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Singular Potentials

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This article attempts to present a comprehensive review of the literature dealing with singular potentials through March 1969. The treatment is confined principally to nonrelativistic quantum mechanics, i.e., to solutions of the partial-wave Schrödinger equation with a singular potential. Some general physical and mathematical properties are given. Exact solutions are presented for those potentials for which they are available. Techniques which have been used in obtaining approximate solutions are outlined. Formal and physical applications of singular potentials are presented. Formal applications are those for which singular potentials have served as mathematical models illustrating concepts in elementary particle physics. These include applications to the Regge pole formalism, quantum field theories, and the peratization approximation. Physical applications entail those which have been made to molecular physics and to high-energy phenomenology.

CONTENTS

I. INTRODUCTION

This article is concerned primarily with singular potentials in nonrelativistic quantum mechanics. Singular potentials within the context of nonrelativistic physics are those having a singularity at least as great as the inverse square at the center of force. What distinguishes such potentials as a class and how are such potentials relevant in physics? One answer to these questions can be found in a simple example.

The familiar Coulomb potential $V(r)=Z_1Z_2e^2r^{-1}$ is well known to describe the motion of two charged particles with charges Z_1 and Z_2, in the absence of other interactions. Thus one can compute scattering as well as bound-state problems in classical and quantum mechanics in excellent agreement with experiment. However other charge configurations lead to different forms for the potential. For example, the interaction between a charge Z and an induced dipole of polarizability α has the form $V(r)=-\frac{1}{2}Z\alpha e^2r^{-4}$. This attractive potential is singular according to our opening statement. In sharp contrast to the Coulomb case, it does not admit physically meaningful solutions with the usual boundary conditions. In classical mechanics a particle moving under the influence of such a potential falls to the center with infinite velocity. In the absence of additional physical assumptions, an inbound trajectory can be connected with an infinite number of outbound trajectories. In quantum mechanics one finds that both solutions of the Schrödinger equation satisfy the physical boundary condition of the vanishing of the wave function at the origin. There is no apparent way to determine the arbitrary phase factor between these solutions.

One might hasten to note that the r^{-4} potential in the example above represents, of course, only the long-range part of the potential between the two particles. However this restriction is also true for the Coulomb potential which, nevertheless, leads to well-defined solutions. The force between two charges presumably does not become infinite as the separation decreases, the force eventually being cut off by some unspecified mechanism. Thus, in the singular case, the long-range part of the force between particles does not alone suffice to determine their behavior; some cutoff mechanism apparently must be provided.

In contrast to the attractive potential, the repulsive singular potential offers no problems regarding physical interpretation. Physical solutions can be determined uniquely. However in both attractive and repulsive cases, mathematical difficulties are inherent as regards

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techniques of solution. This is fundamentally because the solutions of thc Schrödinger equation for a singular potential $gV(r)$ are not analytic functions of g at $g=0$. Hence the usual technique for obtaining scattering amplitudes—the Born approximation—fails, aud some other calculational method must be devised.

Because of the problems concerning the physical interpretation of attractive singular potentials, physicists concluded for a long time that no significance could be given to any singular potential as regards the singularity at the center of force. Furthermore the mathematical difficulties made the problem of understanding singular potentials even more formidable. But it was finally realized that many aspects of singular potentials are physically meaningful, and that certain of their properties are relevant. In addition, mathematical techniques were developed for handling the calculational problems of singular potentials.

Singular potentials first attracted attention in a relativistic context, where in fact the threshold for singular behavior occurs for the physically interesting Coulomb potential when it entails a charge Z satisfying $Z/137\gtrsim 1$. The difficulty involved in the choice of a physical solution for this case was first discussed in detail by Plesset (1932) (see Secs. II.C and II.G). The problem of nonunique solutions was also noted by Case (1950). He suggested that the arbitrariness can be resolved by specifying one bound-state energy, and determining the rest of the bound-state spectrum by imposing orthogonality on the wave functions. Other papers dealing with this problem had appeared from time to time, but there had been no sustained interest for quite a while.

An important paper by Predazzi and Regge (1962) helped to stimulate interest in singular potential theory. In this work the authors argued that as inasmuch as real world interactions were likely to be highly singular, the study of singular rather than regular (nonsingular) potentials might be more relevant physically. In addition they showed that the Regge behavior for singular potentials was much simpler than for regular potentials (see Sec. IV.A). This concept, though not explored in great depth by them, was very important in generating interest in many aspects of singular potentials.

A further stimulus to the study of singular potentials emerged from efforts to find effective potentials to describe field-theoretic interactions by means of the Bethe–Salpeter or quasipotential equations. Needless to say, such correspondences were made in approximations which entailed retention of only selected diagrams, usually those of the lowest order. A fair amount of work has gone into calculating such effective potentials for field-theoretic interactions. A most intriguing observation was made by Bastai, Bertocchi, Fubini, Furlan, and Tonin (1963a) who found a correspondence between the renormalizability attributes of a field theory and the regularity or singularity of the effective potential. The effective potential for a superrenormalizable theory was found to be regular, that for a renormalizable theory was found to be transitional (see definition in Sec. II.A), and that for a nonrenormalizable theory was found to be singular (see Sec. IV.B).

The work of Feinberg and Pais (1963, 1964) which introduced the peratization approximation in connection with a field-theoretic study of weak interactions aroused much interest in singular potentials. In this approximation it was found by Feinberg and Pais that in a particular case the sum of the most singular parts (as a function of a cutoff parameter) of each term of an infinite series yields a finite result in the limit in which the cutoff was removed. This procedure had no mathematical justification. It was soon realized that a nearly identical mathematical technique could be tried on the Born series for scattering by a singular potential, where in some cases the exact answer is known. Such situations were studied as a test of the peratization idea (see Sec. IV.D).

The preceding examples illustrate those applications in which singular potentials serve as mathematical models for certain concepts in field theory and elementary particle theory. The interest in such applications arises because of the resemblance of a singular potential to the light cone singularity of the propagator function characteristic of a nonrenormalizable quantum field theory. In addition, they illustrate the mathematical difficulties entailed by nonanalyticity in the coupling constant g at $g=0$. However singular potentials have also found applications in various physical problems. They have appeared extensively in molecular physics for quite some time (Sec. V.A). They have also been used occasionally in descriptions of the scattering of elementary particles (Sec. V.B).

In molecular physics, potentials are used as descriptions of the interatomic or intermolecular force. This force consists of a long-range and a short-range part. The long-range force arises from the electrostatic interaction between two atoms or molecules. Such forces include purely electrostatic forces between polar molecules, as well as induction forces between a polar and nonpolar molecule, and dispersion forces between two nonpolar molecules. However, as the interparticle separation decreases, a repulsive force develops because of the overlap of the electron clouds of the particles. This force is frequently represented by a repulsive singular potential whose parameters are determined phenomenologically. The Lennard-Jones potential which describes the interaction between two nonpolar molecules is an example.

In applications involving elementary particle scattering, repulsive singular potentials have been used to simulate the strong repulsion which characterizes the short-range part of the interaction between such particles. The use of a repulsive singular potential as opposed to a simple hard core provides a greater

generality in the parameterization of physical situations.

In this review we have attempted to give a comprehensive and critical survey of the published physics literature about both the mathematical and physical aspects of singular potentials. The review covers the literature up to March, 1969. We have indicated the problems that have been considered and have tried to draw attention to significant features. We have in our survey found a seemingly large number of inaccuracies, some of which are rather basic, as well as some conflicting results, which have not previously been reported. These are pointed out and discussed. The level of presentation of this review should suit anyone with a general background in quantum mechanics. However, some applications in Sec. IV and V presuppose a familiarity with their context. The larger portion of this article deals with nonrelativistic quantum mechanics. We have touched only briefly upon aspects pertaining to many-body theory or to physical chemistry. There exist several fine review articles which discuss both calculations and mathematical techniques in this area (see, e.g., Bernstein, 1966).

Section II surveys the general mathematical and physical aspects of nonrelativistic singular potentials. It also includes a brief outline of some of the features of singular potentials in the contexts of classical mechanics and relativistic quantum mechanics. Section III deals with purely mathematical aspects pertaining to solutions of equations with singular potentials. Section IV deals with formal applications of singular potentials, i.e., applications where singular potentials serve as mathematical models for certain concepts in field theory and elementary particle theory. Section V deals with physical applications such as calculations in molecular physics and phenomenological applications to elementary particle physics. A few concluding remarks are presented in Sec. VI. A consistent notation has been employed throughout the article with little exception. Some of the notations are indicated in an Appendix. The reader should take cognizance of the fact that the notation generally does not agree with that of the papers being reviewed, and the same symbol may have a different meaning in this article and in the paper under discussion.

II. GENERAL PROPERTIES

A. Definition and Characterization

The present article deals with singular potentials as a distinguished subclass within a large class of potentials. The larger (universal) class consists of local, velocity-independent, time-independent, spherically symmetric potentials. This class is restricted for convenience by the conditions that the potentials be finite[1] for all $r>0$, and have at most a finite number of changes of sign.

Nonrelativistic Case. The domain of nonrelativistic quantum mechanics is the principal concern of this article. The motion of a single nonrelativistic particle under the influence of a spherically symmetric potential, $V(|\mathbf{r}|)\equiv V(r)$, obeys the three-dimensional, time-independent Schrödinger equation

$$-(\hbar^2/2M)\nabla^2\psi(\mathbf{r})+[V(r)-E]\psi(\mathbf{r})=0.$$

In this article we generally regard $\hbar=2M=1$, although these factors can easily be recovered if the coupling constant g associated with the potential $V(r)$ is understood to contain the factor $2M/\hbar^2$. The wave function $\psi(\mathbf{r})$ can be resolved into a sum of products of an r-dependent factor and an angular factor. Thus, in the case of scattering one obtains the familiar partial-wave expansion,

$$\psi(\mathbf{r})=(k/\pi)^{1/2}(4\pi kr)^{-1}\sum_{l=0}^{\infty}(2l+1)i^lP_l(\cos\theta)u_l(r),$$

where k is related to the energy E by $k=(2ME/\hbar^2)^{1/2}$, and $P_l(\cos\theta)$ are the Legendre polynomials. In the case of bound states of angular momentum l, only one term, appropriately normalized, would appear from this summation. The radial wave function $u_l(r)$ satisfies the differential equation

$$d^2u_l/dr^2+\{k^2-V(r)-[l(l+1)/r^2]\}u_l(r)=0.$$

Much of the investigation of singular potentials is concerned with a study of this equation and its solutions.

In addition to the restrictions mentioned above, we further demand that the behavior of the potential for asymptotically large r be subject to one of a number of alternative conditions as suits the context:

$$\int_C^{\infty}dr\,|\,V(r)\,|<\infty,\qquad\text{(a)}$$

$$\int_C^{\infty}drr\,|\,V(r)\,|<\infty,\qquad\text{(b)}$$

$$\int_C^{\infty}drr^2\,|\,V(r)\,|<\infty,\qquad\text{(c)}$$

with C some fixed positive constant. Note that the class of potentials satisfying condition (c) includes the class satisfying (b) which includes the class obeying the condition (a). The condition (a) is necessary for the finiteness of the partial-wave phase shift $\delta_l(k)$, for any

[1] This would exclude potentials with a hard core, i.e., potentials which are infinite for $r<r_c$, where $r_c>0$. Hard core potentials share many of the properties of singular potentials, and we may include them by allowing the universal class to include potentials which are infinite for $r<r_c$ for some $r_c>0$, and finite for all $r>r_c$.

physical l and $k\neq 0$.[2] The condition (b) specifies the necessary large r behavior for the finiteness of the total cross section,[3] as well as of the zero-energy phase shift.[4] The condition (c) is the necessary large r behavior for the finiteness of the S-wave scattering length[5] as well as of the forward differential cross section.[6]

A potential $V(r)$ in the universal class is said to be regular at $r=0$ if

$$\lim_{r\to 0} r^2V(r)=0, \tag{2.1}$$

and singular[7] at $r=0$ if

$$\lim_{r\to 0} r^2V(r)=\pm\infty. \tag{2.2}$$

If the limiting value in Eqs. (2.1) and (2.2) is finite, the potential is said to be a transition potential. A singular potential is called repulsively or attractively singular according to whether the limiting value in Eq. (2.2) is, respectively, $+\infty$ or $-\infty$, regardless of whether the potential maintains one sign for $r>0$. We shall call a potential a transitional potential (N.B. the suffix) if

$$\lim_{r\to 0} r^{2+\epsilon}V(r) \tag{2.3}$$

is zero for all $\epsilon>0$, and infinite for all $\epsilon<0$. Examples of transitional potentials are of the form

$$V(r)=[\,|\ln r|^p/r^2]+V'(r), \tag{2.4}$$

where $V'(r)$ is not as singular at $r=0$ as the first term in Eq. (2.4). When $p<0$, the potential of Eq. (2.4) is a regular transitional potential, while when $p>0$, the potential is a singular transitional potential. The class of transitional potentials can also be characterized by the condition

$$\lim_{r\to 0}\{-[\ln V(r)]/\ln r\}=2. \tag{2.5}$$

Potentials of the form in Eq. (2.4) are discussed in Sec. II.B.

The physical characteristics of singular potentials are to be found in the attractive case. In the context of classical mechanics, a particle moving under the influence of an attractive singular potential undergoes well-defined scattering only if its impact parameter is greater than a certain critical value. In other cases, which include bounded motion, the particle falls to the center of force with infinite velocity. In addition, the scattering or bound-state trajectory is not well defined unless trajectory tangents are matched, and conservation of energy and angular momentum is assumed at the origin (see Sec. II.F). In the context of quantum mechanics, the scattering problem for an attractive singular potential is not resolved as in the classical case, and is never well defined without a further physical assumption. An arbitrary (phase) parameter remains. In addition, an attractive singular potential supports a nonunique bound-state spectrum with an infinite number of bound states, and with no lower bound on the energy[8] (Sec. II.C). Classical scattering is, in some instances, well defined without additional assumptions because the centrifugal barrier may prevent the particle from falling under the influence of that portion of a potential which is attractive singular. On the other hand, in quantum-mechanical scattering, a particle can always "sense" an attractive singular potential. Thus the basic feature of an attractive singular potential is seen to lie in the fact that physical processes are not uniquely determined. This gives rise to the possibility of imposing unusual or unconventional boundary conditions in physical problems as a means of representing particular physical processes. An example of a process of this type is provided by particle absorption or capture. Another example is contained in the work of Vogt and Wannier (1954) (see Secs. III.A.1, V.A).

[2] For regular potentials (as we presently define them), the existence of the phase shift for $k>0$ is inferred from the existence and nonvanishing of the Jost function if condition (a) is satisfied. See Newton (1966), p. 334, Eq. (12.18), for the S-wave case, and pp. 371–374 for the case of higher partial waves. The condition (a) excludes potentials with Coulomb-like tails, for which the phase shift in its conventional sense is known not to exist. The existence of the partial-wave phase shift for a repulsive singular potential can be concluded from a limiting procedure on a regularized potential. See Frank (c).

[3] For regular potentials see e.g., Landau and Lifshitz (1960), pp. 399–400, and Wu and Ohmura (1962), pp. 8–9. The conditions reported there imply a slightly weaker condition for finite total cross section in the form

$$\int_A^\infty dr r^3\,|\,V^2(r)\,|<\infty,$$

for some $A>0$. The conclusion remains valid for repulsively singular potentials as is easily verified by a limiting procedure (see Frank c).

[4] The finiteness of the zero-energy phase shift for regular potentials satisfying condition (b) is an immediate consequence of Levinson's theorem and the Bargmann–Schwinger inequality, Newton (1966), pp. 355–357. The conclusion is valid for repulsively singular potentials, as can be verified by a limiting procedure (See Frank c). Levinson's theorem does not hold directly for singular potentials. See Aly and Okubo (1967).

[5] See e.g., Newton (1966), p. 322. N.B. that condition (c) does not guarantee the finiteness of the scattering length, due to the possibility of bound states.

[6] For regular potentials see e.g., Landau and Lifshitz (1960), pp. 399–400, and Wu and Ohmura (1962), p. 9. The imaginary part of the forward cross section is of course finite, by the optical theorem, if condition (b) is obeyed. The dispersion relation for the forward scattering amplitude is discussed for powerlike potentials by Martin (1966).

[7] There are differing definitions of singular in the literature; see e.g., Guttinger and Pfaffelhuber (1966), Footnote, p. 427, and Aly and Taylor (1968). These authors call a potential singular if it does not possess a Fourier transform. In Sec. IV. C we conform to this definition for the sake of the context. For another definition, see Wu and Ohmura (1962), p. 47.

[8] This fact can be inferred from the infinite number of nodes of the zero-energy radial wave function as found from the WKB method. For the relation between nodes of the zero-energy wave function and bound states, see e.g., Calogero (1967), p. 168. One can easily verify the lack of a lower bound on the spectrum from the fact that the expectation value of the Hamiltonian can be made arbitrarily negative by choosing appropriate trial wave functions whose support is restricted to a small neighborhood of the origin.

In contrast to the attractive potential, the repulsive case poses no problem as regards physical interpretation. The solutions to physical problems are uniquely given. Certain features of the transition inverse-square case differ somewhat and are discussed in Secs. II.B, II.C, and II.F.

The reason why r^{-2} is the transition point in nonrelativistic quantum mechanics can be seen by the following argument due to Landau and Lifshitz (1960). Consider a particle confined to a small region of radius r_0 about the origin. The uncertainty in the position of the particle is of the order of r_0, and hence the uncertainty in its momentum is of the order of $1/r_0$ ($\hbar=2M=1$). For potentials of the form gr^{-m}, the average value of the energy is approximately

$$E\approx(1/r_0^2)+(g/r_0^m).$$

For g negative, corresponding to an attractive potential, one sees that if $m>2$ there is no lower bound on the energy. The particle in this case "falls" to the center in seeking its ground state. However, if $m<2$, the energy cannot take arbitrarily large negative values, and the discrete spectrum must have a lower bound. This behavior is seen to correspond to that discussed above. Similar considerations show that the transition from regular to singular in the relativistic case occurs with the Coulomb potential.

There are many mathematical characterizations which set singular potentials apart from regular potentials. When the potential is singular, the radial wave equation for any partial wave has a non-Fuchsian singularity, i.e., an irregular singular point, at $r=0$. This means that there is no indicial equation and that the radial wave function has an essential singularity at $r=0$. In the case of regular and transition potentials, the indicial equation has solutions which specify a powerlike behavior of the wave function in the neighborhood of $r=0$. For repulsive singular potentials, a unique (to within normalization) physical solution exists to the partial-wave radial equation. For attractive singular potentials, no unique physical solution exists. This is discussed in greater detail in Secs. II.B and II.C.

In addition, one encounters fine points pertaining to the application of various techniques. For example the WKB approximation for singular potentials gives the correct radial behavior of the wave function in the neighborhood of $r=0$, whether or not the Langer transformation is employed (Langer, 1937 and Sec. III.B.2). This is in contrast to potentials, including transition potentials, which are unbounded and not singular at $r=0$. The WKB solution for singular potentials also supplies the correct nonanalytic coupling-constant dependence (to within a normalization) of the wave function near $r=0$. In a second instance the variable phase equation, even in the case of repulsively singular potentials, does not yield a unique solution with the usual boundary condition of the vanishing of the variable phase at $r=0$.[9] (See Sec. III.B.3.)

A phase shift can be uniquely defined for repulsive, but not attractive, singular potentials when the condition (a) is obeyed. Only repulsive singular potentials are considered in the ensuing discussion. The behavior of the scattering phase shift for repulsive singular potentials as a function of energy, angular momentum, and the coupling constant is distinguished from the behavior for regular potentials. The high-energy limit of a partial-wave phase shift for a singular potential is infinite in the convention where the phase shift is defined as a continuous function of energy which vanishes for all energies at zero coupling (see Frank, c). This contrasts with the result for regular potentials. Levinson's theorem fails to hold for repulsively singular potentials (which by definition may be attractive for nonzero values of r) (Aly and Okubo, 1967). The scattering phase shift as well as the S-wave scattering length for a singular potential is not analytic in the coupling constant g (Sec. II.D.2). The S-matrix for a singular potential at any given energy, as a function of the complex angular momentum $\lambda=l+\frac{1}{2}$, displays the symmetry property in the complex λ plane (Sec. IV.A)

$$S(\lambda, k)=\exp(2\pi i\lambda)S(-\lambda, k). \tag{2.6}$$

This provides a simple relation between the Regge poles in diagonally opposite quadrants of the complex plane. This contrasts with the Regge behavior for regular potentials which displays no such relationship.

Relativistic Case. The kinematics of relativistic motion lead to a different criterion for the singularity of a potential. The characterizations of singularity in the nonrelativistic case have their analogy in the relativistic case in the class of potentials whose behavior near $r=0$ gives an infinite value for the limit

$$\lim_{r\to 0} r\,|\,V(r)\,|, \tag{2.7}$$

in contrast to Eq. (2.2). The class of regular potentials would be defined by a vanishing value of Eq. (2.7). A finite value of this limit defines the class of transition potentials, which includes all potentials with Coulomb-like behavior at $r=0$. In contrast to the nonrelativistic case, all singular potentials appear as attractively singular in the equations of motion. As a result the singular potentials in the classical case all give rise to orbits which spiral into the origin. In the quantum-mechanical case, singular potentials appearing in relativistic equations such as the Klein–Gordon, Dirac, and the spin-one equations give rise both to an infinite number of bound states and to non-Fuchsian singularities of the wave function at $r=0$. (See Secs. II.C, II.F, and II.G for further discussion.)

The description of the scattering of particles interacting via a field-theoretic interaction can be cast into

[9] See Calogero (1967), Chap. 15.

the form of a relativistic wave equation with an effective potential, as for example, the Bethe–Salpeter equation. The characterization of these effective potentials in these relativistic equations as singular potentials depends on the order of the differential equation which, in turn, depends on the spin of the scattering particles. For example, the description of the scattering of spin-zero particles leads to fourth-order radial partial-wave equations (Sec. IV.B).[10] The characteristics of singular potentials in this case would apply to potentials defined as singular by the infinite value of the limit

$$\lim_{r\to 0} r^4 \mid V(r) \mid. \tag{2.8}$$

Thus the wave function has non-Fuchsian singularities at $r=0$ for such singular potentials. Regular potentials correspond to a vanishing value of the limit in Eq. (2.8). Potentials with inverse fourth-power behavior at $r=0$ are transition potentials. In the equations deriving from a Bethe–Salpeter equation, repulsive as well as attractive potentials are possible.

B. Stages of Singularity

The transition from regular to singular potentails takes place in stages marked by differing thresholds for different aspects of singularity. These thresholds generally lie in the range of transitional potentials which are characterized by Eq. (2.5).[11] This class has also been referred to as the Class II case in another context (Sec. IV.B). Various transition points in this range are conveniently indexed by the value of p in potentials of the form of Eq. (2.4).

$p<0$. The transitional potentials for which $p<0$ are not singular according to our definition, but certain singular aspects are anticipated as p increases. When $p=-2$, a change in the strong coupling limit of the Jost function is found which is precursive of singularity. [The Jost function is defined as the Wronskian[12] of the Jost and regular solutions, where the Jost solution $f(k, r)$ is defined by its asymptotic behavior $f(k, r)\sim \exp(-ikr)$.] The partial-wave Jost function for a regular potential is finite and is an entire function of the coupling constant g, while for a singular potential this quantity is infinite.[13] It has been found (Frank, 1967, 1968) for a general regular potential (unless $p>-2$) that the Jost function is an entire function of g of exponential order $\frac{1}{2}$.[14] This means, roughly speaking, that the Jost function for large g grows in certain directions in the complex g plane (in the positive g direction, in particular, when the potential is purely repulsive) like $\exp \tau \mid g \mid^{1/2}$. The quantity τ, the type for a purely repulsive, nonincreasing potential is equal to

$$\int_0^\infty dr \mid V(r) \mid^{1/2},$$

which has a threshold of divergence when $p=-2$. When $q=-p<2$, one finds that the Jost function is still an entire function of g, but that the exponential order of the Jost function is no longer fixed at $\frac{1}{2}$ but becomes $1/q$.

In the range $0<q<1$ a change takes place in the radial behavior of the solutions near $r=0$. The small r behavior of the solution of the radial differential equation for the lth partial wave is no longer of the power form $r^{\frac{1}{2}+\epsilon(l+\frac{1}{2})}$ (where $\epsilon=1$ for the regular solution, and $\epsilon=-1$ for a singular solution). The behavior is more complicated and depends on whether $1/p$ is an integer. When $1/q\equiv -1/p$ is not an integer, and $N<1/q<N+1$ ($N=1, 2, 3, \cdots$), one finds that the small r behavior of the lth partial-wave function has the form (Cornille and Predazzi, 1965b)

$$u_l(r)\sim r^{\frac{1}{2}+\epsilon(l+\frac{1}{2})} \exp\left[-\epsilon(l+\tfrac{1}{2}) \sum_{j=1}^{N} d_j (\ln r^{-1})^{1-jq}\right], \tag{2.9}$$

where the constants d_j are given in Cornille and Predazzi. When $1/q=N$, one finds

$$u_l(r)\sim \frac{r^{\frac{1}{2}+\epsilon(l+\frac{1}{2})}}{[\ln (1/r)]^{\epsilon(l+\frac{1}{2})b_N}} \times \exp\left[-\epsilon(l+\tfrac{1}{2}) \sum_{j=1}^{N-1} d_j (\ln r^{-1})^{1-jq}\right], \tag{2.10}$$

where the constant b_N is given in Cornille and Predazzi. For the special case $q=-p=1$ (Charap and Dombey, 1964), one finds that the regular solution behaves for small r like

$$u_l(r)\sim \frac{r^{l+1}}{[\ln (1/r)]^{\frac{1}{2}g(l+\frac{1}{2})}}. \tag{2.11}$$

One notes, in particular, that this leading term depends on both coupling constants, l and g, and is not analytic in $\lambda=l+\frac{1}{2}$ at $\lambda=0$, which is also true of the general expressions, Eqs. (2.9) and (2.10). One also finds that $q=1$ is the threshold for the divergence of the coupling-

[10] The restriction of a universal class to velocity-independent potentials is to be understood in the present case. One might conceivably encounter velocity-, and higher derivative, dependent interactions in a Bethe–Salpeter equation which may make the differential equation higher than fourth order for scattering of scalar particles.

[11] One might note that the high-energy limit of the S-wave phase shift, which can be found by the Born approximation, shows a characteristic change from $1/k$ behavior for potentials which are already as singular as r^{-1} near $r=0$. The behavior of the high-energy limit of the phase shift is conveniently tabulated in Calogero (1967), Table I, p. 220.

[12] The Wronskian is defined by $W(F, G)\equiv FG'-F'G$.

[13] By the partial-wave Jost function here is meant the value at $r=0$ of the ratio of the Jost solution to the scaling function $(2l-1)!!(kr)^{-l}$. For a regular potential this definition agrees with the definition by means of the Wronskian.

[14] The function $f(z)$ is said to be an entire function of z, if it is analytic in the whole finite plane. An entire function $f(z)$ is said to have exponential order ρ if, for every positive but no negative ϵ, $\mid f(z) \mid \leq \exp \mid z \mid^{\rho+\epsilon}$ for all sufficiently large $\mid z \mid$. If $f(z)$ has exponential order ρ, its type τ is defined by

$$\lim_{\mid z\mid\to\infty} \sup z^{-\rho} \mid \ln f(z) \mid.$$

constant expansion of the partial-wave Jost function as derived from iteration of the integral equation for the Jost solution. The Jost function for $0<q\leq 1$ remains an entire function of g with exponential order $1/q$, as for $1<q<2$. However the coupling-constant expansion requires modification through introduction of Weierstrass convergence factors. A higher-order factor is required as $q\to 0$ each time $1/q$ reaches an integer. The exceptional behavior of Eq. (2.10) for the case that $1/q$ is an integer is related to the corresponding modification of the coupling-constant expansion of the Jost function necessitated by the introduction of a Weierstrass convergence factor (Frank and Land d).

$p=0$. One finds the characteristics of the solutions for the transition potential gr^{-2} to depend critically on the value of the coupling constant g. A solution has a powerlike behavior r^s near $r=0$, with the two possible values of the exponent $s_\pm=\frac{1}{2}\pm(g+\frac{1}{4})^{1/2}$ as determined by an indicial equation which depends on g. (The two values coincide at $\frac{1}{2}$ when $g=-\frac{1}{4}$, and the two solutions behave near $r=0$ like $r^{1/2}$ and $r^{1/2}\ln r$.) For the range $g>\frac{3}{4}$, only one square integrable solution is possible. In the range $-\frac{1}{4}<g<\frac{3}{4}$, both solutions to the indicial equation lead to square integrability at $r=0$. In the "weakly" repulsive case $0<g<\frac{3}{4}$, the less singular solution vanishes at $r=0$, while the more singular solution is infinite. For the "weakly" attractive case $-\frac{1}{4}<g<0$, both solutions vanish at $r=0$. In both cases it is customary to select the solution with the less singular behavior at $r=0$ as physical. The Hilbert space aspects of this situation are presented in Sec. II.C. In the "weakly" attractive case, there are only a finite number of bound states. The high-energy limit of the phase shift for transition potentials with $g\geq-\frac{1}{4}$ is finite and nonvanishing. At $g=-\frac{1}{4}$, the characteristic aspects of attractive singular potentials appear. The two solutions of the indicial equation for $g<-\frac{1}{4}$ are complex conjugates, with the common real part $\frac{1}{2}$. This has the consequence that all solutions are square integrable at the origin, and there is no evident manner of selecting any linear combination of these two solutions. All the solutions moreover have an infinite number of oscillations in the neighborhood of the origin, which reflects the presence of an infinite number of bound states. Similar considerations apply, of course, to other potentials with r^{-2} behavior near the origin, such as $V(r)=g\sinh^{-2}\beta r$ (see Sec. III.A.1). Further details on this situation are discussed in Sec. II.C.

The radial behavior of the physical solution near $r=0$ for a potential with leading inverse-square behavior can be obtained by employing the WKB method in the Langer form (Sec. III.B.2). One finds that the S-matrix for such potentials possesses a branch cut in the complex angular-momentum plane.

$p>0$. Almost all of the characteristics of singular potentials enumerated in Sec. II.A hold for singular transitional potentials with $p>0$. The radial behavior of the wave function for r near zero is obtainable from either the Langer or the non-Langer form of the WKB approximation.[15] The phase shifts now diverge in the high-energy limit. One also finds, in contrast to more strongly singular potentials, that the Born series for the scattering length has finite coefficients, but is asymptotic (Calogero and Cassandro, 1964; Frank and Land d). If one regularizes the potential by cutting it off in some small neighborhood $r<\alpha$, one finds that the wave function is factorable in the $\alpha\to 0$ limit into a product of an r-dependent and an α-dependent factor. This "wave function renormalization" is discussed further in Sec. IV.C.

C. Attractive Case

Perhaps the most manifest difference between singular potentials and regular potentials is found in the properties of the attractive potential. Historically, Case (1950) was the first to notice that singular attractive potentials do not lead to physically reasonable results. One finds that both solutions of the radial Schrödinger equation for all energies lead to radial wave functions which go to zero at the origin while oscillating infinitely rapidly. This behavior is in fact given by (see Sec. III.B.2)

$$u^\pm(r)\underset{r\to 0}{\sim}|V(r)|^{-1/4}\times\exp\left(\pm i\int^r[-V(r)]^{1/2}\,dr\right). \quad (2.12)$$

By way of contrast, one may note from this equation that, if $V(r)$ is repulsive or has a negative imaginary part (to guarantee probability less than or equal to unity) in the neighborhood of the origin, there is a unique solution which vanishes at the origin.

For positive energies, Eq. (2.12) implies that no unique scattering may be defined since any linear combination of u^+ and u^- forms an acceptable solution at the origin. In consideration of various problems, workers such as Vogt and Wannier (1954) and Aly and Müller (1966a) have made particular choices of boundary conditions to correspond to the physical situation (see Secs. IV.A and V.B).

Equation (2.12) also implies that bound states exist for all negative energies.[8] In order to approach this problem, Case utilized orthogonality of solutions to produce a discrete bound-state spectrum as described in Sec. III.A.1. Thus, from an arbitrarily selected bound state, one can by orthogonality construct a spectrum of bound states. One interprets this arbitrariness by thinking of the attractive potentials as being physically cut off at some point near the origin by some unspecified mechanism, the formal attractive potential providing a physical description outside of the cutoff region. For singular attractive and strongly attractive

[15] One may note the similarity in form of the small-r behavior of the radial wave function for $p<0$ and $p>0$. See Eqs. (15), (16), (19), and (20) in Cornille and Predazzi (1965b).

(i.e., $g<-\frac{1}{4}$) transition potentials there is a continuum of bound states which is unbounded from below, i.e., there is no ground state. The Case procedure produces a discrete spectrum with no lower bound.

Case (1950) and Meetz (1964) have explicitly found such discrete bound-state spectra for the strongly attractive transition potential gr^{-2}. Scarf (1958) found the same for the potential $gr^{-2}+ar^{-1}$. Scarf also found an infinite number of bound states with a lower bound when $-\frac{1}{4}<g<0$, but the infinitude of states in this case originates in the Coulomb tail. Tietz (1959) claimed to have derived for the same potential "correct eigenvalues without any extra boundary conditions," but did not mention that this spectrum is not unique. Scarf (1958) claimed to have found an infinite number of bound states with a lower bound for the potential $g\csc^2 r$ considered in the interval $0<r<\pi/2$ with a repulsive wall at $r=\pi/2$. He concluded from this that unbounded spectra are not a general feature of the Schrödinger equation for singular potentials. Such a result, which would be quite surprising, is, in fact, unfounded and arises from a mathematical error in the analysis of the potential $g\csc^2 r$.[16]

Scarf has also attempted to obtain bound-state spectra by reflecting the wave function past the singularity at $r=0$ into negative r by analytic continuation. In this way he found complex spectra which are unbounded for the potentials $g\csc^2 r$ $(-\pi/2<r<\pi/2)$, $gr^{-2}+ar^{-1}$ $(-\infty<r<\infty)$, and for the relativistic Coulomb case. Not much physical significance can be attached to these states (see Sec. III.A.1).

Even more surprising is the fact that, for singular attractive potentials at least as singular as $-gr^{-4}$ at the origin, the full three-dimensional Schrödinger equation *has no solutions at all* (in the space of distributions in which partial differential equations are usually solved). First suggested by Newton (1966) and subsequently proved generally by Spector (1967a), the trouble occurs because at the origin the radial part of the formal solutions which behave like $\psi^{\pm}=u^{\pm}/r$ does not satisfy the differential equation. Simply illustrated, the difficulty is the same as occurs if one claims r^{-1} to be a solution to the three-dimensional Laplace equation. It is not, of course, since

$$\nabla^2(r^{-1})=-4\pi\delta(\mathbf{r}).$$

In fact, the correct radial portion of the Laplacian operator must then be written as (Spector, 1967a)

$$\nabla^2\rightarrow(\partial^2/\partial r^2)+(2/r)(\partial/\partial r)+\delta(r)(\partial/\partial r). \quad (2.13)$$

The last term in Eq. (2.13) gives a contribution at the origin when operating on $\psi^{\pm}(r)$ for $m\geq 4$. We note that by adding a suitable delta-function term to the potential, the last term in Eq. (2.13) may be canceled. A similar approach in which higher derivatives of the delta function appear as compensating terms in the potential, has been investigated in detail by Guttinger and Pfaffelhuber (1966) (see Sec. IV.C). We note that if the potential is repulsive or has a negative imaginary part in the neighborhood of the origin, solutions of the three-dimensional equation do exist (Spector, 1967a).

The contrast between attractive and repulsive singular and/or transition potentials can be characterized in the language of operators in Hilbert space. For physical interpretation one requires an operator to be self-adjoint.[17] The condition of self-adjointness of an operator guarantees the existence of a real spectrum, and the corresponding resolution of the identity operator into a family of projection operators spanning the Hilbert space. The requirement of self-adjointness may be replaced by the weaker condition of essential self-adjointness, whereby a unique self-adjoint extension is possible. It can also be replaced by the condition of the existence of a physically reasonable self-adjoint extension. Self-adjointness or essential self-adjointness fails to hold when there is a subspace of the Hilbert space which belongs to a complex (i.e., nonreal) eigenvalue of the operator. The deficiency indices (m, n) of the operator are defined as the dimensionalities of these eigenspaces belonging to a complex eigenvalue in the upper and lower open half-planes, respectively. The deficiency index of each half-plane is characteristic of the half-plane; i.e., it is the same for all complex numbers in that half-plane. The deficiency indices of a self-adjoint or essentially self-adjoint operator are $(0, 0)$. If $m=n<\infty$, self-adjoint extensions are characterized essentially by m independent parameters. If $m=n=\infty$, self-adjoint and non-self-adjoint extensions are possible. If $m\neq n$, no self-adjoint extensions are possible. For real potentials, the Hamiltonian commutes with complex conjugation, and self-adjoint extensions must exist.

Limic (1963) has proved the essential self-adjointness of a Hamiltonian entailing a repulsive singular, twice differentiable potential $V(r)$ satisfying condition (a). He considered the Hamiltonian $-\nabla^2+V(r)$ over the Hilbert space $L^2(R^3)$ of square integrable functions in Euclidean three space, as well as the partial-wave Hamiltonian

$$H_l=-(d^2/dr^2)+[l(l+1)/r^2]+V(r) \quad (2.14)$$

[16] The error consists of the improper evaluation of the argument of the quantity $B_\sigma(\eta)=\exp\{2[\gamma_\sigma(\eta)]\}$ of Eq. (12) in Scarf's paper. Scarf claimed that $\gamma(\eta)$ is bounded as $\eta\rightarrow\infty$. One can verify, from the asymptotic behavior of the gamma function, that $\gamma_\sigma(\eta)$ is unbounded and behaves like $\ln\eta$ as in Eq. (10) of Scarf.

[17] An operator A (unbounded in the instances of interest) is said to be symmetric (or Hermitian) if $(Ag,f)=(g,Af)$, when f, g range over everywhere dense domains of the Hilbert space. An extension of an operator is an extension of the everywhere dense domain over which the operator is defined to act, and the extension is symmetric if the symmetry property is maintained. A symmetric operator is said to be self-adjoint (or hypermaximal) if it is maximal, i.e., if it has no proper symmetric extension. This means that the everywhere dense domains of the Hilbert space in which f and g lie are identical. A self-adjoint extension is a maximal symmetric extension. See e.g., Akhieser and Glazman (1961).

over the Hilbert space $L^2(0, \infty)$ of square integrable functions over the interval $0 \leq r < \infty$. In this paper, Limic also proved the existence and unitarity of the scattering operator for repulsively singular, spherically symmetric potentials. This is proven from the existence of the wave operators[18] by first deriving the validity of the expansion of functions in $L^2(0, \infty)$ in terms of eigenfunctions of the Hamiltonian operator H_l. In a second paper, Limic (1965) has extended these results to repulsively singular, nonspherically symmetric potentials where partial-wave methods do not apply.

The corresponding situation for transition potentials was studied by Meetz (1964) with respect to the Hilbert space $L^2(0, \infty)$. He found the deficiency indices for the pure inverse-square potential gr^{-2} to be (0, 0) when $g > \frac{3}{4}$, corresponding to essential self-adjointness. When $g < \frac{3}{4}$, all solutions are square integrable in the neighborhood of $r=0$. In this case the deficiency indices are (1, 1), and one parameter is sufficient to specify a condition on the domain of the operator in order to determine a self-adjoint extension. The situation that all solutions of a differential equation in the neighborhood of a singular point are square integrable is known as the "limit circle" case, in contrast with the "limit point" case (Akhiezer and Glasman, 1961).[19] There is, however, a difference between the ranges $-\frac{1}{4} < g < \frac{3}{4}$ and $g < -\frac{1}{4}$. Only in the former range can a self-adjoint extension be selected by means of a regularization procedure (Sec. III.B.6), whereby the potential is first made regular by cutting it off in some small neighborhood $r < \alpha$, and the limit of the solution as $\alpha \to 0$ is then taken. This procedure selects the solution which is less singular in the neighborhood of $r=0$, which has been the intuitive procedure applied in both the nonrelativistic and relativistic cases.[20] This would be an example of a self-adjoint extension specified by "physical" considerations. This consideration is of practical importance in the selection of the physical wave function of the Dirac equation in a Coulomb field (Sec. II.G).[21] Behncke (1968) has in fact pointed out that the deficiency indices are (1, 1) for the S-wave with a regular potential, but not for the higher partial waves and not for a repulsive singular potential. The self-adjoint extension which selects the solution vanishing linearly with r near the origin is determined in this case by the full three-dimentional Schrödinger equation. For transition potentials the spectral resolution of the unit operator has been explicitly constructed by Meetz for the radial operator in the self-adjoint case ($g > \frac{3}{4}$), and for the cases where a one-parameter family of extensions is necessary. The dependence of the bound-state energies on the extension parameter is found, and is in agreement with the resuls of Case (1950).

The situation for attractive singular potentials has been considered by Meetz for powerlike singularities in the Hilbert space $L^2(0, \infty)$, and by Behncke (1968) generally, for the Hilbert space $L^2(R^3)$. The deficiency indices are shown by Meetz to be (1, 1), and the explicit form of the self-adjoint extensions is indicated. The parameter specifying the extension is related to the phase parameter introduced by Case (1950) in his discussion of singular attractive and transition potentials. Meetz found that regularization does not lead to any self-adjoint extensions. Behncke has analyzed the equation from the point of view of the three-dimensional Schrödinger equation and found the deficiency indices for an attractive singular potential in the Hilbert space to be (∞, ∞). These correspond to the infinite number of parameters necessary to specify the self-adjoint extensions; one parameter for each value of l and m which specifies a partial wave. The parameters might be chosen as Case phase parameters for each value of l and m. Since the deficiency indices are infinite, non-self-adjoint extensions are possible which would correspond to boundary conditions which describe inelastic scattering. In the context of Hilbert space theory, self-adjoint extensions do lead to solutions of the three-dimensional Schrödinger equation for attractive singular potentials.

D. Energy and Coupling-Constant Behavior

The singular nature of the potential manifests itself conspicuously in the energy and coupling constant dependence of the scattering functions, partial-wave phase shifts, and (S-wave) scattering length. Specifically, the singularity of the potential at $r=0$ is reflected in the high-energy behavior of the phase shift, and in the small g behavior of the phase shift and the scattering length. In contrast, the low-energy behavior of the phase shift and the large g behavior of the phase shift and scattering length reflect the "tail," i.e., the large r behavior, of the potential. The association of small (large) r behavior of the potential with high-(low-) energy behavior of the phase shift is well known, and qualitatively reflects properties of the Fourier transform. One readily recognizes that the strong-coupling limit emphasizes the effect of the tail of the potential which is first encountered by the incident particle, especially for a potential which is repulsively singular. In the weak-coupling limit, on the other hand, the particle is not much affected by the tail of the potential, and for a repulsive singular potential registers the effect of the strong repulsion near the origin. One can readily recognize this correspondence between the influence of the origin and tail of the potential and the

[18] The wave operators are also termed the Moller wave matrices. The scattering operator is the unitary operator defined over the whole Hilbert space and is generally termed the S-matrix. See e.g., Wu and Ohmura, Chap. S. Note that this reference somewhat confusingly employs the indices $\pm$, respectively, for limits taken as $t \to \mp \infty$.

[19] See Scarf (1958), Footnote 7.

[20] Landau and Lifshitz (1960), Sec. 35.

[21] References on the treatment both of this problem and the case of S waves for regular potentials can be found in Scarf, Footnotes 1, 3, 4, and 5. See also, Armstrong and Power (1963).

weak- and strong-coupling limits from the scattering length for the potential $V(r)=gr^{-4}\exp(\lambda/r)$ as calculated by Calogero and Cassandro (1965), Eq. (3.70). The strong-coupling limit of the scattering length is $-g^{1/2}$ which is identical to the scattering length for the potential r^{-4} [Eq. (2.18)], while the weak-coupling limit of the scattering length, $\lambda[\ln(g/\lambda^2)+2\gamma]^{-1}$, which is larger than that for any pure power potential, clearly reflects the singularity at the origin. The association of the weak-coupling limit with the behavior at the origin has been noted by Tiktopoulos and Treiman (1964).

Because of the fact that the pure power potential is the most commonly used illustration of a singular potential, we are including a discussion of the low-energy behavior of phase shifts. This will help the reader to discern which aspects of phase-shift behavior for pure power potentials originate in the singularity at the origin, and which in the nature of the tail. We shall also comment, for the same reason, on the weak- and strong-coupling limits for the pure power potential. A brief review of the large l behavior of phase shifts is also included, though this is not associated with the singular nature of the potential.

For pure power potentials $V(r)=gr^{-m}$, one finds the dimensionless parameter $\chi\equiv k^{1-2/m}g^{1/m}$ as the natural argument. This parameter for $m>2$ is a monotonically increasing function of k and g, so that, for example, large values of χ may correspond to large values of k or g. One notes, however, that the k dependence for $k>1$ is a monotonically increasing function of m, while the g dependence for $g>1$ is a monotonically decreasing function of m. This situation reflects the fact that the large k behavior is determined by the singularity at $r=0$ which becomes stronger with increasing m, while the large g behavior is determined by the tail which becomes weaker with increasing m. An analogous situation applies to the m dependence of χ for $k<1$ and $g<1$. It should therefore not be surprising in the case of a sum of two power potentials that the regime of small k and small g do not correspond as pointed out by del Giudice and Galzenati (1965b) at the end of their article.

1. *Energy Behavior*

Low-Energy Behavior

The low-energy behavior of the phase shifts is not an aspect of singular potentials since this behavior is dependent on the potential tail (Landau and Lifshitz, 1960). For pure singular inverse powers, however, completeness demands that the low-energy nature of the phase shifts be examined.

The familiar low-energy phase-shift behavior is such that

$$\lim_{k\to 0} k^{2l+1}\cot\delta_l(k)\approx -(A_l)^{-1}+\tfrac{1}{2}r_l k^2+\cdots, \quad (2.15)$$

where A_l is the scattering length, and r_l the effective range. [We adopt the usual convention that $\delta_l(0)=0$.] In fact, the right-hand side of Eq. (2.15) is a power series in k^2. It was long ago realized (O'Malley, Spruch, and Rosenberg, 1961) that this series breaks down somewhere for potentials with powerlike tails. The exact place of breakdown and its nature (whether fractional powers of k or $\ln k$ terms appear) depend on whether $2l+3-m$ is positive, negative, or zero. A general discussion for powerlike tails is given by Levy and Keller (1963).

The general breakdown of the expansion of Eq. (2.15) may be described in the following way. Let us write

$$k^{2l+1}\cot\delta_l(k)=\sum_{j=0}^{\infty}a_j k^j \quad (2.16)$$

for regular potentials where only even j appears. For potentials with powerlike tails this series breaks down at $j=J=m-2l-3$, where, if J is an even positive integer or zero, an exceptional term in $k^J\ln k$ appears. If $J<0$, the series begins with the positive power k^{-J}. If $J>0$ is not an integer, an exceptional term in the form of a fractional power of k causes Eq. (2.16) to break down. Higher fractional powers also appear. Finally, if J is an odd positive integer, exceptional log terms appear. For $J>0$, the even integer powers preceding the exceptional term can be obtained from the Born approximation.

Pure power potentials have been studied by a number of workers. The methods used to derive low-energy expressions for $\delta_l(k)$ are varied and are generally complicated. (See Secs. III.B.1 and III.B.3.) Bertocchi, Fubini, and Furlan (1965b) have employed an iterated Voltera equation combined with certain reflection properties of the wave function to find the low-energy behavior of $\delta_l(k)$ for the r^{-4} potential. del Giudice and Galzenati (1965a) have solved the $k\neq 0$ integral equation for the wave function by an iterative matching method which is a generalization of the Bertocchi, Fubini, and Furlan method. They were able to find the Jost functions and hence the phase shifts to any desired order. Their method is valid only for pure powers. They noted that for $m<2l+3$ a few terms of the Born approximation are valid for the phase shift. This is not true for $m\geq 2l+3$. Explicit results are given for $m=2l+3$, $m=2l+5$, and m not integral.

Stanciu (1967) used a method which is simpler than those just mentioned, and is not necessarily restricted to pure powers. He employed a variable-phase method (see Sec. III.B.3) which involves the variable-phase function $T_l(r)=\tan\delta_l(r,k)$ which satisfies a differential equation with certain boundary conditions [see e.g., Eq. (3.102)]. Here $T_l(\infty)$ gives $\delta_l(\infty,k)$ which is the same as $\delta_l(k)$. For r smaller than some r_0, $T_l(r)$ may be expanded in a power series, while for $r\geq r_0$, the phase equation for $T_l(r)$ may be linearized and solved. The solutions can be matched at $r=r_0$ and then $T_l(\infty)$

may be found. His results are similar to those of del Giudice and Galzenati but less comprehensive.

Finally, Handelsman, Pao, and Lew (1968) have used a variable-phase method applicable only to pure powers, and which involves finding $T(r)$ in the two regions $kr\ll 1$ and $g^{1/(2-m)}r\gg 1$. These regions overlap, and the solution may be joined to any desired order. Their explicit results hold for m not integral.

All the above papers reach consistent expressions for the low-energy $\delta_l(k)$ behavior. Each deals with a slightly different set of restrictions on m and l, and they carry out their expressions to differing numbers of terms.

For $m>2l+3$, the usual low-energy leading term is obtained as (Carter, 1952)

$$\tan\delta_l(k)\sim k^{2l+1}, \tag{2.17a}$$

while for $m<2l+3$, unusual behavior sets in, and we find

$$\tan\delta_l(k)\sim k^{m-2}. \tag{2.17b}$$

The results of Eqs. (2.17a), and (2.17b) may be easily obtained as done by Landau and Lifshitz (1960).[22] For $m=2l+3$, we have

$$\tan\delta_l(k)\sim k^{2l+1}\ln k. \tag{2.17c}$$

The S-wave scattering lengths may be easily found from the exact $k=0$ solutions (see Sec. III.A.2). These are (Khuri and Pais, 1964)

$$A_0=-(\nu g^{1/2})^{2\nu}[\Gamma(1-\nu)/\Gamma(1+\nu)],\qquad \nu=(m-2)^{-1}, \tag{2.18}$$

where $m>3$ is required since the scattering length does not exist otherwise. For $2<m<3$, the asymptotic form of the wave function has the fractional power r^{3-m} appearing between the usual linear and constant terms. For $l\neq 0$, of course, the asymptotic wave function does not behave as r plus a constant, but has a term in r^{l+1} plus a term in r^{-l} (for $m>2l+3$).

In terms of the dimensionless variable $\chi^{1-2/m}g^{1/m}$, we quote the $l=0$ results in three cases. These hold for $\chi\ll 1$ which means low energy or weak coupling. These are (γ is the Euler–Mascheroni constant)

$$\tan\delta_0(k)=[3\gamma-\tfrac{3}{2}+\ln 2]\chi^3+3\chi^3\ln\chi+O(\chi^6),\qquad m=3,$$

$$\tan\delta_0(k)=-\chi^2+\tfrac{1}{3}\pi\chi^4+(\tfrac{8}{3}\gamma-\tfrac{28}{9})\chi^6+\tfrac{8}{3}\chi^6\ln 2\chi+O(\chi^8),\qquad m=4,$$

$$\tan\delta_0(k)=3^{-2/3}[\Gamma(-\tfrac{1}{3})/\Gamma(\tfrac{1}{3})]\chi^{5/3}+[\tfrac{13}{36}+\tfrac{1}{18}\ln 3-\tfrac{1}{3}\ln 2-\tfrac{5}{9}\gamma-(\pi/6\sqrt{3})]\chi^5-\tfrac{5}{9}\chi^5\ln\chi-3^{-5/3}[\Gamma(-\tfrac{1}{3})/\Gamma(\tfrac{1}{3})]\chi^{20/3}+O(\chi^{25/3}),\qquad m=5. \tag{2.19}$$

For $l=0$, and $m=3$, the quantity $k\cot\delta_0(k)$ begins like $(\ln k)^{-1}$, and the scattering length does not exist.

del Giudice and Galzenati (1965a) give the more complicated expression for the case when m is not an integer. Expressions for $l\geq 1$, for m not integral, may be found to several orders in χ in their second paper (1965b). Handelsman, Pao, and Lew (1968) point out that a term of order k^{2m-4} which they include is omitted. The general expression is not given here. A breakdown in the power series occurs because of the appearance of fractional powers of χ. No logarithmic terms appear; such terms occur only when m is an integer. They also give lengthy expansions in the two cases $m=2l+3$, and $m=2l+5$, where only $\log\chi$ terms and integral powers of χ occur.

High-Energy Behavior

One manifestation of the special properties of singular potentials is, not surprisingly, the nonvanishing high-energy limit of the phase shifts. For the repulsive centrifugal potential $l(l+1)r^{-2}$, one sees this in the energy-independent phase shift $-\frac{1}{2}l\pi$. The constancy of the high-energy limit is easily seen to hold for any locally repulsive r^{-2} potential singularity. For a potential ever so slightly more singular than r^{-2} at the origin, the phase shift diverges as $k\to\infty$ (Frank, b). For large k, $|\delta(k)|$ can never grow faster than linearly in k. This is consistent with the weak causality condition of Wigner (Wigner, 1955)

$$d\delta/dk\geq -R,$$

where R is a measure of the range of the potential. In fact, if condition (a) holds, the phase shift will have weaker than linear growth as $k\to\infty$ (Frank, b).

The high-energy limit of the scattering phase shift has been calculated by essentially two methods: the WKB method and the variable-phase method (Secs. III.B.2 and III.B.3). Jabbur (1965) has calculated the high-energy scattering limit for pure power potentials by constructing asymptotic (in k) solutions to the scattering integral equation, which is equivalent to an approximation of the WKB method (see discussion in Sec. III.B.2). Bertocchi, Fubini, and Furlan (1965b), and Paliov and Rosendorff (1967) have explicitly worked with the WKB method. The former have worked only with pure power potentials, while the latter have also considered forms $e^{-\mu r}r^{-m}$ for small μ. Paliov and Rosendorff have also used energy-dependent coupling constants (see Secs. III.B.2, III.V.A). The variable-phase method has been employed by Calogero (1964, 1967) to determine the leading behavior in k for a wide class of potentials. Either method allows the development of a high-energy expansion.

While all authors are agreed on the leading k dependence of the high-energy phase shift for power potentials, there is no such agreement on the coefficient of this term. For $V(r)=gr^{-m}$, the high-energy phase shift behaves like ($\zeta=m^{-1}$)

$$\delta_l(k)\underset{k\to\infty}{\approx}-a_0g^{\zeta}k^{1-2\zeta}=-a_0\chi, \tag{2.20}$$

[22] See Landau and Lifshitz (1960), p. 405.

where a_0 is independent of l, and of other less singular terms which may be added to gr^{-m}. The following values of a_0 are offered:

$$a_0=[2^{1-2\zeta}/\Gamma(3-2\zeta)](\pi\zeta/\sin\pi\zeta), \quad \text{(Calogero)}$$

$$a_0=\tfrac{1}{2}\pi^{1/2}[\Gamma(1-\zeta)/\Gamma(\tfrac{3}{2}-\zeta)], \quad \text{(Bertocchi, } et\ al.\text{, Paliov and Rosendorff)}$$

$$a_0=(1-\tfrac{1}{2}\zeta)/(1-\zeta). \quad \text{(Jabbur)}$$

These three expressions are indeed in conflict, though they do give very similar numerical values (see Table I). Jabbur's calculation is based on an approximation to the WKB method and is discussed in more detail in Secs. III.B.1 and III.B.2. However, between the WKB calculations of Bertocchi *et al.*, and Paliov and Rosendorff, and the variable-phase calculation of Calogero, the reviewers can find no clear cut preference.[23] The reader might take note of certain features of these two methods of calculation. The WKB method provides an explicit dependence of the phase shift on the local behavior of the potential in the allowed region. The phase shift depends on the potential in the forbidden region in higher approximations through the coefficients of a Taylor series expansion of the potential about the turning point. The variable-phase method, on the other hand, accumulates the phase shift outward from $r=0$, and depends on the local behavior of the potential everywhere. However the approximation used by Calogero in obtaining the phase shift breaks down in the allowed region. No error bounds have yet been provided for this method. Calogero in his Table II makes numerical comparisons for a number of potentials of the form $V(r)=gr^{-m}\exp(-\mu r)$ for several energies between the variable-phase approximation, the WKB approximation, and the exact result, computed numerically. Both approximation schemes give generally good agreement. The variable-phase approximation is most accurate for the smallest value of m considered ($m=4$), where the forbidden region is largest. The WKB method yields very accurate results for all potentials at high energies where the criterion for its validity is best satisfied, but fails badly at low energies.

Schemes for the calculation of higher-order terms are offered in Calogero, Bertocchi *et al.*, and Paliov and Rosendorff, and the second and sometimes third terms of the expansion are explicitly exhibited. The reader should be warned that different physical effects have been considered by different authors in obtaining these higher-order correction terms. The expansion of Bertocchi *et al.* is based on higher-order WKB corrections (see Bertocchi *et al.*, 1965a, in which the details of the WKB method are presented), while the corrections of Paliov and Rosendorff receive their contribution only from the Langer centrifugal term $-(4r^2)^{-1}$ (see Sec. III.B.2). The former considered only the S wave, while the latter also considered higher angular momenta. In the S-wave case, both expansions are of the form

$$\chi^{-1}[\delta(k)-\tfrac{1}{4}\pi]=-a_0+\sum_{n=1}^{\infty}a_n\chi^{-2n}. \quad (2.21)$$

The a_1 coefficients of Bertocchi *et al.*, and Paliov and Rosendorff are similar but not the same. The expansion of Eq. (2.21) does not apply directly to the calculation of Calogero because the correction terms exhibited there are due in part to the addition of a potential of the form $C'r^{1-m}$ to the original gr^{-m} potential. We note that the $-\pi/4$ is obtained by Calogero as the second term in his expansion when $m<4$.[24]

The leading k dependence for potentials with the behavior for small r,

$$V(r)\approx gr^{-m}(|\ln r|)^n(\ln|\ln r|)^p, \quad (2.22)$$

was found by Calogero (1967) (in his Appendix III) to have the form

$$\delta(k)\sim k^{1-(2/m)}(\ln|k|)^{n/m}(\ln|\ln|k||)^{p/m}, \quad (2.23)$$

where at least one of the three numbers $m-2$, n or p is positive. For $V(r)\sim\exp(cr^{-q})$, $(c>0)$, one finds that $\delta(k)\sim k(\ln|k|)^{-q}$.

TABLE I. Comparison of numerical values of a_0 as given by different authors.

m	Jabbur	Bertocchi *et al.*, Paliov and Rosendorff	Calogero
2	1.50	$\pi/2$	$\pi/2$
3	1.25	1.2934	1.2795
4	1.1667	1.1977	1.1811
6	1.10	1.1195	1.1049
∞	1.0	1.0	1.0

Complex k Behavior

Not much has been systematically determined regarding the behavior of the wave functions or the S-matrix as a function of complex k for repulsive singular potentials. The soundest and most systematic study was made by Limic (1962). He showed that the regular solution of the partial-wave Schrödinger equation is an entire function of k as in the nonsingular case. The S-matrix for a potential which has essentially an exponential or powerlike tail is shown to have cuts along

[23] If the WKB results truly constitute asymptotic expansions, then the WKB method ought to provide the correct high-energy behavior.

[24] The $-\pi/4$ was found by Calogero for the case $m<4$ only, since the next to the leading term due to the additional potential $C'r^{1-m}$ grows faster than a constant when $m>4$. Calogero's Eq. (I.14) would give, in fact, $-\pi/8$, and not $-\pi/4$, but this is due to a typographical error which resulted in $[1+2(1+\epsilon)/m]!$ having been written in place of $[1-2(1+\epsilon)/m]!$ in this equation.

the imaginary axis, and poles at the zeros of the Jost function as for nonsingular potentials. The uniqueness of the singular potential lies in the asymptotic behavior along different rays in the complex k plane. He studied, for illustrative purposes, potentials of the form $r^{-m}W(r)$ with $2<m\leq 4$, $W(0)>0$, and $W(r)$ analytic and bounded in the right-half r plane. For the case $W(r)\equiv 1$, Limic showed an exp $(-2ik\mid \chi/k\mid \text{const})$ behavior $(\chi\equiv g^{1/m}k^{1-2/m})$ of the S-matrix for large $\mid k\mid$. Jabbur (1965) has calculated for pure power potentials a high-energy asymptotic expression for the S-matrix having the behavior exp $(-2i\chi$ const). We note that this behavior differs from that derived by Limic. Both Jabbur, and Aly and Wit (1967) have tried to conclude from this behavior the exponential growth of the S-matrix in some complex k direction. Such a result is quite plausible. However, no discussion has been provided to show in what region of the complex k plane this form of the S-matrix holds. Exponential growth would not follow if it were valid only for real k.[25] One may, however, correctly conclude the presence of an essential singularity at $k=\infty$.

Martin (1966) and Aly and Wit (1967) have addressed themselves to the question of the validity of dispersion relations for repulsive singular potentials. Martin showed that for potentials with power singularities[26] which cut off beyond a finite range R, the forward scattering amplitude is an analytic function of k^2 in a cut plane. For potentials with singularities no stronger than r^{-3}, the forward amplitude was shown to diverge no faster than linearly in any direction of the cut k^2 plane. Thus a once-subtracted dispersion relation can be written in this case. Aly and Wit considered only partial wave dispersion relations, and concluded that an infinite number of subtractions would be necessary for singular potentials. They reached this conclusion on the basis of exponential growth for the partial-wave amplitude deduced from the behavior of the phase shift on the real k axis. However, as discussed above, this conclusion need not follow in general. Moreover, it was not shown in what direction exponential growth occurred. Exponential growth on the unphysical sheet of the k^2 plane would not prevent one from writing a dispersion relation for the scattering amplitude on the physical sheet, if the amplitude were polynomially bounded there. Hence the conclusion of an infinite number of subtractions seems not to be justified. The employment by Aly and Wit of Phragmen–Linelöf methods seems promising, however. For potentials with r^{-2} singularities they correctly concluded that one subtraction is sufficient.

[25] It does not follow from $f(g)\sim e^{iz}$ for large real z that $f(z)$ must have exponential growth in some complex direction. Two counterexamples are $f(z)=[z^{-1}+\exp(-iz)]^{-1}$, and $f(z)=z\ln(1+e^{iz}/z)$.

[26] Martin works explicitly with singularities for which

$$\int_0^R drr^2V(r)<\infty,$$

but notes in the conclusion that the method can be extended to include any powerlike singularities.

2. *Coupling-Constant Behavior*

It has been noted earlier that the singular nature of the potential manifests itself in the small g behavior of quantities of interest. One can conclude from Poincare's theorem[27] that the Jost solution is analytic in the entire g plane for all real energies, and for r real and unequal to zero. However the physical wave function is not analytic in the entire g plane. It can be shown to be analytic in the first sheet of a multisheeted surface with a cut along the entire negative real axis. See, e.g., Eq. (3.26) for pure power potentials at zero energy. Tiktopoulos and Treiman (1964) have argued these results for $k\neq 0$ explicitly on the basis of a Volterra integral equation for potentials whose $k=0$ wave function is known. Meetz (1964) has constructed the spectral representation in the complex g plane of the Green's function (resolvent) for a fixed negative energy. He has shown the resolvent for a singular potential to be analytic in the half-plane Re $[+(g^{1/2})]>0$ and to have a cut along the negative real axis, and possibly as well some discrete poles along the negative real axis corresponding to bound states. The discontinuity across the cut is expressible in terms of the Jost function, and a dispersion relation can be written for the resolvent. For the transition inverse-square potential, the cut extends along the negative real axis up to $g=-\frac{1}{4}$. The existence of this type of dispersion relation is conjectured by Meetz for positive energy.

For pure power potentials, the complete k and g behavior of the S-wave phase shift is exactly determined by the same function in view of the dependence on the single variable $\chi=k^{1-2/m}g^{1/m}$. The low-energy expansion obtained for the phase shift for pure power potentials which has been previously described implicitly contains the small g dependence in view of the correspondence of the regime of small χ to low k and small g [see Eqs. (2.19)]. For higher partial waves when $J=m-2l-3>0$, one can find the terms of the expansion preceding the exceptional term from the Born approximation. These are also expressed in terms of χ, and so the g dependence is determined. Bertocchi *et al.* (1965b) have discussed the particular case of the S wave for $m=4$. Jabbur (1965) has deduced a branch point or essential singularity at $g=0$ in the scattering amplitude for pure power potentials by showing that this quantity does not possess a power series expansion about $g=0$. The general scheme of the small g expansion for pure power potentlals has been giveu by Tiktopoulos and Treiman (1964).

The small g dependence of the scattering length has been explicitly determined for several potentials. The exact scattering length for the pure inverse-power potentials gr^{-m} is given by Eq. (2.18). An approximate

[27] See e.g., R. G. Newton (1960), Footnotes 8, 9.

expression, valid in the small g limit for the scattering length due to the logarithmically singular potentials of the form

$$V(r)=gr^{-n}\ln r^{-1}$$

has been suggested by Wu (1964). It was found that (Sec. III.A.2)

$$A=-(g\nu^2/\epsilon)^\nu[\Gamma(1-\nu)/\Gamma(1+\nu)]$$
$$\times\{1-\epsilon\nu^2[2\ln\nu+2-\psi(1+\nu)-\psi(1-\nu)]+O(\epsilon^2)\},$$

where

$$\nu=(m-2)^{-1},\qquad \epsilon=\ln^{-1}\tau,$$

and τ is given by the solution of the transcendental equation

$$g\tau^{m-2}\ln\tau=1.$$

From this equation, one finds, to a second approximation,

$$\epsilon=\ln^{-1}\tau=[\ln(1/g^\nu)-\ln\ln^\nu(1/g^\nu)]^{-1}.$$

The small g dependence of the scattering length has also been determined for the exponentially singular potential $r^{-4}\exp(\lambda/r)$ [see Eq. (3.71)]. This result has been generalized by Land (a) to the potentials $r^{-m}\exp(\lambda/r)$, $m>3$, by means of an analysis which is similar to that of Wu (1964) for the logarithmically singular potentials discussed above. The first approximation for the scattering length was found to be [cf. Eq. (3.71)]

$$A\approx(\lambda/2\tau)[K_0'(\tau^{-1})/K_0(\tau^{-1})]\approx\tfrac{1}{2}\lambda/[\ln(2\tau)^{-1}+\gamma],$$

where τ is again determined by the solution to a transcendental equation

$$(2/\lambda)^{m-2}g\tau^2\ln^{m-4}\tau=1.$$

In Sec. III.A.2 an outline of the method of Wu is given along with a comment on this analysis. However the approximate expressions presented above for the scattering lengths corresponding to these logarithmically and exponentially singular potentials are consistent with some inequalities for the scattering length discussed by Frank and Land (c).

3. *Real l Behavior*

We consider in this section the real l behavior, since l plays the role of a coupling constant (see Sec. II.E). The calculation of the total cross section from the partial waves entails a knowledge of all the phase shifts or of a finite number, if one knows from the high l limit how many terms of the series are sufficient to achieve a specified accuracy. We mention here some of the work pertaining to the real large l behavior of the phase shifts despite the fact that this behavior depends on the tail of the potential and does not reflect the singular nature of the potential. The behavior of the scattering amplitude for singular potentials in the complex angular momentum plane is discussed in Sec. IV.A.

Bertocchi, Fubini, and Furlan (1965b) have claimed that the phase shift for a singular potential ought to yield a resonance in each angular-momentum state at sufficiently high energy. Dombey (1965) has argued that there should be no resonances for purely repulsive potentials, and that the phase shifts should damp for sufficiently large angular-momentum states. Dombey suggested a very crude estimate for the high l behavior of the phase shift for a pure power potential by analogy with a repulsive hard core potential and found the total high-energy cross section to behave as $k^{-4/m}$. For a potential with an exponential attenuation factor he conjectured that the total high-energy cross section behaves like a constant. Tiktopoulos (1965) has found the differential cross section for pure power potentials by applying the WKB approximation to determine the high l behavior of the phase shifts. He confirmed the $k^{-4/m}$ behavior of the cross section for large k in the absence of absorption (Secs. III.B.2 and V.A).

Paliov and Rosendorff (1967) have derived expansions for the large k behavior of the phase shift valid for the complementary regimes of low and high angular momenta. They have considered potentials with energy-dependent coupling constants as a model for high-energy scattering. The potentials they considered have the form

$$V(r)=gk^sF(r)/r^m, \tag{2.24}$$

with $F(r)=1$, $e^{-\mu r}$ or $e^{-(\mu r)^2}$, and $s=1$ favored as the value suggested by experiment. Their starting point is the WKB expression for the phase shift [Eq. (3.93), see Sec. III.B.2]

$$\delta_{\rm WKB}=\int_{r_0}^{\infty}dr\left(k^2-\frac{\lambda^2}{r^2}-V(r)\right)^{1/2}-\int_{\rho}^{\infty}dr\left(k^2-\frac{\lambda^2}{r^2}\right)^{1/2},$$

which they expanded alternatively in powers of λ ("the λ representation") or g ("the ρ representation"). Here r_0 is the classical turning point, and $\rho=\lambda/k$ is the classical impact parameter. The expansion in the λ representation is valid for sufficiently small λ with a radius of convergence proportional to ω [$\omega\equiv(gk^{s+m-2})^{1/m}$]. In the ρ representation one obtains an expansion in the parameter $(\omega/\lambda)^m$, and the expansion is valid for λ constant for sufficiently large k. The leading term in the phase shift for large λ in the $s=0$ case is found to be proportional to $\chi^m\lambda^{1-m}$.

An interesting classification of three cases according to the sign of $q\equiv m+s-2$ was found. For the case $q>0$, termed the "strong interaction" case, the classical turning point at high energy is determined by the potential, while for the case $q<0$, termed the "weak interaction" case, the turning point at high energy is determined by the centrifugal barrier. The $q=0$ case is termed the "intermediate interaction" case. In the strong interaction case such as for singular potentials with $0\leq s<2$, the expansion parameter ω is an in-

creasing function of k, in the weak interaction case ω is a decreasing function of k, while in the intermediate interaction case the phase shift is entirely independent of energy.

Paliov and Rosendorff have constructed the form of the double series in λ or g and the parameter $t=\mu/k$ for the case $F(r)=e^{-\mu r}$ and for the particular potential $V(r)=gkr^{-2}e^{-(\mu r)2}$. The expansion in t contains fractional powers and sometimes a logarithmic term.

E. Sum of Two Singular Potentials

While the failure of analytic coupling-constant expansions for singular potentials has been well known (Sec. II.D), there has been interest in the analytic properties in the coupling constant of the weaker singular (or transition) potentials for the case of the sum of two singular potentials. We consider a singular potential $gV_1(r)+fV_2(r)$, where the condition (a) or (b) is assumed to hold for both $V_1(r)$ and $V_2(r)$, and

$$\lim_{r\to 0} V_2(r)/V_1(r)=0. \tag{2.25}$$

Regge and Predazzi (1962) have considered this question with particular interest for the case in which the weaker potential is the inverse-square potential whose coupling constant depends on the total angular momentum. Their object was to ascertain the validity of the maximum analyticity principle of Chew for the angular-momentum variable. They investigated in detail the specific potential

$$V(r)=gr^{-4}+fr^{-3}+l(l+1)r^{-2}+W(r),$$

where $W(r)$ is a nonsingular potential which can be represented as a superposition of Yukawa potentials. They set up the Volterra integral equations for the radial wave function (regular solution), and the Jost solution for this potential with energy. They concluded from standard considerations that the wave function is an entire function of f and $\lambda^2=(l+\frac{1}{2})^2$, and that the Jost solution as well as the Jost function are entire in f and λ and analytic in the k plane cut along Im $k>\frac{1}{2}\mu$ (μ^{-1} is the range of the Yukawa potential). From this they concluded the meromorphy of the S-matrix in the angular-momentum plane and in the cut k plane, as well as the symmetry relation Eq. (2.6). Their demonstration of the analyticity in λ^2 and f from the integral equations for the regular and Jost solutions can be carried out for any potential for which a zero-energy solution is known explicitly as pointed out by Tiktopoulos and Treiman (1964) with regard to λ^2. del Giudice and Galzenati (1965b), who have also used the integral equation method, have noted that analyticity of the wave function in the weaker coupling constant can also be inferred directly from Poincare's theorem.[27]

Pais and Wu (1964a) have considered analytic properties in the stronger and weaker coupling constants with regard to application of the peratization procedure to a double series expansion in these coupling constants (Sec. IV.D). They studied the specific potential $gr^{-(2+2\tau)}+fr^{-(2+\tau)}$ ($\tau>1$), and found the scattering length to be expressible as an asymptotic series in g for fixed f, and as an analytic power series in f for fixed g. del Giudice and Galzenati (1965b) have studied the energy dependence of the S-wave phase shift for the potential considered by Pais and Wu by means of the integral equations for the regular and Jost solutions. They calculated the first few terms of the asymptotic series in g which entails fractional powers and in some cases logarithmic terms and of course only integer powers in f. Another singular potential containing two independent coupling constants [Eq. (3.33c) with $g=0$] has been used by Gale (1967), and his calculations for the wave function and scattering length verify the analyticity in the weaker coupling constant. Other solvable potentials entailing two independent coupling constants can be found in Sec. III.A. See Eqs. (3.33c) and (3.72).

Frank (a) has proved the analyticity in the weaker coupling constant. This proof utilizes the concept of the "relative Jost function." In analogy to the lth partial-wave Jost function for a potential $V(r)$ which is equal to

$$\lim_{r\to 0}[(2l-1)!!]^{-1}(kr)^l f(k,r),$$

the Jost function for the potential $gV_1(r)+fV_2(r)$ "relative" to $gV_1(r)$ is defined by

$$\lim_{r\to 0}[f(k,r)/\varphi(k,r)],$$

where $f(k,r)$ is the Jost solution for $gV_1(r)+fV_2(r)$, and $\varphi(k,r)$ is the Jost solution for $gV_1(r)$. The relative Jost function, which is proportional to the Jost function, is shown to be an entire function of the weaker coupling constant when the condition

$$\int_0^A dr\,\frac{V_2(r)}{|V_1(r)|^{1/2}} \times\left[1\Big/\left(\int_r^A ds\,|V_1(s)|^{1/2}\right)^{\beta}\right]<\infty \tag{2.26}$$

is satisfied for some β with $0<\beta<1$. The meromorphy in f of the S-matrix would follow from this. [The quantity A in Eq. (2.26) is some finite positive number; condition (a) is to hold for $gV_1(r)+fV_2(r)$.]

F. Classical Singular Potentials

For the sake of completeness we summarize here the most important properties of classical nonrelativistic singular potentials. These are discussed in Newton (1966) and in Landau and Lifshitz (1958).

For all repulsive singular potentials, the classical scattering is well defined since there is a distance of closest approach. By well defined we mean that the

integral for the scattering angle,[28]

$$\theta_{\text{scatt}}=\pi-2\int_{r_{\min}}^{\infty} drr^{-2} \times\{[E-V(r)]2Ml^{-2}-r^{-2}\}^{-1/2}, \quad (2.27)$$

exists, where M is the mass of the scattering particle, and $r_{\min}$ is the distance of closest approach. The forward cross section diverges unless the potential becomes rigorously zero beyond some finite r. This phenomenon is familiar from classical Coulomb scattering. In quantum mechanics, the potential need only fall off faster than r^{-3} at infinity to make the forward cross section finite.

For the attractive inverse-square potential $V(r)=-g/r^2$, with $g>0$, the scattering is well defined for $l>(2Mg)^{1/2}$, which corresponds to an impact parameter b satisfying $b>(g/E)^{1/2}$. However for $l<(2Mg)^{1/2}$, the particle spirals into the center, and the scattering angle given by Eq. (2.27) is infinite. These results are somewhat analogous to those discussed in Sec. II.B for the weakly attractive ($g>-\frac{1}{4}$) and strongly attractive ($g<-\frac{1}{4}$) transition potentials in quantum mechanics. The former display nonsingular characteristics, while the latter have essentially all the difficulties of singular potentials.

For an attractive potential more singular than an inverse square at the origin, there are several contrasts between the classical and quantum-mechanical situations. As mentioned previously, there is an arbitrariness in the quantum-mechanical case in that the boundary condition does not specify a unique solution. It is even possible to select a boundary condition which describes complete absorption (Vogt and Wannier, 1954; see Secs. III.A.1, V.A). By contrast, in the classical case, the scattering problem presents no particular difficulties if the impact parameter exceeds a certain critical value b_{crit}. If the impact parameter is less than this value, the particle falls to the center in a finite time. However, the scattering angle at the origin is well defined, since the integral for the scattering angle in Eq. (2.27) is finite. However, as discussed by Behncke (1968), there is an ambiguity in that infinitely many orbits go through the origin with a given tangent. One can connect the incoming orbit with a unique outgoing orbit if one assumes energy and angular-momentum conservation as the particle passes through the origin.

We may understand this general behavior in the following way. The effective potential, composed of the potential itself and the repulsive centrifugal barrier, has a repulsive hump at a value $r=r_0$. For energies below the top of this hump, there is a distance of closest approach, and the scattering is well defined. For energies above this hump, the particle falls to the center, but the scattering is still well defined. If the energy is exactly equal to the top of the hump, the integral in Eq. (2.27) does not exist. In this case the particle spirals in a trajectory which asymptotically approaches, but never attains, a circular orbit with radius r_0. This behavior, called spiral scattering, is not peculiar to singular potentials (see Newton, 1966). For the attractive inverse power potential $V(r)=-gr^{-m}$, $m>2$, for example, the critical value of the impact parameter is given by

$$b_{\text{crit}}=(gm/2E)^{1/2}[2E/g(m-2)]^{(m-2)/2m},$$

and the radius r_0 by $(mgM/l^2)^{1/(m-2)}$.

An interesting feature for potentials that behave like $-gr^{-m}$ at the origin, with $4>m>2$, should be noted. Newton (1966) has observed that in these cases the curvature of the particle trajectory does not exist at the origin, and concludes from this that one cannot define a scattering angle unambiguously. This should not affect the scattering, however, because the tangent to the trajectory, and hence its direction, do exist.[29]

Scarf (1958) has applied the virial theorem to determine the permissible range of energies for periodic orbits for a given potential in the classical case. This theorem states that for periodic or asymptotic orbits the energy is given by

$$E=\langle V\rangle+\tfrac{1}{2}\langle r(dV/dr)\rangle,$$

where the angle brackets denote time averages. For the one-dimensional potentials $-V_0(1+\alpha x)x^{-2}$, $\alpha>0$, and $-gx^{-m}$, $m>2$, he has shown that singular periodic orbits have $-\infty<E<0$ and $E=(\frac{1}{2}m-1)V_0\langle x^{-m}\rangle$, respectively. For the potential $-V_0\csc^2 x$, with $0<x<\pi/2$, and $V(\pi/2)=\infty$, singular periodic orbits have $-\frac{1}{3}V_0\leq E\leq\infty$.

Felder (1968) has numerically evaluated the classical differential cross section for the class of potentials gr^{-m} and has presented some approximate forms for this. Other references to such calculations can be found in this article.

For relativistic classical scattering, singular behavior sets in for r^{-1} potentials because of the appearance of $[E-V(r)]^2$ in the relativistic version of Eq. (2.27). The same result is true in the quantum-mechanical case as discussed below.

G. Relativistic Quantum-Mechanical Case

Our discussion of relativistic quantum-mechanical singular potentials is limited to the Klein–Gordon and Dirac equations with nonzero mass. The first investigation of the relativistic case is due to Plesset (1932) who studied singular power potentials in the Dirac equation. He observed for this situation the general property of the relativistic quantum-mechanical case that the essential distinction between attractive and repulsive potentials is lost: all potentials look attractive near the origin. Both attractive and repulsive potentials produce

[28] See Newton (1966), Eq. (5.4).

[29] For example, the curve $y=x^{3/2}$ is continuous with a continuous tangent in the neighborhood of the origin, but it has infinite curvature there.

large and small components of the wave function that behave very near the origin like a power of r times a factor

$$\exp\left[\pm i\int^{r} V(r)dr\right].$$

This contrasts with the nonrelativistic case in that the quantity $V(r)$, rather than $V^{1/2}(r)$, appears in the exponent. The same situation occurs with the Klein–Gordon equation due to the appearance of the dominant term $-V^2(r)$ in the differential equation.

This general behavior is a consequence of the advent of negative energy states which become important in regions of strong potentials which are either attractively *or* repulsively singular. The single particle Dirac or Klein–Gordon equations lose their meaning in this region because of these negative energy states. Thus it is impossible to meet the usual boundary conditions at the origin. The existence of negative energy states also has the consequence that discrete bound states must lie between $\pm mc^2$, in contrast to the nonrelativistic case. We also note, from the foregoing considerations, that r^{-1} is the transition potential for these relativistic wave equations.

The solutions in the case of the Dirac equation were shown by Plesset to be power series in r times the oscillatory factor given above. Spector (1967a) explicitly computed the coefficients in the power series. These power series are asymptotic and not convergent (see Sec. III.A.1). Rose and Newton (1951) have pointed out that these solutions violate unitarity because of the nonreality of the wave functions.

Case (1950) has investigated the Dirac, Klein–Gordon, and Gunn (spin-one) equations for the Coulomb potential. The potential in the Dirac equation becomes singular when $\alpha Z>j+\frac{1}{2}$, and in the Klein–Gordon equation when $\alpha Z>l+\frac{1}{2}$. Here α is the fine structure constant, Z is the nuclear charge, j is the total spin, and l is the orbital angular momentum. In these cases the solutions are subject to the same maladies at the origin as we have just discussed. Case imposes orthogonality on the wave functions for bound states to obtain a discrete but nonunique spectrum, in a manner identical to our treatment of the inverse square in the Schrödinger equation in Secs. II.C and III.A.1. Scarf (1958) has also discussed bound-state solutions for the singular Coulomb potential in the Dirac equation by a procedure of analytic continuation of the wave function past the origin to negative r in coordinate space, as outlined in Sec. II.C. He obtained complex energy levels of the form $E_n=W_n\pm i(\frac{1}{2})\Gamma_n$ representing nonstationary states, with energy W_n, and lifetime $\tau_n=h/\Gamma_n$.

Because relativistic quantum-mechanical equations are intrinsically unrealistic for both attractive and repulsive singular potentials due to the appearance of negative energy states, they are of little interest.

III. SOLUTIONS AND CALCULATIONAL TECHNIQUES

A. Exact Solutions

1. $k\neq 0$

Unfortunately the completely general radial Schrödinger equation with a singular potential can be solved exactly in only two cases for nonzero energy and for all angular momenta. In one case, the inverse square, the potential is not truly singular. In the other case, the inverse fourth, the solutions are in terms of the very complicated modified Mathieu functions of generally complex argument. In addition, the potential $V(r)=g\sinh^{-2}\beta r$ $(g>0)$ may be solved for S waves. It is also possible to find energy-dependent potentials which are exactly solvable, for example

$$V(r)=A^2r^{-4}-2Ar^{-3}+2iAkr^{-2}.$$

The inverse square does possess some features in common with true singular potentials and is deserving of study. We follow Morse and Feshbach (1953) to develop the solution in this case. The equation for the radial wave function of the lth partial wave is

$$u''+\{k^2-[l(l+1)/r^2]-V(r)\}u=0, \tag{3.1}$$

where $\hbar=2M=1$. Putting $V(r)=g/r^2$, gives

$$u''+[k^2-(\lambda/r^2)]u=0, \tag{3.2}$$

with $\lambda=l(l+1)+g$.

This is the equation for a free particle with nonintegral (possibly negative) angular momentum. For a repulsive potential ($g>0$ and $k^2>0$), the unique solution which vanishes at $r=0$ is

$$u(r)=r^{1/2}J_p(kr), \tag{3.3}$$

with $p=(\lambda+\frac{1}{4})^{1/2}$. This case is clearly not singular, but for an attractive potential ($g<0$) the situation changes when g is sufficiently negative so that p is imaginary. (See Secs. II.B, II.C.) This occurs for $\lambda<-\frac{1}{4}$ or $g<-\frac{1}{4}-l(l+1)$. Then, for any energy, there are two linearly independent solutions, both of which vanish at the origin. The general solution now contains two arbitrary constants, and becomes

$$u(r)=Ar^{1/2}J_{iq}(kr)+Br^{1/2}J_{-iq}(kr), \tag{3.4}$$

where now $q=(-\lambda-\frac{1}{4})^{1/2}>0$.

At the origin we have

$$r^{-1/2}u(r)\underset{r\to 0}{\sim}A(kr)^{iq}+B(kr)^{-iq}$$
$$=A\exp(iq\ln kr)+B\exp(-iq\ln kr), \tag{3.5}$$

and $u(r)$ oscillates infinitely rapidly while going to zero. The wave function is square integrable at the origin. This feature is characteristic of all truly singular attractive potentials regardless of the magnitude of g. We conclude that since no unique wave function exists

at the origin, no unique scattering may be defined. We will return to this point.

The case of $\lambda<-\frac{1}{4}$ and $k^2<0$ produces a most interesting aspect of Eq. (3.2). We choose the solution that vanishes exponentially fast at infinity which is (with $k=i\kappa$)

$$u(r)=r^{1/2}H_{iq}^{(1)}(i\kappa r)$$
$$=(r^{1/2}/\sinh \pi q)[e^{\pi q}J_{iq}(i\kappa r)-J_{-iq}(i\kappa r)], \quad (3.6)$$

where

$$u(r) \underset{r\to\infty}{\sim} (2/\pi i\kappa)^{1/2} \exp[-\kappa r - i(\pi/2)(iq+\tfrac{1}{2})],$$

and

$$u(r) \underset{r\to 0}{\sim} (e^{\pi q/2}/\kappa \sinh \pi q)[r^{1/2}/|\Gamma(1+iq)|] \times \sin[q \ln(\tfrac{1}{2}\kappa r)-\Phi_q],$$

with $\Gamma(1+iq)=|\Gamma(1+iq)|\exp(i\Phi_q)$. We see that for *any* negative energy there is a solution which is square integrable at the origin and goes to zero at infinity. Thus there is a continuum of bound states. This again is a characteristic feature of all bound-state singular potential problems.

If we impose the further requirement (Case, 1950) that solutions for different bound states be orthogonal, then we obtain a discrete, though nonunique, spectrum. This requirement gives

$$(\kappa_1^2-\kappa_2^2)\int_0^\infty u_1^* u_2\, dr = 4g\left[\frac{e^{\pi q}}{|\Gamma(1+iq)|^2 \sinh^2 \pi q} \sin q \ln\frac{\kappa_2}{\kappa_1}\right], \quad (3.7)$$

and orthogonality obtains only when $q \ln(\kappa_2/\kappa_1)$ is π times an integer. If one bound level is fixed at $-\kappa_0^2$, then the other levels are given by

$$E_n=-\kappa_0^2 \exp(2\pi n/q),$$
$$n=\cdots,-2,-1,0,+1,+2,\cdots.$$

The most significant feature of this spectrum is that it is not bounded below. Generally, singular attractive potentials have no ground state and also lack a unique scattering phase shift for positive energies. Both result from not having a unique square integrable solution at the origin.

The orthogonality method was utilized by Case for the strong Coulomb potential, which is also a singular potential in the Dirac, Klein–Gordon, and spin-one equations, to find discrete, nonunique spectra. These three spectra are bounded below, unlike the normal singular potential problem (Case, 1950).

By imposing other constraints, the bound-state spectrum for a general attractive singular potential may be made unique, and, possibly, bounded below. Scarf (1958) proposed an additional set of (artificial) conditions on the behavior of the wave functions in an analytically continued **r** space. The energy levels so generated are unique and are complete. He cautioned that such states are not to be considered physically meaningful, however. A method suggested by Tietz (1959) to find the bound-state spectrum does not produce a unique spectrum (Sec. II.C).

There exist, of course, regular potentials which can be solved with angular momentum in the presence of energy, such as the Coulomb potential or the square well. We do not list these here.

Turning to the case of the inverse fourth power, the problem separates into only two cases, attractive and repulsive. To solve this potential exactly, we follow Spector (1964), and draw on the results of Vogt and Wannier (1954), and Aly and Müller (1966a). Putting $u(r)=r^{1/2}\varphi(r)$ in Eq. (3.1) with $V(r)=g/r^4$, we obtain

$$\varphi''(r)+r^{-1}\varphi'(r) + \{k^2-[l(l+1)+\tfrac{1}{4}/r^2]-(g/r^4)\}\varphi(r)=0. \quad (3.8)$$

We make the further substitution $x=\lambda r$, where λ is to be determined and set

$$x=e^{-z}, \qquad 0\le|x|\le 1,$$
$$x=e^{z}, \qquad 1\le|x|\le\infty,$$

to obtain

$$\varphi''(z)+[(k/\lambda)^2 e^{\pm 2z}-g\lambda^2 e^{\mp 2z}-(l+\tfrac{1}{2})^2]\varphi(z)=0,$$

where we always have Re $z\ge 0$. Setting

$$k^2/\lambda^2=-g\lambda^2=h^2, \quad \text{and} \quad a=(l+\tfrac{1}{2})^2,$$

we obtain

$$\varphi''(z)-(a-2h^2\cosh 2z)\varphi(z)=0. \quad (3.9)$$

This equation is the modified Mathieu equation, i.e., the Mathieu equation of pure imaginary variable (McLachlan, 1947; Meixner and Schäfke, 1954; Wannier, 1953). The theory of Mathieu functions, though extensive, is very complicated, since the solutions to Eq. (3.9) involve three-term recursion relations for the coefficients appearing in the infinite Bessel series solutions. These recursion relations (given below) contain a parameter ν (the Floquet parameter, often denoted β) which is a complicated function of a and h, $\nu=\nu(a, h^2)$. The determination of ν is a major problem and can be done only by approximation for certain ranges of a and h. Once ν is determined, the solution to Eq (3.9) may be written in conventional notation as

$$M_\nu^{(3)(4)}(z,h)=[me_\nu(0,h^2)]^{-1} \times \sum_{n=-\infty}^{\infty}(-1)^n C^\nu_{2n}(h^2)H_{(\nu+2n)}^{(3)(4)}(2h\cosh z), \quad (3.10)$$

where

$$[a-(\nu+2n)^2]C^\nu_{2n}-h^2(C^\nu_{2n-2}+C^\nu_{2n+2})=0,$$

and

$$me_\nu(0, h^2) = \sum_{n=-\infty}^{\infty} C^\nu_{2n}(h^2).$$

In Eq. (3.10), $H^{(3)}(y) \equiv H^{(1)}(y)$ and $H^{(4)}(y) \equiv H^{(2)}(y)$ are Hankel functions which behave at infinity like e^{iy} and e^{-iy}, respectively. These series are known to converge for $|\cosh z| \geq 1$, but uniformly so only for $|\cosh z| > 1$. Since the asymptotic behavior in z of each term in Eq. (3.10) is independent of n, we see that $M^{(3)}$ and $M^{(4)}$ have the same asymptotic behavior for large z as do $H_\nu^{(3)}(2h\cosh z)$ and $H_\nu^{(4)}(2h\cosh z)$.

We consider first the attractive case ($g<0$) and $k^2>0$ so that

$$\lambda = [k^{1/2}/(-g)^{1/4}] > 0 \quad \text{and} \quad h^2 = k(-g)^{1/2} > 0,$$

and then $z = \pm \ln \lambda r$ is real. The general solution becomes

$$u(r) = Ar^{1/2} M_\nu^{(3)}(\pm \ln \lambda r, h) + Br^{1/2} M_\nu^{(4)}(\pm \ln \lambda r, h), \quad (3.11)$$

where the plus sign is taken if $|\lambda r| \geq 1$, and the minus when $|\lambda r| \leq 1$. Note that the behavior of $u(r)$ at *both* the origin and at infinity comes from the large z behavior of $M_\nu^{(3)(4)}(z, h)$. Actually the solution given by Eq. (3.11) is valid only for $|\lambda r| > 1$ or $|\lambda r| < 1$, since the derivative of Eq. (3.11) does not exist at $r = \lambda^{-1}$ as mentioned earlier. We may connect across this point by matching the logarithmic derivative with another form of solution valid for all finite z which is

$$Me_{\pm\nu}(z, h^2) = \sum_{n=-\infty}^{\infty} C^\nu_{2n}(h^2) \exp[\pm(2n+\nu)z].$$

When this is done we obtain the following expressions:

$$u(r) \underset{r\to 0}{\sim} A\left[\frac{2}{\pi(-g)^{1/2}}\right]^{1/2} r \exp\left[i\frac{(-g)^{1/2}}{r} - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right] + B\left[\frac{2}{\pi(-g)^{1/2}}\right]^{1/2} r \exp\left[-i\frac{(-g)^{1/2}}{r} - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right],$$

$$u(r) \underset{r\to\infty}{\sim} A'(2/\pi k)^{1/2} \exp\{i[kr - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi]\} + B'(2/\pi k)^{1/2} \exp\{-i[kr + \tfrac{1}{2}\nu\pi + \tfrac{1}{4}\pi]\}. \quad (3.12)$$

In general, $A \neq A'$ and $B \neq B'$ because the small r and large r solutions must be joined across $r = \lambda^{-1}$ as described. Note that in Eq. (3.12) all solutions are square integrable at the origin. In fact they go to zero while oscillating infinitely rapidly.

Vogt and Wannier (1954) chose a boundary condition $B=0$ at the origin to correspond to a totally incoming wave, so that they could calculate the quantum-mechanical "capture" cross section. They wished to compare this quantity with the classical cross section for fall to the center, which they chose to call the classical "capture" cross section. This problem is of interest physically and we discuss their results in Sec. V. We merely indicate their method here.

With this boundary condition, the usual scattering cross section contains complex phase shifts (due to the complex nature of the chosen wave function), and is not unitary. Vogt and Wannier did *not* calculate this cross section, but rather the "capture" cross section, defined as the flux entering the sink divided by the flux density of the incoming plane wave. By decomposing the wave function into a partial-wave expansion, they adjusted the coefficient in each term so that the incoming wave part agrees with their chosen boundary condition at the origin. The total wave function then has the form

$$\psi(\mathbf{r}) \underset{r\to 0}{\sim} \exp[i(-g)^{1/2}/r] g(\theta).$$

The "capture" cross section becomes

$$\sigma_{\text{cap}}^{QM} = \frac{(-g)^{1/2}}{k} \int |g(\theta)|^2 \, d\Omega = \frac{\pi}{k^2} \sum_{l=0}^{\infty} \frac{2l+1}{1+e^{2\Phi}}, \quad (3.13)$$

where $e^\Phi = i \sin \pi\gamma / \sin \pi\nu$, with γ another parameter which is a complicated function of a and h^2. It can be expressed as

$$e^{i\pi\gamma} = M_{-\nu}^{(1)}(0)/M_\nu^{(1)}(0),$$

with

$$M_{\pm\nu}^{(1)}(0) = (C_0^{\pm\nu})^{-1} \times \sum_{n=-\infty}^{\infty} (-1)^n C^{\pm\nu}_{2n} J_n(i^{1/2}h) J_{\pm\nu+n}(i^{1/2}h). \quad (3.14)$$

The evaluation of Eq. (3.13) for various energies is given in Sec. V.

No one has as yet investigated the bound-state properties of this potential.

In the case of the repulsive ($g>0$) potential for $k^2>0$, it is easily seen from Eq. (3.12) that a solution that goes to zero at the origin occurs only for $B=0$. This solution is unique. By means of the matching described earlier the partial-wave S-matrix may be calculated. Note that in this case $z = \pm \ln \lambda r \mp (i\pi/4)$. After a lengthy calculation, the result is

$$S(k, l) = \exp[2i\delta_l(k)] = i(-1)^l\{(R^2-1)/[R^2 - \exp(-2i\pi\nu)]\}, \quad (3.15)$$

with

$$R = M_{-\nu}^{(1)}(0)/M_\nu^{(1)}(0).$$

The sign in the numerator and denominator in Eq. (3.15) is incorrectly given as plus in Spector (1964), while Aly and Müller (1966a) are missing an over-all factor of $-(-1)^l$. The correct sign in the fraction may be determined by examining the Regge pole structure as given by Challifour and Eden (1963). Only with the

minus sign does Eq. (3.15) have the correct pole structure. Alternately, the S-matrix may be written as

$$S=i(-1)^l[\sin \pi\gamma/\sin \pi(\gamma+\nu)],$$

or, for the partial-wave amplitude,

$$a(l, k)=ie^{i\pi l}/\{2[i\cos \pi\nu-(e^{-2\Phi}+\sin^2 \pi\nu)^{1/2}]\}. \quad (3.16)$$

None of these forms is very simple. However Eq. (3.16) may be used to examine the Regge pole structure of this potential (Challifour and Eden, 1963). (See Sec. IV.A.)

Aly and Müller (1966b) have solved the repulsive potential $V(r)=g\sinh^{-2}\beta r$ exactly for S waves only. This potential is similar to the inverse-square potential as regards the existence of a critical value of the coupling constant at the point at which the potential becomes attractively singular (Sec. II.B). The solution is readily found to be

$$u(r)=(\sinh \beta r)^P F(a, b; c; -\sinh^2 \beta r), \quad (3.17)$$

where

$$g=\beta^2 P(P-1), \qquad K^2=k^2/\beta^2,$$

and

$$a=\tfrac{1}{2}(P\pm iK), \qquad b=\tfrac{1}{2}(P\mp iK), \qquad c=(P+\tfrac{1}{2}).$$

Here $F(a, b, c; z)$ is the hypergeometric function. The S-matrix in this case is

$$S_0(k)=[\Gamma(P-iK)\Gamma(1+iK)/\Gamma(P+iK)\Gamma(1-iK)]. \quad (3.18)$$

Since this potential has an exponential tail, the small k phase shift behaves like $\delta_0\sim k$, rather than like the $\ln k$ characteristic of the pure inverse square.

Scarf (1958) has considered the potential $V(r)=g\csc^2 r$ in the presence of energy in connection with the bound-state problem (see Sec. II.C).

One other exactly solvable case has been briefly discussed by Vasudevan, Venkatesan, and Jagannathan (1967).[30] The Schrödinger equation for the complex, energy-dependent potential,

$$V(r)=(A^2/r^4)-(2A/r^3)+(2iAk/r^2), \quad (3.19)$$

has the general solution,

$$u(r)=e^{-A/r}e^{ikr}\left\{a+b\int^r \exp[(2A/x)-2ikx]\,dx\right\},$$

with a and b arbitrary constants. The regular solution is obtained by putting $b=0$. This potential is of some interest in view of its similarity to the potentials in Eqs. (3.8) and (3.33b). Other singular potentials with energy-dependent strengths can probably be constructed but are not of interest to us here.

This exhausts the singular potentials familiar to us for which solutions with energy can be found. However (Spector, 1967a), some further insight may be gained as to the general nature of the solutions for other singular potentials when $k\neq 0$. In the three cases of r^{-4}, r^{-6}, and $r^{-4}e^{-\mu r}$ there exists an asymptotic power-series solution in a sense to be specified. It is always true that we may write for inverse powers, gr^{-m},

$$u_\pm(r)=r^{m/4}\exp\{\pm[2g^{1/2}/(m-2)]r^{1-m/2}\}\chi(r). \quad (3.20)$$

A similar form is true even for more general potentials (see Sec. III.B.2 on WKB methods). The function $\chi(r)$ satisfies

$$\chi''(t)+\left[\frac{2}{t}+\frac{4}{m-2}g^{1/2}t^{-2}\right]\chi'(t)+\left(\frac{2}{m-2}\right)^2\left(k^2t^\delta-\frac{\lambda}{t^2}\right)\chi(t)=0,$$

where $\lambda=l(l+1)+\frac{1}{4}m(\frac{1}{4}m-1)$, $\delta=(8-2m)/(m-2)$, and $r=t^{2/(m-2)}$. Clearly, near $t=0$ we have

$$\chi(t)\underset{t\to 0}{\approx}1+[\lambda/g^{1/2}(m-2)]t+\cdots.$$

However the power series in t continues only for $m=4$ or 6 [for $r^{-4}e^{-\mu r}$ such a power series for $\chi(t)$ exists with the multiplying function in Eq. (3.20) somewhat different]. For all other values of m, the next term in $\chi(t)$ is *not* a power of t. The general nature of $\chi(t)$ in such cases is indicated by Spector (1967a), and is extremely complicated. In the Dirac case, however, all inverse powers have such an asymptotic (not convergent) power-series solution.

The analyticity properties in k of the wave functions and phase shifts of the inverse square and inverse fourth power are the same as other singular potentials and have already been dealt with (Sec. II.D). The general l (Regge) properties will be discussed in Sec. IV.A.

We note again that there are no solutions to the three-dimensional problem for the purely attractive case, though there are solutions for a complex-coupling constant with a negative imaginary part, as discussed in Sec. II.C.

2. $k=0$

Unlike the case of $k\neq 0$, there exist a variety of potentials for which one can solve the Schrödinger equation exactly in the limit of zero energy, some for all partial waves, and some for the S wave only. This is certainly not unexpected, because considerable simplicity is gained by dropping a term (the k^2 term) from the radial Schrödinger equation, Eq. (3.1).[31] The importance of this case lies not only in the inherent interest of having exact solutions for a variety of

[30] Vasudevan *et al.* claim that the potential

$$V(r)=-(\theta_2/16\alpha^2r^4)+(\theta_1/8i\alpha r^3)+(i\alpha\theta_1/2r),$$

for which exact solutions are available from solutions of the Hill equation, can be made real and energy independent. The energy term k^2 occurs in the Schrödinger equation in the form $\alpha^2\theta_2$. It can easily be seen that the assertion of the energy independence of the potential is not correct.

[31] Fubini and Stroffolini (1965) point out that this simplicity arises because the Hill determinant associated with the zero energy Schrödinger equation is triangular (see Sec. III.B.7).

singular potentials, but also in the fact that exact solutions serve as a standard with which to compare the results of calculational techniques. In particular, the calculation of the scattering length, which can be obtained from the zero-energy wave function, is frequently used as a testing ground for the regularization and peratization procedures; the latter concept having been introduced originally in a field-theoretic application. Solutions to certain of these potentials have not been discussed in the literature to the knowledge of the present authors. Table II contains a list of the potentials solvable at zero energy and is given at the end of this section.

The simplest and most frequently discussed singular potential which can be solved exactly at $k=0$ is the pure inverse power potential (Khuri and Pais, 1964),

$$V(r)=g/r^m. \tag{3.21}$$

In fact, this potential can be solved for all partial waves. The zero-energy radial Schrödinger equation for the lth partial wave is

$$(d^2u_l/dr^2)-[l(l+1)/r^2]u_l-(g/r^m)u_l=0. \tag{3.22}$$

By making the transformation

$$u_l(r)=r^{1/2}\varphi_l(y), \qquad y=\beta r^\sigma,$$

we obtain the standard form of the Bessel equation [Bateman, Vol. 2, p. 5 (11)],

$$(d^2\varphi_l/dy^2)+y^{-1}(d\varphi_l/dy)-[1+(\nu^2/y^2)]\varphi_l=0, \tag{3.23}$$

where the constants ν, β, σ are given by

$$\nu=(2l+1)/(m-2), \qquad \beta=2g^{1/2}/(m-2),$$
$$\sigma=1-(m/2). \tag{3.24}$$

The general solution of Eq. (3.22) is

$$u_l(r)=r^{1/2}[\alpha_1 K_\nu(\beta r^\sigma)+\alpha_2 I_\nu(\beta r^\sigma)], \tag{3.25}$$

where $K_\nu(x)$ and $I_\nu(x)$ are the modified Bessel functions. The physical solution must satisfy the boundary condition $u_l(0)=0$. This implies that $\alpha_2=0$ in Eq. (3.25). The small r behavior of the wave function is found from the asymptotic expansion of the Bessel function [Bateman, Vol. 2, p. 23 (1)]:

$$u_l(r)=r^{1/2}K_{(2l+1)/(m-2)}\{[2g^{1/2}/(m-2)]r^{-(m-2)/2}\} \tag{3.26}$$
$$\underset{r\to 0}{\approx} \tfrac{1}{2}\pi^{1/2}\left(\frac{m-2}{g^{1/2}}\right)^{1/2} r^{m/4}$$
$$\times\exp\left(-\frac{2g^{1/2}}{m-2}r^{-(m-2)/2}\right). \tag{3.27}$$

We see from Eq. (3.27), as well as from Eq. (3.32) below, that neither the wave function nor the scattering length is an analytic function of g at $g=0$. This fact illustrates a general property of singular potentials discussed in Sec. II.D. The Wronskian (see Footnote 12) of the two solutions of Eq. (3.22) is given by

$$W\{r^{1/2}K_\nu(\beta r^\sigma), r^{1/2}I_\nu(\beta r^\sigma)\}=\sigma. \tag{3.28}$$

From the zero-energy wave function, one can compute the scattering length. The reader is referred to Sec. II.D.1 for a general discussion of the low-energy behavior of the phase shift. For potentials which vanish sufficiently rapidly at large r (e.g., short-range potentials), the low-energy lth partial-wave phase shift has the behavior

$$k^{2l+1}\cot\delta_l(k)\underset{k\to 0}{\approx}(A_l)^{-1}+\tfrac{1}{2}r_l k^2+\cdots, \tag{3.29}$$

where A_l is the "scattering length" ($A_l>0$ for purely attractive potentials in the absence of bound states; $A_l<0$ for purely repulsive potentials), and r_l the effective range. In the asymptotic r region, the wave function for these potentials is given by the solution of Eq. (3.1) with the potential term absent:

$$u_l(r)\underset{r\to\infty}{\sim}(kr)^{1/2}[J_{l+\frac{1}{2}}(kr)+(-1)^l\tan\delta_l J_{-(l+\frac{1}{2})}(kr)],$$

where $J_\nu(x)$ is the Bessel function of the first kind. In the limit of zero energy, the wave function, expanded in descending powers of r, becomes

$$u_l(r)\underset{k\to 0}{\sim}r^{l+1}+\frac{\tan\delta_l(k)}{k^{2l+1}}\frac{[(2l)!]^2(2l+1)}{2^{2l}(l!)^2}r^{-l}+\cdots$$
$$\sim r^{l+1}+C_l(k)r^{-l}+\cdots, \tag{3.30}$$

where $C_l(k)$ denotes the coefficient of the r^{-l} term when the r^{l+1} term is normalized to unity. The scattering length is therefore obtained from the second term of the expansion and is given by the expression

$$A_l=\lim_{k\to 0}[2^{2l}/(2l+1)]\{(l!)^2/[(2l)!]^2\}C_l(k).$$

We note that for the S wave, Eq. (3.30) becomes simply

$$u_0(r)\underset{r\to\infty}{\sim}r+A_0. \tag{3.31}$$

We now calculate the scattering length for the pure inverse-power potential of Eq. (3.21). An expansion of the zero-energy wave function for this potential gives

$$u_l(r)=r^{1/2}K_\nu\left(\frac{2g^{1/2}}{m-2}r^{-[(m-2)/2]}\right)$$
$$\underset{r\to\infty}{\approx}[\Gamma(1-\nu)]^{-1}\left(\frac{g^{1/2}}{m-2}\right)^{-\nu}\left[r^{l+1}-\left(\frac{g^{1/2}}{m-2}\right)^{2\nu}\right.$$
$$\left.\times\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}r^{-l}+(1-\nu)^{-1}\frac{g}{(m-2)^2}r^{l-m+3}+\cdots\right],$$

where ν is given in Eq. (3.24). It is clear that the r^{-l} term is the second leading term in the expansion only when $m>2l+3$; i.e., only when this condition is satisfied is the potential sufficiently well cut off for a given value of l that the usual phase shift behavior of Eq. (3.29) is obtained. In this case the scattering

length for the lth partial wave is written

$$A_l = -\frac{2^{2l}(l!)^2}{(2l+1)[(2l)!]^2}\left(\frac{g^{1/2}}{m-2}\right)^{2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}. \quad (3.32)$$

When $m \leq 2l+3$, the phase-shift behavior of Eq. (3.29) does not hold. We do not discuss this case but mention that one could make an alternative definition of the scattering length, based on the behavior given in Sec. II.D.1, and relate this to the asymptotic wave function. For the S wave, the condition which gives rise to the normal behavior of the phase shift for the potentials gr^{-m}, Eq. (3.29), reduces to $m>3$. When evaluating the scattering length in this section, we shall consider only those potentials and orbital angular-momentum states which entail the usual behavior of Eq. (3.29). Finally, for the S wave, Eq. (3.32) clearly reduces to the expression obtained by Khuri and Pais (1964), with $\nu=(m-2)^{-1}$.

There are several potentials which are given by a sum of inverse powers with arbitrary coefficients which are exactly solvable at $k=0$. These include

$$V(r)=(g/r^{2+2\tau})+(f/r^{2+\tau}), \quad (3.33a)$$

$$V(r)=(g/r^4)+(f_1/r^3)+(f_2/r^2), \quad (3.33b)$$

$$V(r)=(g/r^6)+(f_1/r^5)+(f_2/r^4). \quad (3.33c)$$

These are discussed in turn.

The first of these potentials was studied by Pais and Wu (1964a), whose particular interest was to determine the analytic properties of the scattering length in the two coupling constants g and f. As usual, g must be positive, while f may be of either sign. We follow here the paper of Pais and Wu who considered only the S wave; however the solution is later generalized to all partial waves. Furthermore, we restrict ourselves to $\tau>1$ in the present discussion in order that the usual behavior for low-energy phase shift be obtained. The Schrödinger equation is

$$(d^2u/dr^2)-[(g/r^{2+2\tau})+(f/r^{2+\tau})]u=0. \quad (3.34)$$

We set

$$u(r)=e^{-y/2}\varphi(y), \qquad y=2g^{1/2}/\tau r^{\tau}.$$

Then $\varphi(y)$ satisfies the equation

$$y(d^2\varphi/dy^2)+[(1+1/\tau)-y](d\varphi/dy) -\tfrac{1}{2}[1+1/\tau+f/(g^{1/2}\tau)]\varphi=0. \quad (3.35)$$

We put

$$c=1+1/\tau, \qquad a=\tfrac{1}{2}[1+1/\tau+f/(g^{1/2}\tau)], \quad (3.36)$$

so that Eq. (3.35) reads

$$y(d^2\varphi/dy^2)+(c-y)(d\varphi/dy)-a\varphi=0. \quad (3.37)$$

This is the standard form of the confluent hypergeometric equation having the solution (Bateman, Vol. 1, p. 258)

$$\varphi(y)=\alpha_1\Psi(a, c; y)+\alpha_2 e^y\Psi(c-a, c; -y). \quad (3.38)$$

We write the solution of this equation in terms of the Ψ functions for convenience in later use, since these functions exist for all values of the parameter c. If c is not an integer or zero, which is the case with $\tau>1$, the Ψ function may be expressed in terms of another set of functions $\Phi(a, c; y)$ by the relation [Bateman, Vol. 1, p. 257, (7)]

$$\Psi(a, c; y)=[\Gamma(1-c)/\Gamma(a-c+1)]\Phi(a, c; y) +[\Gamma(c-1)/\Gamma(a)]y^{1-c}\Phi(a-c+1, 2-c; y), \quad (3.39)$$

where the Φ functions have the familiar small y expansion [Bateman, Vol. 1, p. 248, (1)]

$$\Phi(a, c; y)=1+\frac{a}{c}\frac{y}{1!}+\frac{a(a+1)}{c(c+1)}\frac{y^2}{2!}+\cdots. \quad (3.40)$$

The physical solution satisfying $u(0)=0$ is obtained by setting $\alpha_2=0$, since the Ψ function, for r small or y large, has the asymptotic behavior [Bateman, Vol. 1, p. 278 (1)]

$$\Psi(a, c; y) \underset{y\to\infty}{\approx} y^{-a}. \quad (3.41)$$

The wave function may be written

$$u(r)=\exp(-g^{1/2}/\tau r^{\tau})\left\{\frac{\Gamma(-1/\tau)}{\Gamma\{\frac{1}{2}[1-(1/\tau)+(f/g^{1/2}\tau)]\}} \times\Phi\left[\frac{1}{2}\left(1+\tau^{-1}+\frac{f}{g^{1/2}\tau}\right), 1+\tau^{-1}; \frac{2g^{1/2}}{\tau r^{\tau}}\right] + \frac{\Gamma(1/\tau)}{\Gamma\{\frac{1}{2}[1+(1/\tau)+(f/g^{1/2}\tau)]\}}\left(\frac{2g^{1/2}}{\tau}\right)^{-1/\tau} \times r\Phi\left[\frac{1}{2}\left(1-\tau^{-1}+\frac{f}{g^{1/2}\tau}\right), 1-\tau^{-1}; \frac{2g^{1/2}}{\tau r^{\tau}}\right]\right\}. \quad (3.42)$$

Using Eq. (3.40) to determine the asymptotic behavior of the wave function for large r, and recalling Eq. (3.31), one finds the S-wave scattering length to be

$$A_0=\left(\frac{2g^{1/2}}{\tau}\right)^{1/2}\frac{\Gamma(-1/\tau)}{\Gamma(1/\tau)} \times\frac{\Gamma\{\frac{1}{2}[1+(1/\tau)+f/(g^{1/2}\tau)]\}}{\Gamma\{\frac{1}{2}[1-(1/\tau)+f/(g^{1/2}\tau)]\}}. \quad (3.43)$$

It is easily seen that, for $f=0$, this expression reduces to Eq. (3.32) for the pure inverse power [$m=2\tau+2$, $\nu=(m-2)^{-1}=(2\tau)^{-1}$]:

$$A_0(f=0)=(4\nu g^{1/2})^{2\nu}[\Gamma(-2\nu)/\Gamma(\tfrac{1}{2}-\nu)] \times[\Gamma(\tfrac{1}{2}+\nu)/\Gamma(2\nu)] =-(\nu g^{1/2})^{2\nu}[\Gamma(1-\nu)/\Gamma(1+\nu)], \quad (3.44)$$

where we have used the identity [Bateman, Vol. 1, p. 5 (15)]

$$\Gamma(2\nu)=2^{2\nu-1}\pi^{-1/2}\Gamma(\nu)(\nu+\tfrac{1}{2}).$$

The analytic properties of A_0 can be seen at once[32] from Eq. (3.43). The function $1/\Gamma(z)$ is an entire function of z, with simple zeros at $z=0, -1, -2, \cdots$. Thus A_0 cannot be analytic in g because of the factor $g^{1/(2\tau)}$. However, it is analytic in f about $f=0$ in a circle limited by the first pole arising from

$$\Gamma\{\tfrac{1}{2}[1+(1/\tau)+f/(g^{1/2}\tau)]\};$$

its radius is given by

$$|f|<g^{1/2}(1+\tau).$$

When g is negative (attractive potential), the scattering length is not well defined; when g is positive, one may use Stirling's formula for the Γ function of large argument to obtain an asymptotic expansion of A_0 about $g=0$.

The potential

$$V(r)=(g/r^4)+(f_1/r^3)+(f_2/r^2), \qquad (3.33b)$$

where g, f_1, and f_2 are arbitrary (but $g>0$), is clearly solvable for all partial waves, since we may set $f_2=l(l+1)$. This potential was, in fact, studied by Predazzi and Regge (1962) precisely to determine the analytic properties of the scattering amplitude in the angular-momentum plane in the case of a singular potential (see Sec. IV.A). The Schrödinger equation

$$\frac{d^2u}{dr^2}-\left(\frac{g}{r^4}+\frac{f_1}{r^3}+\frac{f_2}{r^2}\right)u=0, \qquad (3.45)$$

under the transformation

$$u(r)=y^\rho e^{-y/2}\varphi(y), \qquad y=2\alpha/r,$$

reduces again to the confluent hypergeometric equation, Eq. (3.37), whose solution is given by Eq. (3.38). We must choose

$$\alpha=\pm g^{1/2},$$
$$\rho=-\tfrac{1}{2}\pm\tfrac{1}{2}(1+4f_2)^{1/2},$$

and the parameters a and c of the confluent hypergeometric equation are given by the expressions

$$c=1\pm(1+4f_2)^{1/2},$$
$$a=\tfrac{1}{2}\pm\tfrac{1}{2}(1+4f_2)^{1/2}\pm[f_1/(2g^{1/2})]. \qquad (3.46)$$

Despite the apparent ambiguity in signs, the physical solution is nevertheless unique. If we choose $\alpha=+g^{1/2}$, the boundary condition $u(0)=0$ is satisfied by setting $\alpha_2=0$, so that

$$u(r)=r^{\frac{1}{2}[1-(1+4f_2)^{1/2}]}\exp(-g^{1/2}/r)$$
$$\times\Psi\{\tfrac{1}{2}[1+(1+4f_2)^{1/2}+(f_1/g^{1/2})],$$
$$[1+(1+4f_2)^{1/2}];(2g^{1/2}/r)\}. \qquad (3.47)$$

Here we have taken the plus signs in Eq. (3.46) for a and c. However, from the identity [Bateman, Vol. 1, p. 257 (6)]

$$\Psi(a,c;x)=x^{1-c}\Psi(a-c+1,2-c;x),$$

$u(r)$ may also be written

$$u(r)=r^{\frac{1}{2}[1+(1+4f_2)^{1/2}]}\exp(-g^{1/2}/r)$$
$$\times\Psi\{\tfrac{1}{2}[1-(1-4f_2)^{1/2}+(f_1/g^{1/2})],$$
$$[1-(1-4f_2)^{1/2}];(2g^{1/2}/r)\}. \qquad (3.48)$$

This expression could have been obtained in place of Eq. (3.47) by choosing the minus signs in Eq. (3.46) for a and c. Similarly the choice $\alpha=-g^{1/2}$ would require $\alpha_1=0$ in Eq. (3.38) in order to obtain the physical solution of Eq. (3.47) or (3.48). Finally we mention that the solution is analytic in the parameters f_1, f_2, a, c, but not in g.[32]

With the foregoing solution available, one can now generalize the potential of Eq. (3.30a), considered by Pais and Wu (1964a), to all partial waves. In addition, the following technique allows one to generate further potentials that are exactly solvable at $k=0$. Suppose one can solve the Schrödinger equation

$$\frac{d^2u}{dr^2}-\left(\frac{g}{r^m}+\frac{f}{r^n}+\frac{\lambda^2-\frac{1}{4}}{r^2}\right)u=0. \qquad (3.49)$$

If we put

$$u(r)=(r/y)^{1/2}\varphi(y), \qquad y=r^{1/\tau}, \qquad (3.50)$$

then Eq. (3.49) becomes

$$\frac{d^2\varphi}{dy^2}-\left(\frac{\tau^2 g}{y^{\tau(m-2)+2}}+\frac{\tau^2 f}{y^{\tau(n-2)+2}}+\frac{\tau^2\lambda^2-\frac{1}{4}}{y^2}\right)\varphi=0, \qquad (3.51)$$

and the point $y=0$ corresponds to $r=0$. Thus, if we can solve the Schrödinger equation for the potential

$$V(r)=(g/r^m)+(f/r^n)+[(\lambda^2-\tfrac{1}{4})/r^2], \qquad (3.52)$$

we have the solution for the potential

$$\tilde{V}(y)=\frac{g'}{y^{\tau(m-2)+2}}+\frac{f'}{y^{\tau(n-2)+2}}+\frac{\lambda'^2-\frac{1}{4}}{y^2}, \qquad (3.53)$$

where

$$g'=\tau^2 g, \qquad f'=\tau^2 f, \qquad \lambda'=\tau\lambda. \qquad (3.54)$$

In particular, by choosing $m=4$, $n=3$ in Eq. (3.49), the potential Eq. (3.53) reduces simply to the potential Eq. (3.33a) considered by Pais and Wu but for arbitrary angular momentum, while the potential Eq. (3.52) is simply the potential of Eq. (3.33b) which was studied by Predazzi and Regge, and whose solution is given in Eq. (3.47) above. Thus, upon recalling the transformation of Eqs. (3.50) and (3.54), one finds that the wave function for the potential Eq. (3.53), which satisfies the Schrödinger equation Eq. (3.51) and the boundary condition $\varphi(0)=0$, is given by

$$\varphi(y)=y^{\frac{1}{2}-\lambda'}\exp[-g'^{\frac{1}{2}}/(\tau y^\tau)]$$
$$\times\Psi\left[\frac{1}{2}\left(1+\frac{2\lambda'}{\tau}+\frac{f'}{\tau g'^{\frac{1}{2}}}\right),1+\frac{2\lambda'}{\tau};\frac{2g'^{\frac{1}{2}}}{\tau y^{\tau'}}\right]. \qquad (3.55)$$

[32] We mention that one can determine immediately by Poincare's theorem that the wave function and scattering parameters of the potentials of Eq. (3.33) are analytic functions of all the coupling constants except for g, the coupling constant associated with the most singular term in the potential near the origin (see Sec. II.E).

It is readily verified that, for S-wave scattering for which $\lambda'=\frac{1}{2}$, Eq. (3.55) is identical to Eq. (3.42).

The Schrödinger equation for the third of the power potentials of Eq. (3.33c) is written (Vasudevan *et al.*, 1967)

$$\frac{d^2u}{dr^2}-\left(\frac{g}{r^6}+\frac{f_1}{r^5}+\frac{f_2}{r^4}\right)u=0. \tag{3.56}$$

We set

$$u(r)=r\exp\left[-\tfrac{1}{2}g^{1/2}r^{-2}-\tfrac{1}{2}(f_1/g^{1/2})r^{-1}\right]\varphi(y),$$
$$y=g^{1/2}[r^{-1}+(f_1/2g)]^2.$$

Then $\varphi(y)$ satisfies the confluent hypergeometric equation, Eq. (3.37) with

$$a=\tfrac{1}{4}-(f_1^2/16g^{3/2})+(f_2/4g^{1/2}),\qquad c=\tfrac{1}{2}. \tag{3.57}$$

The physical solution satisfying $u(0)=0$ is given by

$$u(r)=r\exp\left[-(\tfrac{1}{2})g^{1/2}(1/r^2)-\tfrac{1}{2}(f_1/g^{1/2})r^{-1}\right]$$
$$\times\Psi\{a,\tfrac{1}{2};g^{1/2}[r^{-1}+(f_1/2g)]^2\}. \tag{3.58}$$

The foregoing solution simplifies for the potential of Eq. (3.33c) with $g=0$. The resulting potential, now written as

$$V(r)=(g/r^5)+(f/r^4), \tag{3.33d}$$

has been discussed by Gale (1967) in connection with peratization, and by Vasudevan *et al.* (1967). The Schrödinger equation,

$$(d^2u/dr^2)-[(g/r^5)+(f/r^4)]u=0, \tag{3.59}$$

under the transformation

$$u(r)=r\varphi(y),\qquad y=g^{1/3}[(1/r)+(f/g)], \tag{3.60}$$

becomes

$$(d^2\varphi/dy^2)-y\varphi=0. \tag{3.61}$$

This is the Airy equation with the solution (National Bureau of Standards Handbook, pp. 446–7)

$$\varphi(y)=\alpha_1 Ai(y)+\alpha_2 Bi(y). \tag{3.62}$$

The Airy functions $Ai(y)$ and $Bi(y)$ are simply related to the modified Bessel functions of order $\pm\frac{1}{3}$ (cf. the National Bureau of Standards Handbook, p. 447). Using asymptotic expansions for the Bessel functions, we see that the physical solution satisfying $u(0)=0$ must have $\alpha_2=0$, so that

$$u(r)=rAi[g^{1/3}(1/r+f/g)]. \tag{3.63}$$

The scattering length is readily found to be

$$A_0=g^{1/3}\frac{Ai'(f/g^{2/3})}{Ai(f/g^{2/3})}. \tag{3.64}$$

Neither the wave function nor the scattering length is analytic in g about $g=0$, but both are analytic in f about $f=0$; the region of analyticity in f being limited by the occurrence of the first bound state for f sufficiently negative.

So far we have considered only those singular potentials which have as their dominant singularity a pure inverse power. We turn next to a class of potentials which are exponentially singular at the origin:

$$V(r)=g(e^{\lambda/r}/r^4)+(f/r^4),\qquad \lambda>0. \tag{3.65}$$

This potential is discussed by Newton (1966) and, with $\lambda=2$, by Aly, Riazuddin, and Zimerman (1964b) and by Calogero and Cassandro (1965) in connection with the peratization approximation (see Sec. IV.C). The S-wave Schrödinger equation is written

$$(d^2u/dr^2)-[g(e^{\lambda/r}/r^4)+(f/r^4)]u=0. \tag{3.66}$$

Put

$$u(r)=r\varphi(y),\qquad y=(2/\lambda)g^{1/2}e^{\lambda/2r}.$$

Then Eq. (3.66) is transformed into the Bessel equation Eq. (3.23), where

$$\nu=(2/\lambda)f^{1/2}.$$

The general solution is

$$u(r)=r\{\alpha_1 I_\nu[(2/\lambda)g^{1/2}e^{\lambda/2r}]+\alpha_2 K_\nu[(2/\lambda)g^{1/2}e^{\lambda/2r}]\}, \tag{3.67}$$

and the solution which satisfies $u(0)=0$ is obtained by setting $\alpha_1=0$. The scattering length is given by

$$A_0=g^{1/2}\frac{K_\nu'[(2/\lambda)g^{1/2}]}{K_\nu[(2/\lambda)g^{1/2}]}. \tag{3.68}$$

Two cases are of particular interest in connection with the application of this potential in the peratization program. When $f=\frac{1}{4}$, the scattering length becomes simply

$$A_0(f=\tfrac{1}{4})=-\tfrac{1}{2}-g^{1/2}; \tag{3.69}$$

when $f=0$, we have

$$A_0(f=0)=g^{1/2}\frac{K_0'[(2/\lambda)g^{1/2}]}{K_0[(2/\lambda)g^{1/2}]} \tag{3.70}$$

$$\underset{g\to 0}{\approx}\frac{\lambda}{\ln\,(g/\lambda^2)+2\gamma}, \tag{3.71}$$

where $\gamma=0.5772$ is the Euler–Mascheroni constant.

This class of potentials is especially interesting because, apart from the power potentials, it is the only potential for which the peratization program appears to be successful.

There is one further general class of singular potentials, those having a logarithmic singularity at the origin. There are three potentials of this type which are exactly solvable at $k=0$ (in addition, these are solvable for all partial waves). Let us consider the potential

$$V(r)=g_1[\ln\,(r^{-1})^p/r^2]+g_2[\ln\,(r^{-1})^q/r^2]. \tag{3.72}$$

This potential with $p=1$, and $g_2=0$ is discussed by Arbuzov, Filippov, and Khrustalev (1964) in connection with both renormalization (Sec. IV.C) and peratization (Sec. IV.D).

The Schrödinger equation for the lth partial wave for

the general potential of Eq. (3.72) is

$$(d^2u_l/dr^2)-\{g_1[\ln (r^{-1})^p/r^2]+g_2[\ln (r^{-1})^q/r^2]+[l(l+1)/r^2]\}u_l=0. \quad (3.73)$$

Under the transformation (Cornille and Predazzi, 1965b)

$$u_l(r)=r^{1/2}y^{-1}\varphi_l(y), \qquad y=[\ln (1/r)]^{-1},$$

one obtains

$$(d^2\varphi_l/dy^2)-\{(g_1/y^{p+4})+(g_2/y^{q+4})+[(l+\tfrac{1}{2})^2/y^4]\}\varphi_l=0. \quad (3.74)$$

This equation is nothing but the S-wave Schrödinger equation for the potential

$$\tilde{V}(y)=(g_1/y^{p+4})+(g_2/y^{q+4})+[(l+\tfrac{1}{2})^2/y^4].$$

With $p=2$, $q=1$, $g_1\neq 0$, and $g_2\neq 0$, $\tilde{V}(y)$ is the potential of Eq. (3.33d). Furthermore, the boundary condition $u_l(0)=0$ is equivalent to the boundary condition $\varphi_l(0)=0$. Hence, upon making the obvious substitutions, one obtains the physical solutions. For example, for $p=1$, $g_1\neq 0$, and $g_2=0$, one obtains from Eq. (3.63)

$$u_l(r)=r^{1/2}\left(\ln r^{-1}+\frac{(l+\frac{1}{2})^2}{g}\right)^{1/2}\times K_{1/3}\left[\frac{2g^{1/2}}{3}\left(\ln r^{-1}+\frac{(l+\frac{1}{2})^2}{g}\right)^{3/2}\right]; \quad (3.75)$$

while for $p=2$, $g_1\neq 0$, $g_2=0$, one has from Eq. (3.58)

$$u_l(r)=r^{1/2}\ln r^{-1}\exp[-\tfrac{1}{2}g^{1/2}\ln^2(r^{-1})]\times\Psi\left[\frac{1}{4}\left(3+\frac{(l+\frac{1}{2})^2}{g^{1/2}}\right),\tfrac{3}{2};g^{1/2}\ln^2 r^{-1}\right]. \quad (3.76)$$

We omit the explicit expression for the more general case ($p=2$, $q=1$) because of its complexity. The foregoing technique also allows one to obtain solutions for some nonsingular potentials. For example, if $p=-1$, and $q=-2$ in Eq. (3.72), the potential in Eq. (3.74) reduces to the potential of Eq. (3.33b) whose solution has been given above [Eq. (3.47)]. See Charap and Dombey (1964).

The foregoing discussion exhausts the potentials (known to us) exactly solvable at zero energy that may be written either as a single term or as the sum of terms, each term multiplied by a coupling strength which is independent of the other coupling strengths. There are certainly many potentials which are exactly solvable at $k=0$ which are written as sums of terms where the coupling strengths are related.[33] One potential of this type which has been discussed by Gale (1966a, b), and by Aly, Riazuddin, and Zimerman (1965) is

$$V(r)=[g(\ln^2 r/r^4)-g^{1/2}(1/r^3)]\theta(R-r). \quad (3.77)$$

[33] Some potentials of this kind are given by Vasudevan, Venkatesan, and Jagannathan (1967).

The S-wave Schrödinger equation has the solution

$$u(r)=\alpha_1 re^{F(r)}+\alpha_2 re^{F(r)}\int_R^r dx\,\frac{e^{-2F(x)}}{x^2},$$

where

$$F(r)=g^{1/2}[(\ln r+1)/r].$$

The condition $u(0)=0$ implies that $\alpha_2=0$. The exact scattering length is given by

$$A_0=g^{1/2}\ln R[1-g^{1/2}R^{-1}\ln R]^{-1}. \quad (3.78)$$

Another potential of this type may be generated from Eq. (3.33d) and is given by

$$V(r)=(g^{(2+m)/3}/r^4)[r^{-1}+(f/g)]^m.$$

The transformation of Eq. (3.60) reduces the S-wave Schrödinger equation to

$$(d^2\varphi/dy^2)-y^m\varphi=0,$$

and this equation has a solution in terms of Bessel functions:

$$\varphi(y)=y^{1/2}[\alpha_1 K_{1/(m+2)}\{[2/(m+2)]y^{(m+2)/2}\}+\alpha_2 I_{1/(m+2)}\{[2/(m+2)]y^{(m+2)/2}\}].$$

These potentials are not particularly interesting or useful because one cannot distinguish the functional dependence in the wave function or scattering parameters of the coupling strengths associated with the more singular and less singular terms in the potential. In particular, the potential of Eq. (3.77) has been used as an example of the peratization procedure applied to a logarithmically singular potential which is not transitionally singular (Gale, 1966). (The potentials $gr^{-2}\ln^p(1/r)$ are transitionally singular; see Sec. II.B.) However the interpretation of the results is not without ambiguities because of the above difficulty.

We conclude this section with a discussion of an interesting class of potentials for which only an approximate treatment is possible. The potentials

$$V(r)=gr^{-m}\ln r^{-1}, \qquad m>3,$$

have been studied by Wu (1964) who suggested an approximate expression for the S-wave scattering length. The Schrödinger equation

$$(d^2u/dr^2)-gr^{-m}\ln r^{-1}\,u=0,$$

under the transformation

$$u(r)=r\varphi(y),$$
$$y=(\tau r)^{-1},$$

becomes

$$(d^2\varphi/dy^2)-y^{m-4}(1+\epsilon\ln y)\varphi=0.$$

Here we have set

$$g\tau^{m-2}\ln\tau=1,$$
$$\epsilon=(\ln\tau)^{-1}.$$

This differential equation was treated perturbatively by Wu who found the S-wave scattering length to be

$$A=-(g\nu^2/\epsilon)^\nu[\Gamma(1-\nu)/\Gamma(1+\nu)]$$
$$\times\{1-\epsilon\nu^2[2\ln\nu+2-\psi(1+\nu)-\psi(1-\nu)]+O(\epsilon^2)\},$$

where $\nu=(m-2)^{-1}$, and $\psi(x)$ is the logarithmic derivative of the gamma function. The lowest-order result is simply the scattering length for the pure-inverse-power potential $g'r^{-m}$, where g' is to be interpreted as $g'=g/\epsilon$. This result of Wu is quite plausible. However, one aspect of the perturbative treatment of this problem was not clarified. The nature of the solution of the differential equation for $\varphi(y,\epsilon)$ for large y is qualitatively modified under a change of sign of ϵ. This suggests that the solution is not analytic in ϵ about $\epsilon=0$. Moreover, the second term in the solution for $\varphi(y,\epsilon)$, obtained by iteration, is not uniformly small with respect to the first term over the positive real axis. It is not clear to the reviewers whether the scattering length, obtained as an integral of the iterative solution (in ϵ) over the entire positive real axis, is indeed given by a valid asymptotic series.

TABLE II. Singular potentials solvable at zero energy. The general and/or physical solutions are given by the equation number in the Table. Discussion of these solutions is found in the text accompanying these equations.

Potential	Solution (general and/or physical)
$(g/r^m)+[l(l+1)/r^2]$	3.25, 3.26
$(g/r^{2+2\tau})+(f/r^{2+\tau})+[l(l+1)/r^2]$	3.38, 3.42, 3.55
$(g/r^4)+(f_1/r^3)+(f_2/r^2)$	3.38, 3.47
$(g/r^6)+(f_1/r^5)+(f_2/r^4)$	3.38, 3.58
$(g/r^5)+(f/r^4)$	3.62, 3.63
$g(e^{\lambda/r}/r^4)+(f/r^4)$	3.67
$g_1(\ln^2 r^{-1}/r^2)+g_2(\ln r^{-1}/r^2)+[l(l+1)/r^2]$	3.38, 3.75, 3.76

B. Calculational Techniques

1. Matching Methods

When an exact solution to a quantum-mechanical equation is not known, one generally expands in terms of the solutions to a solvable "part" of the problem. Such a procedure is naturally suggested for the study of the energy dependence of phase shifts for power potentials, in view of the known exact solutions to the $k=0$ case (see last section). In this way, the regular solution $u(r,k)$ for power potentials can be written as the solution to the Volterra integral equation

$$u(r,k)=r^{1/2}K_\nu(\beta r^\sigma)+k^2\int_0^r dr'G(r,r')u(r',k), \quad (3.79)$$

where $K_\nu(\beta r^\sigma)$ is the regular solution at zero energy [Eq. (3.25), with $\alpha_2=0$], and

$$G(r,r')=\sigma^{-1}(rr')^{1/2}$$
$$\times[K_\nu(\beta r^\sigma)I_\nu(\beta r'^\sigma)-K_\nu(\beta r'^\sigma)I_\nu(\beta r^\sigma)], \quad (3.80)$$

in the notation of Sec. III.A.2. Iteration of this integral equation will lead to a convergent expression for $u(r,k)$ in the form of a power series in k^2. This expression is not very useful as it is extremely difficult to determine the phase of the asymptotic oscillating r dependence of $u(r,k)$ from such a representation.

One may, on the other hand, set up integral equations which are based on $\tilde{G}_l(r,r';k)$, the free particle (with angular momentum l) Green's function, and which determine wave functions based on a boundary condition at $r=\infty$. For example, the integral equation for the Jost solution is written[34]

$$f_l(k,r)=krh_l^{(-)}(kr)$$
$$+gk\int_r^\infty dr'rr'[h_l^{(-)}(kr)j_l(kr')-h_l^{(-)}(kr')j_l(kr)]$$
$$\times r'^{-m}f_l(k,r'). \quad (3.81)$$

The physical solution can be constructed from this function. This integral equation may be iterated to give a convergent series for $r\neq 0$, but gives a divergent quantity at $r=0$. A similar situation obtains with regards to solution of the variable-phase equation (Sec. III.B.3) where solutions which can be constructed for small (large) r prove useless when extended to large (small) r. One copes with this situation by constructing solutions beginning from both extremes and matching them at some intermediate point, frequently the classical turning point or some nearby point. We refer to these as "matching methods." One generally expands the solutions in terms of a parameter, e.g., the dimensionless parameter $\chi=g^{1/m}k^{1-2/m}$ for the power potentials, and matches up to a given order in the parameter to find an expansion of the physical quantity ($\tan\delta$ or $e^{2i\delta}$) to that order.

The method was applied by Jabbur (1965) to study the high-energy limit of scattering for S-wave power potentials. The two integral equations for the physical wave function, Eq. (3.79), and an equation based on Eq. (3.81), were considered, and solutions were presented which were purported to be correct to the neglect of terms of relative order χ^{-1}. They were matched as to value and slope at the classical turning point. The mechanics of solution were not exhibited. In fact, the

[34] An alternative integral equation may be written based on the Green's function for the $l=0$ case, where the potential $l(l+1)r^{-2}+gr^{-m}$ appears in the kernel, see e.g., Giffon and Predazzi (1964).

[35] One is aware that a higher-order term in an expansion based on a parameter may become very large for special values of the argument and therefore dominate. It would therefore not be correct to extrapolate a WKB solution for large values of energy (where it ought to be good) out to a turning point.

solutions found represent an approximate form of WKB solution (see end of Sec. III.B.2), corresponding to the neglect of terms which are really $O(\chi)$. The results do not agree with those of other calculations (Sec. II.D.1), but seem nevertheless to be a good approximation (Table I). The matching seems to have been an attempt to continue a WKB solution through a turning point in an approximation in which the critical behavior disappears.[35] This example illustrates the pitfalls inherent in matching expressions which are improperly expanded.

Bertocchi, Fubini and Furlan (1965b), and del Giudice and Galzenati (1965a,b) have derived the low-energy expansion of the phase shift by a matching approach. They availed themselves of a symmetry of the radial wave equation to relate the regular solution for small r to the Jost solution for large r. Thus by calculating only one of these solutions (the Jost solution), they automatically obtained the other to the same accuracy. They found $\tan\delta$ from the Wronskian of these two solutions. The radial wave equation for power potentials with energy and angular momentum has the form

$$(d^2u/dr^2)+\{k^2-(g/r^m)-[(\lambda^2-\tfrac{1}{4})/r^2]\}u=0. \quad (3.82)$$

The quantity $\varphi(\rho)=r^{-1/2}u(r)$, in terms of the dimensionless variable $\rho=r/r_0$ $[r_0=(g/k^2)^{1/m}]$, obeys the equation

$$\frac{d^2\varphi}{d\rho^2}+\rho^{-1}\frac{d\varphi}{d\rho}+\left(\chi^2-\frac{\chi^2}{\rho^{2+2/\beta}}-\frac{\lambda^2}{\rho^2}\right)\varphi=0. \quad (3.83)$$

The notation $\beta=2/(m-2)$ has been used. The solution $\varphi^{(-)}(\rho)$ of Eq. (3.83), which corresponds to the Jost solution $u^{(-)}(r)$ of Eq. (3.82), behaves for large ρ like

$$\varphi^{(-)}(\rho)\sim\rho^{-1/2}\exp(-i\chi\rho), \quad (3.84)$$

while the solution $\varphi_{\rm reg}(\rho)$, which corresponds to the regular solution of Eq. (3.83) [Eq. (3.27)], behaves for small ρ like

$$\varphi_{\rm reg}(\rho)\sim^{1/(2\beta)}\exp(-\beta\chi\rho^{-1/\beta}). \quad (3.85)$$

The substitution $\xi=\rho^{-1/\beta}$ transforms Eq. (3.83) into the differential equation

$$\frac{d^2\varphi}{d\xi^2}+\xi^{-1}\frac{d\varphi}{d\xi}+\left(-\beta^2\chi^2+\frac{\beta^2\chi^2}{\xi^{2\beta+2}}-\frac{\beta^2\lambda^2}{\xi^2}\right)=0, \quad (3.86)$$

which shows Eq. (3.83) to be invariant under the substitutions

$$\rho\to\rho^{-1/\beta}, \quad \beta\to\beta^{-1}, \quad \chi\to-i\beta\chi, \quad \lambda\to\beta\lambda. \quad (3.87)$$

One finds that these substitutions transform the regular and Jost solutions into each other, leading to the identity

$$\varphi_{\rm reg}(\rho,\chi,\lambda,\beta)=\varphi^{(-)}(\rho^{-1/\beta},-i\beta\chi,\beta\lambda,\beta^{-1}). \quad (3.88)$$

By means of this expression, del Giudice and Galzenati were able to derive the terms of an expansion in χ of $\varphi_{\rm reg}$ from that for $\varphi^{(-)}$. Bertocchi *et al.* originally applied this technique to the case $m=4$ $(\beta=1)$, where the symmetry was well known, while del Giudice and Galzenati extended it to general powers. The matching was done through evaluation of the Wronskian at an arbitrary point r.

A matching technique was employed by Stanciu (1967) and Handelsman, Pao, and Lew (1968) in studying the low-energy expansion of the phase shift via the tangent equation, Eq. (3.112) of the variable-phase formalism. Again solutions for small and large r are feasible, but may not be extended to the complementary asymptotic limit. Stanciu and Handelsman *et al.* demonstrated the technique for the pure power case. They expanded the tangent function for small r in powers of k, and explicitly evaluated the two leading terms. They used slightly different techniques to solve the tangent equation in the region of large r. Stanciu linearized the nonlinear equation, while Handelsman *et al.* constructed a solution by iteration. Stanciu matched corresponding terms at the classical turning point, on the basis of the intuition that this would be the most logical place at which to match solutions. However, he found that changing this point somewhat does not affect the results. Handelsman *et al.* proved that the two expansions have a large region of overlap. Both results agree.

2. *The WKB Method*

The most commonly applied technique in the study of singular potentials is the familiar semiclassical or WKB method.[36] As the method is discussed in virtually every quantum mechanics text, it is not necessary to present the fundamentals in this article, so much as to point out the reasons for its usefulness as well as some lesser known aspects regarding its applicability to singular potentials. We are concerned with the method only as applied to the solution of partial-wave radial equations. The method is well suited to the study of effects which depend on the singular nature of the potential, as the criterion for the validity of the approximation is well satisfied in the neighborhood of the singularity. The interest of physicists in singular potentials derives to a large extent from considerations in high-energy physics, and it is fortunate that the high-energy regime validates WKB calculations, not only in the neighborhood of small r, but over the entire range $r=0$ to ∞, provided the potential is smooth. The method is also good for high angular momenta, even at low energy, and is well suited to the study of large complex angular momenta (Jaksic and Limic, 1966a, b). It is not applicable as a low-energy calculational technique.

[36] The approximation known in the physical literature as the WKB method is also known as the JWKB method (J for Jeffreys) and as the Liouville–Green method. The name WKB approximation sometimes refers only to the connection formula at a turning point. A history of the approximation is presented in the first chapter of Heading's monograph (1962).

The criterion for the applicability of the WKB method is usually expressed as the requirement that the local (radial) wave length λ be slowly varying, i.e.,

$$|\, d\lambda/dr\,| = |\, (d/dr)[k^2 - V(r)]^{-1/2}\,| = |\, \tfrac{1}{2}V'(r)/[k^2 - V(r)]^{3/2}\,| \ll 1. \quad (3.89)$$

For the moment we disregard turning points. For large r, this condition is automatically satisfied if $k \neq 0$ and $V'(r) \to 0$. For large k, the criterion tends to be satisfied over intermediate ranges. For small r, the criterion is ideally satisfied for very flat potentials $[V'(0)=0]$ and, paradoxically, for singular potentials also.[37] For example, if

$$V(r) \underset{r\to 0}{\approx} gr^{-m}, \quad (3.90)$$

one finds

$$(d\lambda/dr) \underset{r\to 0}{\sim} g^{-1/2} r^{1/(2m)-1}, \quad (3.91)$$

which goes to zero for $m > 2$, and is small at $m = 2$ if g is large. The approximation is not valid near $r = 0$ for $0 < m < 2$. The correct behavior of the radial wave function for small r, for a singular repulsive potential, is easily calculated from the WKB expression

$$u(r) \sim u_{\mathrm{WKB}}(r) = [V(r)]^{-1/4} \times \exp\left[-\int_r^c dx V(x)^{1/2}\right], \quad (3.92)$$

($c > r$) which is independent of k and l. The irregular solution would have the positive sign in the exponential, and for an attractive potential (Sec. II.C) one would have imaginary exponents or their linear combinations (Case, 1950).

The criterion, $|\, d\lambda/dr\,| \ll 1$, one knows is necessarily violated in the neighborhood of a classical turning point, where $\lambda \to \infty$. It is well known that there is a prescription for continuing around such a point,[38] and converting in a unique fashion the oscillating solution on one side to the exponentially damped solution on the other. This prescription leads, for example, to the WKB approximation for the phase shift due to a repulsive potential with, say, one turning point,

$$\delta_{\mathrm{WKB}}(k) = \int_{r_0}^{\infty} dr \left(k^2 - V(r) - \frac{(l+\frac{1}{2})^2}{r^2}\right)^{1/2} - \int_{r_1}^{\infty} dr \left(k^2 - \frac{(l+\frac{1}{2})^2}{r^2}\right)^{1/2}$$
$$= \tfrac{1}{2}\pi(l+\tfrac{1}{2}) - kr_0 + \int_{r_0}^{\infty} dr \left[\left(k^2 - V(r) - \frac{(l+\frac{1}{2})^2}{r^2}\right)^{1/2} - k\right], \quad (3.93)$$

where $r_1 = (l+\frac{1}{2})/k$, and r_0 is the turning point, i.e., the point where a classical particle would have zero kinetic energy $[V(r_0) = k^2 + (l+\frac{1}{2})^2 r^{-2}$; see below for a discussion of the $(l+\frac{1}{2})^2$ term$]$.

We note certain aspects of the applications of the WKB method to scattering problems. As $k \to \infty$, r_0 clearly approaches zero; however, in the singular case, *unlike the regular case*, the quantity kr_0 does not go to zero as $k \to \infty$, and in fact its contribution is of the same order of magnitude as the integral. For $V(r) = gr^{-m}$, we find

$$kr_0 = g^{1/m} k^{1-2/m} \equiv \chi \underset{k\to\infty}{\to} \infty, \quad (3.94)$$

while the contribution of the integral in Eq. (3.93) is also proportional to χ. In the S-wave case, the quantity kr_0 is, in fact, just the S-wave phase shift due to an infinitely repulsive core extending out to $r = r_0$. Thus, apart from the constant term $\pi/4$ (in the S-wave case), the lowest-order or semiclassical approximation is classical in that it treats a classically forbidden region almost as an impenetrable core. There are corrections to the phase shift from higher-order WKB approximations. These depend on the behavior of the potential within the forbidden region only through the coefficients of a Taylor series expansion of the potential about the turning point (see Bertocchi *et al.*, 1965a, b, for an evaluation of some of these correction terms).

We call attention to the existence of two forms of the WKB approximation which we term the "standard" and "Langer" forms (see Bertocchi *et al.*, 1965a and Froman and Froman, 1965). These differ with regard to the treatment of the centrifugal barrier term in the potential. If the potential $V(r)$ in the WKB wave function of Eq. (3.92) should contain the centrifugal barrier term, the coefficient of this r^{-2} term in the Langer form would be $(l+\frac{1}{2})^2$, while in the standard form it would be the familiar $l(l+1)$. The original motivation for the Langer prescription was that the expression for the WKB physical wave function in the presence of a regular potential have the correct behavior r^{l+1}, while the expression corresponding to the irregular solution has the behavior r^{-l}. Langer (1937) has argued the necessity of this prescription for the case of regular potentials with angular momentum (including $l=0$) from the condition that the relative error term due to the WKB solution be small. To see this, we note that the WKB wave function $u_{\mathrm{WKB}}(r)$ of Eq. (3.92), with Q^2 replacing V, satisfies the differential equation

$$d^2 u_{\mathrm{WKB}}(r)/dr^2 + [Q^2(r) - \omega(r)] u_{\mathrm{WKB}}(r) = 0, \quad (3.95)$$

where

$$Q^2(r) = k^2 - V(r) - [l(l+1)/r^2], \quad (3.96a)$$

$$\omega(r) = \tfrac{3}{4}\{[Q'(r)]^2/Q^2(r)\} - \tfrac{1}{2}[Q''(r)/Q(r)]. \quad (3.96b)$$

In order that the WKB solution of Eq. (3.92) be a satisfactory solution to Eq. (3.95) with the ω term absent, it is at least required that the relative error

[37] It is also always assumed that the singular potentials approach their singularity monotonically within some small neighborhood of $r = 0$. The situation is expected to be different when the potential oscillates infinitely rapidly near $r = 0$.

[38] Connection formulas are presented in standard texts, e.g., Schiff (1955), Sec. 28, or Landau and Lifshitz (1960), Sec. 47. More rigorous treatments can be found in Heading (1962), Chap. 3.4, Froman and Froman (1965), or Bertocchi *et al.* (1965a).

term satisfy

$$\omega(r)/Q^2(r)\ll 1. \tag{3.97}$$

This leads to a more refined criterion than the condition of Eq. (3.89). For regular potentials with angular momentum, this quantity is not small. Langer has shown that, by means of the variable changes $r=e^x$, $u=e^{x/2}\varphi$, which transform the interval $0<r<\infty$ to the interval $-\infty<x<\infty$, the relative error in terms of x does become small. This transformation gives rise to the Langer prescription $l(l+1)\rightarrow(l+\frac{1}{2})^2$ (see Langer, 1937, or Morse and Feshbach, 1953). The Langer prescription leads to Eq. (3.93) for the phase shift.

The reasoning in the case of singular potentials is different. Unlike the situation for regular potentials, both the standard and Langer terms for the WKB wave function in Eq. (3.92) give the correct behavior at the origin, since the replacement $l(l+1)\rightarrow(l+\frac{1}{2})^2$ does not affect the behavior there. This can be seen, for example, for the pure inverse power potential $V(r)=gr^{-m}$ from Eq. (3.27). In addition, one notes that for singular potentials, the relative error term ω/Q^2 is vanishingly small at $r=0$. For the pure inverse power potential, the relative error is

$$\omega/Q^2 \underset{r\rightarrow 0}{\approx} [m(m-4)/16g]r^{m-2}, \tag{3.98}$$

which vanishes identically at the origin, so that the criterion is ideally satisfied. The two forms for the WKB wave function generally differ for large r, since the Langer form saddles the potential with an additional $(4r^2)^{-1}$ tail. One may note that for $m=4$ with $k^2=0$ and $l=0$, the quantity ω vanishes identically, and the standard form gives the exact solution for all r. One in fact finds that both forms have been used for the wave function in the study of singular potentials.

The Langer form is nevertheless to be preferred for the WKB expression for the phase shifts, Eq. (3.93), in the case of singular potentials. The expression of Eq. (3.93) is obtained by taking the difference of the phase of the wave function for large r, in the presence and absence of the potential $V(r)$. The Langer form is necessary in the second term of Eq. (3.93) to represent the phase of the wave function in the absence of the singular potential, but in the presence of the centrifugal barrier. The condition that the WKB phase shift vanish in the absence of an interaction implies the Langer form in the first term of Eq. (3.93).

For the high l behavior there is very little difference between the two forms. Limic (1962), in fact, has employed the standard form in his study of the large l behavior of the S-matrix, while Paliov and Rosendorff (1967) used the Langer form. For large k, the leading term, which is independent of l, can be found from either form. However, higher-order terms depend on l and, as mentioned above, the Langer form is to be preferred, at least for nonzero values of l.

For singular potentials in the case of S-waves there appears to be no clear-cut choice between the two forms. The wave function in the absence of interaction does not require the Langer form; furthermore the standard form is preferable at the origin. The S-wave WKB phase shift in the standard form is given by

$$\delta_{\mathrm{WKB}}(k)=-kr_0+\tfrac{1}{4}\pi + \int_{r0}^{\infty} dr\{[k^2-V(r)]^{1/2}-k\}. \tag{3.99}$$

This expression, however, does not vanish at $g=0$, and hence is valid for large k with $g\neq 0$.

Higher-order corrections to the WKB approximation have been considered in connection with singular potentials. Schemes for calculating higher-order terms generally have been given by Bertocchi, Fubini, and Furlan (1965a). They found corrections to the leading WKB contribution for singular power potentials at high energy in the form of an asymptotic series in negative odd powers of the dimensionless parameter χ (see Sec. II.D.1). Limic (1962), and Jaksic and Limic (1966a, b), have written integral equations for the Jost solutions to singular potential radial equations (for real and complex l) in terms of the Green's functions constructed from the exact solution to a suitably chosen equation of the form of Eq. (3.95). In this way, they constructed solutions with WKB-type leading terms and bounds on the corrections to the leading terms.

Tiktopoulos (1965) has studied scattering by a power-type singular potential with a complex energy-dependent coupling constant. He showed that the WKB approximation provides an asymptotic form for the phase shift in certain restricted regions of the complex $\lambda=l+\frac{1}{2}$, g, or k planes, which include the positive real axes in these variables. He found the scattering amplitude in the high-energy strong coupling limit from the WKB values of the phase shifts for complex l, by applying the Watson–Sommerfeld transformation in which the contour integral is estimated by steepest descent techniques. Error bounds on the WKB expressions for the phase shift are determined (see Sec. V.B).

In addition, one also finds that approximations have been made on the WKB expressions. Paliov and Rosendorff (1967) wished to determine the large real l behavior of the phase shift for large k in order to sum the partial-wave series approximately. They therefore expanded the WKB expression for the partial-wave phase shift into two complementary expansions, one appropriate for l^2/gk small (the "λ representation"), and the other for l^2/gk large (the "ρ representation"). In essence, they distinguished between the situations when the turning point is determined by the centrifugal barrier, and when it is determined by the singular potential (see Sec. II.D).

Jabbur (1965) has employed a matching technique intended as appropriate to power potentials in the high-energy limit (see Sec. III.B.1). It is in fact an

approximation on the WKB technique. Thus the asymptotic solution for large $\chi \equiv g^{1/m}k^{1-2/m}$ in Eq. (23) of his paper for the region of small r is merely the WKB solution. The approximations $[V(r)-k^2]^{1/2} \approx V^{1/2}(r) - \frac{1}{2}k^2V^{-1/2}(r)$ and $[k^2-V(r)]^{1/2} \approx k-(2k)^{-1}V(r)$ in the exponents in his Eq. (24) give his asymptotic (in k) solution for large r.

The reader interested in some of the more rigorous and detailed aspects of the WKB approximation may consult the recent monographs by Heading (1962), and Froman and Froman (1965).

3. *The Variable-Phase Method*

A very useful method in scattering theory which is well suited to singular potential investigations is the variable-phase method. Some reviews of this method can be found in Calogero (1963) and Babikov (1967), as well as in Calogero's well written monograph *Variable Phase Approach to Potential Scattering* (1967). The quantity one deals with in this method is the variable phase $\delta_l(\rho, k)$, which represents the phase shift at energy k^2 due to the potential $V(r)\theta(\rho-r)$ [$\theta(x)$ is the step function which vanishes for $x<0$ and is unity otherwise]. It follows by definition that $\delta_l(0, k)=0$, and $\delta_l(\infty, k)=\delta_l(k)$ which is the lth partial-wave phase shift. It obeys a first-order, nonlinear differential equation which for the $l=0$ case is

$$d\delta_0(r, k)/dr = -k^{-1}V(r)\sin^2[kr+\delta_0(r, k)]. \quad (3.100)$$

For nonsingular potentials, the boundary condition $\delta_0(0, k)=0$ determines a unique solution. The phase shift described by Eq. (3.100) is automatically continuous in k, and is zero at $V(r)=0$, and thus has no mod π ambiguity. For higher partial waves, the variable-phase equation has the form

$$d\delta_l(r, k)/dr = -k^{-1}V(r)\hat{D}_l^2(kr) \times \sin^2[\hat{\delta}_l(kr)+\delta_l(r, k)], \quad (3.101)$$

where $\hat{D}_l(x)$, $\hat{\delta}_l(x)$ are, respectively, the modulus and phase of $-xn_l(x)+ixj_l(x)$ [$j_l(x)$, $n_l(x)$ are the familiar spherical Bessel functions]. Forms and properties of these functions can be found in Appendix I of Calogero (1967). One can also write a nonlinear first-order differential equation for various functions of the variable phase.[39] For example, $T_0(r, k)=\tan\delta_0(r, k)$ obeys the "tangent equation"

$$dT_0(r, k)/dr = -k^{-1}V(r)[\sin kr+T_0(r, k)\cos kr]^2, \quad (3.102)$$

and $T_0(0, k)=0$. Similar equations can be written for $T_l(r, k)=\tan\delta_l(r, k)$, $S_l(r, k)=\exp 2i\delta_l(r, k)$, and $A_l(r, k)=+(2i)^{-1}[S_l(r, k)-1]$.[39] An equation for the "variable scattering length" can also be written. For S waves,[40] one has

$$A'(r) = -V(r)[r+A(r)]^2. \quad (3.103)$$

These equations have proven fruitful in theoretical and numerical applications of scattering theory,[41] and have certain advantages as well as disadvantages as compared to the straightforward Schrödinger equation approach. Some advantages of the method are ease of interpretation, convenience for generation of approximations and recursive procedures, and determination of bounds on the error, for all ranges of energy, coupling constant, and angular momentum. It is useful for numerical computation in certain cases, but not when the potential is strongly repulsive, though bounds are available. Levy and Keller (1963) derived the low-energy behavior of regular potentials very conveniently by this method.

The variable-phase equation has also proven useful in the study of singular potentials, and is suited to investigation of both high- and low-energy behaviors. Here, the boundary condition $\delta_l(0, k)=0$ in fact does not uniquely specify a solution. A unique solution is obtained from the condition $\delta_l(r, k) \geq -\hat{\delta}_l(kr)$, where $\hat{\delta}_l(kR)$ is the phase shift for a hard core potential of radius R. One finds that $\delta_l(r, k)$ for small r is expressible as a sum of terms,[42] where each term vanishes more rapidly near $r=0$ than the succeeding term. For the $l=0$ case, one finds

$$\delta_0(r, k) = -kr+k[V(r)]^{-1/2} + \tfrac{1}{4}kV'(r)V(r)^{-2}+\cdots. \quad (3.104)$$

This expression has been used to calculate the high-energy behavior of the phase shift, which is determined by the singularity of the potential at $r=0$ (Sec. II.D.1). For the leading high-energy behavior, one finds

$$\delta_0(k) \underset{k\to\infty}{\approx} -k^{-1}\int_0^\infty drV(r)\sin^2\{k[V(r)]^{-1/2}\}. \quad (3.105)$$

Note that the integral always converges if condition (a) is satisfied. Calogero has shown[43] that Eq. (3.105) gives, in fact, the leading high-energy behavior for all l. One notes, however, that the first two (or even three) terms of Eq. (3.104) clearly do not represent the variable-phase function for large r, since $\delta_l(\infty, k)$, the phase shift, must be finite. Calogero (1964, 1967) has suggested an approximate form for the phase function in place of Eq. (3.104),

$$\delta_0(r, k) = -kr+kr[1+rV(r)^{1/2}]^{-1}, \quad (3.106)$$

which, in turn, leads to the expression for the S-wave

[39] Calogero (1967), Chap. 3, Eqs. (10), (19a), and (19b).

[40] Calogero (1967), Eq. (9) of Chap. 11. The variable scattering length for general l is given in Eq. (7).

[41] Bibliographies may be found in Calogero (1967) and Babikov (1967). An approximate guide to the bibliography by topic is given in Chap. 1, and on p. 238 of Calogero (1967).

[42] Calogero (1964), Sec. II, and Chap. 15 of the book.

[43] Appendix II of Calogero (1964).

phase shift

$$\delta_0(k)=-k^{-1}\int_0^\infty drV(r)\times\sin^2\{kr[1+rV(r)^{1/2}]^{-1}\} \quad (3.107)$$

as an approximation appropriate at all energies for $l=0$. A corresponding expression is constructed for higher partial waves.[44] The expression of Eq. (3.106) reproduces the first two terms of Eq. (3.104) for small r, and agrees in order of magnitude with the third term for power potentials, but has little to recommend it for large r. It might be expected to have some validity for potentials with long tails, since $\delta_0(r, k)$ tends to grow like $-kr$ in strongly repulsive regions. Equation (3.107) leads to the expression[45]

$$A_0=-\int_0^\infty \frac{drr^2V(r)}{[1+rV^{\frac{1}{2}}(r)]^2} \quad (3.108)$$

for the S-wave scattering length. It gives satisfactory numerical agreement[46] which, for power potentials, is best for small values of m where the potential tail is longest.

The low-energy behavior of phase shifts, as was mentioned earlier (Sec. II.D), is not properly an aspect of singular potentials. Additional mathematical considerations, however, are necessary in the study of the low-energy behavior for singular potentials. Stanciu (1967) and Handelsman, Pao, and Lew (1968) have both employed the tangent equation to derive the low-energy expansion for repulsive singular potentials. The method solves the tangent equation in the domains of small and large r by recursive methods. For small r, the solution is expanded in powers of k as in Levy and Keller (1963). For power potentials, the r-dependent coefficients of powers in k can be constructed in terms of the known zero-energy wave function, Eq. (3.26). The nonlinear tangent equation for large r is solved by Stanciu using a sequence of linearizations and by an iterative method in Handelsman *et al.* The coefficients of like powers of k ($\ln k$ terms may also be present) are matched at the classical turning point[47] to give a low-energy expansion of the phase shift. Both authors agree in their results, although the treatment of Stanciu is more concise. Both claim that the method can be extended to nonpower potentials. The results are in agreement with other calculations (Bertocchi *et al.* 1964b; del Giudice and Galzenati, 1965a, b), except for a k^{2m-4} term (see Sec. II.D).

Calogero and Cassandro (1964) have shown from the S-wave variable-phase equation that the coefficients of the Born series for the S-wave scattering length for potentials of the form[48] $g\ln^p(r^{-1})/r^2$ are finite though the series is only asymptotic. Thus upon making the substitutions $A(r)=rB(r)$, $\ln r^{-1}=x$ in Eq. (3.103) one finds

$$(dB/dx)-B=x^p(1+B)^2, \quad (3.109)$$

which is formally solvable in a power-series expansion with finite coefficients.

4. *Jost Function Methods*

The Jost function can be utilized for the calculation of partial-wave scattering by singular repulsive potentials, just as for regular potentials. There are, however, distinct features characteristic of the singular case which call for detailed consideration. The Jost function may be defined in terms of the Wronskian of the regular solution and the Jost solution $f(k, r)$. For regular potentials, the $l=0$ Jost function is identical with the value of the Jost solution at $r=0$. For $l>0$, the Jost solution is infinite at $r=0$, but the Jost function is expressed as the value at $r=0$ of the ratio of the Jost solution to the singular scaling function $(2l-1)!!(kr)^{-l}$.

In the case of potentials which are repulsive and singular at $r=0$, the Jost function is likewise expressible as the value at $r=0$ of the ratio of the Jost solution to an appropriate energy-independent singular scaling function $Z(r)$. It is possible to determine this scaling function to within a positive constant factor, and thus determine the Jost function to within a trivial normalization. This point of view has been extensively dealt with in a series of articles by Cornille and coworkers who have made this the basis of certain approximation techniques. They have considered an "effective" Jost solution $F(k, r)$ defined by

$$F(\pm k, r)=f(\pm k, r)/Z(r), \quad (3.110)$$

for which they constructed iterative approximations, either by Laplace transform techniques or by solving the integral equations in coordinate space. These iterations generally do not result in finite approximate values of $F(k, 0)$, and Cornille and coworkers have devised the method of "limiting dependences" to obtain finite converging approximations from the iterations.

Let $f(\pm k, r)$ be the Jost solutions[49] corresponding to the asymptotic behavior $\exp(\mp ik, r)$, respectively, for large r. The behavior of $f(\pm k, r)$ near $r=0$ is specified by the WKB approximation

$$f(\pm k, r)\sim V^{-1/4}(r)\exp\left\{+\int_r dr'[V(r')]^{1/2}\right\}. \quad (3.111)$$

[44] Equation (6.7) of Calogero (1964), and Chap. 15, Eq. (23), of the book.

[45] Equation (6.8) of Calogero (1964). Note that Eq. (26) in Chap. 15 of the book contains a typographical error.

[46] Calogero (1964), Table I and Fig. 1, p. 110, of the book.

[47] Stanciu chooses r_0, the classical turning point, as a "logical" connection point, but does not prove the validity of both expansions there, though he indicates that connection at $\lambda r_0(\lambda\approx 1)$ yields the same results. Handelsman *et al.* prove the validity of the small-r expansion up to $\chi^{-1}r_0$, and beyond $\chi^{2l(m-2)}r_0$, which for small χ provides generous overlap.

[48] Calogero and Cassandro (1964) consider only $p=1$, but the result is clearly valid for integral and nonintegral positive p (See Frank and Land d).

[49] We follow Jost's convention of defining the Jost solution $f(k, r)$ by the asymptotic behavior e^{-ikr} for large r. The opposite convention is employed by Newton (1966). Most of the present discussion is limited to $l=0$ partial waves. The considerations for higher l values are virtually identical in principle, and are presented in Cornille (1964, 1965a). Most of the ensuing discussion is limited to $k\neq 0$, and the condition (a) is assumed to hold.

It is clear that one can determine an energy-independent $Z(r)$, with the property that $F(\pm k, 0)$ is finite. Such a factor is not unique, and it is convenient to impose on $Z(r)$ further conditions: i.e., that $Z(r)$ be positive and continuous for real r, and $Z(r)\to 1$ as $r\to\infty$. It is further convenient for practical purposes to have a manageable analytic form for $Z(r)$. The form

$$Z(r)=\exp\left[\int_\infty^r dr'\chi(r')\right]$$

has been suggested. Cornille (1964) has applied it to $V(r)$, expressible as a Laplace transform and possessing a r^{-2} singularity. Cornille and Predazzi (1965a, b) have constructed a function $Z(r)$ of this form, when the singular part of $V(r)$ can be expanded in terms of elementary functions. For the singular part of $V(r)$ expressible as a sum of negative powers of r, as powers of a logarithm times a negative power of r, as the exponential of a negative power of r, or as a sum of such terms, they showed how to construct $\chi(r)$ as a finite sum in terms of such singularities. Here $\chi(r)$ is not affected by the addition of any regular potential to the singular part. The quantity $F(k)\equiv F(k, 0)$ serves as an "effective" Jost function in that the S matrix for the partial wave is expressible as[50]

$$\begin{aligned}S(k)&=\lim_{r\to 0}[f(k, r)/f(-k, r)]\\&=F(k, 0)/F(-k, 0),\end{aligned}\qquad(3.112)$$

and the complex zeros of $F(-k)$ locate the bound states. One finds that the "effective" Jost solution $F(k, r)$ obeys the differential equation

$$\begin{aligned}&[(d^2/dr^2)+2\chi(d/dr)+k^2]F(k, r)\\&\quad=(V-\chi'-\chi^2)F(k, r)\equiv WF(k, r).\end{aligned}\qquad(3.113)$$

This differential equation has the merit that its solutions are finite at $r=0$. One may now seek approximation procedures for the solution of this differential equation to determine the value of the effective Jost function, i.e., the value of the effective Jost solution at $r=0$. Cornille and coworkers have prescribed two methods for the construction of solutions to Eq. (3.113) by successive approximation, the Laplace transform method and the coordinate space method.

Laplace Transform Method

The Laplace transform method applies when the potential is expressible as a sum of a singular part $V_1(r)$, and a regular part $V_2(r)$, which can be represented as Laplace transforms whose supports vanish when $0<\alpha<\mu$, i.e.,

$$V_j(r)=\int_\mu^\infty d\alpha e^{-\alpha r}\sigma_j(\alpha),\qquad(3.114)$$

where $\alpha^2\sigma_2(\alpha)\to 0$ as $\alpha\to\infty$ which, expresses the regularity of $V_2(r)$. Cornille (1964), and Cornille and Predazzi (1964, 1965a) have explicitly shown how this method can be conveniently applied to a $V_1(r)$ of the form

$$V_1(r)=e^{-\mu r}\sum_{j=1}^{\nu}a_j r^{-m_j}\qquad(m_j>m_{j+1}\geq 2).\quad(3.115)$$

Application of the Laplace transform method due to Martin (1961) leads to a Volterra integral equation for $\rho(k, \alpha)$, the inverse Laplace transform of $e^{ikr}F(k, r)$,

$$\begin{aligned}\rho(k, \alpha)=\delta(\alpha)+[\alpha(\alpha+2ik)]^{-1}\int_\mu^\infty d\beta\rho(k, \beta)\\\times[2(ik+\beta)\xi(\alpha-\beta)+\omega(\alpha-\beta)],\end{aligned}\qquad(3.116)$$

where

$$\chi(r)=\int_\mu^\infty d\alpha e^{-\alpha r}\xi(\alpha),\qquad(3.117a)$$

and

$$W(r)=\int_\mu^\infty d\alpha e^{-\alpha r}\omega(\alpha).\qquad(3.117b)$$

One may write the solution by iteration to Eq. (3.116) as

$$\rho(k, \alpha)=\delta(\alpha)+\sum_{n=1}^{\infty}\rho_n(k, \alpha),\qquad(3.118)$$

from which one can construct $F(k, r)$ for all $r>0$. The iterations display a range-order correlation where the term obtained from the nth iteration gives the exact value of $\rho(k, \alpha)$ for α in the interval $n\mu\leq\alpha<(n+1)\mu$. In the case in which the sum in Eq. (3.115) contains only one term, then, for $m_1<4$, the individual terms in the expansion of the effective Jost function,

$$\tilde{F}_n(k, 0)=\int d\alpha\rho_n(k, \alpha),\qquad(3.119)$$

are finite, and the sum over n is convergent. When $m_1\geq 4$, the individual terms $\tilde{F}_n(k, r)$ are divergent as $r\to 0$, the singularity becoming stronger with increasing n. In this case, truncation of the series of Eq. (3.118) would not even serve as an approximation for the effective Jost function. For $V_1(r)$ of the form of Eq. (3.115), it is possible to determine the leading behavior of $\rho(k, \alpha)$ for large α which determines the behavior of $F(k, r)$ for small r. When the sum in Eq. (3.115) consists of only a single term $m_1=m$, one finds the behavior of $\rho(k, \alpha)$ for large α to have the form (Cornille and Predazzi, 1965a; Cornille, Burdet and Giffon, 1965)

$$\exp\left(\sum_{i=1}^{[m/2]}c_i\alpha^{1-2i/m}+c\ln\alpha\right)\qquad(3.120)$$

$[m]$ denotes the integral part of m, and the c_i can be calculated. This behavior obtains for large α only when $m\geq 2$ as one would expect. One has $\mathrm{Re}\, c_1<0$ for $m<4$. Despite the divergence of $\tilde{F}_n(k, 0)$ of Eq. (3.119) for $m_1\geq 4$, Cornille and Predazzi (1965a) have demonstrated, by means of steepest descent estimates, the convergence of the effective Jost function when the

[50] This expression for the S-matrix, valid for repulsive singular potentials, is derived in Limic (1962), and Pais and Wu (1964b). See Sec. III.B.6.

large α behavior of $\rho(k, \alpha)$ is given as in Eq. (3.120). A knowledge of the large α behavior of $\rho(k, \alpha)$ might be used to estimate $F(k,0)$. Alternatively, Cornille (1965a), Cornille and Predazzi (1964, 1965a), and Cornille, Burdet, and Giffon (1965) have developed the method of limiting dependences to derive approximations to the effective Jost function from the iterative solution Eq. (3.118). This is discussed subsequently.

Coordinate Space Method

One may approach the calculation of $F(k, r)$ by considering the integral equation in coordinate space based on the Green's function for the differential operator $(d^2/dr^2)+k^2$:

$$F(k,r)=\exp(-ikr)+k^{-1}\int_r^\infty dr' \sin k(r-r') \times\left(-2\chi(r')\frac{dF(k,r')}{dr'}+W(r')F(k,r')\right). \quad (3.121)$$

For $l\neq 0$, one may alternatively treat the centrifugal potential in the potential term $W(r)$, or in the Green's function, as part of the inverted operator (Cornille, 1965a). We may attempt to solve these equations by iteration. For $m_1=2$, the individual iterations are finite at $r=0$, and iteration thus serves as an effective approximation procedure. For $m_1>2$, one finds that the individual iterations diverge at $r=0$, even though $F(k, 0)$ is finite. The degree of divergence as $r\to 0$ increases as $r^{-(m_1-2)n}$, where n is the order of iteration. This reflects the essential singularity of $F(k, r)$ at $r=0$. The iterations can be exploited to obtain approximate answers through the method of limiting dependences.

One can also attempt a calculation of $F(k, 0)$ by using Eq. (3.110), and iteratively solving the integral equation for the Jost solution (for $l=0$, say)

$$f(k,r)=\exp(-ikr) +k^{-1}\int_r^\infty dr' \sin k(r'-r)V(r')f(k,r'). \quad (3.122)$$

One would, however, find that any finite iterative approximation,

$$f^{(p)}(k,r)=\sum_{n=0}^{p} f_n(k,r),$$

to the solution of Eq. (3.122), would have the property $f^{(p)}(k, r)/Z(r)\to 0$ as $r\to 0$. Alternatively, one could employ the iterative approximations to Eq. (3.122) in the ratio given by the middle expression of Eq. (3.112), and go to the limit $r\to 0$. It can be shown that, although this gives a nontrivial answer, it would converge very badly to the true answer. A modification of such an approach which enables one to utilize finite orders of iteration to obtain good convergence to the finite value of $F(k, 0)$ or $S(k)$ is afforded by the method of limiting dependences. This method makes feasible the exploitation of finite orders of iteration for solution of any of the above-mentioned Eqs. (3.116), (3.121), and (3.122).

Method of Limiting Dependences

The finite quantity $F(k, 0)$ is expressible as a double limit

$$F(k,0)=\lim_{r\to 0}\lim_{p\to\infty}\sum_{n=0}^{p} F_n(k,r), \quad (3.123)$$

where $F_n(k, r)$ is the nth term of an expansion of $F(k, r)$ obtained by iteration. One would like to reduce this to a single limiting procedure, where a finite stage of the procedure provides a useful approximation. As we have seen, the finite sum

$$\sum_{n=0}^{p} F_n(k,0)$$

would not do. Instead one evaluates the truncated sum

$$G_p(k)=\sum_{n=0}^{p} F_n[k, r(p)] \quad (3.124)$$

at a point $r(p)$ which is chosen to correspond to the order of truncation p in such a way that

$$\lim_{p\to\infty} G_p(k)=\lim_{p\to\infty}\sum_{n=0}^{p} F_n[k, r(p)]=F(k,0). \quad (3.125)$$

In the situations considered in connection with Eq. (3.122), the iteration procedure starts with the zeroth-order Jost solution $\exp(-ikr)$ valid at asymptotically large values of r. Each successive iteration approximates the exact solution over a larger and larger range of r, extending inward from $r=\infty$; the accuracy improving for larger r. Thus a choice of an order of iteration p provides the solution with a given preassigned error down to some $r(p)$ which goes to zero as $p\to\infty$. There is an approximate "limiting dependence" $r_L(p)$ with the property that, if $r(p)$ in Eq. (3.125) is chosen so that $r(p)/r_L(p)\to\infty$ as $p\to\infty$, then Eq. (3.125) is valid. The same limiting dependences $r_L(p)$ are appropriate to the iterative solution either of Eq. (3.121) or Eq. (3.122), and would also apply to the numerator and denominator functions of a Fredholm solution (Cornille, 1965c). The limiting dependences $r_L(p)$ depend only on the most singular part of the potential. The dependence $r_L(p)$ has been worked out for a variety of singular potentials, those of the power type, the power type with logarithmic factors, and those with exponential type singularities (Cornille, 1965b). Cornille has worked out a numerical example to illustrate the convergence properties of this procedure.

The method of limiting dependences can also be applied to sum the iterative solution of Eq. (3.116) for the inverse Laplace transform of the Jost solution. The iterative solution to Eq. (3.116) was noted to possess a range-order correlation. The behavior of the Jost function at $r=0$ is determined by the value of the Laplace transform for asymptotically large α, which as we see cannot generally be represented satisfactorily by any finite order of iteration. Unlike the iterative expansion of the radial dependence of the Jost solution

discussed previously, finite partial sums of the $\tilde{F}_n(k, 0)$ of Eq. (3.119) give finite results when m_1 of Eq. (3.115)<4 (Cornille and Predazzi, 1965a). However, when $m_1 \geq 4$, the individual terms of the iteration are not finite, and the method of limiting dependences may then be resorted to in order to estimate $F(k, 0)$. Under these circumstances, one would evaluate the truncated sum $F^{(p)}(k, r)$ at the value $r=\bar{r}(p)$ appropriate to the order of iteration p. Once again there is a limiting dependence $\bar{r}_L(p)$ such that $\bar{r}_L(p) \to 0$ as $p \to \infty$, and the $\bar{r}(p)$ should be chosen so that $\bar{r}(p)/r_L(p) \to \infty$ as $p \to \infty$. For potentials where the singular part is of the form of Eq. (3.114), $\bar{r}_L(p)$ is found to be of the form (Cornille and Predazzi, 1965a)

$$\bar{r}_L(p) = (\text{const}) p^{-a} (\ln p)^b. \tag{3.126}$$

By means of the limiting dependence procedure one may alternatively work directly with the Jost solution $f(k, r)$ in order to find the S matrix. Thus one can write

$$S(k) = \lim_{p \to \infty} \{ f^{(p)}[k, r(p)] / f^{(p)}[-k, r(p)] \}, \tag{3.127}$$

where $f^{(p)}(k, r)$ is the Jost solution obtained by iterating Eq. (3.122) p times; $r(p)$ is chosen so that $r(p)/r_L(p) \to \infty$, and $r_L(p)$ is the same as discussed in connection with Eq. (3.125).

Optimum-n Method

The iterative solution of the integral equation, Eq. (3.122), for the Jost solution leads to a power series in g, the coupling parameter of the potential. This power series has been analyzed by Frank (1967, 1968), and by Masson (1967), and shown to represent generally an entire function[14] of g of exponential order $\frac{1}{2}$ and type

$$\int_r^\infty dx \mid V(x) \mid^{1/2},$$

if the potential is nonincreasing and does not change sign, and also $r \neq 0$. A general entire function

$$\varphi(g) = \sum_{n=0}^{\infty} C(n) g^n \tag{3.128}$$

of exponential order ρ and type τ has growth properties[51] in the complex plane which conveniently provide certain types of estimates. One can determine that the index of the maximum terms of the power series Eq. (3.128) for large $\mid g \mid$ lie in the neighborhood of

$$\bar{n}(g) = \rho\tau \mid g \mid^{\rho}. \tag{3.129}$$

We call this value of n the "optimum-n" of the function. It can be shown that the maximum modulus $M(\mid g \mid)$ on a large circle of radius $\mid g \mid$ can be estimated by the "optimum term" in the series

$$m(g) \equiv C(n) g^n \mid_{n=\bar{n}(g)}, \tag{3.130}$$

in the sense that, for asymptotically large $\mid g \mid$, we have

$$\ln M(\mid g \mid) = \ln m(\mid g \mid) + O(\ln \mid g \mid). \tag{3.131}$$

The value of the function for any large value of $\mid g \mid$ is dominated by the terms in a neighborhood of specified width about the optimum term, and the contribution from terms outside of this range is negligible.

These considerations allow very simple determination of the limiting dependences $r_L(p)$ appropriate to the method of Cornille and co-workers. The work of Frank and Land (1970a) has shown that $f_n(k, r)$, the nth term in the iteration of Eq. (3.122) for large n, has the form

$$f_n(k, r) \sim P_n(k, r) \left(\left[\int_r^\infty dx \mid V(x) \mid^{1/2} \right]^{2n} \Big/ (2n)! \right), \tag{3.132}$$

with $P_n(k, r) \leq 1$ for potentials which do not change sign, and are nonincreasing. This, of course, implies that $\rho = \frac{1}{2}$, and

$$\tau = \int_r^\infty dx \mid V(x) \mid^{1/2}$$

as stated. Iteration of Eq. (3.122) p times reproduces the terms of $f(k, r)$ up to the term of order g^p. In view of the above discussion, one would expect iteration p times to well represent the Jost solution for values of r down to r_L which satisfy

$$p = \bar{n}(g) = \tfrac{1}{2} g^{1/2} \int_{r_L}^\infty dx \mid V(x) \mid^{1/2}. \tag{3.133}$$

Inversion of this relation for r as a function of p gives the $r_L(p)$ determined by Cornille and co-workers. When the coefficients of the power series are all positive as in the case $k=0$, the optimum-n method can be used to evaluate the ratio of power series of the same order and type. This method has been applied by Frank and Land (a) in the peratization of the scattering length due to potentials of the form $r^{-m} \exp(\lambda/r)$, when $m \geq 4$. The method also reproduces the scattering length for power potentials at some saving of labor.

5. *Phase-Amplitude Method*

It is well known that the partial-wave phase shifts can be obtained as the phase of the Jost solution[49] $f_0(k, r)$ (in the $l=0$ case, say) at $r=0$. The phase-amplitude approach is another method by which one can extract the phase shift and related scattering quantities from the Jost solution. The method is an exact one and has some very significant features vis-a-vis singular potentials. It provides an expression for the phase shift which is valid without qualification for both regular and singular potentials, and the same expression may be applied for all physical angular momenta. It is also amenable to convenient approximation. It is very convenient for analyzing the strong- and weak-coupling limits (Sec. II.D).

Let $f_l(k, r)$ and $f_l(-k, r)$, respectively, be solutions of

[51] Some of these properties are referenced in Frank and Land (1970a), and can be found in Valiron (1949), Chap. 2.

the partial-wave radial equation obeying the boundary conditions[49] for asymptotically large r,

$$f_l(\pm k, r) \approx \exp\{\mp i[kr-(l\pi/2)]\}. \quad (3.134)$$

One readily verifies the identity[52]

$$\frac{d}{dr}\left(\frac{f_l(k,r)}{f_l(-k,r)}\right) = \frac{-2ik}{f_l(k,r)f_l(-k,r)}\frac{f_l(k,r)}{f_l(-k,r)}, \quad (3.135)$$

where the Wronskian identity is used. One finds by integration that

$$\frac{f_l(k,r)}{f_l(-k,r)} = \frac{f_l(k,0)}{f_l(-k,0)} \times \exp\left(-2ik\int_0^r \frac{dx}{|f_l(k,x)|^2}\right), \quad (3.136)$$

where $f_l(k,x)=f_l^*(-k,x)$ has been employed (k real). Equation (3.136) remains valid for k complex if Eq. (3.135) is integrated without this gratuitous replacement. If we let $r\to\infty$ in Eq. (3.136) we find, considering Eq. (3.134),

$$\exp[2i\delta_l(k)] = \exp(i\pi l)[f_l(k,0)/f_l(-k,0)] = \exp\left[2ik\int_0^\infty dr(|f_l(k,r)|^{-2}-1)\right], \quad (3.137)$$

so that

$$\delta_l(k) = k\int_0^\infty dr[|f_l(k,r)|^{-2}-1] \quad (3.138)$$

where the convention of continuity of $\delta_l(k)$ in k, and the vanishing of $\delta_l(k)$ at zero coupling, have been imposed. This is the phase-amplitude expression for the phase shift. Note that, though condition (a) is sufficient for the phase shift to exist for k real and nonzero, the expression of Eq. (3.138) does not apply unless condition (b) holds. This condition is required for convergence at the upper limit of the integral. An alternative derivation of this expression in the case of the S-wave can be found in Newton (1966), in which expressions for the phase and modulus of the Jost function $f_0(k,r)$ at any r are also derived.[53] A similar expression for the scattering amplitude,

$$A_0(k) = k^{-1}\exp(i\delta_0)\sin\delta_0 = \int_0^\infty dr[f_0^{-2}(-k,r)-\exp(-2ikr)], \quad (3.139)$$

or the related S-matrix element is also derived by Newton (1966) and by Wu (1964).[54] Equations (3.138) and (3.139) both lead to the expression for the S-wave scattering length[55] (Wu, 1964; Calogero, 1965b),

$$A = \int_0^\infty dr\{[f(0,r)]^{-2}-1\}. \quad (3.140)$$

This converges at the upper limit if condition (c) is obeyed. For a repulsive singular potential with $l=0$, or any repulsively singular or regular potential with $l\neq 0$, the Jost solution becomes unbounded at $r=0$. Equation (3.138) shows the contribution from this region to be small, so that the contribution to the phase shift at a fixed energy, or to the S-wave scattering length from the repulsive region, is in fact unimportant. This is as it should be since the particle hardly penetrates strongly repulsive regions. If $f(0,r)=0$ for some $r>0$ in Eq. (3.140), the integral is evaluated by deforming the integration path (Calogero and de Stefano, 1966).

Frank and Land (c) have used Eq. (3.140) to derive inequalities for the scattering length, which are useful in studying its g dependence. Calogero and de Stefano (1966) have indicated how the complex k singularities for the scattering amplitude can be read from Eq. (3.139). Wu (1964), Calogero (1965b), and Calogero and de Stefano (1966) have employed an approximation procedure for calculating the scattering length proceeding by solution of the integral equation

$$f(0,r) = 1+\int_r^\infty ds(s-r)V(s)f(0,s) \quad (3.141)$$

for the zero-energy Jost solution. Equation (3.141) is solved by an iterative approximation procedure which constructs the successive terms in the power-series expression for $f(0,r)$. These iterates, when substituted into Eq. (3.140), provide convergents to the scattering length which is known to be nonanalytic in g at $g=0$. Calogero and de Stefano (1966) have proven the convergence of this procedure for the scattering length as well as for the phase shift. Calogero (1965b) has found that the second iterate yields exceptionally accurate results for the scattering length of r^{-m} potentials, with agreement at $m=3$ and $m=\infty$, and a maximum error of 4% for intermediate values. Wu (1964) has applied Eq. (3.140) toward the calculation of the first two orders of peratization for the scattering length of the potential $-gr^{-4}\ln r$ (Sec. IV.D).

6. *Regularization Methods*

The mathematical problems of quantum field theory have led to the wide use of regulator techniques in calculations. In these situations, the divergent quantities, which appear essentially or inessentially in the intermediate stages of a calculation, are made finite through the introduction of arbitrary cutoffs or

[52] This derivation is found in part in Limic (1962), Sec. 3, and in Jaksic and Limic (1966a), end of Sec. I, where an expression is derived appropriate to all l.

[53] Newton (1966), pp. 348–349, Newton (1966), pp. 395–396, and Wu (1964), Appendix C.

[54] See also Calogero and de Stefano (1966), Eq. (2.3), where another expression of a somewhat similar type, Eq. (2.3a), is presented. The generalization to higher l values is derived and presented in Jaksic and Limic (1966a).

[55] The subscript 0 for $l=0$ is dropped in the following. The quantity $f(0,r)$ exceeds unity for nonattractive potentials, as can be seen from the iteration of Eq. (3.141). This shows the scattering length to be negative. An expression for the scattering "length" for higher l values has not been derived.

regulators. In some cases, the final result has a cutoff-independent finite limit, which is presumed to represent the physical answer. In other cases, the limit is not finite, and judicious physical interpretation enables one to make physical sense of such a calculation. The method has given rise to the technique of peratization as a possible calculational tool. The theory of singular potentials has been employed as a testing ground for these ideas. Clarification of some of these concepts has been possible in the limited context of potential theory. Khuri and Pais (1964) have noted a limitation in the analogy of potential theory to relativistic field theory (RFT), in that a regularized RFT is not a RFT, though a regularized singular potential is still a potential. Further aspects of regularized singular potentials vis-a-vis field-theoretic ideas can be found in Secs. IV.B and IV.D.

Some aspects of the conventional scattering formalism require modification in the case of repulsive singular potentials. As indicated in Secs. II.B and III.B.4 the partial-wave Jost functions do not exist, and the expression for the S-matrix element as the ratio of Jost functions is meaningless. The same applies to expressions for the partial-wave scattering amplitude or scattering length[56] in terms of a ratio. It has been shown by Limic (1962), and a bit more thoroughly by Pais and Wu (1964b), that these relations remain valid if one generalizes these ratios to limits of ratios, as in the relation

$$S(k)=\lim_{r\to 0}[f(k,r)/f(-k,r)]. \qquad (3.142)$$

Pais and Wu found that a desirable aspect of this prescription is that it provides an "intrinsic" cutoff. This is because the physical answer is found as the limit of finite quantities without the introduction of an arbitrary cutoff into the interactions. The intrinsic cutoff idea, however, does not immediately suggest practicable approximation procedures.[57] Spector (1966a) has remarked that a separate proof is required for some of the relations of scattering theory which remain valid for singular potentials.

The standard approximation procedure of scattering theory, the Born expansion, is inoperative for singular potentials because of the divergence of the expansion coefficients.[58] This has led to regularization by means of externally imposed cutoffs. In this procedure, the singular potential $V(r)$ is replaced by a "regularized" potential $V(r,\alpha)$ which depends on an additional real parameter α, and which obeys the conditions: (i) $V(r,\alpha)$ is nonsingular for $\alpha\neq 0$, and (ii) $V(r,\alpha)\to V(r)$ pointwise as $\alpha\to 0$. One generally demands that $V(r,\alpha)$ be real for α real.

The Born series for the regularized potential can in principle be calculated. If it is feasible to calculate and sum the series, the limit of this sum as $\alpha\to 0$ is presumed to represent the physical answer. There has been no real use of this method as a direct calculational technique in potential theory to give answers not already known; rather it is used as a technique to be tested *a posteriori* with otherwise derived answers. Frank and Land (1970b) have however calculated the S-wave scattering length for pure power potentials through explicit calculation and summation of power series for the regularized potentials. They deal with the scattering length as a ratio of two power series in the form[56]

$$A=-\frac{\sum_{n=1}^{\infty}\int_0^{\infty}dr_1\int_0^{r_1}dr_2\cdots\int_0^{r_{n-1}}dr_n r_1(r_1-r_2)\cdots(r_{n-1}-r_n)r_n\prod_{j=1}^{n}V(r_j,\alpha)}{\sum_{n=0}^{\infty}\int_0^{\infty}dr_1\int_0^{r_2}dr_2\cdots\int_0^{r_{n-1}}dr_n(r_1-r_2)\cdots(r_{n-1}-r_n)r_n\prod_{j=1}^{n}V(r_j,\alpha)}. \qquad (3.143)$$

The nature of the singular dependence of the Born terms for $\alpha\neq 0$ has been analyzed for the pure power case without explicit calculation by Tiktopoulos and Treiman (1964) and Khuri and Pais (1964). The nature of the singular dependence on α is also explicitly exhibited by Calogero and Cassandro (1965) for the regularized potential

$$V(r,\alpha)=g(r+\alpha)^{-4}\exp[2/(r+\alpha)].$$

It can, of course, be explicitly determined for any of the exactly solvable potentials (Sec. III.A.2).

The regularizations most commonly employed are: $V(r,\alpha)=\theta(r-\alpha)V(r)$, "$\theta$ regularization" [$\theta(r)$ is the step function which is unity for positive values of the argument, and zero for negative values], and $V(r,\alpha)=V(r+\alpha)$, "+ regularization." A special form of regularization was employed by Khuri and Pais (1964) for pure power potentials of the form $V(r,\alpha)=\alpha^{-m}U(r/\alpha)$, where $U(r)=r^{-m}$ as $r\to\infty$. We shall call this "U regularization."

A number of fundamental questions are raised by the regularization procedure. Under what conditions does it

[56] An expression for the scattering amplitude as a ratio is given in Pais and Wu (1964b), Eq. (1.8). This quantity can be written as a power series in the coupling constant [Newton (1966), p. 334]. The limit as $k\to 0$ of the numerator and denominator in the above formula for the scattering amplitude leads to an expression for the S-wave scattering length as a ratio, which is given in Eq. (3.143).

[57] In fact, the "method of limiting dependences," discussed in Sec. III.B.4, serves as just such an approximation, as pointed out by Cornille.

[58] Strictly speaking, for transitionally singular potentials of the form $(\ln^p r^{-1})r^{-2}$, the Born coefficients are actually finite, but the series is asymptotic (Sec. II.B).

give finite answers? Under what conditions are such answers correct or well-defined approximations? Convergence to the correct answer has been checked for the solvable potentials employed in connection with peratization investigations (Secs. III.A.2 and III.D). A bit of a flurry was raised by Calogero (1965a) who adduced a class of regularized singular potentials whose scattering lengths did not converge to the scattering length of the limiting singular potential. The wave functions of these regularized potentials converged to an irregular solution, not the regular solution, of the limiting potential. Calogero's example consists of the set of regularized potentials.

$$V(r,\alpha)=r^{-4}[g\exp(-2\alpha/r) - (2n+1)g^{1/2}\exp(-\alpha/r)] \quad (3.144)$$

(n integer), which converge pointwise as $\alpha\to 0$ to the singular potential $V(r)=gr^{-4}$. Further counterexamples have been found by Cornille (1966), and by Vasudevan *et al.* (1967).

Following this, several investigations have dealt with attempts to clarify and justify the regularization procedure. Cornille (1965b, c, 1966) has shown that the physical result is always obtained in the limit both for zero and nonzero energy in the θ and $+$ regularizations. A more concise proof of the correct limit of the scattering length in θ and $+$ regularization can be found in Gale (1967).[59]

Some of the counterexamples found by Cornille (1966) and Vasudevan *et al.* (1967) apply to regularization of the Khuri–Pais type. The Khuri–Pais calculation has also been criticized by Calogero (1965a) who showed that an exchange of limits performed by them is not generally justified as in the instance of his counterexample, which does not, however, involve the U regularization. Part of the criticism of the regularization procedure of Khuri and Pais is a result of their having failed to represent their intention clearly. For their purposes it was sufficient that *some* regularization of the form they propose, i.e., some choice of the function $U(r)$, lead to the correct limit. They do not say, though they do suggest, that every choice of $U(r)$ obeying the specified conditions would be satisfactory. They present a specific form of U for which regularization is valid, and briefly indicate that, in fact, if α were chosen complex, one would obtain the wrong limit as $\alpha\to 0$. Their interchange of the $\alpha\to 0$ limit with the gratuitous σ-limiting procedure is not of general validity, as pointed out by Calogero. It is valid for the "right" functions U as illustrated in their example. The introduction of the σ limit in the integrals is actually a form of θ regularization which makes the U regularization vacuous in the $k=0$ calculation. The U regularization plays a role only in the discussion of the Lippmann–Schwinger equation with energy, for which the $k=0$ results are only a background.

The question of conditions under which a regularization procedure gives the correct limit, for other than θ or $+$ regularizations, has been discussed by Cornille (1965b, c, 1966). Cornille (1966) has shown for $k=0$, that if the regularized potential $V(r,\alpha)$ is repulsive in a fixed neighborhood of $r=0$ for sufficiently small α, then the limit $\alpha\to 0$ leads to the correct physical limit. When the regularized potential $V(r,\alpha)$ is attractive in the neighborhood of $r=0$, as it is in the various counterexamples to regularization, then three things may happen: (i) if the number of bound states supported by the elements of the potential sequence is finite, and the $\alpha\to 0$ limit is not the threshold of a new bound state, then regularization in the limit gives the physical wave function, and the correct answer, (ii) if the number of bound states is finite in the limit, and the limit is the threshold of a new bound state, then the limit of the wave function is an irregular solution of the differential equation, and one does not get the correct physical limit,[60] and (iii) if the number of zeros grows without bound in the limit, then the limiting wave function is not a solution of the differential equation in the neighborhood of $r=0$ as in the attractive singular case (Sec. II.C). For $k\neq 0$ Cornille has shown that if the wave function approaches the physical solution in the limit, then so does the scattering amplitude. Frank (c) has drawn attention to the fact that pointwise convergence is the wrong topology in which to discuss limiting procedure for potentials, and that convergence of the potential sequence in appropriate integral norms is required for convergence of the phase shifts and the scattering length.

Meetz (1964) has considered the use of a regularization limit to obtain a proper self-adjoint extension of the radial scattering differential operator (see Sec. II.C). He has shown in the case of repulsive power potentials that a cutoff type of regularization leads to the unique self-adjoint extension. He found that the less singular behavior is selected as the self-adjoint extension in the weakly attractive transition potential (Sec. II.B), while no self-adjoint extension is selected for other attractively singular potentials.

Finally we mention that regularization techniques have been applied by Arsenev (1968) to the scattering by hard core potentials in order to construct a Lippmann–Schwinger equation whose kernal is a bounded integrable function.

7. *Other Methods*

One finds a number of methods which are suggested as applicable to singular potentials, and which are briefly

[59] The reader should be warned of the misleading notation in this paper. The quantity $\Phi'(\infty)$ in Sec. II represents the derivative of $\Phi(r)$ with respect to $1/r$ at the point $r=\infty$. Gale's statement following his Eq. (9) that the more singular of two functions also has a more singular derivative is not true generally, but is true in the present context.

[60] The reader should not be surprised that though the limiting potential is repulsive, the sequence of regularized potentials may develop a new bound state in the limit. Such a situation is easily realized with a sequence of square wells. Such examples are discussed in detail in Frank (c).

developed and applied to a single example by way of illustration. We take note of them here. We recall first that Laplace transforms have been applied to singular potentials, and have been discussed in Sec. III.B.4 in connection with Jost function methods. In Sec. IV.C another method is described based on renormalization ideas and the use of distributions. In the present section, we mention the Mellin transform method due to de Alfaro and Predazzi (1965), the Hill determinant method due to Fubini and Stroffolini (1965), and methods based on higher-order differential equations.

Mellin Transform Method. The Mellin transform has been suggested by de Alfaro and Predazzi (1965) as a convenient technique for solving the singular integral equation arising in connection with singular potentials, where nonanalyticity in the coupling constant is anticipated. The Mellin transform $\varphi(s)$ of the function $u(g)$ is defined by

$$\varphi(s)=\int_0^\infty dg u(g)g^{s-1}, \tag{3.145}$$

and the inverse Mellin transform is given by

$$u(g)=(2\pi i)^{-1}\int_{C-i\infty}^{C+i\infty} ds\varphi(s)g^{-s}, \tag{3.146}$$

where C is a real number, such that $\varphi(s)$ is analytic along $s=C+i\tau$. It is characteristic of the inverse Mellin transform, Eq. (3.146), that it maps a function $\varphi(s)$ $(s=\sigma+i\tau)$ which is analytic in the strip $\sigma_1\leq\sigma\leq\sigma_2$, and obeys $|\varphi(s)|\leq$const exp $(-\theta_0|\tau|)$ into a function $u(g)$ $(g=\rho e^{i\theta})$ which is analytic in the angular sector $|\theta|\leq\theta_0$ and possesses a uniform exponential bound of the form $|u(g)|\leq$const $\rho^{-\sigma_1}$ for $|\rho|<1$. Such a mapping would be appropriate to the study of the coupling-constant dependence of the wave function of a singular potential which shows analytic behavior of uniform exponential bound in angular sectors of the coupling-constant plane. One would choose the path of integration in Eq. (3.146) to lie within a strip which would make the solution of the integral equation yield the free solution as $g\to 0$. The method is intended to determine the nature of the singularity for small r and g. The method is explicitly applied by de Alfaro and Predazzi to pure power singularities, and the results agree with those derived by other methods.

Application of the Mellin transform, Eq. (3.145), to the coupling constant dependence in the integral equation for the radial wave function,

$$u(r,g)=(\sin kr/k)+g\int dr'G(r,r')V(r')u(r',g), \tag{3.147}$$

[$G(r,r')$ is a Green's function] leads to the integral-difference equation

$$\varphi(r,s)=(\sin kr/k)+\int dr'G(r,r')V(r')\varphi(r',s+1). \tag{3.148}$$

For a potential with a pure power singularity, an ansatz of the form

$$\varphi(r,s)=h(s)p(r)[\sum_{i=1}^{\nu}(c_ir)^{-b_i}]^{-s} \tag{3.149}$$

for the small r, large s behavior of $\varphi(r,s)$ leads to solutions with ν depending on the singularity. In particular for $2\leq m\leq 4$, one has $\nu=1$. One finds more than one solution corresponding to differing behaviors in different sectors.

The method is also applied to the solution of an integral equation arising in a field-theoretic problem. This is discussed in Sec. IV.B.

Hill Determinant Method. Fubini and Stroffolini (1965) have made an interesting application of the theory of differential equations with periodic coefficients to singular potentials. The basic formalism, which involves the Hill determinant and its properties, can be found in standard mathematical physics texts such as that of Morse and Feshbach (1953), or that of Whittaker and Watson (1927). The method leads to very interesting insights and provides a calculational technique as well.

One considers the radial differential equation $(\lambda=l+\frac{1}{2})$

$$\{(d^2/dr^2)+k^2-[(\lambda^2-\tfrac{1}{4})/r^2]-V(r)\}u(r)=0. \tag{3.150}$$

One makes the substitution $u(r)=r^{1/2}v(r)$, and finds for $v(r)$ the differential equation

$$[(d^2/dr^2)+r^{-1}(d/dr)-(\lambda^2/r^2)+k^2-V(r)]v(r)=0. \tag{3.151}$$

We now consider this equation for r complex and of the form $r=e^{iz}$, with z real. The differential equation now reads[61]

$$[(d^2/dz^2)+\lambda^2+U(z)]\varphi(z)=0, \tag{3.152}$$

where

$$U(z)=r^2[-k^2+V(r)], \tag{3.153}$$

and $\varphi(z)=v(r)$. For a potential $V(r)$ which is analytically a single-valued function of r, the potential $U(z)$ in Eq. (3.152) is of period 2π in z space. One can apply considerations from the theory of differential equations with periodic coefficients to Eq. (3.152). From Floquet's theorem,[62] known familiarly as Bloch's theorem in solid state physics, one can infer that there are solutions of the general form

$$\varphi_j(z)=\exp(i\sigma_jz)P_j(z) \qquad (j=1,2), \tag{3.154}$$

where the $P_j(z)$ have periodicity 2π. There are well-

[61] The reader should note sign errors in the corresponding Eq. (6) of Fubini and Stroffolini.

[62] For Floquet's theorem, see Morse and Feshbach (1953), pp. 556–557. For Bloch's theorem, see Kittel (1956), pp. 279–280.

known classes[63] for which $\sigma_1=-\sigma_2$ corresponding to solutions of Eq. (3.151) which have the form

$$v(r)=r^{\pm\sigma}\chi_{\pm}(r) \qquad (3.155)$$

($\sigma\equiv\sigma_1$), with $\chi_{\pm}(r)$ single-valued functions of r. The "eigenvalue" σ is determined as a zero of the Hill determinant, based on the Fourier coefficients of the periodic potential $U(z)$, which are, of course, related to the power-series (or generally Laurent series) coefficients of the potential $V(r)$. When the potential is analytic in r, the Hill determinant becomes triangular, and the eigenvalues are trivially determined and coincide with the quantities specified by the indicial equation at regular singular points. For potentials expandable in negative integral powers of r, but excluding the value -1, the differential equation (3.152) at zero energy corresponds again to a Hill determinant which is triangular. It is therefore not surprising that, through appropriate variable changes, the zero-energy equations for such potentials reduce to equations for a regular problem. Not only σ, but also the power-series coefficients of the solutions, can be found in terms of Hill determinants. This leads to an expression for the S-matrix in terms of Hill determinants which is susceptible to a weak-coupling expansion.

The authors have outlined these ideas, suggested a program, and implied that promising results are obtainable by this method. They have not made clear how the method might be adapted to the case when the potentials involve logarithmic singularities or irrational powers of r.

Higher-Order Differential Equations. Arbuzov and Filippov (1964) and Guttinger and Pfaffelhuber (1966) have noted that for negative even power potentials, the coordinate space differential equation can be transformed into a momentum space differential equation of order equal to the power. The Lippmann–Schwinger equation for the lth partial wave is

$$f_l(p,p';k^2)=F_l(p,p')+\int_0^\infty dq\,\frac{F_l(p,q)f_l(q,p';k^2)}{k^2-q^2+i\epsilon}, \qquad (3.156)$$

where

$$F_l(p,p')\equiv\frac{2}{\pi}pp'\int_0^\infty drr^2V(r)j_l(pr)j_l(p'r) \qquad (3.157)$$

[the function $j_l(pr)$ is the familiar spherical Bessel function]. When $V(r)=gr^{-m}$, the integral generally does not exist unless $l>\frac{1}{2}m-\frac{3}{2}$ (Sec. II.D). One can, however, ascribe a value to $F_l(p,p')$ when this inequality is violated, either by working with the power potential as a distribution, or equivalently through analytic continuation in the l variable.[64] One finds that for even m, $F_l(p,p')$ involves the step functions $\theta(p-p'),\theta(p'-p)$. This makes it possible to convert by differentiation the integral equation, Eq. (3.156), into an inhomogeneous linear differential equation. Arbuzov and Filippov find for $m=4$ the differential equation [$f(p)\equiv f_l(p,p';k^2)$, $\lambda=l+\frac{1}{2}$]

$$f^{(iv)}(p)-\frac{2(\lambda^2-\frac{1}{4})}{p^2}f''(p)+\frac{4(\lambda^2-\frac{1}{4})}{p^3}f'(p)+\frac{(\lambda^2-25/4)}{p^4}f(p)-\frac{g}{k^2-p^2}=g\delta(p-p'). \qquad (3.158)$$

One imposes, as boundary conditions, regularity of $f(p)$ at $p=0$, and the requirement that the behavior of $f(p)$ for large p guarantee the convergence of the integral in Eq. (3.156). This suffices to select a solution. Arbuzov and Filippov have presented the solution to Eq. (3.158) for $l=k^2=0$, and have noted that the leading singular dependence on g for small g involves terms with $g^{3/2}$ and $g^2\ln g$. Guttinger and Pfaffelhuber[65] have constructed the differential equation in momentum space from the full Lippmann–Schwinger equation unexpanded in partial waves. They obtained a partial differential equation containing the Laplacian operator to the order $\frac{1}{2}m$. They did not, however, present any solutions.

IV. APPLICATIONS—FORMAL

A. Regge Behavior

Predazzi and Regge (1962) were among the first to realize the usefulness of singular potentials as a formal "laboratory" for investigating many physical and mathematical ideas. They argued that physical interactions among particles in the real world are very likely highly singular in character. The study of regular potentials is unlikely to reflect this situation. Rather, the singular potentials, with their strong repulsion and lack of analyticity in the coupling constant, would be more likely to shed some reliable light on the physics of strong interactions. Others, as we shall see in this section, have used this argument in one variation or another to justify the examination of various features of singular potentials.

Predazzi and Regge confined themselves to the

[63] Such a result follows from the indicial equation for Eq. (3.151), when $V(r)$ is single valued and nonsingular, or possibly transitionally singular. It also follows for $k=0$ and $V(r)$ single valued and singular, in which case the point at infinity is an ordinary or regular singular point, and one can find σ_1 and σ_2 from an indicial equation. The authors claim $\sigma_1=-\sigma_2$ to hold for general single valued $V(r)$ though this is not indicated by their reference, nor by any result familiar to the reviewers.

[64] The essential object of the article by Guttinger and Pfaffelhuber is the demonstration of procedures based on the ideas of distribution theory for interpreting solutions to singular potential problems (attractive and repulsive) in the spirit of renormalization methods in field theory (Sec. IV.C). They dwell at length on the equivalence of the distribution approach, and analytic continuation in l. Arbuzov and Filippov have also noted the possibility of extending the meaning of Eq. (3.157) to $l<\frac{1}{2}m-\frac{3}{2}$ by analytic continuation in l.

[65] Guttinger and Pfaffelhuber (1966), Eq. (3.10), and Sec. 4.

repulsive potential

$$V(r)=(g/r^4)+(f_1/r^3)+(f_2/r^2), \qquad (4.1)$$

which may be solved exactly for zero energy. They were concerned with the analytic nature of the Jost functions and S-matrix in both k and l. At about the same time, Limic (1962) investigated the k and l behavior for singular potentials generally. Their conclusions about the k behavior of the S-matrix have been dealt with in Sec. II.D.1. We recall that the singularity of the potential does not manifest itself in the analytic properties of the S-matrix in the finite k plane. These properties are essentially the same as for regular potentials.

However the l behavior for singular potentials is different and is, in fact, simpler than for regular potentials. In the case of regular potentials, the region $\mathrm{Re}\, l<-\frac{1}{2}$, or $\mathrm{Re}\,\lambda<0$ $(\lambda=l+\frac{1}{2})$ is generally troublesome. Special techniques are needed to analytically continue S into this region, and its properties there depend on the detailed nature of the potential. The reason for this is easily traced to the lack of symmetry in λ of the boundary condition imposed on the physical wave function $u(r)$ at the origin. Though the Schrödinger equation contains only λ^2, the boundary condition demands that one choose the solution which behaves like $r^{\lambda+\frac{1}{2}}$ rather than $r^{-\lambda+\frac{1}{2}}$ near the origin. When $\lambda=0$ $(l=-\frac{1}{2})$, the boundary condition is no longer well defined.

For singular potentials, the potential term dominates the centrifugal term near the origin, and the boundary condition there is *independent* of λ. Hence the Jost functions, and consequently the S-matrix, are functions of λ^2. In fact, it is easily proved that the Jost functions are entire functions of λ^2 in regions where they are analytic in k. This is a consequence of Poincare's theorem.[27] We may write (Predazzi and Regge, 1962)

$$S(\lambda, k)=[F(\lambda^2, k)/F(\lambda^2, -k)] \times\exp[i\pi(\lambda-\tfrac{1}{2})], \qquad (4.2)$$

so that

$$S(-\lambda, k)=[F(\lambda^2, k)/F(\lambda^2, -k)]\exp[i\pi(-\lambda-\tfrac{1}{2})],$$

and therefore one finds

$$S(-\lambda, k)=\exp(-2\pi i\lambda)S(\lambda, k). \qquad (4.3)$$

This symmetry is true for all singular potentials and is the same as for hard core potentials (Brander, 1963). The problem of continuing S into the left-half λ plane is automatically solved. We note that as a result the Regge poles of S occur in pairs at l and $-l-1$.

In a number of respects the detailed nature of the pole structure of S differs from that found for regular potentials. Giffon and Predazzi (1964) investigated whether the Mandelstam representation could hold for singular potentials. Whether it does depends first on whether there exists, at fixed energy, a Regge pole with a maximum real part. This is certainly true if there are a finite number of poles, but could be true even if there were an infinite number suitably distributed. In addition, the existence of the double dispersion relation requires that as $k\to\infty$ the maximum real part of the leading pole remains bounded.

Jaksic and Limic (1966b) have proved that the Jost function $F(\lambda^2, -k)$ has finite exponential order[14] for singular potentials. This fact, together with the standard result (Predazzi and Regge, 1962; Limic, 1962) that the Jost function is an entire function of λ, means that it must have an infinite number of zeros. Such a conclusion would fail only if it were a polynomial times an exponential of a polynomial, which Giffon and Predazzi (1964) and Jaksic and Limic (1966a, 1966b) showed not to be the case. Hence S has an infinite number of poles, but because of the symmetry of Eq. (4.3) and the absence of poles in the fourth quadrant (proved in the same way as for regular potentials), it follows that there are an infinite number of Regge poles in the first quadrant. Giffon and Predazzi (1964) claimed that as $k\to\infty$ the real part of these poles go to infinity. While their proof is not clearly given, it is certainly true for the case of r^{-4} as will be demonstrated below, and it is probably true in general.

Jaksic and Limic (1966a) studied the asymptotic λ behavior of the S-matrix for singular repulsive potentials which fall off exponentially fast at infinity. For positive energies, they concluded that there are an infinite number of poles in the first quadrant. They also concluded, without explicit proof, that the poles accumulate at infinity along the positive imaginary axis and showed that, along any ray in the right-half λ plane,

$$\lim_{\mathrm{Re}\,\lambda\to\infty} S(\lambda, k)=1+O(\lambda^{-1/2}e^{-\alpha\lambda}),$$

$$\cosh\alpha=1+(m^2/2k^2). \qquad (4.4a)$$

The leading λ terms in Eq. (4.4a) may be found in their paper. In their other closely related paper (Jaksic and Limic, 1966b), in which the potentials need not have exponential tails, they studied the same limits for complex energy (but k not pure imaginary). Their conclusion about the particular ray of accumulation of the poles is again not explicitly substantiated, though a proof of this is implied by their results. In the pure power case, the asymptotic form of $S(\lambda, k)$ along any ray in the right-half plane was found to be

$$\lim_{\mathrm{Re}\,\lambda\to\infty} S(\lambda, k)=1. \qquad (4.4b)$$

Equations (4.4a) and (4.4b), assuming they are uniform limits, require that $\arg\lambda=\pi/2$ be the line of accumulation of the poles. It is also known (Dombey, 1965; Paliov and Rosendorff, 1967) that along the real axis

$$\lim_{\lambda\to\infty} S(\lambda, k)=1+O(\lambda^{1-m}).$$

Another important property of the S-matrix should be noted. Using the unitarity relation (for real k),

$$S^*(\lambda^*, k)S(\lambda, k)=1,$$

and the symmetry property of Eq. (4.3), it is easy to derive the exact relation, valid for all singular potentials,

$$|S(\lambda, k)|=\exp(-\pi y), \tag{4.5}$$

for $\lambda=iy$, y real. This behavior implies the unbounded oscillation of $S(\lambda, k)$ as $\lambda\to-i\infty$, and means that the usual background integral in the Sommerfeld–Watson transformation does not converge absolutely for *any* value of $\cos\theta$.

In order to determine the analytic continuation in $\cos\theta$ of the S-matrix, various authors have applied different forms of the Sommerfeld–Watson transformation. Giffon and Predazzi (1964) tried to conclude from Eq. (4.3) that the background integral in their form of this transformation is zero. This conclusion is not correct because the part of the integral coming from the term unity in $[S(\lambda, k)-1]$ does not vanish in their version of the integral.

Bowcock and Contogouris (1964) have tried to find the analytic continuation to unphysical values of $\cos\theta$ for a singular potential scattering amplitude. However, their continuation of the amplitude seems incorrect. In their form of the Sommerfeld–Watson transformation, in order to eliminate an integral which diverges for physical $\cos\theta$, they considered a particular region of the complex $\cos\theta$ plane, where this integral vanishes. However, in this region another integral, which vanishes for physical $\cos\theta$, now diverges.

Aly and Müller (1966a), Challifour and Eden (1963), and Dombey and Jones (1968) have studied the inverse-fourth-power potential for Regge behavior. The first work considers the attractive potential, while the second and third works deal with the repulsive potential.

For the repulsive potential, both Challifour and Eden and Dombey and Jones derived the location of the poles in the λ plane. However, their results are in disagreement. For $E<0$, both groups agreed that an infinite number of poles lie on the positive imaginary axis. As $E\to0$, these poles move into the origin (forming an essential singularity there) and then, as E becomes positive, they jump into the first quadrant. Challifour and Eden claimed that these poles are restricted to lie below the line $\arg\lambda=\pi/4$. However, Dombey and Jones have concluded that these poles lie above $\arg\lambda=\pi/4$. This is illustrated by the expression valid for large k given in Eq. (4.6) below which shows that the poles asymptotically approach $\arg\lambda=\pi/4$. However, we note that the claim of Challifour and Eden is in contradiction with the conclusions of Jaksic and Limic (1966b), where the uniform limit of Eq. (4.4b) is derived. The accumulation of poles (for fixed k) along the positive imaginary axis obviously requires poles above $\arg\lambda=\pi/4$. This resolves the issue in favor of Dombey and Jones. The discrepancy between these groups seems to result from a missing $i^{3/2}$ factor in the pole location formula of Challifour and Eden.

As $k\to\infty$, the pole locations (for $V=g/r^4$, $g>0$) are given by

$$\lambda_m(k)=(2ig^{1/2}k)^{1/2}+\sqrt{2}\,(m+\tfrac{1}{2})i\{1+O[(m+\tfrac{1}{2})/(g^{1/2}k)^{1/2}]\}, \tag{4.6}$$

with $m=0, 1, 2, \cdots$. Equation (4.6) is valid for $|g^{1/2}k|\gg(m+\frac{1}{2})^2$. Note that as $k\to\infty$, the real parts of these poles become infinite; however, for fixed k there is indeed a pole with maximum real part. For all energies, all poles lie outside a circle given by

$$|\lambda|^2=2g^{1/2}|k|.$$

Dombey and Jones analytically continued the r^{-4} scattering amplitude to complex $\cos\theta$ by utilizing the asymptotic λ properties of $S(\lambda, k)$. They improved on the result of Eq. (4.4b) to show that asymptotically in the right-half plane the following holds:

$$\lim_{|\lambda|\to\infty} S(\lambda, k)=\exp(2i\delta_{\mathrm{WKB}})/(1+e^{-2\Phi}),\qquad 0\leq\arg\lambda\leq(\pi/2),$$

$$\lim_{|\lambda|\to\infty} S(\lambda, k)=\exp(2i\delta_{\mathrm{WKB}})[1+\exp(-2\tilde{\Phi})],\qquad -(\pi/2)<\arg\lambda\leq0. \tag{4.7}$$

In Eq. (4.7), $\delta_{\mathrm{WKB}}=-(\pi k^2g/8\lambda^3)$, and Φ, which is defined in Sec. III.A.1, Eq. (3.16) is given by

$$\Phi(\lambda, k)\underset{|\lambda|\to\infty}{\approx}2\lambda\ln[2\lambda/e(ig^{1/2}k)^{1/2}]+O(\lambda^{-1}).$$

These results of Dombey and Jones are consistent with previously derived general results. Along any ray in the right-half plane, one sees that Eq. (4.4b) holds. In the first quadrant between the ray $\arg\lambda=\pi/2-\epsilon$ ($\epsilon>0$) and the positive imaginary axis, $S(\lambda, k)$ declines from unity to nearly zero along an asymptotic arc subtending ϵ. In the fourth quadrant between the ray $\arg\lambda=-\pi/2+\epsilon$ and the negative imaginary axis, $S(\lambda, k)$ increases from unity to a large value along an asymptotic arc. Equation (4.5) is also seen to hold on the imaginary axis.

The form of Dombey and Jones for the Sommerfeld–Watson transformation differs from the usual one in that the background integral does not follow the entire imaginary axis. In the fourth quadrant, where it differs, it follows a path which asymptotically at $\lambda=-i\infty$ begins on the axis, and then advances to the origin by going slightly to the right of the negative imaginary axis in a well-defined way. They also proved that the infinite sum of Regge poles converges. Finally, they indicated, but could not prove, that such a path also works for any singular power potential.

Dombey and Jones gave the residues of the poles as

$$\gamma_m = \lim_{\lambda\to\lambda_m} (\lambda-\lambda_m)S(\lambda, k)$$
$$= e^{i\pi\lambda_m}\cos\pi\beta(\lambda_m)/(d\Phi/dx)_{\lambda=\lambda_m}, \qquad (4.8)$$

where β is defined in Sec. III.A.1. For large m, Eq. (4.8) becomes

$$\gamma_m = 4\ln[2\lambda_m/(ig^{1/2}k)^{1/2}]^{-1}. \qquad (4.9)$$

Aly and Müller (1966a) have derived the Regge pole locations for the attractive r^{-4} potential. However, their results for the small $|k|$ behavior of the poles are not correct and would lead to positive energy bound states. Their formula Eq. (5.3), which they used to obtain this result is not valid at low energies, and also is not valid near integer values of the parameter m which labels the different poles. However, their high-energy behavior is correct and consistent with Eq. (4.6). Since they dealt with an attractive potential, they must arbitrarily choose a wave function at the origin. They made the same choice as Vogt and Wannier (see Sec. III.A.1).

For $k>0$ and large, the poles lie on the imaginary axis and approach the origin as k shrinks. For large negative values of energy, the poles lie in the fourth quadrant but always below the line $\arg\lambda=-\pi/4$. [Note that they used λ to designate what we would call $(\lambda^2-\frac{1}{4})$.] They obtain an asymptotic formula similar to Eq. (4.6) which is (for $V=g/r^4$, $g<0$)

$$\lambda_m^2 \approx -2k(-g)^{1/2}\{1+(2m+1)[(-g)^{1/2}(k)]^{-1/2}\},$$
$$m=0, 1, 2, \cdots. \qquad (4.10)$$

It is interesting that in this case the usual prohibition in Regge theory against poles appearing in the fourth quadrant is violated. This occurs because of the non-unitary choice of boundary condition which makes the usual proofs fail.

We have not reproduced the techniques of Aly and Müller, Challifour and Eden, or Dombey and Jones because they utilized specialized properties of Mathieu functions, and their derivations are not of general interest.

Finally a word may be said about the analyticity in λ of the S-matrix for the inverse square. A number of authors (Sawyer, 1963; Challifour and Eden, 1963) have proved that an energy-independent branch cut appears in the S-matrix between the points $\lambda=\pm i(g)^{1/2}$ for $V=g/r^2$ $(g>0)$. The symmetry, Eq. (4.3), holds provided it is taken to apply to the other Riemann sheet in the left-half plane. In the Klein–Gordon equation for the inverse square, energy-dependent branch points appear.

B. Relation to Field Theory

Numerous authors have investigated various aspects of field theory with the aim of finding similarities in some respects to singular potentials. This is generally done in the hope that the properties of singular potentials, which are more easily studied, may have some significance for the physical relativistic scattering amplitudes. Studies of a variety of field theories in certain approximations by means of the Bethe–Salpeter (B–S) or quasipotential equations have suggested a correspondence in which these interactions may be equivalent, for small interparticle separation, to potentials which in many cases are singular. An interesting correspondence has been found between the classification of field theories as superrenormalizable,[66] renormalizable, or nonrenormalizable, and the character of the equivalent potential as regular, transitionally singular, or singular. An analogy has also been found between the Jost function of potential theory and the vertex renormalization constant Z in field theory. Dispersion relations have been studied briefly for singular potentials as an analogy with dispersion relations in elementary particle theory. Regge behavior for singular potentials has been discussed in Sec. IV.A. The literature on particle theory is exceedingly vast and the reviewers do not pretend to exhaust all instances of comparisons between particle interactions and singular potentials. Rather they attempt to review what to their knowledge seem to be the most interesting efforts to relate field theory to singular potentials.

Bethe–Salpeter Equation. It has been possible to find a correspondence between field-theoretic Lagrangians and local potentials, by studying in certain approximations the B–S equation which describes the scattering of two particles in field theory. While such a correspondence can be at most of conjectural significance, the picture that emerges seems to be sensible and has intuitive value. It should be borne in mind that the order of the B–S equation depends on the spin of the scattering particles, and likewise that the definition of a potential as singular depends on the order of the differential equation (Sec. II.A).

The B–S equation for two particles of equal spin has in configuration space the form (x_i denote four-vectors),

$$K_{x_1}K_{x_2}\psi(x_1, x_2) = V(x_1-x_2)\psi(x_1, x_2), \qquad (4.11)$$

where K_{x_i} denotes the free particle operator for one of the incoming particles with coordinate x_i, Klein–Gordon for spin zero, Dirac for spin $\frac{1}{2}$. The potential $V(x_1-x_2)$ corresponds to the exchange of a particle of specified spin and mass as calculated from a selected set of diagrams. In order to obtain an equivalent potential, one extracts a radial equation from Eq. (4.11). The center-of-mass dependence of Eq. (4.11) is first eliminated. The resulting equation in the relative coordinate variable $x_{12}\equiv x_1-x_2$ is then transformed by means of a Wick rotation to the Euclidean metric.

[66] The term superrenormalizable is applied to a field theory, when either none or, at most, a finite number of primitive diagrams are divergent.

This transformation, which is legitimate for bound-state problems, transforms the light cone into the origin so that the potential function $V(x_{12})$ then shows spherical symmetry. The operator in the equation becomes rotation symmetric when the total energy is zero, and this is the case usually considered. The equation can then be separated in (four-dimensional) hyperspherical harmonics, and one obtains a radial equation for each hyperspherical partial wave corresponding to a value of the total four-dimensional angular momentum. When the scattering particles are of spin zero, the partial-wave radial differential equation is of fourth order, while for spin-$\frac{1}{2}$ particles the corresponding operations give a radial differential equation of second order.

The resulting radial differential equation describes a bound state of specified four-dimensional angular momentum due to the potential $V(x_{12})$ incurred by the exchange of a particle of specified mass and spin according to a selected set of diagrams. The second-order radial equation corresponding to the scattering of spin-$\frac{1}{2}$ particles is similar to the radial scattering equations of conventional potential theory. The fourth-order radial equation for spin-zero particles of mass m scattering with four-dimensional angular momentum ω has the form ($x\equiv|x_{12}|$)

$$\{(d^2/dx^2)+x^{-1}(d/dx)-[(\omega+1)^2/x^2]-m^2\}^2\psi(x) = V(x)\psi(x). \quad (4.12)$$

The boundary conditions for the selection of solutions to Eq. (4.12) are discussed at length at Bastai, Bertocchi, Fubini, Furlan, and Tonin (1963a), and by Bastai, Bertocchi, Furlan, and Tonin (1963b). The potential $V(x)$, corresponding to a given field theory in the appropriate approximation, would be characterized as regular, transitionally singular, or singular, if the leading negative power dependence of the effective potential has exponent less than, equal to, or greater than the order of the differential equation. Logarithmic factors do not affect this classification. Thus an x^{-2}-type potential is transitionally singular for the spin-$\frac{1}{2}$ scattering equation, as it has the same dependence as the centrifugal potential. In the fourth-order equation the centrifugal potential gives rise to an x^{-4} term which is transitionally singular for the spin-zero equation. Bastai *et al.* (1963a) term those theories described by radial equations with regular potentials as the class I case, equations with transitionally singular potentials as the class II case, and equations with singular potentials as the class III case.[67]

Bastai *et al.* (1963a) have found the effective potentials for a variety of field theories in the ladder approximation. The scattering of spin-zero bosons through the exchange of a spin-zero boson as in a φ^3 theory was found to be of class I (x^{-2} singularity). The scattering of spin-zero particles by exchange of boson pairs as in a φ^4 theory is of class II (x^{-4} singularity), while the scattering in a φ^5 theory is of class III (x^{-6} singularity). The scattering of spin-zero particles through the exchange of a spin-one boson is of class II (x^{-4} singularity). The class II character of two scalar boson exchange and vector boson exchange between spin-zero particles is also borne out by calculations of the Regge behavior for these theories in a ladder approximation by Sawyer (1963), and Cosenza, Sertorio and Toller (1964). The amplitudes show a Regge cut characteristic of a centrifugal-type potential. Contogouris (1965) has found that the scattering of scalar particles through exchange of spin-two particles is of class III (x^{-6} singularity). He explicitly verified that some of the characteristic attributes of singular potential solutions hold for the solution to the B–S equation, such as meromorphy of the partial-wave equation in angular momentum, and reflection symmetry of the S matrix, Eq. (4.3), in the l plane.

The scattering of fermions through the exchange of a scalar meson or a photon is of class II, as found by Goldstein (1953). Scattering through exchange of a massive vector boson with nonderivative coupling as in W boson theory is of class III (x^{-4} singularity). If the vector boson propagator, however, is chosen proportion to $g_{\mu\nu}$, which can be done when the coupled current is conserved, the class II case is obtained as found by Guttinger, Penzl, and Pfaffelhuber[68] (1965a). This result has suggestive support in Sawyer (1963) who obtained from the B–S equation a cut in the angular-momentum plane. Scattering of fermions via the four-fermion interaction is of class III, as found by Sawyer (1964), and by Guttinger, Penzl, and Pfaffelhuber (1965a). Sawyer found the four-fermion force to be attractive for S and V coupling, and repulsive for P and A coupling.

Despite the approximate and therefore conjectural nature of these results, a most interesting observation emerges between the class of the potential scattering description of these field theories, and the renormalizability attributes of the theories. Superrenormalizable theories such as φ^3 give rise to class I equations, renormalizable theories such as φ^4, or nonderivative Yukawa coupling theories, give rise to class II equations, while nonrenormalizable field theories such as the four-fermion theory give rise to class III equations. These results are also confirmed both by studies of the quasipotential equation corresponding to various field theories (see below), and by studies of the vertex functions for these theories (see below). Attempts to apply renormalization ideas to singular potentials (Sec. IV.C) are also consistent with the

[67] Aly and Taylor (1968) have further subdivided Class III into IIIa and IIIb, where the former class contains power-type singular potentials, and the latter includes stronger singularities such as exponential ones.

[68] An error should be noted in Guttinger *et al.* (1965a) on p. 253 following Eq. (2.23). Read "where $a'=0$ in V if we neglect the nondiagonal term in the boson propagator."

above classification. This connection follows very simply for the ladder approximation from dimensional considerations. The implications of this correspondence are most interesting. It lends support to the belief that nonrenormalizable field theories, which lead to repulsive interactions, have self-damping mechanisms, and therefore may have solutions.[69] This is analogous to equations with repulsive singular potentials which, though they cannot be solved perturbatively, have well-defined solutions. When the theories give rise to attractive forces there are ambiguities as in the attractive singular case, which may perhaps be renormalized away by proper treatment (see Secs. II.C and IV.C).

Quasipotential Equation. The quasipotential method provides another description of field-theoretic scattering by means of an effective potential. This formalism also derives from a study of the equation for the four-point Green's function as does the B–S equation. The B–S equation is essentially the joint equation of motion for two particles interacting via a field-theoretic interaction, and is a description in terms of the independent coordinates of the two particles. The quasipotential equation describes the system consisting of two particles in interaction as evolving in a common time as described by a Schrödinger-type equation for the time rate of change of the wave function. The formalism for the quasipotential equation is described by Logunov and Tavkhelidze (1963a), and Logunov, Tavkhelidze, Todorov, and Khrustalev (1963b).

The quasipotential equation, usually written in momentum space, has the form

$$(E^2-p^2-m^2)\Psi(p)=(p^2+m^2)^{1/2}\times\int d^3p'V(p,p';E)\Psi(p'), \quad (4.13)$$

where p is the momentum of the scattering particles of equal mass m relative to the center of mass, and E is their energy in this system. The "quasipotential" $V(p, p'; E)$ is generally energy dependent and complex. One can write an integral equation for T matrix constructed in familiar fashion from Eq. (4.13):

$$T(p,p';E)=V(p,p';E)+\int d^3q\,\frac{V(p,q;E)T(q,p';E)}{(E^2-q^2-m^2)(m^2+q^2)^{1/2}}. \quad (4.14)$$

The quasipotential $V(p, p'; E)$ has been so constructed that the T-matrix of Eq. (4.14) coincides with the field-theoretic T-matrix to a specified order, even off the mass shell. By foregoing agreement between the quasipotential and field-theoretic T matrices off the mass shell, one can make the quasipotential "local"[70] [i.e., a function of $(p-p')^2$] though energy dependent. One would thus obtain the correct S matrix to any prescribed order. The quasipotential is derived according to the prescriptions for constructing an optical potential, from the relation

$$\tilde{G}^{-1}(p,p';E)-\tilde{G}_0^{-1}(p,p';E)=V(p,p';E). \quad (4.15)$$

Here $\tilde{G}(p, p'; E)$ is the exact momentum space, four-point, equal-time Green's function in the center-of-mass system, and $\tilde{G}_0(p, p'; E)$ is the corresponding free Green's function.

Logunov *et al.* (1963b) have deduced, from the unsubtracted Mandelstam representation, a spectral representation for the local (i.e., on-the-mass-shell) quasipotential of the form

$$V^{\pm}(r,E)=\pi^{-1}\int d\nu\tau^{\pm}(\nu,E)(e^{-\nu r}/r), \quad (4.16)$$

where the superscripts $\pm$ refer to the potential in even/odd states respectively. The quasipotential deduced from a Mandelstam representation with n subtractions is valid for angular momenta $l\geq n$. The effect of subtractions in the Mandelstam representation on the quasipotential is discussed by Filippov (1964), and Faustov (1964).

One can construct a Schrödingerlike equation in configuration space for the partial-wave amplitudes of the T matrix of Eq. (4.14), in terms of the wave function $u_l(r)$ defined by

$$u_l(r)=(2/\pi)^{1/2}\int dppj_l(pr)\Psi_l(p), \quad (4.17)$$

where $\Psi_l(p)$ is the lth partial-wave amplitude of $\Psi(p)$ of Eq. (4.13). One finds that $u_l(r)$ satisfies the differential equation

$$(d^2/dr^2)u_l(r)+\{k^2-[l(l+1)/r^2]\}u_l(r)=V(r,E)\int dr'K_l(r,r')u_l(r'), \quad (4.18)$$

where

$$K_l(r,r')=4\pi r'\int[dqq^2/(q^2+m^2)^{1/2}]\times j_l(qr)j_l(qr'). \quad (4.19)$$

The kernel $K_l(r, r')$ in Eq. (4.18) merely smears the wave function over a distance of the order of the range of the Compton wave length of the scattering particles. It may be approximated in the nonrelativistic domain by a delta function. The resulting equation is a local Schrödinger equation in configuration space for motion in the presence of the effective potential $V(r, E)$.

Filippov (1964) has calculated the weight function $\tau(\nu, E)$ for the φ^3 and φ^4 coupling theories. He found that the φ^3 coupling theory gives rise to a class I equation, while the φ^4 coupling theory leads to a class II case with $V(r, E)\sim(a+b\ln r)r^{-2}$ for small r. Charap and Fubini (1959) found the φ^4 theory to correspond to a class III case with $V(r)\sim(a+b\ln r)r^{-3}$. In a later paper, however, Charap and Dombey (1964), considering a few low-order exchange diagrams, found the modified result $V(r)\sim(a+b\ln r)r^{-2}$ in agreement with Filippov. They also considered a larger class of exchange

[69] See, e.g., Lee and Yang (1962), Lee (1962), Feinberg and Pais (1963, 1964), Guttinger, Penzl, and Pfaffelhuber (1965a, b), Arbuzov and Filippov (1965, 1966).

[70] The quasipotential does not make a truly local appearance in a differential equation in configuration space due to the relativistic factor $(p^2+m^2)^{-1/2}$ in Eqs. (4.13) and (4.14). See however discussion following Eqs. (4.18) and (4.19) further.

diagrams contributing to $V(r, E)$, namely an infinite sum over bubbles, and found that, although one obtains the class II case, the potential is nonsingular. Specifically, it is of the form

$$V(r)\sim(a+b/(\ln r)+c/(\ln r)^2+\cdots)r^{-2}.$$

One sees that the potential one obtains depends on the diagram scheme chosen, as one ought to expect generally. However, one is encouraged to the extent that the classification of the equation seems to be maintained regardless of the selection of diagrams. The general result is in agreement with the result arrived at from the study of the B–S equation.

Vertex Renormalization. The compositeness condition $Z=0$ of field theory[71] has been studied by a number of authors in terms of a potential-theory analog. This analog also confirms the correspondence between the renormalizability attributes of a field theory and the class of the corresponding potential-theory equation. A noteworthy aspect of this correspondence is the cutoff independence of the bound-state condition $Z=0$ for class II potential equations, and of renormalizable field theories in the ladder approximation.

A number of authors, Bertocchi, McMillan, Predazzi, and Tonin (1964a), Bertocchi, Fubini, and Furlan (1964b), and Guttinger (1965), have pointed out the analogy between the Jost function of potential theory, evaluated at a bound-state energy, and the field-theoretic constant Z, which renormalizes the vertex of an interaction of a particle with "component" particles. The vanishing of either quantity expresses the composite nature of the system. The correspondence between these quantities has been made more intimate in the work of Bertocchi, *et al.* (1964a, b), who have defined unrenormalized and renormalized coupling constants in potential theory, and have shown that the Jost function is indeed the ratio of the former to the latter. The unrenormalized and renormalized coupling constants g, G, respectively, for the coupling of two particles of mass m to, say, a mass-zero particle with four-momentum zero are defined in terms of limiting values of the vertex function $\Gamma(p)$ by

$$g=\lim_{p^2\to\infty}\Gamma(p), \tag{4.20a}$$

$$G=\lim_{p^2\to -m^2}\Gamma(p), \tag{4.20b}$$

where Z is defined by $Z\equiv g/G$. The potential-theory analog of $\Gamma(p)$ is $(E-k^2)\Psi(k)$, where $\Psi(k)$ is the momentum space wave function for a particle of energy E. This correspondence can be seen formally in the comparison of Eq. (4.22) for the bound-state wave function in the presence of an external source term and the integral equation of Eq. (4.24) for the vertex function $\Gamma(p)$ in the ladder approximation. Thus for a bound state with energy $E=-B$, the unrenormalized and renormalized coupling constants are defined respectively by

$$g=\lim_{k^2\to\infty}(-B-k^2)\Psi(k), \tag{4.21a}$$

$$G=\lim_{k\to -B}(-B-k^2)\Psi(k). \tag{4.21b}$$

In order to illustrate this correspondence, we consider the coupling of a particle in potential theory to a delta-function point source of strength g, thus obtaining an inhomogeneous Schrödinger equation. The momentum space integral equation describing a bound state of energy $-B$ in this situation has the form

$$(-B-k^2)\Psi(k)=g+\int d^3k'V(k,k')\Psi(k'). \tag{4.22}$$

Consider the case in which the potential is regular. One finds that the $l=0$ bound state for $g\neq 0$ is described by a singular solution (in configuration space) to the radial Schrödinger equation, the strength of the singular part of the wave function being proportional to g. The renormalized coupling constant G, on the other hand, is related to the coefficient of the asymptotic wave function in configuration space. Bertocchi *et al.* (1964a) have shown that the quantity $Z\equiv g/G$ is exactly the Jost function for the potential at energy $-B$. The presence of a state, bound by the potential at energy $-B$, corresponds to $g=0$; i.e., the Jost function vanishes (since G is nonzero), and the wave function is the regular solution at $r=0$. Bertocchi *et al.* (1964b) have shown how to construct a function $Z(E)$ which is an analytic function of E having a cut on the positive real E axis, and which agrees with Z at $E=-B$. The S-matrix at energy $E>0$ is expressible as

$$\begin{aligned}S(E)&=Z(E-i\epsilon)/Z(E+i\epsilon)\\&=Z^{(-)}(E)/Z^{(+)}(E),\end{aligned} \tag{4.23}$$

and $Z(E+i\epsilon)$ clearly plays the role of the Jost function at energy E.

For a potential with the transitionally singular behavior $V(r)\sim f/r^2$ near $r=0$, the (S wave) Jost function is known not to exist (Secs. II.B, III.B.4) although the S-matrix does. In order to study the $Z=0$ condition in this case, a short-range cutoff at $r=\alpha$ can be introduced which makes Z finite. One finds that the α dependence of Z factors, and the bound states determined by the $Z=0$ condition, are cutoff independent. The S-matrix element likewise has a well-defined limit as $\alpha\to 0$. The factorizability of the α dependence of Z can also be argued to hold for potentials with the behavior $V(r)\sim f(\ln r)^p/r^2$, which implies the cutoff independence of the S-matrix and of the bound-state condition $Z=0$. Unlike the case of the simple f/r^2 behavior near $r=0$, the expansions of Z in the coupling constant in the logarithmic case are asymptotic and not analytic (cf. Sec. II.B).

The cutoff dependence of the $Z=0$ condition for various field theories has been studied by a number of

[71] The renormalization constant Z, in what follows presently, will refer only to the vertex renormalization constant Z_1.

authors. By appealing to the work of Bastai *et al.* (1963a,b) on the class II B–S equation, Bertocchi *et al.* (1964b) concluded the cutoff independence of the S matrix, and the $Z=0$ condition for renormalizable field theories in the ladder approximation. They argued that the result remains valid even after all the field-theoretic renormalizations of the internal masses and coupling constants have been performed. They conjectured that the over-all situation ought to remain unchanged even after inclusion of internal loops on the lines.

The behavior of Z as a function of the cutoff was explicitly studied by Furlan and Mahoux (1965) for a variety of field theories, superrenormalizable, renormalizable, and nonrenormalizable. They considered the vertex function for the interaction of two lines corresponding to particles of mass m coupled to a composite particle of four-momentum zero. They worked in the ladder approximation where the exchanged particle is conveniently chosen to have mass zero. The vertex function $\Gamma(p)$ calculated in the ladder approximation obeys an integral equation of the schematic form

$$\Gamma(p)=Z+\lambda\int d^4p'K(p,p')\Gamma(p'), \quad \lambda=f^2/4\pi, \quad (4.24)$$

where Z is to be identified with the bare coupling constant to which it is proportional, and f is the coupling strength of the exchanged particle. Here $K(p, p')$ is a kernel appropriate to the theory. Z is evaluated from Eq. (4.20a).

As a first example Furlan and Mahoux considered the superrenormalizable coupling $f\varphi^2\Psi$ of three spin-zero bosons which gives rise to a bound state of arbitrary spin. The integral equation, Eq. (4.24), can be converted to a second-order differential equation in the variable $u\equiv p^2$ ($S=$spin)

$$\left[\frac{d^2}{du^2}+\frac{S+2}{u}\frac{d}{du}+\frac{\lambda}{u(u+1)^2}\right]\Gamma(u)=0, \quad (4.25)$$

obeying the conditions

$$\lim_{u\to 0} u^2(d/du)\Gamma(u)=0, \quad (4.26a)$$

$$\lim_{u\to\infty}[\Gamma(u)+u(d/du)\Gamma(u)]=Z. \quad (4.26b)$$

They found Z to be finite (except for an infrared divergence for which a cutoff is introduced).

Furlan and Mahoux also considered two examples of a renormalizable field theory: a four spin-zero boson coupling $f\varphi^2\Psi^2$, and a Yukawa coupling $f\bar{\Psi}\gamma\Psi\varphi$, where $\gamma=\gamma_5$ or 1. They found a fourth-order class II equation in r, as well as an equation of the form of Eq. (4.25). They also found Z can be defined only with a cutoff, though the $Z=0$ condition is independent of the cutoff in the "uncritical" range of the coupling constant. There is a critical range of the coupling constant in which the effective potential is of an attractive singular nature, and in which the cutoff dependence of Z cannot be factored. The limit of the set of cutoff-dependent bound-state coupling values, as the cutoff is removed, was found to merge into a line of values extending to infinity.

The nonrenormalizable interaction $f\bar{\Psi}\gamma\Psi\varphi^2$ ($\gamma=\gamma_5$ or 1) was considered by Furlan and Mahoux, and by Aly and Müller (1966a) who also included an additional renormalizable Yukawa interaction for greater generality. Furlan and Mahoux found the integral equation for the vertex function $\Gamma(p)$ in the ladder approximation,

$$\Gamma(p)=Z-\frac{\lambda\epsilon}{(2\pi)^4}\int\frac{d^4p'\Gamma(p)}{p'^2+m^2}\times\int\frac{d^4k}{k^2(k+p'-p)^2}, \quad (4.27)$$

where $\lambda=f^2/(2\pi)^4m^2$, and $\epsilon=\pm1$ according to the γ_5 or 1 coupling. This equation is transformed to the Euclidean metric. One defines a quantity $F(x^2)$ by

$$\Gamma(p)/(p^2+m^2)=[1/(2\pi)^2]\int d^4xe^{-ipx}F(x^2), \quad (4.28)$$

as suggested by the previously noted correspondence between the vertex function and the quantity $(E-k^2)\Psi(k)$. The quantity $u(r)=rF(r^2)$ satisfies the differential equation ($M=1$)

$$\left[\frac{d^2}{dr^2}+r^{-1}\frac{d}{dr}-\frac{1}{r^2}-1\right]u(r)=\frac{u(r)}{r^4}, \quad (4.29)$$

and the boundary conditions

$$\lim_{r\to\infty} e^{-r}r^{1/2}(u'+u)=0, \quad (4.30a)$$

$$\lim_{r\to 0} r(ru'-u)=-2Z. \quad (4.30b)$$

This equation involves the singular potential r^{-4} and can be solved in terms of Mathieu functions (Sec. III.A.1). de Alfaro and Predazzi (1965) have also studied the same integral equation directly in momentum space, and have found the large momentum behavior of $\Gamma(p)$ by means of Mellin transform methods (Sec. III.B.7).

Furlan and Mahoux, and Aly and Müller have both calculated the quantity Z for this interaction in the presence of a small radial cutoff. While one does not expect to find bound states in the repulsive case, one does find that the cutoff dependence of Z factorizes as the cutoff becomes small. For the attractive case, the cutoff dependence of Z cannot be factorized as the cutoff becomes small. Aly and Müller have proposed a condition that would select a discrete spectrum from the $Z=0$ condition in the attractive case. This condition consists of the vanishing of the wave function at a suitably chosen complex radius. This is in line with the suggestion of Case that one can obtain a discrete spectrum for the attractive singular problem by arbitrarily fixing a parameter in the solution (Sec. II.C). The condition of Aly and Müller is of course quite arbitrary, but they indicated that the condition has the reasonable

attribute that the location of the cutoff shrinks to zero as the energy becomes infinite or as the coupling constant goes to zero.

Dispersion Relations. The validity of dispersion relations for singular potentials has been considered briefly by Martin (1966), and by Aly and Wit (1967). Martin has proved the validity of a dispersion relation in energy for a purely repulsive singular potential which has a power-type singularity and is cut off beyond some finite distance R. Though the proof is explicitly constructed for a potential whose singularity is weaker than r^{-3} in which case one subtraction is sufficient, it appears that the proof can be readily extended to more singular power potentials as outlined briefly in the article.

Aly and Wit have considered the question of the number of subtractions necessary in partial-wave dispersion relations. They concluded that one subtraction is sufficient for an r^{-2}-type singularity, but that no finite number will suffice for any stronger power-type singularity. They have employed Phragmen–Lindelof-type theorems in arriving at the latter result. However, their conclusion, which is in contradiction with Martin, does not seem justified. It is based on the assumption of exponential growth of the scattering amplitude. Such growth, however, has not been established on the physical sheet, where a dispersion relation would be written. (See the discussion in Sec. II.D.1.)

C. Renormalization

The idea of renormalization, which has been eminently successful and of profound significance in the theory of fields and particles, can be applied also to the theory of singular potentials. Arbuzov, Filippov, and Khrustalev (1964), and Ahmed and Fairlie (1965) have shown for a class of transitionally singular potentials, that the wave function for the regularized potential becomes renormalized by a regulator-dependent factor in the singular limit. This is analogous to wave function renormalization in field theory. Guttinger and Pfaffelhuber (1966) and Aly and Taylor (1968) have shown that the ideas and techniques of field-theoretic renormalization can be applied to the Lippmann–Schwinger (L–S) equation describing the scattering by a potential with a power-type singularity.

Wave Function Renormalization of Transitionally Singular Potentials. Arbuzov, Filippov, and Khrustalev have considered the wave function corresponding to the transitionally singular potential

$$V(r)=-gr^{-2}\ln r \tag{4.31}$$

with angular momentum $l=\lambda-\frac{1}{2}$, and at zero energy. It obeys the integral equation

$$u(r)=r^{\lambda+\frac{1}{2}}-(2\lambda)^{-1}\int_0^r dr'(rr')^{1/2}(r/r')^{\lambda}V(r')u(r')$$
$$+(2\lambda)^{-1}\int_0^r dr'(rr')^{1/2}(r'/r)^{\lambda}V(r')u(r'). \tag{4.32}$$

Iterations do not yield finite results as expected, and Arbuzov *et al.* introduced a regularization in the form of a lower-limit cutoff at $r=\alpha\neq 0$ in the integral equation. One can verify that it suffices to introduce the cutoff only in the first integral in order to obtain finite results to every order of iteration, and that the solution regularized in this way maintains the property of vanishing at $r=0$. Arbuzov *et al.* found that the iteration solution takes the form of a product of a factor depending on α only with a factor depending on r only. The latter is identifiable with the asymptotic series in g for the known exact solution of Eq. (4.32) [Sec. III.A.2, Eq. (3.75)]. We note in passing that the calculation of Arbuzov *et al.* also provides an illustration of the fact that by summing leading singularities in each order one can badly misrepresent the behavior of the sum (Sec. IV.D).

The multiplicative renormalization property noted by Arbuzov *et al.* for the potential of Eq. (4.31) was shown by Ahmed and Fairlie (1965) to hold for a potential of the general form

$$V(r)=\sum_{n=1}^{N}[g_n(\ln r)^n/r^2], \tag{4.33}$$

and at nonzero energy, where one may formally employ the representation

$$k^2=\frac{k^2}{r^2}\left[1+\sum_{n=1}^{\infty}\frac{2^n(\ln r)^n}{n!}\right]. \tag{4.34}$$

One can then formally solve Eq. (4.32), where $V(r)$ includes the energy term in the form of Eq. (4.34) if $k^2\neq 0$, through the ansatz

$$u(r)=r^{\lambda+\frac{1}{2}}\sum_{n=0}^{\infty}c_n(\ln r)^n \tag{4.35}$$

which is an asymptotic expansion (Sec. II.B). The coefficients c_n of Eq. (4.35) obey easily derived linear recursion relations. For $k^2=0$ and N finite in Eq. (4.33), the recursion relations have a bounded number of terms for all $n\neq 0$; otherwise the number of terms depends on the recursion relation. One can easily show from the recursion relations that the α dependence of $u(r)$ is factorable. Ahmed and Fairlie have further shown how to choose an α-dependent coefficient for the exact solution to the potential Eq. (4.31) so that the iteration solution to Eq. (4.32) (with cutoff) is reproduced with the proper wave function renormalization. They have also shown that, through the choice of an appropriate relation between α and g, the explicit essential singularity of the solution can be made to disappear in favor of a formal asymptotic series in g. Bertocchi, Fubini, and Furlan (1964b) have concluded, on the basis of the renormalization technique of Dyson, that the potential-theory analog of the vertex renormalization constant (see previous section) ought to have a factorable cutoff dependence for transitionally singular potentials.

Renormalization of Power-Type Singular Potentials. The Lippmann–Schwinger (L–S) equation for a singu-

lar potential[72] does not exist because the Born term is not defined. The Fourier transform for a power-type singular potential can be defined if the potential is understood in the sense of a distribution. This generalized L–S equation which one obtains fails to possess convenient iterative properties. The distribution-theoretic redefinition of singular potentials was presented in detail by Guttinger and Pfaffelhuber (1966). In two earlier articles Guttinger, Penzl, and Pfaffelhuber (1965a, b) presented a more concise version of these ideas, together with a method of solution. Aly and Taylor (1968) also employed a distribution-theoretic redefinition of singular potentials in order to write a L–S equation, which they can solve iteratively, employing subtraction ideas analogous to field-theoretic renormalization.

Guttinger and Pfaffelhuber (1966) defined a regular potential $V_R(r)$ equivalent in a distribution-theoretic sense to a singular potential $V(r)$ by the two conditions:

$$\text{(i)}\quad \int d^3r V(r)\varphi(r) = \int d^3r V_R(r)\varphi(r) \tag{4.36}$$

for all test functions $\varphi(r)$ which vanish along with a sufficiently large (finite) number of derivatives at $r=0$, and (ii) $V_R(r)$ has a finite three-dimensional Fourier transform $\tilde{V}_R(k)$. These two characterizations define $V_R(r)$ up to an arbitrary additive distribution of the form

$$F(r) = \sum_{n=0} c_n(\nabla^2)^n\delta(r) \tag{4.37}$$

(∇^2 is the Laplacian operator). An explicit representation for $V_R{}^m(r)$, the equivalent regular potential for potentials of the form $V^m(r)=gr^{-m}$, can be constructed over the space of test functions $\varphi(r)$ which do not necessarily vanish with derivatives at $r=0$. This is given by the limiting procedure[73]

$$\int d^3r V_R{}^m(r)\varphi(r) = \lim_{\epsilon\to 0}\int d^3r\left[gr^{-m}\theta(r-\epsilon) + 2\pi^{3/2}g\sum_{j=0}^{M}\frac{\epsilon^{2j-m+3}}{2^{2j}j!\Gamma(j+\frac{3}{2})(2j-m+3)}\right]\varphi(r), \tag{4.38}$$

where M is the smallest integer such that $2M>m-3$. One notes that the expression in Eq. (4.38) corresponds to the subtraction from $V(r)$ of a multipole core consisting of a linear combination of derivatives of $\delta(r)$, which compensates for the singularity of $V(r)$ at $r=0$.

[72] The Born term exists for power potentials $V(r)=gr^{-m}$ with $m<3$, though such potentials with $m>2$ would be singular by our definition. In the present considerations regarding the L–S equation, singularity of the potential will mean a singularity at least as strong as an r^{-3} potential. The L–S equation which is defined for $2<m<3$ is however not iterable. In connection with the work of Aly and Taylor who were concerned with iterability properties of the L–S equation, singularity of the potential would mean a singularity at least as strong as r^{-2}. The existence of the Born term is guaranteed when the large r behavior of the potential is characterized by an exponent $m>1$, if $k\neq 0$, and $m>3$ if $k=0$.

[73] The validity of Eq. (4.38) entails sufficiently rapid attenuation of the potential tail for large r. When m is of the form $3+2n$ ($n=0, 1, 2\cdots$), the term with $j=M$ in Eq. (4.38) is modified. See Eqs. (2.4), (2.13b), and (2.15) in Guttinger and Pfaffelhuber.

The Fourier transform of the equivalent regular potential has the form

$$\tilde{V}_R{}^m(k) = \frac{g}{\pi^{3/2}2^m}\frac{\Gamma[(3-m)/2]}{\Gamma(m/2)}k^{m-3}, \tag{4.39}$$

unless m is of the form $3+2n$ (n a nonnegative integer), in which case a different expression applies.[73] Equations corresponding to Eqs. (4.38) and (4.39) are also derived for potentials of the more general form $V^{mp}(r)=gr^{-m}(-\ln r)^p$. When m is not of the form $3+2n$, the quantities in Eqs. (4.38) and (4.39) coincide with the analytic continuations from $\mathrm{Re}\, m<3$, where these quantities are well defined, to $\mathrm{Re}\, m>3$. For m of the form $3+2n$, the quantities in Eqs. (4.38) and (4.39) have a simple pole. Equations (4.38) and (4.39) are then defined in terms of the regular part of the Laurent expansion about this pole. The authors presented a compact expression in the form of a contour integral which automatically represents the quantities of Eqs. (4.38) and (4.39) for all singular potentials.

One can now define the L–S equation in terms of the Fourier transform $V_R(p)$ of the equivalent potential

$$T_R(p,k) = \tilde{V}_R(p-k) + \int d^3q[\tilde{V}_R(p-g)T_R(q,k)/(k^2-q^2+i\epsilon)]. \tag{4.40}$$

(Note that the one-the-energy shell L–S equation has been used.) For repulsive potentials it can be shown that one obtains the same T-matrix element as from the defining expression

$$T(p,k) = [1/(2\pi)^3]\int d^3r V(r)\Psi(r,k), \tag{4.41}$$

where $\Psi(r,k)$ is the wave function. This is due to the very strong damping properties of $\Psi(r,k)$ near $r=0$. These kill the multipole core additive terms of the form in Eq. (4.37), which distinguish $V(r)$ from $V_R(r)$. The prescriptions of Eqs. (4.38) through (4.40) described previously may be taken to define a scattering amplitude for the attractive case too. However, one would not find unique solutions in the attractive case (Sec. II.C), since Eq. (4.41) would be affected by the addition to $V_R(r)$ of terms of the form Eq. (4.37). The lth partial-wave L–S equation obtained from Eq. (4.40) coincides with the result of an analytic continuation in l from the half-plane $\mathrm{Re}\,(2l-m)>-3$, when $2l-m$ is not of the form $-3-2n$ (n a nonnegative integer). In the latter case, the partial-wave equation coincides with the regular part of the Laurent expansion about the simple pole in l. The alternatives of interpretation through analytic continuation in the l or m variables follow from the appearance of the variable $2l-m$ in the pole. The idea of analytically continuing the L–S equation in l for singular potentials was also noted briefly by Arbuzov and Filippov (1964). Guttinger and Pfaffelhuber have also noted that, for pure negative even power potentials, the L–S equation, Eq. (4.40), can be converted into a linear partial differential equation of order equal to the negative of the exponent $-m$ (Sec. III.B.7).

In two earlier papers Guttinger, Penzl, and Pfaffelhuber (1965a, b) studied the radial partial-wave equation for a singular potential using distribution-theoretic ideas, though the analysis was much less thoroughgoing. In these papers they introduced an interesting generalization of the distribution-theoretic approach, and explicitly solved for the scattering length due to a repulsive power potential. The zero-energy radial equation for the radial wave function is

$$u(r)=r+\int_0^r dr'G(r';r)V(r')u(r') + \int_r^\infty dr'G(r;r')V(r')u(r'), \quad (4.42)$$

with $G(r;s)$ proportional to $r^{l+1}s^{-l}$ for the lth partial wave. For the case under discussion, $V(r)$ has a power-type singularity, and may include additional logarithmic factors. One knows that Eq. (4.42) does not lead to finite iterations. In view of the very damped behavior of $u(r)$ near $r=0$, one may treat the singular factor $G(r;r')V(r')$ as a distribution. This entails the subtraction from $G(r;r')V(r')$ of an appropriate linear combination of derivatives of the δ function at $r'=0$. The resulting integral equation has, of course, the correct exact solution. In this paper Guttinger *et al.* have suggested a generalization of the distribution prescription which produces a converging sequence of iterations which is finite at each stage. The prescription based on a distribution-theoretic motivation is equivalent to the simple ansatz

$$\int_0^r dr'G(r';r)V(r')u(r') = Ar^{-l} - \int_r^\infty dr'G(r';r)V(r')u(r'), \quad (4.43)$$

where the constant A is easily recognized as the r' integral over the whole range of positive r'. More generally, the prescription entails both the choice of some convenient constant limit in the integral (which need not lie along the contour) which makes the integral convergent, together with a compensating subtraction of the appropriate constant integral. The latter may be treated as an indeterminate parameter. The quantity A in Eq. (4.43) is merely the scattering amplitude and is to be determined by the condition $u(0)=0$. The integral equation, obtained after substitution of Eq. (4.43) into Eq. (4.42), is iterable and allows the determination of A by successive approximations. A similar treatment of the integral equation, Eq. (4.42), without the elaborate distribution-theoretic rationale, is given by Pais and Wu (1964b). Guttinger *et al.* have employed this method to calculate the scattering length due to a repulsive pure power potential.

Aly and Taylor (1968) have applied to singular potentials procedures, analogous to renormalization in field theory, with the object of obtaining finite results from the iterative solution. They start with the off-the-energy shell L–S equation

$$T(p,p';k^2)=\tilde{V}(p-p') + \int d^3q[\tilde{V}(p-q)T(q,p';k^2)/(k^2-q^2+i\epsilon)]. \quad (4.44)$$

The Born term is a Fourier transform of the singular potential, and does not exist in the conventional sense. It is redefined by appeal to the interpretation of the singular potential as a distribution. One finds $\tilde{V}(p)\sim p^{m-3}$ as $p\to\infty$ for $V(r)\sim r^{-m}$ at $r=0$. The resulting L–S equation is not iterable. A finite number of subtractions can be performed on Eq. (4.44) to produce an iterable form. The subtraction procedure is defined in analogy to the momentum space renormalization of the Green's function equations in a renormalizable field theory, particularly as developed in a series of articles by Taylor (1963). For this purpose, one first differentiates Eq. (4.44) with respect to the external momentum p, and then integrates it back, absorbing the constants of integration into renormalization constants. This is formally achieved by means of an operation $D_p\equiv d/d\,|\,p\,|$. Differentiation and integration of Eq. (4.44) M times, results in[74]

$$T(p,p';k^2)=V'(p,p';k^2) + \int_0^{|p|}d\lambda_1\int_0^{\lambda_1}d\lambda_2\cdots\int_0^{\lambda_{M-1}}d\lambda_M\int d^3q \times\frac{[D_{p''}{}^M\tilde{V}(p''-q)]_{p''=\lambda_M p}T(q,p';k^2)}{k^2-q^2+i\epsilon}, \quad (4.45)$$

where

$$V'(p,p';k^2)=\int_0^{|p|}d\lambda_1\int_0^{\lambda_1}d\lambda_2\cdots \times\int_0^{\lambda_{M-1}}d\lambda_M D_{p''}{}^M[\tilde{V}(p''-q)]_{p''=\lambda_M p} - \sum_{r=0}^{M}\frac{|\,p\,|^r}{r!}\left[\frac{d^rf(p,p';k^2)}{d\,|\,p\,|^r}\right]_{|p|=0}. \quad (4.46)$$

If $m-2=M$, a positive integer, then one can write

$$\tilde{V}(p)=\sum_{j=0}^{M-1}a_j\,|\,p\,|^j+W(p), \quad (4.47)$$

where $W(p)=O(|\,p\,|^{-1})$ for p large. If only even degrees of p appear in the first expression on the right of Eq. (4.47) which Aly and Taylor claimed can always be arranged, then one finds $D_p{}^M[\tilde{V}(p-q)]=D_p{}^MW(p-q)$ in which the polynomial term in p is absent. The quantity $V'(p,p';k^2)$ has large p behavior which is $O(|\,p\,|^{M-1})$, and iteration leads to the same behavior for $T(p,p';k^2)$. Thus one finds that the subtracted L–S equation can be solved by iteration.

[74] Equations (4.45) and (4.46) are intended to be a corrected form of Eq. (8) of these authors, which is clearly in error.

Aly and Taylor have not studied these solutions and their relation to the exact solution.

It is worthwhile to contrast the work of Guttinger and co-workers with that of Aly and Taylor. The primary intention of Guttinger and co-workers was to define a Lippmann–Schwinger equation for scattering by singular potentials. This was done by a careful distribution theoretic analysis. This leads to a L–S equation which produces solutions which agree with correct scattering solutions for a repulsive singular potential, and which provides nonunique solutions for an attractive potential. Aly and Taylor, on the other hand, dealt with the problem of defining a Born term only very briefly, and focused primarily on how to obtain meaningful solutions of the L–S equation by iteration. Their procedure led them to the limitation that m be an even integer. One may also note that Guttinger and Pfaffelhuber have worked with the L–S equation on the energy shell, while Aly and Taylor found it necessary to work with the L–S equation off the energy shell.

D. Peratization

Peratization[75] was suggested by Feinberg and Pais (1963, 1964) as a method of evaluating higher-order correction terms to the scattering amplitude in nonrenormalizable field theories. Basically the method is the following. In a nonrenormalizable theory, the individual terms of the Born series do not exist, and moreover, taken together, cannot be made to exist in terms of a finite number of renormalization constants. A cutoff parameter (regulator) is introduced into the theory so that the terms of the Born series become finite. Of course, they would be expected to diverge as this parameter becomes infinite. A set of the most divergent parts, either from each term of the Born series or from a class of diagrams, are retained and summed, after which the cutoff is made infinite. In some cases this procedure has been shown to yield finite results.

However, in the case of a full-blown field theory, there exist no results which are either exact solutions or good approximations. Hence one had no way of judging the correctness of the results of the peratization procedure as employed by Feinberg and Pais. Peratization was therefore applied by many authors in calculations involving singular potentials, for which exact solutions are known, as a test of the validity of the method itself. It was indicated in an earlier section that some nonrenormalizable field theories effectively give rise in some approximations to singular potentials in configuration space (see Sec. IV.B). Furthermore, the scattering amplitude associated with a singular potential has a singularity such as a branch point in the coupling constant g at $g=0$, and hence the perturbation series expansion about $g=0$ diverges. Thus it is appropriate to investigate peratization by means of singular potentials.

Let us now consider in more detail the peratization idea as applied to potential scattering. As the starting point in the field-theoretic context, an (invariant) cutoff parameter which renders all divergent quantities finite, is introduced into the theory. Analogously, in potential scattering, a regularized (parameterized) nonsingular potential $V(r, \alpha)$ is introduced such that

$$\lim_{\alpha\to 0} V(r, \alpha) = V(r), \tag{4.48}$$

where $V(r)$ is singular. Conditions under which the wave function and scattering parameters, corresponding to the regularized potential, converge to the wave function and scattering parameters of the singular potential as $\alpha\to 0$, have been discussed previously (Sec. III.B.6 under Limiting Procedures). Two forms of regularization commonly employed in peratization calculations are the + regularization,

$$V_+(r, \alpha) = V(r+\alpha),$$

and the θ regularization,

$$V_\theta(r, \alpha) = V(r)\theta(r-\alpha).$$

With divergences thus removed, one now calculates the Born series which has a finite, although small, radius of convergence which depends on α. Insofar as most authors, in their studies of peratization for potential scattering, calculate the zero-energy scattering amplitude, i.e., the scattering length, we apply the notion here to the scattering length also, although the method is applicable at nonzero energies. We write the scattering length $A(\alpha)$ corresponding to the regularized potential $V(r, \alpha)$ as

$$A(\alpha) = \sum_{n=1}^{\infty} g^n b_n(\alpha). \tag{4.49}$$

Because the unregularized potential is singular, the coefficients $b_n(\alpha)$ become infinite as $\alpha\to 0$. These coefficients can generally be expanded in terms of an appropriate set of quantities which are singular in α,

$$b_n(\alpha) = \sum_m y_{nm}(\alpha) b_{nm}. \tag{4.50}$$

Here the set $\{y_{nm}(\alpha)\}$ is a set of singular basis functions such that

$$y_{nm}(\alpha)/y_{n,m+1}(\alpha) \underset{\alpha\to 0}{\longrightarrow} 0.$$

Generally the functions $y_{nm}(\alpha)$ have the form

$$y_{nm}(\alpha) = [x(\alpha)]^n u_{nm}(\alpha).$$

The function $x(\alpha)$ becomes infinite as $\alpha\to 0$, while the functions $u_{nm}(\alpha)$ are usually nonsingular, except perhaps for the first few values of m. It should be emphasized that the choice of singular basis functions $\{y_{nm}(\alpha)\}$ in Eq. (4.50) is not unique. In an actual

[75] The term "peratization" was developed by Feinberg and Pais (1963) from the Greek language; the negative form means boundless, infinite, that "in which one is entangled past escape," while the affirmative means "to cut a long story short."

calculation, one set usually emerges as the most convenient.

At this point we introduce the concept of peratization. This consists first of retaining in each order in g in the Born expansion terms of the highest singularity in α, and then of summing these terms, after which α is allowed to approach zero.[76] By first peratization we mean that only the most singular term in each order in g is retained; by second peratization we mean that the first two most singular terms (in each order in g) are retained, and so forth. We denote by $A^{(s)}$ the result of performing s peratizations in the calculation of A, which in the above context is the scattering length. The first and second peratized scattering lengths are, for example,

$$A^{(1)}=\lim_{\alpha\to 0}\sum_{n=1}^{\infty}[gx(\alpha)]^n u_{n1}(\alpha)b_{n1},$$

$$A^{(2)}=\lim_{\alpha\to 0}\sum_{n=1}^{\infty}[gx(\alpha)]^n[u_{n1}(\alpha)b_{n1}+u_{n2}(\alpha)b_{n2}].$$

This process in general can be continued to give higher steps in the approximation scheme. The problem in investigations of the peratization technique is to determine whether the elements of the set $\{A^{(s)}\}$ so obtained are finite (neither zero nor infinity), and whether the set forms a sequence of approximations to the exact solution A.

We illustrate the notion of summing a series, all the terms of which diverge, by calculating the scattering length for the singular potential gr^{-4}. With θ regularization, the Born expansion is readily found to be

$$A(\alpha)=-[g\alpha^{-1}-\tfrac{1}{3}g^2\alpha^{-3}+(2/15)g^3\alpha^{-5}\cdots]$$
$$=-g^{1/2}\tanh g^{1/2}/\alpha.$$

In the $\alpha\to 0$ limit, we have

$$A=-g^{1/2}.$$

This is the exact scattering length for the potential gr^{-4}.

This potential, and in fact, the entire class of the pure inverse power potentials $V(r)=gr^{-m}$ at zero energy, do not really constitute a test of peratization as an approximation procedure. In each order in g in the Born series for the regularized potential, each term is of the form α^{-p}, where p is a positive number. Thus, with the singular basis functions α^{-p}, there are no lesser singularities in the Born series. Upon summing the series and setting $\alpha\to 0$, one obtains the exact scattering length. The fact that the result is exact should not be surprising since nothing has been neglected. However, these potentials do illustrate the possibility of summing a series of divergent terms and getting a finite result which is related in a definite way to the exact answer.

Before discussing the various investigations of peratization which have been performed for potential scattering, we ought to emphasize the distinction between regularization and peratization. This distinction has been a frequent source of confusion in the literature. Regularization consists essentially of two steps. First, the singular potential is replaced by a parameterized, nonsingular potential, according to Eq. (4.48), and this is then used in the calculation of physical quantities. The second step consists of a limiting process wherein the regularization parameter is allowed to approach zero. Peratization, on the other hand, consists of three steps. First, physical quantities, calculated from a regularized potential, are expanded in the Born series. Second, only leading singularities from the individual terms of the Born series are retained, and these are then summed. Third, as with regularization, the regularization parameter is put equal to zero by a limiting process.

It has been observed by Feinberg and Pais (1963), Khuri and Pais (1964), and Tikotpoulos and Treiman (1964) that, at least in some cases the leading singularities of the Born series for the scattering amplitude at nonzero energy give rise to the scattering amplitude at zero energy, i.e., the scattering length. Several authors have, therefore, calculated the scattering length by a regularization technique, and claimed that such a calculation is a test of peratization. However, as we shall see shortly, the literal application of the peratization procedure as a systematic isolation and summation of successive leading singularities of the Born series usually fails to yield a sensible result. Hence, studies which merely compute the regularized scattering length do not probe the mechanism of peratization. Regularization, if successful, yields exact answers since nothing in the problem is neglected (see Sec. III.B.6). Peratization, on the other hand, if it is successful, would generally be expected to give a sequence of approximations only. Of course, it would certainly appear that the success of regularization is a necessary prerequisite for the success of peratization, but again it is emphasized that these are separate problems.

The above discussion clearly implies what we mean by saying that peratization is successful. Namely, we say that peratization is successful if it leads to a systematic sequence of results which are related in a definite and sensible way to the exact answer. In fact, when successful, peratization gives a sequence of approximations valid in the small-coupling limit. One should not expect an exact result. It is the purpose of the studies of peratization in potential scattering to see when this is indeed the case.

We mention in this connection a general result due to Cornille (1966): If the wave functions corresponding to two different regularizations converge to the proper limit (regular solution) as $\alpha\to 0$, then the first peratiza-

[76] See Eq. (2.17) of Khuri and Pais (1964), as well as Sec. III.B.6 of the present work.

tion of each is the same. It is understood that leading singularities are to be obtained from expansions in the same set of singular basis functions.

Classes of Singular Potentials. There are three classes of singular potentials which are discussed most frequently in connection with peratization. The first class includes potentials having, as the leading singularity at the origin, a pure inverse power

$$V(r)=(g/r^m)+V_2(r).$$

The second class includes potentials which have a logarithmic-type singularity together with an inverse power, e.g.,

$$V(r)=g[(\ln r^{-1})^p/r^m]+V_2(r).$$

This type of singularity will generally be referred to in this section as a logarithmic singularity. The third class includes potentials which have an exponential-type singularity with an inverse power, e.g.,

$$V(r)=g[\exp(\lambda/r^\beta)/r^m]+V_2(r);$$

these are called simply exponential-type singularities. The second and third classes of singular potentials listed above will be generalized in the following. In all cases the second term $V_2(r)$, which may or may not depend on g, is taken to be less singular than the first (it could be zero). We note that condition (c) must be satisfied if we wish to calculate the scattering length. We now discuss peratization as applied to these specific classes.

Inverse power singularity. Among the first authors to study peratization in connection with singular potentials were Khuri and Pais (1964), Tiktopoulos and Treiman (1964), and Pais and Wu (1964a). All of these authors consider potentials which are given exactly by a pure inverse power or whose leading singular behavior of the origin is that of an inverse power.

Khuri and Pais (1964) applied the peratization idea to the pure inverse power potential gr^{-m} at arbitrary energy. They proposed the general regularization

$$V(r,\alpha)=\alpha^{-m}U(r/\alpha), \tag{4.51}$$

where the function $U(\zeta)$ is arbitrary but subject to certain integrability and asymptotic conditions. Since the leading singularities in the Born expansion of the scattering amplitude for nonzero energy are associated with the scattering problem at zero energy, these authors concentrated on this case. It was shown that the zero-energy amplitude is a good approximation to the amplitude for nonzero energy provided the condition $kg^\nu \ll 1$ is satisfied, where k is the momentum of the scattering particle, and $\nu=(m-2)^{-1}$. Khuri and Pais therefore developed a regularization procedure, the σ-limiting procedure, in order to evaluate the scattering length.[76] This procedure, however, has been criticized by several authors [Calogero (1965a), Cornille (1966), and Gale (1967)] because of an unjustified exchange of two limits and has been discussed by us previously under "Regularization Methods," Sec. III.B.6. In addition, the σ-limiting procedure has subsequently been applied by some authors, under the guise of peratization, in calculations of the scattering length for various singular potentials. We emphasize again that this procedure examines only regularization at zero energy, and that nothing can be concluded about peratization.

Tiktopoulos and Treiman (1964) also applied peratization to the pure inverse power potential $V(r)=gr^{-m}$ for arbitrary energy in the weak-coupling limit. They showed that an expansion of $\tan\delta_0(k;g)$ in g about $g=0$ is asymptotic, and that peratization (i.e., first peratization) gives the leading term in this expansion. These conclusions are essentially the same as those reached by Khuri and Pais (1964), but the work of Tiktopoulos and Treiman is a bit more detailed as regards peratization. In addition to the simple power potential, peratization for the more general potential

$$V(r)=g[r^{-m}+V'(r)], \qquad r^mV'(r)\underset{r\to 0}{\longrightarrow}0,$$

is discussed briefly. The authors point out that, in first peratization, the less singular term in the potential does not affect the scattering parameters and they suggest that this is the case in general in the weak-coupling limit.

Pais and Wu (1964a) have discussed peratization in connection with the sum of two power potentials for the particular class

$$V(r)=(g/r^{2+2\tau})+(f/r^{2+\tau}), \qquad \tau>1. \tag{4.52}$$

The condition $\tau>1$ implies our condition (c), and is imposed in order that the scattering length exist. This potential has an exact solution which is given in Sec. III.A.2. Pais and Wu did not explicitly peratize this potential, but rather studied the analytic properties of the exact scattering length as a function of both g and f [Sec. III.A.2, Sec. II.E, and Frank (a)]. Their purpose in this study was to discuss some of the problems that arise in peratization when dealing with two physical interactions, and, in particular, to develop a procedure for handling the double infinite series in g and f. However their discussion is limited to the potential of Eq. (4.52).

Gale (1967) has evaluated the first and second peratized scattering lengths for the potential

$$V(r)=(g/r^5)+(f/r^4).$$

This potential can also be solved exactly and is discussed in Sec. III.A.2. Peratization is performed with θ regularization, and with the potential expressed as

$$V(r)=g(r^{-5}+\varphi r^{-4}), \tag{4.53}$$

where $\varphi=f/g$. It was shown by Gale that the first

peratized scattering length for this potential is exactly the scattering length for the potential with $\varphi=f=0$, and that the second peratized scattering length is equal to the first plus a correction term linear in φ.

Gale's results can easily be extended to the s-peratized scattering length [Frank and Land, (1970a)]. This quantity can be found from the exact solution, and is given by a polynomial in φ of degree $s-1$. In fact, the terms of this polynomial are merely the first s terms in a Taylor series expansion of the scattering length in φ about $\varphi=0$. This comes about because, in each order in g, different orders of singularities differ only by integral powers of α. Furthermore, the parameter α, and coupling constant φ, always appear in the product $\alpha\varphi$, except for an over-all factor α^{-3n}. Hence each order of peratization contributes successively to one and only one degree in φ. We see in this example that the power of the parameter φ labels the order of peratization.

Having examined in detail one particular potential given by the sum of two pure inverse powers, we may safely generalize this behavior to an arbitrary sum of two pure inverse powers,

$$V(r)=g(r^{-m}+\varphi r^{-m'}), \qquad m>m'. \tag{4.54}$$

In the general case, different orders of singularities differ by α^p, where $p=m-m'$, and the coupling constant φ appears in the product $\varphi\alpha^p$. Each peratization thus contributes to only one order in φ, and so we conclude that s peratizations give s terms in a Taylor series expansion in φ of the scattering length.

Logarithmic-type singularity. The second general class of potentials which have been used in the peratization program are potentials having logarithmic-type singular behavior at the origin, together with an inverse power. An example of this class of potentials is given by

$$V(r)=gr^{-m}(\ln r^{-1})^p, \qquad p>0.$$

However, this class may be expressed most generally by the form (Spector, 1966b; Gale, 1966b; Frank and Land, b)

$$V(r)=g[L(r)/r^m],$$

where $L(r)$ is a slowly varying function, defined by (Evgrafov, 1961)[77]

$$L'(r)/L(r)=1/[r\lambda(r)], \qquad |\lambda(0)|=\infty.$$

Examples of functions $L(r)$ include

$$L(r)=(\ln r^{-1})^p,\ \ln\ln r^{-1},\ (1-\ln r)^{-1}, \qquad p>0, \tag{4.55a}$$

$$L(r)=\exp(\lambda r^\mu),\ \cos^\lambda r^\mu, \qquad \mu>0. \tag{4.55b}$$

A significant property of slowly varying functions is that (Evgrafov, 1961)

$$\lim_{\alpha\to0}[L(\alpha\rho)/L(\alpha)]=1.$$

This relation provides the first term in a singularity expansion of a slowly varying function, and enables one to calculate the first peratized scattering length.

It is useful to introduce a very general subclass of these functions, which are called tempered slowly varying functions (Frank and Land, 1970b), by the requirement that the function $\lambda(r)$ be itself slowly varying. The examples of Eq. (4.55a) are tempered slowly varying functions. For these functions, we have the relation

$$\lim_{\alpha\to0}[L(\alpha\rho)/L(\alpha)]=1+[\ln\rho/\lambda(\alpha)][1+o(1)],$$

which provides a second term in the singularity expansion of $L(\alpha\rho)$, and enables one to calculate the second peratized scattered length.

Aly, Riazuddin, and Zimerman (1964) have evaluated the first peratized scattering length for the potential $g\ln^2 r/r^4$ with θ regularization. They calculated the first few terms of the Born series, and kept the leading singularity in each term to obtain

$$\begin{aligned}A^{(1)}(\alpha)&=-[g(\ln^2\alpha^{-1})\alpha^{-1}-\tfrac{1}{3}g^2(\ln^4\alpha^{-1})\alpha^{-3}\\&\qquad+(2/15)g^3(\ln^6\alpha^{-1})\alpha^{-5}+\cdots]\\&=-g^{1/2}(\ln\alpha^{-1})\tanh(g^{1/2}(\ln\alpha^{-1})/\alpha)\\&\underset{\alpha\to0}{\longrightarrow}-g^{1/2}\ln\alpha^{-1}.\end{aligned}$$

This limit is not defined. Hence they concluded that peratization is not successful.

The class of potentials

$$V(r)=gr^{-m}\ln r^{-1}$$

has been studied by Wu (1964). This author has calculated the asymptotic ($\alpha\to0$) limit of every contribution to the infinitely peratized scattering length. For $m=4$, the first two contributions in the asymptotic limit give rise to the first and second peratized scattering lengths

$$A^{(1)}=-g^{1/2}\ln^{1/2}(1/\alpha),$$
$$A^{(2)}=-\tfrac{1}{2}g^{1/2}\ln^{1/2}(1/\alpha).$$

Thus the effect of including the second leading singularities in the Born series is to change only the numerical coefficient and not the functional form of the first peratized scattering length which diverges. The contributions from higher peratizations maintain this property. Wu also showed for the case $m=4$ that the asymptotic contributions, when summed, give zero. Analogous expressions are also given for arbitrary $m>3$. The first peratized scattering length in the

[77] We note that Evgrafov (1961), Chap. II, Sec. 1, denotes the functions $L(r)$ as slowly increasing functions, even if the function actually is decreasing.

general case can be written[78]

$$A^{(1)}=\ln^{(1/(m-2))}(1/\alpha)A_m,$$

where A_m is the scattering length for the potential gr^{-m}.

Further generalizations of these potentials have been made by Cornille (1965b) and by Gale (1967), who considered the forms

$$V(r)=[(\ln r^{-1})^p/r^m], \qquad m>3, \qquad p>0,$$
$$V(r)=g[L(r)/r^4],$$

respectively, and by Spector (1966b) and Frank and Land (1970b) to the entire class of potentials $V(r)=gL(r)r^{-m}$, with $m>3$. The first peratized scattering length, obtained by a different technique by each author, is found to be

$$A^{(1)}=L^{[1/(m-2)]}(0)A_m,$$

where A_m is the scattering length for the power potential gr^{-m}. Thus it is found quite generally that the first peratized scattering length does not exist whenever $L(0)$ does not. The second peratized scattering length has been investigated by Frank and Land for those potentials in which $L(r)$ is a tempered slowly varying function. It is found that $A^{(2)}$ is proportional to $A^{(1)}$, and hence second peratization offers no improvement if $L(0)$ does not exist. These results are in agreement with those of Wu (1964) which were derived for the potentials $gr^{-m}\ln(1/r)$.

For those functions $L(r)$ which are finite at $r=0$, it is not clear how good an approximation is obtained in first peratization. Certainly the two functions $\exp(\lambda r^\mu)$ and $\cos^\lambda r^\mu$ $(\mu>0)$ are quite different, and yet both give rise to the same first peratized scattering length. The validity of first peratization in this case remains a subject for further study.

The preceding investigations of peratization for potentials involving logarithmic-type singularities are somewhat hampered by the fact that for none of these potentials is an exact solution available, though a sequence of approximations was derived by Wu (1964) for the potentials $gr^{-m}\ln r^{-1}$. However, it was shown in Sec. III.A.2 that the potential

$$V(r)=[g(\ln^2 r/r^4)-g^{1/2}(1/r^3)]\theta(R-r) \qquad (4.56)$$

can be solved exactly at zero energy. This potential has, as the leading singularity at the origin, the behavior $r^{-4}\ln^2 r$. It has been used in studies of peratization by Aly, Riazuddin, and Zimerman (1965a), and by Gale (1966b). Aly *et al.*, however, use the Khuri–Pais σ-limiting procedure for the calculation of the scattering length which they consider to be peratization. As discussed above, such a calculation is not a test of peratization.

[78] Equation (E21) of Wu (1964) from which this result is obtained appears to contain a typographical error and should read $B_0(\Lambda)=2\nu^{2\nu-1}[\Gamma(\nu)]^{-2}(g\ln\Lambda)^\nu[K_{\nu-1}(\Omega)/I_{\nu-1}(\Omega)-\frac{1}{2}\pi/\sin\pi\nu]$.

Gale (1966b) presents the results of a calculation of both the first and second peratized scattering lengths for this potential, first peratization giving the approximate result

$$A^{(1)}=g^{1/2}\ln R,$$

and second peratization yielding the exact result of Eq. (3.78)

$$A^{(2)}=A=g^{1/2}\ln R(1-g^{1/2}R^{-1}\ln R).$$

However this potential is given by the sum of two terms, and hence it must be specified how peratization is to be performed. If the potential is expressed in the form

$$V(r)=g[(\ln^2 r/r^4)+(\varphi/r^3)]\theta(R-r),$$

with φ set equal to $g^{-1/2}$ at the end, and peratization is performed with respect to the coupling constant g, then the previous result for tempered slowly varying potentials holds, and neither first nor second peratization yields a finite answer. On the other hand, it might seem most natural to consider a Born series in the variable $g^{1/2}$. However, the present reviewers have not been able to reproduce the results presented in this paper within this context, and hence are skeptical of this result which strikes an unjustifiably optimistic note for the success of peratization.

The limitations of the approximation of summing only leading singularities are well illustrated by a calculation of Arbuzov, Filippov, and Khrustalev (1964). They considered the zero-energy scattering by the marginally singular potential

$$V(r)=gr^{-2}\ln r^{-1}.$$

The wave function for this potential is known exactly (see Secs. III.A.2 and IV.C). They found with the regularization of the singular integral in the integral equation for the wave function that the exact solution has the behavior for small r

$$\psi(r)\sim b(\alpha)(r^{1/2}/|\ln r|^{1/4}) \times\exp\{-\tfrac{2}{3}g^{1/2}|\ln r|^{3/2}\}. \qquad (4.57)$$

while the calculation of the contribution of the leading logarithmic terms to each order implies a behavior of the form

$$\psi(r)\sim c(\alpha)r^{\lambda+\frac{1}{2}}\exp[-\tfrac{1}{4}g\lambda^{-1}\ln^2 r]. \qquad (4.58)$$

Here $b(\alpha)$ and $c(\alpha)$ are factors which become infinite as the regularization parameter is set equal to zero. One notes that the contribution for the leading logarithmic term is analytic in g, while the exact behavior is not. Thus the dependence of the index n of the leading logarithm in the Born series produces a convergent series, while the n dependence of the less singular logarithmic terms in fact corresponds to an asymptotic

series. Thus the n dependence of the coefficients can be more significant than the strength of the singularity with regard to the dominant behavior of the solution.

Exponential-type singularity. We consider next the class of potentials having an exponential-type singularity at the origin. An example of a potential of this type is

$$V(r)=g(e^{\lambda/r}/r^4)+f(1/r^4). \tag{4.59}$$

This potential has been studied by Aly, Riazuddin, and Zimerman (1965b) and by Calogero and Cassandro (1965), both groups setting $\lambda=2$.

Aly *et al.* (1965b) evaluated the scattering length for this potential with $f=\frac{1}{4}$ by means of the Khuri–Pais σ-limiting procedure. However, as commented previously, this formalism tests only regularization and not peratization. In addition, these authors stated that the sum of the leading singularities from each order of the θ-regularized Born series for the scattering length gives zero in the $\alpha\to 0$ limit. They conjectured that the second leading singularities would give the correct result. These statements, however, are ambiguous in that it is not indicated how the leading singularities are to be obtained. In addition, in support of the claim that the first peratized scattering length is zero, the authors used a calculation they performed in which it was found that the scattering length for the potential $ge^{2/r}/r^4$, which is the most singular part of the potential of Eq. (4.59), is zero. This result is clearly incorrect, since a purely repulsive potential cannot have a vanishing scattering length. The claim of these authors that peratization succeeds is thus not justified.

Calogero and Cassandro (1965) presented a careful analysis of the peratization procedure as applied to the potential of Eq. (4.57), also with $\lambda=2$. The scattering length, computed from the $+$-regularized potential $V(r+\alpha)$, and with λ arbitrary, is readily found from the solution presented in Sec. III.A.2 to be given by[79]

$$A(\alpha)=g^{1/2}\times\frac{K_\nu(g^{1/2}e^{\lambda/2\alpha})I_\nu'(g^{1/2})-I_\nu(g^{1/2}e^{\lambda/2\alpha})K_\nu'(g^{1/2})}{K_\nu(g^{1/2}e^{\lambda/2\alpha})I_\nu(g^{1/2})-I_\nu(g^{1/2}e^{\lambda/2\alpha})K_\nu(g^{1/2})}+\alpha, \tag{4.60}$$

and

$$A=\lim_{\alpha\to 0}A(\alpha)=g^{1/2}[K_\nu'(g^{1/2})/K_\nu(g^{1/2})], \tag{4.61}$$

where $\nu=f^{1/2}$.

In the special case where $f=\frac{1}{4}$, Eq. (4.60) simplifies to

$$A(\alpha)=-\tfrac{1}{2}-g^{1/2}\coth(g^{1/2}e^{\lambda/2\alpha}-1)+\alpha. \tag{4.62}$$

[79] Equation (8) of Calogero and Cassandro (1965), which corresponds to our Eq. (4.60), omits the additive term $+\alpha$. This omission is reflected in their Eqs. (9), (10), (15), and (17a), and is corrected in our Eqs. (16), (17), and (18). This error does not affect the conclusions of their work.

By expanding the hyperbolic cotangent, one has

$$A(\alpha)=-\tfrac{1}{2}-\sum_{n=0}^{\infty}c_n g^n[e^{\lambda/2\alpha}-1]^{2n-1}+\alpha, \tag{4.63}$$

and on keeping the most singular term in each order in g, and summing, one reproduces the exact scattering length in the $\alpha\to 0$ limit. Thus first peratization succeeds. This appears to be the only nontrivial case (excluding the pure inverse-power potentials) for which this is true. The success seems to be a fluke arising from the particular choice of parameter $f=\frac{1}{4}$.

We consider next a more interesting situation with $f=0$, and λ still arbitrary. The scattering length, which is given by Eq. (4.61) with $\nu=0$, is most conveniently written as a ratio

$$A(\alpha)=N(\alpha)/D(\alpha),$$

$$N(\alpha)=\sum_{n=1}^{\infty}g^n N_n(\alpha),$$

$$D(\alpha)=\sum_{n=0}^{\infty}g^n D_n(\alpha).$$

Here $D(\alpha)$ is the Jost function for the regularized potential $V(r+\alpha)$. One finds that

$$N_n(\alpha)=-\frac{2\alpha}{\lambda^{2n}}\sum_{l=0}^{n}\frac{e^{l\lambda/\alpha}}{l!^2(n-l)!^2}\times\left[[\psi(l+1)-\psi(n-l+1)]\right]\left(n-l+\frac{\alpha}{\lambda}\right)\left[-\frac{\lambda}{2\alpha}(n-l)\right],$$

$$D_n(\alpha)=\frac{1}{\lambda^{2n}}\sum_{l=0}^{n}\frac{e^{l\lambda/\alpha}}{l!^2(n-l)!^2}\times\left[1-\frac{2\alpha}{\lambda}[\psi(l+1)-\psi(n-l+1)]\right]. \tag{4.64}$$

The first peratized scattering length, obtained by keeping the most singular term in N and D in each order in g, was readily found to be zero. Calogero and Cassandro then proceeded to calculate the second peratized scattering length. They showed that keeping the first two most singular terms in N and D in Eqs. (4.64) is not quite correct, because the second leading singularity in the numerator, which involves the factor $\exp[(n-1)\lambda/\alpha]$, is a less singular basis function than the second leading singularity in the denominator, which involves the factor $\exp(n\lambda/\alpha)$. Therefore, in order to keep only the leading two singularities consistently in the ratio, the term involving the lower-order exponential $\exp[(n-1)\lambda/\alpha]$ must be dropped. They then found that

$$A^{(2)}=\frac{\lambda/2}{\frac{1}{2}\ln(g/\lambda^2)+\gamma}.$$

This result is precisely the first term in a convergent

expansion of the exact scattering length about $g=0$; see Sec. III.A.2, Eq. (3.71).

It is interesting to look at higher peratizations for this potential (Frank and Land, 1970a). One can easily verify that the third peratized scattering length, obtained by keeping terms up to and including the leading term of the coefficient of $\exp[\lambda(n-1)/\alpha]$ equals $+2$. This result is in no sense an approximation to the exact answer. On the other hand, if we keep the complete polynomial factor (in α) of $\exp[\lambda(n-1)/\alpha]$ in $N_n(\alpha)$ and $D_n(\alpha)$, we obtain (in fifth peratization)

$$A^{(5)}=\frac{\lambda/2}{\frac{1}{2}\ln(g/\lambda^2)+\gamma}\times\{1+(g/\lambda^2)[\ln(g/\lambda^2)-2+2\gamma]\}. \quad (4.65)$$

These are the first two terms of a convergent expansion of the exact scattering length Eq. (3.70) about $g=0$.

This example illustrates the concept of the "appropriate singular unit." We see that it is not necessarily true that successive peratizations give successively improved results even for a peratizable potential. One could still get a nonsensical answer, e.g., the value $+2$ noted above. Rather, it seems from this potential, that one must perform successive peratizations with respect to an appropriate singular unit, in this case the exponential factor $e^{\lambda/2\alpha}$, together with a polynomial coefficient, in order to obtain a successive approximation scheme.

The peratization of a generalized class of potentials of the form

$$V(r)=g(e^{\lambda/r}/r^m) \qquad m \text{ an integer, } \geq 4$$

has been performed by Frank and Land (1970a). The scattering length was calculated to third peratization. Only in the case of $m=4$ does peratization (second peratization, as discussed above) give a finite, nonzero result, which is the weak-coupling limit of the exact scattering length. In all other cases, the first three peratizations give zero.

Cornille (1965b) has studied the class of potentials

$$V(r)=g[\exp(\lambda/r^\beta)/r^m], \qquad \lambda>0, \qquad \beta>0,$$

but in first peratization only, and found that the first peratized scattering length always vanishes.

The preceding results have been further generalized by Land (a) to include a wide class of potentials having an exponential-type singularity at the origin. These potentials are written

$$V(r)=g[R(r)/r^m],$$

where $R(r)$, which we call a rapidly varying function, is defined by the expression

$$R'(r)/R(r)=-[1/o(r)]=-[1/r\xi(r)],$$
$$\xi(r)=o(1),$$
$$r\xi'(r)=o(1).$$

Examples of $R(r)$ include $\exp(\lambda/r^\beta)$, $\exp(\lambda\ln^\mu r)$, $\mu>1$, $\exp(e^{\lambda/r^\beta})$, etc. It has been shown that the first peratized scattering length for these potentials vanishes. Thus the result $A^{(1)}=0$, found by Calogero and Cassandro for the potential $ge^{\lambda/r}/r^4$ and by Cornille for the potentials $g\exp(\lambda/r^\beta)r^{-m}$, is indicative of a very general situation. The second peratized scattering length has also been calculated. However, no form of $R(r)$ could be found, apart from $e^{\lambda/r}$ with $m=4$, for which peratization gives a nonzero result to this order.

The foregoing summary of the results of peratization calculations suggests that peratization is not a particularly viable approximation technique in potential theory. It works as a successive approximation scheme for the class of potentials consisting of the sum of two inverse powers, and for the single potential $V(r)=ge^{\lambda/r}r^{-4}$. It also succeeds for the truncated pure inverse power potentials $V(r)=gr^{-m}\theta(R-r)$. First peratization generally gives an approximate result for potentials whose leading behavior is that of a pure inverse power at both zero and nonzero energy.

Fairly general classes of potentials having logarithmic or exponential singularities at the origin have been studied by Frank and Land (1970a, b) and, among these potentials, none could be found (but for the one exceptional case $gr^{-4}e^{\lambda/r}$) for which peratization is successful. There are two general features concerning the customary expansion into singularities of the Born series that seems to emerge from their analysis. The first is that the explicit dependence on the summation index n in the various terms of the Born series contains an implicit dependence on α. In particular, in the $\alpha\to 0$ limit, those terms indexed by larger values of n are the most important ($n\to\infty$, in fact). This effect was discussed above in connection with the singularity structure of the wave function for the potential $gr^{-2}\ln r^{-1}$ as studied by Arbuzov *et al.* (1964). This effect also is felt in the peratization of the potential $gr^{-4}e^{\lambda/r}$. It was shown in Frank and Land (1970a) that a first term with a stronger singularity in α is canceled by a second term which is explicitly less singular in α, but whose n dependence, along with the explicit α dependence, gives rise to a singularity as strong as the first term.

A second feature is that the nth term in the Born series consists in general of a sum of products of a dominating singularity with a series of less singular terms. Consider the series with the greatest singularities. Even if this series is finite with, say, n terms [or more generally $O(n)$ terms], the preceding remarks about the $\alpha\to 0$ limit suggests that these series behave effectively as infinite series, thus implying that an infinite peratization is necessary to gain even a first approximation. The reader might contrast this situation with what actually occurs for the $gr^{-4}e^{\lambda/r}$ potential for which the series described above consists of at most three terms [see Eq. (4.64)].

There has been an attempt by Gale (1966a, 1967)

to draw some general conclusions about whether peratization might be successful. The author claimed to present a general argument for the success of peratization as an approximation scheme. However his specific conclusions are briefly stated, and seem rather cryptic to the reviewers who are unconvinced by his arguments.

V. APPLICATIONS–PHYSICAL

A. Molecular Physics

One of the earliest appearances of singular potentials in physical problems was in the area of molecular physics. Potentials that are singular in form may enter in both the long-range and short-range parts of the interatomic or intermolecular force. The long-range part arises from electrostatic, induction, or dispersion forces (see below). The related potentials are attractive singular when extended to the origin. However, as the atoms or molecules approach each other, their electron clouds overlap, thus creating a strong repulsive force at smaller distances. This force is frequently represented phenomenologically by a repulsive singular potential, but sometimes also by a repulsive nonsingular potential. In some cases, particularly in low-energy calculations, where only the long-range part of the potential should be important, the attractive singular potential is simply cut off.

Because the attractive singular potential arises only incidentally and the repulsive singular potential is a purely phenomenological representation, there appears not to have been much emphasis in molecular physics on unravelling the theoretical difficulties associated with the singularity at the origin. This contrasts with the interest of elementary particle physics in singular potentials. The most frequently employed analytical technique for obtaining solutions to the Schrödinger equation is the WKB method (Sec. III.B.2). However solutions are also obtained numerically with the help of electronic computers. As stated in the introduction, we do not attempt in this review to cover the vast field of applications of singular potentials in molecular physics, but rather we point to a few significant features. The reader interested in calculations which have been performed and in the techniques that are employed is referred to the excellent review article by Bernstein (1966) which contains many references.

An application by Vogt and Wannier (1954) of singular potentials to the scattering of ions in a gas is of particular interest. In this work the authors studied solutions to the Schrödinger equation for an attractive singular potential without imposing a cutoff on the potential, but by choosing an acceptable boundary condition based upon physical considerations. When gaseous ions or electrons move through a dilute gas whose molecules are not too large, the interaction is given by the polarization potential,

$$V(r)=-\tfrac{1}{2}e^2\alpha r^{-4}, \quad \alpha=\text{molecular polarizability}. \tag{5.1}$$

This form holds only away from the origin, and represents the interaction of a charge with an induced dipole. Vogt and Wannier argued that in many physical situations (such as helium ions in helium) the exact close-in repulsion mechanism is unimportant. If the wave function of the moving particle oscillates rapidly in the neighborhood of the molecule, then the location and nature of the cutoff to Eq. (5.1) is washed out by the small energy spread ΔE of the particles. In the case of an electron in helium, however, the details of the cutoff are important.

The quantity which Vogt and Wannier calculated as an application of these ideas is the cross section for a particle to pass through the origin, which they called the "capture" cross section. The physical revelancy of this quantity is derived from the fact that, in the classical case, it should be a good approximation to the total cross section, since particles which do pass through the origin are deflected into random directions, while those that do not are deflected only slightly.[80] The classical cross section for a particle of velocity v to fall to the center or be "captured" by the potential of Eq. (5.1) is

$$\sigma_{\text{cap}}{}^{\text{class}}=2\pi(e^2\alpha/Mv^2)^{1/2}.$$

This expression follows by computing the quantity $\pi b_{\text{crit}}{}^2$, where b_{crit} is the critical impact parameter for fall to the center (see Sec. II.F). Vogt and Wannier argued that the analogous quantum-mechanical cross section is obtained by treating the origin as a sink, and hence they chose a boundary condition to give rise to totally incoming waves. The description of this calculation is given in Sec. III.A.1. Vogt and Wannier found in the low-energy limit

$$\sigma_{\text{cap}}{}^{\text{QM}} \underset{k\to 0}{\approx} 2\sigma_{\text{cap}}{}^{\text{class}},$$

and more generally

$$\sigma_{\text{cap}}{}^{\text{QM}}=\frac{\sigma_{\text{cap}}{}^{\text{class}}}{2k(-g)^{1/2}}\sum_{l=0}^{\infty}\frac{2l+1}{1+e^{2\Phi}}, \tag{5.2}$$

in the notation of Sec. III.A.1. The actual evaluation of Eq. (5.2) is very difficult. They concluded that the quantum-mechanical cross section oscillates about the classical value as the energy increases. The amplitude of deviation becomes monotonically smaller while the period remains nearly constant as the energy increases. They also derived the high-energy limit for the potential

[80] Strictly, one cannot talk of the total cross section for this potential in the classical case, since the total cross section does not exist because the forward cross section diverges (see Sec. II.I). Instead one uses a kinetic cross section, defined by $\sigma_{\text{kin}}=S(1-\cos\theta)d\sigma$ (see Vogt and Wannier, 1954). Particles scattered into random directions have $\langle\cos\theta\rangle=0$, while particles defiected slightly give only a small contribution to σ_{kin}.

of Eq. (5.1) ($g=\frac{1}{2}e^2\alpha$),

$$\frac{\sigma_{\text{cap}}{}^{\text{QM}}}{\sigma_{\text{cap}}{}^{\text{class}}}\underset{k\to\infty}{\longrightarrow}1-\{4/[k(-g)^{1/2}]^{1/2}\}\exp(-\pi\sqrt{2})$$
$$\times\{\sin 2\pi[2k(-g)^{1/2}]^{1/2}+\{(\sqrt{2}\pi+1)/8[k(-g)^{1/2}]^{1/2}\}\}$$
$$+\{4/[k(-g)^{1/2}]^{1/2}\}\exp(-2\pi\sqrt{2})$$
$$\times\{\sin 4\pi[2k(-g)^{1/2}]^{1/2}+O\{1/[k(-g)^{1/2}]^{1/2}\}\}$$
$$+\text{higher order terms in }[k(-g)^{1/2}]^{-1/2}. \quad (5.3)$$

We note again, as in Sec. III.A.1, that this is not the usual scattering cross section as it violates unitary.

Thaler (1959) investigated the induced electric polarization of the neutron in the field of a nucleus. The interaction is again taken to be of the form of Eq. (5.1), with α now the neutron polarizability. In particular, he calculated the low-energy phase shifts by employing a cutoff on the potential.

Singular power potentials are of great importance in describing intermolecular forces (Hirschfelder, Curtiss, and Bird, 1954). Two nonpolar molecules are frequently represented as interacting via the Lennard–Jones potential,

$$V(r)=(d/r^{12})-(c/r^6), \qquad d, c>0,$$

or sometimes via the Buckingham potential,

$$V(r)=be^{-ar}-(c/r^6)-(c'/r^8), \qquad a, b, c, c'>0.$$

Modifications of the latter potential have been proposed to eliminate its undesirable feature of approaching $-\infty$ as $r\to 0$.

The long-range attractive inverse-sixth power is characteristic of the interaction of nonpolar molecules. This form of interaction, first derived by London, is a quantum-mechanical result known as the dispersion contribution to the potential, and originates in the mutual interaction of two instantaneously existing dipoles. The Buckingham potential also includes the contribution from an induced-dipole–induced-quadrupole interaction. On the other hand, the repulsive interaction has been given various phenomenological representations, such as a singular inverse power d/r^δ, with $9\leq\delta\leq 15$, or as a regular exponential be^{-ar}. The general Lennard–Jones potential which employs a repulsive singular interaction with variable exponent is perhaps the most frequently considered intermolecular potential. However, in some instances more satisfactory experimental agreement has been obtained by means of a regular rather than a singular potential (see, e.g., Hartmann and Slawsky, 1967).

We also mention a few long-range interactions between molecules which result from induction and electrostatic forces. Interactions of a polar molecule with a nonpolar one (when averaged over angular orientation with an appropriate Boltzmann weighting factor) are given by

$$V(c, \text{ind }\mu)\sim-1/r^4,$$
$$V(\mu, \text{ind }\mu)\sim-1/r^6,$$

where the first expression represents a charge interacting with an induced dipole, and the second a dipole interacting with an induced dipole. Interactions between two (angular averaged) polar molecules have the forms

$$V(c, \mu)\sim-1/r^4,$$
$$V(c, Q)\sim-1/r^6,$$
$$V(\mu, \mu)\sim-1/r^6,$$
$$V(\mu, Q)\sim-1/r^8,$$
$$V(Q, Q)\sim-1/r^{10},$$

where Q means quadrupole.

The various molecular potentials just discussed do not hold for the interaction of long molecules, excited states, free radicals, or free ions. In such cases, however, the interaction may still be expanded in a power series in r^{-1}, the inverse separation of the molecules. See, for example, Chang (1967).

A considerable amount of effort is presently being devoted both theoretically and experimentally to the determination of intermolecular potentials. There are two basic theoretical approaches to this problem. One method consists of calculating the matrix element of the interaction energy between appropriate molecular states. Needless to say, these results are fairly approximate because molecular wave functions are not well known. A second method consists of the calculation of some macroscopic quantity which is related via statistical mechanics to the intermolecular potential. Such quantities include viscosity coefficients, ionic mobilities, or virial coefficients. The work of Larsen, Witte and Kilpatrick (1966) gives an example of the use of numerical solutions for the Lennard-Jones potential in the calculation of the quantum-mechanical pair-correlation function for ^{4}He gas at low temperatures. For the experimental determination of the potential, one of the most direct techniques is by means of molecular-beam scattering. A discussion of various theoretical and experimental methods currently employed may be found in Hirschfelder (1967). In addition, the review articles of Amdur and Jordan (1966), and Pauly and Toennies (1965), on molecular-beam techniques are also of interest.

B. High Energy and Nuclear

Another physical application of singular potentials has been to the phenomenological interpretation of high-energy scattering. Tiktopoulos (1965) has used singular potentials of complex strength to investigate the sharp forward elastic peak in high-energy scat-

tering. The general form of such scattering in p–p and p–π experiments is

$$(d\sigma/d\Omega)\,|_{\text{elastic}} \approx c \exp\left[-b(\theta, k)\right], \tag{5.4}$$

where the Orear form for $b(\theta, k)$ is $\sim k \sin\theta$, and c is taken to be a slowly varying function of θ and k. Tiktopoulos reasoned that nonforward scattering comes from particles passing very close to the scattering center and that, if these were strongly absorbed (by a complex potential), there would be a sharp dropoff in elastically scattered particles away from the forward direction. Potentials less singular than the inverse square were investigated by Serber (1963, 1964a, b), and do not give the exponential falloff of Eq. (5.4). Tiktopoulos employed a potential of the form

$$V(r) = g e^{-i\Delta} r^{-m}, \quad \text{with} \quad m \geq 2, \quad \Delta > 0,$$

and

$$g = g(k) > 0,$$

where $g(k)$ is assumed to fall off more slowly than k^{2-m} at high energy, since the dimensionless parameter which determines the scattering, $\bar{\chi} = k^{m-2} g e^{-i\Delta}$, would otherwise correspond to weak coupling at high energy (see Sec. II.D). Paliov and Rosendorff (1967) have also used energy-dependent singular potentials in their high k phase shift investigations (see Sec. II.D).

Tiktopoulos derived a WKB expression for the phase shifts at high energy for fixed l which is

$$\delta_{\text{WKB}}(l) = \tfrac{1}{2}\pi(l+\tfrac{1}{2}) - \int_{\rho_0}^{\infty} \left[1 - \left(1 - \frac{(l+\frac{1}{2})^2}{\rho^2} - \frac{\bar{\chi}}{\rho^m}\right)^{1/2}\right] d\rho - \rho_0, \tag{5.5}$$

where ρ_0 is a certain (complex) root of the square root in Eq. (5.5), and the integral is evaluated along a certain ray in the complex ρ plane. By writing the scattering amplitude in a Sommerfeld–Watson form, a steepest descent technique can be employed along with Eq. (5.5) to find useful, though very complicated, expressions for the scattering.

In the case of $\Delta = 0$, the high-energy scattering becomes classical except in the forward and backward directions. Such a result, for energy-independent coupling, was also shown to be true by Spector (1967b, 1969). He derived the large-angle high-energy cross section in this case which may be given as

$$\sin\theta\,\sigma(\theta) = A_1^{-2}(\pi-\theta) + 4A_2A_1^{-5}(\pi-\theta)^3 + O\left[(\pi-\theta)^5\right], \tag{5.6}$$

where

$$A_1 = 2\pi^{1/2}\lambda^{-1/m}\frac{\Gamma(1/m)}{\Gamma\left[(1/m)+\frac{1}{2}\right]}$$

and

$$A_2 = \frac{2\pi^{1/2}}{m}\lambda^{-3/m}\frac{\Gamma(3/m)}{\Gamma\left[(3/m)-\frac{1}{2}\right]},$$

with $\lambda = g/k^2$. Equation (5.6) is valid, for large m, for $0 < (\pi-\theta) \leq 6^{1/2}$.

For the computationally easiest case of $m=2$, Tiktopoulos fitted p–p scattering satisfactorily with

$$V(r) = \eta^2 k^2 e^{-i\Delta} r^{-2},$$

and $\eta = 3.5\times10^{-13}$ cm $(+100\%, -50\%)$,

$$\eta \sin(\Delta/2) = 2.53\times10^{-14} \text{ cm } (\pm 2\%). \tag{5.7}$$

Except at angles greater than 80° in the center-of-mass system (which is symmetrical about 90°), the agreement with experiment is good.

Kouris (1966) extended these calculations to the complex inverse fourth power, and to p–π scattering. The analysis is much more involved and requires a computer. He used the potential

$$V(r) = \eta^4 k^2 e^{-i\Delta} r^{-4}$$

and found a good fit to p–p scattering at angles less than 60° in the center of mass, with

$$\eta = 20.124\times10^{-14} \text{ cm},$$
$$\Delta = 46°14.48'.$$

At $\theta_{\text{cm}} > 60°$, he found a reasonable fit, except near 90°, in two lab momentum ranges

$11 \leq p_{\text{LAB}} \leq 12$ BeV/c	$20 \leq p_{\text{LAB}} \leq 26$ BeV/c
$\eta = 8.727\times10^{-14}$ cm	$\eta = 8.64\times10^{-14}$ cm
$\Delta = 137°44'$	$\Delta = 126°44'$.

Fits to $\pi^- p$ and $\pi^+ p$ scattering at $p_{\text{LAB}} = 8$ BeV/c and 12 BeV/c are not as good as in the p–p case. Kouris also calculated the ratio of elastic to total cross sections by means of a standard optical approximation (impact-parameter method). This model has the unfortunate feature that $\sigma_{\text{total}} \to \infty$ as the energy increases. He found

$$R = \frac{\sigma_{\text{elastic}}}{\sigma_{\text{total}}} = 1 - \frac{2^{-1/2}(\sin\Delta)^{2/3}}{\sin\left[(2\Delta/3)+(\pi/6)\right]}, \tag{5.8}$$

and R is independent of η. For p–p scattering at $p_{\text{LAB}} = 12.8$ BeV/c, the values are

$$R_{\text{theory}} = 0.267,$$
$$R_{\text{exptl}} = 0.265 - 0.276;$$

however for $\pi^- p$, R_{theory} is too large. As indicated by the diverging total cross section, the theoretical basis for the optical model in this case is rather weak. Also, both Tiktopoulos and Kouris neglected exchange effects.

Spector and Chand (1968) have investigated the low-energy interactions of the K and $\bar{K}$ mesons with nucleons, where the K-meson lab momentum is less than 250 MeV/c. Using an energy-independent form

$$V(r) = g e^{-i\Delta} r^{-m},$$

they treated both isospin zero and isospin one for both KN and $\bar{K}N$ systems. By utilizing the experimental scattering length data, they found that only $m=4$ is

allowed. This contrasts to the high-energy papers just discussed where the exponent was arbitrarily chosen. The coupling strengths found for the four physical systems give consistent values for the average interaction energy in each case. Typical results are, for KN,

$$\begin{array}{ll} I=0 & I=1 \\ \Delta=0 & \Delta=0 \\ g=0.29 & g=5.06, \end{array}$$

and for $\bar{K}N$,

$$\begin{array}{ll} I=0 & I=1 \\ \Delta=46°40' & \Delta=170°10' \\ g=200 & g=28.5. \end{array}$$

Treacy has recently investigated the small l phase shifts for low energy α–α scattering by means of a singular potential. He found that a potential of the form $(g>0)$

$$V(r)=(g/r^4)-(B^2/r^2)-V_aY_a(r)+(4e^2/r) \qquad (5.9)$$

produces a reasonable fit to the $l=0$, 2, and $l=4$ phase shifts up to 15 MeV in channel energy. In Eq. (5.9), V_a is a constant, and $Y_a(r)$ is a sum of three Yukawa potentials. The best fit occurs for $g^2=0.412$, $B=1.054$, and $V_a=648.5$. The $l=2$ fit is the poorest, falling below the experimental value by 25% or so at points. However, the general fit is superior to that which can be obtained without the r^{-4} term.

It is of interest in high-energy phenomenology to compute scattering parameters for exponentially attenuated power potentials, i.e., potentials of the form

$$V(r)=g(e^{-\mu r}/r^m).$$

Spector (1967a) has performed such a calculation for the case $m=4$ by applying the Born approximation. He has noted that the scattering length so obtained does not reduce to the gr^{-4} potential in the $\mu\to 0$ limit. This result is, of course, due to the fact that the Born approximation is not valid in the neighborhood of $\mu=0$. This is because the wave function for the exponentially attenuated potential differs sufficiently from that for the pure power potential as to invalidate such an approximation.

VI. CONCLUDING REMARKS

The essential complication associated with singular potentials is seen to lie in the phy.ical interpretation of the attractive case. Though the "usual" boundary conditions are imposed, a unique solution of the Schrödinger equation is not obtained. In Sec. II.C this circumstance was attributed to the mathematical property that the Hamiltonian for an attractive singular potential is not a self-adjoint or essentially self-adjoint operator in the Hilbert space of state vectors in which it acts. Rather it possesses an infinite number of self-adjoint extensions which depend on a single arbitrary parameter. The physical consequences of an operator not being essentially self-adjoint can be illustrated with an example suggested by Wightman (1967).

Consider the operator

$$H=d^2/dx^2+cx^n, \qquad c>0,$$

whose domain in the Hilbert space consists of all infinitely differentiable functions of compact support. Under the action of the potential in this operator, a particle can accelerate from the point x_0 to infinity in a time given by

$$t=\int_{x_0}^{\infty}\frac{dx}{v}=\int_{x_0}^{\infty}dx(E+cx^n)^{-1/2}\approx c^{-1/2}\int_{x_0}^{\infty}dxx^{-n/2}.$$

If $n>2$, the particle reaches infinity in a finite time. Hence it is necessary to specify reflection conditions at infinity to obtain a unique dynamics. For each reflection condition there corresponds a self-adjoint extension of H. On the other hand, if $n<2$, a particle does not reach infinity in a finite time and so reflection conditions at infinity are irrelevant. It can, in fact, be shown that this H is essentially self-adjoint, at least for the case of a linear potential (Stark effect).

The situation for an attractive singular potential is similar, with the origin in this instance playing the role of the point at infinity in the example above. Classical considerations show that a particle indeed reaches the origin in a finite time. (The angular momentum barrier which may be present is of no consequence, as a quantum mechanical particle can penetrate through it.) Thus in this case it is also necessary to specify reflection conditions at the origin, in order to obtain a unique dynamics. The usual boundary condition for the vanishing of the wave function at the origin does not specify the phase of the wave function upon reflection. In the case of any repulsive potential or of a regular potential with angular momentum, a particle never reaches the origin, and so no reflection condition is necessary and a unique solution to the Schrödinger can be obtained. (The reader might recall that the case of the S wave with an attractive regular potential requires a different consideration; see Sec. II.C and Behncke, 1968.)

We mention again the possibility of exploiting the arbitrariness connected with the attractive singular potential in constructing representations or parameterizations of appropriate elementary particle processes. An example of such a process is particle absorption. This type of application has not, to our knowledge, been considered in the literature. The "capture" process of Vogt and Wannier (1954), which in a sense illustrates the technique, was nevertheless intended simply as the quantum mechanical analog of the classical fall to the center for an elastic scattering process.

We conclude finally with a comment on those formal applications in which singular potentials have been

used as models for investigating nonrenormalizable quantum field theories. It is frequently noted by the investigators of rigorous quantum field theory that any connection between nonrenormalizable field theory and singular potential theory is at best very hazy or does not exist at all. This conclusion is reached on the basis that (a) only subsets of diagrams are considered from which the singular potential emerges, or (b) the transformation from the Minkowski space of the quantum field theory to the Euclidean space of the singular potential theory is not a legitimate one. The work of Halpern (1966) might particularly be noted in this respect. The present reviewers do not dispute such criticisms. However we do feel that an intuitive connection between the two theories is sufficiently well-established for the qualitative picture which emerges from singular potential studies might, in some instances, to serve as a useful guideline for rigorous investigations of nonrenormalizable field theory as well as for phenomenological representations of elementary particle processes. After all, to give an oft-cited example, Regge pole theory, which is successfully applied in elementary particle physics, has its origins in nonrelativistic potential theory whose connection with a full relativistic quantum field theory is not entirely unambiguous. Perhaps singular potentials may at some time find a corresponding application.

Note added in proof: Since the completion of this manuscript, several articles of interest dealing with singular potentials have appeared. Aly, Guttinger, and Müller (1970) in two articles have discussed a number of topics on the singular interactions in both nonrelativistic and relativistic quantum mechanics. These articles are more specialized and more detailed than our work. The methode of Padé approximants has been applied recently in the context of singular potentials by Kloet and Zimerman (1970) and by Garibotti, Pellicoro, and Villani (1970).

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VII. APPENDIX

We present in this appendix a summary of the notations used in this article.

Three different notions of limit occur. Given two functions $f(x)$ and $g(x)$, we say that

(i) $$f(x) \underset{x\to x_0}{\longrightarrow} g(x) \tag{A1}$$

implies

$$\lim_{x\to x_0} |f(x)-g(x)|=0;$$

(ii) $$f(x) \underset{x\to x_0}{\approx} g(x) \tag{A2}$$

implies

$$\lim_{x\to x_0} [f(x)/g(x)]=1;$$

(iii) $$f(x) \underset{x\to x_0}{\sim} g(x) \tag{A3}$$

implies

$$\lim_{x\to x_0} [f(x)/g(x)]=c, \qquad 0<|c|<\infty.$$

Clearly Eq. (A1) implies Eq. (A2), which in turn implies Eq. (A3), but the converse is not true.

The "order of" symbols O and o in the limit $x\to x_0$ are defined by the relations

(i) $$f(x)=O[g(x)],$$

if $$f(x) \underset{x\to x_0}{\sim} g(x),$$

(ii) $$f(x)=o[g(x)],$$

if $$\lim_{x\to x_0} |f(x)/g(x)|=0.$$

Standard Notations

This article is concerned mainly with scattering problems in the context of partial waves. The symbol l is attached as a subscript to such quantities as the wave function or phase shift to denote the lth partial wave. When this subscript does not appear, the S wave is usually (but not always) understood.

Three dimensional vectors as denoted as **a**.

The notation and normalization for various functions used in the text are conventional and may be found in the *Bateman Manuscript Project* (1953) or the National Bureau of Standards *Handbook of Mathematical Functions* (1964).

The following is a list of our standard notations:

A	Scattering length
$Ai(z)$	Airy function
$Bi(z)$	Airy function
E	Particle energy
f	Coupling constant
$f(k)$	Jost function
$f(k, r)$	Jost solution
$f'(x_0)$	$df/dx\,\vert_{x=x_0}$
$F(a, b; z)$	Confluent hypergeometric function
$F(a, b; c; z)$	Hypergeometric function
g	Coupling constant
$h^{(1)}(z)$, $h^{(2)}(z)$	Spherical Hankel functions

$H^{(1)}(z)$, $H^{(2)}(z)$	Hankel functions
$I(z)$	Modified Bessel function
$j(z)$	Spherical Bessel function
$J(z)$	Bessel function
k	$(2ME/\hbar^2)^{1/2}$
$k(z)$	Modified Bessel function
l	Orbital angular momentum
m	Exponent in the pure inverse-power potential
M	Particle mass
$M_\nu^{(3)(4)}(z, h)$	Mathieu functions
p	Particle momentum
$P_l(\cos\theta)$	Legendre polynomials
$S(k)$, $S(k, \lambda)$	S matrix
$T(r, k)$	$\tan\delta(r, k)$ (Sec. III.B.3)
$u(r)$	Radial wave function
$V(r)$	Potential function
$W(f, g)$	Wronskian$\equiv fg'-f'g$
α	Regularization parameter (length)
γ	Euler–Mascheroni constant$=$ 0.577216
$\delta(k)$, $\delta(k; g)$	Scattering phase shift
$\delta(r, k)$	Variable-phase function (Sec. III.B.3)
$\delta(x)$	Dirac δ function
λ	$l+\frac{1}{2}$
$\theta(x)$	Step function: $=0$, $x<0$; $=1$, $x>0$
$\Phi(a, b; z)$	Confluent hypergeometric function
χ	$k^{1-2/m}g^{1/m}$
$\psi(\mathbf{r})$	Three-dimensional wave function
$\psi(z)$	Logarithmic derivative of gamma function
$\Psi(a, b; z)$	Confluent hypergeometric function
ω	Four-dimensional angular momentum
∇^2	Laplacian operator
$\Box$	d'Alembertian operator
K_x	Klein–Gordon operator

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