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Analytical Descriptions of Ultrashort Optical Pulse Propagation in a Resonant Medium

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A unified survey is presented of various theoretical approaches that have been developed to account for the novel propagation effects which may take place when extremely short pulses of coherent light interact with matter.

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I. INTRODUCTION

Recent advances in laser technology have led to the production of coherent optical pulses having durations in the picosecond (10^{-12} sec) regime (DSG67, Ma68, DGBM69). Such time intervals are comparable to or shorter than the phase memory times associated with many atomic systems. The high-frequency polarization induced in a medium by such a light pulse can therefore retain a definite phase relationship with the incident pulse. The resonant interaction of radiation and matter on such short time scales gives rise to phenomena which, as a result of the quantum mechanical coherence effects, cannot be described by the rate equation analysis developed previously (BBW63, FN63, BAZKL66) for the treatment of much longer pulses. In gases, where atomic phase memory times may be on the order of nanoseconds, pulses of a correspondingly longer duration may play this same role and, in the present work, any pulse which is shorter than all relevant relaxation times will be referred to as "ultrashort."

The novelty of the effects that may occur as a result of the coherent response of a medium to an optical pulse has been brought out quite strikingly by the recent discovery of self-induced transparency (McCH67, McCH69). In this effect, the leading edge of the pulse is used to invert an atomic population, while the trailing edge returns the population to its initial state by means of stimulated emission. The process is realizable if it takes place in a time that is short compared to the phase memory time of the resonant atomic systems, i.e., to the homogeneous broadening time of the medium, and also if the pulse has sufficient intensity to effect the population inversion. When conditions for the process are met, it is found that a steady-state pulse profile is established, and that this pulse envelope then propagates without attenuation at a velocity that may be considerably less than the phase velocity of light in the medium. Pulses with intensity below the threshold required for this process merely attenuate in the usual manner. Within the theoretical framework that has been used to describe this effect, it has been shown that the above-mentioned steady-state propagation takes place after the profile of the electric field has evolved to the form of a hyperbolic secant (McCH67). Many of the experimental and theoretical aspects of this phenomenon have been considered since its discovery (McCH69, PS67, GS70, HS69). The possibility of analogous effects in semiconductors has also been proposed (PP69). Somewhat similar steady-state pulse propagation has been observed in the study of neuristor waveforms (Sc70a) and is known to occur in the propagation of impulses on the nerve axon (Ka66).

In addition to the anomalous transmission property of ultrashort optical pulses, the amplification of such pulses has also drawn considerable attention. A number of analytical results have been obtained here as well (AB65, AC68, AC69). One expects that ultimately the amplification process will be limited by nonresonant loss mechanisms. If these are introduced in a phenomenological way by means of a conductivity, then for a nondispersive host medium, the *ad hoc* assumption that there is a steady-state pulse propagating at the phase velocity of light in the medium is readily verified. Steady-state propagation has also been demonstrated when host medium dispersion is included (AC69).

Whenever it becomes necessary to extend the range of validity of a theory to encompass new phenomena, it is useful to seek limiting cases of the general formalism (WW66, TS66, McCH69, HS69) which admit to exact solutions of the type referred to above. The present paper is an attempt to summarize the success that has thus far been achieved in describing the novel aspects of ultrashort optical pulse propagation by analytical methods. No attempt has been made to survey the entire field of ultrashort optical pulse propagation. Experimental results as well as numerical computations have been referred to only insofar as they enhance an understanding of the analytical results under consideration. A review which does place emphasis upon experimental topics and is accordingly complementary to the following presentation has appeared recently (KL70). While the experimentalist is frequently unmoved by theoretical descriptions that fail to provide for all facets of a phenomenon as it occurs in the experimental milieu, such as level degeneracy, finite relaxation times, inhomogeneous broadening, etc., it should be emphasized that many of the most interesting aspects of ultrashort pulse propagation appear already in rather highly idealized theoretical contexts and our understanding of the equations that govern even these simpler situations is still far from complete. An increased understanding of these simpler theoretical models will undoubtedly enhance our understanding of the more complete descriptions that require numerical computations. A case in point is the recent discovery that certain nonlinear equations, notably the Korteweg–deVries equation, possess conservation laws in addition to those of field energy and field momentum. An examination of a highly idealized theoretical model of optical pulse propagation has shown that the governing equation is another example of an equation possessing such higher conservation laws. Guided by this result, it has been found that higher conservation laws may also be obtained for one of the more general theoretical formulations that has been used to describe ultrashort optical pulse propagation. Such conservation laws provide an extremely simple means of obtaining certain results which previously required extensive numerical computation, and also provide a guide to the synthesis of these numerical results. The most promising method for attacking such nonlinear problems is clearly through a simultaneous application of both analytical and computational techniques, and the field of optical pulse propagation provides an ideal opportunity for application of this "synergetic" (ZA67) approach. In fact, self-induced transparency was discovered from an analysis of numerical solutions of the equations which describe optical pulse propagation.

In addition to the steady-state results mentioned above, two relatively simple models, one of which has already been alluded to, have been devised to describe a number of the effects that have been observed both experimentally and as output from machine computations. The first model is one in which inhomogeneous broadening is neglected. The physical situation most closely related to such a model is that of propagation under conditions of extreme saturation broadening. Although the problem under consideration involves a coupling between radiation and matter that is too strong to be treated by perturbation theory, a fairly extensive analytical treatment of this model is still possible since it expresses this interaction in terms of a single nonlinear partial differential equation which arose long ago in differential geometry. The techniques developed about the turn of the century for obtaining solutions to this equation may be employed here to considerable advantage (La69b).

Certain other phenomena, notably that of photon echo (AKH65, CLA68, PS68, SSB68, GWPST69), require for their explanation the relative dephasing of atoms that results when inhomogeneous broadening is included. This effect also provides an example of a collective superradiant state in which energy is radiated coherently into the electromagnetic field (Di54, AP69, AMS69). Here again, it is possible to construct a soluble model which takes into account the reaction of stimulated emission upon the incident wave (La69a). If one is willing to forego consideration of the detailed structure of pulse shapes, the time dependence of the pulse may be assumed to be that of a delta function, and interest confined to the spatially dependent amplitude of such delta function pulses. Only the time integral of such pulse envelopes is meaningful, of course, but such time integrals have been shown to be precisely the quantities of interest in the treatment of ultrashort pulses. The area theorem (McCH67), which is so useful in understanding short pulse phenomena, is also found to govern the spatial evolution of the amplitude functions introduced in this model.

Although much of the physical insight required for an understanding of the propagation effects associated with ultrashort optical pulses may be obtained from a consideration of the interaction of a plane monochromatic light wave with a system of nondegenerate two-level atoms, it should be noted that effects thus uncovered do experience modification when transverse mode structure (McCH70), homogeneous and inhomogeneous broadening (HS69, IL69), and especially level degeneracy (McCH69, RSJ68) are taken into account.

In addition, it is now becoming apparent that the restriction to a monochromatic light pulse will have to be relaxed. Ultrashort optical pulses have recently been shown to possess a frequency sweep (Tr68b, GDH68, Tr69b). It is to be expected that future research in the field of ultrashort optical pulse propagation will place increasing emphasis upon the phase characteristics of the pulse (GHS).

II. BASIC EQUATIONS

We begin by summarizing the standard semiclassical description of the interaction of an electromagnetic wave with an assembly of two-level systems. The optical field in the form of a plane polarized electromagnetic pulse may be characterized by its electric field vector $\mathbf{E}(\mathbf{r}, t)$ which satisfies the usual wave equation

$$\nabla^2\mathbf{E}-\frac{4\pi\sigma}{c^2}\frac{\partial\mathbf{E}}{\partial t}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=\frac{4\pi}{c^2}\frac{\partial^2\mathbf{P}}{\partial t^2}, \tag{2.1}$$

where σ is a conductivity that is introduced to simulate nonresonant losses in the medium, c is the phase velocity of light in the medium, and $\mathbf{P}$ is the polarization of the medium that is induced by the electromagnetic wave. For a medium consisting of an assemblage of noninteracting two-level systems distributed with a uniform density n_0, this polarization is $n_0\mathbf{p}$, where $\mathbf{p}$ is the polarization of an individual two-level system.

The polarization $\mathbf{p}$, as well as the difference in population between upper and lower levels, n, are in turn driven by the optical field $\mathbf{E}$. As shown in Appendix A, the quantities $\mathbf{p}$, n, and $\mathbf{E}$ are related through the equations

$$\ddot{\mathbf{p}}+\omega_{ab}^2\mathbf{p}=-(\tfrac{1}{3})(2\omega_{ab}\wp^2/\hbar n_0)\mathbf{E}n, \tag{2.2}$$

$$\dot{n}=(2n_0/\hbar\omega_{ab})\mathbf{E}\cdot\dot{\mathbf{p}}, \tag{2.3}$$

where ω_{ab} is the transition frequency between the upper and lower levels a and b, respectively, and $\wp$ is the dipole matrix element for such a transition. The factor of $\frac{1}{3}$ in parentheses is to be included if all possible spatial orientations of the two-level systems are permitted so that an average over all orientations must be performed (VanV24, KS48).

In Eqs. (2.2) and (2.3) it is customary (BAZKI66) to include terms proportional to the longitudinal and transverse relaxation times T_1 and T_2, respectively. However, for ultrashort pulses it is assumed that pulse widths are much shorter than all relaxation times. Hence contributions from terms proportional to relaxation times will be much smaller than contributions from time derivatives of pulse profile envelopes. All terms containing relaxation times will therefore be ignored.

We now return to Eq. (2.1) and consider the effect of the medium upon the optical wave. For the resonant situation being considered, the term $\partial^2\mathbf{P}/\partial t^2$ may be replaced by $-\omega_0^2\mathbf{P}$, where ω_0 is the carrier frequency of the incident pulse. This follows immediately if ω_{ab} is replaced by ω_0 in Eq. (2.2), and the term on the right-hand side of that equation is then ignored. Neglect of this term is justified if $\omega_0 \gg \wp\,|\,\mathbf{E}\,|/\hbar$. As will be seen in the subsequent development, the term on the right-hand side of this inequality is comparable to the pulse widths being considered. Since they will always contain many optical periods, the above inequality will always be satisfied. Two-photon resonant propagation in which $\omega_0=2\omega_{ab}$ has also been considered (BP69).

Since even the shortest pulses produced to date contain many optical cycles, it is appropriate to write the electric field in terms of a carrier wave, as well as envelope and phase functions $E(\mathbf{r}, t)$ and $\phi(\mathbf{r}, t)$, respectively, which vary slowly on the length and time scales of the carrier wave. Hence one may write

$$\mathbf{E}(\mathbf{r}, t)=E(\mathbf{r}, t)\cos[\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t+\phi(\mathbf{r}, t)], \tag{2.4}$$

and assume $\omega_0 E \gg \partial E/\partial t$, $\mathbf{k}_0 E \gg \nabla E$ plus similar inequalities for ϕ. Because of the assumption that $E(\mathbf{r}, t)$ and $\phi(\mathbf{r}, t)$ vary slowly compared to the carrier wave, Eq. (2.1) may be reduced to a much simpler form. In particular, only the first derivatives of $E(\mathbf{r}, t)$ and $\phi(\mathbf{r}, t)$ need be retained on the left-hand side of this equation when it is expressed in terms of the envelope and phase as defined by Eq. (2.4). The solution thus obtained is customarily referred to as the solution in the slowly varying envelope and phase approximation.

In general, it is appropriate to allow for a continuous distribution of transition frequencies ω about ω_{ab}, the so-called inhomogeneous broadening, rather than a single transition frequency ω_{ab}. It is convenient to analyze the situation in which this frequency distribution is symmetric about ω_{ab}, and the carrier frequency of the incident optical pulse is at this center frequency, i.e., $\omega_0=\omega_{ab}$.

If the spectrum which characterizes the broadening is written as $g(\Delta\omega)$, where $\Delta\omega=\omega-\omega_0$, then the polarization of the medium is given by

$$\mathbf{P}(\mathbf{r}, t)=n_0\int_{-\infty}^{\infty}d\Delta\omega\, g(\Delta\omega)\mathbf{p}(\Delta\omega, \mathbf{r}, t)\equiv n_0\langle\mathbf{p}(\Delta\omega, \mathbf{r}, t)\rangle. \tag{2.5}$$

The spectrum $g(\Delta\omega)$ is assumed to be normalized so that

$$\int_{-\infty}^{\infty}d\Delta\omega\, g(\Delta\omega)=1. \tag{2.6}$$

Equation (2.1), specialized to a plane wave traveling in a positive x direction, then becomes

$$[(dE/dt)+2\pi\sigma E]\sin\Phi(x, t)+E(d\phi/dt)\cos\Phi(x, t) = 2\pi\omega_0 n_0\langle p(\Delta\omega, x, t)\rangle, \tag{2.7}$$

where

$$\Phi\equiv k_0x-\omega_0t+\phi(x, t),$$

and

$$d/dt=(\partial/\partial t)+c(\partial/\partial x). \tag{2.8}$$

When the polarization is decomposed into parts which are in phase and $\pi/2$ out of phase with the electric field, one finds, as shown in Appendix A, that Eq. (2.7) may be written as the pair of relations

$$(d\tilde{E}/dt)+2\pi\sigma\tilde{E}=c\alpha'\langle\mathcal{P}(\Delta\omega,x,t)\rangle, \tag{2.9a}$$

$$\tilde{E}(d\phi/dt)=-c\alpha'\langle\mathcal{Q}(\Delta\omega,x,t)\rangle, \tag{2.9b}$$

where

$$\tilde{E}=(\wp/\hbar)E,$$

and $\mathcal{P}$ and $\mathcal{Q}$ are related to p according to

$$p=\wp[\mathcal{P}(\Delta\omega,x,t)\sin\Phi(x,t)+\mathcal{Q}(\Delta\omega,x,t)\cos\Phi(x,t). \tag{2.10}$$

The constant α' is defined by

$$\alpha'\equiv 2\pi n_0\omega_0\wp^2/\hbar c. \tag{2.11}$$

The transformation to slowly varying quantities given in Eqs. (2.4) and (2.10) is equivalent to the transformation to a rotating frame that is frequently employed by workers in this field (McCH69).

As shown in Appendix A, the two components of the polarization are related to $\tilde{E}$, ϕ, and the normalized population inversion $\mathfrak{N}=n/n_0$ by

$$\partial\mathfrak{N}/\partial t=-\tilde{E}\mathcal{P}, \tag{2.12a}$$

$$\partial\mathcal{P}/\partial t=\tilde{E}\mathfrak{N}+[\Delta\omega+(\partial\phi/\partial t)]\mathcal{Q}, \tag{2.12b}$$

and

$$\partial\mathcal{Q}/\partial t=-[\Delta\omega+(\partial\phi/\partial t)]\mathcal{P}. \tag{2.12c}$$

Multiplication of Eqs. (2.12) by $\mathfrak{N}$, $\mathcal{P}$, and $\mathcal{Q}$, respectively, and summation of the resulting equations yields an exact differential which is equivalent to

$$\mathfrak{N}^2+\mathcal{P}^2+\mathcal{Q}^2=1. \tag{2.13}$$

In this latter result, a constant of integration has been set equal to unity since in the usual applications of the theory one requires $\mathcal{P}(x,-\infty)=\mathcal{Q}(x,-\infty)=0$, $\mathfrak{N}(x,-\infty)=\pm1$. The form of Eq. (2.13) enables one to interpret the response of a two-level system in terms of the motion of a vector on the surface of a sphere in a $\mathcal{P}$, $\mathcal{Q}$, $\mathfrak{N}$ space. This description will be considered in Sec. V.

Equations (2.12) describe how the field amplitude $\tilde{E}$ and phase ϕ determine $\mathcal{P}$, $\mathcal{Q}$, and $\mathfrak{N}$ for a two-level system that is off resonance by an amount $\Delta\omega$. Equations (2.9) show how the total polarization due to an assembly of two-level systems, with transition frequencies distributed according to $g(\Delta\omega)$, reacts back on the amplitude and phase. Equations similar to (2.12), the Bloch equations, also arise in nuclear magnetic resonance studies in which an oscillating magnetic field interacts with an assemblage to two-level systems which possess a magnetic moment. Such studies have been confined to samples that are sufficiently thin that the reaction of the induced field back upon the exciting field may be ignored. In that case, Eqs. (2.12) may be solved for a specified external field. This is not the case in the situation envisioned here. A satisfactory description of optical pulse propagation is achieved only when the field is determined self-consistently by the simultaneous solution of Eqs. (2.9) and (2.12).

For lossless propagation ($\sigma=0$), the conservation of energy follows from Eq. (2.9a) upon multiplication by $\tilde{E}$ and introduction of Eq. (2.12a). The result may be written in the form

$$(\partial/\partial t)(\tfrac{1}{2}\tilde{E}^2+\alpha'c\langle\mathfrak{N}\rangle)+c(\partial/\partial x)(\tfrac{1}{2}\tilde{E}^2)=0. \tag{2.14}$$

It will be shown in Sec. VII that this is the first of a number of conservation laws which are satisfied by the coupled Maxwell and Bloch equations in the slowly varying envelope approximation.

III. SELF-INDUCED TRANSPARENCY AND THE AREA THEOREM

Up to the present time, the full set of equations given by Eqs. (2.9) and (2.12) has received limited attention. A simplified version of these equations that has been treated quite extensively is obtained by adopting the consistent set of assumptions that the phase term ϕ is initially zero, that the carrier frequency is at the center of a symmetrically broadened line [i.e., $g(\Delta\omega)=g(-\Delta\omega)$], and that $\mathcal{Q}$ is an odd function of $\Delta\omega$. From Eq. (2.9b) one then sees that the source term governing variations in ϕ is zero so that ϕ will remain zero.

The stability of this choice of the phase is perhaps most easily inferred from recent numerical calculations by Diels (Di70) in which the mean frequency of a light pulse was found to be pulled toward the center of a Lorentzian broadened line. The form of the theory, which follows from the choice of constant phase, has provided considerable insight into the subject of ultrashort pulse propagation (HS69, McCH69, IL69).

The only analytical solutions of this specialized form of the basic equations that are available to date are steady-state solutions. They include both the solitary wave solution of self-induced transparency and infinite wave train solutions (ADS68, Cr69a, Eb69) which contain the solitary wave as a limiting case. Only the former will be described here; infinite wave train solutions will be considered later in connection with a somewhat more specialized theoretical model.

For a steady-state solution, one may assume that $\tilde{E}$, $\mathcal{P}$, $\mathcal{Q}$, and $\mathfrak{N}$ are functions of a single independent variable $w=(t-x/V)$, where V is the velocity of the pulse.

Equations (2.9a) and (2.12c) may now be combined and integrated to yield

$$[1-(c/V)]\tilde{E}(w)+\alpha'c\langle\mathcal{Q}(\Delta\omega,w)/\Delta\omega\rangle=0. \tag{3.1}$$

An integration constant has been set equal to zero in this result since $\tilde{E}$ and $\mathcal{Q}$ are zero before arrival of the pulse. If this equation is divided by $\tilde{E}(w)$ and differentiated with respect to w, one obtains

$$\int_{-\infty}^{\infty} d\Delta\omega \frac{g(\Delta\omega)}{\Delta\omega} \frac{d}{dw}\left(\frac{\mathcal{Q}}{\tilde{E}}\right)=0. \tag{3.2}$$

Now the function $g(\Delta\omega)$ contains a parameter such as T_2^* that determines the width of the inhomogeneous broadening, e.g.,

$$g(\Delta\omega)=[T_2^*/2(\pi)^{1/2}] \exp[-(\Delta\omega T_2^*/2)^2]. \tag{3.3}$$

Although V, and hence w, depends in an implicit way upon T_2^*, there is no explicit dependence of $\tilde{E}$ upon T_2^*. Since $\mathcal{Q}$ is the response of an individual two-level system, it is also independent of T_2^* except for the implicit dependence contained in w. Consequently, the function in parentheses in the integrand of Eq. (3.2) does not contain explicit dependence upon T_2^*. Since it is known that the theory is still valid for arbitrarily large values of T_2^*, one may invoke Lerch's theorem[1] to justify the conclusion that the term multiplying $g(\Delta\omega)$ in the integrand must itself be equal to zero. It then follows that

$$\mathcal{Q}(\Delta\omega, w)=\chi(\Delta\omega)\tilde{E}(w), \tag{3.4}$$

where $\chi(\Delta\omega)$ is an as yet undetermined function of the detuning. As shown in Appendix B, the factorization of $\mathcal{Q}$ provided by Eq. (3.4) enables one to solve completely for the self-consistent interaction of $\mathcal{P}$, $\mathcal{Q}$, and $\mathfrak{N}$ with the field envelope $\tilde{E}$. One obtains (McCH67)

$$\tilde{E}=(2/\tau_p)\operatorname{sech}(w/\tau_p)=(2/t_p)\sin(\varphi/2), \tag{3.5a}$$

$$\mathfrak{N}=-1+2D\sin^2(\phi/2), \tag{3.5b}$$

$$\mathcal{P}=-D\sin\varphi, \tag{3.5c}$$

$$\mathcal{Q}=2D\tau_p\Delta\omega\sin(\varphi/2), \tag{3.5d}$$

where

$$\varphi=\int_{-\infty}^{w} dw'\tilde{E}(w')=4\tan^{-1}\exp(w/\tau_p), \tag{3.6}$$

$$D=[1+(\tau_p\Delta\omega)^2]^{-1}, \tag{3.7}$$

and, from Eq. (3.4),

$$\chi(\Delta\omega)=D\tau_p^2\Delta\omega. \tag{3.8}$$

The pulse width τ_p may be chosen arbitrarily. It is essential to recognize that this steady-state pulse of self-induced transparency is determined not only by the strength of the field but also by the properties of the medium as expressed through the dipole moment $\wp$.

Various topics of experimental interest can now be considered. In particular, Eq. (3.1) enables one to express the envelope velocity in the form

$$V^{-1}=c^{-1}+\alpha'\tau_p^2\langle D\rangle. \tag{3.9}$$

For $g(\Delta\omega)$ given by Eq. (3.3), one finds that

$$\langle D\rangle=2K\exp(K^2)\operatorname{erfc}(K), \tag{3.10}$$

where

$$K\equiv T_2^*/2\tau_p, \tag{3.11}$$

and erfc (K) is the compliment of the error function (AS64). For $T_2^*\gg\tau_p$, we find that $\langle D\rangle\to1$. As T_2^* becomes much less than τ_p, a much smaller percentage of the atoms are on resonance. One then finds $\langle D\rangle\to0$ and hence $V\to c$. A number of other experimental implications of these results have been considered (McCH69).

Finally, it has been noted (CS68) that if the carrier frequency is not located at the center of a symmetric line but is in fact far from resonance, the expression for the envelope velocity goes over to the usual result for the velocity of a wave in a dispersive medium. In that case, Eq. (3.7) is replaced by

$$D=[1+(\tau_p\delta\omega)^2]^{-1}, \tag{3.12}$$

where $\delta\omega=\omega-\omega_0=\Delta\omega+\omega_{ab}-\omega_0$. Far from resonance, $\omega_{ab}-\omega_0\gg\Delta\omega$, and Eq. (3.9) reduces to

$$V^{-1}=c^{-1}+\alpha'/(\omega_{ab}-\omega_0)^2. \tag{3.13}$$

The above results, along with the previously mentioned infinite wave train solutions are the only analytical solutions of the inhomogeneously broadened version of Eqs. (2.9a) and (2.12) that have been reported to date. However, further analytical progress has been made by confining attention to the area under the envelope curve (McCH67). If we define

$$\theta(x)\equiv\int_{-\infty}^{\infty} dt\tilde{E}(x,t), \tag{3.14}$$

the equation governing the variation in θ is readily obtained by integrating Eq. (2.9a) over all time. The details are contained in Appendix C. The result is

$$(d\theta/dx)+\kappa\theta=\pm(\alpha/2)\sin\theta, \tag{3.15}$$

in which

$$\kappa=2\pi\sigma/c, \tag{3.16}$$

and

$$\alpha=2\pi g(0)\alpha'. \tag{3.17}$$

The constant α' is as defined in Eq. (2.11).

Equation (3.15) is known as the area theorem (McCH67). It contains the key to an understanding of many of the effects which occur in the propagation of ultrashort pulses. Again, if orientational averaging is included, the factor α should be replaced by $\alpha/3$. The physical significance of α follows from the linearized version of Eq. (3.15) in which $\sin\theta$ is replaced by θ. For $\kappa=0$, the field is then seen to amplify or decay in the characteristic length α^{-1}. When the spectrum function given in Eq. (3.3) is used, one finds that α is

[1] For present purposes, the theorem may be stated in the form given by Watson (Wa62, p. 382): If $f(r)$ is a continuous function of r such that

$$\int_0^{\infty}\exp(-r^2t)f(r)dr=0$$

for all sufficiently large positive values to t, then $f(r)$ is identically zero.

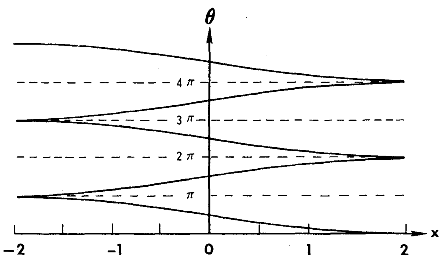

FIG. 1. Schematic representation of solution of Eq. (3.13) for $\kappa=0$.

proportional to T_2^*. In Sec. VI, a simplified theoretical model will be considered in which inhomogeneous broadening is neglected. This may be accomplished by letting $T_2^*\rightarrow\infty$ in the present model. An immediate consequence of this limiting procedure is that this characteristic length tends to zero.

For $\kappa=0$, the solution of Eq. (3.15) which satisfies $\theta=\theta_0$ at $x=x_0$ is

$$\tan(\theta/2)=\tan(\theta_0/2)\exp[\pm(\alpha/2)(x-x_0)] \quad (3.18)$$

and is depicted schematically in Fig. 1. Modifications of this result due to nonvanishing conductivity have been inferred (IL69, Co69) from numerical solutions of Eq. (3.15).

Since Eq. (3.15) contains a choice of signs, it is actually two distinct differential equations. The two solutions are obtained from Fig. 1 by reading the diagram from right to left for the plus sign (amplifier), and from left to right for the minus sign (attenuator). Hence one sees that an infinitesimal area will grow to π in an amplifier, while any area less than π will evolve to zero in an attenuator. This second result allows for not only the well-known decay of a pulse as it propagates in an attenuator, but also for evolution into a nonvanishing zero π pulse, i.e., one in which the total area under the pulse envelope is zero, but the area under the pulse energy ($\sim\tilde{E}^2$) is not zero. This is possible if the positive portions of a pulse envelope are equal in area to the negative portions. Physically, the regions of positive and negative envelope are merely regions in which there is a relative difference of 180 degrees in the phase of the carrier wave. In an attenuator, initial pulse areas between π and 3π will evolve into the steady-state 2π pulse of self-induced transparency: the 2π pulse is unstable in an amplifier and will evolve into either a π or 3π pulse. Figure 1 refers only to the total area of a pulse and gives no information at all about either the possible breakup of a pulse into two or more pulses with the same total area or whether a continually amplifying π pulse will retain an area of π by virtue of pulse narrowing or by developing negative regions in the pulse envelope.

Although the phase has been neglected by most workers in this field, the effect on the phase of a Kerr effect or nonlinearity in the refractive index has been obtained (EM69) for steady-state pulses in an unbroadened medium [i.e., $g(\Delta\omega)=\delta(\Delta\omega)$]. It is found that the frequency has a nonmonotic frequency sweep or "chirp" proportional to $\text{sech}^2\, w$.

IV. STEADY-STATE PULSE IN AN AMPLIFIER

In addition to the self-induced transparency solution in an attenuator, a somewhat similar steady-state result may be obtained in an amplifier if the loss term σ is retained in Eq. (2.9a). This was first recognized by observation of machine computations (WW64) and subsequently described analytically (AB65). Both results have been obtained in the limit of no inhomogeneous broadening. Certain cases in which homogeneous broadening is retained have also been treated (AB65). From Eqs. (2.9), the relevant equations are

$$(d\tilde{E}/dt)+2\pi\sigma\tilde{E}=\alpha' c\mathcal{P} \quad (4.1)$$

and

$$\tilde{E}(d\phi/dt)=-\alpha' c\mathcal{Q}, \quad (4.2)$$

as well as Eqs. (2.12). The *ad hoc* assumption which renders the analysis tractable is that both $\tilde{E}$ and ϕ travel at the velocity c. The differential operators in Eqs. (4.1) and (4.2) then vanish identically and the problem is greatly simplified. It is emphasized that no rigorous justification for this assumption has been presented. However, from a study of numerical computations, it has recently been noted (IL69) that if a steady-state pulse at any velocity $v\neq c$ is assumed, then the resulting numerical solution is unstable and evolves into a pulse propagating at $v=c$.

From Eq. (4.2) one sees that $\mathcal{Q}=0$, and hence from Eq. (2.12c) that

$$\Delta\omega+(\partial\phi/\partial t)=0. \quad (4.3)$$

Consequently, we have

$$\phi=(\omega-\omega_0)t, \quad (4.4)$$

and from Eq. (2.4) it follows that the frequency of the steady-state pulse is always equal to ω the transition frequency of the two-level system (AB65). As previously mentioned a similar frequency shift has been observed in a numerical analysis of the 2π pulses of self-induced transparency (Di70).

The remaining equations are now

$$\tau\tilde{E}=\mathcal{P}, \quad (4.5)$$

$$\dot{\mathcal{P}}=\tilde{E}\mathfrak{N}, \quad (4.6)$$

$$\dot{\mathfrak{N}}=-\tilde{E}\mathcal{P}, \quad (4.7)$$

where the dot signifies differentiation with respect to $t-x/c$, and

$$\tau\equiv 2\pi\sigma/\alpha' c. \quad (4.8)$$

Equations (4.6) and (4.7) have the parametric representation

$$\mathcal{P}=\sin\varphi, \quad (4.9)$$

$$\mathfrak{N}=\cos\varphi, \quad (4.10)$$

with

$$\tilde{E}=\dot{\varphi}. \tag{4.11}$$

Equation (4.5) then provides a differential equation for φ. The solution yields

$$\tilde{E}=\tau^{-1}\,\mathrm{sech}\,[(t-x/c)/\tau]. \tag{4.12}$$

The population is seen to be inverted by the pulse since Eq. (4.10) is now equivalent to

$$\mathfrak{N}=-\tanh[(t-x/c)/\tau]. \tag{4.13}$$

Finally, since

$$\int_{-\infty}^{\infty} dt\tilde{E}=\pi \tag{4.14}$$

and also because the vector whose rectangular components are $\mathcal{P}$ and $\mathcal{Q}$ is rotated through an angle π during passage of the pulse, the result given in Eq. (4.12) is customarily referred to as a π pulse. Preliminary measurements indicate that such pulses may be realized in standing wave geometries (FS67).

Steady-state pulse propagation in an amplifier has also been analyzed without the assumption of a slowly varying envelope and phase (AC68). For a nondispersive medium the assumption of propagation at the phase velocity of light in the medium is retained, however, and again it provides the simplification that is sufficient to permit an exact solution. For pulses that are many optical cycles in duration, there is very little difference between the pulse shape obtained with this more exact treatment and the method described above. What is of great interest, however, is the prediction of a phase variation in the carrier wave. The "chirp" predicted by the theory is proportional to the square of the ratio of optical period to pulse width. Such a result could not be obtained in the slowly varying phase and envelope approximation which is equivalent to an expansion to only first order in this ratio.

The method has subsequently been extended (AC69) to include the effect of dispersion in the host medium. In the limit of large dispersion it was found that a monotonic frequency sweep is predicted. Such chirping of ultrashort pulses has been observed experimentally and offers new opportunities for pulse compression (GDH68, Tr68b, Tr69) and population inversion (Tr68a).

We first consider the case in which no dispersion is present. When the pulse is assumed to propagate at the velocity of light in the host medium, Eq. (2.1) reduces to

$$\sigma\mathbf{E}=-\partial\mathbf{P}/\partial w, \tag{4.15}$$

where $w=t-x/c$. When $\mathbf{E}$ and $\mathbf{P}$ are written in the form of Eq. (2.4) and (2.10), respectively, but without any assumption that $\mathcal{E}$ and ϕ are slowly varying, Eq. (4.15) yields the pair of relations

$$\tau\tilde{E}=\mathcal{P}-(\omega_0)^{-1}[\mathcal{P}(\partial\phi/\partial w)+(\partial\mathcal{Q}/\partial w)] \tag{4.16a}$$

and

$$(\partial\mathcal{P}/\partial w)+\mathcal{Q}[\omega_0-(\partial\phi/\partial w)]=0. \tag{4.16b}$$

Equations (2.12), with $\Delta\omega$ set equal to zero, are also applicable. Combination of Eq. (4.16a) with (2.12c), and of Eq. (4.16b) with (2.12b) leads to

$$\tau\tilde{E}=\mathcal{P}, \tag{4.17a}$$

$$\omega_0\mathcal{Q}=-\tilde{E}\mathfrak{N}. \tag{4.17b}$$

Equations (2.12b) and (2.12c) may be combined to yield

$$\mathcal{Q}^2(\partial/\partial w)(\mathcal{P}/\mathcal{Q})=\tilde{E}\mathfrak{N}\mathcal{Q}+(\partial\phi/\partial w)(1-\mathfrak{N}^2) \tag{4.18}$$

in which Eq. (2.13) has been employed. From Eqs. (4.17), (2.12), and (2.13), one obtains

$$\partial\mathfrak{N}/\partial w=-\tau^{-1}[(1-\mathfrak{N}^2)/(1+a^2\mathfrak{N}^2)]. \tag{4.19}$$

The constant a is defined as

$$a\equiv(\omega_0\tau)^{-1} \tag{4.20}$$

and is the ratio of the period of the carrier wave to the width of the pulse. Finally, Eq. (4.18) yields

$$\frac{\partial\phi}{\partial w}=\frac{a}{\tau}\left\{\frac{\mathfrak{N}^2}{1+(a\mathfrak{N})^2}-\frac{1-\mathfrak{N}^2}{[1+(a\mathfrak{N})^2]^2}\right\}. \tag{4.21}$$

Unfortunately, explicit time dependence for $\mathfrak{N}$ is no longer possible; Eq. (4.19) leads to the implicit relation

$$w/\tau=a^2\mathfrak{N}-(1+a^2)\tanh^{-1}\mathfrak{N}. \tag{4.22}$$

For $a\ll 1$ this reduces to Eq. (4.13).

The "instantaneous" frequency is

$$\omega_{\text{inst}}=\omega_0-(\partial\phi/\partial w). \tag{4.23}$$

For $\mathfrak{N}=\pm 1$, i.e., at both ends of the pulse, this becomes

$$\omega_{\text{inst}}=\omega_0[1-a^2/(1+a^2)], \tag{4.24}$$

while at the center of the pulse $\mathfrak{N}=0$ and

$$\omega_{\text{inst}}=\omega_0(1+a^2). \tag{4.25}$$

The fractional shift in frequency is $2a^2/(1+a^2)$. For a picosecond pulse at 1μ, $a\sim 10^{-3}$, and the fractional frequency shift is 10^{-6}. The absolute frequency shift is 300 Mc/sec.

The calculation outlined above was subsequently (AC69) extended to include dispersion in the host medium. The dispersion was treated by standard methods of linear wave propagation. It was found that in the limit of large dispersion, the chirp becomes proportional to the first power of the ratio of optical period to pulse duration, rather than to the second power as found in the calculation summarized above. As is to be expected, then, this limiting case can be treated in the slowly varying envelope and phase approximation, and this formulation is adopted below.

The wave equation given in Eq. (2.1) is readily modified to include effects arising from the presence of a host medium. Since the effect of the host is merely to provide an additional contribution to the polarization, one need merely introduce an additional polarization term $\mathbf{P}_{nr}$ to describe this nonresonant contribution. The total polarization in Eq. (2.1) is then

$$\mathbf{P}=\mathbf{P}_r+\mathbf{P}_{nr}, \tag{4.26}$$

where the first term $\mathbf{P}_r$ is the previously considered resonant polarization which results from the interaction of the wave with the two-level systems suspended in the host medium. The frequency dependence of the nonresonant polarization is conveniently described in terms of a susceptibility $\chi(\omega)$ by writing

$$\mathbf{P}_{nr}=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{-i\omega t}\chi(\omega)\mathbf{E}(\omega), \tag{4.27}$$

where $\mathbf{E}(\omega)$ is the Fourier transform of the electric field vector

$$\mathbf{E}(\omega)=\int_{-\infty}^{\infty}dte^{i\omega t}\mathbf{E}(t). \tag{4.28}$$

In the neighborhood of the carrier frequency ω_0, the susceptibility may be approximated by

$$4\pi\chi(\omega)=a_0+a_2(\omega_0/\omega)^2, \tag{4.29}$$

where a_0 and a_2 are real. Hence χ is real, and $\chi(\omega)=\chi(-\omega)$ which assures the reality of $\mathbf{P}_{nr}$. Absorption associated with an imaginary part of χ, as required by the Kramers–Kroenig relations, is presumably small and will be ignored.

If it is assumed that $\mathbf{E}$ is plane polarized and of the form of a steady-state pulse with envelope velocity v_e and phase velocity v_p, then one may introduce the scalar electric field

$$E=\mathcal{E}(t-x/v_e)\cos[\omega_0(t-x/v_p)+\phi(t-x/v_e)]. \tag{4.30}$$

The resonant polarization of the medium may be written

$$P_r=n_0\wp[-\mathcal{P}\sin(\psi+\phi)+\mathcal{Q}\cos(\psi+\phi)], \tag{4.31}$$

where

$$\psi\equiv\omega_0(t-x/v_p), \tag{4.32}$$

and $\mathcal{P}$, $\mathcal{Q}$, and ϕ are functions of $t-x/v_e$. As is shown in Appendix D, the insertion of these forms into the wave equation and into the equation for energy conservation leads to

$$c/v_e=(1+a_0)^{1/2}, \tag{4.33}$$

$$c/v_p=(1+a_0)^{1/2}-a_2^{1/2}. \tag{4.34}$$

Also, as shown in Appendix D, one finds

$$\mathcal{P}=\tau\tilde{E}, \tag{4.35a}$$

$$\mathcal{Q}=\gamma\tau\tilde{E}, \tag{4.35b}$$

where

$$\gamma\equiv-(\omega_0/2\pi\sigma)[a_2(1+a_0)]^{1/2}, \tag{4.36}$$

and τ is as defined in Eq. (4.8). From Eqs. (4.35) it is seen that γ represents the ratio of the in-phase component of polarization to the out-of-phase component. For a dispersionless system, $\gamma=0$ and hence $\mathcal{Q}=0$. The effects of dispersion may therefore be considered to be large when $\gamma\geq1$.

From Eq. (2.13)

$$\tilde{E}^2=(1-\mathfrak{N}^2)/\tau^2(1+\gamma^2), \tag{4.37}$$

and from Eq. (2.12a)

$$\partial\mathfrak{N}/\partial t=-[(1+\mathfrak{N}^2)/\tau(1+\gamma^2)]. \tag{4.38}$$

The population is again seen to be inverted by the pulse, for the solution of Eq. (4.38) is

$$\mathfrak{N}=-\tanh(t/\tau_p), \tag{4.39}$$

where

$$\tau_p=\tau(1+\gamma^2). \tag{4.40}$$

The pulse shape, which follows from Eq. (4.37), is

$$\tilde{E}=\tau_p^{-1}\operatorname{sech}(t/\tau_p). \tag{4.41}$$

From Eq. (4.19), (4.35), and (4.37),

$$\partial\phi/\partial t=-(\gamma/\tau_p)\mathfrak{N}, \tag{4.42}$$

and the relative change in the "instantaneous" frequency is

$$(\omega_{\text{inst}}-\omega_0)/\omega_0=-\omega_0^{-1}(\partial\phi/\partial t)=(\gamma/\omega\tau_p)\mathfrak{N}. \tag{4.43}$$

Hence, the chirp is now monotonic and, for γ comparable to unity, is proportional to the first power of the ratio of optical period to pulse width.

For large γ, it may easily be seen that the population inversion takes place by means of adiabatic rapid passage (Tr68a). If we introduce a position vector $\mathfrak{R}$ in a three-dimensional $\mathcal{P}$, $\mathcal{Q}$, $\mathfrak{N}$ space according to

$$\mathfrak{R}=\mathbf{e}_1\mathcal{P}+\mathbf{e}_2\mathcal{Q}+\mathbf{e}_3\mathfrak{N}, \tag{4.44}$$

as well as a vector describing both the electric field and the detuning of an individual two-level system by

$$\mathfrak{E}=\mathbf{e}_2\tilde{E}+\mathbf{e}_3(\partial\phi/\partial t), \tag{4.45}$$

then Eqs. (2.12), for the case of exact resonance, may be written

$$(d\mathfrak{R}/dt)=\mathfrak{E}\times\mathfrak{R}. \tag{4.46}$$

For large γ, the angle between $\mathfrak{R}$ and $\mathfrak{E}$ is given by

$$\cos\theta=\mathfrak{R}\cdot\mathfrak{E}/|\mathfrak{R}||\mathfrak{E}|\underset{\gamma\gg1}{\longrightarrow}1. \tag{4.47}$$

Hence the position vector remains collinear with $\mathfrak{E}$. As the pulse passes a given two-level system and $\partial\phi/\partial t$ goes from γ/τ_p to $-\gamma/\tau_p$, the $\mathbf{e}_3$ component of $\mathfrak{R}$ must proceed from 1 to -1. Such a response of the position vector $\mathfrak{R}$ has been encountered in nuclear magnetic resonance and has come to be referred to as adiabatic rapid passage (Ab61).

V. A TRANSFORMATION OF THE UNDAMPED BLOCH EQUATIONS

In treating ultrashort pulses it has been found useful to observe that the undamped Bloch equations have exactly the same structure as the Frenet–Serret equations of differential geometry (Ei60). The solution of such a set of equations is equivalent to the solution of a Riccati equation (Ei60). To show this, one first recalls from Eq. (2.13) that an integral of Eqs. (2.12) is

$$\mathcal{R}^2+\mathcal{P}^2+\mathcal{Q}^2=1. \tag{5.1}$$

Two new functions may now be introduced by writing

$$(\mathcal{R}+i\mathcal{P})/(1-\mathcal{Q})=(1+\mathcal{Q})/(\mathcal{R}-i\mathcal{P})\equiv\varphi, \tag{5.2a}$$

and

$$(\mathcal{R}-i\mathcal{P})/(1-\mathcal{Q})=(1+\mathcal{Q})/(\mathcal{R}+i\mathcal{P})\equiv-\psi^{-1}=\varphi^*. \tag{5.2b}$$

Equations (5.2) may be inverted to yield

$$\mathcal{R}=(1-\varphi\psi)/(\varphi-\psi)=2\,\mathrm{Re}\,\varphi/(|\varphi|^2+1), \tag{5.3a}$$

$$\mathcal{P}=i(1+\varphi\psi)/(\varphi-\psi)=2\,\mathrm{Im}\,\varphi/(|\varphi|^2+1), \tag{5.3b}$$

$$\mathcal{Q}=(\varphi+\psi)/(\varphi-\psi)=(|\varphi|^2-1)/(|\varphi|^2+1). \tag{5.3c}$$

Equations governing the time dependence of φ and ψ are readily deduced by inserting Eqs. (5.3) into Eqs. (2.12). It is found that φ satisfies the Riccati equation

$$\partial\varphi/\partial t=i\tilde{E}\varphi+(i/2)[\Delta\omega+(\partial\phi/\partial t)](\varphi^2-1), \tag{5.4}$$

and that ψ satisfies the same equation.

One may now employ the usual transformation to convert this Riccati equation to a second-order linear equation. If the phase term is neglected the problem is reduced to that of solving the equation

$$d^2W/dt^2+\tfrac{1}{4}[(\Delta\omega)^2+\tilde{E}^2+2i(d\tilde{E}/dt)]W=0, \tag{5.5}$$

where the dot signifies differentiation with respect to t. The new dependent variable W is related to φ through the transformations

$$W=u\exp\left(-\tfrac{1}{2}i\int_{-\infty}^{t}dt'\tilde{E}\right)$$

and $\varphi=(2i/\Delta\omega)d(\ln u)/dt$. From well-known properties of such second-order differential equations, it follows that in general it is impossible to write φ or ψ explicitly in terms of quadratures of $\tilde{E}$.

Equation (5.5) is particularly instructive since it puts power broadening in evidence and provides immediate contact with results obtainable from the well-known vector model for describing the response of a two-level system to an external field (RRS54, FVH57). For a constant envelope $\tilde{E}=\tilde{E}_0$, Eq. (5.5) is readily solved in terms of the functions

$$W_{1,2}=\exp\{\pm(it/2)[(\Delta\omega)^2+\tilde{E}_0{}^2]^{1/2}\}. \tag{5.6}$$

If the population is initially in the lower level and the pulse is turned on at $t=0$, then the proper initial condition for W is readily found to be $\dot{W}(0)/W(0)=i(\Delta\omega-\tilde{E}_0)/(\Delta\omega^2+\tilde{E}_0{}^2)^{1/2}$, and one obtains

$$\mathcal{R}=-1+2\cos^2\beta\sin^2(\Delta\omega t/2), \tag{5.7}$$

where

$$\beta=\tan^{-1}(\Delta\omega/\tilde{E}_0). \tag{5.8}$$

This constant field result agrees with that obtained from the geometric model. It should be emphasized that although the vector model itself is valid for arbitrarily short pulses, Eq. (5.7) and the form for power broadening expressed by Eq. (5.6) are only applicable if the pulse envelope varies slowly on the time scale $\tilde{E}_0{}^{-1}$. For the ultrashort pulses under consideration here, this condition is violated. An example of a time-dependent pulse profile for which Eq. (5.4) is still soluble in closed form is the steady-state solution for self-induced transparency, namely

$$\tilde{E}=(2/\tau)\,\mathrm{sech}\,(t/\tau). \tag{5.9}$$

The solution of Eq. (5.4) when $\tilde{E}$ has this form and the phase term is neglected may be shown to be

$$\varphi=-[\tau\Delta\omega-i\exp\tfrac{1}{2}(i\sigma)]/[\tau\Delta\omega-i\exp\tfrac{1}{2}(-i\sigma)], \tag{5.10}$$

where

$$\sigma\equiv\int_{-\infty}^{t}dt'\tilde{E}=4\tan^{-1}e^{t/\tau}. \tag{5.11}$$

Equations (5.3) now yield expressions for the response of the system which agree with Eqs. (3.5).

VI. A SOLUBLE MODEL

Although it is possible to obtain a fairly complete analytical description of steady-state pulse propagation in an inhomogeneously broadened medium, most other features of ultrashort pulse propagation have not thus far yielded to analytical treatment when inhomogeneous broadening is included. However, if inhomogeneous broadening is neglected, the analysis may be pursued much further. It has been found that results predicted on the basis of such a model are preserved to a considerable extent when inhomogeneous broadening is included, and the more complete set of equations is investigated by numerical computations (HS69, IL69, HRLS71). Furthermore, the model is not without physical interest in its own right. As might be expected from Eq. (5.6), it may be used as an approximate description of optical pulse propagation under conditions of extreme saturation broadening (RSJ68).

The simplification introduced by the assumption of vanishing bandwidth is immediately evident when it is noted that Eq. (5.4) becomes linear when one continues to neglect the phase term and sets $\Delta\omega=0$. The solution is then

$$\varphi=\pm e^{i\sigma}, \tag{6.1}$$

where σ is as defined by Eq. (5.11), or equivalently,

$$\tilde{E}=\partial\sigma/\partial t. \tag{6.2}$$

The choice of sign in Eq. (6.1) is again related to the two relevant initial conditions $\mathcal{R}(x,-\infty)=\pm1$. From

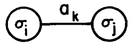

FIG. 2. Symbolic representation of Bäcklund transformation given in Eqs. (6.11).

Eq. (5.3) there follows

$$\mathfrak{N} = \pm \cos\sigma, \tag{6.3a}$$

$$\mathcal{P} = \pm \sin\sigma, \tag{6.3b}$$

and

$$\mathcal{Q} = 0. \tag{6.3c}$$

It is convenient in the subsequent analysis to introduce the dimensionless independent variables

$$\tau = \Omega(t - x/c), \qquad \xi = (\Omega/c)x, \tag{6.4}$$

where

$$\Omega = (\alpha' c)^{1/2}. \tag{6.5}$$

When nonresonant losses are neglected, Eq. (2.9a) takes the form

$$\partial^2\sigma/\partial\xi\partial\tau = \pm \sin\sigma. \tag{6.6}$$

This nonlinear partial differential equation is the fundamental equation of the model (AB65, La67). Fortunately, it has already been studied extensively since it arose long ago in the theory of pseudospherical surfaces, i.e., surfaces of constant negative curvature (Ei60). More recently, it has also arisen in dislocation theory (SDK53), model field theories (SK61, PS62, En63), superconductivity (Jo65, LS67, Sc70), and in mechanical models of nonlinear wave propagation (Sc69).

The general solution of Eq. (6.6) is unknown at the present time. However, a variety of particular solutions have been discovered. One rather large class of solutions is expressible in terms of the variables

$$u = a\tau + \xi/a,$$

$$v = a\tau - \xi/a, \tag{6.7}$$

where a is an arbitrary constant. In terms of these independent variables, Eq. (6.6) becomes

$$(\partial^2\sigma/\partial u^2) - (\partial^2\sigma/\partial v^2) = \pm \sin\sigma. \tag{6.8}$$

The above-mentioned solutions are of the form

$$\sigma(u, v) = 4 \tan^{-1}[F(u)/G(v)]. \tag{6.9}$$

Substitution of this assumed form into Eq. (6.8) leads to the requirement that $F(u)$ and $G(v)$ satisfy the equations

$$F'^2 = -kF^4 + mF^2 + n$$

and

$$G'^2 = kG^4 + (m-1)G^2 - n, \tag{6.10}$$

where k, m, and n are arbitrary constants, and the primes indicate differentiation with respect to the appropriate independent variable. The various pseudospherical surfaces corresponding to such solutions are known as the surfaces of Enneper of constant curvature, and have been exhaustively catalogued by Steuerwald (St36).

Among such solutions are analytical expressions that describe not only the steady-state 2π pulse associated with self-induced transparency, but also solutions that correspond to a 4π pulse as well as pulse envelopes for which the total pulse area is zero—the so-called 0π pulses. As noted previously, such pulse shapes cannot be discarded on any physical basis. The negative part of the envelope in a 0π pulse merely indicates the way in which the present model accommodates a phase change of π that could take place in a more complete theory in which the phase term ϕ of Eq. (2.4) were retained. Small amplitude solutions of Eq. (6.6) which exhibit this ringing have been investigated recently (Cr70). The 4π solution exhibits the pulse breakup phenomenon that has been observed both experimentally (McCH69, GS70) and in numerical computations (McCH69, HS69).

A more general method of obtaining solutions of Eq. (6.6) uses the fact that it is an example of an equation which admits of a Bäcklund transformation (Bä76, Bä82, Cl03, Go18, SDK53, Fo59). Such transformations, which are more general than contact transformations, may be interpreted geometrically as the transformation of a surface that corresponds to a solution of a given partial differential equation, into another surface which is the solution to another, or in some cases, the same equation.

For Eq. (6.6), the transformation equations are (Go18, Ei60, SDK53)

$$\tfrac{1}{2}(\partial/\partial\tau)(\sigma_1 - \sigma_0) = a \sin[(\sigma_1 + \sigma_0)/2], \tag{6.11a}$$

and

$$\tfrac{1}{2}(\partial/\partial\xi)(\sigma_1 + \sigma_0) = \pm a^{-1} \sin[(\sigma_1 - \sigma_0)/2]. \tag{6.11b}$$

These relations may be derived without appealing to their geometric significance by using a technique devised by Clairin (Cl03, La69b).

One may easily show that both σ_0 and σ_1 satisfy Eq. (6.6). Hence, from a given solution σ_0, one may obtain a new solution σ_1 which contains not only the constant a, but also an arbitrary constant of integration γ. This transformation may be used repeatedly to generate a solution σ_2 from σ_1, etc. For extensive calculations of this sort, it is convenient to use a symbolic representation of Eqs. (6.11) in which a transformation from a solution σ_i to a solution σ_j with a constant a_k is represented as shown in Fig. 2. As a first usage of multiple transformations of this sort, one may show quite readily that the four solutions related by the transformation depicted in Fig. 3 satisfy

$$\tan\tfrac{1}{4}(\sigma_3 - \sigma_0) = [(a_1 + a_2)/(a_1 - a_2)] \tan\tfrac{1}{4}(\sigma_1 - \sigma_2). \tag{6.12}$$

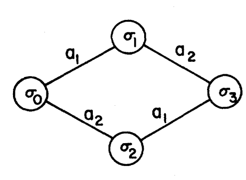

FIG. 3. Diagram for sequence of transformations giving 4π and 0π pulses.

This result, quite remarkably, permits the determination of a new solution σ_3 without the use of quadratures. A simple algebraic manipulation of the eight equations implied by Fig. 3 leads immediately to Eq. (6.12). It will be shown subsequently that this result may be used to construct a 4π pulse, as well as a number of different types of 0π pulses.

If $a_1=a_2$, and the integration constants γ_1 and γ_2 in σ_1 and σ_2, respectively, are different, this relation merely yields

$$\sigma_3=\sigma_0\pm\pi. \tag{6.13}$$

When the integration constants are the same, however, the resulting indeterminate form may be evaluated from the usual Taylor expansion, and one finds

$$\tan\tfrac{1}{4}(\sigma_3-\sigma_0)=\tfrac{1}{2}a_1[(\partial\sigma_1/\partial a_1)+(\partial\sigma_1/\partial\gamma)(\partial\gamma/\partial a_1)], \tag{6.14}$$

where $\gamma=\gamma_1=\gamma_2$.

It will be shown that the compounding of transformations shown in Fig. 4 yields a 6π pulse. An obvious generalization to obtain a $2n\pi$ pulse, and, of course, more complicated 0π pulses suggests itself immediately, but the subject has not been pursued beyond this point.

According to Fig. 1, all of the above solutions represent modes of propagation that are realizable only in an attenuator. It has been found that the invariance of Eq. (6.6) under the one-parameter group of transformations $\bar{\xi}=a\xi$, $\bar{\tau}=a^{-1}\tau$ leads to a similarity solution in terms of a single independent variable $z=\xi\tau$. The solution is a π pulse and hence is realizable in an amplifier. Such solutions have also been considered within the context of differential geometry (Am55), and have been used recently in the attenuator case to describe coherent resonance fluorescence of thin samples of resonant materials (BC69).

A. Specific Pulse Profiles

We now turn to a more detailed consideration of the various solutions of Eq. (6.6) that were mentioned above. The results are, of course, meager in comparison with the complete analytical description of the evolution of arbitrary initial pulse profiles that can be obtained for linear initial value problems. However, the particular solutions that have been found do exhibit many of the important and interesting features of optical pulse propagation.

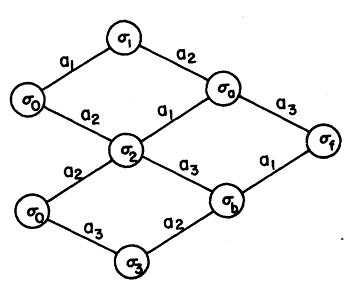

FIG. 4. Diagram for sequence of transformations giving 6π pulses.

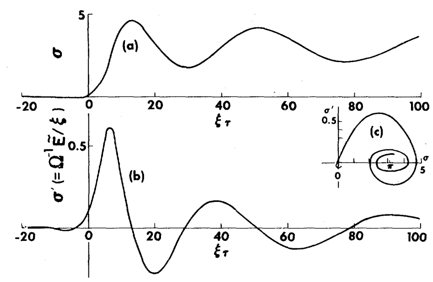

FIG. 5. (a) Numerical solution of Eq. (6.15) satisfying $\sigma(0)=0.1$. (b) Derivative of the solution shown in (a). The pulse envelope is related to this result by $\Omega^{-1}\tilde{E}=\xi\sigma'$. (c) Phase plane diagram of the solution.

1. π Pulse

A π pulse is described by a similarity solution of Eq. (6.6). The fact that the equation is invariant under the transformation $\bar{\xi}=a\xi$, $\bar{\tau}=a^{-1}\tau$ implies the existence of the similarity variable $z=\xi\tau$ (Am65). The solution of the linearized counterpart of Eq. (6.6), obtained by replacing $\sin\sigma$ by σ, is also expressed in terms of this combination of variables (BC69). In terms of this independent variable, Eq. (6.6) reduces to (La69c)

$$z\sigma''+\sigma'-\sin\sigma=0, \tag{6.15}$$

where the prime indicates differentiation with respect to z.

The new dependent variable, W, related to σ by $W=\exp(i\sigma)$, is readily shown to satisfy a special case of the equation which defines the third Painlevé transcendent (In56). Since these functions are not available in convenient form, it is preferable to resort to a direct numerical integration of Eq. (6.15). The result of such a solution is shown in Fig. 5 which also includes the result for $\sigma'=\Omega^{-1}\tilde{E}/\xi$ and a phase plane diagram of the solution. The example shown in Fig. 5 satisfies the condition $\sigma(0)=0.1$ as well as $\sigma(0)=\sin\sigma(0)$ which is required in order for the solution to be finite in the vicinity of the origin (Fo59, Vol. III, p. 193).

Scaling laws for π pulse propagation in a lossless amplifier may be inferred from these results. Since the abscissa for the pulse envelope is $\xi\tau$, the actual pulse envelope narrows linearly with increasing distance of propagation. Also, since $\Omega^{-1}\tilde{E}=\xi\sigma'$, the amplitude of the envelope increases linearly with distance. Since the pulse is moving forward with respect to the retarded time coordinate frame employed above, the pulse profile is actually moving slightly faster than the light velocity. With continued propagation, however, it slows down and asymptotically approaches the light velocity. This is the velocity of a temporal peak, however, and not of a spatial peak. It may be expected that the velocity associated with such temporal profiles will exceed the velocity of light in the medium (IL69).

This spatial evolution of the pulse shape is shown explicitly in Fig. 6. Similar results have been obtained from direct numerical analysis of the partial differential

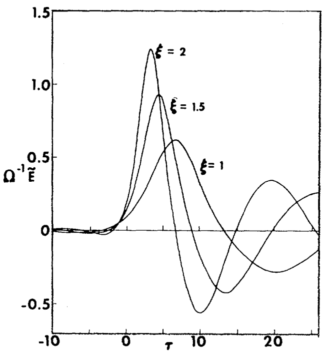

FIG. 6. Spatial evolution of a π pulse in a lossless amplifier.

equations governing optical pulse propagation in a resonant medium (WHG68, BN69, IL69). A comparison of Fig. 6 with the numerical results of these other workers shows that until the signal becomes so large that linear loss is dominant (AB65, AC68, AC69), neglect of the loss term introduces no significant change in propagation in an amplifier.

The fact that the self-consistent interaction of field and resonant matter should give rise to ringing is not unexpected in view of the known (BC64) response of an inverted population to a specified spatial mode of the electric field. The ringing may also be inferred from a theorem concerning solutions of Eq. (6.6). It may be shown (Bl45, St69) that there is no function which satisfies Eq. (6.6) and at the same time remains within the interval $0<\sigma<\pi$.

2. 2π Pulse

As indicated previously, a large number of pulse profiles may be obtained for propagation in an attenuator. Perhaps the most widely known solution of this type is the one related to self-induced transparency. It may be obtained in a number of ways, the simplest being that of merely assuming a steady-state solution of the form $\sigma(t-x/V)$. Such steady-state solutions will be discussed subsequently. The solution may also be obtained by noting that $\sigma=0$ is a solution of Eq. (6.6). This solution may then be used as σ_0 in the Bäcklund transformation given by Eqs. (6.11). When we choose the lower sign in the second of Eqs. (6.11), as is required for propagation in an attenuator, the two resulting first-order differential equations have the solution

$$\sigma_1=4\tan^{-1}e^v, \tag{6.16}$$

where v is as defined in Eq. (6.7). A constant of integration that merely serves to translate the initial location of the solution along the v axis has been neglected in this result. The corresponding electric field follows from Eq. (6.2), and one finds

$$\tilde{E}=2a\Omega\ \mathrm{sech}\ v=2a\Omega\ \mathrm{sech}\ [a\Omega(t-x/V)], \tag{6.17}$$

where

$$V^{-1}=c^{-1}[1+a^{-2}]. \tag{6.18}$$

From Eq. (6.17) it is seen that $(a\Omega)^{-1}$ determines the width of the pulse envelope. Therefore, setting $a\Omega=\tau_p^{-1}$, where τ_p is the pulse half-width, the expression for the electric field envelope becomes

$$\tilde{E}=\frac{2}{\tau_p}\ \mathrm{sech}\left[\frac{t-x/V}{\tau_p}\right]. \tag{6.19}$$

The envelope velocity is given by

$$V^{-1}=c^{-1}+\alpha'\tau_p^2=c^{-1}(1+U_m/U_w), \tag{6.20}$$

where $U_m=n_0\hbar\omega_0$ is the energy density stored in the medium, while $U_w=(\hbar/\wp\tau_p)^2/2\pi$ is the energy density in the wave. The second form for the velocity given in Eq. (6.20) is particularly instructive and has been derived (Co68) on simple physical grounds by equating the average energy of both wave field and medium $V\tau_p(U_w+U_m)$, to $c\tau_pU_w$ the amount of energy that flows through the volume $V\tau_p$ at the light velocity c. Eqs. (6.19) and (6.20) agree with Eqs. (3.5a) and (3.9) in the appropriate limit, namely $g(\Delta\omega)=\delta(\Delta\omega)$.

3. 4π Pulse

It has been observed, both experimentally and from machine computations, that the combination of field strength and magnitude of dipole moment sufficient to induce two inversions in the population of the two-level system, a so-called 4π pulse, does not propagate as a single pulse but rather separates into two separate 2π pulses. Pulse decomposition is a natural by-product of the alternate amplification and attenuation of a pulse that accompanies the coherent oscillations in population and induced polarization of the two-level systems. A portion of the pulse is attenuated if it interacts with the atomic systems when they are in the lower level, while it is amplified by stimulated emission if the population is inverted. Each isolated portion of the pulse becomes a 2π pulse with amplitude and envelope velocity related according to Eqs. (3.5a) and (3.9). Such pulse decomposition is also exhibited by the analytical solution (La67, La69b). The 4π pulse is obtainable as the function σ_3 in Eq. (6.12), when one chooses $\sigma_0=0$. Again choosing the lower sign in Eq. (6.6), as is appropriate for the attenuator, one obtains

$$\sigma_i=4\tan^{-1}(\exp v_i), \qquad i=1,2, \tag{6.21}$$

where

$$v_i=a_i\tau-\xi/a_i. \tag{6.22}$$

The resulting expression for σ_3 may be put in the form

$$\sigma_3=4\tan^{-1}\left[\left(\frac{a_1+a_2}{a_1-a_2}\right)\frac{\sinh\frac{1}{2}(v_1-v_2)}{\cosh\frac{1}{2}(v_1+v_2)}\right]. \tag{6.23}$$

For $a_1>0$, $a_2<0$, the function σ_3 in Eq. (6.23) varies from -2π to 2π as τ proceeds from $-\infty$ to ∞. Since

$$\theta=\int_{-\infty}^{\infty}dt\tilde{E}=\sigma(\infty)-\sigma(-\infty)=4\pi, \tag{6.24}$$

one may expect that the associated electric field will correspond to a 4π pulse and that the envelope will exhibit the pulse decomposition effect. This is found to be the case.

Setting $a_1\Omega=\tau_1^{-1}$, $-a_2\Omega=\tau_2^{-1}$ and using Eq. (6.2), the electric field is found to be

$$\tilde{E}=A\,\frac{(2/\tau_1)\text{ sech }X+(2/\tau_2)\text{ sech }Y}{1-B(\tanh X\tanh Y-\text{sech }X\text{ sech }Y)},\qquad(6.25)$$

where

$$A=(\tau_2^2-\tau_1^2)/(\tau_2^2+\tau_1^2),$$

$$B=2\tau_1\tau_2/(\tau_2^2+\tau_1^2),$$

$$X=\frac{(t-x/V_1)}{\tau_1},$$

and

$$Y=\frac{(t-x/V_2)}{\tau_2}.\qquad(6.26)$$

The velocities V_1 and V_2 are given by

$$V_1^{-1}=c^{-1}(1+\alpha' c\tau_1^2),$$

$$V_2^{-1}=c^{-1}(1+\alpha' c\tau_2^2).\qquad(6.27)$$

A graph of Eq. (6.25) is shown in Fig. 7. As the pulses become completely separated, Eq. (6.25) reduces to the two steady-state pulses

$$\tilde{E}=(2/\tau_1)\text{ sech }(X\pm\beta)+(2/\tau_2)\text{ sech }(Y\mp\beta),\qquad(6.28)$$

where the upper sign is to be included for $\tau_1<\tau_2$, and the lower sign for $\tau_1>\tau_2$, and where

$$\beta=\tanh^{-1}B.\qquad(6.29)$$

In order to obtain a pulse envelope that begins at $\xi=0$ with only one peak as a function to time, one must impose the requirement $\partial^2\tilde{E}/\partial\tau^2<0$ at $\xi=\tau=0$. This condition, along with the requirement $\tilde{E}>0$, leads to

$$(1-r)(1+r^2-3r)>0,\qquad r\equiv\tau_1/\tau_2,\qquad(6.30)$$

which is equivalent to

$$0<r<\tfrac{1}{2}[3-(5)^{1/2}].\qquad(6.31)$$

Figure 7 could, of course, be continued back to negative values of ξ to provide an example of the envelope distortion that takes place when an ultrashort pulse overtakes a slower pulse and passes through it.

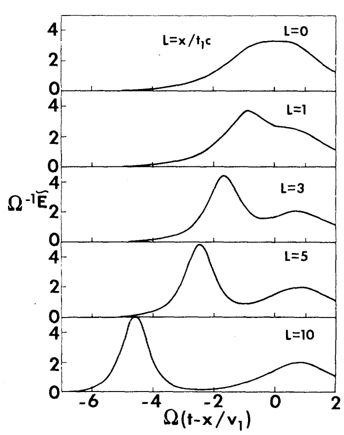

FIG. 7. Breakup of 4π pulse for $\alpha' c\tau_1^2=\frac{1}{2}$, and ratio of final pulse widths $\tau_1/\tau_2=(3+\sqrt{5})/2$.

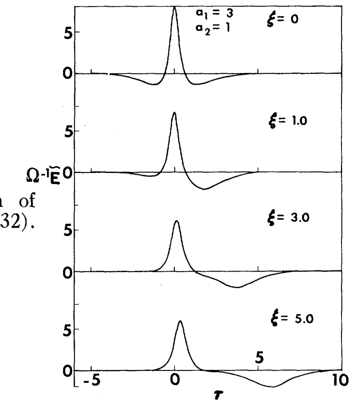

FIG. 8. Propagation of 0π pulse given in Eq. (6.32).

In addition to the above results for specific pulse profiles, certain conclusions relating to 4π pulses of arbitrary initial shape may be given by employing conservation laws that are satisfied by Eq. (6.6). This topic will be considered in Secs. VI.C and VII.

4. 0π Pulses

As mentioned previously, Eq. (6.6) also admits of solutions for which the associated electric field envelope becomes negative. Two distinct types of 0π pulses have been constructed from the solutions described above. The simplest type is obtained by merely choosing $a_2>0$ in the previous solution for the 4π pulse. The electric field envelope is

$$\tilde{E}=A\,\frac{(2/\tau_1)\text{ sech }v_1-(2/\tau_2)\text{ sech }v_2}{1-B(\tanh v_1\tanh v_2+\text{sech }v_1\text{ sech }v_2)}.\qquad(6.32)$$

An example of this result is shown in Fig. 8.

In the limit $a_1=a_2=a_0$, Eq. (6.12) becomes indeterminate. In this case, one may use Eq. (6.14) to obtain a zero π pulse of the form

$$\sigma=4\tan^{-1}(u\text{ sech }v).\qquad(6.33)$$

This yields the field envelope

$$\tilde{E}=(4/\tau_0)\text{ sech }v\left[\frac{(1-u\tanh v)}{(1+u^2\text{ sech}^2 v)}\right],\qquad(6.34)$$

where $\tau_0=(a_0\Omega)^{-1}$. A graph of this result is shown in Fig. 9.

The second, and by far the more interesting, type of 0π pulse is obtained by allowing the parameters a_1 and a_2 in Eq. (6.12) to become complex, and requiring

$$a_1=a_2^*=a=\alpha+i\beta.\qquad(6.35)$$

One then finds

$$\sigma=4\tan^{-1}[(\alpha/\beta)(\sin q/\cosh p)],\qquad(6.36)$$

where

$$p=\alpha(\tau-\xi/|a|^2),\qquad(6.37)$$

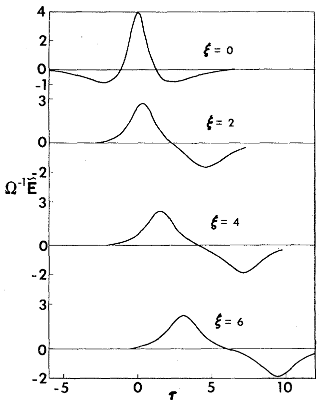

FIG. 9. Propagation of 0π pulse given in Eq. (6.34).

and

$$q=\beta(\tau+\xi/|\,a\,|^2). \qquad (6.38)$$

The electric field envelope is

$$\tilde{E}=(4/\tau_\alpha)\ \text{sech}\ p\left[\frac{\cos q-(\alpha/\beta)\ \sin q\ \tanh p}{1+(\alpha/\beta)^2\sin^2 q\ \text{sech}^2\ p}\right], \qquad (6.39)$$

where $\tau_\alpha=(\alpha\Omega)^{-1}$. A graph of this result is shown in Fig. 10.

Unlike the two previous types of 0π pulses, which separate into two distinct pulses, the envelope given in Eq. (6.44) remains as a single localized disturbance with a half-width equal to τ_α. It provides an alternate, but more complicated, form of self-induced transparency. It has been found (HRLS70) from numerical computations that this pulse shape is remarkably insensitive to variations in inhomogeneous broadening.

5. 6π Pulse

The 6π pulse is obtained from the sequence of transformations depicted in Fig. 4. From this diagram, the corresponding analytical expressions are

$$\sigma_f=\sigma_2+4\tan^{-1}\{K_{13}\tan[(\sigma_a-\sigma_b)/4]\}, \qquad (6.40)$$

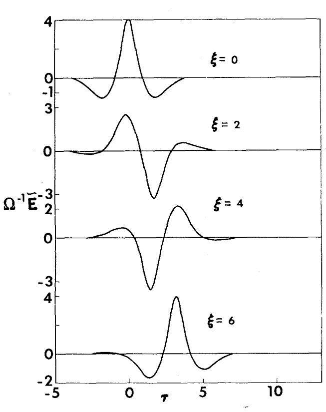

FIG. 10. Propagation of 0π pulse given in Eq. (6.39).

where

$$\sigma_2=4\tan^{-1}\exp(v_2), \qquad (6.41a)$$

$$\sigma_a=4\tan^{-1}\{K_{12}[\sinh\tfrac{1}{2}(v_1-v_2)/\cosh\tfrac{1}{2}(v_1+v_2)]\}, \qquad (6.41b)$$

$$\sigma_b=4\tan^{-1}\{K_{23}[\sinh\tfrac{1}{2}(v_2-v_3)/\cosh\tfrac{1}{2}(v_2+v_3)]\}, \qquad (6.41c)$$

and

$$K_{ij}=(a_i+a_j)/(a_i-a_j). \qquad (6.42)$$

One may immediately impose a number of constraints upon the triad of constants a_1, a_2, a_3. In the first place, for the envelope function corresponding to σ_2 to be positive, one must require $a_2>0$. To obtain a 6π pulse, one may proceed by making σ_a a 4π pulse which requires $a_1<0$. Also, σ_b is made a 0π pulse which requires $0<a_3<a_2$. The three constants a_i may be related to the widths of the three pulses when complete separation has taken place by setting $-a_1\Omega=\tau_1^{-1}$, $a_2\Omega=\tau_2^{-1}$, $a_3\Omega=\tau_3^{-1}$.

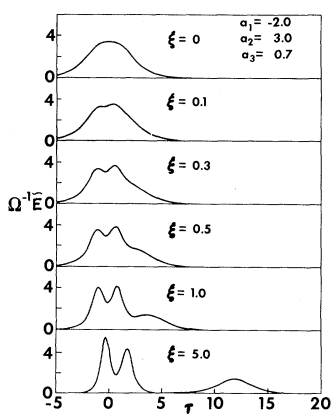

FIG. 11. Propagation of 6π pulse as obtained from sequence of transformations shown in Fig. 4.

As with the 4π pulse, one must impose additional restrictions in order to assure a pulse shape that consists of a single peak at $\xi=0$. The inequality is much more complicated in this case than in Eq. (6.30) and has not been analyzed in detail. By a trial-and-error method the case shown in Fig. 11 has been obtained.

The decomposition of a 6π pulse into three 2π pulses has been observed recently in Rb vapor (GS70), and pulse profiles very similar to those of Fig. 11 have resulted. In particular, amplitudes of final 2π pulses in excess of the initial pulse amplitude, as shown in Figs. 7 and 11, have been obtained.

B. Steady-State Solutions

An example of a steady-state solution has already been given with the discussion of the 2π pulse. This solution is actually a limiting form of a more general oscillatory solution which is now considered. Similar results for propagation in an inhomogeneously broadened medium have also been reported (ADS68, Cr69, Eb69).

Steady-state solutions will be functions of one of the variables defined in Eq. (6.7). If the variable is chosen to be v, one readily shows that the conservation laws given by Eqs. (2.13) and (2.14) take the form

$$-\tfrac{1}{2}\tau_p{}^2\tilde{E}^2+\mathfrak{N}=\mathfrak{N}_1, \tag{6.43}$$

$$\mathcal{P}^2+\mathfrak{N}^2=1, \tag{6.44}$$

where $\mathfrak{N}_1$ is a constant of integration. If one allows for a steady-state solution in which $\tilde{E}$ is nonzero when the entire population is in the ground state, one sees from Eq. (6.44) that the constant $\mathfrak{N}_1$ may be less than -1.

From Eqs. (2.12a), (6.43), and (6.44)

$$(d\mathfrak{N}/dv)^2=2(\mathfrak{N}-\mathfrak{N}_1)(1-\mathfrak{N}^2), \tag{6.45}$$

from which it follows that $\mathfrak{N}$ may be expressed in terms of elliptic functions.

If $-1<\mathfrak{N}_1<1$, a solution for which the population varies between $\mathfrak{N}_1$ and 1 is given by (BF54, p. 79).

$$\mathfrak{N}=1-2k^2sn^2[(v-v_0), k], \tag{6.46}$$

where $k^2=\frac{1}{2}(1-\mathfrak{N}_1)$. From Eq. (6.43), we find

$$\tilde{E}=(2k/\tau_p)cn[(v-v_0), k]. \tag{6.47}$$

A solution for which $-1<\mathfrak{N}<\mathfrak{N}_1$ could also be given, but it requires that $\tau_p{}^2$ be negative. According to Eq. (6.20) this implies envelope function propagation faster than the light velocity. In the limit $k\rightarrow 1$ this solution goes over to one which represents 2π pulse propagation in an amplifier. As will be shown in Sec. VI.D, such a solution is unstable.

For $\mathfrak{N}_1<-1$, it is seen that $k^2>1$. Using the relations (MO54, p. 105)

$$sn(v, k)=k^{-1}sn(kv, k^{-1}),$$

$$cn(v, k)=dn(kv, k^{-1}), \tag{6.48}$$

one finds that the population difference and field envelope may be written

$$\mathfrak{N}=1-2sn^2[k^{-1}(v-v_0), k], \tag{6.49a}$$

$$\mathcal{E}=(2/k\tau_p)dn[k^{-1}(v-v_0), k], \tag{6.49b}$$

where now $k^2=2/(1-\mathfrak{N}_1)$. These latter forms may, of course, be obtained by direct integration of Eq. (6.45). In the limit $\mathfrak{N}_1\rightarrow -1$ both solutions reduce to that for the 2π pulse in an attenuator. It has been conjectured (RN68) that such steady-state solutions may be realizable in self-pulsing situations.

C. Higher Conservation Laws

The hyperbolic secant solution of Eq. (6.6) and the decomposition of pulses into a sequence of such "solitary waves" is quite similar to results obtained in recent investigations of the Korteweg–deVries equation (ZK65, WT66, GGKM67, La68, KS68). In fact, it has been noted (AMS69, Ru70) that for steady-state solutions, the square of the envelope function $\tilde{E}$ satisfies the steady-state Korteweg–deVries equation. This may be seen by writing $\sigma(\xi, \tau)$ in the form $\sigma(v)$. A first integral of Eq. (6.6) is readily obtained. The integral satisfying $\tilde{E}(-\infty)=\sigma(-\infty)=0$ is

$$\cos\sigma=1-\tfrac{1}{2}\tau_p{}^2\tilde{E}^2. \tag{6.50}$$

When this result is solved for σ and differentiated, one obtains

$$\tfrac{1}{4}(dF/dv)^2=F^2-\tfrac{1}{4}F^3, \tag{6.51}$$

where $F=\tau_p{}^2\tilde{E}^2$. Two derivatives of this equation yield the steady-state Korteweg–deVries equation in the form

$$-cf'+ff'+f'''=0, \tag{6.52}$$

where

$$f(x)=\tfrac{3}{4}cF, \tag{6.53}$$

$$x=2c^{-1/2}v, \tag{6.54}$$

and the prime indicates differentiation with respect to x. The solution that vanishes for large values of x may be written in the form (La68)

$$f=3c\,\text{sech}^2\,(\tfrac{1}{2}c^{1/2}x). \tag{6.55}$$

This isolated pulse solution, which has come to be known as a soliton, is equivalent to the result given in Eq. (6.19). A periodic solution of Eq. (6.52) in terms of the cn Jacobian elliptic function may also be obtained. It corresponds to the result given in Eq. (6.17), and is known in the hydrodynamic literature as a cnoidal wave.

Although the multisoliton and oscillatory (BK67) solutions of the Korteweg–deVries equation are similar to results obtained above for $2n\pi$ and π pulses, respectively, and the criterion given in Eq. (6.31) is similar to one appearing in the breakup of two soliton solutions of the Korteweg–deVries equation (La68), to the author's knowledge a quantitative relation between Eq. (6.6) and the time-dependent Korteweg–deVries equation has not been discovered.

An additional similarity between the two equations lies in the fact that both possess conservation laws in addition to the usual ones governing field energy and field momentum. The question of whether or not pulse decomposition may always be inferred from the existance of higher conservation laws has already been raised (Za67). The first two conservation laws, which correspond to field energy and field momentum, follow immediately upon multiplication of Eq. (6.6) by $\partial\sigma/\partial\xi$ or $\partial\sigma/\partial\tau$. They may be written

$$\tfrac{1}{2}(\sigma_\xi{}^2)_\tau+(1-\cos\sigma)_\xi=0, \tag{6.56a}$$

$$(1-\cos\sigma)_\tau+\tfrac{1}{2}(\sigma_\tau{}^2)_\xi=0, \tag{6.56b}$$

where subscripts indicate partial differentiation. Equation (6.56b) follows from (6.56a) by the interchange of ξ and τ. That the conservation laws should appear in such pairs is to be expected in view of the symmetry of Eq. (6.6). The following two conservation laws have also been constructed (La70).

$$(\tfrac{1}{4}\sigma_\xi^4-\sigma_{\xi\xi}^2)_\tau-(\sigma_\xi^2\cos\sigma)_\xi=0, \tag{6.57a}$$

$$(\tfrac{1}{6}\sigma_\xi^6-\tfrac{2}{3}\sigma_\xi^2\sigma_{\xi\xi}^2+(8/9)\sigma_\xi^3\sigma_{\xi\xi\xi}+\tfrac{4}{3}\sigma_{\xi\xi\xi}^2)_\tau -[\cos\sigma(\tfrac{1}{9}\sigma_\xi^4-\tfrac{4}{3}\sigma_{\xi\xi}^2)]_\xi=0. \tag{6.57b}$$

Two additional laws again follow from the interchange of ξ and τ in these expressions.

The conservation laws given in Eqs. (6.57), and presumably even higher ones as well, may be derived by considering Eq. (6.6) within a Hamiltonian framework. A Lagrangian density for Eq. (6.6) is

$$\mathcal{L}=\tfrac{1}{2}\sigma_\xi\sigma_\tau-(1-\cos\sigma), \tag{6.58}$$

as is readily verified by insertion in the appropriate Euler–Lagrange equation, namely,

$$\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{L}}{\partial\sigma_\tau}\right)+\frac{\partial}{\partial\xi}\left(\frac{\partial\mathcal{L}}{\partial\sigma_\xi}\right)-\frac{\partial\mathcal{L}}{\partial\sigma}=0. \tag{6.59}$$

The canonical momentum is $\pi=\partial\mathcal{L}/\partial\sigma_\tau=\frac{1}{2}\sigma_\xi$, and the Hamiltonian density takes the form $\mathcal{H}=(1-\cos\sigma)$. Following the prescription that is well known from the classical theory of fields, one may introduce a function f, defined by $f(\tau)\equiv\int d\xi\mathcal{F}(\sigma,\sigma_\xi,\pi,\pi_\xi)$. The density $\mathcal{F}$ is integrated over the entire range of values accessible to the space variable ξ. The time derivative $\dot{f}$ is given by

$$\dot{f}=\int d\xi[(\delta\mathcal{F}/\delta\sigma)(\delta\mathcal{H}/\delta\pi)-(\delta\mathcal{H}/\delta\sigma)(\delta\mathcal{F}/\delta\pi)], \tag{6.60}$$

where the symbol δ refers to the variational derivative, namely,

$$\delta\mathcal{F}/\delta\sigma\equiv(\partial\mathcal{F}/\partial\sigma)-(\partial/\partial\xi)(\partial\mathcal{F}/\partial\sigma_\xi). \tag{6.61}$$

One sees that in the present instance f will be a conserved quantity (i.e., $\dot{f}=0$) provided

$$\int d\xi(\delta\mathcal{F}/\delta\pi)\sin\sigma=0. \tag{6.62}$$

The conserved density given in Eq. (6.57a) is recovered from this result by first noting that it is of the form $\mathcal{F}(\pi,\pi_\xi)$. Secondly, the vanishing of the integral in Eq. (6.62) will be assured if the integrand is of the form $\partial G/\partial\xi$, where $G(\sigma,\pi,\pi_\xi)$ vanishes for large values of ξ. This leads to the partial differential equation

$$\left[\frac{\partial\mathcal{F}}{\partial\pi}-\frac{\partial}{\partial\xi}\left(\frac{\partial\mathcal{F}}{\partial\pi_\xi}\right)\right]\sin\sigma=\frac{\partial G}{\partial\xi}=\frac{\partial G}{\partial\sigma}\sigma_\xi+\frac{\partial G}{\partial\pi}\pi_\xi+\frac{\partial G}{\partial\pi_\xi}\pi_{\xi\xi}. \tag{6.63}$$

If we recall that $\pi=\frac{1}{2}\sigma_\xi$, the form of Eq. (6.63) is determined by first noting that G must be of the form $G=A(\pi,\pi_\xi)\cos\sigma+B(\pi,\pi_\xi)\sin\sigma$, where A and B are to be determined. Neglecting terms which have the form of an exact divergence or correspond to the lower conservation laws pertaining to field energy or field momentum, one finds

$$\mathcal{F}\sim\pi^4-\pi_\xi^2 \tag{6.64}$$

which is proportional to the density given in Eq. (6.57a). If $\mathcal{F}$ were assumed to be of the form $\mathcal{F}(\pi)$, then the density given by Eq. (6.56a) would be recovered.

It is, of course, unnecessary to confine attention to densities which are merely functions of σ, π and their first spatial derivatives. If $\mathcal{F}$ is of the form $\mathcal{F}(\pi,\pi_\xi,\pi_{\xi\xi})$, then a minor extension of the method outlined above enables one to derive the density given in Eq. (6.57b). A somewhat more direct approach is to avoid the partial integrations employed in the derivation of Eq. (6.60) and merely set

$$\frac{\partial\mathcal{F}}{\partial\tau}=\frac{\partial\mathcal{F}}{\partial\pi}\pi_\tau+\frac{\partial\mathcal{F}}{\partial\pi_\xi}\pi_{\tau\xi}+\frac{\partial\mathcal{F}}{\partial\pi_{\xi\xi}}\pi_{\tau\xi\xi}=\frac{\partial G}{\partial\xi}(\sigma,\pi,\pi_\xi). \tag{6.65}$$

Since $\pi_\tau=-\frac{1}{2}\sin\sigma$, and G is again of the form $G=A(\pi,\pi_\xi)\cos\sigma+B(\pi,\pi_\xi)\sin\sigma$, while $\mathcal{F}$ is at most quadratic in $\pi_{\xi\xi}$, one readily obtains the density given in Eq. (6.57b).

The above conservation laws may be used to determine quite accurately the amplitude of each of the pulses into which a large pulse will decompose as it propagates through an attenuator. This topic will be considered in Sec. VII where the method is extended to inhomogeneously broadened systems.

D. Stability Considerations

When inhomogeneous broadening is present, the stability of the area under the electric field envelope may be inferred (McCH67, McCH69) from the solution of Eq. (3.15). However, an integration over the frequency of the detuning associated with inhomogeneous broadening is a crucial step in the derivation of this result. In the model being considered here, inhomogeneous broadening is neglected, and so one must rely upon other considerations to infer area stability. This is accomplished quite readily by noting that Eq. (6.6) may be written

$$(\partial\tilde{E}/\partial x)=\alpha'\sin\sigma, \tag{6.66}$$

where

$$\sigma(x,t)=\sigma(x,-\infty)+\int_{-\infty}^{t}dt'\tilde{E}(x,t'). \tag{6.67}$$

For a system initially in the lower level, one may take $\sigma(x,-\infty)=-\pi$, for then $\mathfrak{N}(x,-\infty)=\cos(x,-\infty)=-1$. For a system initially in the upper level, one may assume $\sigma(x,-\infty)=0$.

For the hyperbolic secant pulse envelope given in Eq. (6.17),

$$\theta=\int_{-\infty}^{\infty}dt\tilde{E}=2\pi,$$

so that near the trailing edge of this pulse, Eq. (6.66) goes to

$$\frac{\partial\tilde{E}}{\partial x}=\alpha'\sin\left[\begin{pmatrix}-\pi\\0\end{pmatrix}+2\pi\right], \tag{6.68}$$

where the upper choice is made for the attenuator, and the lower choice for the amplifier. Now if there is a perturbation in $\tilde{E}$ such that the total area θ is greater than 2π, then in an attenuator $\partial\tilde{E}/\partial x\sim \sin(\pi+\epsilon)<0$. The field at the trailing edge therefore tends to decrease to recover a total area of 2π. On the other hand, if the perturbation is such that θ is less than 2π, then $\partial\tilde{E}/\partial x>0$, and the field at the trailing edge increases. The total area of such a pulse therefore tends to remain at 2π. In the amplifier, the inequalities are reversed and the hyperbolic secant no longer represents a stable pulse envelope. These results are in agreement with those previously obtained (McCH67, McCH69) for the case in which inhomogeneous broadening is included.

The above considerations refer only to area stability and leave open the possibility of perturbations in which the total area remains unchanged. We now take up this topic and show, by exhibiting a Liapunov functional (Ha67) with vanishing derivative, that pulse shapes are stable but not asymptotically stable, i.e., perturbations remain finite.

Consider first the Liapunov functional $F(u)$ given by

$$F(u)\equiv \int_{-\infty}^{\infty} dv\left[\left(\frac{\partial\sigma}{\partial u}\right)^2+\left(\frac{\partial\sigma}{\partial v}\right)^2+2(1-\cos\sigma)\right] \tag{6.69}$$

which is proportional to the total energy residing in field and medium. Differentiation with respect to u, and a subsequent partial integration yields

$$\frac{dF}{du}=\int_{-\infty}^{\infty} dv\,\frac{\partial\sigma}{\partial u}\left(\frac{\partial^2\sigma}{\partial u^2}-\frac{\partial^2\sigma}{\partial v^2}+\sin\sigma\right)+\frac{\partial\sigma}{\partial u}\frac{\partial\sigma}{\partial v}\Big|_{v=-\infty}^{v=+\infty}. \tag{6.70}$$

For an attenuator the result will vanish by virtue of Eq. (6.8) (in which the lower sign has been chosen as is appropriate for an attenuator) and the boundary conditions that, since σ represents a pulse, both $\partial\sigma/\partial u$ and $\partial\sigma/\partial v$ must vanish at $v=\pm\infty$.

Since dF/du is merely zero rather than negative definite, it is not unexpected that a first-order perturbation analysis of Eq. (6.8) will contain a zero eigenvalue. This is readily seen to be the case. Setting $\sigma=\sigma^{(0)}(v)+\sigma^{(1)}(u,v)$, one finds that $\sigma^{(1)}$ satisfies

$$(\partial^2\sigma^{(1)}/\partial u^2)-(\partial^2\sigma^{(1)}/\partial v^2)-(1-2\,\mathrm{sech}^2\,v)\sigma^{(1)}=0. \tag{6.71}$$

If we express $\sigma^{(1)}$ in the form

$$\sigma^{(1)}(u,v)=V(v)e^{su}, \tag{6.72}$$

$V(v)$ is found to satisfy the "Schrödinger" equation

$$V''+(\lambda-2\,\mathrm{sech}^2\,v)V=0, \tag{6.73}$$

where $\lambda=-(s^2+1)$. For $\lambda=-1$ (and hence $s=0$), one readily finds (MF53) $V(v)=\mathrm{sech}\,v$ which is the solution corresponding to the expected zero eigenvalue.

In an amplifier, the opposite sign in Eq. (6.8) must be used and the above results are no longer applicable. Stability of solutions in terms of ξ and τ may still be shown by considering the Liapunov functional

$$F(\tau)=\int_{-\infty}^{\infty} d\xi(1-\cos\sigma). \tag{6.74}$$

Differentiation and use of Eq. (6.6) yields

$$\dot{F}(\tau)=\pm\tfrac{1}{2}\int_{-\infty}^{\infty} d\xi\,\frac{\partial}{\partial\xi}(\sigma_\tau^2)=\pm\Omega^{-2}\tilde{E}^2\Big|_{-\infty}^{\infty}=0. \tag{6.75}$$

Again, solutions are stable, but not asymptotically stable.

VII. HIGHER CONSERVATION LAWS FOR THE INHOMOGENEOUSLY BROADENED MEDIUM

The decomposition of intense pulses into a number of separate 2π pulses has been described by numerical methods for the inhomogeneously broadened medium (McCH69, HS69). It has been found that the final pulse amplitudes may be fairly accurately determined by using higher conservation laws that are satisfied by the inhomogeneously broadened system. The conservation laws are a generalization of those for the completely unbroadened system that were considered in Sec. VI.C.

Since $\tilde{E}=\Omega\sigma_\tau$, the conservation laws that may be obtained by interchanging ξ and τ in Eqs. (6.56)–(6.58) will express conservation laws obeyed by the electric field envelope. It has been found that the extension of these results to include inhomogeneous broadening is readily accomplished. If the terms in the ξ derivative are retained while those in the τ derivative are suitably modified by employing the (phase-independent) Bloch equations (2.12), one finds the conservation laws

$$(\partial\rho_n/\partial t)+c(\partial F_n/\partial x)=0, \tag{7.1}$$

where

$$\rho_n=F_n+T_n, \tag{7.2}$$

and

$$F_2=\tfrac{1}{2}\Omega^{-2}\tilde{E}^2, \tag{7.3a}$$

$$F_4=\Omega^{-4}(\tfrac{1}{4}\tilde{E}^4-\tilde{E}_t^2), \tag{7.3b}$$

$$F_6=\Omega^{-6}(\tfrac{1}{6}\tilde{E}^6-\tfrac{2}{3}\tilde{E}^2\tilde{E}_t^2+\tfrac{8}{9}\tilde{E}^3\tilde{E}_{tt}+\tfrac{4}{3}\tilde{E}_{tt}^2), \tag{7.3c}$$

$$T_2=\Omega^2\langle 1+\mathfrak{N}\rangle, \tag{7.4a}$$

$$T_4=[\tilde{E}^2\langle\mathfrak{N}\rangle+2\tilde{E}\langle\Delta\omega\mathfrak{Q}\rangle-2\langle(\Delta\omega)^2(1+\mathfrak{N})\rangle], \tag{7.4b}$$

and

$$\begin{aligned}T_6=\Omega^{-2}[&\tfrac{1}{9}\tilde{E}^4\langle\mathfrak{N}\rangle-\tfrac{4}{3}\tilde{E}_t^2\langle\mathfrak{N}\rangle+(4/9)\tilde{E}^3\langle\Delta\omega\mathfrak{Q}\rangle\\&+(8/3)\tilde{E}_t\langle(\Delta\omega)^2\mathcal{P}\rangle-(8/3)\tilde{E}\langle(\Delta\omega)^3\mathfrak{Q}\rangle\\&-\tfrac{4}{3}\tilde{E}^2\langle(\Delta\omega)^2\mathfrak{N}\rangle+(8/3)\langle(\Delta\omega)^4\mathfrak{N}\rangle].\end{aligned} \tag{7.4c}$$

Integration of Eq. (7.4) over the entire length of the host medium, assumed for convenience to be semi-infinite, and over all time, leads to

$$\int_0^{\infty} dx\rho_n(x,t)=c\int_{-\infty}^{\infty} dtF_n(0,t). \tag{7.5}$$

The functions $F_n(0, t)$ may be calculated for a given pulse shape at $x=0$ by using Eqs. (7.3). After the pulse has propagated far enough into the material for the steady-state pulse shapes to evolve, one may write (McCH67, McCH69)

$$\tilde{E}(x, t) = \sum_{i=1}^{N} \left(\frac{2}{\tau_i}\right) \operatorname{sech}\left(\frac{t-t_i-x/v_i}{\tau_i}\right), \qquad (7.6)$$

where the τ_i are the pulse widths, the v_i are the corresponding pulse velocities, and the t_i are merely time delays which separate the various pulses. The value of N is determined by the area under the pulse at $x=0$ in the manner described in Sec. III. For each 2π pulse, the response of the two-level systems in the vicinity of the pulse is given by (McCH67, McCH69)

$$\mathcal{P}_i = -D_i \sin \varphi_i, \qquad (7.7a)$$

$$\mathfrak{N}_i = -1+2D_i \sin^2 (\varphi_i/2), \qquad (7.7b)$$

$$\mathcal{Q}_i = 2D_i\Delta\omega\tau_i \sin (\varphi_i/2), \qquad (7.7c)$$

where

$$\varphi_i = \int_{-\infty}^{t} dt' \tilde{E}_i, \qquad (7.8)$$

and

$$D_i = [1+(\tau_i\Delta\omega)^2]^{-1}. \qquad (7.9)$$

There is, of course, a small region at the edge of the medium within which the pulse evolution into a sequence of 2π pulses takes place. In this region, Eqs. (7.6) and (7.7) are inapplicable. An unknown amount of the population is left in an inverted state, and values of the polarization $\mathcal{P}$ and $\mathcal{Q}$ are similarly unknown. In performing the spatial integration in Eq. (7.5), this fact is neglected, and Eqs. (7.6) and (7.7) are used over the entire range of the spatial integration. The closeness of the results thus obtained to those predicted by the exact numerical calculation shows that, as for the Korteweg–deVries equation (BK67, KS68), this transient region may indeed be ignored. For weaker initial pulses which result in only a single 2π pulse, however, this initial region is no longer negligible. Hence only pulses which have an initial area greater than 3π are considered in the following.

For the initial pulse profile

$$\tilde{E}(0, t) = (\theta_0/\pi t_0) \operatorname{sech} (t/t_0), \qquad (7.10)$$

one finds

$$t_0 \int_{-\infty}^{\infty} dt F_2(0, t) = \left(\frac{\theta_0}{\pi}\right)^2, \qquad (7.11a)$$

$$t_0 \int_{-\infty}^{\infty} dt F_4(0, t) = \frac{2}{3}\left(\frac{\theta_0}{\pi}\right)^2 \left[\frac{1}{2}\left(\frac{\theta_0}{\pi}\right)^2 - 1\right], \qquad (7.11b)$$

$$t_0 \int_{-\infty}^{\infty} dt F_6(0, t) = \left(\frac{56}{45}\right)\left(\frac{\theta_0}{\pi}\right)^2 \left[\frac{1}{7}\left(\frac{\theta_0}{\pi}\right)^4 - \frac{5}{7}\left(\frac{\theta_0}{\pi}\right)^2 + 1\right]. \qquad (7.11c)$$

Also, substitution of Eqs. (7.6) and (7.7) into Eq. (7.2), and integration over the entire length of the medium, yields

$$\int_0^{\infty} dx \rho_2 = 2c \sum a_i, \qquad (7.12a)$$

$$\int_0^{\infty} dx \rho_4 = (1/3) c \sum a_i^3, \qquad (7.12b)$$

$$\int_0^{\infty} dx \rho_6 = (1/15) c \sum a_i^5, \qquad (7.12c)$$

where $a_i = 2/\tau_i = (\tilde{E}_i)_{\max}$. As is known from the area theorem, one expects two pulses for $3\pi<\theta_0<5\pi$. In this case only the first two conservation laws are needed and one finds

$$a_1+a_2 = \tfrac{1}{2}\int dt F_2,$$
$$a_1^3+a_2^3 = 3\int dt F_4. \qquad (7.13)$$

For $5\pi<\theta_0<7\pi$, three pulses will be obtained. Hence one requires the solution of

$$a_1+a_2+a_3 = \tfrac{1}{2}\int dt F_2,$$
$$a_1^3+a_2^3+a_3^3 = 3\int dt F_4,$$
$$a_1^5+a_2^5+a_3^5 = 15\int dt F_6. \qquad (7.14)$$

One sees from Eqs. (7.3) that the higher conservation laws depend upon higher derivatives of the initial pulse shape. Hence, the decomposition of strong pulses is expected to be sensitive to the detailed structure of the initial pulse profile. Such sensitivity has been noted in experimental observations (PS67).

In Fig. 12, the roots of Eqs. (7.13) and (7.14) divided by $\tilde{E}(0, t)_{\max}$ are plotted as a function of θ_0. Equation (7.13) possesses solutions for values of θ_0 for which three pulses are to be expected. The locus of these unphysical solutions is given by the dashed portion of Fig. 12. The crosses are results obtained from numerical solutions of the equations which describe optical pulse propagation and are seen to be in quite good agreement with the approximate values obtained from the conservation laws. The numerical results include cases in which t_0/T_2^* varies from 0.1 to 10. The lack of complete correspondence is due partly to the improper treatment of the initial region in which the isolated pulses have not formed and partly to the fact that the correct amplitudes must satisfy all the higher conservation laws and not merely the lowest two or three that have been used here. Further work on this subject is currently being pursued (LSH).

VIII. INHOMOGENEOUS BROADENING AND PHOTON ECHO

Certain phenomena, notably that of photon echo, require for their explanation the relative dephasing of atoms that results when inhomogeneous broadening is present. The concept of photon echo has been taken over directly from past work on spin echoes (Ha50).

However, since one is primarily interested in the propagation of photon echoes, both the Bloch equations and the Maxwell equations must be solved simultaneously. The associated analytical complexities have only been overcome by numerical methods. However, if one foregoes consideration of the actual pulse shapes and confines attention to the area under the envelope of the pulse, then further analytical progress may be made. In particular, a very simple description may be given of the spatial evolution of the photon echoes that may appear behind two optical pulses as they propagate through an inhomogeneously broadened medium (La69a). This may be carried out by noting that the area theorem, Eq. (3.15), is still satisfied if the pulses are assumed to be infinitely narrow, i.e., of the form

$$\tilde{E}(x,t)=\theta(x)\delta[t-(x/c)], \tag{8.1}$$

where $\delta[t-(x/c)]$ is a delta function. The assumption of propagation at the light velocity is consistent with that of zero pulse width according to Eq. (6.20).

The apparent inconsistency of using a delta function in the slowly varying envelope $\tilde{E}(x,t)$ does no violence to the theory. It merely provides a convenient device for obtaining solutions to Eqs. (2.9a) and (2.12) in the short pulse limit. A derivation of the area theorem for delta function pulses is now given. It is then shown how some of its implications may be readily explored.

The response of a two-level system is governed by the Riccati equation given in Eq. (5.4). If the new complex function $\mu=\mu_r+i\mu_i$ is introduced by the definition $\varphi=\exp(i\mu)$, Eqs. (5.4) and (5.3) become

$$(\partial\mu/\partial t)-i\Delta\omega\sin\mu=\tilde{E}, \tag{8.2}$$

$$\mathcal{P}=\operatorname{sech}\mu_i\sin\mu_r, \tag{8.3a}$$

$$\mathcal{Q}=-\tanh\mu_i, \tag{8.3b}$$

$$\mathcal{N}=\operatorname{sech}\mu_i\cos\mu_r. \tag{8.3c}$$

Using the form for $\tilde{E}$ given in Eq. (8.1) and integrating Eq. (8.2) across the singularity at $t=x/c$, one finds

$$\mu^{>}-\mu^{<}=\theta(x). \tag{8.4}$$

Since θ is real, μ_i is continuous across the pulse, and according to Eq. (8.3b), $\mathcal{Q}$ is also continuous across the pulse. The change in population is

$$\Delta\mathcal{N}=\mathcal{N}^{>}-\mathcal{N}^{<}=\operatorname{sech}\mu_i^{<}(\cos\mu_r^{>}-\cos\mu_r^{<}) \tag{8.5}$$

which may be written as

$$\mathcal{N}^{>}=\mathcal{N}^{<}\cos\theta-\mathcal{P}^{<}\sin\theta. \tag{8.6}$$

Similarly, Eq. (8.3a) yields

$$\mathcal{P}^{>}=\mathcal{P}^{<}\cos\theta+\mathcal{N}^{<}\sin\theta. \tag{8.7}$$

Equations (8.6), (8.7), and the continuity of $\mathcal{Q}$ across the pulse may be summarized in the vector form

$$\begin{pmatrix}\mathcal{P}^{>}\\ \mathcal{Q}^{>}\\ \mathcal{N}^{>}\end{pmatrix}=\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}\mathcal{P}^{<}\\ \mathcal{Q}^{<}\\ \mathcal{N}^{<}\end{pmatrix}. \tag{8.8}$$

The 3×3 matrix represents a rotation about the $\mathcal{Q}$ axis by an angle θ, and may be represented symbolically by $R_{\mathcal{Q}}(\theta)$. Considering $(\mathcal{P},\mathcal{Q},\mathcal{N})$ as the three components of a vector P, Eq. (8.8) may be written

$$P^{>}=R_{\mathcal{Q}}(\theta)P^{<}. \tag{8.9}$$

While the pulse is not acting, the system evolves according to the homogeneous counterpart for Eq. (8.2) which has the solution

$$e^{i\mu}=i\cot\tfrac{1}{2}(\Delta\omega t+\alpha), \tag{8.10}$$

where $\alpha=\alpha_r+i\alpha_i$ is a constant of integration. It is now a simple matter to show that the corresponding evolution of $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{N}$ may be written

$$\begin{pmatrix}\mathcal{P}(t)\\ \mathcal{Q}(t)\\ \mathcal{N}(t)\end{pmatrix}=\begin{pmatrix}\cos\tilde{\omega} & \sin\tilde{\omega} & 0\\ -\sin\tilde{\omega} & \cos\tilde{\omega} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathcal{P}(t_0)\\ \mathcal{Q}(t_0)\\ \mathcal{N}(t_0)\end{pmatrix}, \tag{8.11}$$

where $\tilde{\omega}=\Delta\omega(t-t_0)$. This represents a rotation through an angle $\Delta\omega(t-t_0)$ about the $\mathcal{N}$ axis and may be

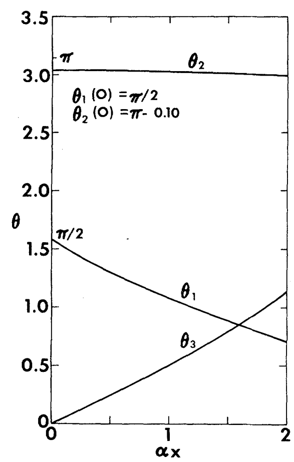

FIG. 13. Spatial development of area functions for two optical pulses and the first photon echo generated by them.

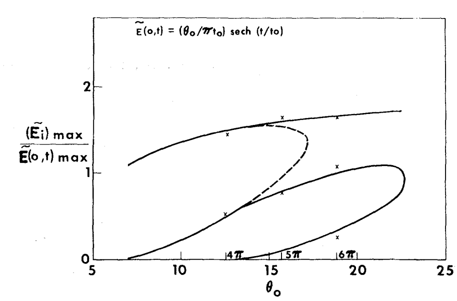

FIG. 12. Pulse amplitudes obtained from conservation laws. Crosses indicate results of preliminary numerical calculations provided by M. O. Scully.

written

$$P(t)=R_{\mathcal{N}}(t-t_0)P(t_0). \tag{8.12}$$

If a transverse relaxation time were retained in the analysis, so that between pulses $\mathcal{P}$ and $\mathcal{Q}$ satisfied by

$$(\partial\mathcal{P}/\partial t)+(1/T_2)\mathcal{P}=\Delta\omega\mathcal{Q}, \tag{8.13a}$$

and

$$(\partial\mathcal{Q}/\partial t)+(1/T_2)\mathcal{Q}=0, \tag{8.13b}$$

Eqs. (8.12) would be replaced by

$$P(t)=\exp[-(t-t_0)/T_2]R_{\mathcal{N}}(t-t_0)P(t_0). \tag{8.14}$$

At a time t after interaction with a pulse of area θ, the state of a system that was initially in the lower level is given by

$$P(t)=R_{\mathcal{N}}(t)R_{\mathcal{Q}}(\theta)P(0)=\begin{pmatrix}-\sin\theta\cos\Delta\omega t\\ \sin\theta\sin\Delta\omega t\\ -\cos\theta\end{pmatrix}. \tag{8.15}$$

Equation (2.9a), with $g(\Delta\omega)=g(0)$ to accommodate all spectral components of the delta function, takes the form

$$\delta(t-x/c)[(d\theta/dx)+\kappa\theta-\pi\alpha' g(0)\sin\theta]=0 \tag{8.16}$$

which is the area theorem given in Eq. (3.15).

This scheme may now be used repeatedly to describe the response of the medium to a sequence of pulses. The response due to two pulses of area θ_1 and θ_2 a time T apart is found to have a contribution at $t=2T$. Evaluating P just beyond the time $t=2T$, one finds

$$P(t)=R_{\mathcal{N}}(t-2T)R_{\mathcal{Q}}(\theta_3)R_{\mathcal{N}}(T)\times R_{\mathcal{Q}}(\theta_2)R_{\mathcal{N}}(T)R_{\mathcal{Q}}(\theta_1)P(0). \tag{8.17}$$

Carrying out the indicated multiplications for a system that is initially in the ground state, one obtains

$$\mathcal{P}(x,t)=-\sin\theta_1\cos\Delta\omega t-\sin\theta_2\cos\theta_1\cos\Delta\omega(t-T)-[\sin\theta_3\cos\theta_2\cos\theta_1-\cos\theta_3\sin^2(\theta_2/2)\sin\theta_1]\times\cos\Delta\omega(t-2T). \tag{8.18}$$

If a relaxation time T_2 were included, this entire expression would merely be multiplied by $\exp(-2T/T_2)$.

If one substitutes this result into Eq. (2.9a) with σ set equal to zero, one obtains

$$d\theta_1/dx=-(\alpha/2)\sin\theta_1, \tag{8.19}$$

$$d\theta_2/dx=-(\alpha/2)\sin\theta_2\cos\theta_1, \tag{8.20}$$

and

$$d\theta_3/dx=\tfrac{1}{2}\alpha[\sin\theta_3\cos\theta_2\cos\theta_1\cos\theta_3\sin^2(\theta_2/2)\sin\theta_1]. \tag{8.21}$$

Equations (8.19) and (8.20) have solutions

$$\tan(\theta_1/2)=\exp(-\tfrac{1}{2}\alpha x+\beta) \tag{8.22}$$

and

$$\tan(\theta_2/2)=\gamma\operatorname{sech}(\tfrac{1}{2}\alpha x-\beta)=\gamma\sin\theta_1, \tag{8.23}$$

where

$$e^{\beta}=\tan[\theta_1(0)/2]$$

and

$$\gamma=\tan[\theta_2(0)/2]\csc\theta_1(0). \tag{8.24}$$

Now, if $\theta_1(0)=\pi/2$, $\theta_2(0)\approx\pi$, the optimum case for photon echo experiments, then $|\gamma|\gg 1$ and, from the solution for θ_2, one sees that θ_2 remains nearly equal to its initial value until θ_1 decays to a value equal to γ^{-1}. Until this final state in the pulse evolution is reached, one may set $\theta_2=\pi$ in Eq. (8.21). The resulting equation may then be transformed to

$$d\theta_3/d\theta_1+\cos\theta_3+\cot\theta_1\sin\theta_3=0. \tag{8.25}$$

Upon substituting $y=\tan(\theta_3/2)$, this becomes a Riccati equation which is converted to a second-order linear equation by the substitution $y=-2(du/d\theta_1)/u$. The substitution $k=\sin(\theta_1/2)$ leads to

$$k(k^2-1)u''+(3k^2-1)u'+ku=0, \tag{8.26}$$

where the prime indicates differentiation with respect to k. Equation (8.26) has the solution (Ka59)

$$u=aK(k)+bK(k'), \tag{8.27}$$

where $K(k)$ is the complete elliptic integral of modulus k, while k' is the complementary modulus. Finally, the solution for θ_3 may be written

$$\tan(\theta_3/2)=[(k'/k)B(k')-(k'/k)B(k)]/[K(k)+K(k')], \tag{8.28}$$

where $B(k)$ is a tabulated function (JE45) related to the complete elliptic integral by

$$B(k)=[(1-k^2)/k](dK/dk). \tag{8.29}$$

Figure 13 contains a graph of Eq. (8.28), as well as the variations in θ_1 and θ_2. It is seen that the area of the echo increases at a rate approximately equal to that at which the first pulse decreases. This is consistent with experimental observations (PS68). A completely satisfactory comparison of the above results with experimental observations is not possible since the experiments measure the area under $\tilde{E}^2$.

A similar analysis of subsequent echoes produced by, say, the second pulse and the first echo could also be carried out by the method described above.

IX. LEVEL DEGENERACY

It has been pointed out (RSJ68, McCH69) that, due to level degeneracy, pulse propagation under conditions which prevail experimentally may lead to results that are considerably different from those predicted here. It has also been shown that level degeneracy has a marked effect on the direction of polarization of the electric field vector of the echo pulse in a photon echo experiment (GWPST69). Thus far, however, only the source term for the echo pulse has been calculated when degeneracy is present. No consideration has as yet been given to the complete problem in which the spatial evolution of the photon echo is followed in the presence of level degeneracy, and so this latter topic will not be considered here.

To avoid detailed consideration of specific molecular models, level degeneracy will merely be expressed in terms of a simple jm scheme. The two states previously denoted by a and b, are now characterized by angular momentum quantum numbers $j'm'$ and jm, respectively. Each element in the 2×2 matrices of Eq. (A9) now becomes a $(2j+1)\times(2j'+1)$ submatrix itself with elements $\langle j'm' \mid \mathfrak{P} \mid jm\rangle$. As is well known (CS57), transitions in j are restricted to $\Delta j = j'-j = -1$, 0, 1, the three alternatives frequently being referred to as P, Q and R branch transitions, respectively. In addition, if the quantization axis is aligned parallel to the electric field polarization vector, then only the $\mathfrak{P}_z$ matrix elements need be calculated. All such matrix elements vanish unless $m'=m$. One then finds that $\wp_{mj} = \langle j'm \mid \mathfrak{P} \mid jm\rangle = \kappa_m \wp$, where $\wp$ is now the largest value of $\wp_{mj}$ in each of the three cases, and

$$\kappa_m = (j^2-m^2)^{1/2}/j, \qquad \Delta j = -1, \quad (9.1a)$$

$$\kappa_m = m/j, \qquad \Delta j = 0, \quad (9.1b)$$

$$\kappa_m = [(j+1)^2-m^2]^{1/2}/(j+1), \qquad \Delta j = 1. \quad (9.1c)$$

Since the submatrices of $\mathcal{V} = -\mathbf{E}\cdot\mathfrak{P} = -E\mathfrak{P}_z$ are now diagonal in m, the various pairs of levels designated by different m values are not coupled by the interaction, and may be treated separately.

Hence, for each value of m one may write

$$\ddot{p}_m + \omega_{ab}^2 p = -(2\omega_{ab}/\hbar) n_m E \mid \langle jm \mid \mathfrak{P}_z \mid j'm\rangle \mid^2, \quad (9.2)$$

$$\dot{n}_m = (2n_0 E/\hbar\omega_{ab}) \dot{p}_m. \quad (9.3)$$

If we assume that all sublevels of the lower state are equally populated initially, then

$$n_m = [n_0/(2j+1)](\langle j'm \mid \rho \mid jm\rangle - \langle jm \mid \rho \mid j'm\rangle). \quad (9.4)$$

The electric field is governed by

$$(\partial\mathcal{E}/\partial t) + c(\partial\mathcal{E}/\partial x) = 2\pi n_0\omega_0 \int_{-\infty}^{\infty} d\Delta\omega g(\Delta\omega) \sum_m p_m. \quad (9.5)$$

When this relation is integrated over the duration of the pulse, one obtains, in analogy with the derivation of Eq. (3.15),

$$d\theta/dx = [\alpha/2(2j+1)] \sum_m \kappa_m \sin(\kappa_m\theta), \quad (9.6)$$

where

$$\theta = (\wp/\hbar) \int_{-\infty}^{\infty} dt' \mathcal{E}(x, t'). \quad (9.7)$$

For transparency to take place, it is necessary that the right-hand side of Eq. (9.6) vanish. For Q branch transitions this will be possible for $\theta = 2n\pi$ just as in the nondegenerate case, since the various κ_m are integrally related. For P and R branch transitions, however, the irrational ratios of the various κ_m prevent a simultaneous vanishing of all κ_m except in the few cases in which there is only one nonvanishing value of κ_m. This takes place for $j=0, \frac{1}{2}$.

However, it has been noted (RSJ68) that the right-hand side of Eq. (9.6) will also vanish if $\int dt'\mathcal{E} = 0$. Such 0π pulses should exhibit transparency independently of the values of the κ_m, and this is borne out by recent numerical solutions (HLRS71). Although profiles of 0π pulses have been described in Sec. VI.A, it should be emphasized that they have been obtained for a nondegenerate two-level system, and are not directly applicable to the present situation.

For large values of j, the summation may be approximated by an integration. Setting $m = j\cos\alpha$, the results quoted in Eqs. (9.1) may be replaced by those for a continuous variable κ given by

$$\begin{aligned} \kappa &= \sin\alpha, \qquad \Delta j = \pm 1 \\ &= \cos\alpha, \qquad \Delta j = 0. \end{aligned} \quad (9.8)$$

For $\Delta j = 0$, we have

$$(2j+1)^{-1} \sum \kappa_m \sin(\kappa_m\theta) \to \frac{1}{2}\int_0^{\pi} d\alpha \sin\alpha\cos\alpha \sin(\theta\cos\alpha) = \frac{\sin\theta - \theta\cos\theta}{\theta^2} = j_1(\theta), \quad (9.9)$$

where $j_1(\theta)$ is a spherical Bessel function. For $\Delta j = \pm 1$, we have

$$(2j+1)^{-1} \sum \kappa_m \sin(\kappa_m\theta) \to \frac{1}{2}\int_0^{\pi} d\alpha \sin^2\alpha \sin(\theta\sin\alpha) = \tfrac{1}{2}\pi\left(\mathbf{H}_0(\theta) - \frac{\mathbf{H}_1(\theta)}{\theta}\right), \quad (9.10)$$

where the $\mathbf{H}_n(\theta)$ are Struve functions (AS64).

The two forms for the area theorem in the presence of degeneracy for large j are now obtained by combining Eq. (9.6) with Eqs. (9.9) and (9.10). It is again evident from these results that a 0π pulse should exhibit transparency. For small θ, the area theorem for Q branch transitions reduces to the small-θ form of Eq. (3.15) when orientational averaging is included.

As in Sec. VI, pulse shapes may be obtained in the limit of extreme saturation broadening. Setting $g(\Delta\omega) = \delta(\Delta\omega)$, Eq. (6.6) now becomes

$$\partial^2\sigma/\partial x\partial t' = [\alpha'/(2j+1)] \sum_m \kappa_m \sin(\kappa_m\sigma), \quad (9.11)$$

where $t' = t - x/c$, and α' is as defined in Eq. (2.11).

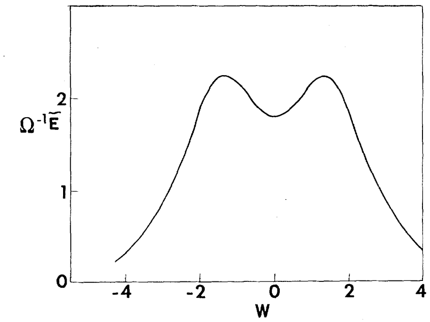

FIG. 14. Steady-state pulse profile for $Q(2)$ transition.

Equation (9.11) has also arisen in dislocation theory (Se55).

Examples of steady-state pulse profiles have been obtained numerically (RSJ68). For the Q-branch transition with $j=2$, the result may be given in a simple closed form (Se55). One obtains

$$\sigma=-4\tan^{-1}[(5)^{1/2}\operatorname{csch} w/\tau], \tag{9.12}$$

where $w=t-x/V$ and

$$V^{-1}=c^{-1}+\tfrac{1}{2}\alpha'\tau^2. \tag{9.13}$$

The electric field is

$$\tilde{E}=\frac{4(5)^{1/2}\operatorname{sech}(w/\tau)}{\tau[1+4\operatorname{sech}^2(w/\tau)]} \tag{9.14}$$

which is shown in Fig. 14.

ACKNOWLEDGMENTS

The author is indebted to Marlan Scully for many informative discussions on the subject of optical pulse propagation, and for providing the numerical solutions shown in Fig. 12. The suggestion by Fred Otter, Jr., of the relevance of previous analytical results in dislocation theory to the material described in Sec. VI is greatly appreciated, for it served to initiate much of the author's work on this topic. The many numerical computations carried out by Boyce Burdick are also gratefully acknowledged.

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APPENDIX A

The polarization of an individual system may be obtained from its microscopic description by the usual trace operation

$$\mathbf{p}=\operatorname{Tr}(\rho\mathfrak{P}), \tag{A1}$$

where ρ is the density matrix of the two-level system (La64), and $\mathfrak{P}$ is the polarization operator. The time dependence of ρ is given by the quantum mechanical Liouville equation

$$i\hbar(\partial\rho/\partial t)+[\rho,\mathcal{H}]=0, \tag{A2}$$

where $\mathcal{H}$ is the total Hamiltonian of an individual two-level system. The time dependence of an arbitrary operator, $\mathfrak{O}$, is governed by the relation

$$i\hbar(d\mathfrak{O}/dt)=i\hbar(\partial\mathfrak{O}/\partial t)+[\mathfrak{O},\mathcal{H}]. \tag{A3}$$

For later use, it proves convenient to recognize that operators not containing explicit time dependence also satisfy

$$\hbar^2(d^2\mathfrak{O}/dt^2)+[[\mathfrak{O},\mathcal{H}],\mathcal{H}]=i\hbar[(\partial\mathcal{H}/\partial t),\mathfrak{O}]. \tag{A4}$$

The Hamiltonian of a two-level system interacting with a classical electromagnetic field may be adequately represented by

$$\mathcal{H}=\mathcal{H}_0+\mathcal{V}, \tag{A5}$$

where $\mathcal{H}_0$ is the Hamiltonian of the isolated two-level system, and

$$\mathcal{V}=-\mathbf{E}\cdot\mathfrak{P} \tag{A6}$$

is the interaction energy in dipole approximation.

The wave function for the isolated two-level system may be written

$$\psi(\mathbf{r},t)=a_a(t)u_a(\mathbf{r})+a_b(t)u_b(\mathbf{r}), \tag{A7}$$

where u_a and u_b are eigenfunctions of the system and satisfy

$$\mathcal{H}_0u_\alpha=E_\alpha u_\alpha, \qquad \alpha=a,b. \tag{A8}$$

Such energy eigenfunctions provide the specific representations

$$\mathcal{H}_0=\begin{pmatrix}E_a & 0\\ 0 & E_b\end{pmatrix},\quad \mathfrak{P}=\begin{pmatrix}0 & \mathbf{p}_{ab}\\ \mathbf{p}_{ba} & 0\end{pmatrix},\quad \rho=\begin{pmatrix}\rho_{aa} & \rho_{ab}\\ \rho_{ba} & \rho_{bb}\end{pmatrix}, \tag{A9}$$

where

$$\mathbf{p}_{\alpha\beta}=-e\int d^3\mathbf{r}u_\alpha^*\mathbf{r}u_\beta,$$
$$\rho_{\alpha\beta}=a_\alpha^*a_\beta, \tag{A10}$$

and the levels are labeled such that $E_a>E_b$. The vanishing of the diagonal elements in $\mathfrak{P}$ signifies the assumed absence of any permanent dipole moment in the system under consideration.

In addition to the polarization $\mathbf{p}$, the difference in population between upper and lower states, n, is also of interest and may be conveniently expressed in the form

$$n=n_0(\rho_{aa}-\rho_{bb})=n_0\operatorname{Tr}(\rho\sigma_z), \tag{A11}$$

where σ_z is the Pauli spin matrix

$$\sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{A12}$$

The time dependence of $\mathbf{p}$ can be conveniently obtained by taking the trace of the operator equation

$$\hbar^2(d^2\mathfrak{P}/dt^2)+[[\mathfrak{P},\mathcal{H}],\mathcal{H}]=i\hbar[(\partial\mathcal{H}/\partial t),\mathfrak{P}] \quad (A13)$$

which follows from Eq. (A4). With the representations given above, the right-hand side of this equation vanishes while

$$[\mathfrak{P},\mathcal{H}]=[\mathfrak{P},\mathcal{H}_0]=-\hbar\omega_{ab}\begin{pmatrix}0 & \mathbf{p}_{ab}\\ -\mathbf{p}_{ba} & 0\end{pmatrix}, \quad (A14a)$$

$$[[\mathfrak{P},\mathcal{H}],\mathcal{H}_0]=\hbar\omega_{ab}^2\mathfrak{P}, \quad (A14b)$$

and

$$[[\mathfrak{P},\mathcal{H}],\mathcal{V}]=2\hbar\omega_{ab}\mathbf{E}\cdot\begin{pmatrix}\mathbf{p}_{ab}\mathbf{p}_{ba} & 0\\ 0 & -\mathbf{p}_{ba}\mathbf{p}_{ab}\end{pmatrix}, \quad (A14c)$$

where

$$\hbar\omega_{ab}=E_a-E_b. \quad (A15)$$

Equation (A11) therefore reduces to

$$(d^2\mathfrak{P}/dt^2)+\omega_{ab}^2\mathfrak{P}=-(2\omega_{ab}/\hbar)\mathbf{E}\cdot\mathbf{p}_{ab}\mathbf{p}_{ba}\sigma_z. \quad (A16)$$

Application of the trace operation converts this operator equation into

$$(d^2\mathbf{p}/dt^2)+\omega_{ab}^2\mathbf{p}=-(\tfrac{1}{3})(2\omega_{ab}\wp^2/\hbar n_0)\mathbf{E}n, \quad (A17)$$

where it has been assumed that $\mathbf{p}_{ab}=\mathbf{p}_{ba}=\wp$. The factor of $\frac{1}{3}$ in parentheses is to be included if all possible spatial orientations of the two-level systems are permitted (KS48, VaV24) so that an average over all such orientations must be performed.

From a direct multiplication of the quantities involved, there follows

$$i\hbar(d\mathfrak{P}/dt)=[\mathfrak{P},\mathcal{H}]=-\tfrac{1}{2}\hbar\omega_{ab}[\sigma_z,\mathfrak{P}]. \quad (A18)$$

On the other hand, multiplication of Eq. (A2) by $n_0\sigma_z$, and application of the trace operation, yields

$$i\hbar(dn/dt)=n_0\mathbf{E}_0\,\mathrm{Tr}\,\{\rho[\mathfrak{P},\sigma_z]\}. \quad (A19)$$

Calculation of the trace of Eq. (A18) then finally leads to the equality

$$(dn/dt)=(2n_0/\hbar\omega_{ab})\mathbf{E}\cdot(d\mathbf{p}/dt). \quad (A20)$$

Equations (A17) and (A20) provide a starting point for describing the response of a two-level system to an external electromagnetic field. Equation (A17) with ω_{ab} now replaced by $\omega=\omega_{ab}+\Delta\omega$, may be solved in terms of a Green's function. The causal Green's function for this equation, i.e., the solution of

$$d^2G/dt^2+\omega^2G=-\delta(t-t') \quad (A21)$$

which satisfies $G=0$ for $t<t'$ is

$$G(t\mid t')=-\omega^{-1}u(t-t')\sin\omega(t-t'), \quad (A22)$$

in which $u(t)$ is the unit step function. Multiplication of Eqs. (A17) and (A22) by $G(t\mid t')$ and $\mathbf{p}$, respectively, subtraction, and integration over all time yields

$$p(\Delta\omega,\mathbf{r},t)=-2\wp\int_{-\infty}^{t}dt' \times\sin\omega(t-t')\mathfrak{N}(\Delta\omega,\mathbf{r},t')\tilde{E}(\mathbf{r},t')\cos\Phi(\mathbf{r},t'), \quad (A23)$$

where $\mathfrak{N}=n/n_0$. The factor of $\frac{1}{3}$ due to orientational averaging is neglected. This expression for the polarization may now be decomposed into parts which are in phase and $\pi/2$ out of phase with the electric field. One may write

$$p=\wp[\mathcal{P}(\Delta\omega,\mathbf{r},t) \times\sin\Phi(\mathbf{r},t)+\mathcal{Q}(\Delta\omega,\mathbf{r},t)\cos\Phi(\mathbf{r},t)]. \quad (A24)$$

When the carrier frequency is at the center of the inhomogeneously broadened transition, i.e., $\omega_0=\omega_{ab}$, Eq. (A23) leads to

$$\mathcal{P}=\int_{-\infty}^{t}dt'\tilde{E}(\mathbf{r},t')\mathfrak{N}(\Delta\omega,\mathbf{r},t') \times\cos[\Delta\omega(t-t')+\phi(\mathbf{r},t)-\phi(\mathbf{r},t')], \quad (A25)$$

$$\mathcal{Q}=-\int_{-\infty}^{t}dt'\tilde{E}(\mathbf{r},t')\mathfrak{N}(\Delta\omega,\mathbf{r},t') \times\sin[\Delta\omega(t-t')+\phi(\mathbf{r},t)-\phi(\mathbf{r},t')]. \quad (A26)$$

In obtaining this result, terms near the second harmonic of ω_0 have been discarded. However, it should be emphasized that there has been no assumption that $\mathcal{P}$, $\mathcal{Q}$, and $\mathfrak{N}$ vary slowly compared to the carrier wave.

Differentiation of Eqs. (A25) and (A26) shows that $\mathcal{P}$ and $\mathcal{Q}$ satisfy the differential equations

$$\partial\mathcal{P}/\partial t=\tilde{E}\mathfrak{N}+[\Delta\omega+(\partial\phi/\partial t)]\mathcal{Q} \quad (A27)$$

and

$$\partial\mathcal{Q}/\partial t=-[\Delta\omega+(\partial\phi/\partial t)]\mathcal{P}. \quad (A28)$$

When time dependence near the second harmonic of ω_0 is also neglected in Eq. (A20), it is equivalent to

$$\partial\mathfrak{N}/\partial t=-\tilde{E}\{\mathcal{P}-[(1/\omega_0)(\partial\mathcal{Q}/\partial t)]\}. \quad (A29)$$

When the slowly varying envelope approximation is used, the second term on the right-hand side is discarded.

APPENDIX B

To obtain the form of the envelope function $\tilde{E}$, it is first noted that Eqs. (2.12a) and (2.12b), with the phase term ϕ set equal to zero, are equivalent to the linear equation

$$d\lambda/dt+i\tilde{E}\lambda=\Delta\omega\mathcal{Q}, \quad (B1)$$

where $\lambda=\mathcal{P}+i\mathfrak{N}$. Setting

$$\varphi(w)=\int_{-\infty}^{w}dw'\tilde{E}(w') \quad (B2)$$

and introducing Eq. (3.4), one finds that the solution of Eq. (B1) which reduces to $\mathcal{P}(-\infty)=0$, $\mathfrak{N}(-\infty)=-1$, i.e., atoms initially in the ground state, is

$$\mathcal{P}=-(1-\chi\Delta\omega)\sin\varphi, \tag{B3}$$

$$\mathfrak{N}=-\chi\Delta\omega-(1-\chi\Delta\omega)\cos\varphi. \tag{B4}$$

Equation (2.12c) now reduces to

$$d^2\varphi/dw^2=\tau_p^{-2}\sin\varphi, \tag{B5}$$

where Eqs. (3.4) and (B2) have been employed and

$$\tau_p^{-2}=\Delta\omega(1-\chi\Delta\omega)/\chi. \tag{B6}$$

The solution of Eq. (B5) that vanishes as $w\to-\infty$ is

$$\varphi=4\tan^{-1}\exp(w/\tau_p). \tag{B7}$$

From Eq. (B2), the electric field envelope is then

$$\tilde{E}=(2/\tau_p)\operatorname{sech}(w/\tau_p). \tag{B8}$$

From Eq. (B6), we have

$$\chi(\Delta\omega)=\Delta\omega\tau_p^2/[1+(\tau_p\Delta\omega)^2]. \tag{B9}$$

APPENDIX C

Integrating Eq. (2.9a) over all time and using Eqs. (A26) and (A28), with ϕ again set equal to zero, one finds

$$\frac{d\theta}{dx}+\kappa\theta=\alpha'\int_{-\infty}^{\infty}d\Delta\omega g(\Delta\omega)(\Delta\omega)^{-1}$$

$$\times\int_{-\infty}^{\infty}dt'\tilde{E}(x,t')\mathfrak{N}(\Delta\omega,x,t')\lim_{t\to\infty}\sin\Delta\omega(t-t'), \tag{C1}$$

where $\kappa=2\pi\sigma/c$. To carry out the limiting process, one may first introduce the well-known forms

$$\lim_{t\to\infty}\sin\Delta\omega t/\Delta\omega=\pi\delta(\Delta\omega),$$

$$\lim_{t\to\infty}\cos\Delta\omega t/\Delta\omega=[(1/\Delta\omega)-(P/\Delta\omega)], \tag{C2}$$

where P denotes a principal value. Using another standard representation for the delta function and the principal value, one obtains

$$\lim_{t\to\infty}\cos\Delta\omega t/\Delta\omega=\lim_{\epsilon\to0}((\Delta\omega)^{-1}-\{\Delta\omega/[\epsilon^2+(\Delta\omega)^2]\})$$

$$=\pi\delta(\Delta\omega)\lim_{\epsilon\to0}(\epsilon/\Delta\omega). \tag{C3}$$

When these results are used in Eq. (C1), one obtains

$$\frac{d\theta}{dx}+\kappa\theta=\tfrac{1}{2}\alpha\int_{-\infty}^{\infty}dt'\tilde{E}(x,t')\mathfrak{N}(0,x,t'), \tag{C4}$$

where

$$\alpha=2\pi g(0)\alpha'. \tag{C5}$$

As given by Eq. (6.3a), the population of on-resonance atoms, i.e., those represented by $\mathfrak{N}(0,x,t)$, may be expressed as

$$\mathfrak{N}(0,x,t)=\pm\cos\left(\int_{-\infty}^{t}dt'\tilde{E}(x,t')\right), \tag{C6}$$

where the upper sign is to be used if the population is initially inverted, while the lower sign is used if the population is initially in the lower level. Hence Eq. (C4) takes the form

$$d\theta/dx+\kappa\theta=\pm\tfrac{1}{2}\alpha\sin\theta \tag{C7}$$

and Eq. (C7) is the area theorem.

APPENDIX D

From Eqs. (4.27) and (4.29), the nonresonant polarization satisfies

$$4\pi(\partial\mathbf{P}_{nr}/\partial t)=a_0(\partial\mathbf{E}/\partial t)-a_2\omega_0^2\int_{-\infty}^{t}dt'\mathbf{E}(t'). \tag{D1}$$

For an electric field polarized in the Y direction and traveling in the positive X direction, it follows from the Maxwell equations that the associated magnetic field vector is

$$\mathbf{H}=-i\mathbf{k}\int_{-\infty}^{t}dt'\frac{\partial|\mathbf{E}|}{\partial x}, \tag{D2}$$

where $\mathbf{k}$ is a unit vector in the Z direction.

Energy conservation is expressed by

$$\nabla\cdot(\mathbf{E}\times\mathbf{H})+(2c)^{-1}\frac{\partial}{\partial t}(E^2+H^2)+\frac{4\pi}{c}\mathbf{E}\cdot\frac{\partial\mathbf{P}_{nr}}{\partial t}$$

$$=-\frac{4\pi}{c}\left(\sigma E^2+\tfrac{1}{2}(\hbar\omega)\frac{\partial n}{\partial t}\right), \tag{D3}$$

where Eq. (A20) has been employed. If one assumes that a steady-state pulse is maintained by a balance between ohmic losses and resonant gain, then the right-hand side of Eq. (D3) will vanish. The left-hand side may be simplified and one finally obtains

$$\mathbf{E}\cdot[c^{-1}(\partial\mathbf{E}/\partial t)-\nabla\times\mathbf{H}+(4\pi/c)(\partial\mathbf{P}_{nr}/\partial t)]=0. \tag{D4}$$

Employing Eq. (4.30) for $\mathbf{E}$, Eq. (D2) yields

$$\nabla\times\mathbf{H}=c\mathbf{j}\left\{-v_e^{-1}\frac{\partial|\mathbf{E}|}{\partial x}+\frac{\omega_0}{v_e}(v_e^{-1}-v_p^{-1})\right.$$

$$\left.\times(\tilde{E}_c\sin\psi-\tilde{E}_s\cos\psi)-\omega_0^2\left(\frac{1}{v_e^2}-\frac{1}{v_p^2}\right)\int_{-\infty}^{t}dt'E\right\}, \tag{D5}$$

where

$$\tilde{E}_c=\tilde{E}\cos\phi,$$

$$\tilde{E}_s=\tilde{E}\sin\phi, \tag{D6}$$

and

$$\psi=\omega_0(t-x/v_p). \tag{D7}$$

When second harmonic terms are neglected, we have

$$\partial E^2/\partial x = -v_e^{-2}(\partial E^2/\partial t) \qquad \text{(D8)}$$

and Eq. (D4) takes the form

$$(2c)^{-1}\left(1+a_0-\frac{c^2}{v_e^2}\right)\frac{\partial E^2}{\partial t} + \omega_0^2 c[(v_e^{-2}-v_p^{-2})-a_2c^{-2}]\mathbf{E}\cdot\int_{-\infty}^{t} dt'\mathbf{E}=0. \qquad \text{(D9)}$$

Armstrong and Courtens (AC69) make plausible the choice of satisfying this equation by setting each coefficient equal to zero separately. This yields

$$c/v_e = (1+a_0)^{1/2},$$
$$c/v_p = (1+a_0)^{1/2} - a_2^{1/2}. \qquad \text{(D10)}$$

Turning now to the wave equation for the medium under consideration, one has

$$\frac{\partial^2\mathbf{E}}{\partial x^2} - \frac{1+a_0}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \frac{a_2\omega_0^2}{c^2}\mathbf{E} = \frac{4\pi}{c^2}\frac{\partial}{\partial t}\left(\sigma\mathbf{E}+\frac{\partial\mathbf{P}_r}{\partial t}\right). \qquad \text{(D11)}$$

With the velocities as determined above, the left-hand side of this equation reduces to a perfect time derivative, and one finds

$$\omega_0 A(\tilde{E}_s\cos\psi - \tilde{E}_c\sin\psi) = \sigma E + \partial P/\partial t, \qquad \text{(D12)}$$

where

$$2\pi A = -[a_2(1+a_0)]^{1/2}. \qquad \text{(D13)}$$

Writing the resonant polarization as given in Eq. (4.31) and employing the slowly varying envelope approximation, we find

$$\partial P_r/\partial t = -\omega_0 n_0 \wp[\mathcal{P}\sin(\psi+\phi)+\mathcal{Q}\cos(\psi+\phi)] \qquad \text{(D14)}$$

Equation (D12) then yields

$$(\tau\gamma\tilde{E}-\mathcal{Q})\sin\phi - (\tau\tilde{E}-\mathcal{P})\cos\phi = 0,$$
$$(\tau\tilde{E}-\mathcal{P})\sin\phi + (\tau\gamma\tilde{E}-\mathcal{Q})\cos\phi = 0. \qquad \text{(D15)}$$

Two expressions may now be formed for $\tan\phi$. When they are equated, one finds

$$(\tau\tilde{E}-\mathcal{P})^2 = -(\gamma\tau\tilde{E}-\mathcal{Q})^2, \qquad \text{(D16)}$$

where τ is as defined in Eq. (4.8) and

$$\gamma = \omega_0 A/\sigma. \qquad \text{(D17)}$$

Since all terms are real, each side of Eq. (D16) must vanish, and hence one obtains

$$\mathcal{P} = \tau\tilde{E},$$
$$\mathcal{Q} = \gamma\tau\tilde{E}. \qquad \text{(D18)}$$

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