interchanges of identical particles are also important, and one would like to deal optimally with this aspect of the

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problem. The methods of implementing these properties into the basis solid harmonics comprise the main theme of this paper.

In Secs. II–V, we present the conceptual framework for the subsequent developments. These sections are self-contained and comprise a review of existing literature.

The 3-particle problem is considered in Sec. VI. Here the global role of the unitary group $SU(4)$ is emphasized. The discrete symmetries of the $SU(3)$ state vectors are given in detail. We believe these results to be new. The multiplicity problem is discussed from a different viewpoint.

The N-particle problem is discussed in Sec. VII from a viewpoint which is compatible with the standard angular momentum coupling methods, but which is particularly well adapted to the 4-particle problem.

In the last and most difficult section, Sec. VIII, we realize the imbedding of the orthogonal group $SO(n)$ in the unitary group $U(n)$ in a way which reveals the full structure of all the single-valued irreducible representations of $SO(n)$. This same imbedding is then used to give explicit formulas for constructing all solid harmonics up to degree four which are labeled by good orbital angular momentum quantum numbers for an arbitrary number of particles.

We emphasize again that the construction of solid harmonics of good total orbital angular momentum is a first, but nontrivial, step toward the solving of actual physical problems. For the detailed methods of implementing these functions into the three- and four-nucleon wave functions, we refer the reader to the following papers: *Three nucleons*; Badalyan and Simonov (Ba66), Simonov and Badalyan (Si67), Brayshaw and Buck (Br70). *Four nucleons*; Badalyan *et al.* (Ba67), Beam (Be67), Galbraith (Ga72).

The vast amount of literature relating to Laplace's equation, the orthogonal groups, and the unitary groups prohibits us from referencing all but those works which we have found to be most directly related to the methods presented in this paper.

II. ORTHOGONAL AND UNITARY GROUPS

In this section, we discuss certain aspects of the orthogonal and unitary groups which are required for our later work dealing with N-particle state vectors. We lay the background for finding solutions to Laplace's equation in n-space. Here n is unspecified, and the coordinates need not relate in any fashion to particle coordinates. Particular choices of n which do relate to the N-particle problems are made in subsequent sections. The techniques used are significant for all the subsequent developments.

A. Orthogonal Groups

We use the notation $O(n)$ to denote the group of $n\times n$ real orthogonal matrices ($n=2, 3, \cdots$):

$$O(n)=\{R: \tilde{R}R=I_n, R \text{ real}\}. \tag{2.1}$$

The notation $SO(n)$ denotes the subgroup of $O(n)$ whose elements have determinant equal to $+1$.

All irreducible matrix representations of the Lie algebra of $O(n)$ were given by Gel'fand and Zetlin (Ge50). More detailed derivations of their results and the development of related concepts have been the basis of several subsequent investigations (Lo60a, Pa67, Wo67).

Our interest in the orthogonal groups derives from the fact that the Laplacian operator in Euclidean n-space, R^n, is invariant under orthogonal transformations; hence, the study of the orthogonal groups is pertinent to any investigation of the solutions to Laplace's equation. In this section, we discuss in detail only the simplest aspects of the orthogonal groups, introducing more elaborate and related techniques as they are needed in the later sections. We introduce and discuss the general Gel'fand–Zetlin notation for the abstract basis vectors of an abstract carrier space of an (irreducible representation) IR of $O(n)$. This notation is not utilized until Sec. VIII, but is included here for completeness of presentation, and, more significantly, because the conceptual structure of these general vectors can be easily comprehended as extensions of properties which are explicit in the basis which is given.

Let $\mathbf{x}$ denote a vector which has components relative to a Cartesian basis of R^n given by $(x_1, x_2, \cdots, x_n)$. We find it convenient to associate with each such point of R^n a column matrix x:

$$x=\text{col }(x_1, x_2, \cdots, x_n). \tag{2.2}$$

We are interested only in polynomial solutions to Laplace's equation (the analog of the solid harmonics $\mathcal{Y}_{lm}(x)$ in 3-space), and will accordingly impose this restriction (although many results have a larger domain of validity). Indeed, we will be even more specific. We begin by introducing the space $\mathfrak{L}_p$ of complex polynomial functions f which are homogeneous of degree p in x, and which solve Laplace's equation:

$$\mathfrak{L}_p=\{f: f(\lambda x)=\lambda^p f(x), \nabla_n^2 f(x)=0\}, \tag{2.3}$$

where ∇_n^2 denotes the Laplacian in n-space, i.e.,

$$\nabla_n^2=\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i}\right)^2. \tag{2.4}$$

The first thing we would like to do is to make $\mathfrak{L}_p$ into a *Hilbert space*, i.e., to define a scalar product $\langle f \mid f'\rangle$ for each pair of functions belonging to $\mathfrak{L}_p$. Conventionally, we do this by defining the scalar product as

$$\langle f \mid f'\rangle=\int dSf^*(x)f'(x), \tag{2.5}$$

where the integration is carried out over the *unit* n-sphere. [Despite the fact that f has well-defined values on all finite regions of R^n, the scalar product (2.5) is defined in terms of the values which the functions have on the unit sphere.] However, there is another definition of scalar product which is of considerable utility when one is dealing with polynomial spaces. We introduce this scalar product now, and discuss later why it is useful. Let f and f' be *arbitrary polynomials*. We define the complex number (f,f') by

$$(f,f')=[f^*(\partial/\partial x)f'(x)]_{x=0}, \tag{2.6}$$

where $f^*(\partial/\partial x)$ is the *differential operator* defined by

$$f^*(\partial/\partial x)=\sum_{(\alpha)} a_{(\alpha)}^*(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n} \tag{2.7}$$

for $f(x)$ given by

$$f(x)=\sum_{(\alpha)} a_{(\alpha)}x_1^{\alpha_1}\cdots x_n^{\alpha_n}. \tag{2.8}$$

Thus, this differential operator acts on $f'(x)$ in the right-hand side of Eq. (2.6) to produce a new polynomial which is evaluated at $x=(x_1, x_2, \cdots, x_n)=(0,0,\cdots,0)$, thus yielding the complex number (f,f').

It is easily verified that definition (2.6) satisfies all the requirements of a scalar product. We can use this definition of scalar product to make the space $\mathcal{L}_p$ into a (finite dimensional) Hilbert space. *Observe that $\partial/\partial x_i$ is now the operator which is Hermitian conjugate to x_i.* We denote this Hermitian conjugate by a bar:

$$\bar{x}_i=\partial/\partial x_i. \tag{2.9}$$

We will find it very convenient in the subsequent sections to use the scalar product (2.6). It will be amply demonstrated in this section that in constructing polynomial solutions to Laplace's equation it makes no essential difference which scalar product we use.

There is nothing mysterious about the scalar product (2.6). It is clearly just the adaptation to real variables of the customary scalar product for bosons. Indeed, under the mapping $x_i\rightarrow a_i$, $\bar{x}_i\rightarrow\bar{a}_i$, where a_i and $\bar{a}_i$ are boson creation and annihilation operators, respectively, we have

$$(f,f')=\langle 0\mid f^*(\bar{a})f'(a)\mid 0\rangle, \tag{2.10}$$

where $|0\rangle$ is the vacuum ket. The scalar product in this form has been used by other authors (Su67).

Next, let us see how we obtain an *operator representation* of the group $O(n)$ on the space $\mathcal{L}_p$. For each $R\in O(n)$, we define the linear operator T_R on the space $\mathcal{L}_p$ by the rule as follows:

$$(T_Rf)(x)=f(\tilde{R}x), \quad \forall f\in\mathcal{L}_p, \tag{2.11}$$

i.e., T_Rf denotes the function which has for its value at point x the value of f at the point $\tilde{R}x$. It is not difficult to show that: (a) T_R is a unitary operator; (b) the correspondence $R\rightarrow T_R$ is a representation of $O(n)$ on the space $\mathcal{L}_p$ by a group of unitary operators $\{T_R: R\in O(n)\}$. [These statements are valid for either scalar product, Eq. (2.5) or (2.6).]

One obtains the Lie algebra of the representation $R\rightarrow T_R$ by calculating the *infinitesimal operators* or *generators* which correspond to a basic set of one-parameter subgroups of $SO(n)$.[1] The standard basic set of such subgroups is given by

$$R_{ij}(\varphi)=(e_{ii}+e_{jj})\cos\varphi-(e_{ij}-e_{ji})\sin\varphi \tag{2.12}$$

for $i<j=1, 2, \cdots, n$, where e_{ij} is the $n\times n$ matrix unit (it has 1 in row i and column j and 0 elsewhere). If we define

$$T_{ij}(\varphi)=T_{R_{ij}}(\varphi), \tag{2.13}$$

then the generator $\mathcal{L}_{ij}$ is defined to be

$$\mathcal{L}_{ij}\equiv i[dT_{ij}(\varphi)/d\varphi]_{\varphi=0}. \tag{2.14}$$

(The complex number i is not to be confused with index i.)

The explicit calculation of the generators proceeds as follows:

$$(\mathcal{L}_{ij}f)(x)=\left(i\frac{d}{d\varphi}f[x'(\varphi)]\right)_{\varphi=0}=i\left[\left(\frac{d\tilde{x}'(\varphi)}{d\varphi}\right)_{\varphi=0}\frac{\partial}{\partial x}\right]f(x), \tag{2.15}$$

where $x'(\varphi)$ is the column matrix

$$x'(\varphi)=\tilde{R}_{ij}(\varphi)x, \tag{2.16}$$

and $\partial/\partial x$ is the column matrix

$$\partial/\partial x=\text{col}\,(\partial/\partial x_1, \cdots, \partial/\partial x_n).$$

Noting that

$$[d\tilde{x}'(\varphi)/d\varphi]_{\varphi=0}=-\tilde{x}(e_{ij}-e_{ji}), \tag{2.17}$$

we obtain the following concise matrix form of the generators:

$$(\mathcal{L}_{ij}f)(x)=-i[\tilde{x}(e_{ij}-e_{ji})(\partial/\partial x)]f(x) \tag{2.18}$$

for $i<j=1, 2, \cdots, n$. Equivalently, this result is expressed as

$$\begin{aligned}(\mathcal{L}_{ij}f)(x)&=-i[x_i(\partial/\partial x_j)-x_j(\partial/\partial x_i)]f(x)\\&=-i(x_i\bar{x}_j-x_j\bar{x}_i)f(x).\end{aligned} \tag{2.19}$$

It is convenient to define $\mathcal{L}_{ii}=0$ and $\mathcal{L}_{ji}=-\mathcal{L}_{ij}$ for $i<j=1, 2, \cdots, n$, remembering, of course, that we actually have only $n(n-1)/2$ generators.

It follows immediately from Eq. (2.19) that the

[1] Even though we consider global $O(n)$ transformations, the infinitesimal transformations are "close" to the identity, i.e., using the generators alone, we can only "generate" $SO(n)$ transformations. [The Lie algebra which obtains explicitly on the polynomial spaces considered in this paper will always be that of generators of *single-valued* representations of $SO(n)$.] However, once the explicit (polynomial) basis of the carrier space of an irreducible representation (IR) of $SO(n)$ has been obtained, we can consider its properties under global $O(n)$ transformations. We simply state the results which will obtain: If n is odd, the carrier space of an IR of $SO(n)$ is the carrier space of an IR of $O(n)$; if n is even ($n=2r$), then the carrier space of IR $\{l_{2r,1}\cdots l_{2r,r-1}0\}$ of $SO(2r)$ [see Eq. (2.44)] is the carrier space of an IR (of the same labels) of $O(2r)$; if $l_{2r,r}\neq0$, the carrier space of IR $\{l_{2r,1}\cdots l_{2r,r-1}, |l_{2r,r}|\}$ of $O(2r)$ is the sum of the two spaces which carry IR's $\{l_{2r,1}\cdots l_{2r,r}\}$ and $\{l_{2r,1}\cdots -l_{2r,r}\}$ of $SO(2r)$.

generators are *Hermitian operators* on the space $\mathcal{L}_p$ when equipped with the scalar product (2.6). [They are, of course, also Hermitian with respect to the scalar product (2.5)]:

$$(\mathcal{L}_{ij})^\dagger = \mathcal{L}_{ij}, \tag{2.20}$$

where the dagger denotes Hermitian conjugation (as does the over bar).

We are now in a position to explain why we can use the scalar product of Eq. (2.6) in place of the scalar product of quantum mechanics, Eq. (2.5), without altering the form of the basis vectors which we obtain. The simple case $n=3$ is sufficient to make the whole general process clear. In this case, the operators

$$L_1 = \mathcal{L}_{23}, \qquad L_2 = \mathcal{L}_{31}, \qquad L_3 = \mathcal{L}_{12} \tag{2.21}$$

are the usual orbital angular momentum operators. The standard basis of the space $\mathcal{L}_l$ $(p=l)$ is the well-known set of solid harmonics, which we note explicitly:

$$\mathcal{Y}_{lm}(x) = [(2l+1)(l+m)!(l-m)!/4\pi]^{1/2} \times \sum_k \frac{(-x_1-ix_2)^{k+m}(x_1-ix_2)^k x_3^{l-m-2k}}{2^{2k+m}(k+m)!k!(l-m-2k)!}, \tag{2.22}$$

where for each $l=0, 1, \cdots$ the values of m are $m=l, l-1, \cdots, -l$. These standard solid harmonics are orthonormalized in the usual way on the 3-sphere

$$\langle \mathcal{Y}_{l'm'} \mid \mathcal{Y}_{lm} \rangle = \delta_{l'l}\delta_{m'm}. \tag{2.23}$$

They are the simultaneous eigenvectors of the *Hermitian operators* L^2 and L_3, where the Hermiticity property now refers to the scalar product (2.5).

But now recall that the orthogonality property of the solid harmonics depends, in fact, only on the Hermitian property of the operator $\mathbf{L}$, i.e., the fact that L^2 and L_3 are Hermitian is what guarantees the orthogonality. But the Hermitian property has already been demonstrated on the space $\mathcal{L}_p$ when equipped with the scalar product (2.6): The solid harmonics (2.22) are also orthogonal in the sense of the scalar product (2.6)

$$(\mathcal{Y}_{l'm'}, \mathcal{Y}_{lm}) = N_l \delta_{l'l}\delta_{m'm}. \tag{2.24}$$

The only thing which can change is the over-all normalization (Mo69).

This feature will be recognized by the reader to be very general and applicable in the more complicated structures to follow, and will not require further detailed comment. The practical advantages to using the scalar product (2.6) are many: (a) One need not worry about introducing polar coordinates in n-space (this can be done in the final state vectors, if desired); (b) orthogonality can be checked by a glance for simple eigenvectors; and (c) many results in unitary group theory using bosons become immediately significant for the orthogonal groups. (We exploit this later.)

We emphasize that introducing the scalar product (2.6) is to be considered as a *useful device* for dealing with polynomial spaces—it in no way replaces the physically defined scalar product of quantum mechanics. Having made clear the role of the scalar product (2.6), we now continue the discussion of the orthogonal groups.

The generators defined by Eq. (2.19) satisfy the commutation relations as follows:

$$[\mathcal{L}_{ij}, \mathcal{L}_{kl}] = i(\delta_{ik}\mathcal{L}_{jl} + \delta_{jl}\mathcal{L}_{ik} - \delta_{jk}\mathcal{L}_{il} - \delta_{il}\mathcal{L}_{jk}) \tag{2.25}$$

for $i, j, k, l = 1, 2, \cdots, n$. They also satisfy the relations

$$\mathcal{L}_{ijkl} = \mathcal{L}_{ij}\mathcal{L}_{kl} - \mathcal{L}_{ik}\mathcal{L}_{jl} + \mathcal{L}_{jk}\mathcal{L}_{il} \equiv 0 \tag{2.26}$$

for $i \neq j \neq k \neq l$. The commutation relations (2.25) are general, i.e., a set of Hermitian operators (on some abstract Hilbert space) satisfying these relations may be taken as a basis of the (abstract) Lie algebra of $SO(n)$; relation (2.26) is *particular* to the realization on the space $\mathcal{L}_p$, and already forecasts that the representations of $O(n)$ which can be obtained on the space $\mathcal{L}_p$ will also be particular.

In so far as the *representations* of $O(n)$ are concerned, we consider the properties of the Lie algebra as a useful means of introducing a *basis* into the space $\mathcal{L}_p$ through the standard techniques of quantum mechanics, i.e., by using *complete sets of commuting Hermitian operators*. Once this basis is completely labeled, we go back to the *global definition* (2.11) to obtain a matrix representation of the group by letting the operators $\{T_R\}$ act on the basis vectors. This viewpoint allows one to sidestep any parametrization of R, and similarly avoids the complicated considerations of evaluating matrix elements of "exponentiated" infinitesimal operators. One can, of course, use this technique only when the *carrier* or *representation space*, e.g., $\mathcal{L}_p$, is explicit, and likewise when the operator representation is explicit.

A complete set of independent commuting Hermitian operators whose simultaneous eigenvectors span each irreducible representation (IR) space of $SO(n)$ is known abstractly, i.e., the construction (LO60a) is based only on the commutation relations (2.25) and the assumption that there exists a Hilbert space on which the generators are Hermitian operators. As remarked earlier, the matrix elements of the generators on this basis are completely known.

Fortunately, we need not yet enter into the general theory alluded to above. It suffices to remark that on the space $\mathcal{L}_p$ the identical vanishing of the operator $\mathcal{L}_{ijkl}$ of Eq. (2.26) has the effect of reducing the number of independent commuting operators to $n-1$. [In the general abstract realization this number is r^2 and $r(r+1)$, respectively, for $SO(2r)$ and $SO(2r+1)$.] The operators which remain are quadratic in the generators, and are given explicitly by

$$\Lambda_k^2 = \tfrac{1}{2} \sum_{i,j=1}^{k} (\mathcal{L}_{ij})^2 \tag{2.27}$$

for $k=2, 3, \cdots, n$. For $k=2$ we use the operator $\mathcal{L}_{12}$ in place of its square.

It is remarkably simple to give the explicit construction of the homogeneous polynomials of degree p which satisfy Laplace's equation *and simultaneously diagonalize* $\mathcal{L}_{12}$ *and* Λ_k^2, $k=3, 4, \cdots, n$. We call any homogeneous polynomial which satisfies Laplace's equation in n-space a *solid harmonic on the n-sphere.* [The phrase "on the n-sphere" is a slight misnomer since the coordinates $(x_1 x_2 \cdots x_n)$ need not satisfy $x_1^2+x_2^2+\cdots+x_n^2=1$, but when this condition is imposed we obtain *spherical harmonics* on the n-sphere, and the phrase "on the n-sphere" is intended to remind us of this fact.] Let us indicate how this construction proceeds.

Observe that the solid harmonics of Eq. (2.22) have the form

$$\mathcal{Y}_{lm}(x)=(x_1+ix_2)^m f_{lm}(x_3, \zeta_2) \qquad (2.28)$$

for $m\geq 0$, where $\zeta_2=(x_1^2+x_2^2)/4$. (A similar form obtains for $m<0$.) Notice that the first factor solves Laplace's equation in 2-space. This factorization into a product is quite general. Thus, in 4-space, it must be possible to solve Laplace's equation by a product function (Lo60, Ca65).

$$\mathcal{Y}_{lm}(x_1x_2x_3)f_{pl}(x_4, \zeta_3), \qquad (2.29)$$

where $\zeta_3=(x_1^2+x_2^2+x_3^2)/4$, and where f_{pl} is of degree $p-l$ in $(x_1x_2x_3x_4)$, i.e., l can have any value $0, 1, \cdots, p$. Furthermore, since

$$\sum_{l=0}^{p}(2l+1)=(p+1)^2=\dim \mathcal{L}_p, \qquad (2.30)$$

we see that the basis functions of the form (2.29) must span $\mathcal{L}_p(n=4)$. That the form of each basis vector *must* be that given by Eq. (2.29) follows from the fact that the $\mathcal{Y}_{lm}(x)$ are the simultaneous eigenvectors of the commuting operators $\mathcal{L}_{12}$ and Λ_3^2, formed from the subalgebra $\{\mathcal{L}_{12}, \mathcal{L}_{23}, \mathcal{L}_{31}\}$: The only functions which can multiply the $\mathcal{Y}_{lm}(x)$ in 4-space are functions which are *invariants* with respect to this subalgebra, i.e., polynomials in x_4 and ζ_3. We now follow through with this observation for arbitrary n.

Assume that we have solved the problem of constructing the solid harmonics of degree l_{k-1} on the $(k-1)$-sphere which simultaneously diagonalize $\mathcal{L}_{12}$ and Λ_s^2 $(s=3, 4, \cdots, k-1)$. Denote an arbitrary one of these functions by $f_{l_{k-1}}(x_1x_2\cdots x_{k-1})$. Then each solution to Laplace's equation on the k-sphere, which simultaneously diagonalizes $\mathcal{L}_{12}$ and Λ_s^2 $(s=3, 4, \cdots, k)$, must have the form

$$f(x_k, \zeta_{k-1})f_{l_{k-1}}(x_1x_2\cdots x_{k-1}), \qquad (2.31)$$

where

$$\zeta_{k-1}=\tfrac{1}{4}\sum_{i=1}^{k-1}x_i^2. \qquad (2.32)$$

Since $f_{l_{k-1}}$ solves Laplace's equation in $(k-1)$-space, and since

$$\nabla_k^2=(\partial/\partial x_k)^2+\nabla_{k-1}^2, \qquad (2.33)$$

we easily find the condition that the product function (2.31) satisfies Laplace's equation in k-space to be

$$[(\partial/\partial x_k)^2+\zeta_{k-1}(\partial/\partial\zeta_{k-1})^2 + (l_{k-1}+\tfrac{1}{2}k-\tfrac{1}{2})(\partial/\partial\zeta_{k-1})]f(x_k, \zeta_{k-1})=0. \qquad (2.34)$$

The polynomial solutions to this equation which are homogeneous of degree l_k-l_{k-1} are easily found:

$$f_{l_kl_{k-1}}(x_k, \zeta_{k-1})=\sum_s \frac{(x_k)^{l_k-l_{k-1}-2s}(-\zeta_{k-1})^s}{(l_k-l_{k-1}-2s)!\,s!\,(s+l_{k-1}+\tfrac{1}{2}k-\tfrac{3}{2})!}, \qquad (2.35)$$

where $a!=\Gamma(a+1)$ for half-integral a, and where the sum is over all values of s for which the factorials are non-negative. Since l_{k-1} is integral (non-negative), l_k is any integer such that $l_k\geq l_{k-1}$. Replacing $f(x_k, \zeta_{k-1})$ in Eq. (2.31) with the explicit functions (2.35), we obtain the solutions to Laplace's equation on the k-sphere which are homogeneous of degree l_k, where we note that for *prescribed* $l_k=0, 1, 2, \cdots$, the values of l_{k-1} are $0, 1, 2, \cdots, l_k$.

The eigenvalue of Λ_k^2 is obtained as follows. By direct algebraic manipulations, we establish the identity

$$(\Lambda_k^2 f)(x)=[-4\zeta_k\nabla_k^2+\mathcal{J}_k(\mathcal{J}_k+k-2)]f(x), \qquad (2.36)$$

where $\mathcal{J}_k$ is the homogeneous Euler operator

$$(\mathcal{J}_k f)(x)=[\sum_{i=1}^{k}x_i(\partial/\partial x_i)]f(x). \qquad (2.37)$$

The Euler operator has eigenvalue l_k on a homogeneous function of degree l_k. Thus, we have

$$(\Lambda_k^2 f)(x)=l_k(l_k+k-2)f(x) \qquad (2.38)$$

for each $f\in\mathcal{L}_{l_k}$.

Iterating the preceding construction upward from Eq. (2.29), i.e., by taking $k=4, 5, \cdots, n$, in turn, we obtain the following general eigenvectors as a basis of the solid harmonics of degree l_n on the n-sphere:

$$\mathcal{Y}_{l_nl_{n-1}\cdots l_2}(x)=(N_{l_nl_{n-1}\cdots l_2})^{-1/2}\mathcal{Y}_{l_3l_2}(x_1x_2x_3) \times\prod_{k=4}^{n}f_{l_kl_{k-1}}(x_k, \zeta_{k-1}), \qquad (2.39)$$

where for prescribed $l_n=0, 1, \cdots$, the remaining labels can assume all integral values consistent with

$$l_n\geq l_{n-1}\geq\cdots\geq l_3\geq|l_2|\geq 0, \qquad (2.40)$$

in which m has now been renamed l_2.

The eigenvalues of the complete set of commuting Hermitian operators which characterize the basis (2.39) are

$$\Lambda_k^2\rightarrow l_k(l_k+k-2) \qquad (2.41)$$

and

$$\mathcal{L}_{12}\rightarrow l_2. \qquad (2.42)$$

The basis vectors themselves are *orthogonal* in the labels

$(l_n l_{n-1} \cdots l_2)$ in *either* definition of scalar product, Eq. (2.5) or Eq. (2.6).

The very method of constructing the basis vectors of Eq. (2.39) assures that they span the space $\mathfrak{L}_{l_n}$. We note, but do not derive, the formula for the dimension of the space $\mathfrak{L}_p$ $(p=l_n)$:

$$\dim \mathfrak{L}_p = \frac{2p+n-2}{n-2} \binom{p+n-3}{p}, \qquad (2.43)$$

where

$$\binom{x}{\alpha}$$

denotes a binomial coefficient. One easily checks the dimension formula for simple cases by counting the number of labels which satisfy Eq. (2.40).

We have not bothered to normalize the basis vectors (2.39) explicitly because we will not make direct use of them. Our purpose in giving them has been: (a) to point out the simplicity of their derivation; (b) to emphasize their orthogonality under the scalar product (2.6); and (c) to have available at least one complete orthogonal basis of the space $\mathfrak{L}_p$—the principal subject of this paper. [One can, of course, go further and obtain quite easily the matrix elements of the generators $\mathfrak{L}_{ij}$ on the (normalized) basis of Eq. (2.39). These particular results have, however, been noted previously (Lo60,60a).

It is important to understand the subgroup structure of the basis (2.39). One can then comprehend quite readily the subgroup structure of the abstract basis vectors which are generalizations of this particular basis, and one can understand why this basis (and its abstract generalization) fails to be the complete answer for many of the applications in physics.

Since Λ_n^2 has the single, fixed eigenvalue $l_n(l_n+n-2)$ on the space $\mathfrak{L}_{l_n}$, this space is the carrier space for an IR of $O(n)$.[1] This IR is labeled by the single integer l_n, and is of dimension $\dim \mathfrak{L}_{l_n}$. More generally, an IR of $SO(n)$ is labeled by a set of ordered integers,[2] r in number for either $n=2r$ or $n=2r+1$: For $n=2r$, these integers are denoted by

$$\{l_{2r,1} l_{2r,2} \cdots l_{2r,r}\} \qquad (2.44)$$

and satisfy

$$l_{2r,1} \geq l_{2r,2} \geq \cdots \geq l_{2r,r-1} \geq | l_{2r,r} | \geq 0; \qquad (2.45)$$

for $n=2r+1$, these integers are similarly denoted by

$$\{l_{2r+1,1} l_{2r+1,2} \cdots l_{2r+1,r}\} \qquad (2.46)$$

and satisfy

$$l_{2r+1,1} \geq l_{2r+1,2} \geq \cdots \geq l_{2r+1,r} \geq 0. \qquad (2.47)$$

The basis (2.39) *is the carrier space for the simplest type of representation of* $O(n)$—the IR denoted by $\{l_n 0 \cdots 0\}$. The representation matrices themselves are obtained directly from the transformations of the normalized basis:

$$(T_R \mathcal{Y}_{l_n(l)})(x) = \mathcal{Y}_{l_n(l)}(\tilde{R}x)$$
$$= \sum_{(l')} D^{l_n}{}_{(l')(l)}(R) \mathcal{Y}_{l_n(l')}(x), \qquad (2.48)$$

where $(l) = (l_{n-1} l_{n-2} \cdots l_2)$. The set of matrices

$$\{D^{l_n}(R), R \in O(n)\} \qquad (2.49)$$

is a unitary matrix representation, designated by $\{l_n 0 \cdots 0\}$, of the group of orthogonal matrices.

The subgroup structure of the basis (2.39) is displayed vividly by Eq. (2.48): If we consider R to be of the restricted form

$$R \rightarrow \left(\begin{array}{ccc|c} & R' & & 0 \\ & & & \cdot \\ & & & \cdot \\ & & & \cdot \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array} \right), \qquad R' \in O(n-1), \qquad (2.50)$$

it is clear that the sum over (l') in Eq. (2.48) becomes a sum over $(l') = (l_{n-1} l_{n-2}' \cdots l_2')$, since the $O(n-1)$ invariant functions $f_{l_n l_{n-1}}(x_n, \zeta_{n-1})$ do not participate in the restricted transformations. Correspondingly, each matrix in the IR (2.49) reduces into a direct sum of matrices

$$D^{l_n}(R) \rightarrow \sum_{l_{n-1}} \oplus D^{l_{n-1}}(R'), \qquad (2.51)$$

where the sum is over $l_{n-1}=0, 1, \cdots, l_n$, and where, for each l_{n-1}, the set of matrices

$$\{D^{l_{n-1}}(R') : R' \in O(n-1)\} \qquad (2.52)$$

is an IR of type $\{l_{n-1} 0 \cdots 0\}$ of $O(n-1)$. [The dimension of these matrices is obtained from Eq. (2.43) upon setting $p=l_{n-1}$ and replacing n by $n-1$.]

We continue the restriction procedure by letting $R' \rightarrow R'' \oplus 1$, where $R'' \in O(n-2)$. Then *each* matrix IR (2.52) in turn reduces into a direct sum of matrix IR's of $O(n-2)$, each such IR being labeled by l_{n-2} for $0 \leq l_{n-2} \leq l_{n-1}$ (each value in this interval occurring exactly once). It thus becomes clear that the labels $(l_{n-1} l_{n-2} \cdots l_2)$ are just the labels of the IR's of the subgroups in the chain

$$O(n) \supset O(n-1) \supset \cdots \supset O(3) \supset SO(2) \qquad (2.53)$$

which arise through the chain of subgroup restrictions of the form (2.50), and which we symbolize by writing

$$O(n) \rightarrow O(n-1) \rightarrow \cdots \rightarrow O(3) \rightarrow SO(2). \qquad (2.54)$$

Let us now see how one generalizes this structure. Gel'fand and Zetlin (Ge50) realized that an *abstract*

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[2] The abstract algebra (2.25) also admits half-integers, but these correspond to the so-called double-valued representations of $SO(n)$ and do not occur in this paper.

basis of the carrier space of each IR of $SO(n)$ could be characterized completely by the sets of labels which are associated with the IR's of the (proper) subgroups which occur in the chain (2.53). The *Weyl branching law* then provides the constraints which these labels must satisfy. Thus, an abstract basis vector is labeled by $n-1$ rows of labels of the types (2.44) and (2.46): the bottom (first) row comes from (2.44) with $r=1$, the second row from (2.46) with $r=1$, the third row from (2.44) with $r=2, \cdots$, the top row (row $n-1$) is either (2.44) with $2r=n$ [for $SO(2r)$] or (2.46) [for $SO(2r+1)$]. The resulting *Gel'fand pattern* is displayed below for $SO(2r)$:

$$\begin{bmatrix} l_{2r,1} & l_{2r,2} & \cdots & l_{2r,r-1} & l_{2r,r} \\ & l_{2r-1,1} \quad l_{2r-1,2} & \cdots & l_{2r-1,r-1} & \\ & \vdots & & \vdots & \\ & l_{51} & & l_{52} & \\ & l_{41} & & l_{42} & \\ & & l_{31} & & \\ & & l_{21} & & \end{bmatrix}. \tag{2.55}$$

For $SO(2r+1)$, one simply includes the row (2.46) directly above the top row of the displayed pattern.

The Weyl branching laws for the orthogonal groups are stated as follows: (a) On restricting $SO(2k+1)$ to $SO(2k)$, the IR $\{l_{2k+1,1}\, l_{2k+1,2} \cdots l_{2k+1,k}\}$ of $SO(2k+1)$ reduces into the (direct) sum of all those representations $\{l_{2k,1}\, l_{2k,2} \cdots l_{2k,k}\}$ of $SO(2k)$ for which

$$l_{2k+1,1} \geq l_{2k,1} \geq l_{2k+1,2} \geq l_{2k,2} \geq \cdots \geq l_{2k,k-1} \geq l_{2k+1,k} \geq |\, l_{2k,k} \,| \geq 0, \tag{2.56}$$

each of these constituents appearing exactly once; (b) on restricting $SO(2k)$ to $SO(2k-1)$, the IR $\{l_{2k,1}\, l_{2k,2} \cdots l_{2k,k}\}$ of $SO(2k)$ reduces into the sum of all those representations $\{l_{2k-1,1}\, l_{2k-1,2} \cdots l_{2k-1,k-1}\}$ of $SO(2k-1)$ for which

$$l_{2k,1} \geq l_{2k-1,1} \geq l_{2k,2} \geq l_{2k-1,2} \geq \cdots \geq l_{2k,k-1} \geq l_{2k-1,k-1} \geq |\, l_{2k,k} \,| \geq 0, \tag{2.57}$$

each of these constituents appearing exactly once.

For a prescribed top row of a Gel'fand pattern, the labels in the remaining rows can assume just those values which accord with the Weyl branching laws.

We denote an $SO(n)$ Gel'fand pattern by (l) and the corresponding abstract basis vector by $|\,(l)\,\rangle$. As previously remarked, the complete set of commuting Hermitian operators which characterize this basis is known; furthermore, the matrix elements of the abstract generators are also completely known on this basis.

The solid harmonics of Eq. (2.39) are labeled in the Gel'fand–Zetlin notation by the pattern which has $l_{21}=l_2$, $l_{31}=l_3, \cdots$, $l_{2r,1}=l_{2r}$ $(n=2r)$, $l_{2r+1,1}=l_{2r+1}$ $(n=2r+1)$, *all other* l_{ij} *being zero.*

The abstract basis $|\,(l)\,\rangle$ is completely characterized by the *highest weight* vector, and it is known (Pa67, Wo67) how to generate the remaining vectors in the basis by the application of lowering operators. The highest weight vector is the one whose Gel'fand pattern has all the l_{ij} chosen as large as possible. In particular, the highest weight vector in the space $\mathcal{L}_p$ is the one having $l_2=l_3=\cdots=l_n\equiv p$, i.e.,

$$(x_1+ix_2)^p. \tag{2.58}$$

One can, in principle, generate the basis (2.39) from the vector (2.58) by using the lowering operator technique.

We conclude this section by noting why the classification of the basis vectors of $\mathcal{L}_p$ through the subgroup chain (2.53) is not directly useful for many physical applications. *The $SO(3)$ subgroup in this chain is not the physical $SO(3)$ group corresponding to the total orbital angular momentum of a set of particles.* (This statement will be made more explicit in Sec. IV.) We are accordingly forced to consider alternative techniques for solving Laplace's equation, i.e., for finding a basis of the space $\mathcal{L}_p$ which is labeled by the total angular momentum quantum numbers LM (among others). We will, however, see in Sec. IV that the orthogonal groups make their appearance in the N-particle problem in still another context, and in this context the classification of basis vectors through the chain (2.53) can be used.

B. Unitary Groups

We use the notation $U(n)$ to denote the group of $n\times n$ unitary matrices $(n=2, 3, \cdots)$

$$U(n)=\{U\colon U^\dagger U=I_n\}. \tag{2.59}$$

The notation $SU(n)$ denotes the subgroup of $U(n)$ whose elements have determinant equal to $+1$. We will not attempt to reference the large number of researches relating to the unitary groups, but rather note a recent review (Lo70) where many such references are given.

Our interest in the unitary groups in this paper derives from the particular fact that $SU(4)$ is *homomorphic* to $SO(6)$, and from the general fact that $O(n)$ is a subgroup of $U(n)$. *The unitary groups themselves can serve as a useful starting point for the construction of solutions to Laplace's equation*, and we wish to formulate, in its simplest context, the manner in which this property can be made explicit.

Let $\mathbf{z}$ denote a vector which has components relative to a Cartesian basis of the complex space C^n given by $(z_1, z_2, \cdots, z_n)$. We associate the column matrix z with this point. We next introduce the space $\mathcal{H}_p$ of complex polynomials F which are homogeneous of degree p in z:

$$\mathcal{H}_p = \{F: F(\lambda z) = \lambda^p F(z)\}. \tag{2.60}$$

This space can be made into a Hilbert space by introducing the following scalar product [in analogy to Eq. (2.6)]:

$$(F, F') = [F^*(\partial/\partial z)F'(z)]_{z=0}, \tag{2.61}$$

where $F^*(\partial/\partial z)$ is the differential operator defined by

$$F^*(\partial/\partial z) = \sum_{(\alpha)} a_{(\alpha)}{}^*(\partial/\partial z_1)^{\alpha_1}\cdots(\partial/\partial z_n)^{\alpha_n} \tag{2.62}$$

for $F(z)$ given by

$$F(z) = \sum_{(\alpha)} a_{(\alpha)} z_1{}^{\alpha_1}\cdots z_n{}^{\alpha_n}. \tag{2.63}$$

Observe that the Hermitian conjugate to z_i is

$$\bar{z}_i = \partial/\partial z_i, \tag{2.64}$$

and that again this is just the adaptation to complex variables of the familiar boson scalar product, Eq. (2.10).

We obtain an operator representation of the group $U(n)$ on the space $\mathcal{H}_p$ as follows: For each $U \in U(n)$, we define the linear operator T_U by the rule

$$(T_U F)(z) = F(\tilde{U} z). \tag{2.65}$$

Then: (a) T_U is a unitary operator on the space $\mathcal{H}_p$; and (b) the correspondence $U \rightarrow T_U$ is a representation of $U(n)$ on the space $\mathcal{H}_p$ by a group of unitary operators $\{T_U: U \in U(n)\}$.

The procedure for calculating the infinitesimal operators or generators of a representation of $U(n)$ has been reviewed in detail (Lo70). Here we need only note that the *Weyl generators* of the representation (2.65) take the very simple form as follows:

$$(\mathcal{E}_{ij}F)(z) = z_i \bar{z}_j F(z) \tag{2.66}$$

for $i, j = 1, 2, \cdots, n$. These generators have the properties

$$(\mathcal{E}_{ij})^\dagger = \mathcal{E}_{ji}, \tag{2.67}$$

$$[\mathcal{E}_{ij}, \mathcal{E}_{kl}] = \delta_{jk}\mathcal{E}_{il} - \delta_{il}\mathcal{E}_{kj}, \tag{2.68}$$

$$\mathcal{E}_{ij}\mathcal{E}_{kl} - \mathcal{E}_{il}\mathcal{E}_{kj} = \delta_{jk}\mathcal{E}_{il} - \delta_{kl}\mathcal{E}_{ij}. \tag{2.69}$$

The first two of these relations are general, i.e., are the relations satisfied by the Weyl generators of any (abstract) unitary representation of $U(n)$—hence, define a basis of the abstract Lie algebra of $U(n)$. Relation (2.69) is *particular* to the realization of these generators on the space $\mathcal{H}_p$.

Relations (2.69) imply (Lo65) that a complete set of commuting Hermitian operators whose simultaneous eigenvectors characterize a basis of $\mathcal{H}_p$ is given by the set

$$\{\mathcal{G}_k: k = 1, 2, \cdots, n\}, \tag{2.70}$$

where

$$(\mathcal{G}_k F)(z) = (\sum_{i=1}^{k} z_i \bar{z}_i) F(z), \tag{2.71}$$

i.e., $\mathcal{G}_k$ is the homogeneous Euler operator in the variables $z_1, z_2, \cdots, z_k$.

The corresponding orthonormalized basis of $\mathcal{H}_p$ can be set down immediately:

$$F_{(m_n m_{n-1}\cdots m_1)}(z) = \prod_{k=1}^{n} \left(\frac{(z_k)^{m_k - m_{k-1}}}{[(m_k - m_{k-1})!]^{1/2}} \right), \tag{2.72}$$

in which $m_0 \equiv 0$ and $m_n = p$, and the labels can assume any integral values consistent with

$$m_n \geq m_{n-1} \geq \cdots \geq m_1 \geq 0. \tag{2.73}$$

The dimension of the space $\mathcal{H}_p$ is

$$\dim \mathcal{H}_p = \begin{pmatrix} p+n-1 \\ p \end{pmatrix}. \tag{2.74}$$

The eigenvalues of $\mathcal{G}_k$ are given by

$$\mathcal{G}_k \rightarrow m_k - m_{k-1} \tag{2.75}$$

for $k = 1, 2, \cdots, n$.

The significance of the labels $(m_n\, m_{n-1} \cdots m_1)$ is induced from the chain of subgroups

$$U(n) \supset U(n-1) \supset \cdots \supset U(1) \tag{2.76}$$

in exact parallel to the procedure used for the orthogonal groups, Eqs. (2.48)–(2.54). Under the subgroup restriction $U \rightarrow U' \oplus 1$, $U' \in U(n-1)$, the IR of $U(n)$ labeled by m_n reduces into a sum of the IR's of $U(n-1)$ labeled by m_{n-1}, where $0 \leq m_{n-1} \leq m_n$, each such representation of $U(n-1)$ occurring exactly once, etc.

Again Gel'fand and Zetlin (Ge50a) recognized that one could label an abstract basis of a carrier space of each IR of $U(n)$ by employing the sets of IR labels associated with each IR of the subgroups in the chain (2.76). Each IR of $U(k)$ is characterized by a set of ordered integers (positive, negative, or zero)

$$m_{1k} \geq m_{2k} \geq \cdots \geq m_{kk}. \tag{2.77}$$

A $U(n)$ Gel'fand pattern is a triangular set of integers of n rows, the integers in row k being a set of IR labels of $U(k)$:

$$(m) = \begin{pmatrix} m_{1n} & & m_{2n} & & \cdots & & m_{nn} \\ \cdot & & & & & & \cdot \\ & \cdot & & & & \cdot & \\ & & \cdot & & \cdot & & \\ & m_{13} & & m_{23} & & m_{33} & \\ & & m_{12} & & m_{22} & & \\ & & & m_{11} & & & \end{pmatrix}. \tag{2.78}$$

Once more the Weyl branching law (We31) provides the constraints on the entries in this array: On restricting $U(k)$ to $U(k-1)$ the IR $[m_{1k}\ m_{2k}\ \cdots\ m_{kk}]$ of $U(k)$ reduces into the sum of all those IR's $[m_{1k-1}\ m_{2k-1}\ \cdots\ m_{k-1k-1}]$ of $U(k-1)$ for which

$$m_{1k} \geq m_{1k-1} \geq m_{2k} \geq m_{2k-1} \geq \cdots \geq m_{k-1k-1} \geq m_{kk}, \quad (2.79)$$

each of these representations appearing exactly once.

Thus, for prescribing $U(n)$ labels in the top row, the remaining labels can assume all values consistent with the Weyl branching law.

An abstract basis vector is denoted by $|\,(m)\,\rangle$. The complete set of commuting Hermitian operators, $n(n+1)/2$ in number, which characterize this basis is known (Lo65), including their eigenvalues (Lo70, Lo70a); furthermore, the matrix elements of the abstract Weyl generators are also completely known on this basis (Ge50a, Lo70a).

In terms of the Gel'fand–Zetlin notation, the basis vectors (2.72) are denoted by the Gel'fand pattern which has $m_{11}=m_1$, $m_{12}=m_2$, $\cdots$, $m_{1n}=m_n$, all other m_{ij} being zero.

Despite the simplicity of the basis (2.72), one can, upon replacing n by n^2, i.e., by considering the IR space $[m_{1n^2}\ 0\ \cdots\ 0]$ of $U(n^2)$, obtain not only a basis of a carrier space for IR $[m_{1n}\ m_{2n}\ \cdots\ m_{nn}]$ $(m_{nn}\geq 0)$ of $U(n)$, but one can obtain the IR matrices themselves (Lo70). This structure has been the source of many developments in unitary group theory (Lo70a, Bi67, 68). [The analogous result for $SO(n)$ is developed in Sec. VIII.]

The abstract basis $|\,(m)\,\rangle$ is completely characterized by the highest weight vector, and it is known how to generate the general basis vector from the highest weight by applying known lowering operators (Na65).

Let us now see how, in the simplest case, the unitary group can provide us with solutions to Laplace's equation. The basis vectors (2.72) are well-defined for all complex values of the variables z_k $(k=1, 2, \cdots, n)$. In particular, we can *restrict* these variables to be real: $z_k \rightarrow x_k$. If at the same time we switch to the scalar product (2.6), then the polynomials

$$F_{(m_n m_{n-1} \cdots m_1)}(x) \quad (2.80)$$

remain orthonormalized in the labels $(m_n m_{n-1} \cdots m_1)$, and they span the space of all complex polynomials in x which are homogeneous of degree $p=m_n$. In particular, the space $\mathcal{L}_p$ must occur as a subspace. We can, however, do much better. We can restrict the complex variables z_k to the form

$$z_{2j-1}=(x_{2j-1}+ix_{2j})/\sqrt{2}, \quad (2.81)$$

$$z_{2j}=(x_{2j-1}-ix_{2j})/\sqrt{2}, \quad (2.82)$$

for $j=1, 2, \cdots, n/2$ or $(n-1)/2$, and

$$z_n=x_n, \qquad (n \text{ odd}). \quad (2.83)$$

The resulting functions

$$F_{(m_n m_{n-1} \cdots m_1)}(z) \quad (2.84)$$

still remain orthonormal *under the scalar product* (2.6), since the z's are related to the x's by a unitary transformation (the familiar boson property). But now observe that the highest weight vector in the space $\mathcal{H}_p$ $[m_1=m_2=\cdots=m_n=p$ in Eq. (2.72)$]$ takes the form

$$(x_1+ix_2)^p \quad (2.85)$$

under the restriction (2.81)–(2.83), i.e., the highest weight vectors of the space $\mathcal{L}_p$ and $\mathcal{H}_p$ *coincide*. Using this highest weight vector, which is obtained from $\mathcal{H}_p$ by restricting the domain of definition of the variables, we can now proceed to generate the basis (2.39) of $\mathcal{L}_p$ by using the lowering operators appropriate to $SO(n)$.

Wong (Wo69) has observed that the above property generalizes to a certain class of vectors from a $U(n)$ representation space.[3] We consider this structure in greater detail in Sec. VIII.

We next turn to the task of implementing the concepts of this section into the physical problem of N identical particles in 3-space, beginning with a discussion of the center of mass coordinate problem and related properties of the symmetric group.

III. RELATIVE POSITION VECTORS

Let $\mathbf{r}^1$, $\mathbf{r}^2$, $\cdots$, $\mathbf{r}^N$ denote, respectively, the position vectors in Euclidean 3-space of N identical particles labeled 1, 2, $\cdots$, N, each vector being referred to a common origin. Let the permutation operator P, denoted by

$$P=\begin{pmatrix} 1 & 2 & \cdots & N \\ \alpha_1 & \alpha_2 & \cdots & \alpha_N \end{pmatrix}, \quad (3.1)$$

where $\alpha_1, \alpha_2, \cdots, \alpha_N$ is a rearrangement of $1, 2, \cdots, N$, be defined by

$$P: \mathbf{r}^1 \rightarrow \mathbf{r}^{\alpha_1}, \mathbf{r}^2 \rightarrow \mathbf{r}^{\alpha_2}, \cdots, \mathbf{r}^N \rightarrow \mathbf{r}^{\alpha_N}. \quad (3.2)$$

It is convenient to consider the position vectors as the elements of a $1\times N$ row matrix, $[\mathbf{r}^1\ \mathbf{r}^2\ \cdots\ \mathbf{r}^N]$. Then the transformation P can be described by

$$P: [\mathbf{r}^1\ \mathbf{r}^2\ \cdots\ \mathbf{r}^N] \rightarrow [\mathbf{r}^{\alpha_1}\ \mathbf{r}^{\alpha_2}\ \cdots\ \mathbf{r}^{\alpha_N}] = [\mathbf{r}^1\ \mathbf{r}^2\ \cdots\ \mathbf{r}^N] D(P), \quad (3.3)$$

where $D(P)$ is the $N\times N$ real orthogonal matrix defined by

$$D(P)=[e^{\alpha_1}\ e^{\alpha_2}\ \cdots\ e^{\alpha_N}], \quad (3.4)$$

in which e^α $(\alpha=1, 2, \cdots, N)$ denotes the $N\times 1$ column matrix having 1 in row α, and 0's elsewhere. One now easily verifies that the correspondence

$$P \rightarrow D(P), \qquad \forall P \in S_N \quad (3.5)$$

[3] Wong's analysis is, however, incomplete from a structural viewpoint—it is the subgroup property $U(n) \supset O(n)$ which underlies his construction [see Sec. VIII].

is a representation of the group of permutations, S_N, by the set of $N\times N$ orthogonal matrices, $\{D(P)\}$.

The problem of defining a set of relative position vectors for N identical particles is closely related to the problem of reducing the matrix representation of S_N given by Eq. (3.5) into its irreducible constituents. To see how this comes about, let us first give a precise meaning to the term "relative position vector" in the general case of arbitrary masses.

Let the vector $\mathbf{R}$ defined by

$$M\mathbf{R}=\sum_{\alpha=1}^{N} m_\alpha \mathbf{r}^\alpha, \qquad M=\sum_{\alpha=1}^{N} m_\alpha, \qquad (3.6)$$

denote the center of mass vector of N particles, where m_α is the mass of particle α.

Definition 1: A vector in the set $\{\mathbf{y}^1, \mathbf{y}^2, \cdots, \mathbf{y}^{N-1}\}$ is called a relative position vector (with respect to $\mathbf{R}$) if and only if the set of vectors $\{\mathbf{y}^1, \mathbf{y}^2, \cdots, \mathbf{y}^{N-1}, \lambda\mathbf{R}\}$ (λ a real nonzero number) is related to the set of position vectors $\{\mathbf{r}^1, \mathbf{r}^2, \cdots, \mathbf{r}^N\}$ by a real nonsingular linear transformation of the following general form

$$[\mathbf{y}^1\, \mathbf{y}^2 \cdots \mathbf{y}^{N-1}\, \lambda\mathbf{R}]=[\mathbf{r}^1\, \mathbf{r}^2 \cdots \mathbf{r}^N]C, \qquad (3.7)$$

where (a) C is an $N\times N$ real nonsingular matrix; (b) the Nth column of C is (λ/M) col $[m_1\, m_2 \cdots m_N]$; and (c) the remaining columns of C, column 1 to column $N-1$, are perpendicular to the column matrix, col $[1\, 1 \cdots 1]$.

Remark. A relative position vector $\mathbf{y}^\alpha$ is *not*, in general, the position vector of particle α as seen from the center of mass. Rather, the term "a set of relative position vectors" refers to any set of real vectors which together with a multiple of the center of mass vector $\mathbf{R}$ can be used to replace (by the invertibility of C) the actual position vectors in the description of the motion of the particles. Furthermore, we insist that in the description by the new set of vectors the center of mass motion should "separate off." This means that the kinetic energy, the linear momentum, and the angular momentum of the system of particles should each assume the generic form $A=A_{\text{c.m.}}+A'$ under the transformation $[\mathbf{r}^1\, \mathbf{r}^2 \cdots \mathbf{r}^N]=[\mathbf{y}^1\, \mathbf{y}^2 \cdots \mathbf{y}^{N-1}\, \lambda\mathbf{R}]C^{-1}$, where $A_{\text{c.m.}}$ denotes the value of the particular physical quantity associated with the center of mass motion, and A' is a function only of the vectors $\mathbf{y}^1, \mathbf{y}^2, \cdots, \mathbf{y}^{N-1}$ (and their time derivatives). A' is then appropriately called the value of A relative to the center of mass. Condition (c) on the matrix C is the necessary and sufficient condition that the center of mass motion separates off. (We omit the elementary proof.)

Using the definition of relative position vectors given above, we can now prove:

Lemma 1. Each set of relative position vectors $\{\mathbf{y}^1, \mathbf{y}^2, \cdots, \mathbf{y}^{N-1}\}$ can be written in the form

$$[\mathbf{y}^1\, \mathbf{y}^2 \cdots \mathbf{y}^{N-1}\, \lambda\mathbf{R}]=[\mathbf{x}^1\, \mathbf{x}^2 \cdots \mathbf{x}^N]C_0, \qquad (3.8)$$

where $\mathbf{x}^N\equiv(N)^{1/2}\mathbf{R}$, and where $\{\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^{N-1}\}$ is *a new set of relative position vectors* of a very particular type

$$[\mathbf{x}^1\, \mathbf{x}^2 \cdots \mathbf{x}^N]=[\mathbf{r}^1\, \mathbf{r}^2 \cdots \mathbf{r}^N]DA. \qquad (3.9)$$

In these relations, C_0 is a nonsingular lower triangular matrix having Nth column given by

$$\text{col}\,[0\, 0 \cdots 0\, \lambda/(N)^{1/2}],$$

D is a diagonal matrix having Nm_α/M for its αth diagonal element, A is a proper, real orthogonal matrix having Nth column given by col $[1\, 1 \cdots 1]/(N)^{1/2}$. Finally, the matrices C_0 and A are independent of the masses.

Proof. Given any matrix C of the type prescribed in Eq. (3.7), it can be decomposed into the form $C=DAC_0$. A specific technique for effecting this decomposition is to form the matrix $C'=D^{-1}C$ and then perform the Schmidt orthonormalization procedure on the columns of C', starting with the Nth column and proceeding across the columns from right to left. The result of this procedure is to decompose C' into the form $C'=AC_0$, where the columns of A are the new columns obtained from the Schmidt process. Then A and C_0 will have the properties described above (A can always be made proper by an appropriate choice of normalization signs).

Since an arbitrary set of relative position vectors can be obtained from a set $\{\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^{N-1}\}$ by an appropriate transformation of type C_0, we henceforth restrict our attention to sets of relative position vectors defined by Eq. (3.9), *where A is an arbitrary real, proper orthogonal matrix, the Nth column, however, always being specified to be col* $[1\, 1 \cdots 1]/(N)^{1/2}$. In particular, in the case of equal masses, D becomes the identity matrix, and we have

$$[\mathbf{x}^1\, \mathbf{x}^2 \cdots \mathbf{x}^N]=[\mathbf{r}^1\, \mathbf{r}^2 \cdots \mathbf{r}^N]A. \qquad (3.10)$$

It will be noted that transformations of the type, Eq. (3.10), leave invariant the form of the kinetic energy and angular momentum of the system of identical particles. We henceforth will also consider only the case of identical particles.

Next, we examine the transformation properties of a set $\{\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^{N-1}\}$ of relative position vectors under the action of the permutation operators, $P\in S_N$, given by Eq. (3.3). First, let us note an easy consequence of Eqs. (3.3) and (3.4): Namely, the character (trace) of the matrix $D(P)$ is given by

$$\text{Tr}\, D(P)=\alpha, \qquad (3.11)$$

where α is the number of cycles of length 1 in the cycle notation (Ha62) for the permutation P. We can now prove (Le66):

Lemma 2. Each specified set of relative position vectors $\{\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^{N-1}\}$ of a set of identical particles is a basis for an irreducible representation $\Gamma(P)$ of the group of permutations $\{P\}$ of the type associated with partition $[N-1\,1\,0\cdots 0]$

$$P:[\mathbf{x}^1\,\mathbf{x}^2\cdots\mathbf{x}^{N-1}]\rightarrow[\mathbf{x}^1\,\mathbf{x}^2\cdots\mathbf{x}^{N-1}]\Gamma(P), \quad (3.12)$$

where $\Gamma(P)$ is real orthogonal. Conversely, if $\{\Gamma(P): P\in S_N\}$ is an arbitrary, but specified, real orthogonal IR of S_N of the type $[N-1\,1\,0\cdots 0]$, there exists a set of relative position vectors $\{\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^{N-1}\}$ which undergoes the transformation, Eq. (3.12), and this set is unique for N even, and is unique up to $\pm$ sign for N odd.

Proof. Under the action of $P\in S_N$ specified by Eq. (3.3), we have

$$P:[\mathbf{x}^1\,\mathbf{x}^2\cdots\mathbf{x}^N]\rightarrow[\mathbf{r}^1\,\mathbf{r}^2\cdots\mathbf{r}^N]D(P)A$$
$$=[\mathbf{x}^1\,\mathbf{x}^2\cdots\mathbf{x}^N]\tilde{A}D(P)A. \quad (3.13)$$

But one easily proves that $\tilde{A}D(P)A$ has the form

$$\tilde{A}D(P)A=\left(\begin{array}{ccc|c} & \Gamma(P) & & \begin{array}{c}0\\ \vdots\\ 0\end{array}\\ \hline 0 & \cdots & 0 & 1\end{array}\right), \quad \forall P\in S_N, \quad (3.14)$$

in consequence of the fact that the Nth column of A is col $[1\,1\cdots 1]/(N)^{1/2}$. Furthermore, $\Gamma(P)$ is real orthogonal and has trace equal to $\alpha-1$. Thus, the set of matrices, $\{\Gamma(P): P\in S_N\}$, is a representation of S_N of dimension $N-1$ having character set $\{\alpha-1\}$. It is therefore an IR of type $[N-1\,1\,0\cdots 0]$. Finally, we have the result

$$P:[\mathbf{x}^1\,\mathbf{x}^2\cdots\mathbf{x}^{N-1}]\rightarrow[\mathbf{x}^1\,\mathbf{x}^2\cdots\mathbf{x}^{N-1}]\Gamma(P).$$

To prove the converse, we must find the set of all proper, real orthogonal matrices, $\{A\}$, having Nth column given by col $[1\,1\cdots]/(N)^{1/2}$ such that Eq. (3.14) holds, where each $\Gamma(P)$ is specified. At least one such A must exist since $D(P)$ is real orthogonal, $\Gamma(P)$ is real orthogonal, and $\{D(P)\}$ and $\{\Gamma(P)\oplus 1\}$ are equivalent representations of S_N. Assume there exists a second member, A', of the set $\{A\}$. Then the product $\tilde{A}A'$ has the form

$$\tilde{A}A'=\left(\begin{array}{ccc|c} & B & & \begin{array}{c}0\\ \vdots\\ 0\end{array}\\ \hline 0 & \cdots & 0 & 1\end{array}\right),$$

where $\Gamma(P)B=B\Gamma(P)$, $\forall P\in S_N$. Therefore (Schur's lemma), we have $B=bI_{N-1}$, where b is real. Furthermore, B is proper, and we must have $b=1$ for N even, and $b=\pm 1$ for N odd. Thus, we have $A'=A$ for N even, and

$$A'=A\left(\begin{array}{ccc|c} & \pm I_{N-1} & & \begin{array}{c}0\\ \vdots\\ 0\end{array}\\ \hline 0 & \cdots & 0 & 1\end{array}\right), \quad N \text{ odd}.$$

The essential contents of this section are contained in Definition 1 and Lemmas 1 and 2 which set forth the definition and significant properties of relative position vectors. We refrain from introducing any specific set, but note that the Jacobi coordinates are a popular choice (Kr66).

IV. THE GROUP OF THE SCHRÖDINGER EQUATION

An alternative title of this paper could be: *The Wave Functions of a System of N Identical, Noninteracting, Structureless Particles in an Angular Momentum Basis.* This title indeed describes precisely the problem we are attempting to solve through the use of group theoretical techniques. (We have preferred a less specific title because of the more general usefulness of these same functions.) Despite the fact that this problem is one of the simplest and most fundamental which can be posed for many-particle systems, its *general* solution has yet to be given in anything like a fully satisfactory form. In this section, we examine the general properties of Schrödinger's equation for this simple system, and describe the manner in which various orthogonal groups make their appearance.

Under the transformation, Eq. (3.10), Schrödinger's equation for a system of N free spinless particles separates into a part describing the center-of-mass motion and a part describing the motion relative to the center of mass, the latter equation being (in units with $\hbar=1$)

$$-\sum_{\alpha=1}^{N-1}(\boldsymbol{\nabla}^\alpha\cdot\boldsymbol{\nabla}^\alpha)\Psi(\mathbf{x}^1\,\mathbf{x}^2\cdots\mathbf{x}^{N-1})$$
$$=2mE\Psi(\mathbf{x}^1\,\mathbf{x}^2\cdots\mathbf{x}^{N-1}), \quad (4.1)$$

where $\boldsymbol{\nabla}^\alpha$ is the gradient operator corresponding to the vector $\mathbf{x}^\alpha$.

It is useful at this point to introduce a right-handed Cartesian coordinate frame in Euclidean 3-space. In such a frame, the vector $\mathbf{x}^\alpha$ is represented by three components $(x_1^\alpha\,x_2^\alpha\,x_3^\alpha)$, and we choose to let the notation x^α denote the column matrix

$$x^\alpha=\text{col }(x_1^\alpha\,x_2^\alpha\,x_3^\alpha). \quad (4.2)$$

It is also convenient to introduce the column matrices X_i defined by

$$X_i=\text{col }(x_i^1\,x_i^2\cdots x_i^{N-1}). \quad (4.3)$$

There are two instructive ways of organizing the collection of $3N-3$ components of the $N-1$ vectors $\{\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^{N-1}\}$. In the first arrangement, we associate an $(N-1)\times 3$ matrix X with the components:

$$X\equiv[X_1\,X_2\,X_3]. \quad (4.4)$$

This arrangement is particularly suitable for displaying the properties of the particle coordinates under the permutation operators P

$$P: X \to \tilde{\Gamma}(P)X. \tag{4.5}$$

In the second arrangement, we define

$$x_{3(\alpha-1)+i} = x_i^\alpha \tag{4.6}$$

for $i=1, 2, 3$, and $\alpha=1, 2, \cdots, N-1$. With these components, we associate the column matrix x having $3N-3$ elements:

$$x \equiv \text{col}\ (x_1\, x_2\, x_3 \cdots x_{3N-3}). \tag{4.7}$$

This form is particularly suitable for displaying the full symmetry properties of Eq. (4.1), which we now write as

$$-\nabla^2\Psi(x) = 2mE\Psi(x), \tag{4.8}$$

where ∇^2 is the Laplace operator in a Cartesian space of dimension $3N-3$.

The Laplace operator ∇^2 is invariant under real orthogonal transformations $x'=Rx$, $R\in O(3N-3)$, where $O(3N-3)$ denotes the group of real orthogonal matrices of dimension $3N-3$. Accordingly, Eq. (4.8) separates further into a part which is invariant under orthogonal transformations, $R\in O(3N-3)$, and a part made up of *homogeneous polynomials of some fixed degree, say p, which are solutions to Laplace's equation*:

$$\Psi(x) = g(\rho)f(x), \tag{4.9}$$

where

$$\rho \equiv \Big(\sum_{i=1}^{3N-3} x_i^2\Big)^{1/2}, \tag{4.10}$$

$$f(\lambda x) = \lambda^p f(x), \tag{4.11}$$

$$\nabla^2 f(x) = 0. \tag{4.12}$$

Equation (4.8) now reduces to the radial differential equation

$$\{(\partial/\partial\rho)^2 + [(2p+3N-4)/\rho](\partial/\partial\rho) + 2mE\}g(\rho) = 0. \tag{4.13}$$

We shall not consider the solutions to Eq. (4.13), although they are readily written out. Our concern is with the polynomials f which satisfy Eqs. (4.11) and (4.12). It is these polynomials which comprise the elements of a finite dimensional Hilbert space of the type discussed fully in Sec. II—these are the solid harmonics on the $(3N-3)$-sphere:

$$\mathfrak{L}_p = \{f : f(\lambda x) = \lambda^p f(x),\ \nabla^2 f(x) = 0\} \tag{4.14}$$

and

$$\dim \mathfrak{L}_p = [(2p+3N-5)/(3N-5)]\binom{p+3N-6}{3N-6}. \tag{4.15}$$

The transformation properties of the state vectors Ψ under $R\in O(3N-3)$ are carried fully by the polynomials f.

The group of the Schrödinger equation, Eq. (4.8), is now identified to be $O(3N-3)$; the group is realized on the invariant subspace $\mathfrak{L}_p$ by the unitary operators T_R

$$(T_R f)(x) = f(\tilde{R}x), \qquad \forall R\in O(3N-3), \qquad \forall f\in \mathfrak{L}_p. \tag{4.16}$$

Various properties of this IR of $O(3N-3)$ were discussed in Sec. II. In particular, a set of infinitesimal operators of the representation is given by

$$(\mathfrak{L}_{ij}f)(x) = -i[x_i(\partial/\partial x_j) - x_j(\partial/\partial x_i)]f(x) \tag{4.17}$$

for $i, j=1, 2, \cdots, 3N-3$, where $\mathfrak{L}_{ij} = -\mathfrak{L}_{ji}$.

As already pointed out in Sec. II, a basis for the solid harmonics on the $(3N-3)$-sphere is completely characterized by specifying that each basis solid harmonic is a simultaneous eigenvector of the set of quadratic Casimir operators and $\mathfrak{L}_{12}$. In this classification scheme, the eigenvectors in the basis for IR $\{p\,0\,0 \cdots 0\}$ of $O(3N-3)$ are labeled *completely* by the Gel'fand scheme, i.e., by the labels of the irreducible representations of the subgroups which appear in the chain of subgroup restrictions

$$O(3N-3) \to O(3N-2) \to \cdots \to O(3) \to O(2). \tag{4.18}$$

Observe that the $O(3)$ which appears in this chain is just the set of orthogonal transformations on the vector, $\mathbf{x}^1$, and this is *not* the $O(3)$ which has the total (relative) orbital angular momentum of the system of particles for its infinitesimal operators.

Thus, while the Gel'fand scheme gives a complete labeling of a basis of $\mathfrak{L}_p$, these basis vectors are not eigenvectors of sharp orbital angular momentum quantum numbers. One is thus led to consider subgroup decompositions of $O(3N-3)$ which are alternative to Eq. (4.18). Let us see how the orbital angular momentum operators come into the problem. Reverting to the particle index notation of Eq. (4.6), we can enumerate the set of infinitesimal operators $\{\mathfrak{L}_{ij}: i, j=1, 2, \cdots, 3N-3\}$ in the form

$$\{\Lambda_{ij}^{\alpha\beta}: \alpha, \beta=1, 2, \cdots, N-1;\ i, j=1, 2, 3\}, \tag{4.19}$$

where

$$(\Lambda_{ij}^{\alpha\beta}f)(x) = -i[x_i^\alpha(\partial/\partial x_j^\beta) - x_j^\beta(\partial/\partial x_i^\alpha)]f(x), \tag{4.20}$$

$$\Lambda_{ij}^{\beta\alpha} = -\Lambda_{ji}^{\alpha\beta}. \tag{4.21}$$

The components of total relative orbital angular momentum $(L_1, L_2, L_3) = (L_{23}, L_{31}, L_{12})$ are now easily identified:

$$L_{ij} = \sum_{\alpha=1}^{N-1} \Lambda_{ij}^{\alpha\alpha}, \qquad L_{ij} = -L_{ji}, \qquad i, j=1, 2, 3. \tag{4.22}$$

The global origin of the infinitesimal operators, Eq. (4.22), is easily found. It is clear that the $SO(3)$ which has the orbital angular momentum as its infinitesimal operators is the one which rotates all the vectors $\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^{N-1}$ simultaneously, i.e., as a rigid body. This characteristic is best exhibited by using the notation of Eq. (4.4). Thus, for each $R_3 \in O(3)$, we define the unitary operator D_{R_3} by

$$(D_{R_3} f)(X) = f(XR_3). \tag{4.23}$$

Then the correspondence $R_3 \to D_{R_3}$ is a representation of $O(3)$ by unitary operators on the space $\mathfrak{L}_p$. Furthermore, the three infinitesimal operators of the representation are precisely those given by Eq. (4.22).

Now observe that one can make a second type of orthogonal transformation on the matrix X: Namely, for each $R_{N-1} \in O(N-1)$, we define the unitary operator $\mathfrak{D}_{R_{N-1}}$ by

$$(\mathfrak{D}_{R_{N-1}} f)(X) = f(\tilde{R}_{N-1} X). \tag{4.24}$$

Then the correspondence $R_{N-1} \to \mathfrak{D}_{R_{N-1}}$ is a representation of $O(N-1)$ by unitary operators on the space $\mathfrak{L}_p$. One observes trivially that

$$D_{R_3} \mathfrak{D}_{R_{N-1}} = \mathfrak{D}_{R_{N-1}} D_{R_3}, \quad \forall R_3 \in O(3), \quad \forall R_{N-1} \in O(N-1). \tag{4.25}$$

The infinitesimal operators of the representation, Eq. (4.24), are easily calculated by the technique of Sec. II. They are:

$$L^{\alpha\beta} = \sum_{i=1}^{3} \Lambda_{ii}^{\alpha\beta}, \qquad L^{\beta\alpha} = -L^{\alpha\beta}, \qquad \alpha, \beta = 1, 2, \cdots, N-1. \tag{4.26}$$

Each operator in the set $\{L^{\alpha\beta}: \alpha < \beta = 1, 2, \cdots, N-1\}$ commutes with the orbital angular momentum, as expected from Eq. (4.25).

The relation between the transformations $D_{R_3} \mathfrak{D}_{R_{N-1}}$ and T_R, $R \in O(3N-3)$, is established as follows:

We have

$$(\mathfrak{D}_{R_{N-1}} D_{R_3} f)(X) = f(\tilde{R}_{N-1} X R_3). \tag{4.27}$$

But it is a simple exercise in matrix algebra to verify that the matrix transformation

$$X' = \tilde{R}_{N-1} X R_3 \tag{4.28}$$

is precisely the same as the column matrix transformation

$$x' = (\tilde{R}_{N-1} \otimes \tilde{R}_3) x, \tag{4.29}$$

and conversely, where $\otimes$ designates the direct product of matrices. Thus, we have

$$\mathfrak{D}_{R_{N-1}} D_{R_3} = T_{R_{N-1} \otimes R_3}. \tag{4.30}$$

The set of product operators

$$\{\mathfrak{D}_{R_{N-1}} D_{R_3}: R_{N-1} \in O(N-1), R_3 \in O(3)\}, \tag{4.31}$$

is a unitary representation on the space $\mathfrak{L}_p$ of the direct product group

$$O(N-1) \times O(3) = \{R_{N-1} \otimes R_3: R_{N-1} \in O(N-1), R_3 \in O(3)\} \subset O(3N-3). \tag{4.32}$$

The $SO(3) \subset O(3)$ in this subgroup has the orbital angular momentum as its infinitesimal operators.

Note the special cases of Eq. (4.30):

$$D_{R_3} = T_{I_{N-1} \otimes R_3}, \tag{4.33}$$

$$\mathfrak{D}_{R_{N-1}} = T_{R_{N-1} \otimes I_3}. \tag{4.34}$$

One technique, then, for getting N-particle states of sharp total orbital angular momentum quantum numbers into the state vector labeling problem is to find those invariant subspaces of $\mathfrak{L}_p$ with respect to which the product operators $\{D_{R_3} \mathfrak{D}_{R_{N-1}}\}$ are irreducible, i.e., decompose the space $\mathfrak{L}_p$, which is a carrier space for IR $\{p\, 0 \cdots 0\}$ of $O(3N-3)$, into a direct sum of perpendicular subspaces such that each subspace is the carrier space for an IR of $O(N-1) \times O(3)$. We can, of course, still use the subgroup restriction chain

$$O(N-1) \to O(N-2) \to \cdots \to O(3) \to O(2) \tag{4.35}$$

to label the basis vectors of the carrier space of IR's of the group $O(N-1)$.

The difficulty, of course, with this approach is that the group $O(N-1) \times O(3)$ is not *multiplicity free* in $O(3N-3)$: there will, in general, be several perpendicular subspaces of $\mathfrak{L}_p$, each of which is the carrier space for the *same* IR of $O(N-1) \times O(3)$. The group of operators, $\{D_{R_3} \mathfrak{D}_{R_{N-1}}\}$, does not distinguish between such subspaces, and this implies that we cannot induce a complete labeling scheme of a basis of $\mathfrak{L}_p$ by using only properties of the subgroup of operators, $\{D_{R_3} \mathfrak{D}_{R_{N-1}}\}$. Thus, the labeling scheme induced by the subgroup restriction chain as follows is necessarily incomplete

$$O(3N-3) \to O(N-1) \otimes O(3) \to O(N-2) \otimes O(3) \to \cdots \to O(2) \otimes O(3). \tag{4.36}$$

Equivalently stated: The angular momentum quantum numbers LM together with the Gel'fand–Zetlin labels of the basis vectors of the IR spaces of $O(N-1)$ contained in $\mathfrak{L}_p$ are, in general, insufficient to label a basis of $\mathfrak{L}_p$.

Nonetheless, we will see later (Sec. VIII) that the chain (4.36) is a useful way to view the problem.

Next, let us see how the permutation operators P of the symmetric group are realized on the space $\mathfrak{L}_p$. Comparing Eq. (4.5) with Eq. (4.24), we obtain $P \to \mathfrak{D}_{\Gamma(P)}$ is a representation of S_N on the space $\mathfrak{L}_p$ by unitary operators. Note also from Eq. (4.34) that we can write

$$\mathfrak{D}_{\Gamma(P)} = T_{O(P)}, \tag{4.37}$$

where

$$O(P) \equiv \Gamma(P) \otimes I_3. \tag{4.38}$$

The group of matrices $\{\Gamma(P): P\in S_N\}$ is a subgroup of $O(N-1)$, and the group of matrices $\{O(P): P\in S_N\}$ is a subgroup of $O(3N-3)$.

V. DEMOCRATIC SUBGROUPS

In the previous section, we have introduced two subgroup chains, Eqs. (4.18) and (4.36), which arose in a rather natural way from the study of the orthogonal transformations of the relative coordinates. The first subgroup chain, Eq. (4.18), may be used to label completely a basis of the space $\mathfrak{L}_p$. However, from a physical viewpoint, this basis is inappropriate because the basis states are not states of good angular momentum. The second subgroup chain, Eq. (4.36), seems to possess the desired properties with respect to angular momentum, but is, in fact, deficient because it affords us no means of distinguishing between several perpendicular subspaces of $\mathfrak{L}_p$ which carry the same representation of $O(N-1)\times O(3)$.

For these reasons, one is led to seek new schemes which are in some sense more appropriate. We now wish to establish four Lemmas (3, 4, 5, 8) which will subsequently have an important bearing on this problem. Lévy–Leblond (Le66) arrived at results which are less specific but nonetheless equivalent to these lemmas, and Dragt (Dr65) obtained the generators of the unitary group of Lemma 5. A closely related discussion has also been given by Galbraith (Ga71a).

Lemma 3. The operators in the group $\{T_R: R\in O(3N-3)\}$ which commute with the subgroup of operators $\{T_{O(P)}: P\in S_N\}$ are those contained in the subgroup $\{D_{R_3}: R_3\in O(3)\}$.

Proof. We seek the set of $R\in O(3N-3)$ such that

$$T_R T_{O(P)} = T_{O(P)} T_R, \qquad \forall P\in S_N, \tag{5.1}$$

i.e., we must find the set of $\{R\}$ such that

$$RO(P) = O(P)R, \qquad \forall P\in S_N, \tag{5.2}$$

where $O(P)$ is given by Eq. (4.38).

We make use of the general matrix identity as follows: Let A_3 and B_{N-1} be arbitrary matrices of dimensions 3 and $N-1$, respectively. Then we have

$$\tilde{R}_0(A_3\otimes B_{N-1})R_0 = B_{N-1}\otimes A_3, \tag{5.3}$$

where R_0 is the real orthogonal matrix defined by

$$R_0 = [e_1 e_N e_{2N-1};\, e_2 e_{N+1} e_{2N};\, \cdots;\, e_{N-1} e_{2N-2} e_{3N-3}], \tag{5.4}$$

and where e_i $(i=1, 2, \cdots, 3N-3)$ is the column matrix of length $3N-3$ having 1 in row i and 0's elsewhere. We omit the proof, noting only that the similarity transformation by R_0 simply effects the necessary row and column operations required to reorder the factors in the direct product.

Using Eq. (5.3), we now transform Eq. (5.2) to the form as follows:

$$R'[I_3\otimes\Gamma(P)] = [I_3\otimes\Gamma(P)]R', \qquad \forall P\in S_N, \tag{5.5}$$

where

$$R' \equiv R_0 R \tilde{R}_0. \tag{5.6}$$

Next, the matrix R' is partitioned into block matrices of dimension $N-1$, i.e.,

$$R' = \begin{pmatrix} R_{11}' & R_{12}' & R_{13}' \\ R_{21}' & R_{22}' & R_{23}' \\ R_{31}' & R_{32}' & R_{33}' \end{pmatrix}. \tag{5.7}$$

Equation (5.5) now yields the set of conditions,

$$R_{ij}'\Gamma(P) = \Gamma(P)R_{ij}', \qquad \forall P\in S_N. \tag{5.8}$$

Since $\{\Gamma(P): P\in S_N\}$ is an IR of S_N, we must have (Schur's lemma)

$$R_{ij}' = a_{ij} I_{N-1}, \tag{5.9}$$

where a_{ij} $(i, j=1, 2, 3)$ is a real number. Thus R' has the form $R_3\otimes I_{N-1}$, where R_3 is the 3×3 real orthogonal matrix with arbitrary elements a_{ij}. Using the definition, Eq. (5.6), and another application of Eq. (5.3), we conclude that each R which satisfies Eq. (5.2) has the form

$$R = I_{N-1}\otimes R_3, \qquad R_3\in O(3). \tag{5.10}$$

Since these are precisely the matrices which define the transformations D_{R_3}, the lemma is proved.

For $N\geq 4$, Lemma 3 can be refined:

Lemma 4. For $N\geq 4$, the operators in the group $\{T_R: R\in O(3N-3)\}$ which commute with the subgroup of operators $\{T_{O(P)}: P\in A_N\}$, where A_N is the alternating subgroup of S_N, are those contained in the subgroup $\{D_{R_3}: R_3\in O(3)\}$.

Proof. For $N\geq 4$, the representation of A_N, $\{\Gamma(P): P\in A_N\}$, is irreducible. Hence, the proof follows in the same manner as the one given for Lemma 3.

Let us sketch a proof that $\{\Gamma(P): P\in A_N\}$ is an IR of A_N for $N\geq 4$. The proof follows immediately if we can show that $A\Gamma(P) = \Gamma(P)A$, $\forall P\in A_N$ implies $A\Gamma(P) = \Gamma(P)A$, $\forall P\in S_N$, since then we must have $A = \lambda I_{N-1}$. To show that this is the case, let $P_1, P_2\in S_N$, but $P_1, P_2\notin A_N$. Then, we have $P_1^{-1}, P_2\in A_N$ and $\tilde{\Gamma}(P_1)A\Gamma(P_1) = \tilde{\Gamma}(P_2)A\Gamma(P_2)$. In particular, if P_1 and P_2 belong to the same class $\mathfrak{C}$, we see that $\tilde{\Gamma}(P_1)A\Gamma(P_1) = A'$, where A' is independent of the class $\mathfrak{C}$, i.e., $A\Gamma(P_1) = \Gamma(P_1)A'$. Now sum this relation over all $P_1\in\mathfrak{C}$, using

$$\sum_{P_1}\Gamma(P_1) = [h(\alpha-1)/(N-1)]I_{N-1},$$

where h is the number of elements in class $\mathfrak{C}$, and α is the number of cycles of length one in class $\mathfrak{C}$. We obtain $A' = A$ for $\alpha\neq 1$. Thus, $A\Gamma(P) = \Gamma(P)A$, $\forall P\in A_N$ implies

$$A\Gamma(P) = \Gamma(P)A, \qquad \forall P\in S_N, \qquad P\notin\mathfrak{C}_1,$$

where $\mathfrak{C}_1$ is a class not in A_N which has exactly one cycle of length one. To show that also

$$A\Gamma(P_1)=\Gamma(P_1)A, \qquad \forall P_1\in \mathfrak{C}_1, \qquad N\geq 4,$$

we must proceed differently. This step of the proof hinges on the peculiar fact that for $N\geq 4$ each $P_1\in\mathfrak{C}_1$ can be written as a product, $P_1=QQ'$, where neither Q nor Q' belongs to a class of type $\mathfrak{C}_1$ (we omit the proof). Therefore, we have $A\Gamma(P_1)=A\Gamma(Q)\Gamma(Q')=\Gamma(Q)\Gamma(Q')A=\Gamma(P_1)A$, $\forall P_1\in\mathfrak{C}_1$, $N\geq 4$.

Remark. Lemma 4 is not correct for $N=3$. Its failure for $N=3$ is attributed to the fact that the A_3 representation $\{\Gamma(P):P\in A_3\}$ is *reducible.* This fact itself can be attributed to a *distinguishing* structural property of S_3, namely: No element in the class

$$\{(1)(23),\ (2)(31),\ (3)(12)\}$$

can be written as the product of two elements not in the class, whereas in $S_N(N>3)$ each element in a class $\mathfrak{C}_1$ which contains precisely one cycle of length one *can* be written as a product QQ', where neither Q nor Q' belongs to a class of type $\mathfrak{C}_1$.

The distinguishing feature of S_3 discussed above permits a completely new structure, the unitary group $U(3)$, to enter into the 3-particle problem.

Lemma 5. The operators in the group $\{T_R: R\in O(6)\}$ which commute with the subgroup of operators $\{T_{O(P)}:P\in A_3\}$ are those contained in the subgroup $\{T_R: R\in G\}$, where $G\subset O(6)$ is the group of proper, real orthogonal matrices of the following form

$$G=\{R\in O(6):R=AMA^\dagger\}, \tag{5.11}$$

where

$$A=U_0\otimes I_3, \qquad U_0=(\sqrt{2})^{-1}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}, \tag{5.12}$$

$$M=\left(\begin{array}{c|c}U & 0\\ \hline 0 & U^*\end{array}\right), \tag{5.13}$$

in which U is an arbitrary 3×3 *unitary matrix*, i.e., $U\in U(3)$. *G is isomorphic to $U(3)$.*

Proof: G is the subgroup of real orthogonal matrices defined by

$$G=\{R\in O(6):RO(P)=O(P)R,\ \forall P\in A_3\}. \tag{5.14}$$

The general results, Eqs. (5.1)–(5.8), are valid when particularized to $N=3$ and $P\in A_3$. We must therefore determine the set of 2×2 matrices R_{ij}' $(i,j=1,2,3)$ which satisfy

$$R_{ij}'\Gamma(P)=\Gamma(P)R_{ij}', \qquad \forall P\in A_3, \tag{5.15}$$

where $\{\Gamma(P)\}$ is a real orthogonal 2×2 matrix IR of S_3 having character set $\{2,-1,0\}$. In particular, if we let $P_1=(123)$, then the elements of A_3 are $\{P_1, P_1^2, P_1^3=\text{identity}\}$. Thus, Eq. (5.15) reduces to the single condition

$$R_{ij}'\Gamma(P_1)=\Gamma(P_1)R_{ij}', \qquad P_1=(123). \tag{5.16}$$

We do not specify explicitly the representation $\{\Gamma(P):P\in S_3\}$ other than requiring it to be a *real orthogonal* IR of S_3 having character set $\{2,-1,0\}$. The properties

$$\operatorname{Tr}\Gamma(P_1)=-1, \qquad \det\Gamma(P_1)=1 \tag{5.17}$$

then necessarily follow. (*Lemma 5 is not particular to the choice of relative position vectors.*)

We next observe that an arbitrary proper, real orthogonal 2×2 matrix is diagonalized by the unitary matrix

$$U_0=(\sqrt{2})^{-1}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}. \tag{5.18}$$

In particular, for arbitrary $\Gamma(P_1)$, we have

$$U_0^\dagger\Gamma(P_1)U_0=\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon^2\end{pmatrix}, \tag{5.19}$$

where $\epsilon^2+\epsilon+1=0$ and $\epsilon^3=1$. Transforming Eq. (5.16) by the similarity transformation, Eq. (5.19), we easily find that the most general real R_{ij}' which satisfies Eq. (5.16) is

$$R_{ij}'=U_0M_{ij}'U_0^\dagger, \tag{5.20}$$

where

$$M_{ij}'=\begin{pmatrix}u_{ij} & 0\\ 0 & u_{ij}^*\end{pmatrix}, \tag{5.21}$$

in which u_{ij} is an arbitrary complex number. Thus, we have

$$R'=(I_3\otimes U_0)M'(I_3\otimes U_0^\dagger), \tag{5.22}$$

where M' is the 6×6 matrix constructed from the 2×2 matrices M_{ij}' in the manner of Eq. (5.7). Using Eqs. (5.6) (for $N=3$) and (5.3), we find that R has the form given in the lemma, where U is the matrix with elements (u_{ij}). The requirement that R be orthogonal implies that U is unitary. Indeed, we see that R is *proper*, real orthogonal.

To complete the proof, we note that U_0 of Eq. (5.18) is unique up to arbitrary phase factors of its columns, i.e., the matrix

$$U_0'=U_0\begin{pmatrix}e^{i\alpha} & 0\\ 0 & e^{i\beta}\end{pmatrix} \tag{5.23}$$

also diagonalizes $\Gamma(P_1)$. However, defining A' in terms of U_0' by Eq. (5.12), we see that $A'M(A')^\dagger=AMA^\dagger$, i.e., we get the same group G independent of the choice of U_0'. Finally, it is clear that G is *isomorphic* to $U(3)$, since $I_6\leftrightarrow I_3$.

The matrix group G contains the group $\{I_2\otimes R_3: R_3\in O(3)\}$ as a subgroup, since $R=I_2\otimes R_3$ implies $U_3=R_3$, and conversely. Thus, G is a subgroup of $O(6)$ with the property

$$O(6)\supset G\supset\{I_2\otimes R_3: R_3\in O(3)\}. \tag{5.24}$$

Thus, for $N=3$, we have obtained a subgroup scheme which is alternative to that of Eq. (4.36) for labeling a basis of $\mathcal{L}_p$. Furthermore, we will see in considerable detail in a later section that the IR spaces of G arise in a multiplicity free way on the space $\mathcal{L}_p$, i.e., at the level of G, a basis of $\mathcal{L}_p$ is *completely* labeled by the quantum numbers which label the basis vectors of the IR spaces of G.

Note that the group G has the following simple properties with respect to the subgroup $\{O(P): P\in S_3\}$ of $O(6)$

$$\tilde{O}(P)RO(P)=R, \qquad \forall R\in G, \qquad \forall P\in A_3, \quad (5.25)$$

$$\tilde{O}(P)RO(P)\in G, \qquad \forall R\in G, \qquad \forall P\in S_3. \quad (5.26)$$

As a consequence of these properties, Dragt (Dr65) termed the group G a *democratic subgroup* of $O(6)$.

Encouraged by the enormous simplifications which occur in the 3-particle problem, Lévy-Leblond and Lurcat (Le65) undertook the task of generalizing the notion of a democratic subgroup to the N-particle problem. The consequences of this generalization are disappointing. The concept of a democratic subgroup is elegant, but the structure of the symmetric group is too tight to let any dramatic new structures through. Let us see how this comes about.

Definition 2: A subgroup $G\subset O(3N-3)$ is called *democratic* if $R\in G$ implies

$$\tilde{O}(P)RO(P)\in G, \qquad \forall P\in S_N. \quad (5.27)$$

This definition is clearly an appropriate generalization of the one introduced by Dragt. However, two lemmas, proved by Lévy-Leblond (Le66), forecast the limited structure of $O(3N-3)$ which will be unveiled.

Lemma 6. Let G be a democratic subgroup of $O(3N-3)$. Then the permutations belonging to the set of permutation operators $\{P\}$ which satisfy

$$RO(P)=O(P)R, \qquad \forall R\in G \quad (5.28)$$

are the elements of an *invariant subgroup* of S_N.

Lemma 7. Let $\{P\}$ denote the set of elements of an invariant subgroup of S_N. Then the real orthogonal matrices belonging to the set $\{R\}$ which satisfy

$$RO(P)=O(P)R, \qquad \forall P\in\{P\} \quad (5.29)$$

are the elements of a democratic subgroup G of $O(3N-3)$.

Lévy-Leblond has given the simple proofs of these lemmas, and we will not repeat them.

Since each democratic subgroup of $O(3N-3)$ has an invariant subgroup of S_N associated with it (Lemma 6), one can clearly find all such democratic subgroups which contain $\{I_{N-1}\otimes R_3: R_3\in O(3)\}$ by systematically determining the various groups G associated with the various invariant subgroups of S_N (Lemma 7). It is precisely here where the properties of S_N limit the implications of the democracy concept. For $N\neq 4$, the only invariant subgroups of S_N are the identity, A_N, and S_N itself. But for $N\geq 4$, Lemmas 3 and 4 already show that the democratic subgroup associated with either S_N or A_N is simply

$$L=\{I_{N-1}\otimes R_3: R_3\in O(3)\}. \quad (5.30)$$

For $N=3$, Lemma 3 again shows that the democratic subgroup associated with S_3 is L; however, now one rediscovers Dragt's democratic subgroup G associated with A_3.

The only structure, possibly new, which the generalized democracy concept points out (in addition to giving Dragt's result a general setting) is the democratic subgroup associated with the extra invariant subgroup $\mathcal{V}\subset A_4$ which S_4 possesses. The precise nature of this democratic subgroup is given by the next lemma.

Lemma 8: The operators in the group $\{T_R: R\in O(9)\}$ which commute with the subgroup of operators $\{T_{O(P)}: P\in\mathcal{V}\}$ are those contained in the subgroup $\{T_R: R\in G\}$, where $G\subset O(9)$ is the group of real orthogonal matrices of the following form:

$$G=\{R\in O(9): R=AM\tilde{A}\}, \quad (5.31)$$

where

$$A=S_0\otimes I_3, \quad (5.32)$$

in which S_0 is a 3×3 proper, real orthogonal matrix *which depends on the choice of relative position vectors*;

$$M=\begin{pmatrix} R_3 & 0 & 0 \\ 0 & R_3' & 0 \\ 0 & 0 & R_3'' \end{pmatrix}, \quad (5.33)$$

where R_3, R_3', $R_3''\in O(3)$.

Proof. G is the subgroup of real orthogonal matrices defined by

$$G=\{R\in O(9): RO(P)=O(P)R, \forall P\in\mathcal{V}\}. \quad (5.34)$$

The general results, Eqs. (5.1)–(5.8) are valid when particularized to $N=4$ and $P\in\mathcal{V}$. We must therefore determine the set of 3×3 matrices R_{ij}' $(i, j=1, 2, 3)$ which satisfy

$$R_{ij}'\Gamma(P)=\Gamma(P)R_{ij}', \qquad \forall P\in\mathcal{V}, \quad (5.35)$$

where $\{\Gamma(P)\}$ is a real orthogonal 3×3 matrix IR of S_4 having character set $\{\alpha-1\}=\{3, 1, -1, 0, -1\}$.

The elements of $\mathcal{V}$ are

$$\mathcal{V}=\{E, P_1=(12)(34), P_2=(23)(14), P_3=(13)(24)\}. \quad (5.36)$$

Note that $P_i^2=E$ $(i=1, 2, 3)$, and $P_iP_j=P_k$ for (ijk) any arrangement of (123). These relations imply that the 3×3 real orthogonal matrices $\Gamma(P_i)$ $(i=1, 2, 3)$, which have $\mathrm{Tr}\,\Gamma(P_i)=-1$, are also *symmetric and proper* [the proper condition is implied by the fact that each $\Gamma(P_i)$ is similar to the diagonal matrix with diagonal elements $(1, -1, -1)$]. The most general forms for the $\Gamma(P_i)$ are given by[4]

$$\Gamma(P_i) = -I_3 + 2a_i\tilde{a}_i, \qquad i=1, 2, 3, \tag{5.37}$$

where each $a_i = \text{col}\,(a_{1i}a_{2i}a_{3i})$ is a column matrix, and $\tilde{a}_i a_j = \delta_{ij}$, i.e., (a_{1i}, a_{2i}, a_{3i}) are the components of a triad of unit vectors which are mutually perpendicular, and which can be chosen, without loss of generality, to be a right-handed triad. Then $\{I_3, \Gamma(P_i), i=1, 2, 3\}$ is a representation of $\mathfrak{V}$, and it is the *most general representation* which can be obtained from $\{\Gamma(P): P\in S_4\}$, where as always $\Gamma(P)$ is an IR [3100] of S_4 which is real orthogonal. The proper, real orthogonal matrix S_0 defined by

$$S_0 = [a_1\, a_2\, a_3] \tag{5.38}$$

now diagonalizes each $\Gamma(P_i)$

$$\tilde{S}_0\Gamma(P_i)S_0 = D(P_i), \qquad i=1, 2, 3, \tag{5.39}$$

where

$$D(P_1) = \text{diag}\,(1, -1, -1).\ D(P_2) = \text{diag}\,(-1, 1, -1),$$
$$D(P_3) = \text{diag}\,(-1, -1, 1).$$

Here S_0 is unique up to $(\pm)$ signs of its columns, and these signs are irrelevant (see the proof of Lemma 5).

Following the steps analogous to those in going from Eqs. (5.19)–(5.22) in proving Lemma 5, the proof of Lemma 8 is easily completed, and we omit these details.

Let us observe that by a simple redefinition of relative position vectors, we can always get rid of the matrix A in Eq. (5.31). Thus, suppose we have chosen a particular set of relative position vectors, call them $[\mathbf{y}^1\,\mathbf{y}^2\,\mathbf{y}^3]$. Then, under permutations, these vectors undergo the transformation

$$P: [\mathbf{y}^1\,\mathbf{y}^2\,\mathbf{y}^3] \rightarrow [\mathbf{y}^1\,\mathbf{y}^2\,\mathbf{y}^3]\Gamma(P), \tag{5.40}$$

where $\{\Gamma(P): P\in S_4\}$ is a specific IR [3100] of S_4. We can then determine a_1, a_2, and a_3 (up to irrelevant signs). We then define a new set of relative position vectors $[\mathbf{x}^1\,\mathbf{x}^2\,\mathbf{x}^3]$ by

$$[\mathbf{x}^1\,\mathbf{x}^2\,\mathbf{x}^3] = [\mathbf{y}^1\,\mathbf{y}^2\,\mathbf{y}^3]S_0. \tag{5.41}$$

Then we have

$$P: [\mathbf{x}^1\,\mathbf{x}^2\,\mathbf{x}^3] \rightarrow [\mathbf{x}^1\,\mathbf{x}^2\,\mathbf{x}^3]\Gamma'(P), \tag{5.42}$$

where

$$\Gamma'(P) = \tilde{S}_0\Gamma(P)S_0, \tag{5.43}$$

and, in particular, $\Gamma'(P_i) = D(P_i)$, $i=1, 2, 3$, where the $D(P_i)$ are the diagonal matrices following Eq. (5.39). Indeed, the relative position vectors $[\mathbf{x}^1\,\mathbf{x}^2\,\mathbf{x}^3]$ are uniquely determined (Lemma 2) to be as follows:

$$\mathbf{x}^1 = (\mathbf{r}^1 + \mathbf{r}^2 - \mathbf{r}^3 - \mathbf{r}^4)/2,$$
$$\mathbf{x}^2 = (\mathbf{r}^2 + \mathbf{r}^3 - \mathbf{r}^1 - \mathbf{r}^4)/2, \tag{5.44}$$
$$\mathbf{x}^3 = (\mathbf{r}^1 + \mathbf{r}^3 - \mathbf{r}^2 - \mathbf{r}^4)/2.$$

Finally, we can transform each $R\in O(9)$ by the matrix A of Lemma 8

$$R' = \tilde{A}RA, \qquad A = S_0 \otimes I_3. \tag{5.45}$$

Then it is straightforward to verify that

$$(T_{R'}f)(x) = (T_R g)(y), \qquad y = Ax, \tag{5.46}$$

where, for each given function $g\in\mathfrak{L}_p$, the function $f\in\mathfrak{L}_p$ is defined by

$$f(x) = f(\tilde{A}y) = (T_A f)(y) = g(y), \tag{5.47}$$

that is,

$$f = T_{\tilde{A}}g. \tag{5.48}$$

Thus, it is no restriction to use the relative position vectors, Eq. (5.44), from the start. The group G of Lemma 8 then becomes

$$G = \left\{ R\in O(9): R = \begin{pmatrix} R_3 & 0 & 0 \\ 0 & R_3' & 0 \\ 0 & 0 & R_3'' \end{pmatrix} \right\}, \tag{5.49}$$

where $R_3, R_3', R_3'' \in O(3)$. The transformation $x' = \tilde{R}x$ in

$$(T_R f)(x) = f(\tilde{R}x), \qquad R\in G \tag{5.50}$$

is easily identified in column matrix form to be

$$[x^1\,x^2\,x^3] \rightarrow [\tilde{R}_3 x^1, \tilde{R}_3' x^2, \tilde{R}_3'' x^3]. \tag{5.51}$$

Thus, the group G is the group $O^{(1)}(3)\times O^{(2)}(3)\times O^{(3)}(3)$, where $SO^{(i)}(3)$ is the real orthogonal group whose infinitesimal generators are the orbital angular momentum operators associated with the relative position vector $\mathbf{x}^i$.

The group $O^{(1)}(3)\times O^{(2)}(3)\times O^{(3)}(3)$ is, unfortunately, not multiplicity free in $O(9)$. In particular, the quantum numbers $(l_1m_1; l_2m_2; l_3m_3)$ which label the basis vectors of the IR spaces of this direct product group do not label completely a basis of the space $\mathfrak{L}_p$.

The concept of a democratic subgroup offers a systematic technique for determining subgroups of $O(3N-3)$ which have specific properties with respect to the permutation subgroup

$$\{O(P): P\in S_N\} \subset O(3N-3).$$

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[4] In the correspondence of the points of the solid sphere in 3-space onto the elements of $SO(3)$, the points on the surface of the sphere map, two-to-one, onto the symmetric elements of $SO(3)$. Equation (5.37) is simply the most general form of a symmetric element of $SO(3)$.

The concept is, however, too limited to admit any significantly new structural schemes for classifying N-particle states ($N>4$). There exist, of course, subgroup chains of $O(3N-3)$ other than those brought out by the democratic subgroup concept which conceivably might be useful. However, such schemes necessarily relinquish the property expressed by Eq. (5.27). It is not our purpose here to pursue this point further; rather, we now turn to the systematic development of what can be learned about the classification of N-particle states ($N=3, 4, \cdots$) within the framework already discussed in this paper.

VI. 3-PARTICLE STATES

The problem of constructing completely labeled total angular momentum states on the 6-sphere has been studied by many authors: Smith (Sm60, 62), Kramer (Kr63, 65), Dragt (Dr65), Zickendraht (Zi65), Chacón and Moshinsky (Ch65), Simonov (Si66), and Whitten and Smith (Wh68). More recently, Castilho Alcarás and Leal Ferreira (Ca71), and Éfros (Ef71) have considered this problem in great detail from various viewpoints, often listing tables of specific state vectors. What we offer is of a somewhat different nature.

First of all, we wish to make clear for the 3-particle problem the role of the *homomorphism* of the group $SU(4)$ onto $SO(6)$. This is accomplished in Secs. A and B. [The *isomorphism* of the Lie algebras was discussed by Dragt (Dr65) and used extensively by Chacón and Moshinsky (Ch65).]

In Sec. C we generalize our development to the group $U(4)$, thereby gaining a deeper insight into the underlying structure of the problem. The group $U(4)$ is then realized as a set of transformations on the space $\mathcal{H}_p$ of homogeneous complex polynomials of degree p defined over C^6. The relation of the space $\mathcal{H}_p$ to $\mathcal{L}_p$ is discussed extensively, and, in particular, it is noted how one recovers a basis for the solid harmonics on the 6-sphere from the explicitly given (Sec. D) Gel'fand state vectors of $U(4)$.

In Sec. E, it is demonstrated that by a simple change of basis one can already diagonalize the z component of the total orbital angular momentum. The discrete symmetry properties of our general basis functions are given in Sec. F.

With respect to the 3-particle problem, the results of Secs. A–F may be summarized by the statement that a basis of the solid harmonics of degree p on the 6-sphere has been given which (1) is a basis of IR $[pq0]$ of $U(3)$; (2) possesses sharp z component of total orbital angular momentum; and (3) is classified "democratically" with respect to the permutations of identical particles.

We still have not achieved the desired goal of obtaining a basis of $\mathcal{L}_p$ which also has sharp angular momentum L. This difficult problem is the subject of Sec. G.

A. The Homomorphism of $SU(4)$ onto $SO(6)$

The fact that the group $U(3)$ enters into the 3-particle problem could have been foreseen (Dr65) as a consequence of the well known homomorphism of $SU(4)$ (the group of 4×4 unitary unimodular matrices) onto $SO(6)$, without the aid of the democratic subgroup concept. It would, however, be difficult to foresee the precise way to establish this homomorphism if one did not have available the specific result, Lemma 5.

The idea is to establish the homomorphism between $SU(4)$ and $SO(6)$ in such a way that when $SU(4)$ is restricted to one of its $U(3)$ subgroups there obtains in a simple way the isomorphism between the $U(3)$ of Lemma 5 and the subgroup $G\subset SO(6)$. To this end, let $V\in SU(4)$ be partitioned as follows:

$$V=\left(\begin{array}{c|c} V_3 & \alpha \\ \hline \tilde{\beta} & \gamma \end{array}\right), \tag{6.1}$$

where V_3 is a 3×3 complex matrix, α and β are complex column matrices,

$$\alpha=\text{col}\ (\alpha_1\,\alpha_2\,\alpha_3), \qquad \beta=\text{col}\ (\beta_1\,\beta_2\,\beta_3), \tag{6.2}$$

and γ is a complex number. We assert:

Lemma 9. Let $V\in SU(4)$ and $R\in SO(6)$. The following relation is a 2 to 1 homomorphism of $SU(4)$ onto $SO(6)$:

$$R=AQA^\dagger, \tag{6.3}$$

where

$$A=(\sqrt{2})^{-1}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\otimes I_3, \tag{6.4}$$

$$Q=\begin{pmatrix} Q_1 & Q_2^* \\ Q_2 & Q_1^* \end{pmatrix}, \tag{6.5}$$

in which Q_1 and Q_2 are 3×3 complex matrices which are related to V; specifically, we have

$$Q_1=\gamma V_3-\alpha\tilde{\beta}, \tag{6.6}$$

$$Q_2=\Gamma_\alpha V_3, \tag{6.7}$$

$$\Gamma_\alpha=\begin{pmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix}, \tag{6.8}$$

in which V_3, α, β, and γ are the quantities which appear in the partitioning, Eq. (6.1), of V.

The proof of Lemma 9 is given in Appendices 1 and 2.

Remark. The content of Lemma 9 is simply this: For each $V\in SU(4)$, the matrix R calculated from the definitions, Eqs. (6.3)–(6.8), is proper, real orthogonal. Conversely, for each $R\in SO(6)$, one can find *exactly* two unitary unimodular matrices which when used in Eqs. (6.3)–(6.8) will yield the specified R. Finally, the property $V\rightarrow R$, $V'\rightarrow R'$ implies $VV'\rightarrow RR'$ establishes the fact that the rule, Eqs. (6.3)–(6.8), for relating elements of $SU(4)$ to elements of $SO(6)$ is a homomorphism. Note that $V\rightarrow R$ implies $-V\rightarrow R$ (the 2 to 1 correspondence of the homomorphism).

Now consider the subgroup $\mathcal{G}$ of $SU(4)$ defined as follows:

$$\mathcal{G}=\left\{\begin{pmatrix} & V_3 & & \begin{matrix}0\\0\\0\end{matrix} \\ 0 & 0 & 0 & \gamma\end{pmatrix} : V_3\in U(3), \det V_3=\gamma^*, \gamma\gamma^*=1\right\}. \tag{6.9}$$

Under the restriction of $SU(4)$ to $\mathcal{G}$, the matrix Q of Lemma 9 becomes a matrix M of the type in Lemma 5, where

$$U=\gamma V_3\in U(3). \tag{6.10}$$

Note, in particular, that we obtain the *same* $U\in U(3)$ *from* $V\in\mathcal{G}$ *and* $-V\in\mathcal{G}$: *The subgroup* $\mathcal{G}\subset SU(4)$ *is* 2 *to* 1 *homomorphic onto the subgroup* $G\subset SO(6)$.

It is of interest to determine which matrices of $SU(4)$ correspond to the permutation subgroup, $\{O(P):P\in A_3\}$, of $SO(6)$. It is sufficient to determine the elements $\pm V_1$ of $SU(4)$ which correspond to $O(P_1)$, $P_1=(123)$ [see Eq. (5.19)]. This simple calculation gives

$$V_1=\pm\begin{pmatrix} & \epsilon I_3 & & \begin{matrix}0\\0\\0\end{matrix} \\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{6.11}$$

The subgroup of $SU(4)$ which commutes with V_1 is then seen to be precisely the group $\mathcal{G}$, defined by Eq. (6.9), which maps to the democratic subgroup G of $SO(6)$.

There is another useful way (Es64) to realize the homomorphism of $SU(4)$ onto $SO(6)$ given in Lemma 9. We define a 4×4 complex skew–symmetric matrix Z_x as follows:

$$Z_x=(\sqrt{2})^{-1}\begin{pmatrix} 0 & -x_3+ix_6 & x_2-ix_5 & x_1+ix_4 \\ x_3-ix_6 & 0 & -x_1+ix_4 & x_2+ix_5 \\ -x_2+ix_5 & x_1-ix_4 & 0 & x_3+ix_6 \\ -x_1-ix_4 & -x_2-ix_5 & -x_3-ix_6 & 0\end{pmatrix}. \tag{6.12}$$

[The column matrix x is used as a subscript on Z_x to make explicit the fact that the elements of the matrix Z_x depend on the elements of x.]

Lemma 10.[5] Let $V\rightarrow R$ in the homomorphism of Lemma 9. Then

$$Z_{\tilde{R}x}=\tilde{V}Z_xV. \tag{6.13}$$

The proof is given in Appendix 3.

The significance of the relation, Eq. (6.13), is that through it we can make contact with some standard results of unitary group theory; the significance of the homomorphism in the form of Lemma 9 is the elucidation of the relation of $SU(4)$ to the democratic subgroup $G\subset SO(6)$.

B. The Significance of the $SU(4)\rightarrow SO(6)$ Homomorphism for the 3-Particle Problem

Now let us see how the group $SU(4)$ arises in the 3-particle problem. It is convenient to make a change in notation. We have previously seen the usefulness of organizing the six components of the relative position vectors $\mathbf{x}^1$ and $\mathbf{x}^2$ into a 2×3 matrix X, or a 6×1 column matrix $x=\text{col}(x_1x_2x_3x_4x_5x_6)$, where $(x_1^1\, x_2^1\, x_3^1)=(x_1x_2x_3)$, and $(x_1^2\, x_2^2\, x_3^2)=(x_4x_5x_6)$. We now have a third way of organizing these components, namely, in the manner in which they appear in the definition of the 4×4 complex skew–symmetric matrix Z_x. It is now convenient to let the notation $f(Z_x)$ denote the value of the function f at the point $(x_1\, x_2\cdots x_6)$, i.e., $f(Z_x)\equiv f(x)$. The transformation T_R on the space $\mathcal{L}_p$, previously described by Eq. (4.16), is defined in terms of the new notation by

$$(T_Rf)(Z_x)=f(Z_{\tilde{R}x}),\qquad R\in SO(6). \tag{6.14}$$

Next, consider any complex skew–symmetric matrix Z of the form

$$Z=\begin{pmatrix} 0 & -z_3^* & z_2^* & z_1 \\ z_3^* & 0 & -z_1^* & z_2 \\ -z_2^* & z_1^* & 0 & z_3 \\ -z_1 & -z_2 & -z_3 & 0\end{pmatrix}. \tag{6.15}$$

We can always identify the coordinates $(x_1\, x_2\cdots x_6)$ by the rule

$$x_j=(z_j+z_j^*)/\sqrt{2},\qquad x_{j+3}=-i(z_j-z_j^*)/\sqrt{2} \tag{6.16}$$

for $j=1, 2, 3$. *The set of points of Euclidean* 6-*space is in one-to-one correspondence with the set of* 4×4 *complex skew–symmetric matrices of the form* (6.15):

$$(x_1\, x_2\cdots x_6)\leftrightarrow Z. \tag{6.17}$$

Quite generally, then, we use the notation $f(Z)$ to

[5] This method of realizing the homomorphism is patterned after the results given by Esteve and Sona (Es64).

denote the value of the function f at the point x which corresponds to Z

$$f(Z)=f(x) \quad \text{for} \quad x \leftrightarrow Z. \tag{6.18}$$

We write Z_x only when Z is considered to be *explicitly* in the form of Eq. (6.12). (We can, of course, think of x as a column matrix.)

With the above notational conventions, we can write Eq. (6.14) as

$$(T_R f)(Z_x)=f(\tilde{V} Z_x V), \tag{6.19}$$

where $V \rightarrow R$ in the homomorphism of $SU(4)$ onto $SO(6)$ (Lemma 10).

But now observe that, for each $V \in SU(4)$, we can define an operator S_V on the space $\mathfrak{L}_p$ by

$$(S_V f)(Z)=f(\tilde{V} Z V) \quad \text{for} \quad x \leftrightarrow Z. \tag{6.20}$$

One easily verifies that the set of operators $\{S_V: V \in SU(4)\}$ is a *group of unitary operators* which is a representation of $SU(4)$ on the space $\mathfrak{L}_p$ under the correspondence $V \rightarrow S_V$. Furthermore, since, by definition, Eqs. (6.19) and (6.20) hold for each point x and each $f \in \mathfrak{L}_p$, we have the following operator identity on the space $\mathfrak{L}_p$:

$$T_R = S_V \quad \text{for} \quad V \rightarrow R. \tag{6.21}$$

Also, the operator identity

$$S_{-V} = S_V \tag{6.22}$$

holds on $\mathfrak{L}_p$, and the *general homomorphism $\pm V \rightarrow R$ of groups collapses on the space $\mathfrak{L}_p$ to the equality of operators.*

The seemingly trivial identity, Eq. (6.21), *is the expression of the basic structure of the* 3-*particle problem.* Let us examine its content. The space $\mathfrak{L}_p$ is a carrier space for IR $\{p00\}$ of $SO(6)$. It is therefore also a carrier space for an IR (yet to be identified) of $SU(4)$. But now instead of classifying a basis of $\mathfrak{L}_p$ by using subgroup chains of $SO(6)$, we can use the subgroup chain of $SU(4)$:

$$SU(4) \supset U(3) \supset U(2) \supset U(1). \tag{6.23}$$

But the canonically labeled basis vectors—the so-called Gel'fand basis—of each IR space of $SU(4)$ corresponding to this chain are *completely and generally known* (abstractly and specifically). The significant aspect of the *particular* homomorphism we have given is that under the restriction

$$V \rightarrow V' = \left(\begin{array}{ccc|c} & & & 0 \\ & V_3 & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & \gamma \end{array}\right), \quad \gamma^* = \det V_3, \tag{6.24}$$

we have

$$R \rightarrow R' = A \left(\begin{array}{c|c} \gamma V_3 & 0 \\ \hline 0 & \gamma^* V_3^* \end{array}\right) A^{\dagger}. \tag{6.25}$$

The group of operators $\{T_{R'}\} = \{S_{V'}\}$ is just the unitary representation on $\mathfrak{L}_p$ of the democratic subgroup G of $O(6)$. On the other hand, it is precisely the subgroup restriction, Eq. (6.24), which leads to the assignment of the $U(3)$ subgroup labels of the $SU(4)$ Gel'fand basis vectors. In other words, the canonical Gel'fand basis vectors already provide us with a completely labeled basis of $\mathfrak{L}_p$ such that the subspaces of $\mathfrak{L}_p$ which are spanned by those basis vectors having fixed $U(3)$ IR labels are the IR spaces of the democratic subgroup. *The identification of operators, Eq.* (6.21), *points out the path to the solution of the* 3-*particle problem in terms of standard results of unitary group theory.*

C. Generalization to $U(4)$

We wish now to consider the group $U(4)$ in place of $SU(4)$. It turns out that at the level of $U(4)$, one can understand fully the relation between the space of homogeneous polynomials defined on six arbitrary complex variables and the space $\mathfrak{L}_p$ of homogeneous polynomials on six real variables which solve Laplace's equation. [Abstractly, this is the same problem (Dr65) as extracting solutions to Laplace's equation in 6-space from the harmonic oscillator functions in 6-space.] Quite aside from this important connection, the IR spaces of $U(4)$ which we thereby obtain are of considerable interest in themselves.

In the generalization to $U(4)$, it would *not* be correct to regard the matrix V in Eq. (6.20) as simply belonging to $U(4)$. The compatibility of the transformations on the variables z_i and z_i^* *requires* that V be unimodular (see Appendix 3). *If we let V belong to $U(4)$, we must at the same time let Z be of a more general form.* The way to do this is, however, quite obvious. We define the complex skew-symmetric matrix W as follows:

$$W = \begin{pmatrix} 0 & -\zeta_3 & \zeta_2 & \eta_1 \\ \zeta_3 & 0 & -\zeta_1 & \eta_2 \\ -\zeta_2 & \zeta_1 & 0 & \eta_3 \\ -\eta_1 & -\eta_2 & -\eta_3 & 0 \end{pmatrix}, \tag{6.26}$$

where η_i and ζ_i are arbitrary complex numbers. With each such matrix W, we associate the point

$$(\eta_1\, \eta_2\, \eta_3\, \zeta_1\, \zeta_2\, \zeta_3)$$

of complex 6-space, C^6. The matrix W corresponds to a point of Euclidean 6-space, if we use the correspondence, Eq. (6.16), if and only if $\zeta_i = \eta_i^*$, but we do not impose this restriction.

The results given in Appendix 3 now show that the transformation

$$W' = \tilde{U} W U, \qquad U \in U(4), \tag{6.27}$$

is the same as the transformation

$$\begin{pmatrix}\eta'\\ \zeta'\end{pmatrix}=\tilde{Q}\begin{pmatrix}\eta\\ \zeta\end{pmatrix}, \qquad (6.28)$$

where[6]

$$Q=\begin{pmatrix}Q_1 & Q_2^*(\det U)\\ Q_2 & Q_1^*(\det U)\end{pmatrix}, \qquad (6.29)$$

in which Q_1 and Q_2 are defined *exactly* as before in terms of the partition matrices of U, i.e., by Eqs. (6.6)–(6.8) for U written in the form (6.1).

Next, we introduce the notation $F(W)$ to designate the value of the function F at the point (η, ζ) of C^6

$$F(W)=F(\eta,\zeta). \qquad (6.30)$$

We also restrict ourselves to the set of polynomial functions which are homogeneous of degree p

$$\mathcal{H}_p=\{F: F(\lambda W)=\lambda^p F(W)\}. \qquad (6.31)$$

Using the extended definition of scalar product discussed in Sec. II, we make this function space a Hilbert space.

For each $U\in U(4)$, we now define the operator S_U on $\mathcal{H}_p$ by

$$(S_UF)(W)=F(\tilde{U}WU). \qquad (6.32)$$

The set of operators $\{S_U: U\in U(4)\}$ is then a representation of $U(4)$ on the space $\mathcal{H}_p$ by unitary operators.

The calculation of the Weyl infinitesimal operators of the representation, Eq. (6.32), now proceeds along the lines described in (Lo70). A preliminary calculation which simplifies these calculations is as follows: Let $U(t)\in U(4)$ have the property $U(0)=I_4$. Then, from

$$(S_{U(t)}F)(W)=F[\tilde{U}(t)WU(t)], \qquad (6.33)$$

we calculate straightforwardly (being careful to account for the skew–symmetry of W) the result

$$[(dS_{V(t)}/dt)_{t=0}F](W)$$
$$=\mathrm{Tr}\,\{\tilde{W}[dU(t)/dt]_{t=0}(\partial/\partial W)\}F(W), \qquad (6.34)$$

where $\partial/\partial W$ denotes the matrix of the same form as W, but with η_i replaced by $\partial/\partial\eta_i$, and ζ_i replaced by $\partial/\partial\zeta_i$. With this result, we obtain the following expressions for the Weyl generators, $\{E_{ij}\}$:

$$(E_{ij}F)(W)=\mathrm{Tr}\,[\tilde{W}e_{ij}(\partial/\partial W)]F(W), \qquad (6.35)$$

where e_{ij} is the 4×4 matrix unit having 1 in row i and column j and zeros elsewhere, and where $i, j=1, 2, 3, 4$.

Equation (6.35) is a single, concise expression for the Weyl generators of the representation $U\to S_U$ of $U(4)$. To obtain these generators in a more explicit form, we must perform the matrix multiplication indicated in Eq. (6.35) and take the trace. We note that on the space $\mathcal{H}_p$ the Hermitian conjugates to η_i and ζ_i are, respectively, given by

$$\bar{\eta}_i=\partial/\partial\eta_i, \qquad \bar{\zeta}_i=\partial/\partial\zeta_i, \qquad (6.36)$$

i.e.,

$$\partial/\partial W=\bar{W}. \qquad (6.37)$$

The following set of Weyl generators now obtains from Eq. (6.35):

$$(E_{ij}F)(W)=(\eta_i\bar{\eta}_j-\zeta_j\bar{\zeta}_i)F(W) \qquad (6.38)$$

for $i\neq j=1, 2, 3$;

$$(E_{ii}F)(W)=(\eta_i\bar{\eta}_i-\zeta_i\bar{\zeta}_i+\sum_{k=1}^{3}\zeta_k\bar{\zeta}_k)F(W) \qquad (6.39)$$

for $i=1, 2, 3$;

$$(E_{44}F)(W)=(\sum_{k=1}^{3}\eta_k\bar{\eta}_k)F(W); \qquad (6.40)$$

$$(E_{4i}F)(W)=(\eta_j\bar{\zeta}_k-\eta_k\bar{\zeta}_j)F(W), \qquad (6.41)$$

$$(E_{i4}F)(W)=(\zeta_k\bar{\eta}_j-\zeta_j\bar{\eta}_k)F(W), \qquad (6.42)$$

for i, j, k cyclic in 1, 2, 3 in Eqs. (6.41) and (6.42).

The generators E_{ij} necessarily satisfy the following commutation relations from the method of deriving them:

$$[E_{ij}, E_{kl}]=\delta_{jk}E_{il}-\delta_{il}E_{kj}, \qquad (6.43)$$

and

$$(E_{ij})^\dagger=E_{ji}. \qquad (6.44)$$

These properties can, of course, be verified explicitly.

The relations of the Weyl generators $\{E_{ij}: i, j=1, 2, 3, 4\}$ of $U(4)$ to the $SO(6)$ generators $\{\mathcal{L}_{ij}: i, j=1, 2, \cdots, 6\}$ of Eq. (4.17) are obtained by restricting W to Z, i.e., by introducing the constraint $\zeta_j=\eta_j^*=(x_j-ix_{j+3})/\sqrt{2}$. We then have

$$F(W)=f(Z)=f(Z_x)=f(x) \qquad (6.45)$$

for $W=Z\leftrightarrow x$. The relations follow:

$$E_{ij}=\tfrac{1}{2}(i\mathcal{L}_{ij}+i\mathcal{L}_{i+3,j+3}+\mathcal{L}_{i,j+3}+\mathcal{L}_{j,i+3}) \qquad (6.46)$$

for $i\neq j=1, 2, 3$;

$$E_{ii}=\tfrac{1}{2}\mathcal{I}+\mathcal{L}_{i,i+3}-\tfrac{1}{2}\sum_{k=1}^{3}\mathcal{L}_{k,k+3} \qquad (6.47)$$

for $i=1, 2, 3$;

$$E_{44}=\tfrac{1}{2}\mathcal{I}+\tfrac{1}{2}\sum_{k=1}^{3}\mathcal{L}_{k,k+3}; \qquad (6.48)$$

$$E_{4i}=\tfrac{1}{2}(i\mathcal{L}_{jk}+i\mathcal{L}_{k+3,j+3}+\mathcal{L}_{k+3,j}+\mathcal{L}_{k,j+3}), \qquad (6.49)$$

$$E_{i4}=\tfrac{1}{2}(-i\mathcal{L}_{jk}-i\mathcal{L}_{k+3,j+3}+\mathcal{L}_{k+3,j}+\mathcal{L}_{k,j+3}), \qquad (6.50)$$

for i, j, k cyclic in 1, 2, 3. The operator $\mathcal{I}$ appearing in these equations is

$$(\mathcal{I}f)(x)=[\sum_{i=1}^{6}x_i(\partial/\partial x_i)]f(x), \qquad (6.51)$$

i.e., is the homogeneous operator of Euler. It has fixed

[6] Replacing Q of Eq. (6.5) by the Q of Eq. (6.29) does *not* give a homomorphism of $U(4)$ onto $SO(6)$. To obtain such a homomorphism, one must combine the mapping of $U(4)$ onto $SU(4)$ and the homomorphism of Lemma 9.

value p on the space $\mathcal{L}_p$. Note also that

$$2\mathcal{I}=E_{11}+E_{22}+E_{33}+E_{44}, \tag{6.52}$$

i.e., $2\mathcal{I}$ is the first-order Casimir invariant of $U(4)$.

One should note very carefully the structure of Eqs. (6.46)–(6.50). They are actually expressions of the *isomorphism* between the Lie algebra of $SU(4)$ and the Lie algebra of $SO(6)$: *These equations express invertible relations among the fifteen generators of the unimodular group* $SU(4)$, *say*, $\{E_{ij},\ i\neq j=1,2,3;\ E_{ii}-E_{44},\ i=1,2,3\}$, *and the fifteen generators* $\{\mathcal{L}_{ij},\ i<j=1,2,\cdots,6\}$ of $SO(6)$.

The representation and Lie algebra of $U(4)$ which we have obtained on the space $\mathcal{H}_p$ through the sequence of results, Eqs. (6.31)–(6.44), is of interest in its own right in unitary group theory. It is of significance for the 3-particle problem because upon restricting $U(4)$ to $SU(4)$ and upon restricting the domain of definition of the polynomials of $\mathcal{H}_p$ in the manner of Eq. (6.45), we are led precisely to the relation of $SU(4)$ to $SO(6)$ which is relevant to the 3-particle problem.

Let us examine the relation of the spaces $\mathcal{H}_p$ and $\mathcal{L}_p$ more carefully. First consider the space $\mathcal{H}_p$ which is the carrier space for a *reducible* representation of $U(4)$. The irreducible constituents are, however, easily found by determining the set of *highest weight vectors* belonging to $\mathcal{H}_p$. A highest weight vector is one which is annihilated by all the raising generators E_{ij}, $i<j=1,2,3,4$. One easily verifies that each of the vectors

$$F_{p\nu}(\eta,\zeta)=(\eta_1\zeta_1+\eta_2\zeta_2+\eta_3\zeta_3)^\nu\zeta_3^{p-2\nu}/[\nu!(p-\nu)!]^{1/2} \tag{6.53}$$

for $\nu=0,1,2,\cdots p/2$ or $(p-1)/2$ is a normalized highest weight vector belonging to $\mathcal{H}_p$. The weight of the vector $F_{p\nu}$ is the set of eigenvalues of the diagonal generators E_{ii}, $i=1,2,3,4$

$$(E_{11},E_{22},E_{33},E_{44})\rightarrow(p-\nu,p-\nu,\nu,\nu). \tag{6.54}$$

Note that the factor $(\eta_1\zeta_1+\eta_2\zeta_2+\eta_3\zeta_3)$ is an $SU(4)$ invariant.

Highest weights are the partition labels which are used to denote an IR of $U(4)$, and, quite generally, each IR of $U(4)$ is labeled by a partition, $[\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4]$, of ordered integers $\lambda_1\geq\lambda_2\geq\lambda_3\geq\lambda_4$. Thus, if we let $\mathcal{H}_{p,\nu}$ designate the subspace of $\mathcal{H}_p$ which contains the highest weight vector of Eq. (6.53), then $\mathcal{H}_{p,\nu}$ is the carrier space for an IR of $U(4)$ denoted by the labels $[p-\nu,p-\nu,\nu,\nu]$. If we further note that (We31)

$$\dim\mathcal{H}_{p,\nu}=(p-2\nu+1)(p-2\nu+2)^2(p-2\nu+3)/12, \tag{6.55}$$

$$\sum_\nu\dim\mathcal{H}_{p,\nu}=\dim\mathcal{H}_p=\binom{p+5}{5}, \tag{6.56}$$

it follows that the space $\mathcal{H}_p$ decomposes into a direct sum of the subspaces $\mathcal{H}_{p,\nu}$ (since the spaces $\nu=0,1,\cdots$ are perpendicular):

$$\mathcal{H}_p=\sum_\nu\oplus\mathcal{H}_{p,\nu}. \tag{6.57}$$

Correspondingly, $\mathcal{H}_p$ is the carrier space for the representation

$$\sum_\nu\oplus[p-\nu,p-\nu,\nu,\nu] \tag{6.58}$$

of $U(4)$.

To see what all this has to do with the space $\mathcal{L}_p$, we introduce the operator $\mathcal{L}^\dagger$, defined on an arbitrary polynomial $G(\eta,\zeta)$ by

$$(\mathcal{L}^\dagger G)(\eta,\zeta)=(\sum_k\bar{\eta}_k\bar{\zeta}_k)G(\eta,\zeta). \tag{6.59}$$

Clearly, if $\mathcal{L}^\dagger$ does not annihilate a vector of $\mathcal{H}_p$, it carries such a vector into a vector of $\mathcal{H}_{p-2}$. The reason for introducing $\mathcal{L}^\dagger$ is quite transparent: Under the restriction of Eq. (6.45), $\mathcal{L}^\dagger$ simply becomes the Laplacian in Euclidean 6-space

$$(\mathcal{L}^\dagger f)(x)=\tfrac{1}{2}\nabla^2 f(x). \tag{6.60}$$

We can now understand fully the relation of the space $\mathcal{H}_p$ to $\mathcal{L}_p$. We observe that $\mathcal{L}^\dagger$ is an $SU(4)$ invariant, i.e., it commutes with all the generators of $SU(4)$. *Furthermore*, $\mathcal{L}^\dagger$ *annihilates each vector belonging to the space* $\mathcal{H}_{p,0}$, *and it annihilates no vector belonging to the spaces* $\mathcal{H}_{p,\nu}$, $\nu>0$. This statement requires proof.

That $\mathcal{L}^\dagger$ annihilates each vector of $\mathcal{H}_{p,0}$ is evident from the fact that the invariant $\eta_1\zeta_1+\eta_2\zeta_2+\eta_3\zeta_3$ is missing from the vectors of $\mathcal{H}_{p,0}$. To show that $\mathcal{L}^\dagger$ annihilates no vector of $\mathcal{H}_{p,\nu}$ $(\nu>0)$, let $\mathcal{L}$ denote the Hermitian conjugate to $\mathcal{L}^\dagger$ on the polynomial space $\mathcal{H}_0\oplus\mathcal{H}_1\oplus\cdots$. Then $\mathcal{L}$ is just the operator defined by

$$(\mathcal{L}G)(\eta,\zeta)=(\sum_k\eta_k\zeta_k)G(\eta,\zeta), \tag{6.61}$$

which under the restriction, Eq. (6.45), simply multiplies $f(x)$ by $(\sum_i x_i^2)/2$. Now consider the extended version of the operator identity, Eq. (2.36) of Sec. II for $k=6$. It takes the following form on the space $\mathcal{H}_p$:

$$\Lambda^2=-4\mathcal{L}\mathcal{L}^\dagger+\mathcal{I}(\mathcal{I}+4), \tag{6.62}$$

where $\mathcal{I}$ is the $U(4)$ Casimir operator of Eq. (6.52). The significant point to note next is: *The space* $\mathcal{H}_{p,\nu}$, *which is the carrier space for* IR $[p-\nu,p-\nu,\nu,\nu]$ *of* $U(4)$, *is the carrier space for* IR $[p-2\nu,p-2\nu,0,0]$ *of* $SU(4)$. That is, the spaces $\mathcal{H}_{p,\nu}$ and $\mathcal{H}_{p-2\nu,0}$ are carrier spaces for exactly the same IR of $SU(4)$, hence, of $SO(6)$. Since $\mathcal{L}^\dagger$ annihilates $\mathcal{H}_{p-2\nu,0}$, we see from Eq. (6.62) that Λ^2 has eigenvalue $(p-2\nu)(p-2\nu+4)$ on $\mathcal{H}_{p-2\nu,0}$, and since $\mathcal{H}_{p,\nu}$ and $\mathcal{H}_{p-2\nu,0}$ yield the same IR of $SO(6)$, this same eigenvalue of Λ^2 obtains on the space $\mathcal{H}_{p,\nu}$. We now apply Eq. (6.62) to an arbitrary vector of $\mathcal{H}_{p,\nu}$ noting that the eigenvalue of $\mathcal{I}$ is p. The result is

$$\mathcal{L}\mathcal{L}^\dagger F=\nu(p-\nu+2)F \tag{6.63}$$

for each $F \in \mathcal{H}_{p,\nu}$. Thus, we have

$$\mathcal{L}^\dagger F = 0 \tag{6.64}$$

if and only if $F \in \mathcal{H}_{p,0}$.

We have now proved: $\mathcal{H}_{p,0}$ *is the unique subspace of* $\mathcal{H}_p$ *such that*

$$\mathcal{H}_{p,0} \rightarrow \mathcal{L}_p \tag{6.65}$$

under the restriction

$$\eta_j = (x_j + ix_{j+3})/\sqrt{2}, \qquad \zeta_j = (x_j - ix_{j+3})/\sqrt{2}. \tag{6.66}$$

What happens to the subspaces $\mathcal{H}_{p,\nu}$ ($\nu > 0$) under the restriction of Eq. (6.66)? The obvious answer is that a function of $\mathcal{H}_{p,\nu}$ simply becomes a solid harmonic on the 6-sphere of degree $p-2\nu$, multiplied by $(\sum_i x_i^2)^\nu$, and one would remark that it is obvious that these functions do not satisfy Laplace's equation, unless the factor $(\sum_i x_i^2)^\nu$ is removed. Why then all the seemingly elaborate discussion of the full space $\mathcal{H}_p$? The motivation for this derives firstly from the satisfaction of obtaining a better understanding of how the 3-particle state vector space enters as a *substructure* into the more general space of homogeneous polynomials of six arbitrary complex variables, this space providing a natural setting for the group $U(4)$. Secondly, quite aside from this relation to 3-particle states, the space $\mathcal{H}_p$ is the carrier space for quite general representations of $U(3)$, and this result is of interest in itself. Most of this structure would have been passed by had we restricted our discussion to the space $\mathcal{L}_p$.

Finally, let us note where the Lie algebra of the democratic subgroup G fits into the above scheme. This Lie algebra generates the subgroup of $SU(4)$ corresponding to $SU(4)$ matrices of the type, Eq. (6.24). This algebra is identified as that of

$$\{E_{ij},\ i \neq j = 1, 2, 3;\ E_{ii} - E_{44},\ i = 1, 2, 3\}, \tag{6.67}$$

it being sufficient for the proof to note that this is the algebra of traceless Hermitian matrices which generate unimodular unitary matrices of the required form. One readily confirms that the orbital angular momentum algebra is contained as a sub-algebra of the algebra, Eq. (6.67).

D. The Gel'fand Basis of $\mathcal{H}_p$

The general Gel'fand basis vectors of the subspace $\mathcal{H}_{p,\nu}$ can be obtained from the highest weight by the application of known lowering operators (Na65). In terms of the Gel'fand notation, the normalized highest weight vector, Eq. (6.53), is denoted by

$$F\begin{pmatrix} p-\nu & & p-\nu & & \nu & & \nu \\ & p-\nu & & p-\nu & & \nu & \\ & & p-\nu & & p-\nu & & \\ & & & p-\nu & & & \end{pmatrix}. \tag{6.68}$$

The lowering operators commute with the factor $\eta_1\zeta_1+\eta_2\zeta_2+\eta_3\zeta_3$ appearing in the highest weight vector, and the relation between the normalized Gel'fand basis vectors of $\mathcal{H}_{p,\nu}$ and $\mathcal{H}_{p-2\nu,0}$ is therefore given by

$$F\begin{pmatrix} p-\nu & & p-\nu & & \nu & & \nu \\ & p-\nu & & q-\nu & & \nu & \\ & & \alpha-\nu & & \beta-\nu & & \\ & & & \gamma-\nu & & & \end{pmatrix}(W) = \left(\frac{(p-2\nu)!}{\nu!(p-\nu)!}\right)^{1/2}(\eta_1\zeta_1+\eta_2\zeta_2+\eta_3\zeta_3)^\nu$$
$$\times F\begin{pmatrix} p-2\nu & & p-2\nu & & 0 & & 0 \\ & p-2\nu & & q-2\nu & & 0 & \\ & & \alpha-2\nu & & \beta-2\nu & & \\ & & & \gamma-2\nu & & & \end{pmatrix}(W). \tag{6.69}$$

It is therefore sufficient to determine the Gel'fand basis vectors of the spaces $\mathcal{H}_{p,0}$ for arbitrary p. The procedure for generating these vectors from the highest weight vector is given in Appendix 4. We state the result here, and note some of its structural features:

$$F\begin{pmatrix} p & & p & & 0 & & 0 \\ & p & & q & & 0 & \\ & & \alpha & & \beta & & \\ & & & \gamma & & & \end{pmatrix}(W) = D^{\frac{1}{2}(\alpha-\beta)}{}_{\gamma-\frac{1}{2}(\alpha+\beta),\,-q+\frac{1}{2}(\alpha+\beta)}\begin{pmatrix} \eta_1 & \zeta_2 \\ \eta_2 & -\zeta_1 \end{pmatrix} G_{p\alpha\beta}(\eta,\zeta), \tag{6.70}$$

where the definitions of the functions D and G are given fully in Appendix 4 [see Eqs. (IV.14)–(IV.20)].

The factorized form of the basis vectors, Eq. (6.70), presents a rather interesting structure. First of all, the function $G_{p\alpha\beta}$ depends, in fact, only on the variables η_3, ζ_3 and on

$$\det\begin{pmatrix} \eta_1 & \zeta_2 \\ \eta_2 & -\zeta_1 \end{pmatrix} = -(\eta_1\zeta_1+\eta_2\zeta_2). \tag{6.71}$$

From the explicit forms of the generators, one sees immediately that the function $G_{p\alpha\beta}$ is an *invariant* with respect to the two commuting $SU(2)$ subalgebras corresponding to the generators

$$\{E_{12}=J_+,\ E_{21}=J_-,\ \tfrac{1}{2}(E_{11}-E_{22})=J_3\} \qquad (6.72)$$

and

$$\{E_{34}=J_-',\ E_{43}=J_+',\ \tfrac{1}{2}(E_{44}-E_{33})=J_3'\}. \qquad (6.73)$$

This brings us to the second point. *The transformation of the basis* (6.70) *corresponding to the subalgebras* (6.72) *and* (6.73) *are carried entirely by the D functions.* We will give these transformations explicitly. In order to illustrate familiar aspects, it is convenient to introduce new labels as follows:

$$j=\tfrac{1}{2}(\alpha-\beta), \qquad m=\gamma-\tfrac{1}{2}(\alpha+\beta), \qquad m'=-q+\tfrac{1}{2}(\alpha+\beta), \qquad (6.74)$$

where we note that j is either integral or half-integral, and, for each prescribed j, the range of m or m' is $\{j, j-1, \cdots, -j\}$. *The canonical Gel'fand transformations become*:

$$J_+D^j_{mm'}=[(j-m)(j+m+1)]^{1/2}D^j_{m+1,m'},$$

$$J_-D^j_{mm'}=[(j+m)(j-m+1)]^{1/2}D^j_{m-1,m'},$$

$$J_3D^j_{mm'}=mD^j_{mm'}; \qquad (6.75)$$

$$J_+'D^j_{mm'}=[(j-m')(j+m'+1)]^{1/2}D^j_{m,m'+1},$$

$$J_-'D^j_{mm'}=[(j+m')(j-m'+1)]^{1/2}D^j_{m,m'-1},$$

$$J_3'D^j_{mm'}=m'D^j_{mm'}. \qquad (6.76)$$

In order to understand fully the above infinitesimal transformations, let us note briefly their global origins. For this purpose, it is sufficient to consider transformations on the variables η_1, η_2, ζ_1, ζ_2. Define the matrix $\mathfrak{N}$ by

$$\mathfrak{N}=\begin{pmatrix}\eta_1 & \zeta_2\\ \eta_2 & -\zeta_1\end{pmatrix}, \qquad (6.77)$$

and let g denote a polynomial whose values on the variables η_1, η_2, ζ_1, ζ_2 is denoted by $g(\mathfrak{N})$. For each pair U, $U'\in SU(2)$, we define *commuting* operators $\mathcal{O}_U$ and $\mathcal{O}'_{U'}$ by

$$(\mathcal{O}_Ug)(\mathfrak{N})=g(\tilde{U}\mathfrak{N}), \qquad (6.78)$$

$$(\mathcal{O}'_{U'}g)(\mathfrak{N})=g(\mathfrak{N}U'). \qquad (6.79)$$

Each of the correspondences, $U\rightarrow\mathcal{O}_U$ and $U'\rightarrow\mathcal{O}'_{U'}$, is a unitary representation of $SU(2)$ on the space of polynomials (made into a Hilbert space in the standard way). Furthermore, the sets of infinitesimal operators of these representations are given, respectively, by $\{J_+, J_-, J_3\}$ and $\{J_+', J_-', J_3'\}$ [the operators defined by Eqs. (6.72) and (6.73), respectively].

In particular, the operators $\mathcal{O}_U$ and $\mathcal{O}'_{U'}$ transform the functions $D^j_{mm'}$ in the following manner:

$$\mathcal{O}_UD^j_{mm'}=\sum_{\mu} D^j_{\mu m}(U)D^j_{\mu m'}, \qquad (6.80)$$

$$\mathcal{O}'_{U'}D^j_{mm'}=\sum_{\mu'} D^j_{\mu' m'}(U')D^j_{m\mu'}, \qquad (6.81)$$

in which $D^j_{\mu m}(U)$ and $D^j_{\mu' m'}(U')$ denote *precisely* the functions defined by Eq. (4.18) of Appendix 4, now defined on the elements of a 2×2 unitary unimodular matrix. In deriving Eqs. (6.80) and (6.81), we have used several properties of the D functions, namely,

$$D^j(A)D^j(A')=D^j(AA'), \qquad (6.82)$$

$$D^j(\tilde{A})=\tilde{D}^j(A), \qquad (6.83)$$

for A and A' *arbitrary* matrices.

The set of product operators $\{\mathcal{O}_U\mathcal{O}'_{U'}=\mathcal{O}'_{U'}\mathcal{O}_U,\ \forall \text{ pair } U, U'\in SU(2)\}$ is a unitary representation of $SU(2)\times SU(2)$. The functions $D^j_{mm'}$ ($m, m'=j, j-1, \cdots, -j$) are the basis vectors of a carrier space for the IR

$$D^j(U)\otimes D^j(U') \qquad (6.84)$$

of $SU(2)\times SU(2)$.

The full structure of the Gel'fand basis vectors of the space $\mathcal{H}_{p,0}$, given explicitly by Eq. (6.70), has now been revealed: *These basis vectors factorize into an* $SU(2)\times SU(2)$ *invariant part and a part which comprises a basis for the* IR $D^j(U)\otimes D^j(U')$ *of* $SU(2)\times SU(2)$.

It is useful to rewrite these basis vectors in terms of the notation of Eq. (6.74):

$$F\begin{pmatrix} p & & p & & 0 & & 0 \\ & p & & j-m'+\beta & & 0 & \\ & & 2j+\beta & & \beta & & \\ & & & j+m+\beta & & & \end{pmatrix}(W)$$

$$=D^j_{mm'}(\mathfrak{N})G_{p,2j+\beta,\beta}(\eta_3, \zeta_3, \det\mathfrak{N}). \qquad (6.85)$$

Observe that the $SU(2)$ group associated with the transformations (6.72) is *isomorphic* to the $SU(2)$ group which occurs in the Gel'fand chain, Eq. (6.23). However, the $SU(2)$ group associated with the transformations (6.73) entails transformations on the $U(3)$ label $q=j-m'+\beta$ in the Gel'fand pattern.

We have given explicitly the transformations of the basis, Eq. (6.70), for certain of the $U(4)$ generators E_{ij}, i.e., relations (6.75) and (6.76). The transformations corresponding to the remaining $U(4)$ generators can be calculated from these given ones by use of the commutation relations (6.43), the Hermitian conjugate relation, Eq. (6.44), *and the transformations of* E_{23}. For completeness, we note this latter result and the eigenvalues of the diagonal generators:

$$E_{23}F\begin{pmatrix}2j+\beta & \beta\\ j+m+\beta\end{pmatrix}$$
$$=\left[\frac{(j-m+1)(j+m'+1)(p-2j-\beta)(2j+\beta+2)}{(2j+1)(2j+2)}\right]^{1/2}\times F\begin{pmatrix}2j+1+\beta & \beta\\ j+m+\beta\end{pmatrix}$$
$$+\left[\frac{(j+m)(j-m')(p-\beta+1)(\beta+1)}{2j(2j+1)}\right]^{1/2}\times F\begin{pmatrix}2j+\beta & \beta+1\\ j+m+\beta\end{pmatrix}, \quad (6.86)$$

$$(E_{11}, E_{22}, E_{33}, E_{44})\rightarrow(j+m+\beta, j-m+\beta, p-j-m'-\beta, p-j+m'-\beta). \quad (6.87)$$

The top two rows of the Gel'fand pattern have been omitted in Eq. (6.86) for convenience of expression.

Finally, let us also note that the Gel'fand basis vectors, Eq. (6.70), are homogeneous of degree $p-q$ in (η_1, η_2, η_3) and degree q in $(\zeta_1, \zeta_2, \zeta_3)$

$$F\begin{pmatrix}p & p & 0 & 0\\ p & q & 0\\ \alpha & \beta\\ \gamma\end{pmatrix}(\lambda\eta, \mu\zeta)=\lambda^{p-q}\mu^{q}F\begin{pmatrix}p & p & 0 & 0\\ p & q & 0\\ \alpha & \beta\\ \gamma\end{pmatrix}(\eta, \zeta). \quad (6.88)$$

This result implies that the eigenvalues of the operators $\mathcal{G}_1$ and $\mathcal{G}_2$, defined by

$$(\mathcal{G}_1F)(\eta, \zeta)=(\sum_k \eta_k\bar{\eta}_k)F(\eta, \zeta), \quad (6.89)$$

$$(\mathcal{G}_2F)(\eta, \zeta)=(\sum_k \zeta_k\bar{\zeta}_k)F(\eta, \zeta), \quad (6.90)$$

are given, respectively, by

$$(\mathcal{G}_1, \mathcal{G}_2)\rightarrow(p-q, q) \quad (6.91)$$

on the space $\mathcal{H}_{p,0}$. A further implication of this result is that the $U(3)$ Gel'fand invariants can be expressed as simple polynomials in the operators $\mathcal{G}_1$ and $\mathcal{G}_2$, while the $U(4)$ Gel'fand invariants are polynomials in $\mathcal{G}_1+\mathcal{G}_2$, these statements applying, of course, to the particular realization of these operators on the space $\mathcal{H}_{p,0}$. Thus, the two operators, $\mathcal{G}_1$ and $\mathcal{G}_2$, completely specify the $U(4)$ and $U(3)$ transformation properties of the space $\mathcal{H}_{p,0}$.

While the preceding results are of considerable intrinsic significance in the study of the unitary groups, of particular interest here is the relation of these results to the 3-particle problem which obtains through the identification, Eq. (6.66). The Gel'fand basis vectors, Eq. (6.70), then become solid harmonics on the 6-sphere (span the space $\mathcal{L}_p$), and the $SO(6)$ generators relate explicitly to the $SU(4)$ generators through Eqs. (6.46)–(6.50). In particular, let us now see how the orbital angular momentum operators come into the scheme.

Using Eq. (4.22), we identify the orbital angular momentum operators as

$$L_1=\mathfrak{L}_{23}+\mathfrak{L}_{56},$$
$$L_2=\mathfrak{L}_{31}+\mathfrak{L}_{64},$$
$$L_3=\mathfrak{L}_{12}+\mathfrak{L}_{45}. \quad (6.92)$$

Using Eq. (6.46), we identify the orbital angular momentum operators, in turn, with the $U(3)$ generators as follows:

$$L_i=-i(E_{jk}-E_{kj}), \quad (6.93)$$

where i, j, k are cyclic in 1, 2, 3, i.e., the angular momentum algebra is a subalgebra of the Lie algebra of the democratic subgroup—a structural feature which was assured intrinsically by our particular homomorphism of $SU(4)$ onto $SO(6)$.

The Gel'fand basis vectors of $\mathcal{L}_p$ do not, of course, achieve the full goal of characterizing the basis of $\mathcal{L}_p$ through the angular momentum quantum numbers. One must still reduce each of the $U(3)$ IR spaces—the subspaces of $\mathcal{H}_{p,0}$ spanned by the basis vectors having fixed q—into its $SO(3)$ irreducible constituents. It is this part of the problem which is difficult to effect, in general. *It is, however, possible by a simple change of the basis of the Lie algebra of $SU(4)$ to bring L_3 to diagonal form.* This is the subject of the next subsection.

E. Change of Basis

We regard the relative position vectors $\mathbf{x}^1$ and $\mathbf{x}^2$ with components $(x_1x_2x_3)$ and $(x_4x_5x_6)$, respectively, as given, and we do not wish to change their definition. Neither do we wish to alter the explicit $U(4)$ generators of Eqs. (6.38)–(6.42) and the corresponding Gel'fand basis vectors of the preceding section (as expressed in terms of η and ζ). Each of the goals is achieved by changing the *relation* of η, ζ to x. *This entails giving a new homomorphism of $SU(4)$ onto $SO(6)$*. However, in the new homomorphism, we wish also to preserve the structural relation between the $U(3)$ subgroup of $SU(4)$ and the democratic subgroup G of $SO(6)$. These stringent conditions imply that we should consider transformations of the group $O(6)$ of the form $R\rightarrow\tilde{R}_0RR_0$,

$\forall R \in O(6)$, where R_0 is a fixed element of G, for then the democratic subgroup is mapped onto itself.

We simply state and then verify that the following relations between η, ζ, and x satisfy the above criteria in addition to bringing L_3 to diagonal form on the Gel'fand basis of the last section:

$$\begin{aligned}
\eta_1 &= \tfrac{1}{2}[(x_1+ix_4)+i(x_2+ix_5)],\\
\eta_2 &= \tfrac{1}{2}[(x_1+ix_4)-i(x_2+ix_5)],\\
\eta_3 &= (\sqrt{2})^{-1}(x_3+ix_6),\\
\zeta_1 &= \tfrac{1}{2}[(x_1-ix_4)-i(x_2-ix_5)],\\
\zeta_2 &= \tfrac{1}{2}[(x_1-ix_4)+i(x_2-ix_5)],\\
\zeta_3 &= (\sqrt{2})^{-1}(x_3-ix_6).
\end{aligned} \tag{6.94}$$

We re-emphasize the viewpoint that we have only changed the relations of η, ζ to x. We now define the matrix W_x to be the matrix obtained by the explicit substitution of expressions (6.94) into Eq. (6.26). Conversely, if W is any matrix of the form (6.26) with $\zeta_i=\eta_i^*$, we associate with it the point of Euclidean 6-space obtained by inverting Eqs. (6.94). In this way, we obtain a one-to-one correspondence between the set of matrices of the type W (with $\zeta_i=\eta_i^*$) and the points of Euclidean 6-space

$$(x_1\, x_2 \cdots x_6) \leftrightarrow W. \tag{6.95}$$

Observe that this new correspondence is distinct from the one given by Eqs. (6.17) and (6.45). Furthermore, we now let $F(W_x)$ denote the value of F at the point x of the new correspondence

$$F(W_x)=F(x). \tag{6.96}$$

Starting with the generators of Eqs. (6.38)–(6.39), one now verifies directly the following relations for $W=W_x$:

$$\begin{aligned}
E_{11}-E_{22} &= \mathcal{L}_{12}+\mathcal{L}_{45}=L_3,\\
\sqrt{2}(E_{32}-E_{13}) &= (\mathcal{L}_{23}+\mathcal{L}_{56})+i(\mathcal{L}_{31}+\mathcal{L}_{64})=L_+,\\
\sqrt{2}(E_{23}-E_{31}) &= L_-,
\end{aligned} \tag{6.97}$$

where $L_\pm = L_1 \pm iL_2$. Thus, the assertion that L_3 is now diagonal is verified. Indeed, the eigenvalue of L_3 on the Gel'fand basis vector, Eq. (6.70), is just

$$L_3 \rightarrow 2m. \tag{6.98}$$

We still must demonstrate that we have not upset the democratic subgroup in the process of making the identifications, Eqs. (6.94), i.e., *that we have simply established a new homomorphism of* $SU(4)$ *onto* $SO(6)$ *which leaves the democratic subgroup invariant.* To establish this fact, we first note that

$$W_x = Z_{\tilde{R}_0 x} = \tilde{V}_0 Z_x V_0, \tag{6.99}$$

where Z_x is the matrix of Eq. (6.12), R_0 is the proper orthogonal matrix

$$R_0=(\sqrt{2})^{-1}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & \sqrt{2} & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 0\\ -1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}\end{pmatrix}, \tag{6.100}$$

and V_0 is either of the $SU(4)$ matrices which corresponds to R_0 in the *old homomorphism* of Lemma 9:

$$\pm V_0=\exp\left[\tfrac{1}{4}(i\pi)\right]\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0\\ i/\sqrt{2} & -i/\sqrt{2} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -i\end{pmatrix}. \tag{6.101}$$

We further note that

$$R_0 = AM_0A^\dagger, \tag{6.102}$$

where A is the matrix of Eq. (6.4), and M_0 is the matrix

$$M_0=\begin{pmatrix} Q_0 & 0\\ 0 & Q_0^*\end{pmatrix}, \tag{6.103}$$

and where Q_0 is the unitary matrix

$$Q_0=(\sqrt{2})^{-1}\begin{pmatrix} 1 & 1 & 0\\ i & -i & 0\\ 0 & 0 & \sqrt{2}\end{pmatrix}. \tag{6.104}$$

Thus, R_0 belongs to the democratic subgroup G of $SO(6)$.

Next, let V now denote the $SU(4)$ matrix which corresponds not to R, but rather to $\tilde{R}_0RR_0$ in the *old homomorphism*

$$\pm V \rightarrow \tilde{R}_0RR_0. \tag{6.105}$$

Then, from Eq. (6.99), we obtain

$$\tilde{V}W_xV=\tilde{V}Z_{\tilde{R}_0x}V=Z_{(\tilde{R}_0\tilde{R}R_0)\tilde{R}_0x}=Z_{\tilde{R}_0\tilde{R}x}=W_{\tilde{R}x},$$

that is,

$$W_{\tilde{R}x}=\tilde{V}W_xV. \tag{6.106}$$

Relation (6.106) *is a new homomorphism of* $SU(4)$ *onto* $SO(6)$ *which maps the matrix* V *directly to the matrix* R:

$$V \rightarrow R. \tag{6.107}$$

Indeed, the *new rule* for relating $SU(4)$ matrices and $SO(6)$ matrices is now given by

$$R=A(M_0QM_0^\dagger)A^\dagger, \tag{6.108}$$

where A is the matrix of Eq. (6.4). Here M_0 is the matrix defined by Eqs. (6.103) and (6.104), and, finally, *Q is defined in terms of the partition matrices of V just as in the old homomorphism*, i.e., *by Eqs.* (6.5)–(6.8). Clearly,

$$V \to R \text{ (new homomorphism)} \tag{6.109}$$

implies

$$V_0^\dagger V V_0 \to R \text{ (old homomorphism)}. \tag{6.110}$$

Our new homomorphism, expressed equivalently in either the form (6.108) *or the form* (6.106), *now leads directly to a set of relations between generators of $SU(4)$ and $SO(6)$ such that* Eqs. (6.97) *obtain.* The complete set of relations, analogous to Eqs. (6.46)–(6.50), can, of course, be obtained by making the identifications, Eqs. (6.94), in Eqs. (6.38)–(6.42). [Note that Eqs. (6.19)–(6.22) remain valid upon replacing Z_x by W_x.] We regard the new homomorphism as changing the matrices of $SU(4)$ which correspond to specified matrices R. For example, the subgroup $\{I_2 \otimes R_3 : R_3 \in O(3)\}$ is still generated by the orbital angular momentum operators (L_1, L_2, L_3), just as before, but the new homomorphism associates these generators with the subalgebra of $SU(4)$ given by Eqs. (6.97). Similarly, the democratic subgroup $G \subset SO(6)$ is still the set of matrices defined by Eqs. (5.11)–(5.13), it being clear that the new homomorphism merely effects a unitary similarity transformation on the submatrix U of M.

Summary. When we identify the variables η, ζ in the Gel'fand basis vectors, Eq. (6.70), with x through Eqs. (6.94), we obtain a basis of the space $\mathcal{L}_p$, the basis vectors being enumerated by the $U(3)$ Gel'fand patterns just as before, but now the generator $E_{11}-E_{22}$ has the significance of being the third component, L_3, of the orbital angular momentum. The two matrices of $SU(4)$ which correspond to $O(P_1) = \Gamma(P_1) \otimes I_3$, $P_1 = (123)$, in the new homomorphism, Eq. (6.108), are still given by Eq. (6.11). $\pm V_1$ commutes with the group $\mathcal{G}$ of Eq. (6.9), this group still being the subgroup of $SU(4)$ which corresponds to the democratic subgroup G of $O(6)$. Furthermore, on $\mathcal{L}_p$ we have the operator identity $T_{O(P_1)} = S_{\pm V_1} = S_{V_1}$, and the operator S_{V_1} commutes with the set of generators of Eq. (6.67). *Accordingly, S_{V_1} must be diagonal on each Gel'fand basis vector*, Eq. (6.70), *which has fixed p and q.* We defer giving the explicit diagonal form of S_{V_1} until after investigating the more general question of the discrete symmetry properties of the Gel'fand basis.

F. Certain Discrete Symmetry Properties of the Gel'fand Basis of $\mathcal{H}_{p,0}$

There are three very simple symmetries of the D functions defined by Eq. (18) of Appendix 4. These are the symmetries associated with the following operations on the argument matrix A: (a) interchange the rows of A; (b) interchange the columns of A; and (c) transpose A. (For our purposes, we define a *discrete symmetry operator* to be an operator which carries a D function into a *single* new D function, multiplied at most by a numerical factor.)

For simplicity let us write A as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{6.111}$$

and let $F(A)$ denote an arbitrary polynomial defined on the variables a, b, c, d. We define the operators $\mathfrak{R}$, $\mathfrak{C}$, and $\mathfrak{I}$ by the following rules, respectively:

$$(\mathfrak{R}F)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = F\begin{pmatrix} c & d \\ a & b \end{pmatrix}, \tag{6.112}$$

$$(\mathfrak{C}F)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = F\begin{pmatrix} b & a \\ d & c \end{pmatrix}, \tag{6.113}$$

$$(\mathfrak{I}F)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = F\begin{pmatrix} a & c \\ b & d \end{pmatrix}. \tag{6.114}$$

One should note very carefully the product rule, e.g.,

$$(\mathfrak{C}\mathfrak{I}F)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\mathfrak{I}F)\begin{pmatrix} b & a \\ d & c \end{pmatrix} = F\begin{pmatrix} b & d \\ a & c \end{pmatrix}. \tag{6.115}$$

The operators $\mathfrak{R}$, $\mathfrak{C}$, and $\mathfrak{I}$ generate a group of order eight, the elements of this group being

$$H = \{1, \mathfrak{R}, \mathfrak{C}, \mathfrak{I}, \mathfrak{R}\mathfrak{C} = \mathfrak{C}\mathfrak{R}, \mathfrak{I}\mathfrak{R} = \mathfrak{C}\mathfrak{I}, \mathfrak{I}\mathfrak{C} = \mathfrak{R}\mathfrak{I}, \mathfrak{R}\mathfrak{C}\mathfrak{I}\}, \tag{6.116}$$

where we note that

$$\mathfrak{R}^2 = \mathfrak{C}^2 = \mathfrak{I}^2 = 1,$$
$$\mathfrak{I}\mathfrak{R}\mathfrak{C} = \mathfrak{I}\mathfrak{C}\mathfrak{R} = \mathfrak{R}\mathfrak{C}\mathfrak{I} = \mathfrak{C}\mathfrak{R}\mathfrak{I},$$
$$\mathfrak{R}\mathfrak{I}\mathfrak{C} = \mathfrak{C}\mathfrak{I}\mathfrak{R} = \mathfrak{I}. \tag{6.117}$$

[This group is clearly isomorphic to a subgroup of S_4; a realization of the isomorphism is $\mathfrak{R} \to (13)(24)$, $\mathfrak{C} \to (12)(34)$, $\mathfrak{I} \to (1)(4)(23)$.]

The action of the above operators on the D functions is easily determined by direct use of the definitions and the explicit form of the D functions

$$\mathfrak{R}D^j_{m,m'} = D^j_{-m,m'}, \tag{6.118}$$

$$\mathfrak{C}D^j_{m,m'} = D^j_{m,-m'}, \tag{6.119}$$

$$\mathfrak{I}D^j_{m,m'} = D^j_{m',m}. \tag{6.120}$$

On the basic D functions, the group of operators, H, induces the eight transformations on the quantum labels (m, m') corresponding to any number of sign changes and transposition.[7]

[7] The transposition symmetry (6.120) is closely related to the Regge symmetry of the $SU(2)$ Wigner coefficients [see Bincer (Bi70)].

There is still another operation of considerable importance—complex conjugation. However, we do not necessarily mean complex conjugation in the literal sense, but rather an operation on the space of polynomials which in some sense has the properties of complex conjugation, and, in particular cases, may even be complex conjugation. Looking back at either of the restrictions of the variables, Eq. (6.66) or Eq. (6.94), we are led to associate an operation, which we call *conjugation*, with the interchange $(\eta_1, \eta_2)\leftrightarrow(\zeta_1, \zeta_2)$. Furthermore, comparing the form of the matrix A, Eq. (6.111), with the matrix $\mathfrak{N}$, Eq. (6.77), it is seen that an appropriate definition of a conjugation operator $\mathcal{K}$ which will be relevant to the 3-particle problem is

$$(\mathcal{K}F)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = F\begin{pmatrix} -d & c \\ b & -a \end{pmatrix}. \quad (6.121)$$

Again to make clear the product rule, we give another example

$$(\mathfrak{I}\mathcal{K}F)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\mathcal{K}F)\begin{pmatrix} a & c \\ b & d \end{pmatrix} = F\begin{pmatrix} -d & b \\ c & -a \end{pmatrix}. \quad (6.122)$$

In particular, the function $\mathcal{K}D^j_{m,m'}$ is easily verified to be

$$\mathcal{K}D^j_{m,m'} = (-1)^{m+m'}D^j_{-m,-m'}. \quad (6.123)$$

Remark. From the point of view of the *representations* of $SU(2)$, we can consider the restriction of the matrix A to $U\in SU(2)$. Then, Eq. (6.123) becomes

$$(\mathcal{K}D^j_{m,m'})(U) = (-1)^{2j}[D^j_{m,m'}(U)]^*, \quad (6.124)$$

and $\mathcal{K}$ is, except for an over-all phase, the operation of complex conjugating an IR of $SU(2)$ (Wi59). However, in this work, the D functions arise as pieces of *state vectors* and are defined on arbitrary variables $(\eta_1, \eta_2, \zeta_1, \zeta_2)$. The definition of $\mathcal{K}$ which we have given is more appropriate to this latter situation, and, in fact, is a part of the operator (we still must consider the variables ζ_3, η_3) which complex conjugates a solid harmonic on the six sphere.

We can now adjoin the operator $\mathcal{K}$ to the group H of Eq. (6.116). The generators of the new group are now

$$\mathfrak{R}, \mathfrak{C}, \mathfrak{I}, \mathcal{K}, \quad (6.125)$$

and a set of defining relations of the group is as follows:

$$\mathfrak{R}^2 = \mathfrak{C}^2 = \mathfrak{I}^2 = \mathcal{K}^2 = 1,$$
$$\mathfrak{R}\mathfrak{C} = \mathfrak{C}\mathfrak{R}, \quad \mathfrak{R}\mathfrak{I} = \mathfrak{I}\mathfrak{C}, \quad \mathfrak{C}\mathfrak{I} = \mathfrak{I}\mathfrak{R}, \quad \mathcal{K}\mathfrak{I} = \mathfrak{I}\mathcal{K},$$
$$\mathcal{K}\mathfrak{R}\mathfrak{C} = \mathfrak{R}\mathfrak{C}\mathcal{K}, \quad (\mathfrak{R}\mathcal{K})^2\mathfrak{R} = \mathfrak{R}(\mathfrak{R}\mathcal{K})^2,$$
$$(\mathfrak{R}\mathcal{K})^2\mathfrak{C} = \mathfrak{C}(\mathfrak{R}\mathcal{K})^2, \quad (\mathfrak{R}\mathcal{K})^2\mathfrak{I} = \mathfrak{I}(\mathfrak{R}\mathcal{K})^2,$$
$$(\mathfrak{R}\mathcal{K})^2\mathcal{K} = \mathcal{K}(\mathfrak{R}\mathcal{K})^2. \quad (6.126)$$

A second set of defining relations is obtained by interchanging $\mathfrak{R}$ and $\mathfrak{C}$ in the above relations.

The generators (6.125) satisfying relations (6.126) generate a group of order *thirty-two*. The elements are conveniently enumerated as follows:

$$K = \{H, H\mathcal{K}, H\mathcal{K}\mathfrak{R}, H\mathcal{K}\mathfrak{R}\mathcal{K}\}. \quad (6.127)$$

This enumeration clearly demonstrates that on the basic D functions, we obtain a representation of the group K which has the structure given by the set equalities[8]:

$$\{H\mathcal{K}D^j_{mm'}\} = (-1)^{m'+m}\{HD^j_{mm'}\},$$
$$\{H\mathcal{K}\mathfrak{R}D^j_{mm'}\} = (-1)^{m'-m}\{HD^j_{mm'}\},$$
$$\{H\mathcal{K}\mathfrak{R}\mathcal{K}D^j_{mm'}\} = (-1)^{2j}\{HD^j_{mm'}\}. \quad (6.128)$$

Observe that

$$(-1)^{2m} = (-1)^{2m'} = (-1)^{2j}, \quad (6.129)$$

since m and m' are integral or half-integral with j. Thus, the transformations of the basic D functions corresponding to the group K differ only by phases from those of the group H. Nonetheless, *the group K is the smallest group which contains the elementary transformations $\mathfrak{R}$, $\mathfrak{C}$, and $\mathfrak{I}$ together with the conjugation operator $\mathcal{K}$.*

Relations (6.126) can be used to write some of the operators in K, as expressed by Eq. (6.127), in simpler forms. Indeed, we find it useful to write out all thirty-two elements of K:

$$K' = \{1, \mathfrak{I}, \mathcal{K}, \mathfrak{R}\mathfrak{C}, \mathfrak{I}\mathcal{K}, \mathfrak{R}\mathfrak{C}\mathfrak{I}, \mathfrak{R}\mathfrak{C}\mathcal{K}, \mathfrak{R}\mathcal{K}\mathfrak{R}, \mathfrak{R}\mathcal{K}\mathfrak{C}, \mathfrak{R}\mathfrak{C}\mathfrak{I}\mathcal{K},$$
$$\mathfrak{R}\mathfrak{I}\mathcal{K}\mathfrak{R}, \mathfrak{C}\mathfrak{I}\mathcal{K}\mathfrak{R}, (\mathfrak{R}\mathcal{K})^2, \mathfrak{C}\mathcal{K}\mathfrak{R}\mathcal{K}, \mathfrak{I}(\mathfrak{R}\mathcal{K})^2, (\mathfrak{R}\mathcal{K})\mathfrak{I}(\mathfrak{R}\mathcal{K})\}, \quad (6.130)$$

$$K-K' = \{\mathfrak{R}, \mathfrak{C}, \mathfrak{R}\mathfrak{I}, \mathfrak{C}\mathfrak{I}, \mathfrak{R}\mathcal{K}, \mathcal{K}\mathfrak{R}, \mathfrak{C}\mathcal{K}, \mathcal{K}\mathfrak{C}, \mathfrak{R}\mathfrak{I}\mathcal{K}, \mathfrak{C}\mathfrak{I}\mathcal{K},$$
$$\mathfrak{I}\mathcal{K}\mathfrak{R}, \mathcal{K}\mathfrak{R}\mathfrak{I}, \mathcal{K}\mathfrak{R}\mathcal{K}, \mathcal{K}\mathfrak{C}\mathcal{K}, \mathfrak{I}\mathcal{K}\mathfrak{R}\mathcal{K}, \mathcal{K}\mathfrak{R}\mathfrak{I}\mathcal{K}\}. \quad (6.131)$$

Then we have

$$K = K'\cup(K-K'). \quad (6.132)$$

But now observe that K' is an *invariant subgroup* of K. The proof is very simple and follows immediately from the fact that a transformation in K' does not change the sign of the determinant, $\det A = ad-bc$, while a transformation in $K-K'$ does change the sign of $\det A$. (This fact will be of considerable importance later.)

There are additional discrete symmetries (in the limited sense of our definition) of the D functions which are identified with their *general homogeneity properties.* Thus, for arbitrary (nonzero) complex numbers λ, μ, and ν, we define the operator $\mathcal{P}_{\lambda\mu\nu}$ on the space of polynomials by

$$(\mathcal{P}_{\lambda\mu\nu}F)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = F\begin{pmatrix} \lambda a & \mu b \\ \lambda\nu c/\mu & \nu d \end{pmatrix}. \quad (6.133)$$

---

[8] These symmetries are closely related to the classical symmetries of the hypergeometric functions [see Whittaker and Watson (Wh46)].

In particular, on the D functions, the operator $\mathcal{P}_{\lambda\mu\nu}$ is *diagonal*

$$\mathcal{P}_{\lambda\mu\nu}D^j_{mm'}=\lambda^{j+m'}\mu^{m-m'}\nu^{j-m}D^j_{mm'}. \qquad (6.134)$$

Furthermore, one easily verifies the following operator identities:

$$\mathcal{P}_{-1,1,-1}=\mathcal{R}\mathcal{C}\mathcal{K},$$
$$\mathcal{P}_{1,-1,1}=\mathcal{R}\mathcal{K}\mathcal{C},$$
$$\mathcal{P}_{-1,-1,-1}=(\mathcal{R}\mathcal{K})^2. \qquad (6.135)$$

However, none of the other $\mathcal{P}$ operators (except the identity) is in the group K. (This is easily seen because the only sign changing generator of K is $\mathcal{K}$, and it effects sign changes only on diagonally opposite elements.)

One could, of course, adjoin certain of the $\mathcal{P}$ operators to K, but there appears to be little motivation for doing so.

The preceding properties of the D functions lead to a rich structure of discrete symmetries of the Gel'fand basis of $\mathcal{H}_{p,0}$. To see how this comes about, we must still consider discrete transformations of the variable pair (η_3, ζ_3). To this end, we introduce the group having eight elements as follows:

$$\{\pm\alpha_0, \pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}, \qquad (6.136)$$

where α_0 is the 2×2 identity matrix and

$$\alpha_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \alpha_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad \alpha_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \qquad (6.137)$$

We then define operators $\mathcal{S}_{\pm\alpha_i}$ by the rule

$$(\mathcal{S}_{\pm\alpha_i}G)(\eta_3, \zeta_3)=G(\eta_3', \zeta_3'), \qquad (6.138)$$

where

$$\begin{pmatrix}\eta_3'\\ \zeta_3'\end{pmatrix}=\pm\tilde{\alpha}_i\begin{pmatrix}\eta_3\\ \zeta_3\end{pmatrix}. \qquad (6.139)$$

The group of operators,

$$S=\{\mathcal{S}_{\pm\alpha_i}: i=0, 1, 2, 3\}, \qquad (6.140)$$

then comprises a representation of the group (6.136) on the space of polynomial functions of two variables.

The idea now is to consider the direct product of the groups K and S, where the variables of the D functions are now identified to be $a=\eta_1$, $c=\eta_2$, $b=\zeta_2$, $d=-\zeta_1$. (Observe that the two operator groups commute, since they effect transformations on different variables.) However, the Gel'fand basis vectors of Eq. (6.85) are not quite product functions of the type DG, where G is defined only on the variables η_3, ζ_3, since $\det\mathfrak{N}$ also enters as a variable. This implies that the simple discrete symmetries of the Gel'fand basis vectors do not involve the full direct product group, but rather a subgroup. Indeed, as already noted, the operators $\mathcal{R}$ and $\mathcal{C}$ induce a sign change of $\det\mathfrak{N}$, whereas $\mathfrak{J}$ and $\mathcal{K}$ leave $\det\mathfrak{N}$ invariant. Accounting for this feature and the induced shifts of the labels in the Gel'fand patterns, we are led to introduce the subgroup of the direct product group which is generated by the following four elements:

$$\mathcal{R}\mathcal{S}_{\alpha_2},\ \mathcal{C}\mathcal{S}_{\alpha_2},\ \mathfrak{J}\mathcal{S}_{\alpha_1},\ \mathcal{K}\mathcal{S}_{\alpha_1}. \qquad (6.141)$$

The group D generated by the generators (6.141) contains 128 *elements* as follows:

$$D'=\{K'(\mathcal{S}_{\pm\alpha_0}, \mathcal{S}_{\pm\alpha_1})\}, \qquad (6.142)$$

$$D-D'=\{(K-K')(\mathcal{S}_{\pm\alpha_2}, \mathcal{S}_{\pm\alpha_3})\}, \qquad (6.143)$$

where the notation means that each of the four elements in (,) multiplies each of the elements in K' or $K-K'$, as indicated. Observe that the elements $(\mathcal{S}_{\pm\alpha_0}, \mathcal{S}_{\pm\alpha_1})$ comprise an invariant subgroup of S, so that the elements (6.142) comprise an invariant subgroup D' of the group D,

$$D=D'\cup(D-D'). \qquad (6.144)$$

The finite group D is the only group which we can construct from K and S such that the elements of D carry one Gel'fand state vector into a phase times another. We now list explicitly the transformations on the Gel'fand basis corresponding to the generators (6.141) [omitting the fixed top row in Eq. (6.70)]:

$$\mathcal{R}\mathcal{S}_{\alpha_2}F\begin{pmatrix}p\ q\ 0\\ \alpha\ \beta\\ \gamma\end{pmatrix}=(-1)^{\beta}F\begin{pmatrix}p \quad p+q-\alpha-\beta \quad 0\\ p-\beta \qquad p-\alpha\\ p-\gamma\end{pmatrix},$$

$$\mathcal{C}\mathcal{S}_{\alpha_2}F\begin{pmatrix}p\ q\ 0\\ \alpha\ \beta\\ \gamma\end{pmatrix}=(-1)^{\beta}F\begin{pmatrix}p \quad p-q \quad 0\\ p-\beta \qquad p-\alpha\\ p+\gamma-\alpha-\beta\end{pmatrix},$$

$$\mathfrak{J}\mathcal{S}_{\alpha_1}F\begin{pmatrix}p\ q\ 0\\ \alpha\ \beta\\ \gamma\end{pmatrix}=(-1)^{p+\alpha+\beta}F\begin{pmatrix}p \quad p-\gamma \quad 0\\ p-\beta \qquad p-\alpha\\ p-q\end{pmatrix},$$

$$\mathcal{K}\mathcal{S}_{\alpha_1}F\begin{pmatrix} p \; q \; 0 \\ \alpha \; \beta \\ \gamma \end{pmatrix} = (-1)^{p+q+\alpha+\beta+\gamma}F\begin{pmatrix} p \quad p-q \quad 0 \\ p-\beta \quad p-\alpha \\ p-\gamma \end{pmatrix}. \quad (6.145)$$

Note that the operator $\mathcal{K}\mathcal{S}_{\alpha_1}$ is

$$(\mathcal{K}\mathcal{S}_{\alpha_1}F)(\eta,\zeta)=F(\zeta,\eta), \quad (6.146)$$

and is just complex conjugation on the 6-sphere.

There are *sixteen* distinct final Gel'fand patterns associated with the transformations of the group D, i.e., each of the final Gel'fand state vectors is labeled by one of these sixteen patterns and a phase. Let us note explicitly these patterns by giving the final values $(\alpha'\beta';q'\gamma')$ which can occur in

$$F\begin{pmatrix} p \; q' \; 0 \\ \alpha' \; \beta' \\ \gamma' \end{pmatrix}.$$

For $(\alpha'\beta')=(\alpha\beta)$, we have the following set of values for the labels $(q'\gamma')$:

$$(q,\gamma),\ (\gamma,q),\ (q,\alpha+\beta-\gamma),\ (\alpha+\beta-\gamma,q),\ (\alpha+\beta-q,\gamma),$$
$$(\gamma,\alpha+\beta-q),\ (\alpha+\beta-q,\alpha+\beta-\gamma),\ (\alpha+\beta-\gamma,\alpha+\beta-q). \quad (6.147)$$

For $(\alpha'\beta')=(p-\beta,p-\alpha)$, we have the following set of values for the labels $(q'\gamma')$:

$$(p-q,p-\gamma),\ (p-\gamma,p-q),\ (p-q,p+\gamma-\alpha-\beta),$$
$$(p+\gamma-\alpha-\beta,p-q),\ (p+q-\alpha-\beta,p-\gamma),$$
$$(p-\gamma,p+q-\alpha-\beta),\ (p+q-\alpha-\beta,p+\gamma-\alpha-\beta),$$
$$(p+\gamma-\alpha-\beta,p+q-\alpha-\beta). \quad (6.148)$$

There are additional discrete symmetries of the Gel'fand basis which are identified with their homogeneity properties. Thus, for arbitrary (nonzero) complex numbers λ, μ, ν and ρ, we define the operator $\mathcal{P}_{\lambda\mu\nu\rho}$ on the space of polynomials by

$$(\mathcal{P}_{\lambda\mu\nu\rho}F)(\eta_1\eta_2\eta_3\zeta_1\zeta_2\zeta_3)=F(\lambda\eta_1,\ \lambda\nu\eta_2/\mu,\ \rho\eta_3,\ \nu\zeta_1,\ \mu\zeta_2,\ \lambda\nu\zeta_3/\rho). \quad (6.149)$$

$\mathcal{P}_{\lambda\mu\nu\rho}$ is diagonal on the Gel'fand basis and has the following eigenvalue:

$$(\mathcal{P}_{\lambda\mu\nu\rho}F)\begin{pmatrix} p \; q \; 0 \\ \alpha \; \beta \\ \gamma \end{pmatrix} = \lambda^{\alpha+\beta-q}\mu^{q+\gamma-\alpha-\beta}\nu^{\alpha+\beta-\gamma}\rho^{p-\alpha-\beta}F\begin{pmatrix} p \; q \; 0 \\ \alpha \; \beta \\ \gamma \end{pmatrix}. \quad (6.150)$$

It is interesting to observe that each of the transformations of D as well as the general operator $\mathcal{P}_{\lambda\mu\nu\rho}$ carries the function

$$f(\eta,\zeta)\equiv(\eta_1\zeta_1+\eta_2\zeta_2)/\eta_3\zeta_3 \quad (6.151)$$

into itself.

To each discrete symmetry of a Gel'fand basis vector of $\mathcal{H}_{p,0}$, there corresponds a discrete symmetry of the Gel'fand basis of $\mathcal{L}_p$ which obtains through, say, the restriction of Eqs. (6.94). In particular, consider the operator

$$\mathcal{P}_{(123)}\equiv\mathcal{P}_{\epsilon,\epsilon^*,\epsilon^*,\epsilon}, \quad (6.152)$$

where ϵ is the complex number occurring in Eq. (5.19). (Note that $\epsilon^*=\epsilon^2=\epsilon^{-1}$.) Then

$$\mathcal{P}_{(123)}F\begin{pmatrix} p \; q \; 0 \\ \alpha \; \beta \\ \gamma \end{pmatrix} = \epsilon^{p+q}F\begin{pmatrix} p \; q \; 0 \\ \alpha \; \beta \\ \gamma \end{pmatrix}, \quad (6.153)$$

and under the restriction, Eqs. (6.94), we have

$$\mathcal{P}_{(123)}\rightarrow S_{V_1}=T_{O(P_1)} \quad (6.154)$$

for $P_1=(123)$. Hence, $T_{O(P_1)}$ is diagonal on the space $\mathcal{L}_p$ as expected from its role in defining the democratic subgroup.

In order to discuss the properties of the Gel'fand basis vectors of $\mathcal{L}_p$ under the remaining permutations of S_3 (the permutation group associated with the identical particles), it is convenient to introduce an explicit set of relative coordinates:

$$\mathbf{x}^1=(\mathbf{r}^1-\mathbf{r}^2)/\sqrt{2},$$
$$\mathbf{x}^2=(\mathbf{r}^1+\mathbf{r}^2-2\mathbf{r}^3)/(6)^{1/2}. \quad (6.155)$$

[Recall that $\mathbf{x}^1$ has components $(x_1x_2x_3)$, while $\mathbf{x}^2$ has components $(x_4x_5x_6)$.] Then we have

$$(123):[\mathbf{x}^1\mathbf{x}^2]\rightarrow[\mathbf{x}^1\mathbf{x}^2]\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}, \quad (6.156)$$

$$(12)(3):[\mathbf{x}^1\mathbf{x}^2]\rightarrow[\mathbf{x}^1\mathbf{x}^2]\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (6.157)$$

so that

$$\Gamma(P)=\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}, \qquad P=(123), \quad (6.158)$$

$$\Gamma(P)=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P=(12)(3). \quad (6.159)$$

Since S_3 is generated by (123) and (12)(3), it is sufficient to consider only these two elements.

Note that ϵ in Eqs. (6.152) and (6.153) now becomes the specific number [cf. Eq. (5.19)]

$$\epsilon=\exp(4\pi i/3). \quad (6.160)$$

The transformation (6.157) corresponds to the transformation on the variables (η, ζ) of Eq. (6.94) as follows:

$$(12)(3): (\eta, \zeta)\rightarrow(-\zeta_2, -\zeta_1, -\zeta_3, -\eta_2, -\eta_1, -\eta_3). \quad (6.161)$$

Correspondingly, the operator

$$\mathcal{P}_{(12)(3)}\equiv\mathcal{P}_{-1,-1,1,-1}\mathcal{C}\mathcal{S}_{\alpha_2} \quad (6.162)$$

is the operator on $\mathcal{H}_{p,0}$ which represents the transformation (6.161). (This operator is not in the group D.) Under $\mathcal{P}_{(12)(3)}$, a Gel'fand basis vector undergoes the transformation as follows:

$$(\mathcal{P}_{(12)(3)}F)\begin{pmatrix} p \; q \; 0 \\ \alpha \; \beta \\ \gamma \end{pmatrix} = (-1)^{\beta+\gamma}F\begin{pmatrix} p & & p-q & & 0 \\ & p-\beta & & p-\alpha & \\ & & p+\gamma-\alpha-\beta & & \end{pmatrix}. \quad (6.163)$$

Again, since $\mathcal{P}_{(12)(3)}$ does not commute with the Lie algebra [Eq. (6.67)] of the democratic subgroup, we find, as expected, that the action of $\mathcal{P}_{(12)(3)}$ on a basis vector carrying $SU(3)$ IR labels $[p\, q\, 0]$ is to carry the vector out of the subspace (the new vector, of course, must belong to $\mathcal{H}_{p,0}$). However, $\mathcal{P}_{(12)(3)}$ does commute with the orbital angular momentum subalgebra (as do all the permutation operators), and the angular momentum content of a state vector must be preserved under $\mathcal{P}_{(12)(3)}$. Specifically, we see that each of the initial and final states in Eq. (6.163) has $L_3\rightarrow 2\gamma-\alpha-\beta$.

There is an important exception to the preceding observations. *For $p=2k$ (k integral) and $q=k$, the IR subspace $[2k\; k\; 0]$ of $SU(3)$ is also a representation space for the group S_3.* These are the so-called self-conjugate IR spaces of $SU(3)$—the IR spaces which are mapped onto themselves under the action of the $SU(3)$ conjugation operator, *here identified as* $\mathcal{K}\mathcal{S}_{\alpha_1}$. This subspace of $\mathcal{H}_{2k,0}$ enjoys the additional property that the basis vectors can be further classified (by taking appropriate linear combinations) as states having definite permutation symmetry with respect to the full group S_3. This property is unique to the self-conjugate subspaces, i.e., it is impossible to obtain a complete basis of $\mathcal{H}_{p,0}$ with sharp $SU(3)$ IR labels *and* definite permutational symmetry with respect to S_3; only the self-conjugate subspace of $\mathcal{H}_{2k,0}$ enjoys this property.

Summary. We have obtained a large set of discrete symmetries of a specific set of $SU(3)$ state vectors, some of which have been identified with a finite symmetry group D. There would seem to be little additional information obtained by adjoining to D further operators of the diagonal type, since the important transformations on the state vectors are those associated with the set of sixteen new state vector labels of Eqs. (6.147) and (6.148). We believe this is the first time that a reasonably complete discussion of the symmetries of $SU(3)$ state vectors has been given. These symmetries have been discussed because of their relevance to $SU(3)$ group theory, in general. As a particular application, we obtain a large set of symmetries of the solid harmonics on the 6-sphere, and have indicated how the symmetric group S_3 fits into the scheme.

G. The Reduction Problem SO(3)⊂SU(3)

The problem of reducing an IR space, specified by $[p\, q\, 0]$, of $SU(3)$ into those subspaces which are IR spaces for $SO(3)$ has not been solved in a completely satisfactory manner.

The difficulty is that no clear and general ***structural principle*** has been given which suggests a satisfactory resolution of the problem associated with the multiple occurrence of an IR of $SO(3)$ in $SU(3)$. The final solution to this problem may, indeed, entail the use of an additional invariant (Ba60, 61; Ra62) which distinguishes those subspaces which carry the same $SO(3)$ representation, but one surely must take into account the group of automorphisms of $SU(3)$ (Bi69).

We wish to examine some of the aspects of this problem as they relate specifically to the Gel'fand basis of $\mathcal{H}_{p,0}$. In particular, we suggest a new approach to the multiplicity problem—one which has already had success in a somewhat different, but related context (Lo70a).

The recognition that the orbital angular momentum operators [generators of $SO(3)$] can be related to the $U(3)$ generators through Eqs. (6.97) and the restriction (6.94) greatly simplifies the *counting* in the multiplicity problem. (We will use the notation L_3, $L_\pm$ for the operators on the left-hand side of Eqs. (6.97) even though we do *not* restrict η, ζ to x, this restriction being irrelevant to the structure of the problem.) On a Gel'fand basis vector, the eigenvalue spectrum of L_3 is then just the set of numbers $\{2\gamma-\alpha-\beta\}$, where α, β, and γ run over all values consistent with their positions in an $SU(3)$ Gel'fand pattern specified by $[p\, q\, 0]$.

One simply examines this set of numbers to deduce the tables of allowed angular momenta L (Lo65, Ra49).

One then finds that, for *prescribed* $[p\ q\ 0]$, L always has a value in the interval

$$0 \leq L \leq p, \qquad (6.164)$$

and that it occurs with some multiplicity $\mathfrak{M}(L)$, *which may be zero.*

We do not duplicate the tables for L, since $\mathfrak{M}(L)$ can be given in closed form, thus obviating the need for tables.

Let L be any selected value in the interval (6.164) and consider the $SU(4)$ Gel'fand state vector (we will always omit the top row of labels)

$$F\begin{pmatrix} p & & q & & 0 \\ & \alpha & & \alpha-L-2\sigma & \\ & & \alpha-\sigma & & \end{pmatrix}. \qquad (6.165)$$

This state vector clearly has L_3 eigenvalue L for all α and σ consistent with the allowed entries in the Gel'fand pattern, i.e., for

$$p \geq \alpha \geq q,$$
$$q+L \geq \alpha - 2\sigma \geq L. \qquad (6.166)$$

Furthermore, each Gel'fand basis vector having $L_3 \to L$ can be written in the form (6.165) [if there exist no α and σ satisfying Eqs. (6.166) for a prescribed L, then $\mathfrak{M}(L) = 0$].

We note, without giving the uninteresting proofs, the following formulas for $\mathfrak{M}(L)$. First, *let* $N(L)$ *denote the number of Gel'fand patterns* (6.165) *having* $L_3 \to L$ *for* $0 \leq L \leq p$. Then we have

$$\mathfrak{M}(L) = N(L) - N(L+1). \qquad (6.167)$$

Second, let the numbers $\mathfrak{M}_p(L)$ and $\sigma_q(L)$ be defined as follows:

$$\mathfrak{M}_p(L) = [(p-L+2)/2,\ (p-L+1)/2],$$
$$\sigma_q(L) = [(q-L)/2,\ (q-L+1)/2], \qquad (6.168)$$

where the square bracket signifies that one is to choose the integer from the set of two numbers. The multiplicity is one of the following four numbers depending on the relation of L to q and $p-q$:

$$\begin{aligned} &L \geq q, \quad L \geq p-q; \quad && \mathfrak{M}(L) = \mathfrak{M}_p(L), \\ &L \geq q, \quad L \leq p-q; && \mathfrak{M}(L) = \mathfrak{M}_p(L) - \sigma_{p-q}(L), \\ &L \leq q, \quad L \geq p-q; && \mathfrak{M}(L) = \mathfrak{M}_p(L) - \sigma_q(L), \\ &L \leq q, \quad L \leq p-q; && \mathfrak{M}(L) = \mathfrak{M}_p(L) - \sigma_{p-q}(L) - \sigma_q(L). \end{aligned} \qquad (6.169)$$

The standard procedure for reducing any space which is invariant with respect to $SO(3)$ into its irreducible subspaces is to determine the set of highest weight vectors which belong to the space. Here this is the problem of determining all vectors of the form

$$F_{pq;L} = \sum_{\alpha,\sigma} C\begin{pmatrix} p & & q & & 0 \\ & \alpha & & \alpha-L-2\sigma & \\ & & \alpha-\sigma & & \end{pmatrix} \times F\begin{pmatrix} p & & q & & 0 \\ & \alpha & & \alpha-L-2\sigma & \\ & & \alpha-\sigma & & \end{pmatrix}, \qquad (6.170)$$

such that

$$L_+ F_{pq;L} = 0, \qquad (6.171)$$

since $F_{pq;L}$ already satisfies

$$L_3 F_{pq;L} = L F_{pq;L}. \qquad (6.172)$$

Here L is, of course, any preselected angular momentum value, $0 \leq L \leq p$. The $SO(3)$ basis vectors $F_{pq;LM}$ are then obtained by the standard lowering with L_-.

Let us observe that when L_+ operates on $F_{pq;L}$ it carries the Gel'fand vector in the summation to the form

$$F\begin{pmatrix} p & & q & & 0 \\ & \alpha' & & \alpha'-(L+1)-2\sigma' & \\ & & \alpha'-\sigma' & & \end{pmatrix} \qquad (6.173)$$

for certain new labels α', σ', but in particular, the new value of angular momentum is necessarily $L+1$. However, the number of Gel'fand patterns of form (6.173) is $N(L+1)$, and all of these final state vectors will enter into the new summation of Eq. (6.170) (after L_+ has acted). This signifies that Eq. (6.171) will lead to precisely $N(L+1)$ relations among the set of coefficients, $\{C\}$, which contains $N(L)$ coefficients. In consequence of Eq. (6.167), there must be precisely $\mathfrak{M}(L)$ coefficients in the set $\{C\}$ which can be chosen independently, thus determining $\mathfrak{M}(L)$ linearly independent highest weight vectors of the form (6.170). It is possible to prove that the set of coefficients

$$\left\{ C\begin{pmatrix} p & & q & & 0 \\ & p & & p-L-2\sigma & \\ & & p-\sigma & & \end{pmatrix} : \sigma = \sigma_{p-q}(L),\ \sigma_{p-q}(L)-1,\ \sigma_{p-q}(L)-2,\ \cdots,\ \sigma_{p-q}(L) - \mathfrak{M}(L)+1 \right\} \qquad (6.174)$$

is such an independent set. (The proof is very complicated, and we omit it, since we make no use of the result here.)

It is more instructive to consider special cases, thereby gaining some insight into various aspects of the multiplicity problem. For this purpose, as well as for the general problem above, we note the following generator transformations:

$$E_{13}F\begin{pmatrix} p\ q\ 0 \\ \alpha\ \beta \\ \gamma \end{pmatrix} = \left[\frac{(p-\alpha)(\alpha-q+1)(\alpha+2)(\gamma-\beta+1)}{(\alpha-\beta+1)(\alpha-\beta+2)}\right]^{1/2} \times F\begin{pmatrix} p & q & 0 \\ \alpha+1 & \beta \\ \gamma+1 \end{pmatrix} - \left[\frac{(\alpha-\gamma)(p-\beta+1)(q-\beta)(\beta+1)}{(\alpha-\beta)(\alpha-\beta+1)}\right]^{1/2} \times F\begin{pmatrix} p & q & 0 \\ \alpha & \beta+1 \\ \gamma+1 \end{pmatrix}, \tag{6.175}$$

$$E_{32}F\begin{pmatrix} p\ q\ 0 \\ \alpha\ \beta \\ \gamma \end{pmatrix} = \left[\frac{(p-\beta+2)(q-\beta+1)\beta(\gamma-\beta+1)}{(\alpha-\beta+1)(\alpha-\beta+2)}\right]^{1/2} F\begin{pmatrix} p & q & 0 \\ \alpha & \beta-1 \\ \gamma \end{pmatrix} + \left[\frac{(\alpha-\gamma)(p-\alpha+1)(\alpha-q)(\alpha+1)}{(\alpha-\beta)(\alpha-\beta+1)}\right]^{1/2} F\begin{pmatrix} p\ q & 0 \\ \alpha-1\ \beta \\ \gamma \end{pmatrix}. \tag{6.176}$$

Equations (6.175) and (6.176) are the relations required to effect the application of $L_+/\sqrt{2}=E_{32}-E_{13}$ to Eq. (6.170). Condition (6.171) then clearly leads in the general case to a *four-term recursion relation* on the coefficients, $\{C\}$.

Let us look at some special cases: (Additional unique highest weight vectors are given in Sec. VIII.E):

(1) $L=p$

The unique[9] highest weight is clearly

$$F\begin{pmatrix} p\ q\ 0 \\ p\ 0 \\ p \end{pmatrix}, \tag{6.177}$$

since this state is annihilated by E_{13} and E_{32} separately.

(2) $L=p-1$

The unique (normalized) highest weight is easily determined to be

$$[(p-q)/p]^{1/2}F\begin{pmatrix} p\ q\ 0 \\ p\ 1 \\ p \end{pmatrix} + [q/p]^{1/2}F\begin{pmatrix} p & q & 0 \\ p-1 & 0 \\ p-1 \end{pmatrix}. \tag{6.178}$$

(Observe that if $q=p$ this vector becomes the zero vector, since the coefficient of the first term vanishes, and the second term vanishes in consequence of the violation of the conditions on the entries of the Gel'fand pattern—for $q=p$, we cannot have $L=p-1$.)

(3) $L=p-2$

The Gel'fand patterns which enter into the right-hand side of Eq. (6.170) are

$$\begin{pmatrix} p\ q\ 0 \\ p\ 2 \\ p \end{pmatrix}, \begin{pmatrix} p & q & 0 \\ p & 0 \\ p-1 \end{pmatrix}, \begin{pmatrix} p & q & 0 \\ p-1 & 1 \\ p-1 \end{pmatrix}, \begin{pmatrix} p & q & 0 \\ p-2 & 0 \\ p-2 \end{pmatrix}, \tag{6.179}$$

and from Eq. (6.167) we see that there are two highest weight vectors for q in the interval $2\leq q\leq p-2$, and one for $q=0$, 1, $p-1$, or p. The two general relations among four coefficients resulting from condition (6.171) are as follows:

$$[2p(q-1)]^{1/2}C\begin{pmatrix} p\ q\ 0 \\ p\ 2 \\ p \end{pmatrix} + q^{1/2}C\begin{pmatrix} p & q & 0 \\ p & 0 \\ p-1 \end{pmatrix} - [(p+1)(p-q)]^{1/2}C\begin{pmatrix} p & q & 0 \\ p-1 & 1 \\ p-1 \end{pmatrix} = 0 \tag{6.180}$$

[9] We use "unique" rather loosely to mean "unique up to a phase."

for $p\geq q\geq 1$; and

$$[p-q]^{1/2}C\begin{pmatrix} p & q & 0 \\ & p & 0 \\ & p-1 & \end{pmatrix}+[q(p+1)]^{1/2}C\begin{pmatrix} p & q & 0 \\ & p-1 & 1 \\ & p-1 & \end{pmatrix}-[2p(p-q-1)]^{1/2}C\begin{pmatrix} p & q & 0 \\ & p-2 & 0 \\ & p-2 & \end{pmatrix}=0 \quad (6.181)$$

for $p-1\geq q\geq 0$.

Note that for $q=0$, 1, $p-1$, or p Eqs. (6.180) and (6.181) reduce either to a single equation relating two coefficients or to two equations relating three coefficients. In each of these cases, the solution is unique, and we list these four special solutions explicitly for reasons which will soon become clear:

$$F_{p,0;p-2}=\left(\frac{2(p-1)}{2p-1}\right)^{1/2}F\begin{pmatrix} p & 0 & 0 \\ & p & 0 \\ & p-1 & \end{pmatrix}+\left(\frac{1}{2p-1}\right)^{1/2}F\begin{pmatrix} p & 0 & 0 \\ & p-2 & 0 \\ & p-2 & \end{pmatrix}, \quad (6.182)$$

$$F_{p,1;p-2}=\left(\frac{2(p+1)(p-2)}{p(2p-1)}\right)^{1/2}F\begin{pmatrix} p & 1 & 0 \\ & p & 0 \\ & p-1 & \end{pmatrix}+\left(\frac{2(p-2)}{p(p-1)(2p-1)}\right)^{1/2}F\begin{pmatrix} p & 1 & 0 \\ & p-1 & 1 \\ & p-1 & \end{pmatrix}+\left(\frac{(p+1)}{(p-1)(2p-1)}\right)^{1/2}F\begin{pmatrix} p & 1 & 0 \\ & p-2 & 0 \\ & p-2 & \end{pmatrix}, \quad (6.183)$$

$$F_{p,p-1;p-2}=-\left(\frac{p+1}{(p-1)(2p-1)}\right)^{1/2}F\begin{pmatrix} p & p-1 & 0 \\ & p & 2 \\ & p & \end{pmatrix}+\left(\frac{2(p+1)(p-2)}{p(2p-1)}\right)^{1/2}F\begin{pmatrix} p & p-1 & 0 \\ & p & 0 \\ & p-1 & \end{pmatrix}-\left[\frac{2(p-2)}{p(p-1)(2p-1)}\right]^{1/2}F\begin{pmatrix} p & p-1 & 0 \\ & p-1 & 1 \\ & p-1 & \end{pmatrix}, \quad (6.184)$$

$$F_{pp;p-2}=-[2p-1]^{-1/2}F\begin{pmatrix} p & p & 0 \\ & p & 2 \\ & p & \end{pmatrix}+\left[\frac{2(p-1)}{2p-1}\right]^{1/2}F\begin{pmatrix} p & p & 0 \\ & p & 0 \\ & p-1 & \end{pmatrix}. \quad (6.185)$$

We now turn to the discussion of the general solution to Eqs. (6.180) and (6.181), and begin by raising some questions. Can one introduce in an arbitrary manner any pair of solutions to these equations? Is there some additional structure which indicates how these equations are to be solved? It seems to us that the answers to these questions are no and yes, respectively, for the following reasons. Suppose one does by some arbitrary procedure, find two orthonormal solutions, say, $F_{pq;p-2}^{(1)}$ and $F_{pq;p-2}^{(2)}$. Then certainly for arbitrary $2\leq q\leq p-2$ there can be no ambiguities in these solutions, and they would appear to be quite acceptable. But it is a meaningful question to ask what happens to these solutions when we set $q=0$, 1, $p-1$, or p in them, in which case, we know there is a unique solution to the problem. Does either or both solutions become undefined? Rather than answering this question, we pose a more positive problem. Does there exist a pair of orthonormal vectors which solve Eq. (6.171), and which are defined for all integral values $0\leq q\leq p$, such that for $q=0$, 1, $p-1$, or p one solution becomes the corresponding unique solution of Eqs. (6.182)–(6.185), the remaining one becoming the zero vector? *Yes, as we shall demonstrate.* The existence of such a property in the nontrivial special case under study suggests that, despite the fact that p and q assume discrete values, *there exists a sort of "continuity" principle* on the state vectors, considered as functions of p and q, which regulates their behavior under a sudden change in the multiplicity, which, in our example, is a jump from two to one when q assumes the value 0, 1, $p-1$, or p. We believe this behavior to be an example of a *general reduction principle* which must be reckoned with in dealing with the multiplicity problem. We will state this principle precisely in a moment, after giving the aforementioned solutions and discussing them.

A pair of orthonormal vectors possessing the appropriate reduction properties is as follows:

$$F_{pq;p-2}{}^{(1)}=\left[\frac{(p+1)(p-q)(p-q-1)}{A_{pq}}\right]^{1/2}F\begin{pmatrix} p & & q & & 0 \\ & p & & 2 & \\ & & p & & \end{pmatrix}$$

$$+\left[\frac{2p(q-1)(p-q-1)}{A_{pq}}\right]^{1/2}F\begin{pmatrix} p & & q & & 0 \\ & p-1 & & 1 & \\ & & p-1 & & \end{pmatrix}$$

$$+\left[\frac{(p+1)q(q-1)}{A_{pq}}\right]^{1/2}F\begin{pmatrix} p & & q & & 0 \\ & p-2 & & 0 & \\ & & p-2 & & \end{pmatrix}, \qquad (6.186)$$

$$F_{pq;p-2}{}^{(2)}=-(3p-2q-1)\left[\frac{pq(q-1)}{B_{pq}}\right]^{1/2}F\begin{pmatrix} p & & q & & 0 \\ & p & & 2 & \\ & & p & & \end{pmatrix}$$

$$+(p-2q)\left[\frac{2(p+1)q(p-q)}{B_{pq}}\right]^{1/2}F\begin{pmatrix} p & & q & & 0 \\ & p-1 & & 1 & \\ & & p-1 & & \end{pmatrix}$$

$$+A_{pq}\left[\frac{2}{B_{pq}}\right]^{1/2}F\begin{pmatrix} p & & q & & 0 \\ & p & & 0 & \\ & & p-1 & & \end{pmatrix}+(p+2q-1)$$

$$\times\left[\frac{p(p-q)(p-q-1)}{B_{pq}}\right]^{1/2}F\begin{pmatrix} p & & q & & 0 \\ & p-2 & & 0 & \\ & & p-2 & & \end{pmatrix}, \qquad (6.187)$$

where

$$A_{pq}=p(p-1)^2-2q(p-q), \qquad (6.188)$$

$$B_{pq}=pq(q-1)(3p-2q-1)^2+2(p+1)q(p-q)(p-2q)^2$$
$$+2A_{pq}{}^2+p(p-q)(p-q-1)(p+2q-1)^2. \qquad (6.189)$$

In particular, we have

$$A_{p0}=A_{pp}=p(p-1)^2,$$
$$A_{p1}=A_{p,p-1}=(p+1)(p-1)(p-2),$$
$$B_{p0}=B_{pp}=p^2(p-1)^3(2p-1),$$
$$B_{p1}=B_{p,p-1}=(p+1)p(p-1)^2(p-2)(2p-1). \qquad (6.190)$$

Remark. The coefficients A_{pq} and B_{pq} are, respectively, the sums of squares of the numerators of the coefficients appearing in Eqs. (6.186) and (6.187). However, only for $2\leq q\leq p-2$ are these factors the norms of the vectors one would obtain by deleting A_{pq} and B_{pq} from $F^{(1)}$ and $F^{(2)}$. This follows from the fact that a Gel'fand basis vector may vanish, while its coefficient does not.

We now observe that the vector defined by Eq. (6.186) becomes the zero vector for each $q=0, 1, p-1, p$ (either the coefficient or the Gel'fand vector vanishes). On the other hand, the vector defined by Eq. (6.187) reduces for each $q=0, 1, p-1, p$ to the appropriate unique vector from Eqs. (6.182)–(6.185). Thus, there exists a pair of orthonormal highest weight vectors possessing the reduction properties that we promised to demonstrate.

Still another important property must be mentioned. Noting that $A_{pq}=A_{p,p-q}$ and $B_{pq}=B_{p,p-q}$ and using the property of the operator $\mathcal{P}_{(12)(3)}$ given by Eq. (6.163), we easily derive the following relations:

$$\mathcal{P}_{(12)(3)}F_{pq;p-2}{}^{(1)}=(-1)^pF_{p,p-q;p-2}{}^{(1)}, \qquad (6.191)$$

$$\mathcal{P}_{(12)(3)}F_{pq;p-2}{}^{(2)}=(-1)^{p-1}F_{p,p-q;p-2}{}^{(2)}. \qquad (6.192)$$

In particular, on self-conjugate states where $p=2k$ and $q=k$, the operator $\mathcal{P}_{(12)(3)}$ is diagonal and has eigenvalue $+1$ on the first vector, and eigenvalue -1 on the second. For $q=0, 1, p-1$, or p, the properties expressed by Eq. (6.192) are unique, since the vectors are unique. What is remarkable is the fact that these same properties hold for arbitrary q on our multiplicity two states.

This concludes the detailed discussion of the $L=p-2$ case, except for some general comments made later relating to uniqueness. We have displayed a pair of orthonormal highest weight vectors with some rather intriguing properties. These properties suggest that there exists an underlying structure in the multiplicity problem of a very general nature, which we now discuss.

One point clearly emerges from the study of the preceding particular case. One learns nothing about the structure of highest weight vectors by considering isolated values of q—for prescribed p and L, one must examine the behavior of highest weight vectors for the full set of allowed values $0, 1, \cdots, p$ of q. For this purpose, it is essential to know how the multiplicity, which we now denote by $\mathfrak{M}_{pq}(L)$, varies with q. These results are easily obtained from Eqs. (6.169). For prescribed p and L $(0\leq L\leq p)$, we have the following multiplicities of L associated with q:

(1) $2L\geq p$

$$p-L\leq q\leq L; \qquad \mathfrak{M}_{pq}(L)=\mathfrak{M}_p(L),$$

$$0\leq q\leq p-L; \qquad \mathfrak{M}_{pq}(L)=\mathfrak{M}_p(L)-\sigma_{p-q}(L),$$

$$L\leq q\leq p; \qquad \mathfrak{M}_{pq}(L)=\mathfrak{M}_p(L)-\sigma_q(L). \qquad (6.193)$$

(2) $2L \leq p$

$$L \leq q \leq p-L; \qquad \mathfrak{M}_{pq}(L) = \mathfrak{M}_p(L) - \sigma_q(L) - \sigma_{p-q}(L),$$

$$0 \leq q \leq L; \qquad \mathfrak{M}_{pq}(L) = \mathfrak{M}_p(L) - \sigma_{p-q}(L),$$

$$p-L \leq q \leq p; \qquad \mathfrak{M}_{pq}(L) = \mathfrak{M}_p(L) - \sigma_q(L). \qquad (6.194)$$

For $2L \geq p$, $\mathfrak{M}_p(L)$ is clearly just the *maximum multiplicity* of L which can occur for any q and does occur for a particular q. Furthermore, as q runs over the values $0, 1, \cdots, p$, the multiplicity of L actually assumes one of the values $1, 2, \cdots, \mathfrak{M}_p(L)$ for at least one value of q.

For $2L \leq p$, the maximum multiplicity is $\mathfrak{M}_p(L) - \sigma_{p-L}(L)$, and again the multiplicity $\mathfrak{M}_{pq}(L)$ assumes each integral value from one to the maximum for appropriate choices of q. In either case, $2L \geq p$ or $2L \leq p$, the maximum multiplicity always obtains for $q=L$, i.e., $\mathfrak{M}_{pL}(L)$ is the maximum multiplicity of L for any choice of q, and each multiplicity $1, 2, \cdots, \mathfrak{M}_{pL}(L)$ obtains for some choice of q.

We now conjecture that the following fundamental result is valid, emphasizing that we have, in fact, not proved it, in general:

Multiplicity Reduction Theorem. There exists a set of highest weight vectors

$$\{F_{pq;L}^{(\lambda)} : \lambda = 1, 2, \cdots, \mathfrak{M}_{pL}(L)\},$$

with properties as follows: (a) The set is unambiguously defined for each $q=0, 1, \cdots, p$; (b) for each particular choice of q, the set contains $\mathfrak{M}_{pL}(L) - \mathfrak{M}_{pq}(L)$ zero vectors and $\mathfrak{M}_{pq}(L)$ nonzero orthonormal highest weight vectors; (c) each time the multiplicity is reduced by one, in consequence of choosing q to belong to the set of q values which yields the lesser multiplicity, a single vector goes to zero and remains zero for all subsequent q values which yield still smaller multiplicities.

This conjecture is just the assertion that there exists a *branching law* in the multiplicity space: Each time the multiplicity is reduced by one, a vector "splits off" by becoming (and remaining) the zero vector. (The analog to subgroup reduction is obvious.)

It seems premature to initiate any discussion of uniqueness until one can either affirm or deny the general validity of the Multiplicity Reduction Theorem. If it proves to be correct, then one can proceed to introduce those additional concepts (Lo70a) which will be essential to such a discussion.

Summary. The problem of reducing $SU(3)$ into its $SO(3)$ irreducible constituents has been formulated precisely in terms of the Gel'fand basis. By considering in some detail a special multiplicity two case, we have been led to conjecture the existence of a Multiplicity Reduction Law, which, if valid, signifies a rich structure of the multiplicity space. In any event, the multiplicity problem must be given careful thought, and the search for a meaningful structure continued until it unfolds fully.

VII. N-PARTICLE STATES: COUPLING METHOD

In this section, we consider a construction of the solid harmonics on the $3(N-1)$-sphere which is particularly appropriate to the democratic subgroup structure of the 4-particle problem (Ga71) [see Lemma 8 and Eqs. (5.31)–(5.51)]. We carry through the construction for arbitrary $N=3, 4, \cdots$, since it requires no more effort than does the particular case[10] $N=4$. (For $N=3$ the technique is valid, but it is not particularly useful for elucidating the *democratic subgroup structure* of the 3-particle problem.)

Let $\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^{N-1}$ denote a set of relative position vectors, and let $f_{(l)}(x) \equiv f_{(l)}(\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^{N-1})$, where $(l) = (l_1, l_2, \cdots, l_{N-1})$, denote a polynomial which is homogeneous of degrees $l_1, l_2, \cdots, l_{N-1}$, respectively, in the position vectors $\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^{N-1}$:

$$f_{(l)}(\lambda_1\mathbf{x}^1, \lambda_2\mathbf{x}^2, \cdots, \lambda_{N-1}\mathbf{x}^{N-1}) = \lambda_1^{l_1}\lambda_2^{l_2}\cdots\lambda_{N-1}^{l_{N-1}} f(\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^{N-1}). \qquad (7.1)$$

We also require that $f_{(l)}$ be a solution to Laplace's equation:

$$\nabla^2 f_{(l)}(x) = 0, \qquad (7.2)$$

where

$$\nabla^2 = \sum_{\alpha=1}^{N-1} \boldsymbol{\nabla}^\alpha \cdot \boldsymbol{\nabla}^\alpha. \qquad (7.3)$$

For example, $f_{(l)}$ may be a product of solid harmonics of the form

$$f_{(l)} = \prod_{\alpha=1}^{N-1} \mathcal{Y}_{l_\alpha m_\alpha}, \qquad (7.4)$$

where $\mathcal{Y}_{l_\alpha m_\alpha}$ $(m_\alpha = l_\alpha, l_\alpha - 1, \cdots, -l_\alpha)$ is the set of simultaneous eigenvectors of $\mathbf{L}^\alpha \cdot \mathbf{L}^\alpha$ and L_3^α, where $\mathbf{L}^\alpha$ is the orbital angular momentum associated with relative position vector $\mathbf{x}^\alpha$. Here $f_{(l)}$ may also be any *arbitrary coupling* of the solid harmonics which preserves the angular momentum labels $l_1, l_2, \cdots, l_{N-1}$, hence, the homogeneity property (7.1).

The only *polynomials* in the x_i^α which commute with the angular momentum operators $\mathbf{L}^\alpha$ $(\alpha = 1, 2, \cdots, N-1)$ are polynomials in the variables $\xi_1, \xi_2, \cdots, \xi_{N-1}$, where

$$\xi_\alpha \equiv (\mathbf{x}^\alpha \cdot \mathbf{x}^\alpha)/4. \qquad (7.5)$$

It follows therefore that it must be possible to span the space $\mathcal{L}_p$ [the set of homogeneous polynomials of degree p which solve Laplace's equation on the $3(N-1)$-sphere] by basis vectors of the general form

$$F_\lambda(\xi) f_{(l)}(x), \qquad (7.6)$$

where $F_\lambda(\xi)$ is a polynomial which is homogeneous of degree λ in $\xi_1, \xi_2, \cdots, \xi_{N-1}$:

$$F_\lambda(\mu\xi_1, \mu\xi_2, \cdots, \mu\xi_{N-1}) = \mu^\lambda F_\lambda(\xi_1, \xi_2, \cdots, \xi_{N-1}). \qquad (7.7)$$

Thus, the function (7.6) is homogeneous of degree p

[10] The 4-particle problem has also been considered from the viewpoint of the structure $O(9) \supset O(3) \times O(3)$ (Su67).

as follows in x

$$p=2\lambda+l_1+l_2+\cdots+l_{N-1}. \quad (7.8)$$

The condition that the product function (7.6) satisfies Laplace's equation is

$$\nabla^2 F_\lambda(\xi) f_{(l)}(x)=[\nabla^2, F_\lambda(\xi)]f_{(l)}(x)=0, \quad (7.9)$$

where we easily calculate

$$[\nabla^2, F_\lambda(\xi)]=\sum_\alpha\left[\xi_\alpha\frac{\partial^2 F_\lambda(\xi)}{(\partial\xi_\alpha)^2}+\frac{\partial F_\lambda(\xi)}{\partial\xi_\alpha}(\mathbf{x}^\alpha\cdot\boldsymbol{\nabla}^\alpha+\tfrac{3}{2})\right]. \quad (7.10)$$

Using the homogeneity property (7.1), we find that $F_\lambda(\xi)$ must satisfy

$$\sum_\alpha[\xi_\alpha(\partial/\partial\xi_\alpha)^2+(l_\alpha+\tfrac{3}{2})(\partial/\partial\xi_\alpha)]F_\lambda(\xi)=0. \quad (7.11)$$

We proceed now to solve Eq. (7.11). We put

$$F_\lambda(\xi)=\sum_{(\mu)}C(\mu)[\prod_{\alpha=1}^{N-1}\xi_\alpha{}^{\mu_\alpha}/(\mu_\alpha)!(\mu_\alpha+l_\alpha+\tfrac{1}{2})!], \quad (7.12)$$

where half-integral factorials are gamma functions $a!=\Gamma(a+1)$, and where the sum is over all nonnegative integral μ_α such that

$$\mu_1+\mu_2+\cdots+\mu_{N-1}=\lambda. \quad (7.13)$$

The number of terms in Eq. (7.12), i.e., the number of coefficients $C(\mu)$ is therefore just

$$n_\lambda(N)=\binom{\lambda+N-2}{N-2}. \quad (7.14)$$

The requirement that $F_\lambda(\xi)$ solves Eq. (7.11) now yields the following conditions on the coefficients:

$$\sum_\alpha C(\mu_1, \mu_2, \cdots, \mu_\alpha+1, \cdots, \mu_{N-1})=0, \quad (7.15)$$

where the relation is to be applied to all μ_α such that

$$\sum_\alpha \mu_\alpha=\lambda-1. \quad (7.16)$$

Thus, we have $n_{\lambda-1}(N)$ relations among $n_\lambda(N)$ coefficients, and the number of independent solutions to these equations is

$$n_\lambda(N)-n_{\lambda-1}(N)=n_\lambda(N-1). \quad (7.17)$$

One can, for example, choose the following set of $n_\lambda(N-1)$ coefficients independently:

$$\{C(0, \nu_2, \nu_3, \cdots, \nu_{N-1}): \sum_{\alpha=2}^{N-1}\nu_\alpha=\lambda\}. \quad (7.18)$$

Indeed, the zero in the coefficients in the set (7.18) can be chosen to be in any fixed position.

Corresponding to the set of independent coefficients (7.18), we have the following solution to Eqs. (7.15):

$$C(\mu_1, \mu_2, \cdots, \mu_{N-1})=(-1)^{\mu_1}(\mu_1)!\times\sum_{(\nu)}[C(0, \nu_2, \cdots, \nu_{N-1})/\prod_{\alpha=2}^{N-1}(\nu_\alpha-\mu_\alpha)!], \quad (7.19)$$

where, as usual, $1/(\nu_\alpha-\mu_\alpha)!=0$ for $\mu_\alpha>\nu_\alpha$. One verifies these results by substituting directly into Eq. (7.15).

We now choose all but one of the coefficients in the set (7.18) to be zero, and we choose the value of the nonzero coefficient to be $\prod_\alpha(\nu_\alpha)!$. Since there are $n_\lambda(N-1)$ ways of making this choice, we obtain $n_\lambda(N-1)$ independent solutions to Eq. (7.11):

$$F_{\lambda;(\nu)(l)}(\xi)=\sum_{s=0}^{\lambda}\frac{(-\xi_1)^s}{(s+l_1+\frac{1}{2})!}\times\sum_{(\mu)}\left[\prod_{\alpha=2}^{N-1}[(\nu_\alpha-\mu_\alpha+l_\alpha+\tfrac{1}{2})!]^{-1}\binom{\nu_\alpha}{\mu_\alpha}\xi_\alpha{}^{\nu_\alpha-\mu_\alpha}\right], \quad (7.20)$$

in which $(\mu)=(\mu_2\mu_3\cdots\mu_{N-1})$, and the sum on (μ) is over all nonnegative integral values such that, for each s, we have $\sum_\alpha\mu_\alpha=s$. *For each set of nonnegative integers* $(\nu)=(\nu_2\nu_3\cdots\nu_{N-1})$ *such that* $\sum_\alpha\nu_\alpha=\lambda$, *we obtain an independent solution to Eq.* (7.11).

We chose the coefficients (7.18) as the independent set for the simple reason that the set of solutions (7.20) contains as a subset, the complete set of (polynomial) solutions to Eq. (7.11) for N replaced by $N-1$. These are just the functions from Eq. (7.20) which have $\nu_{N-1}=0$ and accordingly have no dependence on ξ_{N-1}.

The number of independent polynomial solutions, homogeneous of degree p, which we obtain by combining the functions of Eq. (7.20) with $f_{(l)}$ [given, for example, by Eq. (7.4)] is

$$\sum_{\lambda,(l)}\prod_\alpha(2l_\alpha+1)\binom{\lambda+N-3}{N-3}=\frac{2p+3N-5}{3N-5}\binom{p+3N-6}{3N-6}=\dim\mathfrak{L}_p, \quad (7.21)$$

where the sum is over all nonnegative integers $\lambda, l_1, \cdots, l_{N-1}$ which satisfy

$$2\lambda+\sum_\alpha l_\alpha=p. \quad (7.22)$$

These product functions comprise a basis for the solid harmonics of degree p on the $3(N-1)$*-sphere.* [While it is reassuring that Eq. (7.21) checks for simple cases, we need not prove it generally as a separate problem—it must hold in consequence of the fact that we have found a basis for *all* solutions to Laplace's equation which are homogeneous polynomials.] There are two difficulties with this basis: (a) the physical significance of the labels (ν) is not clear; and (b) the basis functions are not orthogonal, in general, for distinct labels (ν) and (ν'). Despite these difficulties, there appears to be no simple alternative to these solutions when one attempts to find solutions to Laplace's equation by coupling single particle solid harmonics.

Let us note that the set of functions (7.20) has the following simple property under permutations of the variables $\xi_2, \xi_3, \cdots, \xi_{N-1}$:

$$F_{\lambda;(\nu)(l)}(\xi') = F_{\lambda;(\nu')(l')}(\xi), \tag{7.23}$$

where

$$(\xi') = (\xi_1 \xi_{i_2} \xi_{i_3} \cdots \xi_{i_{N-1}}),$$
$$(\nu') = (\nu_{i_2} \nu_{i_3} \cdots \nu_{i_{N-1}}),$$
$$(l') = (l_1 l_{i_2} l_{i_3} \cdots l_{i_{N-1}}), \tag{7.24}$$

in which $i_2 i_3 \cdots i_{N-1}$ is an arbitrary arrangement of $2\,3 \cdots N-1$. The permutations involving ξ_1 induce a transformation of considerably more complexity. [The fact that ξ_1 is singled out goes back to the selection of the coefficients (7.18) as independent. If we had chosen the zero in position ρ, then variable ξ_ρ would have been singled out.]

One can understand the origin of the complexities of the democratic subgroup structure in the 3-particle problem from an elementary viewpoint by examining the present construction of solid harmonics. For $N=3$, the solution (7.20) is completely specified by λ; $(l_1 l_2)$, i.e., there is no multiplicity:

$$F_{\lambda;(l_1 l_2)}(\xi_1\xi_2) = \sum_{s=0}^{\lambda} \binom{\lambda}{s} \frac{(-\xi_1)^s(\xi_2)^{\lambda-s}}{(s+l_1+\frac{1}{2})!(\lambda-s+l_2+\frac{1}{2})!}. \tag{7.25}$$

An orthogonal basis for the solid harmonics of degree p on the 6-sphere which has the total angular momentum L^2 and L_3 diagonal is given by

$$F_{\lambda;l_1l_2LM}(\mathbf{x}^1\mathbf{x}^2) = F_{\lambda;(l_1l_2)}(\xi_1\xi_2) f_{l_1l_2LM}(\mathbf{x}^1\mathbf{x}^2), \tag{7.26}$$

where

$$f_{l_1l_2LM} = \sum_{m_1,m_2} C(l_1l_2L;\, m_1m_2M)\mathcal{Y}_{l_1m_1}\mathcal{Y}_{l_2m_2}, \tag{7.27}$$

in which the coefficient is a standard $SU(2)$ Wigner coefficient, and the sum is over all m_1, m_2 such that $m_1+m_2=M$. The specification that the degree is p requires

$$2\lambda+l_1+l_2=p, \tag{7.28}$$

i.e., a basis of $\mathcal{L}_p$ is enumerated by the set of labels $\{\lambda l_1 l_2\colon 2\lambda+l_1+l_2=p\}$ and the set of total angular momentum quantum numbers which is associated with the various pairs (l_1, l_2) by the angular momentum coupling rules.

This new basis spans, of course, the same space as do the Gel'fand basis vectors of Eq. (6.70) [under, say, the restriction (6.66)]. The trouble with the new basis from the viewpoint of the democratic subgroup structure is that the permutation operators corresponding to the permutations in the invariant subgroup A_3 are not diagonal. (The three identical particles are not treated on an equal footing.) This will be the case only for basis vectors which are homogeneous of degree $p-q$ and q, respectively, in the variables η_k and ζ_k of Eq. (6.66). One might attempt to construct this basis by diagonalizing the operator $\mathcal{g}_2$ of Eq. (6.90) directly on the basis (7.26) ($\mathcal{g}_1+\mathcal{g}_2$ is already diagonal). One sees immediately, however, that this requires summing the basis vectors (7.26) over the various (l_1, l_2) pairs which can couple to a prescribed $L(0\leq L\leq p)$. [$\mathcal{g}_2$ does not commute with $\mathbf{L}^1$ and $\mathbf{L}^2$ individually.] In gaining a single new label q, we lose the two labels l_1 and l_2—the multiplicity problem simply reappears in a form entirely equivalent to that discussed in Sec. VI.E.

One can, of course, construct 3-particle states which transform irreducibly under permutations of particles by starting with the solid harmonics (7.26), using, for example, the method described by Moshinsky (Mo69) [see also Éfros (Ef71)]. This method, however, relinquishes any attempt to describe the three particles on an equal footing.

For $N=4$, the 9-dimensional solid harmonics obtained by combining the functions (7.20) with an arbitrary coupling of three ordinary 3-space solid harmonics *do* diagonalize the permutation operators corresponding to the permutations in the invariant subgroup $\mathcal{V}$ of Eq. (5.36). This follows immediately upon noting that the relative position vectors of Eq. (5.44) undergo the simple transformations $\mathbf{x}^\alpha \to \pm\mathbf{x}^\alpha$ ($\alpha=1, 2, 3$) under the permutation of $\mathcal{V}$: The functions of Eq. (7.20) (for $N=4$) are left invariant, while any coupled functions which preserve the homogeneity in $\mathbf{x}^1$, $\mathbf{x}^2$, $\mathbf{x}^3$ simply undergo a transformation of phase.

The properties of this basis under the remaining permutations belonging to S_4 are quite complicated. It turns out (Mo69) that this problem is equivalent to the determination of the transformations of the basis which are induced by the permutations of the relative position vectors, i.e., by the elements of a permutation group S_3. Thus, as an ancillary task, one already encounters the nonstandard problem of constructing total angular momentum states (by the coupling of three solid harmonics $\mathcal{Y}_{lm}$) in such a way as to exhibit a basis which treats the three angular momenta on the same footing, and, in particular, which is completely reduced with respect to S_3 when all three angular momenta are equal. Fortunately, this problem has been considered in some detail (Le65a, Ch64). However, one must carry out a corresponding construction on the $\mathcal{V}$-invariant functions [$N=4$ in Eq. (7.20)], and the general aspects of this problem remain open, although some progress has been made (Ga71).

Summary. We have obtained a general basis for the solid harmonics of degree p on the $3(N-1)$-sphere. Using standard coupling techniques on the individual solid harmonics $\mathcal{Y}_{l_\alpha m_\alpha}$, this basis can be made into one of good total orbital angular momentum states. The basis is nonorthogonal (for $N>3$), and its general properties under permutations of identical particles have not been studied (the 4-particle problem indicates that nonstandard couplings may be appropriate). It is the basis to which one is led rather directly via the democratic subgroup concept in the 4-particle problem. Its usefulness for N-particle states ($N>4$) has not been studied.

VIII. N-PARTICLE STATES: UNITARY GROUP METHOD

In this section, we consider the problem of obtaining solutions to Laplace's equation by using the property $O(n)\subset U(n)$.

In Sec. A, the properties of the unitary groups[11] are reviewed and developed in a form suitable for the later application to N-particle states. In Sec. B, we restrict the particular realization of the group $U(n)$ of Sec. A to its orthogonal subgroup $O(n)$ in such a way that the diagonal generators of the (proper) orthogonal group are given in terms of the diagonal generators of the unitary group. This property is then abstracted to obtain some new results (Sec. C) relating to the general reduction problem $SO(n)\subset U(n)$. These results are then used in Sec. D to give a constructive determination of all the single-valued IR's of $SO(n)$.

The results of these first four sections are quite abstract, but of considerable intrinsic interest from the general viewpoint of group theory.

The results of Secs. A–C are also essential to our discussion in Secs. E and F of a method for obtaining N-particle states of good orbital angular momentum which solve Laplace's equation.

A. Representations of $U(n)$

In analogy to the matrix X of Eq. (4.4), we introduce the complex matrix Z having ρ rows and n columns:

$$Z=(z_i^\alpha), \tag{8.1}$$

where α and i are *row* and *column* indices,[12] respectively, having the ranges $\alpha=1, 2, \cdots, \rho$ and $i=1, 2, \cdots, n$. (We will later choose $\rho=N-1$ and $n=3$, but since it is just as easy to consider general ρ and n, we do so.)

Next, we introduce the space $\mathcal{H}_p$ of polynomials which are homogeneous of degree p in the ρn variables (z_i^α), and we designate the value of such a polynomial F at the point Z by $F(Z)$. For each $U\in U(n)$ and each $V\in U(\rho)$, we define the operators as follows:

$$(\mathcal{O}_UF)(Z)=F(ZU), \qquad U\in U(n), \tag{8.2}$$

$$(\mathcal{O}'_VF)(Z)=F(\tilde{V}Z), \qquad V\in U(\rho). \tag{8.3}$$

Then (a) The operators $\{\mathcal{O}_U: U\in U(n)\}$ and $\{\mathcal{O}'_V: V\in U(\rho)\}$ are unitary operators on the space $\mathcal{H}_p$ [made into a Hilbert space by introducing the scalar product of type (2.61)]; (b) the correspondences $U\rightarrow\mathcal{O}_U$ and $V\rightarrow\mathcal{O}'_V$ are, respectively, representations of $U(n)$ and $U(\rho)$ by unitary operators.

It follows immediately from

$$(\mathcal{O}'_V\mathcal{O}_UF)(Z)=F(\tilde{V}ZU) \tag{8.4}$$

that the two groups of operators commute:

$$\mathcal{O}'_V\mathcal{O}_U=\mathcal{O}_U\mathcal{O}'_V. \tag{8.5}$$

[11] See (Lo70) for a listing of those papers most directly related to this approach.

[12] This is opposite to the convention used previously (Lo65).

Furthermore, in analogy to Eqs. (4.28) and (4.29), we find that the transformation

$$Z'=\tilde{V}ZU \tag{8.6}$$

is the same as the column matrix transformation

$$z'=(\tilde{V}\otimes\tilde{U})z, \tag{8.7}$$

where z (and analogously z') is identified in terms of the elements of Z as follows:

$$z=\text{col}\ (z_1\, z_2 \cdots z_{\rho n}), \tag{8.8}$$

where

$$z_{n(\alpha-1)+i}\equiv z_i^\alpha. \tag{8.9}$$

Thus, Eq. (8.4) is equivalently written as

$$(\mathcal{O}'_V\mathcal{O}_UF)(z)=F[(\tilde{V}\times\tilde{U})z]. \tag{8.10}$$

Comparing with Eq. (2.65) (with n now replaced by ρn), we obtain the following operator identity on the space $\mathcal{H}_p$ of homogeneous polynomials in ρn complex variables:

$$\mathcal{O}'_V\mathcal{O}_U=T_{V\otimes U}, \tag{8.11}$$

where $V\otimes U\in U(\rho n)$.

The group of product operators

$$\{\mathcal{O}'_V\mathcal{O}_U: V\in U(\rho),\ U\in U(n)\} \tag{8.12}$$

is a unitary representation on the space $\mathcal{H}_p$ *of the subgroup*

$$U(\rho)\times U(n)\subset U(\rho n). \tag{8.13}$$

But we have already observed in Sec. II.B that the space $\mathcal{H}_p$ is the carrier space for the IR of $U(\rho n)$ which is specified by the set of labels $[p\,0\cdots 0]$ containing $\rho n-1$ zeros. The space $\mathcal{H}_p$ is the carrier for a reducible representation of $U(\rho)\times U(n)$. However, we know precisely how this reduction occurs (Lo65): *Each* IR $[p\,0\cdots 0]$ *of* $U(\rho n)$ *reduces into a (direct) sum of* IR's

$$[m_{1n}\, m_{2n}\cdots m_{nn}\, 0\cdots 0]\otimes[m_{1n}\, m_{2n}\cdots m_{nn}] \tag{8.14}$$

of $U(\rho)\times U(n)$ (for $\rho\geq n$), *where each representation such that*

$$m_{1n}\geq m_{2n}\geq\cdots\geq m_{nn}\geq 0, \tag{8.15}$$

$$m_{1n}+m_{2n}+\cdots+m_{nn}=p \tag{8.16}$$

occurs exactly once. (A similar statement obtains for $\rho\leq n$.)

The normalized highest weight vector belonging to $\mathcal{H}_p$ which determines the basis of the carrier space for the IR of $U(\rho)\times U(n)$ which is specified by the labels (8.14) is known explicitly. It is given as follows:

$$\mathfrak{M}^{-1/2}([m])\prod_{k=1}^{n}(z_{12\cdots k}{}^{12\cdots k})^{m_{k,n}-m_{k+1,n}}, \tag{8.17}$$

where $m_{n+1,n}\equiv 0$, and $z_{12\cdots k}{}^{12\cdots k}$ is the determinant of the $k\times k$ matrix formed from (z_i^α) for $\alpha, i=1, 2, \cdots, k$. Here $\mathfrak{M}([m])$ is a normalized factor which we need not note explicitly (Ba63).

The basis vectors of the carrier space for the IR (8.14) of $U(\rho)\times U(n)$ may be classified by the IR labels of the subgroups in the *two* subgroup chains

$$U(n)\supset U(n-1)\supset\cdots\supset U(1), \tag{8.18}$$

$$U(\rho)\supset U(\rho-1)\supset\cdots\supset U(1), \tag{8.19}$$

i.e., by a *pair* of Gel'fand patterns.

We next note the notation for the general basis vector in the carrier space specified by the labels (8.14), i.e., the space having highest weight vector (8.17), the basis vectors being classified according to the two chains (8.18) and (8.19). It is convenient to let $[m]_n$ denote the set of $U(n)$ IR labels, $[m]_n=[m_{1n}\, m_{2n}\cdots m_{nn}]$, and also to let (m) denote the triangular array of $n-1$ rows of the form (2.78), i.e., a $U(n)$ Gel'fand pattern is now written as

$$\begin{pmatrix}[m]_n\\ (m)\end{pmatrix} \tag{8.20}$$

in order to display explicitly the labels $[m]_n$. It is also convenient to introduce *inverted* Gel'fand patterns. By the notation (8.20), we designate a normal (uninverted) pattern of the form (2.78), whereas we use the notation

$$\begin{pmatrix}(m)\\ [m]_n\end{pmatrix} \tag{8.21}$$

to designate the same pattern (8.20) turned upside down.

Consider first the case $\rho=n$. The notation for a normalized Gel'fand basis vector of the carrier space specified by $[m]_n\otimes[m]_n$ is

$$F\begin{pmatrix}(m')\\ [m]_n\\ (m)\end{pmatrix}, \tag{8.22}$$

in which the two Gel'fand patterns

$$\begin{pmatrix}[m]_n\\ (m)\end{pmatrix},\quad \begin{pmatrix}(m')\\ [m]_n\end{pmatrix} \tag{8.23}$$

share the same IR labels $[m]_n$, but otherwise (m) and (m') run independently over all sets of values which accord with the Weyl branching law (the so-called lexical patterns). We arbitrarily associate the lower patterns with the transformations of type (8.2) and the upper patterns with transformations of the type (8.3).

The highest weight vector (8.17) is the one designated by the notation

$$F\begin{pmatrix}(\max)\\ [m]_n\\ (\max)\end{pmatrix}(Z),\qquad Z \text{ is } n\times n, \tag{8.24}$$

where (max) denotes that the entries in (m) and (m') are taken to have their largest values, i.e., $m_{ij}=m'_{ij}=m_{in}$.

It is a remarkable (and simply proved) fact (Lo70) that when we restrict the n^2 variables in Z (for $\rho=n$) to be the elements of a unitary matrix, i.e., $Z\rightarrow U\in U(n)$, then the functions

$$D\begin{pmatrix}(m')\\ [m]_n\\ (m)\end{pmatrix}(U)\equiv\mathfrak{M}^{1/2}([m])F\begin{pmatrix}(m')\\ [m]_n\\ (m)\end{pmatrix}(U) \tag{8.25}$$

are the elements of a unitary matrix IR of $U(n)$. More precisely, if we let

$$D^{[m]_n}(U) \tag{8.26}$$

denote the matrix whose *rows are enumerated by the patterns* (m'), *and whose columns are enumerated by the patterns* (m),[12] then

$$U\rightarrow D^{[m]_n}(U) \tag{8.27}$$

is an IR of $U(n)$ by unitary matrices.

Next, consider the case $\rho\geq n$. We introduce the notation

$$[m]_n[0]_{\rho-n}=[m_{1n}\, m_{2n}\cdots m_{nn}\, 0\, 0\cdots 0] \tag{8.28}$$

to designate a set of IR labels of $U(\rho)$ containing $\rho-n$ zeros. The Gel'fand basis vector belonging to the carrier space specified by $[m]_n[0]_{\rho-n}\otimes[m]_n$ and having highest weight vector (8.17) is denoted by

$$F\begin{pmatrix}(m')\\ [m]_n\quad [0]_{\rho-n}\\ (m)\end{pmatrix}. \tag{8.29}$$

The pattern

$$\begin{pmatrix}[m]_n\\ (m)\end{pmatrix} \tag{8.30}$$

is a U(n) Gel'fand pattern associated with transformations of the type (8.2), and the pattern

$$\begin{pmatrix}(m)'\\ [m]_n\quad [0]_{\rho-n}\end{pmatrix} \tag{8.31}$$

is an inverted $U(\rho)$ Gel'fand pattern associated with transformations of the type (8.3). Here (m') is any arbitrary pattern containing $\rho-1$ rows which is compatible with the set of IR labels which contains $\rho-n$ zeros. The highest weight vector (8.17) is denoted by

$$F\begin{pmatrix}(\max)\\ [m]_n\quad [0]_{\rho-n}\\ (\max)\end{pmatrix}(Z),\qquad Z \text{ is } \rho\times n. \tag{8.32}$$

The general vectors, either those of Eq. (8.22) or

(8.29), can be obtained, in principle, by a double application of the lowering operators of Nagel and Moshinsky (Na65).

The lowering operators are polynomials in the generators of the representations (8.2) and (8.3). The Weyl generators of the representations of $U(n)$ and $U(\rho)$ given by Eqs. (8.2) and (8.3), respectively, are easily calculated by the technique of Louck (Lo70):

$$(E_{ij}F)(Z)=(\sum_{\alpha} z_i^{\alpha}\bar{z}_j^{\alpha})F(Z), \tag{8.33}$$

$$(E^{\alpha\beta}F)(Z)=(\sum_{i} z_i^{\alpha}\bar{z}_i^{\beta})F(Z), \tag{8.34}$$

where $i, j=1, 2, \cdots, n$ and $\alpha, \beta=1, 2, \cdots, \rho$. The sum over α is from 1 to ρ, while the sum over i is from 1 to n. The generators E_{ij} induce transformations on the lower labels (8.30) in accordance with the standard Gel'fand–Zetlin matrix elements, while the generators $E^{\alpha\beta}$ have the analogous role with respect to the upper patterns (8.31).

The case $\rho\leq n$ is obtained simply by replacing n by ρ in the highest weight vector (8.17) and interchanging the structure of the upper and lower patterns in (8.29), letting also $n\leftrightarrow\rho$ in the IR labels (Z is, however, still of dimension $\rho\times n$).

Let us note, in particular, the case $\rho=N-1$ $(N\geq 4)$ and $n=3$. The highest weight vector in the carrier space for IR $[m]_3[0]_{N-4}\otimes[m]_3$, $[m]_3=[m_{13}m_{23}m_{33}]$, of $U(N-1)\times U(3)$ is

$$\left[\frac{(m_{13}+2)!(m_{23}+1)!(m_{33})!}{(m_{13}-m_{23}+1)(m_{13}-m_{33}+2)(m_{23}-m_{33}+1)}\right]^{-1/2} \times (z_1^1)^{m_{13}-m_{23}}(z_{12}^{12})^{m_{23}-m_{33}}(z_{123}^{123})^{m_{33}}. \tag{8.35}$$

The general vector in this basis is denoted by

$$F\begin{pmatrix} & & (m') & & & \\ m_{13} & m_{23} & m_{33} & 0 & \cdots & 0 \\ & m_{12} & m_{22} & & & \\ & & m_{11} & & & \end{pmatrix}(Z), \quad Z \text{ is } (N-1)\times 3, \tag{8.36}$$

where the upper pattern (m') is given by

$$(m')=\begin{pmatrix} & & & m'_{11} & & & \\ & & m'_{12} & & m'_{22} & & \\ & m'_{13} & & m'_{23} & & m'_{33} & \\ m'_{14} & & m'_{24} & & m'_{34} & & 0 \\ \vdots & & \vdots & & \vdots & & \ddots \\ m'_{1,N-2} & & m'_{2,N-2} & & m'_{3,N-2} & 0 & \cdots\ 0 \end{pmatrix}. \tag{8.37}$$

For $N=3$, the highest weight vector is obtained from Eq. (8.35) by setting $m_{33}=0$. The general vector in the carrier space for IR $[m_{13}m_{23}]\otimes[m_{13}m_{23}0]$ of $U(2)\times U(3)$ is denoted by

$$F\begin{pmatrix} & m'_{11} & \\ m_{13} & m_{23} & 0 \\ m_{12} & m_{22} & \\ & m_{11} & \end{pmatrix}(Z), \qquad Z \text{ is } 2\times 3. \tag{8.38}$$

B. The Orthogonal Subgroups

We obtain a unitary operator representation of the group $O(\rho)\times O(n)$ on the space $\mathcal{H}_\rho$ simply by restricting $U\rightarrow R\in O(n)$ and $V\rightarrow S\in O(\rho)$ in Eqs. (8.2) and (8.3), respectively. We may at the same time restrict Z to be a real $\rho\times n$ matrix X and switch to the scalar product of type (2.6). However, we choose to restrict Z to the following form:

$$Z\rightarrow\tilde{A}_\rho X A_n, \tag{8.39}$$

where A_n is the $n\times n$ unitary matrix defined as follows. Let U_0 denote the 2×2 unitary matrix

$$U_0=(\sqrt{2})^{-1}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \tag{8.40}$$

Let $[n/2]$ denote $n/2$ (n even) or $(n-1)/2$ (n odd). Then A_n is the unitary matrix which has U_0 repeated $[n/2]$ times along the principal diagonal with zero elements elsewhere, except for n odd in which case 1 is the nn element. A_ρ has precisely this same structure. Thus, if x_i^α denotes the element in row α and column i of X, then the element η_i^α in row α and column i of XA_n is given by

$$\eta_{2j-1}^{\alpha}=(x_{2j-1}^{\alpha}+ix_{2j}^{\alpha})/\sqrt{2}, \tag{8.41}$$

$$\eta_{2j}^{\alpha}=(x_{2j-1}^{\alpha}-ix_{2j}^{\alpha})/\sqrt{2}, \tag{8.42}$$

for $j=1, 2, \cdots, [n/2]$, and

$$\eta_n^{\alpha}=x_n^{\alpha}, \qquad (n \text{ odd}) \tag{8.43}$$

for $\alpha=1, 2, \cdots, \rho$. The element z_i^α in row α and column i of $\tilde{A}_\rho XA_n$ is similarly expressed in terms of the η variables by

$$z_i^{2\beta-1}=(\eta_i^{2\beta-1}+i\eta_i^{2\beta})/\sqrt{2}, \tag{8.44}$$

$$z_i^{2\beta}=(\eta_i^{2\beta-1}-i\eta_i^{2\beta})/\sqrt{2}, \tag{8.45}$$

for $\beta=1, 2, \cdots, [\rho/2]$, and

$$z_i^{\rho}=\eta_i^{\rho}, \qquad (\rho \text{ odd}) \tag{8.46}$$

for $i=1, 2, \cdots, n$.

We next determine the relation between the generators of $U(\rho)\times U(n)$ and $SO(\rho)\times SO(n)$ which obtains upon making the restriction (8.39). Define $f(X)=F(\tilde{A}_\rho XA_n)$. Then the generators which correspond, respectively, to the transformations

$$(T_R f)(X)=f(XR), \quad \forall R\in O(n), \tag{8.47}$$

$$(T'_S f)(X)=f(\tilde{S}X), \quad \forall S\in O(\rho), \tag{8.48}$$

are as follows:

$$(L_{ij}f)(X)=-i\sum_{\alpha}(x_i^{\alpha}\bar{x}_j^{\alpha}-x_j^{\alpha}\bar{x}_i^{\alpha})f(X), \tag{8.49}$$

$$(L^{\alpha\beta}f)(X)=-i\sum_{i}(x_i^{\alpha}\bar{x}_i^{\beta}-x_i^{\beta}\bar{x}_i^{\alpha})f(X), \tag{8.50}$$

where $i,j=1,2,\cdots,n$ and $\alpha,\beta=1,2,\cdots,\rho$, and where, as usual, $L_{ji}=-L_{ij}$ and $L^{\beta\alpha}=-L^{\alpha\beta}$.

We now restrict our attention to the Lie algebra of $SO(n)$ defined by the generators $\{L_{ij}\}$, noting that similar results also hold for the $\{L^{\alpha\beta}\}$. It is convenient to define a new basis (essentially the Cartan basis) of the algebra as follows[13]:

$$\begin{pmatrix} K_{j,k} \\ K_{-j,-k} \\ K_{-j,k} \\ K_{j,-k} \end{pmatrix} \equiv \tfrac{1}{2} \begin{pmatrix} -i & 1 & 1 & i \\ i & 1 & 1 & -i \\ i & 1 & -1 & i \\ -i & 1 & -1 & -i \end{pmatrix} \begin{pmatrix} L_{2j-1,2k-1} \\ L_{2j,2k-1} \\ L_{2j-1,2k} \\ L_{2j,2k} \end{pmatrix}, \tag{8.51}$$

for $j<k=1,2,\cdots,[n/2]$, and for n odd we also define

$$\begin{pmatrix} K_j \\ K_{-j} \end{pmatrix} = (\sqrt{2})^{-1} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \begin{pmatrix} L_{2j-1,n} \\ L_{2j,n} \end{pmatrix} \tag{8.52}$$

for $j=1,2,\cdots,(n-1)/2$. Including also the commuting generators

$$L_{2j-1,2j}, \qquad j=1,2,\cdots,[n/2], \tag{8.53}$$

we obtain the desired new basis which is related to the old basis by a nonsingular transformation. Note the Hermitian conjugation relations as follows:

$$\begin{aligned} (K_{j,k})^{\dagger}&=K_{-j,-k}, \\ (K_{-j,k})^{\dagger}&=K_{j,-k}, \\ (K_j)^{\dagger}&=K_{-j}. \end{aligned} \tag{8.54}$$

The relations which obtain upon making the restriction (8.39) in the $U(n)$ generators of Eq. (8.33) now take the forms as follows:

$$L_{2j-1,2j}=E_{2j-1,2j-1}-E_{2j,2j} \tag{8.55}$$

for $j=1,2,\cdots,[n/2]$,

$$\begin{aligned} K_{j,k}&=E_{2k-1,2j}-E_{2j-1,2k}, \\ K_{-j,-k}&=E_{2j,2k-1}-E_{2k,2j-1}, \\ K_{j,-k}&=E_{2k,2j}-E_{2j-1,2k-1}, \\ K_{-j,k}&=E_{2j,2k}-E_{2k-1,2j-1}, \end{aligned} \tag{8.56}$$

for $j<k=1,2,\cdots,[n/2]$, and for n odd

$$\begin{aligned} K_j&=E_{n,2j}-E_{2j-1,n}, \\ K_{-j}&=E_{2j,n}-E_{n,2j-1}, \end{aligned} \tag{8.57}$$

for $j=1,2,\cdots,(n-1)/2$.

Relations of *exactly* the same forms, Eqs. (8.51)–(8.57) hold also for $SO(\rho)\subset U(\rho)$, it being necessary only to elevate the subscripts to Greek superscripts and to change n to ρ.

The verification of Eqs. (8.55)–(8.57) is most easily accomplished by making the restriction of variables in the right-hand side. The derivation is further simplified by noting that one need only consider the part of the transformation given by Eqs. (8.41)–(8.43). This follows because the transformation (8.44)–(8.46) *commutes* with the E_{ij} generators, i.e., the generators E_{ij} have the same *form* in the η variables as they do in the z variables.

The advantage of relating Z to X in the particular way, Eq. (8.39), is now apparent: The commuting generators (8.55) of the Cartan basis of the Lie algebra of $SO(n)$ are already diagonal on the $U(\rho)\times U(n)$ Gel'fand basis vectors (8.29). Indeed, this property has been simultaneously realized for each orthogonal group in $SO(\rho)\times SO(n)$.

We have established relations (8.55)–(8.57) by making use of very particular realizations of the generators of $U(n)$ and $SO(n)$. However, if $\{E_{ij}\}$ is a set of *abstract generators* of $U(n)$, then we can use relations (8.51)–(8.57) to define a set of *abstract generators* $\{L_{ij}\}$ of $SO(n)$. Each abstract carrier space for an IR of $U(n)$ with Gel'fand basis $|(m)\rangle$ is also a carrier space for a representation of $SO(n)$. The transformations of the basis $|(m)\rangle$ induced by the $SO(n)$ generators $\{L_{ij}\}$ may be obtained directly from relations (8.55)–(8.57). We can now make use of this result to obtain some insights into the *general* reduction problem $SO(n)\subset U(n)$.[14]

C. The Reduction Problem $SO(n)\subset U(n)$

Any vector in the carrier space of a representation of $U(n)$ which simultaneously diagonalizes the generators $E_{11}, E_{22}, \cdots, E_{nn}$ defines a *weight* W. The weight W is defined to be the row vector whose elements are the eigenvalues $w_1, w_2, \cdots, w_n$, respectively, of $E_{11}, E_{22}, \cdots, E_{nn}$:

$$W=[w_1\, w_2 \cdots w_n]. \tag{8.58}$$

In particular, each $U(n)$ Gel'fand basis vector $|(m)\rangle$ [see Eq. (2.78)] has associated with it the weight (8.58) where

$$w_i=\sum_{j=1}^{i} m_{ji}-\sum_{j=1}^{i-1} m_{j,i-1} \tag{8.59}$$

with $w_1=m_{11}$.

[13] This basis is essentially the one given by Pang and Hecht (Pa67).

[14] For a consideration of this problem from the viewpoint of Young diagrams, see (Ha62), p. 399.

Similarly, any vector in the carrier space of a representation of $SO(n)$ which simultaneously diagonalizes the generators $L_{12}, L_{34}, \cdots, L_{2r-1,2r}$ ($r=[n/2]$) defines a weight Ω. The weight Ω is defined to be the row vector whose elements are the eigenvalues $\omega_1, \omega_2, \cdots, \omega_r$, respectively, of $L_{12}, L_{34}, \cdots$:

$$\Omega=\{\omega_1\,\omega_2\cdots\omega_r\}. \tag{8.60}$$

The $SO(n)$ Gel'fand basis vectors associated with the general Gel'fand patterns (2.55) do *not* define weights since these basis vectors do not diagonalize $L_{12}, L_{34}, \cdots$.

We observe, however, that each $U(n)$ Gel'fand basis vector $|\,(m)\,\rangle$ in the carrier space for IR $M\equiv[m]_n$ of $U(n)$ is, in consequence of relation (8.55), a simultaneous eigenvector of $L_{12}, L_{34}, \cdots$;

Lemma 10. The $U(n)$ basis vector $|\,(m)\,\rangle$ has the $SO(n)$ weight

$$\Omega=\{\omega_1\,\omega_2\cdots\omega_r\}, \tag{8.61}$$

where

$$\omega_i=w_{2i-1}-w_{2i} \tag{8.62}$$

for $i=1, 2, \cdots, r$.

Lemma 10 is the basic result which is needed to generalize Eq. (6.167) of Sec. VI. We next derive this generalized formula.

Let $M=[m_{1n}\,m_{2n}\cdots m_{nn}]$ specify an IR of $U(n)$, and let $L=\{l_{n1}\,l_{n2}\cdots l_{nr}\}$ ($r=[n/2]$) specify an IR of $SO(n)$. Then, under the restriction of $U(n)$ to $SO(n)$, the IR of $U(n)$ reduces into a (direct) sum of IR's L of $SO(n)$

$$M=\sum_L \oplus\mathfrak{M}(L)L, \tag{8.63}$$

where $\mathfrak{M}(L)$ denotes the multiplicity of L in M.

The objective is to find a formula analogous to Eq. (6.167) for the non-negative integers $\mathfrak{M}(L)$.

Next, let $N(\Omega)$ denote the number of times an $SO(n)$ weight Ω is repeated when we let the labels in the $U(n)$ basis vector of Lemma 10 run over the set of Gel'fand patterns having IR labels M, i.e., when we let $|\,(m)\,\rangle$ run over the basis of the carrier space for IR M of $U(n)$. Observe that *the numbers $N(\Omega)$ are, in principle, known:* For each specified M, we can write out all the Gel'fand patterns, calculate the $U(n)$ weights, and finally calculate the corresponding set of $SO(n)$ weights contained in M. [This procedure is, of course, substantially sinplified when one accounts for the fact that *equivalent weights* for either $U(n)$ or $SO(n)$ are repeated an equal number of times.] Then we must have

$$N(\Omega)=\sum_L \mathfrak{M}(L)\gamma_L(\Omega), \tag{8.64}$$

where $\gamma_L(\Omega)$ is the number of times the $SO(n)$ weight Ω is repeated in the carrier space of IR L of $SO(n)$ (inner multiplicity).

The inner multiplicities[15] satisfy the following relation[16]:

$$\sum_S \delta_S\gamma_L(\Omega+R-SR)=\delta_{L\Omega}, \tag{8.65}$$

where the sum is over all elements S of the Weyl reflection group, and δ_S is the parity of the Weyl operation S. Here R is given by

$$R=\{r-\tfrac{1}{2}, r-\tfrac{3}{2}, \cdots, \tfrac{3}{2}, \tfrac{1}{2}\} \tag{8.66}$$

for $SO(2r+1)$, and by

$$R=\{r-1, r-2, \cdots, 1, 0\} \tag{8.67}$$

for $SO(2r)$.

We now replace Ω by $\Omega+R-SR$ in Eq. (8.64), multiply by δ_S, sum over S, and use property (8.65) to obtain the following result:

$$\sum_S \delta_S N(\Omega+R-SR)=\delta_{L\Omega}\mathfrak{M}(L). \tag{8.68}$$

[Note that $\sum_{L'}\mathfrak{M}(L')\delta_{L'\Omega}=0$ unless Ω is a set of IR labels of $SO(n)$ contained in M; $\sum_{L'}\mathfrak{M}(L')\delta_{L'\Omega}=\mathfrak{M}(L)$ for $\Omega=L=$ a set of IR labels of $SO(n)$ contained in M; hence, the Kronecker delta is properly included in Eq. (8.68).] If we define

$$\mathfrak{M}(\Omega)\equiv\delta_{L\Omega}\mathfrak{M}(L), \tag{8.69}$$

then

$$\mathfrak{M}(\Omega)=\sum_S \delta_S N(\Omega+R-SR). \tag{8.70}$$

Formula (8.70) will then automatically give $\mathfrak{M}(\Omega)=0$ unless Ω is a set of IR labels L belonging to M.

The use of Eq. (8.70) is best explained by an example. For $n=4$, Eq. (8.70) becomes

$$\mathfrak{M}(\omega_1,\omega_2)=N(\omega_1,\omega_2)+N(\omega_1+2,\omega_2)$$
$$-N(\omega_1+1,\omega_2+1)-N(\omega_1+1,\omega_2-1). \tag{8.71}$$

Consider the reduction of IR [4200] of $U(4)$ into its $SO(4)$ constituents. The $U(4)$ *dominant weights* are [4200], [4110], [3300], 2[3210], 3[3111], 3[2220], 4[2211], the multiplicity of each dominant weight being easily found directly from the Gel'fand patterns. The set of all weights of IR [4200] then consists of these dominant weights together with their equivalents (the distinct permutations of the dominant weights), there being 126 weights in all (the dimension of the representation). Using Lemma 10, we find that the only possible dominant weights of $SO(4)$ contained in [4200] are $\{4, \pm2\}$, $\{4, 0\}$, $\{3, \pm3\}$, $\{3, \pm1\}$, $\{2, \pm2\}$, $\{2, 0\}$, $\{1, \pm1\}$, $\{0, 0\}$. The factors $N(\omega_1, \omega_2)$ are next easily calculated from the weights of [4200] and Lemma 10. For example, the dominant weight $\{2, 0\}$ can only come from [4200], [3111], and [2022]; hence, there are $1+3+3=7$ Gel'fand patterns yielding dominant weight $\{2, 0\}$, i.e., $N(2, 0)=7$. In this manner, we easily determine $N(4, \pm2)=1$, $N(4, 0)=1$, $N(3, \pm3)=1$, $N(3, \pm1)=3$, $N(2, \pm2)=4$, $N(2, 0)=7$, $N(1, \pm1)=8$,

[15] For a more complete discussion of the various multiplicity formulas see (Gr70).

[16] Formula (8.65) is due to Racah (Ra62).

$N(0, 0)=10$. Using these numbers in Eq. (8.71), we find $\mathfrak{M}(4, \pm 2)=1$, $\mathfrak{M}(4, 0)=1$, $\mathfrak{M}(3, \pm 3)=0$, $\mathfrak{M}(3, \pm 1)=1$, $\mathfrak{M}(2, \pm 2)=1$, $\mathfrak{M}(2, 0)=2$, $\mathfrak{M}(1, \pm 1)=0$, $\mathfrak{M}(0, 0)=1$. Thus, we have

$$[4200]=\{4, 2\}\oplus\{4, -2\}\oplus\{4, 0\}\oplus\{3, 1\}\oplus\{3, -1\} \oplus\{2, 2\}\oplus\{2, -2\}\oplus 2\{2, 0\}\oplus\{0, 0\}. \quad (8.72)$$

There is another significant result which obtains from Lemma 10. In the (abstract) $U(n)$ IR representation space specified by the labels $[m_{1n}\, m_{2n} \cdots m_{nn}]$, consider the vector which has $U(n)$ weight given by

$$W_+=[m_{1n}m_{nn};\, m_{2n}m_{n-1,n};\, \cdots;\, m_{rn},\, m_{r+2,n};\, m_{r+1,n}] \quad (8.73)$$

for $n=2r+1$, and by either of the following forms for $n=2r$:

$$W_+=[m_{1n}m_{nn};\, m_{2n}m_{n-1,n};\, \cdots;\, m_{rn}m_{r+1,n}], \quad (8.74)$$

$$W_-=[m_{1n}m_{nn};\, m_{2n}m_{n-1,n};\, \cdots;\, m_{r+1,n}m_{rn}], \quad (8.75)$$

where W_- differs from W_+ only by the interchange of the last pair of numbers.

Since the weight [(8.73)–(8.75)] is just a permutation of the IR labels, it is equivalent to the highest weight. The Gel'fand pattern corresponding to such a weight is *uniquely* determined by the specification of the weight. The resulting pattern is called an *extremal* pattern.[17] For example, the pattern having weight (8.73) for $n=5$ is

$$\begin{pmatrix} m_{15} & & m_{25} & & m_{35} & & m_{45} & & m_{55} \\ & m_{15} & & m_{25} & & m_{45} & & m_{55} & \\ & & m_{15} & & m_{25} & & m_{55} & & \\ & & & m_{15} & & m_{55} & & & \\ & & & & m_{15} & & & & \end{pmatrix};$$

for $n=4$, the pattern having weight (8.74) is

$$\begin{pmatrix} m_{14} & & m_{24} & & m_{34} & & m_{44} \\ & m_{14} & & m_{24} & & m_{44} & \\ & & m_{14} & & m_{44} & & \\ & & & m_{14} & & & \end{pmatrix}.$$

We denote the unique Gel'fand basis vector in the representation space $[m]_n$ which is determined by one of the weights $W_\pm$ of Eqs. (8.73)–(8.75) by

$$\left|\begin{pmatrix} [m]_n \\ (\mathrm{ext})_\pm \end{pmatrix}\right\rangle, \quad (8.76)$$

where "ext" denotes extremal.

The significance of the vector (8.76) is: *it is the highest weight vector in the carrier space for an* IR *of* $SO(n)\subset U(n)$. The IR labels of this carrier space are as follows:

$$\{l_{n1}, l_{n2}, \cdots, l_{nr}\}, \quad (8.77)$$

where

$$l_{ni}=m_{in}-m_{n-i+1,n}, \quad (8.78)$$

for $i=1, 2, \cdots, r$ ($r=[n/2]$) for weight W_+. For weight W_- ($n=2r$), the label $l_{2r,r}$ in (8.77) is altered to

$$l_{2r,r}=m_{r+1,2r}-m_{r,2r}\leq 0. \quad (8.79)$$

The proof of this result is very simple: The weight (8.77) is just the $SO(n)$ weight which is associated to the $U(n)$ weight of Eqs. (8.73)–(8.75) by Lemma 10. Furthermore, one easily sees from Lemma 10 that the weight (8.77) corresponding to W_+ is higher than any other weight of $SO(n)\subset U(n)$; similarly, the weight (8.77) corresponding to W_- is higher than any other weight having $l_{2r,r}<0$. Hence, the weight (8.77) is a set of IR labels of a representation of $SO(n)\subset U(n)$. The highest weight vector in the carrier space of this IR of $SO(n)$ is the vector (8.76).

Thus, given the Gel'fand basis of *any* carrier space for an IR $[m]_n$ of $U(n)$, we can always identify in that space a unique $U(n)$ *extremal* vector which is the *highest weight vector* of the carrier space of the IR (8.77) of $SO(n)$.[18] Using now the lowering operators of $SO(n)$, one can generate the general $SO(n)$ Gel'fand vector classified by the chain $O(n)\supset O(n-1)\supset\cdots\supset O(2)$. Observe that these lowering operators can be expressed through Eqs. (8.55)–(8.57) in terms of the $U(n)$ generators. In this way, one obtains each abstract $SO(n)$ Gel'fand basis vector in the IR space carrying the labels (8.77) and (8.78) as linear combinations of the abstract $U(n)$ Gel'fand basis vectors carrying IR labels $[m]_n$.

The preceding procedure is, of course, only the first and simplest step in the general reduction problem $SO(n)\subset U(n)$. One must still find the carrier spaces of those IR's of $SO(n)\subset U(n)$ which have IR labels which are lower than the weight (8.77). Nonetheless, the single step we have made in the reduction is significant: Combining this abstract result with the explicit $U(n)\times U(n)$ representation space of Sec. A ($\rho=n$), we can determine all the single-valued IR's of $SO(n)$. This is the subject of the next subsection.

D. The Single-Valued IR's of $SO(n)$

The results of the preceding section are abstract, i.e., must hold in any unitary representation of $SO(n)\subset U(n)$. Those results can now be taken over to the explicit realization of $SO(n)\times SO(n)\subset U(n)\times U(n)$ of Secs. A and B (set $\rho=n$). We now demonstrate how one may obtain *all* the single-valued IR's of $SO(n)$ by the procedure already discussed for the unitary groups.

[17] The properties of extremal patterns are discussed in greater detail in (Bi68).

[18] This is the full structure which underlines Wong's results (Wo69).

First, we particularize the $U(n)$ IR labels in Sec. C to the form

$$[m]_r[0]_{n-r}=[m_{1n}\, m_{2n} \cdots m_{rn}\, 0 \cdots 0]. \quad (8.80)$$

The $SO(n)$ IR labels (8.77) then take the form

$$\{l\}_r=\{l_{n1}\, l_{n2} \cdots l_{nr}\}, \quad (8.81)$$

where now

$$l_{ni}=m_{in}, \qquad i=1, 2, \cdots, r, \quad (8.82)$$

for $U(n)$ weight W_+, and where for $O(2r)$ and weight W_-, the rth label is replaced by

$$l_{2r,r}=-m_{r,2r}\leq 0. \quad (8.83)$$

Since the m_{in} are arbitrary integers which satisfy

$$m_{1n}\geq m_{2n}\geq \cdots \geq m_{rn}\geq 0, \quad (8.84)$$

we see that the labels (8.81) may assume all sets of values which correspond to all the single-valued IR's of $SO(n)$.

Now consider the $U(n)\times U(n)$ Gel'fand basis vector F which is labeled by the pair of extremal patterns of the vector (8.76). It is given explicitly as follows:

$$F\begin{pmatrix} (\text{ext})_+ \\ [m]_r \quad [0]_{n-r} \\ (\text{ext})_+ \end{pmatrix}(Z)=G\begin{pmatrix} (\max) \\ \{l\}_r \\ (\max) \end{pmatrix}(Z) = \prod_{k=1}^{r} (z_{13\cdots 2k-1}{}^{13\cdots 2k-1})^{\,l_{n,k}-l_{n,k+1}}, \quad (8.85)$$

$$F\begin{pmatrix} (\text{ext})_- \\ [m]_r \quad [0]_r \\ (\text{ext})_- \end{pmatrix}(Z)=G\begin{pmatrix} (\max) \\ \{l\}_r \\ (\max) \end{pmatrix}(Z) = \prod_{k=1}^{r-1} (z_{13\cdots 2k-1}{}^{13\cdots 2k-1})^{\,l_{2r,k}-l_{2r,k+1}} \times (z_{13\cdots 2r-3,2r}{}^{13\cdots 2r-3,2r})^{-l_{2r,r}}, \quad (8.86)$$

in which we have made the identification of labels given by Eqs. (8.82) and (8.83). The vector (8.85) is simultaneously a *highest weight vector* for each $SO(n)$ in $SO(n)\times SO(n)$ in the carrier space of IR $\{l\}_r\otimes\{l\}_r$, $l_{n,r}\geq 0$; similarly, the vector (8.86) is the highest weight vector in the carrier space of IR $\{l\}_r\otimes\{l\}_r$, $l_{2r,r}\leq 0$, of $SO(2r)\times SO(2r)$. These facts are denoted by introducing the $SO(n)\times SO(n)$ basis vectors G which are labeled by a *pair* of $SO(n)$ Gel'fand patterns, the upper one being inverted in complete analogy to the $U(n)\times U(n)$ notation. The notation (max) in Eqs. (8.85)–(8.86) then designates that the labels in the two Gel'fand patterns are chosen as large as possible for the prescribed IR labels $\{l\}_r$. Quite generally, $z_{i_1 i_2\cdots i_t}{}^{\alpha_1\alpha_2\cdots\alpha_t}$ denotes the $t\times t$ determinant of the matrix which has $z_{i_k}{}^{\alpha_j}$ (j, $k=1, 2, \cdots, t$) in row j and column k.

We now apply the Pang and Hecht (Pa67) $SO(n)$ *lowering operators* [a set for each $SO(n)$ in $SO(n)\times SO(n)$] to the appropriate highest weight vector, Eq. (8.85) or (8.86), to obtain the general basis vector

$$G\begin{pmatrix} (l') \\ \{l\}_r \\ (l) \end{pmatrix}(Z), \qquad Z \text{ is } n\times n \quad (8.87)$$

in the carrier space of IR $\{l\}_r\otimes\{l\}_r$ of $SO(n)\times SO(n)$. We arbitrarily associate the upper patterns with the first $SO(n)$, hence, with the transformations generated by the $\{L^{\alpha\beta}\}$. The lower patterns are then associated with the second $SO(n)$, hence, with the transformations generated by the $\{L_{ij}\}$.

In generating the basis vectors (8.87), it is *very important* that we express the two sets of lowering operators of $SO(n)\times SO(n)$ in terms of the $U(n)\times U(n)$ generators through the use of Eqs. (8.55)–(8.57) and the identical set of equations which carry superscripts. We are then able, by using Eqs. (8.33) and (8.34), to generate the functions (8.87) directly in terms of the n^2 complex variables $z_i{}^\alpha$ (α, $i=1, 2, \cdots, n$) *on which there are no restrictions.* Furthermore, *the lowering and raising operators entail the $U(n)$ generators only in the combinations of $U(n)$ generators which occur in Eqs.* (8.55)–(8.57) *in which the complex number i does not appear.* This implies: *The functions $G(Z)$ of Eq.* (8.87) *are real functions of the n^2 complex numbers $z_i{}^\alpha$*

$$[G(Z)]^*=G(Z^*). \quad (8.88)$$

Furthermore, since the highest weight vector (8.85) or (8.86) is invariant under the exchange of superscripts and subscripts, and since the two sets of lowering operators are also interchanged under the exchange of superscripts and subscripts, we have the additional important property

$$G\begin{pmatrix} (l') \\ \{l\}_r \\ (l) \end{pmatrix}(\tilde{Z})=G\begin{pmatrix} (l) \\ \{l\}_r \\ (l') \end{pmatrix}(Z). \quad (8.89)$$

We emphasize again that Z is an *arbitrary* $n\times n$ complex matrix in Eqs. (8.88) and (8.89).

The vectors (8.87) are orthogonal on the space $\mathcal{H}_p$ ($\sum_i l_{n,i}+|l_{n,r}|=p$) with the scalar product of type (2.61). They are not normalized (but may easily be), since we purposely did not normalize the highest weight vector (8.85) or (8.86). [The vectors (8.87) do all have the *same* norm which is just the norm of the highest weight vector.]

We emphasize again that the $SO(n)\times SO(n)$ generators $\{L^{\alpha\beta}\}$ and $\{L_{ij}\}$ have, even when expressed in terms of *complex variables* through our mapping of $SO(n)$ generators onto $U(n)$ generators, *the standard Gel'fand–Zetlin matrix elements on the basis* (8.87) [the same set of matrix elements for either set of generators].

We can now consider the functions obtained from Eq. (8.87) upon setting $Z=\tilde{A}_nXA_n$ [see Eq. (8.39)]:

$$f\begin{pmatrix}(l')\\ \{l\}_r\\ (l)\end{pmatrix}(X)\equiv G\begin{pmatrix}(l')\\ \{l\}_r\\ (l)\end{pmatrix}(\tilde{A}_nXA_n). \quad (8.90)$$

The relations (8.55)–(8.57) between the $SO(n)\times SO(n)$ generators (8.49) and (8.50) and the $U(n)\times U(n)$ generators (8.33)–(8.34) now obtain explicitly. Furthermore, the vectors are orthogonal in the scalar product of type (2.6), and each vector has the norm of the highest weight vector. The vectors (8.90) are a basis of the carrier space of IR $\{l\}_r\otimes\{l\}_r$ of $SO(n)\times SO(n)$, and the $SO(n)$ generators, now expressed in the form of Eqs. (8.49) and (8.50) have the standard Gel'fand–Zetlin matrix elements on this basis.

While we will see in the next subsection that the restriction of Z to the form in Eq. (8.90) is useful for obtaining solutions to Laplace's equation, we are completely free to relate Z in Eq. (8.87) to a set of real variables *in any manner that we choose.* In particular, we wish next to answer the question: What is the significance of the functions (8.87) when we restrict $Z\rightarrow R\in SO(n)$?

To answer this question, we consider the representation of $SO(n)\times SO(n)$ on the space $\mathcal{H}_p$ defined, *not* by transformation of the form (8.47) and (8.48), but rather by the *direct* restriction of U and V to $SO(n)$ in Eqs. (8.2) and (8.3):

$$(\mathcal{O}_RF)(Z)=F(ZR), \quad (8.91)$$

$$(\mathcal{O}'_SF)(Z)=F(\tilde{S}Z), \quad (8.92)$$

for each pair $R, S\in SO(n)$. Clearly, $R\rightarrow\mathcal{O}_R$ and $S\rightarrow\mathcal{O}'_S$ is a representation on the space $\mathcal{H}_p$ of $SO(n)\times SO(n)$ by unitary operators.

The generators of this representation of $SO(n)\times SO(n)$ are easily verified to relate to the generators of $U(n)\times U(n)$ given by Eqs. (8.33)–(8.34) by the relations as follows:

$$L'_{ij}=-i(E_{ij}-E_{ji}), \quad (8.93)$$

$$(L^{\alpha\beta})'=-i(E^{\alpha\beta}-E^{\beta\alpha}), \quad (8.94)$$

where $i, j, \alpha, \beta=1, 2, \cdots, n$. [One simply replaces x by z and f by F in Eqs. (8.49) and (8.50).]

The relations between the transformations (8.91) and (8.92) to those of Eqs. (8.47) and (8.48), extended to Z, are determined as follows: The function f in Eqs. (8.47) and (8.48) with the extended domain of definition Z is related to F by $f(Z)=F(\tilde{A}_nZA_n)$, since for $Z=X$ we obtain the correct relation $f(X)=F(\tilde{A}_nXA_n)$. Thus, $f=\mathcal{O}'_{A_n}\mathcal{O}_{A_n}F$ follows from Eqs. (8.2) and (8.3). Using this relation and replacing X by Z in Eq. (8.47), we obtain $(T_R\mathcal{O}'_{A_n}\mathcal{O}_{A_n}F)(Z)=(\mathcal{O}'_{A_n}\mathcal{O}_{A_n}F)(ZR)$. Since $\mathcal{O}'_{A_n}$ commutes with T_R and the operators are unitary, this relation becomes $(\mathcal{O}_{A_n}{}^\dagger T_R\mathcal{O}_{A_n}F)(Z)=F(ZR)$, that is,

$$\mathcal{O}_R=\mathcal{O}_{A_n}{}^\dagger T_R\mathcal{O}_{A_n}. \quad (8.95)$$

Similarly, we obtain

$$\mathcal{O}'_S=\mathcal{O}'_{A_n}{}^\dagger T'_S\mathcal{O}'_{A_n}. \quad (8.96)$$

The representation (8.91)–(8.92) *is unitarily equivalent to the one obtained by extending X to Z in Eqs.* (8.47)–(8.48). [It is, of course, this latter extension which led us directly to the basis (8.87).]

Relations (8.93)–(8.96) are, of course, irrelevant to the fact that the transformations (8.91) and (8.92) are unitary representations of $SO(n)\times SO(n)$, but we have noted them for completeness. The functions (8.87) are a basis for the IR $\{l\}_r\otimes\{l\}_r$ of $SO(n)\times SO(n)$. In particular, to the transformations (8.91) and (8.92) of this basis there corresponds a *unitary matrix* IR of $SO(n)\times SO(n)$. These transformations take the following forms[19]:

$$G\begin{pmatrix}(l')\\ \{l\}_r\\ (l)\end{pmatrix}(ZR)=\sum_{(l'')} D^{\{l\}_r}{}_{(l'')(l)}(R)G\begin{pmatrix}(l')\\ \{l\}_r\\ (l'')\end{pmatrix}(Z), \quad (8.97)$$

$$G\begin{pmatrix}(l')\\ \{l\}_r\\ (l)\end{pmatrix}(\tilde{S}Z)=\sum_{(l'')} D^{\{l\}_r}{}_{(l'')(l')}(S)G\begin{pmatrix}(l'')\\ \{l\}_r\\ (l)\end{pmatrix}(Z), \quad (8.98)$$

in which the D function appearing in the second equation is *exactly* the same one appearing in the first equation [because the generators $(L^{\alpha\beta})'$ induce the same transformations on the upper patterns as do the generators L'_{ij} on the lower patterns despite the fact that the matrix elements of these generators are *not* those of Gel'fand and Zetlin]. Then

$$R\rightarrow D^{\{l\}_r}(R) \quad (8.99)$$

is an IR of $SO(n)$ by unitary matrices. The Gel'fand patterns (l'') and (l) in Eq. (8.97) designate rows and columns, respectively, of the matrix (8.99).

If we correspondingly in the G functions let upper patterns and lower patterns label, respectively, rows and columns of a matrix G, then Eqs. (8.97) and (8.98) can be written as matrix equations as follows (IR labels

[19] It is essential here that we consider $R\in SO(n)$ (see Footnote 1).

have been suppressed):

$$G(ZR) = G(Z)D(R), \qquad (8.100)$$

$$G(\tilde{S}Z) = \tilde{D}(S)G(Z). \qquad (8.101)$$

We put $S=\tilde{R}$ in Eq. (8.101) and use $\tilde{D}(\tilde{R}) = D^*(R)$ (unitary property) to obtain

$$G(RZ) = D^*(R)G(Z). \qquad (8.102)$$

We next set $Z=I_n$ (unit matrix) in Eqs. (8.100) and (8.102) and take the complex conjugate of the second result, using property (8.88). The result is

$$G(R) = G(I_n)D(R) = D(R)G(I_n) \qquad (8.103)$$

for each $R\in SO(n)$. Schur's lemma requires that $G(I_n)$ be a multiple of the unit matrix, but, in fact,

$$G\begin{pmatrix} (\max) \\ \{l\}_r \\ (\max) \end{pmatrix}(I_n) = 1, \qquad (8.104)$$

so that $G(I_n)$ *is* the unit matrix. Thus, we have

$$D(R) = G(R); \qquad (8.105)$$

that is,

$$D^{\{l\}_r}{}_{(l')(l)}(R) = G\begin{pmatrix} (l') \\ \{l\}_r \\ (l) \end{pmatrix}(R). \qquad (8.106)$$

The basis functions (8.87) *are precisely the elements of the unitary matrix* IR $\{l\}_r$ *of* $SO(n)$ *when evaluated at* $Z=R\in SO(n)$.

Furthermore, this representation is *real* [property (8.88)]

$$[D^{\{l\}_r}(R)]^* = D^{\{l\}_r}(R), \qquad (8.107)$$

so that the representation is real orthogonal.

We emphasize that the preceding procedure obtains *all* the single-valued matrix IR's of $SO(n)$ directly as homogeneous polynomials in the elements R_{ij} of a proper orthogonal matrix R without any need for parametrizing (*any* suitable parameters can be introduced in the final IR's, if desired).

Summary. We have given an explicit constructive procedure for determining all the single-valued IR's of $SO(n)$. The procedure utilizes the fact that it is possible to identify in the carrier space $\mathcal{H}_p$ of IR $[p\,0\cdots 0]$ of $U(n^2)\supset U(n)\times U(n)$, an explicit vector which is the highest weight vector of the carrier space of IR $\{l\}_r\otimes\{l\}_r$ of $SO(n)\times SO(n)$, where $\sum_i l_{2r+1,i}=p$ for $O(2r+1)$, and $\sum_i l_{2r,i}+|l_{2r,r}|=p$ for $O(2r)$. The general basis vector in this IR space is then generated by using a double (upper and lower) application of the $SO(n)$ lowering operator technique. These general vectors are defined on the n^2 complex variables Z of the polynomials of $\mathcal{H}_p$. The restriction of these variables to the elements of a real, proper orthogonal matrix in the $\{l\}_r\otimes\{l\}_r$ basis vectors then gives directly the real orthogonal matrix IR $\{l\}_r$ of $SO(n)$ in terms of the elements (R_{ij}) of the $R\in SO(n)$ being represented.

E. $SO(\rho)\times SO(3)$ Basis Vectors

It was convenient for proving the important relation (8.105) to introduce the representation of $SO(n)\times SO(n)$ defined by Eqs. (8.91) and (8.92). However, *it is the homogeneous polynomials of the* x_i^α *obtained by restricting* Z *to the form* (8.39) *in* Eq. (8.36) *which leads us most readily to solutions to Laplace's equation—hence, to N-particle states of good orbital angular momentum.* We henceforth consider only the representation (8.47) and (8.48) of $O(\rho)\times O(3)$ which led to the relations (8.55)–(8.57) between the generators of $SO(\rho)\times SO(3)$ and those of $U(\rho)\times U(n)$. Explicitly, the generators of $SO(3)$ (total orbital angular momentum group) relate to those of $U(3)$ through Eqs. (6.97) [Eqs. (8.55)–(8.57) for $n=3$], while the generators of $SO(\rho)$ relate to those of $U(\rho)$ through Eqs. (8.51)–(8.57) when we evaluate all subscripts j, k to superscripts α, β and change n to ρ. We will always consider the $SO(\rho)\times SO(3)$ generators to be expressed directly in terms of the $U(\rho)\times U(3)$ generators, thereby obtaining $SO(\rho)\times SO(3)$ basis vectors in terms of the complex variables z_i^α (no restriction). Only in the final results do we make the implicit restriction $Z=\tilde{A}_\rho XA_3$ in which the x_i^α are the relative coordinates of the N-particle problem ($\rho\equiv N-1$ hereafter).

We write out again the notation for the general $U(\rho)\times U(3)$ basis vectors [highest vector given by Eq. (8.35)]:

$$F\begin{pmatrix} & & (m') & & & \\ m_{13} & m_{23} & m_{33} & 0 & \cdots & 0 \\ & m_{12} & m_{33} & & & \\ & & m_{11} & & & \end{pmatrix}(Z), \qquad Z \text{ is } \rho\times 3. \qquad (8.108)$$

The procedure for constructing $(\rho+1)$-particle states, classified as basis vectors of the carrier space for an IR of $SO(\rho)\times SO(3)$ is threefold: (1) carry out the reduction $SO(3)\subset U(3)$ on the lower patterns of the basis vectors (8.108) (thus constructing basis vectors of good orbital angular momentum L); (2) carry out the reduction $SO(\rho)\subset U(\rho)$ on the upper patterns (thus constructing carrier spaces of reducible representations of the permutation group $S_{\rho+1}\subset O_\rho$); (3) determine the linear combinations of these $SO(\rho)\times SO(3)$ basis vectors which satisfy the Laplace equation. We deal with the first two steps of this procedure in this section. It is not as impossible to carry out for cases of practical interest as it might appear.

First, consider the problem $SO(3)\subset U(3)$. This problem (lower labels) is abstractly the same as the problem discussed in Sec. VI.G. Equations (6.175) and (6.176) are valid on the abstract Gel'fand basis vectors

$$\left|\begin{pmatrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{pmatrix}\right\rangle \tag{8.109}$$

upon making the replacements $p=m_{13}-m_{33}$, $q=m_{23}-m_{33}$, $\alpha=m_{12}-m_{33}$, $\beta=m_{22}-m_{33}$, $\gamma=m_{11}-m_{33}$ throughout, followed by adding m_{33} to each label appearing in the basis vectors F. Thus, in general, one encounters all the difficulties discussed in Sec. VI.G. We can, however, make some practical progress.

The reduction of the abstract carrier space for IR $[m_{13}\, m_{23}\, m_{33}]$ of $U(3)$ into its $SO(3)$ IR subspaces can be carried out explicitly for those general cases in which there is *no multiplicity* (hence, these results may also be used in Sec. VI.G). This is the case for the following sets of IR labels:

$$\begin{aligned} &[k+\lambda, \lambda, \lambda], && L=k, k-2, \cdots, 1 \text{ or } 0, \\ &[\lambda, \lambda, \lambda-k], && L=k, k-2, \cdots, 1 \text{ or } 0, \\ &[k+\lambda, \lambda+1, \lambda], && L=k, k-1, \cdots, 2, 1, \\ &[\lambda, \lambda-1, \lambda-k], && L=k, k-1, \cdots, 2, 1. \end{aligned} \tag{8.110}$$

The (un-normalized) basis vectors of the carrier space for IR L of $SO(3)$ are given in terms of the abstract $U(3)$ basis vectors as follows (these vectors are obtained directly from the requirement that $E_{32}-E_{13}$ annihilate them):

$$\sum_\sigma A_\sigma \left|\begin{pmatrix} k+\lambda & & \lambda & & \lambda \\ & L+2\sigma+\lambda & & \lambda & \\ & & L+\sigma+\lambda & & \end{pmatrix}\right\rangle, \tag{8.111}$$

$$\sum_\sigma (-1)^\sigma A_\sigma \left|\begin{pmatrix} \lambda & & \lambda & & \lambda-k \\ & \lambda & & \lambda-L-2\sigma & \\ & & \lambda-\sigma & & \end{pmatrix}\right\rangle, \tag{8.112}$$

$$\sum_\sigma A_{\sigma,0} \left|\begin{pmatrix} k+\lambda & & \lambda+1 & & \lambda \\ & L+2\sigma+\lambda & & \lambda & \\ & & L+\sigma+\lambda & & \end{pmatrix}\right\rangle + \sum_\sigma A_{\sigma,1} \left|\begin{pmatrix} k+\lambda & & \lambda+1 & & \lambda \\ & L+2\sigma+\lambda+1 & & \lambda+1 & \\ & & L+\sigma+\lambda+1 & & \end{pmatrix}\right\rangle, \tag{8.113}$$

$$\sum_\sigma (-1)^\sigma A_{\sigma,0} \left|\begin{pmatrix} \lambda & & \lambda-1 & & \lambda-k \\ & \lambda & & \lambda-L-2\sigma & \\ & & \lambda-\sigma & & \end{pmatrix}\right\rangle + \sum_\sigma (-1)^\sigma A_{\sigma,1} \left|\begin{pmatrix} \lambda & & \lambda-1 & & \lambda-k \\ & \lambda-1 & & \lambda-L-2\sigma-1 & \\ & & \lambda-\sigma-1 & & \end{pmatrix}\right\rangle, \tag{8.114}$$

where

$$A_\sigma=[2^\sigma/\{[(k-L)/2]-\sigma\}!] \times[(L+\sigma)!(k-L-2\sigma)!/\sigma!k!]^{1/2}, \tag{8.115}$$

$$A_{\sigma,0}=A_\sigma/(L+2\sigma)^{1/2}, \tag{8.116}$$

$$A_{\sigma,1}=A_{\sigma,0}[(k-L-2\sigma)(L+2\sigma)/(k+1)(L+2\sigma+2)]^{1/2}, \tag{8.117}$$

for $k-L$ even,

$$A_{\sigma,1}=A_{\sigma,0}[(k+1)(L+2\sigma)/(k-L-2\sigma)(L+2\sigma+2)]^{1/2}, \tag{8.118}$$

for $k-L$ odd. In the definition of A_σ, the quantity $[(k-L)/2]$ is $(k-L)/2$ ($k-L$ even) or $(k-L-1)/2$ ($k-L$ odd). Here k and λ are non-negative integers.

The way in which we use the abstract results, Eqs. (8.111)–(8.114), is as follows: First, consider the upper pattern in Eq. (8.108) to be arbitrary, as indicated. We then form the same linear combinations on the lower patterns of the F's as appear in Eqs. (8.111)–(8.114). *Each of these vectors* (four types) *is then an* $SO(3)$ *highest weight vector having orbital angular momentum* L. The sum of the $U(3)$ IR labels appearing in Eqs. (8.111)–(8.114) now is identified as the degree p of the functions F, e.g., $p=k+3\lambda$ in Eq. (8.111). If we now enumerate all $U(3)$ labels such that $m_{13}+m_{23}+m_{33}=p\leq 5$, we find in each instance that $[m_{13}\, m_{23}\, m_{33}]$ is of the form of one of the sets of labels (8.110). For $p\leq 5$, *there is no multiplicity of* $SO(3)$ *in* $U(3)$. Indeed, even for $p=6$, 7, we see that the only labels not of the form (8.110) are [420] and [520], respectively. [420] contains $L=0$, 2, 2, 3, 4; [520] contains 1, 2, 3, 3, 4, 5. In the [420] case, the explicit construction of the $L=4$, 3, 2, 2 states was given in Sec. VI.G [the F of Sec. VI.G is now to be replaced by the *new* F of Eq. (8.108)], and the $L=0$ state is easily constructed. Similarly, in the [520] case, Sec. VI.G contains the explicit construction of the $L=5$, 4, 3, 3 states, and the $L=1$, 2 states are easily constructed. *Thus, we can claim to have obtained all good angular momentum states which are contained in the spaces* $\mathcal{H}_p$ *for* $p=0, 1, 2, \cdots, 7$) *and for arbitrary* ρ.

We also have available one other general category of good angular momentum states: These are the states having general $U(3)$ labels $[m_{13}\, m_{23}\, m_{33}]$ and having $L=m_{13}-m_{33}$, $m_{13}-m_{33}-1$, or $m_{13}-m_{33}-2$. We need only change the results of Eqs. (6.177)–(6.178), (6.182)–(6.185), and (6.186)–(6.187) according to the rules given in the paragraph containing Eq. (8.109) above. We then form the same combinations of the $U(\rho)\times U(3)$ state vectors (8.108).

The basis vectors of $\mathcal{H}_p$ obtained by the preceding methods are properly termed $U(\rho)\times SO(3)$ basis vectors, since the arbitrary upper patterns still enumerate the basis vectors of the carrier space of IR

$$[m_{13}\, m_{23}\, m_{33}\, 0 \cdots 0]$$

of $U(\rho)$. The explicit tabulation of the particular $U(\rho)\times SO(3)$ basis vectors considered in the preceding paragraphs is entirely mechanical—it entails only the working out of the relevant $U(\rho)\times U(3)$ Gel'fand basis vectors by known procedures. [These $U(\rho)\times SO(3)$ states are also just the harmonic oscillator states of good angular momentum when the z_i^α are properly identified as the creation operators of the oscillator states.]

The next step toward obtaining N-particle states of good angular momentum is to carry out the reduction $SO(\rho)\subset U(\rho)$. The problem now becomes more difficult. We ignore the lower patterns in the vector (8.108) and first consider only those upper extremal patterns which are determined by the weights of Eqs. (8.73)–(8.75).

We enumerate explicitly these extremal patterns:

(1) $\rho=3$ $(N=4)$

$$\begin{pmatrix} & m_{13} & \\ & m_{13} \quad m_{33} & \\ m_{13} & m_{23} & m_{33} \end{pmatrix} \tag{8.119}$$

having $SO(3)$ weight $m_{13}-m_{33}$.

(2) $\rho=4$ $(N=5)$

$$\begin{bmatrix} m_{13} \\ m_{13} \quad 0 \\ m_{13} \quad m_{23} \quad 0 \\ m_{13} \quad m_{23} \quad m_{33} \quad 0 \end{bmatrix}, \quad \begin{bmatrix} m_{13} \\ m_{13} \quad 0 \\ m_{13} \quad m_{33} \quad 0 \\ m_{13} \quad m_{23} \quad m_{33} \quad 0 \end{bmatrix} \tag{8.120}$$

having $SO(4)$ weights

$$\{m_{13},\, m_{23}-m_{33}\} \quad \text{and} \quad \{m_{13},\, m_{33}-m_{23}\},$$

respectively.

(3) $\rho=5$ $(N=6)$

$$\begin{bmatrix} m_{13} \\ m_{13} \quad 0 \\ m_{13} \quad m_{23} \quad 0 \\ m_{13} \quad m_{23} \quad 0 \quad 0 \\ m_{13} \quad m_{23} \quad m_{33} \quad 0 \quad 0 \end{bmatrix} \tag{8.121}$$

having $SO(5)$ weight $\{m_{13},\, m_{23}\}$.

(4) $\rho=6$ $(N=7)$

$$\begin{bmatrix} m_{13} \\ m_{13} \quad 0 \\ m_{13} \quad m_{23} \quad 0 \\ m_{13} \quad m_{23} \quad 0 \quad 0 \\ m_{13} \quad m_{23} \quad m_{33} \quad 0 \quad 0 \\ m_{13} \quad m_{23} \quad m_{33} \quad 0 \quad 0 \quad 0 \end{bmatrix}, \quad \begin{bmatrix} m_{13} \\ m_{13} \quad 0 \\ m_{13} \quad m_{23} \quad 0 \\ m_{13} \quad m_{23} \quad 0 \quad 0 \\ m_{13} \quad m_{23} \quad 0 \quad 0 \quad 0 \\ m_{13} \quad m_{23} \quad m_{33} \quad 0 \quad 0 \quad 0 \end{bmatrix}, \tag{8.122}$$

having $SO(6)$ weights

$$\{m_{13},\, m_{23},\, m_{33}\} \quad \text{and} \quad \{m_{13},\, m_{23},\, -m_{33}\},$$

respectively.

(5) $\rho>6$ $(N>7)$

$$\begin{bmatrix} m_{13} \\ m_{13} \quad 0 \\ m_{13} \quad m_{23} \quad 0 \\ m_{13} \quad m_{23} \quad 0 \quad 0 \\ m_{13} \quad m_{23} \quad m_{33} \quad 0 \quad 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ m_{13} \quad m_{23} \quad m_{33} \quad 0 \quad 0 \cdots 0 \end{bmatrix} \tag{8.123}$$

having $SO(\rho)$ weight $\{m_{13}, m_{23}, m_{33}, 0, \cdots, 0\}_r$ $(r=[\rho/2])$.

The basis vectors (8.108) which are labeled by the *extremal upper patterns* (8.119)–(8.123) and which have *maximal lower patterns* can be given explicitly [it is only necessary to permute the upper indices in Eq. (8.35)]:

$$F\begin{pmatrix} & (\text{ext}) & \\ m_{13} & m_{23} \quad m_{33} \quad 0 \quad \cdots & 0 \\ & (\max) & \end{pmatrix}(Z)$$

$$=(z_1{}^1)^{m_{13}-m_{23}}(z_{12}{}^{1\alpha})^{m_{23}-m_{33}}(z_{123}{}^{1\beta\gamma})^{m_{33}}, \quad (8.124)$$

where the superscripts assume the following values for the respective cases:

$\rho=3$, $(\alpha\beta\gamma)=(323)$, weight $m_{13}-m_{33}$;

$\rho=4$, $(\alpha\beta\gamma)=(334)$ for weight $\{m_{13}, m_{23}-m_{33}\}$ and $(\alpha\beta\gamma)=(434)$ for weight $\{m_{13}, m_{33}-m_{23}\}$;

$\rho=5$, $(\alpha\beta\gamma)=(335)$, weight $\{m_{13}, m_{23}\}$;

$\rho=6$, $(\alpha\beta\gamma)=(335)$ for weight $\{m_{13}, m_{23}, m_{33}\}$ and $(\alpha\beta\gamma)=(336)$ for weight $\{m_{13}, m_{23}, -m_{33}\}$;

$\rho>6$, $(\alpha\beta\gamma)=(335)$, weight $\{m_{13}, m_{23}, m_{33}, 0 \cdots 0\}$.

[The vector (8.124) is, of course, an $SO(\rho)$ *highest weight vector* having these various weights for its IR labels.]

Next, we combine the preceding results for upper and lower patterns. In order to give a precise description of the procedure, it is convenient to introduce two special notations for certain vectors belonging to the carrier space of IR $[m_{13}\, m_{23}\, m_{33}\, 0 \cdots 0]\otimes[m_{13}\, m_{23}\, m_{33}]$ of $U(\rho)\times U(3)$ [the space with basis vectors (8.108)]:

$$F(\{l_{\rho1}\, l_{\rho2}\, l_{\rho3}\}/[m_{13}\, m_{23}\, m_{33}]), \quad (8.125)$$

$$F_{[m_{13}m_{23}m_{33}]}(\{l_{\rho1}\, l_{\rho2}\, l_{\rho3}\}/L). \quad (8.126)$$

The notation (8.125) designates[20] the *highest weight vector* in the carrier space of IR $\{l_{\rho1}\, l_{\rho2}\, l_{\rho3}\, 0 \cdots 0\}\otimes[m_{13}\, m_{23}\, m_{33}]$ of $SO(\rho)\times U(3)\subset U(\rho)\times U(3)$, where $\{l_{\rho1}\, l_{\rho2}\, l_{\rho3}\, 0 \cdots 0\}$ specifies an IR of $SO(\rho)$ contained in IR $[m_{13}\, m_{23}\, m_{33}\, 0 \cdots 0]$ of $U(\rho)$. The vectors (8.124) are of this type. The notation (8.126) designates the *highest weight vector* in the carrier space of IR $\{l_{\rho1}\, l_{\rho2}\, l_{\rho3}\, 0 \cdots 0\}\otimes L$ of $SO(\rho)\times SO(3)\subset U(\rho)\times U(3)$, where $\{l_{\rho1}\, l_{\rho2}\, l_{\rho3}\, 0 \cdots 0\}$ specifies an IR of $SO(\rho)$ contained in IR $[m_{13}\, m_{23}\, m_{33}\, 0 \cdots 0]$ of $U(\rho)$, and L specifies an IR of $SO(3)$ contained in IR $[m_{13}\, m_{23}\, m_{33}]$ of $U(3)$. For $\rho=3$, 4, or 5, the notation $\{l_{\rho1}\, l_{\rho2}\, l_{\rho3}\}$ must be modified to $l_{31}=L'$, $\{l_{41}\, l_{42}\}$, and $\{l_{51}\, l_{52}\}$, respectively.

We will call the vectors (8.125) and (8.126) simply $SO(\rho)\times U(3)$ and $SO(\rho)\times SO(3)$ highest weight vectors, respectively.

In the 4-particle $(\rho=3)$ case, we obtain the class of highest weight vectors as follows: We replace the abstract vectors in Eqs. (8.111)–(8.114) by the F's of Eq. (8.108), leaving the upper $U(3)$ labels arbitrary. Now form from these vectors the linear combinations (8.111)–(8.114) on the upper patterns (using an index σ' and an angular momentum L'). For example, one obtains

$$\sum_{\sigma,\sigma'} A_\sigma A_{\sigma'} F\begin{pmatrix} & & L'+\sigma'+\lambda & & \\ & L'+2\sigma'+\lambda & & \lambda & \\ k+\lambda & & \lambda & & \lambda \\ & L+2\sigma+\lambda & & \lambda & \\ & & L+\sigma+\lambda & & \end{pmatrix}$$

from Eq. (8.111). This is the $SO(3)\times SO(3)$ highest weight vector denoted by

$$F_{[k+\lambda\,\lambda\,\lambda]}(L'/L).$$

Thus, we obtain the following set of $SO(3)\times SO(3)$ highest weight vectors:

$$F_{[m_{13}m_{23}m_{33}]}(L'/L), \quad (8.127)$$

where $[m_{13}\, m_{23}\, m_{33}]$ is any one of the four types of IR labels of the form (8.110). *These four general types already yield all $SO(3)\times SO(3)$ highest weight vectors contained in the $U(3)\times U(3)$ basis up to and including degree 5 polynomials. There is no multiplicity of $L'\otimes L$ in any $U(3)\times U(3)$ IR $[m]\otimes[m]$ for $m_{13}+m_{23}+m_{33}\leq 5$.*

Using the constructions of Sec. VI.E, we can also obtain the general vectors (8.127) corresponding to L', $L=m_{13}-m_{33}$, $m_{13}-m_{33}-1$, $m_{13}-m_{33}-2$.

For $\rho>3$, we cannot progress quite as far. It is clear that the $SO(\rho)\times U(3)$ highest weight vectors (8.125) are the key to solving the problem: Using the relatively simple $U(3)$ lowering operators on (8.125), we can generate the $U(3)$ basis vectors

$$F[\{l_{\rho1}\, l_{\rho2}\, l_{\rho3}\}/(m)], \quad (8.128)$$

where

$$(m)=\begin{pmatrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{12} & \\ & & m_{11} & & \end{pmatrix},$$

and where for $m_{11}=m_{12}=m_{13}$, $m_{12}=m_{23}$ this vector is the highest weight vector (8.125). The construction of the $SO(\rho)\times SO(3)$ highest weight vectors then proceeds along the lines of Sec. VI.G.

We can, in fact, carry out the above construction explicitly for the following cases:

(1) Let the $SO(\rho)$ labels in the vector (8.125) be

[20] Only IR's of $SO(\rho)$ of the form $\{l_{\rho1}l_{\rho2}l_{\rho3}0 \cdots 0\}$ can occur in an IR of $U(\rho)$ of the type $[m_{1\rho}\, m_{2\rho}\, m_{3\rho}\, 0 \cdots 0]$ [see (Su67) and (Br70a)].

identified as any of those following Eq. (8.124) so that the vector (8.125) becomes the explicit highest weight vector (8.124). It is easy to generate the particular vectors (8.128) which occur in the linear combinations which one must form (Sec. VI.G) to obtain $L=m_{13}-m_{33}$, $m_{13}-m_{33}-1$, or $m_{13}-m_{33}-2$. Thus, with very little effort, we can obtain the general $SO(\rho)\times SO(3)$ highest weight vectors (8.126) whenever $\{l_{\rho 1}\, l_{\rho 2}\, l_{\rho 3}\}$ is the largest highest weight (of its type) contained in IR $[m_{13}\, m_{23}\, m_{33}\, 0 \cdots 0]$ of $U(\rho)$ and for $L=m_{13}-m_{33}$, $m_{13}-m_{33}-1$, or $m_{13}-m_{33}-2$.

(2) Again let the $SO(\rho)$ labels in the vector (8.125) be identified as in (1), but now particularize further to any one of the sets of $U(3)$ labels (8.110). Once again one can generate explicitly from the corresponding $SO(\rho)\times U(3)$ highest weight vector (8.125) all those vectors (8.128) which occur in the linear combinations (8.111)–(8.114). Thus, we can obtain explicit expressions for all $SO(\rho)\times SO(3)$ highest weight vectors of the following types:

$$\begin{aligned} &(a) \quad F_{[k+\lambda,\lambda,\lambda]}(\{\cdots\}/L),\\ &(b) \quad F_{[\lambda,\lambda,\lambda-k]}(\{\cdots\}/L),\\ &(c) \quad F_{[k+\lambda,\lambda+1,\lambda]}(\{\cdots\}/L),\\ &(d) \quad F_{[\lambda,\lambda-1,\lambda-k]}(\{\cdots\}/L), \end{aligned} \tag{8.129}$$

where the $SO(\rho)$ labels $\{\cdots\}$ can have the following values:

$SO(4)$: (a) $\{k+\lambda, 0\}$, (b) $\{\lambda, \pm k\}$, (c) $\{k+\lambda, \pm 1\}$, (d) $\{\lambda, \pm(k-1)\}$;

$SO(5)$: (a) $\{k+\lambda, \lambda\}$, (b) $\{\lambda, \lambda\}$, (c) $\{k+\lambda, \lambda+1\}$, (d) $\{\lambda, \lambda-1\}$;

$SO(6)$: (a) $\{k+\lambda, \lambda, \pm\lambda\}$, (b) $\{\lambda, \lambda, \pm(\lambda-k)\}$, (c) $\{k+\lambda, \lambda+1, \pm\lambda\}$, (d) $\{\lambda, \lambda-1, \pm(\lambda-k)\}$

For $SO(\rho)$ $(\rho>6)$: $SO(\rho)$ labels coincide with the $U(3)$ labels. L can be any of the values given in (8.110).

We will not write out explicitly the highest weight vectors (8.129). These results serve to indicate a somewhat general, but nonetheless limited set of highest weight vectors. While the $U(3)$ labels appearing in (8.129) are sufficient to enumerate all cases up to degree 5, the reduction of $SO(\rho)\subset U(\rho)$ is very limited—it gives only those $SO(\rho)$ labels of the largest highest weight contained in $U(\rho)$ [including the $\pm$ sign of the last label for $SO(4)$ and $SO(6)$].

Clearly, if we are to make any useful progress, the problem of reducing $SO(\rho)\subset U(\rho)$ must be dealt with more completely—we must find the full set of $SO(\rho)\times U(3)$ vectors (8.125) up to some practical degree $p=m_{13}+m_{23}+m_{33}$, say, $p=4$ or 5. This we now do.

In order to construct highest weight vectors in the reduction $SO(\rho)\subset U(\rho)$, we must be able to recognize one. All the $SO(\rho)$ generators

$$\begin{aligned} K^{\alpha,\beta}&=E^{2\beta-1,2\alpha}-E^{2\alpha-1,2\beta},\\ K^{\alpha,-\beta}&=E^{2\beta,2\alpha}-E^{2\alpha-1,2\beta-1}, \end{aligned} \tag{8.130}$$

for $\alpha<\beta=1, 2, \cdots, [\rho/2]\equiv r$;

$$K^{\alpha}=E^{\rho,2\alpha}-E^{2\alpha-1,\rho} \qquad (\rho \text{ odd}) \tag{8.131}$$

for $\alpha=1, 2, \cdots, r$ *increase* the weight of an $SO(\rho)$ basis vector. A highest weight vector is thus partially characterized by being annihilated by the set of above *raising generators*. However, these raising generators can all be obtained by the repeated commutation of the following ones (r in number):

$$E^{2\alpha+2,2\alpha}-E^{2\alpha-1,2\alpha+1} \tag{8.132}$$

for $\alpha=1, 2, \cdots, r-1$ together with either

$$E^{2r-1,2r-2}-E^{2r-3,2r} \tag{8.133}$$

for $SO(2r)$, *or*

$$E^{2r+1,2r}-E^{2r-1,2r+1} \tag{8.134}$$

for $SO(2r+1)$.

Thus, we arrive at the following result: The necessary (and sufficient) conditions that a vector belonging to the abstract carrier space for IR $[m_{1\rho}\, m_{2\rho} \cdots m_{\rho\rho}]$ of $U(\rho)$ is an $SO(\rho)$ highest weight vector labeled by $\{l_{\rho 1}\, l_{\rho 2} \cdots l_{\rho r}\}$ are two-fold: First, the vector must be an eigenvector of the set of operators

$$E^{2\alpha-1,2\alpha-1}-E^{2\alpha,2\alpha}, \qquad \alpha=1, 2, \cdots, r \tag{8.135}$$

and the set of eigenvalues must be $l_{\rho 1}, l_{\rho 2}, \cdots, l_{\rho r}$, respectively, where $l_{\rho 1}\geq l_{\rho 2}\geq \cdots \geq l_{\rho r}\geq 0$ for ρ odd, and $l_{\rho 1}\geq l_{\rho 2}\geq \cdots \geq l_{\rho, r-1}\geq |l_{\rho r}|\geq 0$ for ρ even. Second, the vector must be annihilated by the raising generators (8.132)–(8.134).

We now list explicitly some $SO(\rho)\times U(3)$ highest weight vectors of the type (8.125), and then explain, by example, how they were obtained [the notation is a slight variation of (8.125) and is explained below]:

$$f_{[1]}{}^{\{1\}}(Z)=z_1{}^1,$$

$$f_{[2]}{}^{\{0\}}(Z)=\sum_{\alpha=1}^{r} z_1{}^{2\alpha-1}z_1{}^{2\alpha}+(\rho \text{ odd})\tfrac{1}{2}(z_1{}^{\rho})^2,$$

$$f_{[11]}{}^{\{11\}}(Z)=z_{12}{}^{13},$$

$$f_{[21]}{}^{\{1\}}(Z)=\sum_{\alpha=1}^{r}(z_1{}^{2\alpha-1}z_{12}{}^{1,2\alpha}+z_1{}^{2\alpha}z_{12}{}^{1,2\alpha-1})+(\rho \text{ odd})z_1{}^{\rho}z_{12}{}^{1\rho},$$

$$f_{[111]}{}^{\{111\}}(Z)=z_{123}{}^{135},$$

$$f_{[22]}{}^{\{2\}}(Z)=\sum_{\alpha=1}^{r} z_{12}{}^{1,2\alpha-1}z_{12}{}^{1,2\alpha}+(\rho \text{ odd})\tfrac{1}{2}(z_{12}{}^{1\rho})^2,$$

$$f_{[22]}{}^{\{0\}}(Z)=\sum_{\alpha=1}^{r}(z_{12}{}^{2\alpha-1,2\alpha})^2-2\sum_{\alpha<\beta=1}^{r}(z_{12}{}^{2\alpha,2\beta-1}z_{12}{}^{2\alpha-1,2\beta}$$

$$+z_{12}{}^{2\alpha-1,2\beta-1}z_{12}{}^{2\alpha,2\beta})-(\rho\ \text{odd})\,2\sum_{\alpha=1}^{r}z_{12}{}^{2\alpha-1,\rho}z_{12}{}^{2\alpha,\rho},$$

$$f_{[211]}{}^{\{11\}}(Z)=\sum_{\alpha=1}^{r}(z_1{}^1z_{123}{}^{3,2\alpha-1,2\alpha}-z_1{}^3z_{123}{}^{1,2\alpha-1,2\alpha}$$

$$-2z_1{}^{2\alpha}z_{123}{}^{1,3,2\alpha-1})-(\rho\ \text{odd})\,z_1{}^{\rho}z_{123}{}^{13\rho}, \tag{8.136}$$

where $\rho=4, 5, \cdots$. Additional highest weight vectors not in the above list which we require are:

$SO(4)$

$$f_{[11]}{}^{\{1,-1\}}(Z)=z_{12}{}^{14},$$

$$f_{[111]}{}^{\{1\}}(Z)=z_{123}{}^{134},$$

$$f_{[211]}{}^{\{1,-1\}}(Z)=z_1{}^1z_{123}{}^{124}+z_1{}^4z_{123}{}^{134}; \tag{8.137}$$

$SO(5)$

$$f_{[111]}{}^{\{11\}}(Z)=z_{123}{}^{135}; \tag{8.138}$$

$SO(6)$

$$f_{[111]}{}^{\{1,1,-1\}}(Z)=z_{123}{}^{136}. \tag{8.139}$$

The above vectors are $SO(\rho)\times U(3)$ highest weight vectors of the type (8.125). We have merely regarded the $SO(\rho)$ and $U(3)$ IR labels as superscripts and subscripts, respectively, and at the same time adopted the practice of omitting the unnecessary zeros in these IR labels. For example, $f_{[21]}{}^{\{1\}}$ is the highest weight vector in the carrier space of IR $\{10\cdots 0\}\otimes[210]$ of $SO(\rho)\times U(3)$, where IR $\{10\cdots 0\}$ of $SO(\rho)$ is contained in IR $[210\cdots 0]$ of $U(\rho)$. The integer r is, as always, $\rho/2$ for ρ even and $(\rho-1)/2$ for ρ odd. The notation (ρ odd) preceding a particular term indicates that such a term is included only when ρ is odd.

The calculation of, say, $f_{[21]}{}^{\{1\}}$ proceeds as follows: First, we determine which Gel'fand patterns having $U(\rho)$ IR labels $[210\cdots 0]$ give the $SO(\rho)$ weight $\{10\cdots 0\}$ (Lemma 10). The $U(\rho)$ *dominant weights* in IR $[210\cdots 0]$ are just $[210\cdots 0]$ and $2[1110\cdots 0]$. Therefore, the weight $\{10\cdots 0\}$ arises only from those $[210\cdots 0]$ Gel'fand patterns which have weights $[210\cdots 0]$, (ρ odd) $[10\cdots 02]$, $2[10110\cdots 0]$, $2[1000110\cdots 0]$, $\cdots$, (ρ even) $2[10\cdots 011]$, (ρ odd) $2[10\cdots 110]$. The 2 indicates that there will be two Gel'fand patterns having the indicated weight. Next, we work out the Gel'fand basis vectors (8.108) which have IR labels $[210\cdots 0]$, are *maximal* in their lower patterns, and which have upper patterns corresponding to the preceding sequence of $U(\rho)$ weights which yield $SO(\rho)$ weight $\{10\cdots 0\}$. We then determine the linear combination of these vectors which is annihilated by the raising generators (8.132)–(8.134). [This procedure is simpler than it appears, since it is actually necessary to work out the three Gel'fand basis vectors having weights $[210\cdots 0]$, $2[1110\cdots 0]$—the remaining ones are obtained by permuting the superscripts.]

One need not be too concerned with the derivation of the results (8.136)–(8.139). It is easy to verify directly that the given vectors possess the properties conveyed by their labels.

We can now construct an *enormous number* of more general $SO(\rho)\times U(3)$ highest weight vectors simply by forming arbitrary products of the vectors listed above. The IR labels of the resulting highest weight vector is obtained simply by *adding* the individual IR labels of each of the factors (since each factor is a highest weight vector). We will now note explicitly a set of $SO(\rho)\times U(3)$ highest weight vectors which is sufficiently general to include all cases for which $m_{13}+m_{23}+m_{33}\leq 4$. We now revert to the notation (8.125) dropping, however, unnecessary zeros in $\{\cdots\}$ and $[\cdots]$:

$$F(\{k-2s\}/[k])=(f_{[1]}{}^{\{1\}})^{k-2s}(f_{[2]}{}^{\{0\}})^{s}$$

for $s=0, 1, \cdots, [k/2]$,

$$F(\{k1\}/[k1])=(f_{[1]}{}^{\{1\}})^{k-1}f_{[11]}{}^{\{11\}},$$

$$F(\{k-2, 1\}/[k1])=(f_{[1]}{}^{\{1\}})^{k-3}f_{[2]}{}^{\{0\}}f_{[11]}{}^{\{11\}},$$

$$F(\{k-1\}/[k1])=(f_{[1]}{}^{\{1\}})^{k-2}f_{[21]}{}^{\{1\}},$$

$$F(\{k, 2\}/[k2])=(f_{[1]}{}^{\{1\}})^{k-2}(f_{[11]}{}^{\{11\}})^2,$$

$$F(\{k\}/[k2])=(f_{[1]}{}^{\{1\}})^{k-2}f_{[22]}{}^{\{2\}},$$

$$F(\{k-2\}/[k2])=(f_{[1]}{}^{\{1\}})^{k-2}f_{[22]}{}^{\{0\}},$$

$$F(\{k11\}/[k11])=(f_{[1]}{}^{\{1\}})^{k-1}f_{[111]}{}^{\{111\}},$$

$$F(\{k-1, 1\}/[k11])=(f_{[1]}{}^{\{1\}})^{k-2}f_{[211]}{}^{\{11\}}. \tag{8.140}$$

In addition, we need the following results which are particular to $\rho=4, 5, 6$ and which are not in the preceding list:

$SO(4)$

$$F(\{k, -1\}/[k1])=(f_{[1]}{}^{\{1\}})^{k-1}f_{[11]}{}^{\{1,-1\}},$$

$$F(\{k-2, -1\}/[k1])=(f_{[1]}{}^{\{1\}})^{k-3}f_{[2]}{}^{\{0\}}f_{[11]}{}^{\{1,-1\}},$$

$$F(\{k, -2\}/[k2])=(f_{[1]}{}^{\{1\}})^{k-2}(f_{[11]}{}^{\{1,-1\}})^2,$$

$$F(\{k\}/[k11])=(f_{[1]}{}^{\{1\}})^{k-1}f_{[111]}{}^{\{1\}},$$

$$F(\{k-1, -1\}/[k11])=(f_{[1]}{}^{\{1\}})^{k-2}f_{[211]}{}^{\{1,-1\}}; \tag{8.141}$$

$SO(5)$

$$F(\{k1\}/[k11])=(f_{[1]}{}^{\{1\}})^{k-1}f_{[111]}{}^{\{11\}}; \tag{8.142}$$

$SO(6)$

$$F(\{k, 1, -1\}/[k11])=(f_{[1]}{}^{\{1\}})^{k-1}f_{[111]}{}^{\{1,1,-1\}}. \tag{8.143}$$

The next step is to apply the $U(3)$ lowering operators to the vectors (8.140)–(8.143), thus generating the $U(3)$ basis vectors designated by the notation (8.128). In the last step, we form the linear combinations of these $U(3)$ vectors which appear in Eqs. (8.111)–(8.114). We can, in fact, rewrite Eqs. (8.111)–(8.114) in a form which explicitly contains the required lowering operators,[21] acting on the highest weight vector. Adapting these abstract results to the present notations then gives the following set of (un-normalized) $SO(\rho)\times SO(3)$ highest weight vectors:

$$F_{[k+\lambda,\lambda,\lambda]}(\{\cdots\}/L)=\Omega(k,L)F(\{\cdots\}/[k+\lambda,\lambda,\lambda]) \tag{8.144}$$

for $L=k, k-2, \cdots, 1$ or 0;

$$F_{[\lambda,\lambda,\lambda-k]}(\{\cdots\}/L)=\Lambda(k,L)F(\{\cdots\}/[\lambda,\lambda,\lambda-k]) \tag{8.145}$$

for $L=k, k-2, \cdots, 1$ or 0;

$$F_{[k+\lambda,\lambda+1,\lambda]}(\{\cdots\}/L)=\Omega'(k,L)F(\{\cdots\}/[k+\lambda,\lambda+1,\lambda]) \tag{8.146}$$

for $L=1, 2, \cdots, k$;

$$F_{[\lambda,\lambda-1,\lambda-k]}(\{\cdots\}/L)=\Lambda'(k,L)F(\{\cdots\}/[\lambda,\lambda-1,\lambda-k]) \tag{8.147}$$

for $L=1, 2, \cdots, k$.

In these equations, Ω and Λ are operators to be described. The F's on the right-hand side may be chosen to be any $SO(\rho)\times U(3)$ highest weight vector, where $\{\cdots\}$ designates any $SO(\rho)$ labels which are compatible with the $U(3)$ labels, i.e., $\{\cdots\}$ is contained in IR $[\cdots, 0\cdots 0]$ of $U(\rho)$.

The Ω and Λ operators are described as follows: Define

$$\Omega_\sigma(k,L)=\frac{2^\sigma(L+\sigma)!E_{32}{}^{k-L-2\sigma}E_{21}{}^{k-L-\sigma}}{([(k-L)/2]-\sigma)!k!(k-L-\sigma)!}, \tag{8.148}$$

$$\Lambda_\sigma(k,L)=\frac{(-1)^\sigma 2^\sigma(L+\sigma)!(k-L-2\sigma)!E_{21}{}^\sigma E_{32}{}^{L+2\sigma}}{([(k-L)/2]-\sigma)!k!\sigma!(L+2\sigma)!} \tag{8.149}$$

for $\sigma=0, 1, 2, \cdots, [(k-L)/2]$. Then we find

$$\Omega(k,L)=\sum_\sigma \Omega_\sigma(k,L), \tag{8.150}$$

$$\Lambda(k,L)=\sum_\sigma \Lambda_\sigma(k,L), \tag{8.151}$$

$$\Omega'(k,L)=\sum_\sigma[\Omega_{\sigma,0}(k,L)+\Omega_{\sigma,1}(k,L)], \tag{8.152}$$

$$\Lambda'(k,L)=\sum_\sigma[\Lambda_{\sigma,0}(k,L)+\Lambda_{\sigma,1}(k,L)], \tag{8.153}$$

where

$$\Omega_{\sigma,0}(k,L)=\Omega_\sigma(k,L)E_{32}/(L+2\sigma), \tag{8.154}$$

$$\Omega_{\sigma,1}(k,L)=a_k(E_{23}E_{12}-E_{13}E_{22})\Omega_{\sigma+1}(k,L-1)E_{32}, \tag{8.155}$$

$$\Lambda_{\sigma,0}(k,L)=\Lambda_\sigma(k,L)(E_{32})^{-1}, \tag{8.156}$$

$$\Lambda_{\sigma,1}(k,L)=b_\sigma(k,L)\Lambda_\sigma(k,L)\times[E_{31}(E_{11}-E_{22})+E_{32}E_{21}]/(L+2\sigma+2), \tag{8.157}$$

in which

$$\begin{aligned} a_k&=1/(k+1) && \text{for } k-L \text{ even}\\ &=\tfrac{1}{2} && \text{for } k-L \text{ odd}, \end{aligned} \tag{8.158}$$

$$\begin{aligned} b_\sigma(k,L)&=1/(k+1) && \text{for } k-L \text{ even}\\ &=1/(k-L-2\sigma) && \text{for } k-L \text{ odd}. \end{aligned} \tag{8.159}$$

[In Eq. (8.156), the factor $(E_{32})^{-1}$ is always preceded by a term containing E_{32} to a power greater than zero, and the notation is intended to denote symbolically the reduction of the preceding power by one and has nothing to do with inverse operators.]

All $SO(\rho)\times SO(3)$ $(\rho\geq 4)$ highest weight vectors of degree four or less can now be generated explicitly by selecting the appropriate $SO(\rho)\times U(3)$ highest weight vector from Eqs. (8.140)–(8.143) and applying the appropriate lowering operation from Eqs. (8.144)–(8.147). There is *no multiplicity* in the reduction of $SO(\rho)\times SO(3)\subset U(\rho)\times U(3)$ for $p\leq 5$. [We have not, however, listed sufficient results in Eqs. (8.140)–(8.143) to include all $p=5$ cases.]

The $SO(\rho)\times SO(3)$ highest weight vectors constructed by the preceding procedure will, in general, *not* solve Laplace's equation. We next examine this problem.

F. Solutions to Laplace's Equation

The operator $2\mathfrak{L}^\dagger$ which reduces to the Laplace operator in Euclidean 3ρ-space under the restriction of Z to the form $\tilde{A}_\rho XA_3$ of Eq. (8.39) is given by

$$(\mathfrak{L}^\dagger F)(Z)=\{\sum_\alpha(\bar{z}_1{}^{2\alpha-1}\bar{z}_2{}^{2\alpha}+\bar{z}_2{}^{2\alpha-1}\bar{z}_1{}^{2\alpha}+\bar{z}_3{}^{2\alpha-1}\bar{z}_3{}^{2\alpha})+(\rho\text{ odd})[\bar{z}_1{}^\rho\bar{z}_2{}^\rho+\tfrac{1}{2}(\bar{z}_3{}^\rho)^2]\}F(Z), \tag{8.160}$$

where the sum over α is from 1 to $[\rho/2]$.

The operator $\mathfrak{L}^\dagger$ carries a vector belonging to the space $\mathcal{H}_p$ into one belonging to the space $\mathcal{H}_{p-2}$. In particular, if $F[m_{13}m_{23}m_{33}]$ denotes an arbitrary vector belonging to the carrier space of IR $[m_{13}m_{23}m_{33}\,0\cdots 0]\otimes[m_{13}m_{23}m_{33}]$ [basis vectors given by Eq. (8.108)], then we have

$$\begin{aligned}\mathfrak{L}^\dagger F[m_{13}m_{23}m_{33}]&=F[m_{13}-2, m_{23}, m_{33}]+F[m_{13}, m_{23}-2, m_{33}]+F[m_{13}, m_{23}, m_{33}-2]\\&+F[m_{13}-1, m_{23}-1, m_{33}]+F[m_{13}-1, m_{23}, m_{33}-1]+F[m_{13}, m_{23}-1, m_{33}-1].\end{aligned} \tag{8.161}$$

[21] The must economical way to generate a general vector from a highest weight vector is not always accomplished by using the lowering operators of (Na65), and the lowering operators appearing in Eqs. (8.144)–(8.147) are not in these standard forms.

Let us introduce the notation $\mathcal{L}^\dagger[m_{13}m_{23}m_{33}]$ to denote the set of six (or less) shifted IR labels appearing in Eq. (8.161). For example, $\mathcal{L}^\dagger[310]=\{[110],[200]\}$. We will say that L is contained in $\mathcal{L}^\dagger[m_{13}m_{23}m_{33}]$ if the IR of $SO(3)$ specified by L is a constituent of at least one of the IR's of $U(3)$ specified by the set of labels $\mathcal{L}^\dagger[m_{13}m_{23}m_{33}]$. Similarly, we will say that $\{l_{\rho1}l_{\rho2}l_{\rho3}\}$ is contained in $\mathcal{L}^\dagger[m_{13}m_{23}m_{33}]$ if the IR of $SO(\rho)$ specified by $\{l_{\rho1}l_{\rho2}l_{\rho3}\}$ is a constituent of at least one of the IR's of $U(\rho)$ specified by the set of labels $\mathcal{L}^\dagger[m_{13}m_{23}m_{33}]$.

Now consider an $SO(\rho)\times SO(3)$ highest weight vector [notation (8.126)]. Then we have

$$\mathcal{L}^\dagger F_{[m_{13}m_{23}m_{33}]}(\{l_{\rho1}l_{\rho2}l_{\rho3}\}/L)=0 \qquad (8.162)$$

if *either* $\{l_{\rho1}l_{\rho2}l_{\rho3}\}$ *or* L is not contained in $\mathcal{L}^\dagger[m_{13}m_{23}m_{33}]$.

Property (8.162) follows from the fact that $\mathcal{L}^\dagger$ commutes with the generators of $SO(\rho)\times SO(3)$: If the vector (8.162) were not the zero vector, it would be an $SO(\rho)\times SO(3)$ highest weight vector of degree $m_{13}+m_{23}+m_{33}-2$, and both $\{l_{\rho1}l_{\rho2}l_{\rho3}\}$ and L would be contained in $\mathcal{L}[m_{13}m_{23}m_{33}]$.

In general, it is necessary to form linear combinations of $SO(\rho)\times SO(3)$ highest weight vectors in order to satisfy Laplace's equation:

$$\sum A_{[m]}F_{[m]}(\{l_{\rho1}l_{\rho2}l_{\rho3}\}/L), \qquad (8.163)$$

where the sum is over all sets of labels $[m]=[m_{13}m_{23}m_{33}]$ such that: (1) $m_{13}+m_{23}+m_{33}=p$; (2) both $\{l_{\rho1}l_{\rho2}l_{\rho3}\}$ and L are contained in each $[m]$; and (3) both $\{l_{\rho1}l_{\rho2}l_{\rho3}\}$ and L are contained in each $\mathcal{L}^\dagger[m]$.

Using Eqs. (8.162) and (8.163), we can now list the form of the solutions to Laplace's equation through degree four:

$p=1$:

$$F_{[100]}(\{1\}/1);$$

$p=2$:

$$F_{[200]}(\{\lambda\}/L)$$

for $\lambda=0$, 2 and $L=0$, 2 but $\lambda=L\neq0$;

$$F_{[110]}(\{11\}/1);$$

$p=3$:

$$F_{[300]}(\{\lambda\}/L)$$

for $\lambda=1$, 3 and $L=1$, 3 but $\lambda=L\neq1$;

$$F_{[210]}(\{21\}/L) \qquad L=1,2,$$

$$F_{[210]}(\{1\}/2),$$

$$aF_{[300]}(\{1\}/1)+bF_{[210]}(\{1\}/1);$$

$p=4$:

$$F_{[400]}(\{\lambda\}/4) \qquad \lambda=0,2,4,$$

$$F_{[400]}(\{4\}/L) \qquad L=0,2,$$

$$F_{[310]}(\{31\}/L) \qquad L=1,2,3,$$

$$F_{[310]}(\{11\}/L) \qquad L=2,3,$$

$$F_{[310]}(\{2\}/L) \qquad L=1,3,$$

$$F_{[220]}(\{22\}/L) \qquad L=0,2,$$

$$F_{[211]}(\{\cdots\}/1)$$

for $\{\cdots\}=\{2\}$ in $SO(4)$, $\{21\}$ in $SO(5)$, $\{2,1,\pm1\}$ in $SO(6)$, and $\{211\}$ thereafter;

$$aF_{[400]}(\{\lambda\}/L)+bF_{[220]}(\{\lambda\}/L)$$

for $\lambda=0$, 2 and $L=0$, 2 but $\lambda=L\neq2$;

$$aF_{[310]}(\{11\}/1)+bF_{[211]}(\{11\}/1),$$

$$aF_{[400]}(\{2\}/2)+bF_{[310]}(\{2\}/2)+cF_{[220]}(\{2\}/2).$$

The preceding list gives the *form* of all $SO(\rho)\times SO(3)$ highest weight vectors of degree four and less which solve Laplace's equation for $\rho>4$. For $\rho=4$, one simply includes those $SO(4)$ IR labels which have a negative sign on the second label, i.e., $\{1,-1\}$ wherever $\{1,1\}$ appears, etc.

The occurrence of two constants a and b in this list means that a unique linear combination of the respective vectors will solve Laplace's equation; the occurrence of three constants a, b, and c means that two independent linear combinations of the respective vectors will solve Laplace's equation. Thus, the highest weight vector labeled by $(\{2\}/2)$ occurs with multiplicity two. This vector is the only one having a multiplicity in the set of all states up to degree four.

The explicit listing of these solid harmonics up to degree four is now entirely mechanical: Using the formulas of Sec. E, we work out the various $SO(\rho)\times SO(3)$ highest weight vectors and determine, in those few cases where required, those linear combinations which are annihilated by $\mathcal{L}^\dagger$. The explicit tabulation of these vectors for arbitrary ρ is entirely feasible.

One must still classify the states with respect to their properties under the permutations of the particles. It appears that this procedure should be carried out for each ρ separately, and we have not attempted to do this.

APPENDIX 1. PROOF OF LEMMA 9

The purpose of this Appendix is to prove that the relation between matrices of $SU(4)$ and $SO(6)$ stated in Lemma 9 is a homomorphism.

Let $V\in U(4)$. We first show that the matrix R of Eq. (6.3), which is defined in terms of V through the sequence of relations, Eqs. (6.3)–(6.8), is proper, real orthogonal. That R is real follows from the form of Q and the explicit expression for A. Thus, $R\in SO(6)$ if Q is unitary unimodular, i.e., if $\det Q=1$ and the matrices Q_1 and Q_2 satisfy

$$Q_1Q_1^\dagger+Q_2^*\tilde{Q}_2=I_3, \qquad (A1.1)$$

$$Q_1Q_2^\dagger+Q_2^*\tilde{Q}_1=0. \qquad (A1.2)$$

We must demonstrate that these relations are indeed satisfied by Q_1 and Q_2 of Eqs. (6.6)–(6.8) for $V\in U(4)$. (The restriction to $SU(4)$ is not necessary at this point.)

Since we are assuming that $V\in U(4)$, the unitary condition $VV^\dagger=I_4$ implies the following relations among the partition matrices appearing in Eq. (6.1):

$$V_3V_3^\dagger+\alpha\alpha^\dagger=I_3, \tag{A1.3}$$

$$V_3\beta^*=-\gamma^*\alpha, \tag{A1.4}$$

$$\tilde{\beta}V_3^\dagger=-\gamma\alpha^\dagger, \tag{A1.5}$$

$$\tilde{\beta}\beta^*+\gamma\gamma^*=1. \tag{A1.6}$$

Using these relations, definitions (6.6)–(6.8), and the property $\Gamma_\alpha\alpha=0$, we easily establish the following identities:

$$Q_1Q_1^\dagger=\gamma\gamma^*I_3+\alpha\alpha^\dagger, \tag{A1.7}$$

$$Q_2^*\tilde{Q}_2=\Gamma_\alpha^*\tilde{\Gamma}_\alpha=(\alpha^\dagger\alpha)I_3-\alpha\alpha^\dagger, \tag{A1.8}$$

$$Q_1Q_2^\dagger=-\gamma\Gamma_\alpha^*, \tag{A1.9}$$

$$Q_2^*Q_1=\gamma\Gamma_\alpha^*. \tag{A1.10}$$

Noting that V being unitary also implies $\alpha^\dagger\alpha+\gamma\gamma^*=1$, we obtain Eqs. (A1.1) and (A1.2) from those above.

Nowhere in deriving the above results have we required V to be *unimodular*. For each $V\in U(4)$, the equations of Lemma 9 define a real orthogonal matrix $R\in O(6)$. We still must prove that $\det R=1$, i.e., that $\det Q=1$. This result follows from the matrix lemma given in Appendix 2 (since $Q_1Q_1^\dagger$ has eigenvalues 1, $|\gamma|^2$, $|\gamma|^2$).

We have thus proved: *For each $V\in U(4)$, the equations of Lemma 9 define a unique $R\in SO(6)$.*

To show that the equations of Lemma 9 give a mapping of $U(4)$ *onto* $SO(6)$, it must be demonstrated that each $R\in SO(6)$ is the image of at least one $V\in U(4)$. Stated less precisely: Given $R\in SO(6)$, we must be able to "solve" the equations of Lemma 9 to find a $V\in U(4)$. This we next do.

Let $R\in SO(6)$. Then the matrix $Q=A^\dagger RA$ is *unitary unimodular*, and has the form

$$Q=\begin{pmatrix} Q_1 & Q_2^* \\ Q_2 & Q_1^* \end{pmatrix}, \tag{A1.11}$$

where

$$Q_1=[R_{11}+R_{22}+i(R_{12}-R_{21})]/2, \tag{A1.12}$$

$$Q_2=[R_{11}-R_{22}+i(R_{12}+R_{21})]/2, \tag{A1.13}$$

in which each R_{ij} is a 3×3 real matrix which comes from partitioning R. Since Q is unitary unimodular, the matrices Q_1 and Q_2 necessarily satisfy

$$Q_1Q_1^\dagger+(Q_2Q_2^\dagger)^*=I_3, \tag{A1.14}$$

$$Q_2Q_1^\dagger+(Q_1Q_2^\dagger)^*=0. \tag{A1.15}$$

Given the two matrices Q_1 and Q_2 and the properties $\det Q=1$ and Eqs. (A1.14) and (A1.15), the problem is to find V_3, α, β, and γ such that: (a) Eqs. (6.6)–(6.8) yield Q_1 and Q_2, and (b) the matrix V defined by Eq. (6.1) belongs to $U(4)$. We now give the complete and general construction of V_3, α, β, and γ such that these properties obtain.

The difficult part of the construction is contained in an ancillary result which we state, deferring the proof until later (Appendix 2): *For $\det Q=1$, the matrix Q_1 in Eq. (A1.11) is such that the Hermitian positive semi-definite matrix $Q_1Q_1^\dagger$ has eigenvalues 1, $|\gamma|^2$, $|\gamma|^2$, where*

$$|\gamma|^2\equiv[-1+\mathrm{Tr}\,(Q_1Q_1^\dagger)]/2\leq1. \tag{A1.16}$$

This is the key property which allows us to carry through the proof.

Recall that a 3×3 Hermitian matrix is related to its orthonormal eigenvectors (column matrices) v_1, v_2, v_3 and its real eigenvalues λ_1, λ_2, λ_3 by

$$H=\sum_i\lambda_iv_iv_i^\dagger, \tag{A1.17}$$

where

$$\sum_i v_iv_i^\dagger=I_3. \tag{A1.18}$$

Using this result, we now deduce the form of $Q_1Q_1^\dagger$:

$$Q_1Q_1^\dagger=|\gamma|^2I_3+\alpha\alpha^\dagger, \tag{A1.19}$$

where α is the eigenvector of $Q_1Q_1^\dagger$ having eigenvalue 1 and normalized such that

$$\alpha^\dagger\alpha+|\gamma|^2=1. \tag{A1.20}$$

(This is always possible, since $0\leq|\gamma|^2\leq1$.)

Equations (A1.16) and (A1.19) are the basic relations which allow us to obtain the inverse solutions to Eqs. (6.6)–(6.8). Let us describe the construction of these solutions, verifying subsequently that they are solutions. First, we select any γ satisfying Eq. (A1.16). Second, we determine the column matrix α up to an over-all phase from

$$\alpha_i\alpha_j^*=(Q_1Q_1^\dagger)_{ij}-|\gamma|^2\delta_{ij}. \tag{A1.21}$$

Third, we form the skew-symmetric matrix Γ_α of Eq. (6.8) and *fix the phase* of α by requiring

$$\Gamma_\alpha Q_1=\gamma Q_2. \tag{A1.22}$$

(We must show that this is possible.) Finally, we define β and V_3 as follows:

$$\beta=-\tilde{Q}_1\alpha^*, \tag{A1.23}$$

$$V_3=\gamma^*Q_1-\Gamma_\alpha^*Q_2. \tag{A1.24}$$

We must demonstrate that: (a) the γ, α, β, and V_3 given above solve Eqs. (6.6) and (6.7); and (b) the matrix V, now defined by Eq. (6.1), belongs to $U(4)$.

First, we verify (a):

$$\begin{aligned}\gamma V_3-\alpha\tilde{\beta}&=|\gamma|^2Q_1-\gamma\Gamma_\alpha^*Q_2+\alpha\alpha^\dagger Q_1\\&=(|\gamma|^2+\alpha^\dagger\alpha)Q_1+\Gamma_\alpha^*(\Gamma_\alpha Q_1-\gamma Q_2)\\&=Q_1,\end{aligned} \tag{A1.25}$$

where we have used the identity

$$\alpha\alpha^\dagger=(\alpha^\dagger\alpha)I_3+\Gamma_\alpha^*\Gamma_\alpha \tag{A1.26}$$

in the second step, and where Eqs. (A1.20) and (A1.22) have been used in the last step,

$$\Gamma_\alpha V_3=\gamma^*\Gamma_\alpha Q_1-\Gamma_\alpha\Gamma_\alpha^*Q_2$$
$$=(|\gamma|^2+\alpha^\dagger\alpha)Q_2-\alpha^*\tilde{\alpha}Q_2=Q_2, \quad (A1.27)$$

in consequence of Eq. (A1.20) and the property $\tilde{\alpha}Q_2=0$ [The property $\tilde{Q}_2\alpha=0$ is an easy result of the definition of α and relation (A1.14)].

Second, we verify (b) [by showing that Eqs. (A1.3)–(A1.6) are satisfied]. The following preliminary relations simplify the proof:

$$\gamma^*Q_1Q_2^\dagger\Gamma_\alpha=-\gamma\Gamma_\alpha^*Q_2Q_1^\dagger=|\gamma|^2(Q_2Q_2^\dagger)^*, \quad (A1.28)$$

$$\Gamma_\alpha^*Q_2Q_2^\dagger\Gamma_\alpha=\Gamma_\alpha^*\Gamma_\alpha-\Gamma_\alpha^*Q_1^*\tilde{Q}_1\Gamma_\alpha$$
$$=\Gamma_\alpha^*\Gamma_\alpha+|\gamma|^2(Q_2Q_2^\dagger)^*. \quad (A1.29)$$

These relations follow quite simply from Eqs. (A1.14) and (A1.15) and property (A1.22). Then

$$V_3V_3^\dagger+\alpha\alpha^\dagger=|\gamma|^2Q_1Q_1^\dagger-\Gamma_\alpha^*Q_2Q_2^\dagger\Gamma_\alpha$$
$$+\gamma^*Q_1Q_2^\dagger\Gamma_\alpha-\gamma\Gamma_\alpha^*Q_2Q_1^\dagger+\alpha\alpha^\dagger$$
$$=|\gamma|^2[Q_1Q_1^\dagger+(Q_2Q_2^\dagger)^*]-\Gamma_\alpha^*\Gamma_\alpha+\alpha\alpha^\dagger$$
$$=(|\gamma|^2+\alpha^\dagger\alpha)I_3=I_3 \quad (A1.30)$$

by using Eqs. (A1.14), (A1.20), and (A1.26);

$$V_3\beta^*=-(\gamma^*Q_1-\Gamma_\alpha^*Q_2)Q_1^\dagger\alpha=-\gamma^*\alpha, \quad (A1.31)$$

since $Q_1Q_1^\dagger\alpha=\alpha$ and $Q_2Q_1^\dagger\alpha=-Q_1^*\tilde{Q}_2\alpha=0$;

$$\tilde{\beta}\beta^*+|\gamma|^2=\alpha^\dagger Q_1Q_1^\dagger\alpha+|\gamma|^2=\alpha^\dagger\alpha+|\gamma|^2=1. \quad (A1.32)$$

We still must demonstrate that α can be chosen such that Eq. (A1.22) is satisfied. The relation

$$(\Gamma_\alpha Q_1)(\Gamma_\alpha Q_1)^\dagger=|\gamma|^2\Gamma_\alpha\Gamma_\alpha^\dagger=|\gamma|^2Q_2Q_2^\dagger \quad (A1.33)$$

is established by multiplying Eq. (A1.19) from the left by Γ_α and from the right by $\Gamma_\alpha^\dagger$, noting that $\Gamma_\alpha\alpha=0$. Equations (A1.26) and (A1.14) are then used to prove $\Gamma_\alpha\Gamma_\alpha^\dagger=Q_2Q_2^\dagger$. If $\gamma=0$, then we obtain $\Gamma_\alpha Q_1=0$ from Eq. (A1.33) and Eq. (A1.22) is therefore correct. (Note that for $\gamma=0$, the column matrix α is determined only up to a phase.) If $\gamma\neq0$, then the matrix Q_1 is nonsingular. Furthermore, the relation

$$|\gamma|^2Q_1^{-1}=Q_1^\dagger(I_3-\alpha\alpha^\dagger) \quad (A1.34)$$

follows from Eq. (A1.19) and the property $Q_1^{-1}\alpha=Q_1^\dagger\alpha$. Since $Q_2Q_1^\dagger\alpha=0$, we obtain

$$|\gamma|^2Q_2Q_1^{-1}=Q_2Q_1^\dagger. \quad (A1.35)$$

The matrix $Q_2Q_1^\dagger$ is skew-symmetric [property (A1.15)], and α is an eigenvector having eigenvalue 0. The form of $Q_2Q_1^\dagger$ therefore must be

$$Q_2Q_1^\dagger=a\Gamma_\alpha \quad (A1.36)$$

for some complex number a. Hence,

$$a\Gamma_\alpha Q_1=|\gamma|^2Q_2, \quad (A1.37)$$

and Eq. (A1.33) requires $|a|=|\gamma|$. We put $a=\gamma^*e^{i\varphi}$, $\alpha'=\alpha e^{i\varphi}$ to obtain

$$\Gamma_{\alpha'}Q_1=\gamma Q_2. \quad (A1.38)$$

Thus, we can always satisfy Eq. (A1.22) by an appropriate choice of phase of the column matrix α.

Observe that the matrix Γ_α, hence α, is determined directly by

$$\Gamma_\alpha=(\gamma^*)^{-1}Q_2Q_1^\dagger \quad (A1.39)$$

for $\gamma\neq0$. (We have, however, been careful to formulate the construction of an inverse solution such that it is valid even for $\gamma=0$.)

The results, Eqs. (A1.25)–(A1.38), prove that for each γ determined by Eq. (A1.16), and for α, β, and V_3 given by Eqs. (A1.21)–(A1.24), we obtain a $V\in U(4)$ which maps to the proper orthogonal matrix R whose elements determine Q_1 and Q_2 by Eqs. (A1.12) and (A1.13): *The equations of Lemma 9 are a mapping of $U(4)$ onto $SO(6)$.*

Note that the arbitrary phase of γ (or the arbitrary phase of α if $\gamma=0$) can always be chosen such that $V\in SU(4)$.

While Lemma 9 gives, in fact, a mapping of $U(4)$ onto $SO(6)$, this mapping need *not* be a homomorphism of $U(4)$ onto $SO(6)$. Indeed, the elements of $U(4)$ which are mapped to the identity I_6 of $SO(6)$ are of the form

$$\left(\begin{array}{ccc|c} & & & 0\\ & e^{-i\varphi}I_3 & & 0\\ & & & 0\\ \hline 0 & 0 & 0 & e^{i\varphi}\end{array}\right)\to I_6 \quad (A1.40)$$

for $0\leq\varphi\leq2\pi$. But these elements are not the elements of an *invariant* subgroup of $U(4)$—hence, the mapping *cannot* be a homomorphism of $U(4)$ onto $SO(4)$. However, if we restrict to $SU(4)$, then we have

$$\pm I_4\to I_6, \quad (A1.41)$$

and since $\{I_4, -I_4\}$ are the elements of an invariant subgroup of $SU(4)$, the mapping of Lemma 9 may be a homomorphism of $SU(4)$ onto $SO(6)$. That this is the case, we now prove.

The mapping of Lemma 9 is a homomorphism if it can be demonstrated that $V\to R$, $V'\to R'$ implies $VV'\to RR'$. *This mapping is a homomorphism if and only if $V\in SU(4)$.* (We have already established the "only if" part.)

The proof that we have a homomorphism is not as trivial as one might think. Taking V to be of the form of Eq. (6.1) and V' to be

$$V'=\left(\begin{array}{c|c} V_3' & \alpha'\\ \hline \tilde{\beta}' & \gamma'\end{array}\right), \quad (A1.42)$$

we obtain

$$V''=VV'=\left(\begin{array}{c|c} V_3'' & \alpha'' \\ \hline \tilde{\beta}'' & \gamma'' \end{array}\right), \tag{A1.43}$$

where

$$V_3''=V_3V_3'+\alpha\tilde{\beta}',$$
$$\alpha''=V_3\alpha'+\gamma'\alpha,$$
$$\beta''=\tilde{V}_3'\beta+\gamma\beta',$$
$$\gamma''=\tilde{\beta}\alpha'+\gamma\gamma'. \tag{A1.44}$$

The product $QQ'=Q''$ is again of the form, Eq. (6.5), where

$$Q_1''=Q_1Q_1'+Q_2^*Q_2', \tag{A1.45}$$

$$Q_2''=Q_2Q_1'+Q_1^*Q_2'. \tag{A1.46}$$

To establish the homomorphism, it must be demonstrated that Eqs. (A1.45) and (A1.46) are *identically* satisfied when Q_1, Q_2, Q_1', Q_2', Q_1'', Q_2'' are expressed in terms of the respective partition matrices of V, V', V'', where we can use the relations (A1.44) between these partition matrices. These substitutions lead to the following results. Equations (A1.45) and (A1.46) are satisfied identically if and only if the following relations hold identically:

$$\Gamma_\alpha^*V_3^*\Gamma_{\alpha'}V_3'=(\tilde{\beta}\alpha')V_3V_3'-V_3\alpha'\tilde{\beta}V_3', \tag{A1.47}$$

$$\Gamma_{V_3\alpha'}V_3V_3'=\gamma^*V_3^*\Gamma_{\alpha'}V_3'-\alpha^*\beta^\dagger\Gamma_{\alpha'}V_3'. \tag{A1.48}$$

[In obtaining Eq. (A1.48), we have used two properties of skew-symmetric matrices of the type Γ_a, namely, $\Gamma_ab=-\Gamma_ba$ and $\Gamma_{a+b}=\Gamma_a+\Gamma_b$ for arbitrary $a=(a_1a_2a_3)$ and $b=(b_1b_2b_3)$.] Since these equations must be identically satisfied, the conditions that our mapping be a homomorphism are reduced to

$$\Gamma_\alpha^*V_3^*\Gamma_{\alpha'}=-V_3\Gamma_\beta\Gamma_{\alpha'}, \tag{A1.49}$$

$$\Gamma_{V_3\alpha'}V_3=\gamma^*V_3^*\Gamma_{\alpha'}-\alpha^*\beta^\dagger\Gamma_{\alpha'}, \tag{A1.50}$$

where we have used the relation

$$\alpha'\tilde{\beta}=\Gamma_\beta\Gamma_{\alpha'}+(\tilde{\beta}\alpha')I_3 \tag{A1.51}$$

in obtaining Eq. (A1.49).

Condition (A1.50) can be reduced still further upon noting that

$$\tilde{V}_3V_3^*=I_3-\beta\beta^\dagger, \tag{A1.52}$$

$$\tilde{V}_3\alpha^*=-\gamma^*\beta. \tag{A1.53}$$

(These relations follow from $V^\dagger V=I_4$.) Multiplying Eq. (A1.50) from the left by $\tilde{V}_3$ now yields

$$\tilde{V}_3\Gamma_{V_3\alpha'}V_3=\gamma^*\Gamma_{\alpha'}. \tag{A1.54}$$

Since each side of this equation is skew-symmetric, we can equate the (23), (31), and (12) elements and obtain a column matrix relation. The result is

$$\tilde{C}V_3\alpha'=\gamma^*\alpha', \tag{A1.55}$$

where C is the cofactor matrix of V_3, i.e., the element C_{ij} in row i and column j of C is the cofactor of the element in row i and column j of V_3. The relation between C and V_3 is

$$\tilde{C}V_3=V_3\tilde{C}=(\det V_3)I_3. \tag{A1.56}$$

Equations (A1.49) and (A1.55) must hold for arbitrary α'. Thus, we finally obtain: *The necessary and sufficient conditions for the mapping of Lemma 9 to be a homomorphism are*

$$\Gamma_\alpha^*V_3^*=-V_3\Gamma_\beta, \tag{A1.57}$$

$$\gamma^*=\det V_3 \tag{A1.58}$$

for each V in the map. We assert: These relations are satisfied for each $V\in SU(4)$.

To prove the above assertion, we start with the unitary relation

$$V^*=\tilde{V}^{-1}=(\det V)^{-1}\mathfrak{C}, \tag{A1.59}$$

where $\mathfrak{C}$ is the cofactor matrix of V [see the definition preceding Eq. (A1.56)]. By direct examination of the cofactors of V, written in the form of Eq. (6.1), we find the following identities:

$$\mathfrak{C}_{ij}=\gamma C_{ij}+(\Gamma_\alpha V_3\Gamma_\beta)_{ij} \qquad i,j=1,2,3,$$
$$\text{row }[\mathfrak{C}_{41}\ \mathfrak{C}_{42}\ \mathfrak{C}_{43}]=-\tilde{\alpha}C,$$
$$\text{col }[\mathfrak{C}_{14}\ \mathfrak{C}_{24}\ \mathfrak{C}_{34}]=-C\beta,$$
$$\mathfrak{C}_{44}=\det V_3. \tag{A1.60}$$

Writing V^* in the form of Eq. (6.1), there obtains the following relations for each $V\in U(4)$:

$$(\det V)V_3^*=\gamma C+\Gamma_\alpha V_3\Gamma_\beta, \tag{A1.61}$$

$$(\det V)\alpha^*=-C\beta, \tag{A1.62}$$

$$(\det V)\beta^*=-\tilde{C}\alpha, \tag{A1.63}$$

$$\gamma^*\det V=\det V_3. \tag{A1.64}$$

From Eq. (A1.64), we obtain the result: $\det V=1$ implies $\gamma^*=\det V_3$.

Next, we multiply Eq. (A1.61) from the right with Γ_β^*, noting that

$$\Gamma_\alpha V_3\Gamma_\beta\Gamma_\beta^*=\Gamma_\alpha V_3(\beta^*\tilde{\beta}-\beta^\dagger\beta I_3)$$
$$=\Gamma_\alpha(-\gamma^*\alpha\tilde{\beta}-\beta^\dagger\beta V_3)$$
$$=-(\beta^\dagger\beta)\Gamma_\alpha V_3,$$
$$C\Gamma_\beta^*=\gamma^*(\det V)V_3^*\Gamma_\beta^*.$$

Thus, we obtain

$$\Gamma_\alpha V_3=-(\det V)V_3^*\Gamma_\beta^*. \tag{A1.65}$$

We can now conclude: $\det V=1$ implies $\Gamma_\alpha^*V_3^*=-V_3\Gamma_\beta$.

The proof that our mapping of $U(4)$ onto $SO(6)$ is a homomorphism if and only if $V\in SU(4)$ has now been completed. Note that the kernel of the homomorphism is

$$\pm I_4 \to I_6. \qquad (A1.66)$$

The homomorphism is thus 2 to 1.

APPENDIX 2. A MATRIX LEMMA

The following matrix lemma is proved in this Appendix: *Each unitary matrix, $Q \in U(6)$, which is of the form*

$$Q = \begin{pmatrix} Q_1 & Q_2^* \\ Q_2 & Q_1^* \end{pmatrix} \qquad (A2.1)$$

either has $\det Q = +1$ *or* $\det Q = -1$. *If* $\det Q = +1$, *then* $Q_1 Q_1^\dagger$ *has eigenvalues* $1, \lambda, \lambda$ *for some* λ *in the interval* $0 \le \lambda \le 1$, *and conversely; if* $\det Q = -1$, *then* $Q_1 Q_1^\dagger$ *has eigenvalues* $0, \lambda, \lambda$ *for some* λ *in the interval* $0 \le \lambda \le 1$, *and conversely.*

The proof that the homomorphism of Lemma 9 from $SU(4)$ to $SO(6)$ is onto depends crucially on the validity of the above result in the case $\det Q = 1$. [Note that the subgroup of $U(6)$ with elements of the form (A2.1) is then isomorphic to $O(6)$, the isomorphism being given by $R = AQA^\dagger$, where A is defined by Eq. (6.4).]

To simplify the notation in the proof, we define the 3×3 matrices as follows:

$$S = Q_2 Q_1^\dagger, \qquad H = Q_1 Q_1^\dagger, \qquad K = (Q_2 Q_2^\dagger)^*. \quad (A2.2)$$

Then H and K are Hermitian positive semidefinite, and S is skew–symmetric.

The conditions that Q be unitary are now expressed as

$$H + K = I, \qquad \tilde{S} = -S. \qquad (A2.3)$$

The first of these relations implies that each eigenvalue (necessarily real) of H is in the interval [0, 1]. The relation

$$H(I-H) = S^\dagger S \qquad (A2.4)$$

is also easily establishing upon using the relation $Q_2^\dagger Q_2 = I - Q_1^\dagger Q_1$. Furthermore, since S is skew–symmetric, it has the form

$$S = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}. \qquad (A2.5)$$

The eigenvalues of $S^\dagger S$ are $0, a^2, a^2$, where $a^2 = |a_1|^2 + |a_2|^2 + |a_3|^2$.

Next, we observe from Eq. (A2.4) that H and $S^\dagger S$ commute. Consequently, these two Hermitian matrices can be simultaneously diagonalized by a unitary matrix. Again using Eq. (A2.4), we deduce the following result: *Each eigenvalue of H belongs to the set*

$$\{0, 1, \lambda, 1-\lambda\}, \qquad (A2.6)$$

where λ is a number which satisfies

$$\lambda(1-\lambda) = a^2. \qquad (A2.7)$$

Note that $0 \le a^2 \le 1/4$ in consequence of the fact that the largest value of a^2 occurs for $\lambda = \frac{1}{2}$. Thus, the roots of Eq. (A2.7) always belong to the interval [0, 1]. Note also that if λ_1 is a root of Eq. (A2.7), then the other root is $1 - \lambda_1$.

We divide the remaining portion of the proof into three parts: (a) Q_1 is nonsingular; (b) Q_2 is nonsingular; and (c) Q_1 and Q_2 are singular.

(a) Q_1 *is Nonsingular.* If Q_1 is nonsingular, we can write

$$Q = \begin{pmatrix} I & B^* \\ B & I \end{pmatrix} \begin{pmatrix} Q_1 & 0 \\ 0 & Q_1^* \end{pmatrix}, \qquad (A2.8)$$

where

$$B = Q_2 Q_1^{-1}, \qquad (A2.9)$$

and

$$S = BH = H^* B. \qquad (A2.10)$$

The second identity in Eq. (A2.10) follows easily upon multiplying $Q_1^\dagger Q_1 + Q_2^\dagger Q_2 = I$ from the left by Q_2 and from the right by Q_1^{-1}. Since S is skew–symmetric and H is Hermitian and nonsingular, it follows from Eq. (A2.10) that B is skew–symmetric

$$B = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}. \qquad (A2.11)$$

Evaluation of $\det Q$ from Eq. (A2.8) now gives

$$\det Q = (1 + b^2)^2 \det H, \qquad (A2.12)$$

where

$$b^2 = |b_1|^2 + |b_2|^2 + |b_3|^2.$$

Next, from Eq. (A2.10), we obtain

$$B^\dagger B = H^{-1} S^\dagger S H^{-1}, \qquad (A2.13)$$

so that

$$2b^2 = \mathrm{Tr}\,(B^\dagger B) = \mathrm{Tr}\,(S^\dagger S H^{-2}). \qquad (A2.14)$$

The unitary transformation which diagonalizes $S^\dagger S$ and H carries $S^\dagger S$ to the diagonal form diag $(0, a^2, a^2)$ and H to the form diag $(1, \lambda, \mu)$. Evaluating the trace of the right-hand side of Eq. (A2.14) now yields the relation

$$2b^2 = a^2(\lambda^2 + \mu^2)/\lambda^2\mu^2. \qquad (A2.15)$$

Since H is nonsingular, its eigenvalues are ***necessarily either*** $(1, \lambda, \lambda)$ *or* $(1, \lambda, 1-\lambda)$, where λ is a root of $\lambda(1-\lambda) = a^2$. Corresponding to the set $(1, \lambda, \lambda)$, we obtain the value of $\det Q$ to be

$$\det Q = 1. \qquad (A2.16)$$

Corresponding to the set $(1, \lambda, 1-\lambda)$, we obtain the value of $\det Q$ to be

$$\det Q = 1/4\lambda(1-\lambda). \qquad (A2.17)$$

Since Q is unitary, its determinant must have absolute value 1. Therefore, the possible eigenvalue set

$(1, \lambda, 1-\lambda)$ must be excluded unless $\lambda=1/2$ in which case it is of the form $(1, 1/2, 1/2)$. Thus, Q_1 nonsingular implies that $\det Q=1$ and that $H=Q_1Q_1^\dagger$ has eigenvalues $(1, \lambda, \lambda)$ for some λ in the interval $[0, 1]$.

(b) Q_2 *is Nonsingular.* We use the following relation and apply the results of Part (a)

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} Q_1 & Q_2^* \\ Q_2 & Q_1^* \end{pmatrix}=\begin{pmatrix} Q_2 & Q_1^* \\ Q_1 & Q_2^* \end{pmatrix}=Q'. \quad \text{(A2.18)}$$

Since Q' is unitary and Q_2 is nonsingular, we must have $\det Q'=1$ and the eigenvalues of $Q_2Q_2^\dagger$ are $(1, 1-\lambda, 1-\lambda)$ for some λ in the interval $[0, 1]$. Therefore, the eigenvalues of $Q_1Q_1^\dagger$ are $(0, \lambda, \lambda)$, and $\det Q=-1$ follows from Eq. (A2.18).

(c) Q_1 *and* Q_2 *are Singular.* Both H and K are singular. Hence, each possesses an eigenvector having eigenvalue 0. It follows from Eq. (A2.3) that the orthonormal eigenvectors, u_1, u_2, and u_3 of H satisfy

$$\begin{aligned} Hu_1&=u_1, & Ku_1&=0, \\ Hu_2&=0, & Ku_2&=u_2, \\ Hu_3&=\lambda u_3, & Ku_3&=(1-\lambda)u_3. \end{aligned} \quad \text{(A2.19)}$$

But, from Eq. (A2.4), we obtain $S^\dagger Su_1=0$, $S^\dagger Su_2=0$. However, $S^\dagger S$ necessarily has eigenvalues $(0, a^2, a^2)$, and the only way that two can be zero is for three to be 0, and therefore S is the zero matrix

$$S=0. \quad \text{(A2.20)}$$

Hence, λ in Eqs. (A2.19) is either 0 *or* 1, and H is idempotent:

$$H^2=H. \quad \text{(A2.21)}$$

Let U denote the 3×3 unitary matrix whose columns are the eigenvectors of H:

$$U=[u_1\, u_2\, u_3]. \quad \text{(A2.22)}$$

Then

$$Q'\equiv\begin{pmatrix} U^\dagger & 0 \\ 0 & \tilde{U} \end{pmatrix}Q^\dagger\begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}=\begin{pmatrix} A & B^* \\ B & A^* \end{pmatrix}, \quad \text{(A2.23)}$$

where

$$A=U^\dagger Q_1^\dagger U, \quad \text{(A2.24)}$$

$$B=\tilde{U}\tilde{Q}_2U. \quad \text{(A2.25)}$$

Now consider the two cases, $\lambda=0$ or 1, taking $\lambda=0$ first. It follows easily from Eqs. (A2.19) that columns 2 and 3 of A are col $(0, 0, 0)$, while column 1 of B is col $(0, 0, 0)$. Multiplying Eq. (A2.23) from the right by the *proper* orthogonal matrix $R_0=[e_1\, e_5\, e_6\, e_4\, e_2\, e_3]$ now gives the form

$$P=Q'R_0=\begin{pmatrix} P_1 & 0 \\ 0 & P_1^* \end{pmatrix}, \quad \text{(A2.26)}$$

where we note that P is unitary, *hence,* P_1 *is unitary.* Thus,

$$\det P=(\det Q)^*=|\det P_1|^2=1. \quad \text{(A2.27)}$$

The cases $\lambda=0$ or 1 are characterized by $\mathrm{Tr}\, H=1$ or $\mathrm{Tr}\, H=2$, respectively. *Thus,* Q_1 *and* Q_2 *singular imply* $\mathrm{Tr}\, H=1$ *or* $\mathrm{Tr}\, H=2$. *Here* $\mathrm{Tr}\, H=1$, *in addition, implies that* $\det Q=1$ *and that* H *has eigenvalues* $(1, 0, 0)$ (from the results of the preceding paragraph).

Finally, we must consider the case $\lambda=1$ ($\mathrm{Tr}\, H=2$). In this case column 2 of A is col $(0, 0, 0)$, while columns 1 and 3 of B are col $(0, 0, 0)$. We can now repeat the argument leading to Eq. (A2.26) replacing R_0 by the improper orthogonal matrix $[e_1\, e_5\, e_3\, e_4\, e_2\, e_6]$. The conclusion is: Q_1 *and* Q_2 *singular and* $\mathrm{Tr}\, H=2$ *imply that* $\det Q=-1$ *and that* H *has the eigenvalues* $(0, 1, 1)$.

Since all possible properties of the submatrices of Q have been covered in Parts (a), (b), and (c), the matrix lemma stated at the beginning of this Appendix is proved. It should be remarked that one can write out explicit matrices of the type Q which exhibit the properties considered in each of the parts, (a), (b), and (c).

APPENDIX 3. PROOF OF LEMMA 10

An alternative form of the homomorphism of Lemma 9 is given in this Appendix.

Let Z denote the following skew-symmetric matrix:

$$Z=\begin{pmatrix} 0 & -z_3^* & z_2^* & z_1 \\ z_3^* & 0 & -z_1^* & z_2 \\ -z_2^* & z_1^* & 0 & z_3 \\ -z_1 & -z_2 & -z_3 & 0 \end{pmatrix}. \quad \text{(A3.1)}$$

Let Z_x denote the matrix Z when z_1, z_2, z_3 are introduced explicitly in the form

$$z_j=(x_j+ix_{j+3})/\sqrt{2}, \qquad j=1, 2, 3. \quad \text{(A3.2)}$$

The transformation

$$x'=\tilde{R}x=(A^*\tilde{Q}\tilde{A})x \quad \text{(A3.3)}$$

can be written as

$$(\tilde{A}x')=\tilde{Q}(\tilde{A}x), \quad \text{(A3.4)}$$

that is,

$$\begin{pmatrix} z' \\ z'^* \end{pmatrix}=\begin{pmatrix} \tilde{Q}_1 & \tilde{Q}_2 \\ Q_2^\dagger & Q_1^\dagger \end{pmatrix}\begin{pmatrix} z \\ z^* \end{pmatrix}, \quad \text{(A3.5)}$$

where z is the column matrix col $(z_1\, z_2\, z_3)$. Thus, the proof that

$$Z_{\tilde{R}x}=\tilde{V}Z_xV \quad \text{(A3.6)}$$

for $V\rightarrow R$ is transcribed to the proof that

$$Z'=\tilde{V}ZV \quad \text{(A3.7)}$$

for $V\rightarrow Q$, where $V\in SU(4)$.

To prove Eq. (A3.7) it is necessary to show explicitly that the transformation from z to z' contained in Eq. (A3.7) is precisely Eq. (A3.5) when Q_1 and Q_2 are expressed in terms of the partition matrices of V according to Eqs. (6.6)–(6.8).

Since Z' and Z are each skew-symmetric, it is sufficient to examine the elements ij, $i<j$, of Eq. (A3.7):

$$Z_{ij}'=\sum_{k<l}{}_1^4 (V_{lj}V_{ki}-V_{kj}V_{li})Z_{kl}. \qquad (A3.8)$$

It is just a question of looking at the various elements of this equation to deduce that the transformation is:

$$z'=(\gamma\tilde{V}_3-\beta\tilde{\alpha})z-(\tilde{V}_3\Gamma_\alpha)z^*, \qquad (A3.9)$$

$$z'^*=(\Gamma_\beta\tilde{V}_3)z+\tilde{C}z^*, \qquad (A3.10)$$

where C is the cofactor matrix of V_3 introduced in Appendix 1. Noting the relations, Eqs. (6.6)–(6.8), we see that Eq. (A3.9) agrees with that obtained from Eq. (A3.5).

To demonstrate the agreement of Eq. (A3.10) with (A3.5), we note two results which are valid for $V\in U(4)$

$$\Gamma_\beta\tilde{V}_3=(\det V)Q_2^\dagger, \qquad (A3.11)$$

$$\tilde{C}=(\det V)Q_1^\dagger. \qquad (A3.12)$$

The first relation follows from the Hermitian conjugate of Eq. (A1.65) and the definition of Q_2. The second relation follows from Eq. (A1.3) upon multiplying by $\tilde{C}$ from the left and using Eqs. (A1.56), (A1.63), and (A1.64), together with the definition of Q_1. In particular, for det $V=1$, i.e., $V\in SU(4)$, we obtain the required agreement between Eqs. (A3.10) and (A3.5).

APPENDIX 4. GEL'FAND BASIS VECTORS

The explicit form, Eq. (6.70), of the Gel'fand basis vectors of the space $\mathcal{H}_{p,0}$ is derived in this Appendix.

We have already observed that $\mathcal{H}_{p,0}$ is the carrier space for IR $[p\,p\,0\,0]$ of $U(4)$ and that the unique highest weight is

$$F\begin{pmatrix} p\ p\ 0\ 0 \\ p\ p\ 0 \\ p\ p \\ p \end{pmatrix}(W)=\zeta_3^p/(p!)^{1/2}. \qquad (A4.1)$$

It is now simply a question of using the generators, $\{E_{ij}\}$, to generate the general basis vector from the highest weight. This procedure requires explicit knowledge of the canonical matrices of certain of the generators on a general basis vector of the form

$$F\begin{pmatrix} p\ p\ 0\ 0 \\ p\ q\ 0 \\ \alpha\ \beta \\ \gamma \end{pmatrix}. \qquad (A4.2)$$

These matrix elements have been noted in various places, (Lo65, Ge50a, Ba63) and we will not repeat them. It is fairly trivial to go from the $U(4)$ highest weight of Eq. (A4.1) to a $U(3)$ highest weight, this step requiring the application of E_{32} $p-q$ times followed by the application of E_{43} $p-q$ times (with phase and normalization appropriate to these operators):

$$F\begin{pmatrix} p\ p\ 0\ 0 \\ p\ q\ 0 \\ p\ q \\ p \end{pmatrix}(W)=\frac{(-\eta_1)^{p-q}(\zeta_3)^q}{[(p-q)!q!]^{1/2}}. \qquad (A4.3)$$

The transition to the general basis vector may now be accomplished by the application of the operator as follows (Lo65):

$$F\begin{pmatrix} p\ p\ 0\ 0 \\ p\ q\ 0 \\ \alpha\ \beta \\ \gamma \end{pmatrix}=N^{1/2}E_{21}{}^{\alpha-\gamma}E_{32}{}^{q-\beta}$$

$$\times[E_{31}(E_{11}-E_{22})+E_{32}E_{21}]^{p-\alpha}F\begin{pmatrix} p\ p\ 0\ 0 \\ p\ q\ 0 \\ p\ q \\ p \end{pmatrix}, \qquad (A4.4)$$

where

$$N=\left[\frac{(\gamma-\beta)!(\beta)!}{(\alpha-\gamma)!(\alpha-\beta)!(q-\beta)!q!}\right]$$

$$\times\left[\frac{(\alpha-\beta+1)!(\alpha-q)!(\alpha+1)!}{(p-\beta+1)!(p-\alpha)!(p-q)!(p+1)!}\right]. \qquad (A4.5)$$

To facilitate the application of the operators in Eq. (A4.4) to the state vector of Eq. (A4.3), it is convenient to define

$$\Lambda=E_{31}(E_{11}-E_{22})+E_{32}E_{21}, \qquad (A4.6)$$

$$F_{abcd}(\eta,\zeta)=\frac{\eta_1{}^a\eta_3{}^b\zeta_3{}^c(\eta_1\zeta_1+\eta_2\zeta_2)^d}{a!b!c!d!(a+d+1)!}, \qquad (A4.7)$$

where a, b, c, d are non-negative integers. We *define* F_{abcd} to be zero if any one of the integers a, b, c, or d becomes negative. Then we have

$$F_{a,0,c,0}(\eta,\zeta)=\eta_1{}^a\zeta_3{}^c/a!c!(a+1)!. \qquad (A4.8)$$

Using the explicit forms of the generators given by Eqs. (6.38)–(6.42), we derive the following identity:

$$\Lambda F_{abcd}=(b+1)F_{a-1,b+1,c,d}-(d+1)F_{a-1,b,c-1,d+1}. \qquad (A4.9)$$

Using this identity, we now establish the following result by induction:

$$(\Lambda^e/e!)F_{a,0,c,0}=\sum_d(-1)^d F_{a-e,e-d,c-d,d}, \quad (A4.10)$$

where the sum over d can be considered as running over all integral values 0, 1, 2, $\cdots$.

The next relation is obtained directly upon expanding $(\eta_3\bar{\eta}_2-\zeta_2\bar{\zeta}_3)^f$ by using the binomial theorem and letting the result act on $F_{abcd}(\eta,\zeta)$:

$$[(E_{32}{}^f/f!)F_{abcd}](\eta,\zeta)=\zeta_2{}^f\sum_s\left[\frac{(-1)^{f-s}(b+s)!(a+d-s)!}{s!(f-s)!b!(a+d+1)!}\right]F_{a,b+s,c-f+s,d-s}(\eta,\zeta). \quad (A4.11)$$

The next step is to operate on Eq. (A4.10) with $E_{32}{}^f$ and use Eq. (A4.11) to evaluate the right-hand side. The double sum which occurs in the resulting expression can be reduced to a single sum by putting $r=d-s$ and replacing the summation over d by a summation over r. The summation over s then gives a numerical coefficient:

$$\sum_s[s!(f-s)!(e-r-s)!(a-e+r+s+1)!]^{-1}=(a+f+1)!/[f!(a+1)!(e-r)!(a+f+r-e+1)!]. \quad (A4.12)$$

The result of this calculation is

$$[(E_{32}{}^f\Lambda^e)F_{a,0,c,0}](\eta,\zeta)=\left[\frac{e!(a+f+1)!}{(a+1)!(a-e)!}\right]\eta_1{}^{a-e}\zeta_2{}^f\times\sum_r\left[\frac{(-1)^{f+r}(r+1)!}{(a+f+r-e+1)!}\right]F_{0,e-r,c-f-r,r}(\eta,\zeta). \quad (A4.13)$$

It is convenient at this point to write out the particular basis vectors having $\gamma=\alpha$ in Eq. (A4.4). This result is contained in Eq. (A4.13) upon identifying $f=q-\beta$, $e=p-\alpha$, $a=p-q$, $c=q$ and inserting the appropriate normalization factors. We write the result in the following form:

$$F\begin{pmatrix} p\ p\ 0\ 0 \\ p\ q\ 0 \\ \alpha\ \beta \\ \alpha \end{pmatrix}(W)=\left[\frac{(\alpha-\beta)!}{(\alpha-q)!(q-\beta)!}\right]^{1/2}\eta_1{}^{\alpha-q}\zeta_2{}^{q-\beta}G_{p\alpha\beta}(\eta,\zeta), \quad (A4.14)$$

where

$$G_{p\alpha\beta}(\eta,\zeta)=(-1)^{p-\beta}A^{1/2}\times\sum_r\frac{(-1)^r\eta_3{}^{p-\alpha-r}\zeta_3{}^{\beta-r}(\eta_1\zeta_1+\eta_2\zeta_2)^r}{(\alpha-\beta+r+1)!(p-\alpha-r)!(\beta-r)!r!}, \quad (A4.15)$$

where

$$A=\left[\frac{\beta!(\alpha+1)!(\alpha-\beta+1)!(p-\alpha)!(p-\beta+1)!}{(p+1)!(\alpha-\beta)!}\right]. \quad (A4.16)$$

In the final step, we must apply E_{21} repeatedly to Eq. (A4.14). Note, however, that E_{21} annihilates $G_{p\alpha\beta}$. Thus, we need to apply E_{21} only to the first factor in Eq. (A4.14). The following result is obtained directly upon expanding the operator on the left-hand side by using the binomial theorem:

$$\left[\frac{(\gamma-\beta)!}{(\alpha-\gamma)!(\alpha-\beta)!}\right]^{1/2}(\eta_2\bar{\eta}_1-\zeta_1\bar{\zeta}_2)^{\alpha-\gamma}\times\left[\frac{(\alpha-\beta)!}{(\alpha-q)!(q-\beta)!}\right]^{1/2}\eta_1{}^{\alpha-q}\zeta_2{}^{q-\beta}=D^{\frac{1}{2}(\alpha-\beta)}{}_{\gamma-\frac{1}{2}(\alpha+\beta),-q+\frac{1}{2}(\alpha+\beta)}\begin{pmatrix}\eta_1 & \zeta_2\\ \eta_2 & -\zeta_1\end{pmatrix}, \quad (A4.17)$$

where the D function is the standard one which occurs in the representations of $SU(2)$, the domain of definition now being extended to the space C^4:

$$D^j{}_{mm'}\begin{pmatrix}\xi_{11} & \xi_{12}\\ \xi_{21} & \xi_{22}\end{pmatrix}=[(j+m)!(j-m)!(j+m')!(j-m')!]^{1/2}\times\sum_s\frac{(\xi_{11})^{j+m'-s}(\xi_{21})^s(\xi_{12})^{m-m'+s}(\xi_{22})^{j-m-s}}{(j+m'-s)!s!(m-m'+s)!(j-m-s)!}. \quad (A4.18)$$

The notation for the argument of $D^j{}_{mm'}$ indicates that the ξ_{ij} are the elements of an arbitrary 2×2 complex matrix A:

$$A=\begin{pmatrix}\xi_{11} & \xi_{12}\\ \xi_{21} & \xi_{22}\end{pmatrix}. \quad (A4.19)$$

Under the restriction $A\rightarrow U\in SU(2)$, these D functions become the elements of the unitary matrix $D^j(U)$ which corresponds to U in the standard matrix representation $U\rightarrow D^j(U)$ of $SU(2)$.

Combining Eq. (A4.17) with Eq. (A4.14), we obtain the final form of the Gel'fand basis vectors:

$$F\begin{pmatrix} p\ p\ 0\ 0 \\ p\ q\ 0 \\ \alpha\ \beta \\ \gamma \end{pmatrix}(W)=D^{\frac{1}{2}(\alpha-\beta)}{}_{\gamma-\frac{1}{2}(\alpha+\beta),-q+\frac{1}{2}(\alpha+\beta)}\begin{pmatrix}\eta_1 & \zeta_2\\ \eta_2 & -\zeta_1\end{pmatrix}G_{p\alpha\beta}(\eta,\zeta). \quad (A4.20)$$

REFERENCES

Ba60 V. Bargmann and M. Moshinsky, Nucl. Phys. **18**, 697 (1960).

Ba61 V. Bargmann and M. Moshinsky, Nucl. Phys. **23**, 127 (1961).

Ba63 G. E. Baird and L. C. Biedenharn, J. Math. Phys. **4**, 1449 (1963).

Ba66 A. M. Badalyan and Yu. A. Simonov, Yad. Fiz. **3**, 1032 (1966) [Sov. J. Nucl. Phys. **3**, 755 (1966)].

Ba67 ——, E. S. Halpern, V. N. Lyakhovitskiĭ, V. V. Pustovalov, Yu. A. Simonov, and E. L. Surkov, Yad. Fiz. **6**, 473 (1967) [Sov. J. Nucl. Phys. **6**, 345 (1967)].

Ba70 A. I. Baz' and M. V. Zhukov, Yad. Fiz. **11**, 779 (1970) [Sov. J. Nucl. Phys. **11**, 435 (1970)].

Be67 J. E. Beam, Phys. Rev. **158**, 907 (1967).

Bi67 L. C. Biedenharn, A. Giovannini, and J. D. Louck, J. Math. Phys. **8**, 691 (1967).

Bi68 ——, and J. D. Louck, Commun. Math. Phys. **8**, 89 (1968).

Bi69 ——, Phys. Letters **28B**, 537 (1969).

Bi70 A. M. Bincer, J. Math. Phys. **11**, 1835 (1970).

Br70 D. D. Brayshaw and B. Buck, Phys. Rev. Letters **24**, 733 (1970).

Br70a M. Brunet and M. Resnikoff, J. Math. Phys. **11**, 1471 (1970).

Ca65 J. A. Castilho Alcarás and P. Leal Ferreira, J. Math. Phys. **6**, 578 (1965).

Ca71 ——, and P. Leal Ferreira, Rev. Brasil Fis. **1**, 63 (1971).

Ch64 A. Chakrabarti, Ann. Inst. Henri Poincare **1**, 301 (1964).

Ch65 E. Chacón and M. Moshinsky, Rev. Mex. Fis. **14**, 119 (1965).

Dr65 A. J. Dragt, J. Math. Phys. **6**, 533 (1965).

Ef71 V. D. Éfros, Yad. Fiz. **13**, 1318 (1971) [Sov. J. Nucl. Phys. **13**, 758 (1971)].

Es64 A. Esteve and P. G. Sona, Nuovo Cimento **32**, 1897 (1964).

Ga71 H. W. Galbraith, J. Math. Phys. **12**, 782 (1971).

Ga71a ——, J. Math. Phys. **12**, 2382 (1971).

Ga72 H. W. Galbraith, "Model Calculation of the Binding Energy of ^{4}He," submitted for publication.

Ge50 I. M. Gel'fand and M. L. Zetlin, Dokl. Akad. Nauk SSSR **71**, 1017 (1950). The results of this article are also listed in an Appendix to: I. M. Gel'fand, R. A. Minlos, Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and their Applications* (Pergamon Press, Oxford, England; Macmillan Co., New York, 1963).

Ge50a I. M. Gel'fand and M. L. Zetlin, Dokl. Akad. Nauk SSSR **71**, 825 (1950). The results of this article are also given in the book cited in (Ge50).

Gr70 B. Gruber, J. Math. Phys. **11**, 1783, 3077 (1970).

Ha62 M. Hamermesh, *Group Theory and its Application to Physical Problems* (Addison-Wesley Publ. Co., Reading, Mass., 1962).

Kr63 P. Kramer, Z. Naturforsch. **18a**, 260 (1963).

Kr65 ——, Ann. Physik (Leipzig) **15**, 278 (1965).

Kr66 ——, and M. Moshinsky, Nucl. Phys. **82**, 241 (1966).

Le65 J.-M. Lévy-Leblond and F. Lurcat, J. Math. Phys. **6**, 1564 (1965).

Le65a ——, and M. Lévy-Nahas, J. Math. Phys. **6**, 1372 (1965).

Le66 ——, J. Math. Phys. **7**, 2217 (1966).

Lo60 J. D. Louck, J. Mol. Spectry. **4**, 298 (1960).

Lo60a ——, "Theory of Angular Momentum in *N*-Dimensional Space", Los Alamos Scientific Lab. Rept. La-2451, 1960.

Lo65 ——, J. Math. Phys. **6**, 1786 (1965).

Lo70 ——, Am. J. Phys. **38**, 3 (1970).

Lo70a ——, L. C. Biedenharn, J. Math. Phys. **11**, 2368 (1970).

Mo69 M. Moshinsky, *The Harmonic Oscillator in Modern Physics: From Atoms to Quarks* (Gordon and Breach, New York, 1969), pp. 21, 67.

Na65 J. G. Nagel and M. Moshinsky, J. Math. Phys. **6**, 682 (1965).

Pa67 S. C. Pang and K. T. Hecht, J. Math. Phys. **8**, 1233 (1967).

Ra49 G. Racah, Rev. Mod. Phys. **21**, 494 (1949).

Ra62 G. Racah, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, edited by F. Gürsey (Gordon and Breach, New York, 1962), p. 31.

Ri69 M. Fabre de La Ripelle, Proceedings of the International School of Theoretical Physics, Predeal, Rumania, **10** (1969).

Si66 Yu. A. Simonov, Yad. Fiz. **3**, 630 (1966) [Sov. J. Nucl. Phys. **3**, 461 (1966)].

Si67 ——, A. M. Badalyan, Yad. Fiz. **5**, 88 (1967) [Sov. J. Nucl. Phys. **5**, 60 (1967)].

Si68 ——, Yad. Fiz. **7**, 1210 (1968) [Sov. J. Nucl. Phys. **7**, 722 (1968)].

Sm60 F. T. Smith, Phys. Rev. **120**, 1058 (1960).

Sm62 ——, J. Math. Phys. **3**, 735 (1962).

Su67 E. L. Surkov, Yad. Fiz. **5**, 908 (1967) [Sov. J. Nucl. Phys. **5**, 644 (1967)].

We31 H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1931), pp. 383, 391.

Wh46 E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis* (Cambridge U. P., New York, 1946), p. 286.

Wh68 R. C. Whitten and F. T. Smith, J. Math. Phys. **9**, 1103 (1968).

Wi59 E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra,* translated by J. J. Griffith, (Academic, New York, 1959), p. 289.

Wo67 M. K. F. Wong, J. Math. Phys. **8**, 1899 (1967).

Wo69 ——, J. Math. Phys. **10**, 1065 (1969).

Zi65 W. Zickendraht, Ann. of Phys. (N.Y.) **35**, 18 (1965).

Zi69 ——, J. Math. Phys. **10**, 30 (1969).