Theorems on High Energy Collisions of Elementary Particles

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This review gives a survey of the following topics: (i) results on analyticity and asymptotic growth that follow from axiomatic quantum field theory, (ii) bounds and inequalities involving cross sections at high energies, (iii) theorems and relations between cross sections requiring special assumptions, and (iv) a discussion of the Pomeranchuk theorem and of experimental and theoretical results that would hold if it were violated.

CONTENTS

1. INTRODUCTION

1.1 Objectives in This Review

This review will be concerned with high energy theorems of two main types:

(a) Theorems that have been rigorously derived from the assumptions of axiomatic field theory of the type first proposed by Araki and Haag (references will be given in later sections).

(b) Theorems whose derivation at present requires special assumptions additional to those in axiomatic quantum field theory (QFT). These will include results that follow rigorously from QFT if one assumes certain experimental conditions, for example, asymptotic constancy of total cross sections. However, they will not include results corresponding to special models except where these are used as illustrations.

High energy theorems are most conveniently derived from some intermediate properties of collision amplitudes that can themselves be derived from QFT. These properties involve:

(i) domains of analyticity,
(ii) unitarity,
(iii) polynomial bounds on the asymptotic growth of an amplitude $F(s, t)$ at fixed t as s tends to infinity.

These intermediate properties and their relation to QFT will be discussed in Sec. 2 of this review. The main emphasis in this and other sections will be on reviewing results rather than describing methods, but in some instances the general idea behind the method of derivation will be qualitatively described, and in other cases some of the most important methods will be described briefly.

In Sec. 3 we will describe axiomatic results, which are those high energy theorems or bounds that are based on QFT and do not require additional special assumptions. These include the Froissart bound on total cross sections, some inequalities involving elastic cross sections, and other inequalities that are concerned with the phase of a forward scattering amplitude and properties of the forward peak.

High energy bounds that require special assumptions that have not (yet) been derived from QFT are described in Sec. 4. The special assumptions range from limitations on the possible oscillatory behavior of amplitudes to assumptions that require the validity of the Mandelstam representation including its subtraction requirements. The results concern improved upper bounds and lower bounds, especially bounds at a fixed angle of scattering.

In Sec. 5 we are also concerned with results arising from special assumptions. These include general results on the phase and the rate of growth of the modulus of an amplitude at large energies. The Pomeranchuk theorem on cross sections for particle–target and antiparticle–target collisions is shown to follow from special assumptions on the asymptotic phases of amplitudes. A number of possibilities that form alternatives to the Pomeranchuk theorem are described. Their relevance is suggested by new experimental results on total cross sections which will be briefly described in Sec. 6. Some of these alternatives to the Pomeranchuk theorem are illustrated by special examples in Sec. 7, and some of the difficulties associated with oscillations and zeros are briefly discussed. Finally, a summary of axiomatic bounds and inequalities based on QFT is given in Sec. 8.

1.2 Historical Background

Research in particle physics takes place in islands. Some of these islands are very stable and have good foundations, but others are quite unstable or even volcanic in character. This instability, which applies to even the most respected areas of research in their infancy, makes the task of a bridge builder very difficult, and the connections between the islands of research are often quite tenuous. In this respect the fact that certain high energy theorems follow rigorously from the axioms of quantum field theory represents a triumph of the first magnitude for the intellectual bridge builders.

In the past twenty years the quantum field theory island of research has changed its form extensively. It began this era as old fashioned renormalized quantum field theory in which the convergence of the perturbation series for the S matrix was only slightly suspect. Then it developed with the Lehmann–Symanzik–Zimmermann (1955) formalism from which the first reasonably rigorous results on analyticity were derived. Next followed the Wightman axioms (see, for example, Streater and Wightman, 1964), from which some analyticity of scattering amplitudes was also derived (Bros, Epstein, and Glaser 1964, 1965). However, the polynomial bound on the growth of a scattering amplitude as the squared energy s tends to infinity appears, in Wightman theory, to depend crucially on the use of tempered distributions in coordinate space. This unsatisfactory feature has now been removed by the proof of analyticity and polynomial boundedness by Epstein, Glaser, and Martin (1969) from the theory of localizable observables due to Araki and Haag (see Araki, 1961–1962, 1968; Haag and Schroer, 1962; and Borchers, 1967).

At no stage before 1966 was the development of quantum field theory adequate for rigorous bridge building to the island concerned with high energy theorems. This island was founded by Pomeranchuk in 1956 when he used isospin invariance and intuitive assumptions to prove asymptotic equality of particle and antiparticle differential cross sections. Two years later Pomeranchuk (1958) used forward dispersion relations to establish a similar result for total cross sections. However, his proof involved the additional assumption that forward scattering amplitudes become pure imaginary at high energies, and it is still not known whether this assumption is correct.

The first high energy bound was derived by Froissart (1961), who used the Mandelstam (1958) representation and unitarity to show that

$$\sigma(\text{total}) < C(\log s)^2,$$

where s denotes the square of the center-of-mass energy. It was later shown by Martin (1963) that this result could be obtained with much weaker assumptions about analyticity, but these still exceeded what had been proved in QFT. At that time the proven analyticity yielded only the Greenberg–Low (1961) bound

$$\sigma(\text{total}) < Cs(\log s)^2.$$

A number of further bounds at fixed energy or fixed angle were obtained within the framework of the Mandelstam (1958) representation by Kinoshita, Loeffel, and Martin (1964) and Cerulus and Martin (1963). Some of these bounds, or slightly weaker bounds, hold under weaker analyticity assumptions on the scattering amplitude, but these are still stronger than the Lehmann (1958) ellipse analyticity of QFT.

In 1966 Martin made the very important step required for building the bridge between axiomatic field theory and the theorems on high energy bounds. By using unitarity and the two-variable analyticity of a scattering amplitude, he extended the Lehmann ellipse analyticity to the kind of domain that had been expected from perturbation theory. This was sufficient to derive the Froissart bound and some other bounds.

However, all was not well at the field theory end of the bridge due to the close connection between the polynomial boundedness result and the use of tempered distributions. As noted earlier, this remaining difficulty was removed in 1969 by Epstein, Glaser, and Martin, who established the polynomial bound from Araki–Haag theory, which does not assume tempered distributions.

Further details of the historical development and further references will be given in context with the results in later sections.

1.3 Experiments at High Energies

This review will not contain a detailed account of experimental results, but it is useful to recall some of their main features since they provide a guide to the types of high energy theorems that are most useful to study.

a. Total Cross Sections

Present results up to 70 GeV/c indicate that total cross sections are either asymptotically constant or slowly varying at high energies. In Sec. 5 we will consider various possible extrapolations from present experimental results with special reference to the Pomeranchuk theorems, which are concerned with the question of whether asymptotic equality holds for particle–target and antiparticle–target cross sections.

b. Differential Cross Sections

When the colliding particles have nonzero spin, the differential cross section will in general depend on several independent invariant amplitudes. It may be possible, as in pion–nucleon scattering, to choose amplitudes so that one of them dominates near the forward direction. However, most of the high energy theorems for nonforward scattering do not apply unless a spin average is taken. Most of our discussion in this review will be concerned with collisions of spinless particles; but where results with spin have been established, these will also be given.

c. Phases of Scattering Amplitudes

Near the forward direction, Coulomb interference provides a method for measuring the phase of a forward scattering amplitude. We will find that this is of great importance in connection with results related to the possible violation of the Pomeranchuk theorem. For nonforward scattering, it would be valuable to have measurements of polarization at asymptotic energies, not only because of their possible use to test high energy theorems, but also in connection with tests of phenomenological models.

1.4 Notation

We will use the conventional notation s, t, u for the relativistic invariants in a two-body collision,

$$1+2\rightarrow3+4, \tag{1.4.1}$$

$$s=(p_1+p_2)^2, \qquad t=(p_1+p_4)^2, \qquad u=\sum_1^4 m_i^2-s-t. \tag{1.4.2}$$

In the case of equal-mass particles and in the center-of-mass system for Process (1.4.1),

$$\begin{aligned} s&=4(m^2+k^2),\\ t&=-2k^2(1-\cos\theta),\\ u&=-2k^2(1+\cos\theta). \end{aligned} \tag{1.4.3}$$

In the case of scalar bosons in collision, there is a single scattering amplitude $F(s, t)$. This case will be used for illustration through most of this review. In the next section we will indicate what analyticity properties and what growth properties have been proved for $F(s, t)$ from QFT.

2. ANALYTICITY, UNITARITY, AND POLYNOMIAL BOUNDS

In this section we will begin by indicating some of the basic assumptions and properties of axiomatic quantum field theory. We will then describe the sequence of results through which one progresses from initially limited analyticity and growth properties to the full results that are required for high energy theorems. We will pay special attention to the result on polynomial boundedness of a scattering amplitude and we will see that this result is now firmly established from minimal axioms in QFT and that it appears to be as fundamental as the properties of analyticity.

2.1 Axiomatic Quantum Field Theory Assumptions

The first application of quantum field theory (QFT) to the problem of analyticity of collision amplitudes was made by Eden (1952), who used perturbation theory to indicate the branch-point character of normal thresholds. Three years later Goldberger (1955) wrote a dispersion relation for pion–nucleon scattering which was the prelude to the golden era for studying analyticity properties. At about the same time, Bogoliubov (1957) and Lehmann, Symanzik, and Zimmermann (1955 and 1957) provided the first reasonably rigorous formulations of QFT. These formulations were used by Bogoliubov, Medvedev, and Polivanov (1958), by Symanzik (1957), and by Bremermann, Oehme, and Taylor (1958), to derive dispersion relations at fixed momentum transfer for forward scattering or near to the forward direction. Also in 1958, Lehmann established a domain of analyticity in the $(\cos\theta)$ variable at fixed energy (the Lehmann ellipse). There is an excellent review of the forgoing work by Froissart (1964).

The next stage in QFT was the development of the Wightman axiomatic formulation (see Streater and Wightman, 1964). Dispersion relations have been derived from the Wightman axioms by Hepp (1964). These axioms can be qualitatively described by the following assumptions (Epstein, 1968):

(1) There exists a unique vacuum Ω and a minimum mass (greater than zero) for all states orthogonal to Ω.

(2a) Each stable particle has an isolated mass hyperboloid (at least in the sector corresponding to its quantum numbers) and the restriction of the representation of the Poincaré group to the corresponding subspace is irreducible.

(2b) Each stable particle is represented by at least one local field (i.e., at least one local field has nonvanishing matrix elements between one-particle states and the vacuum state Ω).

(3) The asymptotic states are complete.

(4) The "generalized retarded functions" can be defined as tempered distributions with "sharp" support properties.

Assumption (4) is not necessary for many of the results on analyticity that have been derived from

Wightman theory. However, it does feature in a significant way in the derivation of asymptotic growth properties of scattering amplitudes, as we will indicate in Sec. (2.3). These properties play an essential role in proving dispersion relations and it is therefore of great importance that the required properties of analyticity and polynomial boundedness of scattering amplitudes have now been derived without an assumption like (4) from the Araki–Haag theory of localizable observables (see Araki, 1961–1962, 1968; Haag and Schroer, 1962; and Borchers, 1967).

Although we will refer to the Araki–Haag theory as axiomatic quantum field theory (QFT), it does not necessarily imply that fields exist in the usual sense. The basic assumptions are outlined by Epstein, Glaser, and Martin (1969); they involve:

(a) physical state vectors are elements of a Hilbert space;

(b) there exists invariance under the Poincaré group, and certain reducibility properties;

(c) there is a unique vacuum state, and states orthogonal to the vacuum have masses larger than a strictly positive mass $m_0>0$;

(d) a local Araki–Haag field is defined by means of a von Neumann algebra in which the property of local causality is defined by commutation of operators that correspond to spacelike separation. It is not necessary to assume the existence of interpolating fields as in the theory of Lehmann, Symanzik, and Zimmermann (1954, 1957; see also Hepp, 1966). Neither is it necessary to make an assumption that restricts expectation values of operators to tempered distributions as in Wightman theory (Streater and Wightman, 1964).

From the assumptions of Araki–Haag QFT, Epstein, Glaser, and Martin (1969) have derived the analyticity and polynomial boundedness properties of two-body collision amplitudes that are required for most high energy theorems and bounds. Their derivation makes use of earlier results, some of which will be indicated in the next subsection.

2.2 Results on Analyticity

General references: Froissart (1964), Hepp (1966), Epstein (1966 and 1968), and Martin (1967a, 1969).

a. Fixed t Dispersion Relations

For $-t_0\leq t\leq 0$, the scattering amplitude $F(s,t)$ is the boundary value of a real analytic function,

$$F(s,t)=F^*(s^*,t), \qquad (2.2.1)$$

such that for $s>(m_1+m_2)^2$,

$$F(\text{physical})=\lim_{\epsilon\to 0+} F(s+i\epsilon,t) \qquad (2.2.2)$$

is the scattering amplitude for the process

$$1+2\to 3+4. \qquad (2.2.3)$$

$F(s,t)$ is regular in the s plane except for poles due to stable particles and cuts along the real axis from the leading threshold in s to $s=+\infty$, and from the leading threshold in u to $s=-\infty$. The limit on to the latter cut (the left-hand cut) gives the antiparticle process

$$F(1+\bar{3}\to\bar{2}+4)=\lim_{\epsilon\to 0+} F(s-i\epsilon,t) \qquad (2.2.4)$$

when $u>(m_1+m_3)^2$.

The proofs of analyticity hold provided the masses of the theory satisfy certain inequalities. These hold for

$$\pi N\to\pi N,\quad \pi\pi\to\pi\pi,\quad \pi\pi\to K\bar{K},\quad \pi K\to\pi K,$$
$$KK\to KK,\quad \pi(\Lambda \text{ or } \Sigma)\to\pi(\Lambda \text{ or } \Sigma).$$

In Wightman theory, polynomial boundedness was derived using the assumption about tempered distributions mentioned in Sec. 2.1, thus establishing dispersion relations (Hepp, 1964, 1966; Bogoliubov *et al.*, 1959). More recently, polynomial boundedness has been derived from Araki–Haag theory (without this temperedness assumption) by Epstein *et al.* (1969) giving for some N

$$|F(s,t)|<|s|^N,\quad \text{for}\quad s>s_0,\quad \text{and}\quad -t_0<t\leq 0. \qquad (2.2.5)$$

The proof by Epstein *et al.* is given only for spinless neutral particles, but the authors note (i) that it can easily be extended to all cases of charge and spin and (ii) that it can be generalized to particles that can be described by regularized products of Wightman fields or Jaffe fields [the latter are discussed by Jaffe (1966, 1968, 1969), who gives further references]. The derivation of both analyticity and polynomial boundedness (2.2.5) will be qualitatively described in Sec. 2.3. Given these results, dispersion relations follow for t real in $-t_0\leq t\leq 0$, (where for equal-mass particles, $t_0=8m^2$, for example),

$$F(s,t)=\frac{s^N}{\pi}\int_{s_0}^{\infty}\frac{dxF_1(x,t)}{x^N(x-s)}+\frac{u^N}{\pi}\int_{u_0}^{\infty}\frac{dyF_2(y,t)}{y^N(y-u)}$$
$$+(\text{pole terms})+(\text{polynomial in } s \text{ and } u), \qquad (2.2.6)$$

where s_0 and u_0 denote threshold values.

b. The Lehmann Ellipse

References: Lehmann (1958, 1959) (see also Lehmann, 1964, and Froissart, 1964).

For elastic scattering (2.2.3) the amplitude $F(s,t)$ is analytic for s (real) $>(m_1+m_2)^2$, in the domain $t=t(\cos\theta)$ corresponding to the Lehmann ellipse. This is an ellipse in the $\cos\theta$ plane, with foci at $\cos\theta=+1$ and -1, and with semimajor axis

$$\cos\theta_0=1+\frac{(m_A-m_1)^2(m_B-m_2)^2}{k^2\{s-(m_A-m_B)^2\}}, \qquad (2.2.7)$$

where m_A denotes the lowest-mass intermediate state having the same quantum numbers as particle 1; i.e., such that the matrix element $(A \mid j_1(0) \mid \Omega)$ is nonzero, where Ω denotes the vacuum, and j_1 is the current operator for particle 1. The mass m_B is similarly defined using $j_2(0)$. Thus if 1 denotes a pion, $m_A = 3m_\pi$ on account of G parity.

It was also proved by Lehmann that the absorptive part of F is analytic inside a larger ellipse. However, in both cases, one finds that the ellipse extends only to include positive values of t, in the range $0 \leq t \leq t_1$ where

$$t_1(s) \sim C/s \quad \text{as} \quad s \to \infty, \tag{2.2.8}$$

where C is a constant ($9m^4$ for equal masses). This may be contrasted with the analyticity out to $t = 4m^2$, which is expected from perturbation theory. This additional analyticity has now been established in axiomatic QFT via the steps indicated below.

c. Crossing and Analyticity for General Masses

Reference: Bros, Epstein, and Glaser (1965).

The results described above in Sec. 2.2.a have been extended to the case of general masses. A general two-body reaction amplitude $T(s, t)$ is analytic for real negative values of t in the complex s plane minus the s and u cuts, and minus a finite (but possibly large) region $|s| < R(t)$.

The above results describe only single-variable analyticity in either s or in t. Analyticity in the two variables simultaneously is required as an essential preliminary to the extension of the above limited domains.

d. Mandelstam and Lehmann Domains

References: Mandelstam (1964) and Lehmann (1966).

Mandelstam used the above results to prove that the $\pi\pi$ scattering amplitude (for example) is analytic in both s and t in the domain

$$|st| < 256m^4 \quad \text{(minus the cuts)}, \tag{2.2.9}$$

where m denotes the pion mass. Lehmann obtained a domain of analyticity for the scattering amplitude $F(s, t)$ in both s and t, which interpolates between the dispersion relations and the Lehmann ellipses.

e. Two-Variable Analyticity near Physical Points

Reference: Bros, Epstein, and Glaser (1964).

Each physical real point has a complex neighborhood (in both variables) in which $F(s, t)$ [or a reaction amplitude $T(s, t)$] is analytic except at those points actually on the relevant s, t, or u cut.

f. Enlargement of the Lehmann Ellipse near Threshold

Reference: Bessis and Glaser (1967).

For $\pi\pi$ scattering, the Lehmann ellipse extends to

$$\cos\theta_0 = (1 + 9m^4/k^2 s)^{1/2}. \tag{2.2.10}$$

Near the threshold $s = 4m^2$, this corresponds to a value t_1 of t given by

$$t_1(s) = -2k^2(1 - \cos\theta_0) \tag{2.2.11}$$

$$\approx 2mk. \tag{2.2.12}$$

The fact that this tends to zero at threshold is a kinematic accident arising from the technique used by Lehmann. Using a more general method, Bessis and Glaser have shown that (for example) in $\pi\pi$ scattering there is a domain of analyticity near threshold that extends to $t = 4m^2$. A similar result holds for πN or $\pi\Lambda$ scattering.

g. Martin's Extension of the Lehmann Ellipse

References: Martin (1966a, 1967, 1969).

Using the analyticity properties noted above, and for negative t the polynomial bound (2.2.5), Martin has extended the known analyticity of $F(s, t)$ to an ellipse that includes a fixed positive value of t, t_0, say, which is independent of s. His method also extends the polynomial boundedness (2.2.5) to the same larger domain. The essential steps in Martin's method are the following:

(1) A dispersion relation defines $F(s_1, t)$ for $s_1 < s_0$, in terms of $\text{Im}\, F(s', t)$ for $s' > s_0$. (The subtraction terms and the left-hand cut cause some technical complications which will not be discussed here.)

(2) Using the partial wave series and unitarity, which gives $0 \leq \text{Im}\, f_l(s') \leq 1$, for $s' > s_0$, restrictions are obtained on $\text{Im}\, F(s', t)$ and its derivatives at $t = 0$.

(3) Combining (1) and (2), Martin justifies the differentiation of $F(s, t)$ in terms of differentiation under the integral sign in the dispersion relation (2.2.6) and hence obtains a bound on the n-fold derivative of $F(s_1, t)$ at $t = 0$, for $s_1 < s_0$. This is then extended to complex s not on the cut along the real axis.

(4) The bound at $t = 0$ on $(d/dt)^n F(s, t)$ is used to set a bound on the Taylor expansion of $F(s, t)$ giving the result:

$$|F(s, t)| < |(s/s_1)^{N+1} F(s_1, t)|, \quad \text{as } s \to \infty. \tag{2.2.13}$$

This result holds for all t in $|t| < t_0$, where t_0 is a fixed constant. It follows that

(i) $F(s, t)$ is analytic in $|t| < t_0$, where t_0 is found from the analyticity domain of $F(s_1, t)$, and is therefore independent of s.

(ii) The scattering amplitude $F(s, t)$ is polynomial bounded, for $|t| < t_0$, as $s \to \infty$.

From the partial wave series for $F(s, t)$, one obtains immediately the result that the domain of analyticity can be extended to the Martin–Lehmann ellipse in the $\cos\theta$ plane, which has foci at $+1$ and -1, and extends so that the fixed point $t = t_0$ is included.

h. Further Extensions and Completions

(i) The optimum value of t_0, giving analyticity of $F(s, t)$ in $|t|<t_0$, can be found for any reaction by a method due to Sommer (1967a). This method subtracts out finite parts of the s cut and u cut so that s_1 in (2.2.13) can be chosen on the real axis above threshold. Then t_0 is found from the analyticity of $F(s_1, t)$ inside the extension of the Lehmann ellipse near threshold, which was found by Bessis and Glaser (1967). By optimizing t_0 by choice of s_1, one obtains $t_0=4m_\pi^2$ for $\pi\pi$, KK, πN, and $\pi\Lambda$ scattering.

(ii) Using the previously found analyticity, together with crossing, Martin has applied methods of analytic completion to enlarge the two-variable domain of analyticity. We will not describe the full domain since it is not required for our later discussion. However, we should note that he makes an additional *ad hoc* hypothesis, namely, that σ(total) is bounded in any subinterval of $4m^2<s<16m^2$; most physicists would accept this as intuitively reasonable. Further details of these extensions of the analyticity domain are described by Martin (1966b, 1967, and 1969). They will not be required for our discussion of high energy theorems.

(iii) The work of Martin extending the Lehmann ellipse has been generalized by Sommer (1967b) to include collisions of particles having nonzero spin (see also Sec. 3.6).

2.3 Polynomial Bounds

In this section we will begin by describing an analogous problem based on the scattering of light. We will then describe the results for relativistic collisions of strongly interacting particles that are based on the Araki–Haag theory (see Sec. 2.1). Finally we will give an S-matrix argument for a bound along the real s axis.

a. A Classical Analog

References: Toll (1956), Wong (1964), and Eden *et al.* (1966).

Let $A(z, t)$ be a wave packet with Fourier component $a(\omega)$ moving with velocity c in the z direction towards a scattering center at $z=0$,

$$A(z,t)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}d\omega a(\omega)\exp\left[i\omega\left(\frac{z}{c}-t\right)\right]. \quad (2.3.1)$$

The scattered wave may be written

$$B(r,t)=r^{-1}(2\pi)^{-1/2}\times\int_{-\infty}^{\infty}d\omega f(\omega)a(\omega)\exp\left[i\omega\left(\frac{z}{c}-t\right)\right], \quad (2.3.2)$$

where r is the distance from the origin.

From the inverse to (2.3.1),

$$a(\omega)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}dtA(0,t)\exp(i\omega t), \quad (2.3.3)$$

we see that, if the incident wave does not reach the origin before $t=0$, that is, if

$$A(0,t)=0,\quad \text{for}\quad t<0, \quad (2.3.4)$$

then $a(\omega)$ is regular in the upper-half complex ω plane.

Causality in this model requires that there is no scattered wave at a distance r, until a time r/c after the incident wave reaches the scatterer at $z=0$. Thus we require that

$$B(r,t)=0,\quad \text{for}\quad (ct-r)<0. \quad (2.3.5)$$

From this condition and the inverse of (2.3.2), analogous to (2.3.3), we see that,

$$f(\omega)a(\omega)\quad \text{is regular in}\quad \text{Im}(\omega)>0. \quad (2.3.6)$$

The following points should be noted:

(i) With sharp causality (2.3.4), the "wave function" $a(\omega)$ involves only a decaying exponential along any ray in the upper-half ω plane. This provides not only the convergence giving analyticity of $a(\omega)$, but also indicates that $a(\omega)$ decreases as $\omega\rightarrow\infty$ along this ray, or at least grows no faster than along the real axis.

(ii) Similar remarks apply to the product $f(\omega)a(\omega)$, and since both $a(\omega)$ and this product are analytic, we see that the "scattering amplitude" $f(\omega)$ is analytic. If both $a(\omega)$ and the product are polynomial bounded as $|\omega|\rightarrow\infty$ in Im $(\omega)\geq0$, so is $f(\omega)$.

b. Axiomatic QFT

References: Epstein, Glaser, and Martin (1969).

Epstein *et al.* prove the result

$$|F(s,t)|<|s|^{N+1},\quad \text{as}\quad s\rightarrow\infty,\quad \text{for}\quad |t|<t_0, \quad (2.3.7)$$

from the axioms of the Araki–Haag theory (see Sec. 2.1). The central difficulty that they have to overcome arises from the fact that this theory does *not* assume the "sharp" support properties of tempered distributions. Consequently they do not have sharp causality relations analogous to (2.3.4) and (2.3.5).

If $A(0, t)$ is nonzero for $t<0$, but decreases faster than an exponential as $t\rightarrow-\infty$, we see from (2.3.3) that, as $\omega\rightarrow\infty$ along a ray in Im $(\omega)>0$,

$$a(\omega)\sim\exp[C\,\text{Im}(\omega)]. \quad (2.3.8)$$

Our assumed fast decrease of $A(0, t)$ as $t\rightarrow-\infty$ ensures convergence of (2.3.3) provided the integral converges over the positive t range, and this gives analyticity of $a(\omega)$ as before.

Epstein *et al.* establish the analog of this required fast decrease of $A(0, t)$ and of B as $t\rightarrow-\infty$, by a method in which Lorentz invariance plays a vital role. They therefore obtain analyticity for their wave function and product function analogous to $a(\omega)$ and $f(\omega)a(\omega)$, in Im $(\omega)>0$. In their theory this gives (i) analyticity of $F(s, t)$ in the cut s plane, for $-t_0\leq t\leq0$. Their result analogous to (2.3.8) has $s^{1/2}$ analogous to ω, giving (ii) $|F(s, t)|$ has a bound $|\exp(Cs^{1/2})|$ as $s\rightarrow\pm\infty$ along any ray in the cut s plane. They also establish that

(iii) $F(s, t)$ has a polynomial bound as $s \to \infty$ along the real axis, with $-t_0 \leq t \leq 0$. It should perhaps also be noted that Epstein *et al.* have to overcome a considerable technical difficulty in disentangling the product of the wave functions and the scattering amplitude to obtain the result (ii). For the result (iii) one scarcely requires use of QFT as we will see below, in Sec. 2.3.c.

By subtracting the discontinuity along the real branch cuts, Epstein *et al.* show that $F(s, t)$ consists of an entire function plus a dispersion integral in the variable s. However, the entire function also satisfies (ii) and (iii) above; therefore it must be a polynomial. This proves that, for t real and $-t_0 < t \leq 0$, $F(s, t)$ has a polynomial bound. Using analyticity, one sees that $F(s, t)$ satisfies a dispersion relation with N (say) subtractions, in $-t_0 \leq t \leq 0$.

This result can then be extended by the method of Martin (1966a) (see Sec. 2.2.g) to establish a polynomial bound and a dispersion relation for $|t| < t_0$, in particular for positive values of t in the range $0 \leq t \leq t_0$. This gives their result (2.3.7).

c. A Comment on Growth for s Real

The derivation of analyticity in the Lehmann ellipse does not require a polynomial bound on growth as $s \to \infty$. Therefore let us suppose that for $t = t_2 = C/s$, inside the ellipse,

$$\operatorname{Im} F(s, t_2) \leq \exp(s^M), \quad \text{as} \quad s(\text{real}) \to \infty. \qquad (2.3.9)$$

From convergence in the Lehmann ellipse, we can write

$$\operatorname{Im} F(s, t) = \frac{s^{1/2}}{k} \sum_0^\infty (2l+1) \operatorname{Im} f_l(s) P_l\left(1 + \frac{2t_2}{s}\right). \qquad (2.3.10)$$

Since each term is positive and the sum satisfies (2.3.9), we deduce that

$$\operatorname{Im} f_l(s) \leq \exp(s^M) \exp(-lC'/s^{1/2}). \qquad (2.3.11)$$

Thus the terms in the partial wave series for $\operatorname{Im} F(s, 0)$ become negligible for $l > L'' = C'' s^{M+1}$. Using $0 \leq \operatorname{Im} f_l(s) \leq 1$, this proves that

$$\operatorname{Im} F(s, t) \leq C s^{2M+2}, \quad \text{as} \quad s(\text{real}) \to \infty, \qquad (2.3.12)$$

for $-t_0 \leq t \leq 0$. By crossing, this also holds for $s \to -\infty$.

Thus a polynomial bound for $s(\text{real}) \to \infty$ must hold for $t \leq 0$ if one does not allow growth faster than $\exp(s^N)$ for arbitrary N when $t > 0$. This means that the amplitude can be separated into a dispersion integral plus an entire function when $t \leq 0$. The entire function must be polynomial bounded along $s(\text{real}) \to \infty$ since $|F|$ as well as $|\operatorname{Im} F|$ can be bounded by using the partial wave series (see the beginning of Sec. 3 for general references on techniques). The important new result of Epstein *et al.* shows that the entire function (for $t \leq 0$) is bounded in complex directions by $\exp|Cs^{1/2}|$ and therefore from (2.3.12) we see that it must be a polynomial in the variable s. One then proceeds to $t > 0$ as noted earlier in Sec. 2.2.g.

3. BOUNDS AND INEQUALITIES BASED ON AXIOMATIC FIELD THEORY

General references: Martin (1963a, 1967, and 1969) and Eden (1966a, 1967).

In this section we will indicate the main results that have been derived from the analyticity and polynomial boundedness established from QFT and described in the previous sections.

3.1 Forward Amplitudes and Total Cross Sections

a. The Froissart Bound

Reference: Froissart (1961), Greenberg and Low (1961), and Martin (1963a, 1963b, 1966a).

From analyticity in the Martin–Lehmann ellipse, we can write the partial wave series,

$$\operatorname{Im} F(s, t) = \frac{8\pi s^{1/2}}{k} \sum_{l=0}^\infty (2l+1) \times \operatorname{Im} f_l(s) P_l\left(1 + \frac{2t}{s - 4m^2}\right), \qquad (3.1.1)$$

where our normalization is chosen so that $\operatorname{Im} F(s, 0) = 2ks^{1/2}\sigma(\text{total})$. Unitarity requires that

$$0 \leq |f_l|^2 \leq \operatorname{Im} f_l \leq 1. \qquad (3.1.2)$$

Hence, with $t = t_0 > 0$, each term in (3.1.1) is positive. The sum of this series of positive terms satisfies the polynomial bound (2.3.12); therefore each term satisfies this bound as $s \to \infty$,

$$(2l+1) \operatorname{Im} f_l(s) P_l\{1 + [2t_0/(s - 4m^2)]\} < Cs^N.$$

For large values of l and s,

$$P_l\{1 + [2t_0/(s - 4m^2)]\} > [C'/(2l+1)^{1/2}] \times \exp\{l[4t_0/(s - 4m^2)]^{1/2}\}.$$

Hence, for large values of l and s,

$$0 \leq \operatorname{Im} f_l(s) \leq C'' \exp\{-2l(t_0/s)^{1/2}[1 - \epsilon(s)] + N \log(s)\},$$

where

$$\epsilon(s) \to 0 \quad \text{as} \quad s \to \infty.$$

From this result one sees that the magnitudes of the partial waves decrease exponentially with l and become negligible when s is large and $l >$ (const)$[s^{1/2} \log(s)]$. It follows that for $t \leq 0$,

$$F(s, t) = \frac{8\pi s^{1/2}}{k} \sum_0^L (2l+1) f_l(s) P_l\left(1 + \frac{2t}{s - 4m^2}\right) + O(s^{-M}), \qquad (3.1.3)$$

where

$$L = Cs^{1/2} \log(s/s_0). \qquad (3.1.4)$$

In deriving (3.1.3) from (3.1.1), the inequality (3.1.2) has also been used. By choice of the constant C in (3.1.4), the inverse power M in (3.1.3) can be made arbitrarily large.

Using (3.1.2) in the sum in (3.1.3), one can bound $\mathrm{Im}\,F(s, 0)$; then using the optical theorem, one obtains the Froissart bound as $s\rightarrow\infty$, namely,

$$\sigma(\text{total})\leq C'[\log (s/s_0)]^2. \tag{3.1.5}$$

The constant C' can also be bounded (Lukaszuk and Martin, 1967),

$$C'\leq\pi/m^2 \tag{3.1.6}$$

(m is the pion mass). This value of C' corresponds to $C=(1/4m)$ in (3.1.4).

b. The Forward Amplitude

Using (3.1.2) and (3.1.3) at $t=0$, one obtains

$$|F(s, 0)|\leq\pi(s/m^2)[\log (s/s_0)]^2. \tag{3.1.7}$$

c. Bounds Involving Total Cross Sections at All Energies

References: Yndurain (1970), Martin (1967b), and Common (1970).

Using proven analyticity out to $t=4m^2$, for $\pi\pi$ scattering, the t-channel D wave scattering length is given by

$$a_2{}^t=\lim_{q^2\rightarrow 0}[f_2(t)/q^5](3\pi/128) \tag{3.1.8}$$

$$=(40m)^{-1}\int_{4m^2}^{\infty} ds s^{-3}\,\mathrm{Im}\,F(s, t=4m^2). \tag{3.1.9}$$

Combining this result with the techniques used in deriving the Froissart bound, Yndurain shows that

$$\frac{1}{s-4m^2}\int_{4m^2}^{s} ds'(s'-4m^2)\sigma_{\text{total}}(s')$$

$$\leq\pi\left(\frac{n+1}{n}\right)^2\frac{s}{m^2}\left(\log\frac{s}{m^2}\right)^2+8\pi\left(\frac{n+1}{n}\right)\left(\frac{s}{m^2}\right)^{1/2}\left(\log\frac{s}{m^2}\right)$$

$$+(1280)\pi m^3\left[\frac{2^n(n!)^2}{(2n)!}\left(\frac{2n}{n+1}\right)^n\right]a_2{}^t s, \tag{3.1.10}$$

where $n\geq 1$. The only unknown constant is the scattering length $a_2{}^t$ (assumed to be finite).

The above result has been generalized by Common to give bounds on moments of $\sigma_{\text{total}}(s)$. For $\pi\pi$ scattering, the Nth moment is defined for $N\geq 1$ by

$$\sigma_{\text{total}}{}^{(N)}(s)=\frac{(N+1)}{(s-4m^2)^{N+1}}$$

$$\times\int_{4m^2}^{s} ds'(s'+4m^2)^N\sigma_{\text{total}}(s). \tag{3.1.11}$$

Common shows that for any $n\geq 1$,

$$\sigma_{\text{total}}{}^{(N)}(s)\leq\left(\frac{N+1}{N}\right)\left(\frac{n+1}{n}\right)^2\left(\frac{\pi}{m^2}\right)\left[\log\left(\frac{s}{m^2}\right)\right]^2$$

$$+\left(\frac{N+1}{N-\frac{1}{2}}\right)\left(\frac{n+1}{n}\right)\left(\frac{4\pi}{ms^{1/2}}\right)\log\left(\frac{s}{m^2}\right)$$

$$+(N+1)30\pi^2M_n{}^{-1}\alpha_2{}^t, \tag{3.1.12}$$

where

$$M_n=[(2n)!(n+1)^n/2^n(n!)^2(2n+1)^n].$$

Taking $N=\log (s/m^2)$ and letting $n\rightarrow\infty$, this equation gives the Froissart bound in the form

$$\sigma_{\text{total}}(s)\leq(\pi/m^2)[\log (s/s_0)]^2. \tag{3.1.13}$$

The above results have been generalized to other scattering processes by Common and Yndurain (1970).

3.2 Elastic Cross Sections and the Forward Peak

References: Martin (1963c), Kinoshita (1966), Eden (1966a, 1966b, 1967), Logunov, Mestvirishvili, Nguyen van Hieu, and Nguyen ngoc Thuan (1968), and Singh and Roy (1970a).

For practical purposes, provided $t<4m^2$, we can cut off the partial wave series at $l=L=Cs^{1/2}\log (s)$. Then, using Cauchy's inequality, we obtain

$$\left[\left(\frac{s}{32\pi}\right)\sigma_{\text{total}}(s)\right]^2\leq\Big[\sum_0^L l\,\mathrm{Im}\,f_l(s)\Big]^2$$

$$\leq\Big[\sum_0^L l\,|f_l(s)|\Big]^2$$

$$\leq\Big[\sum_0^L l\,|f_l(s)|^2\Big]\Big[\sum_0^L l\Big].$$

Hence,

$$[\sigma_{\text{total}}(s)]^2\leq C\sigma(\text{elastic})[\log (s/s_0)]^2. \tag{3.2.1}$$

The constant C can be shown to be π/m^2 [for example, see Łukaszuk (1970) or Singh and Roy (1970a)]. Using similar methods, the following results can be obtained:

$$\frac{|F(s, 0)|^2}{16\pi s^2}=\left(\frac{d\sigma(\text{elastic})}{dt}\right)_{t=0}$$

$$\leq\left(\frac{\sigma(\text{elastic})}{16m^2}\right)\left(\log\frac{s}{s_0}\right)^2, \tag{3.2.2}$$

$$\frac{d}{dt}\log|F(s, t=0)|$$

$$\leq C[\sigma(\text{elastic})]^{1/2}\left(\frac{s[\log (s/s_0)]^3}{|F(s, 0)|}\right). \tag{3.2.3}$$

From (3.2.2) one can obtain a bound on the phase of the forward amplitude:

$$\left|\frac{\text{Re}\,F(s,0)}{\text{Im}\,F(s,0)}\right| \leq \frac{\pi}{m^2}\left(\log\frac{s}{s_0}\right)\frac{[\sigma(\text{elastic})]^{1/2}}{\sigma(\text{total})}, \quad (3.2.4)$$

$$\leq \frac{\pi}{m^2}\left(\log\frac{s}{s_0}\right)[\sigma(\text{total})]^{-1/2}. \quad (3.2.5)$$

For completeness, we note also the obvious unitarity bound, which was used in going from (3.2.4) to (3.2.5),

$$\sigma(\text{elastic}) \leq \sigma(\text{total}). \quad (3.2.6)$$

3.3 Nonforward Scattering

References: Martin (1963b), Jin and Martin (1964), Logunov *et al.* (1968), and Singh and Roy (1970a, 1970c).

Using bounds on $P_l(\cos\theta)$ and the methods of Sec. 3.1, one obtains the following bounds:

For fixed negative transfer, $t<-\epsilon<0$,

$$\frac{d\sigma(\text{elastic})}{dt} \leq \left[\frac{[\log(s/s_0)]\sigma(\text{elastic})}{4\pi m(-t)^{1/2}}\right]. \quad (3.3.1)$$

Using the bounds given in previous sections, this leads to the bound

$$|F(s,t)| \leq \left[\frac{s[\log(s/s_0)]^{3/2}}{2\pi^{1/2}m^{3/2}(-t)^{1/4}}\right], \quad (3.3.2)$$

for $t<-\epsilon<0$. For positive values of t in $(0<t<4m^2)$,

$$|F(s,t)| \leq C(s/s_0)^{2-\epsilon}, \quad \text{where } \epsilon>0. \quad (3.3.3)$$

For fixed angle $0<\theta<\pi$, as $s(\text{real})\rightarrow\infty$,

$$\frac{d\sigma(\text{elastic})}{d\Omega} \leq \left[\frac{s^{1/2}[\log(s/s_0)]\sigma(\text{elastic})}{8\pi^2 m(\sin\theta)}\right]. \quad (3.3.4)$$

Using $\sigma(\text{elastic})\leq\sigma(\text{total})\leq(\pi/m^2)[\log(s/s_0)]^2$, this gives for fixed $\theta>0$, as $s(\text{real})\rightarrow\infty$,

$$|F(s,t(\cos\theta))| \leq \left[\frac{s^{3/4}[\log(s/s_0)]^{3/2}}{\pi^{1/2}2^{3/2}m^{3/2}(\sin\theta)^{1/2}}\right]. \quad (3.3.5)$$

An upper bound at fixed negative t is given by Singh and Roy (1970a, 1970c),

$$|\text{Im}\,F(s,t)/\text{Im}\,F(s,0)|$$

$$<[1-\tfrac{1}{9}x+\tfrac{3}{8}(\tfrac{1}{9}x)^2-\tfrac{21}{320}(\tfrac{1}{9}x)^3+\cdots], \quad (3.3.6)$$

provided $x<2.5$, where

$$x=(-t)[\sigma(\text{total})]^2/4\pi\sigma(\text{elastic}). \quad (3.3.7)$$

It is interesting to note that the bound (3.3.5) cannot be reached in any finite interval $\theta_1<\theta<\theta_2$ since this would give $\sigma(\text{elastic})>\sigma(\text{total})$ for large values of s. This suggests that a rigorous improvement to this bound should be possible, but this has only been achieved by making extra assumptions beyond those based on QFT [see Kinoshita *et al.* (1964) and Sec. 4.1].

3.4 Lower Bounds

a. The Forward Peak

Reference: MacDowell and Martin (1964).

The partial wave series for $\text{Im}\,F(s,t)$ contains terms involving the Legendre polynomials giving the t dependence, and $\text{Im}\,f_l(s)$, which is restricted by unitarity to the range (0, 1). Using this restriction one can minimize the derivatives of $\text{Im}\,F(s,t)$ at $t=0$, by choice of $\text{Im}\,f_l(s)$, assuming that $\sigma(\text{total})$ and $\sigma(\text{el. im.})$ are given. The latter quantity is the value that $\sigma(\text{elastic})$ would take if $\text{Re}\,f_l(s)=0$. This leads to a rigorous lower bound,

$$(d/dt)[\log \text{Im}\,F(s,t)]_{t=0}$$

$$\geq \tfrac{1}{9}\{[\sigma(\text{total})]^2/4\pi\sigma(\text{elastic})-(1/k^2)\}. \quad (3.4.1)$$

b. The Forward Amplitude and Total Cross Section

Reference: Jin and Martin (1964).

From the dispersion relation for a symmetric forward scattering amplitude $F(s,0)$, one can prove that $F(s,0)$ has no more than two zeros in the complex s plane (or on the real axis between the cuts). The proof makes use of the Froissart bound, which limits the number of subtractions, and the positivity of $\text{Im}\,F(s,0)$ on the cut $s>s_0$. This result permits one to relate $F(s,0)$ to a Herglotz function $H(s,0)$ by dividing out the zeros. The asymptotic bounds on Herglotz functions then lead to the following result,

$$|F(s,0)| \geq |C/s^2|. \quad (3.4.2)$$

This bound holds rigorously in complex directions and in an average sense for $s(\text{real})\rightarrow\infty$. In an average sense, this gives

$$\sigma(\text{total}) \geq \frac{\text{const}}{s^6[\log(s/s_0)]^2}. \quad (3.4.3)$$

The reason for the appearance of the extra powers of s and $(\log s)$ in (3.4.3) is that (3.4.2) may be dominantly real, so one cannot use the optical theorem to give (3.4.3) directly. Instead one has to use the inequality (3.2.2) giving

$$\sigma(\text{total}) \geq \sigma(\text{elastic}) \geq \frac{|F(s,0)|^2}{(\pi/m^2)s^2[\log(s/s_0)]^2}; \quad (3.4.4)$$

substituting into (3.4.2), the inequality (3.4.3) is obtained.

c. The Method of Lagrange Multipliers

This method was first used in the present context by MacDowell and Martin (1964) for deriving their lower bound (3.4.1). It can also be used to obtain other inequalities and bounds such as those in Sec. 3.3. A very useful discussion of the method of Lagrange multipliers generalized to include inequality constraints has been given by Einhorn and Blankenbecler (1970). These authors show that their generalized method can be used to derive many of the rigorous inequalities both for $\text{Im}\,F(s,t)$ and for $\text{Re}\,F(s,t)$, and it can also be applied to inequalities based on special assumptions which will be discussed in Sec. 4.

3.5 Two-Body Inelastic Processes

References: Logunov, Mestvirishvili, Nguyen van Hieu, and Nguyen ngoc Thuan (1968), and Roy and Singh (1969).

For two-body inelastic processes, the partial waves are bounded by unitarity

$$|f_l^{ab\to cd}(s)|^2 \leq \operatorname{Im} f_l^{ab\to ab}(s). \qquad (3.5.1)$$

Hence

$$|F_{abcd}(s,t=0)| \leq \sum_0^L (2l+1)|f_l^{abcd}(s)| \leq Cs[\log(s/s_0)]^2, \qquad (3.5.2)$$

giving

$$[d\sigma(ab\to cd)/dt]_{t=0} \leq C[\sigma(ab\to cd)][\log(s/s_0)]^2. \qquad (3.5.3)$$

At fixed angle $0<\theta<\pi$, $s(\text{real})\to\infty$, the bounds are essentially the same as for elastic scattering,

$$d\sigma(ab\to cd)/d(\cos\theta) \leq C' s^{1/2}[\log(s/s_0)] \times[\sigma(ab\to cd)]/\sin\theta. \qquad (3.5.4)$$

Some "nearly" rigorous results have been obtained for many-body inelastic processes. These will be given with references in Sec. 4.3.

3.6 Nonzero Spin

a. Analyticity Properties

References: Sommer (1967), Mahoux and Martin (1968), Mahoux (1969), Bell (1968), and Martin (1969).

The starting point of our discussion was the analyticity obtained by Lehmann (1958) (see Sec. 2.2) and by Bros, Epstein, and Glaser (1964, 1965) (see Sec. 2.2). These results still hold for scattering amplitudes of particles with nonzero spin provided they have no kinematic singularities. It is also necessary to ensure that the amplitudes have the right positivity properties for Im F on both the s cut and the u cut. For example this is achieved for pion–nucleon scattering by

$$F^{\pm}(s,t) = A^{\pm}(s,t) + [(s-u)/4M]B^{\pm}(s,t), \qquad (3.6.1)$$

where A and B are the usual invariant amplitudes (free of kinematic singularities), and $\pm$ denotes symmetric and antisymmetric combinations of π^+p and π^-p scattering. In the general case, the key results can be based on the following theorem (Mahoux 1969):

Given a scattering amplitude $F(s,t)$ for the scattering of two massive particles having nonzero spin, that is free from kinematic singularities and satisfies a dispersion relation in $(t_0\leq t\leq 0)$, this amplitude $F(s,t)$ also satisfies a dispersion relation in $|t|<R$ (where R is a positive constant); the number of subtractions increases by no more than one.

The above theorem is essentially the same as that proved for spinless particles by Martin (1966a) (see Sec. 2.2.g). As with that theorem, full cut-plane analyticity at fixed t is not essential, but one can use instead the analyticity proved by Bros, Epstein, and Glaser (1965) (sce Sec. 2.2.c).

The derivation of the appropriate kinematic singularity-free amplitudes can be achieved by standard methods and is described in the above references [see also Wang (1966), and Cohen-Tannoudji, Morel, and Navelet (1968)].

b. Bounds

References: Yamamoto (1963), Hara (1964), Cornille (1964), Martin (1967, 1969), and Mahoux and Martin (1968).

For a spin-averaged amplitude F, orbital angular momentum is conserved, so $F(s,t)$ will have a similar partial wave expansion to that for nonzero spin. In particular, unitarity will impose the usual condition

$$0\leq |f_l(s)|^2 \leq \operatorname{Im} f_l(s) \leq 1 \qquad (3.6.2)$$

for the partial wave in the expansion of the spin-averaged amplitude. This is sufficient to establish bounds that are similar to those for the case of nonzero spin.

If the spin average is not taken, it may be possible to choose an amplitude that satisfies stronger bounds than in the nonspin case. This may lead to superconvergence relations for such an amplitude [see Mahoux and Martin (1968) and Sec. 4.6].

3.7 Form Factors

References: Martin (1965a) and Jaffe (1966).

Jaffe has established from the general principles of local QFT, that a form factor $F(t)$ is analytic in the t plane cut from t_0 to infinity with the possible exception of a finite region $|t|<R$. He also shows that

$$\int_\Gamma \frac{\log[1+|F(t)|]}{(1+|t|^{3/2})}|dt| < M, \qquad (3.7.1)$$

for some finite M, where Γ is a straight line in Im $t>0$ parallel to the real axis.

Jaffe's assumptions do not include the temperedness of the fields, and are at least as general in character as those discussed in Sec. 2. From the bound (3.7.1) using theorems on growth rates (Boas, 1954), it follows that

$$|F(t)| \geq A\exp[-b|t|^{1/2}], \quad \text{as} \quad t\to-\infty. \qquad (3.7.2)$$

This result had been obtained earlier by Martin (1966a) assuming analyticity of $F(t)$ and a bound on its growth rate. The work of Jaffe shows that it follows rigorously from QFT.

3.8 Zeros of Amplitudes

References: Jin and Martin (1964), Wit (1964), Bessis (1966), and Eden and Lukaszuk (1967).

It is evident, from discussions on the problem of duality of scattering amplitudes, or from the phenomenology of Regge theory, that the distribution of zeros of scattering amplitudes is very important both for consistency and for experimental comparison. The number of zeros of $F(s, t)$ in the s plane for $t=0$ also plays an important role in the discussion of lower bounds as $s\rightarrow\infty$ (see Sec. 3.4). We will see later (Sec. 4.2) that the number of zeros in the t plane for fixed (large) s is also important for the study of fixed angle bounds.

The following results about zeros have been proved rigorously:

(i) $F(s, 0)$ has no more than two zeros in the cut plane provided the cut corresponds only to physically allowed scattering (as in the equal-mass case).

(ii) The zero of Im $F(s, t)$ in $t\leq 0$, which is nearest to $t=0$, cannot be closer than a distance of order $(\log s)^{-2}$, as $s\rightarrow\infty$.

(iii) The number N of zeros of $F(s, t)$, inside a circle of radius $|t|=b<r<t_0$, satisfies the bound

$$N(s, b)\leq\frac{4\log s}{\log(r/b)}. \qquad (3.8.1)$$

4. BOUNDS REQUIRING SPECIAL ASSUMPTIONS

General references: Kinoshita (1964), Kinoshita, Loeffel, and Martin (1964), and Eden (1967).

The special assumptions that are used to give additional results on high energy behavior can all be stated in precise mathematical terms, but not all have a clear physical meaning. Alternative special assumptions include (i) the validity of the Mandelstam representation, (ii) the asymptotic constancy of total cross sections, (iii) analyticity and polynomial bounds in domains where these results have not yet been proved from QFT, (iv) nonoscillatory behavior or some knowledge about the distribution of zeros of an amplitude, (v) restrictions on the asymptotic phase, and (vi) a simple form of a particular model for high energy scattering, such as the Regge pole model.

The reason for studying special assumptions is that their consequences may be subject to experimental tests. Although such tests cannot prove or disprove asymptotic theorems, it is possible for them to appear either as reasonable or unreasonable in comparison with experiments at high, but finite, energies. We will illustrate some of these special assumptions and their consequences in this section.

4.1 Fixed Angle Upper Bounds

Reference: Kinoshita, Loeffel, and Martin (1964).

At a fixed angle θ, differential cross sections appear to decrease experimentally like $\exp(-s^{1/2})$ or $\exp(-s)$. The bounds (3.3.4) and (3.3.5) are very far from this behavior, and they would even violate unitarity if they were saturated for a range $\theta_1>\theta>\theta_2$. Improved bounds have been obtained by Kinoshita *et al.*, who assume the following:

Assumptions

(i) Unitarity for partial waves.

(ii) Analyticity in the cut $z=\cos\theta$ plane for Im $F(s, t)$.

(iii) Polynomial boundedness in the variable z, in the cut z plane.

The last two assumptions are additional to the results obtained from QFT. They are contained in the Mandelstam representation, but assumption (iii) would not hold if, for example, Regge theory was valid with indefinitely rising trajectories.

The main results of Kinoshita *et al.* are the following:

(a) For forward scattering no improvement is possible. An explicit counterexample satisfying (i), (ii), and (iii) was constructed.

(b) For $\theta\neq 0$ or π, they obtained the improved bound

$$|F(s, t(\cos\theta))|\leq\frac{C[\log(s/s_0)]^{3/2}}{\sin^2\theta}. \qquad (4.1.1)$$

For the differential cross section this gives

$$\frac{d\sigma}{d\Omega}<\frac{C[\log(s/s_0)]^3}{s\sin^4\theta}. \qquad (4.1.2)$$

It is possible to obtain these results with less than full cut-plane analyticity in $z=\cos\theta$, but it is necessary by present methods to exceed the analyticity and boundedness that has been proved from QFT.

4.2 Fixed Angle Lower Bounds

References: Cerulus and Martin (1963) and Eden and Tan (1968b).

The method of Cerulus and Martin was originally based on the Mandelstam representation, but it can be used in more restrictive conditions. The essential ingredients are

(i) the lower bound (3.4.1), $|F(s, 0)|>Cs^{-2}$,

(ii) special assumption: $F(s, t(\cos\theta))$ is analytic in a certain bounded domain $D(s)$ in the $\cos\theta$ plane containing the real interval $(-\rho, \rho)$, with $\rho=1+C'/s$,

(iii) special assumption:

$$|F(s, t(\cos\theta))|<M(s), \qquad (4.2.1)$$

on the boundary of $D(s)$, where $M(s)$ is a known function.

Cerulus and Martin take $M(s)=s^N$, which is the boundedness of the Mandelstam representation, and they take a large domain $D(s)$, giving the result, for $0<\theta<\pi$,

$$|F(s, t(\cos\theta))|\geq\exp[-C(\theta)s^{1/2}\log(s/s_0)]. \qquad (4.2.2)$$

This appears to be fairly close to experimental results.

If one either uses a smaller domain for $D(s)$ or allows $M(s)$ to be larger, the bound becomes weaker. For example, using the value $M(s)=\exp(s/s_0)$, that is indicated by rising Regge trajectories, the lower bound will become,

$$|F(s, t(\cos\theta))|>\exp[-C(\theta)s\log(s/s_0)]. \qquad (4.2.3)$$

4.3 Angular Dependence

Reference: Tiktopoulos and Treiman (1968a) and Kinoshita (1964).

If, in addition to the Cerulus–Martin assumptions of Sec. 4.2, one assumes that $F(s, t(\cos\theta))$ has no zeros for $\cos\theta$ in $D(s)$, it is possible to obtain restrictions on the angular dependence [$D(s)$ is a certain bounded domain in the cut $\cos\theta$ plane, whose size depends on s]. Thus, if $F(s, t(\cos\theta))$ is analytic for $\cos\theta$ in the domain $D(s)$, and if

$$|F(s, t(\cos\theta))| < Q(s)[1+(\rho^2-\cos^2\theta)^{1/2}]^{M(s)}, \quad (4.3.1)$$

then, for $\theta_1 > \theta_2$,

$$\phi(s, \cos\theta_1)/(\rho^2-\cos^2\theta_1)^{1/2} < \phi(s, \cos\theta_2)/(\rho^2-\cos^2\theta_1)^{1/2} \quad (4.3.2)$$

and

$$(\rho^2-\cos^2\theta_2)^{1/2}\phi(s, \cos\theta_2) \leq (\rho^2-\cos^2\theta_1)^{1/2}\phi(s, \cos\theta_1), \quad (4.3.3)$$

where

$$\phi(s, \cos\theta) = -\log|F(s, t(\cos\theta))|. \quad (4.3.4)$$

The method can still be used if a limited number of zeros $N(s)$ is allowed inside $D(s)$ as $s\to\infty$, but it breaks down if $N(s)\sim s$ as $s\to\infty$. This is in fact the number of zeros that is to be expected if there are linearly rising Regge trajectories (Eden and Tan, 1968a, 1968b; Chiu, Eden, Green, and Guerin, 1969). The main conclusion here is that information on the distribution of zeros of collision amplitudes plays a vital role in asymptotic behavior.

4.4 Inelastic Processes

Lehmann's derivation of analyticity in the small ellipse has been generalized by Ascoli and Minguzzi (1960) to give rigorously from QFT a domain of analyticity of a production amplitude *in one variable.* However, it has not been established that there is a neighborhood of analyticity in all variables as for two-body scattering (see Sec. 2.2.d). It seems therefore that results on production amplitudes do not have a full rigorous foundation.

Bounds for production amplitudes have been studied by Tiktopoulos and Treiman (1968b) and by Logunov *et al.* (1968). They use unitarity for the partial waves,

$$\int\sum\rho\,|T_l^m(s, t_1\cdots t_p)|^2 \leq k^{-1}\operatorname{Im} f_l(s) \leq k^{-1}. \quad (4.4.1)$$

$f_l(s)$ denotes the partial wave for $a+b\to a+b$. Using Schwarz's inequality, one deduces

$$d\sigma(\text{inelastic})/d(\cos\theta) \leq C_1 s[\log(s/s_0)]^4, \qquad \cos\theta=\pm1, \quad (4.4.2)$$

$$d\sigma(\text{inelastic})/d(\cos\theta) \leq C_2 s^{2/3}[\log(s/s_0)]^{10/3}(\sin 2\theta)^{-1/3}. \quad (4.4.3)$$

4.5 Lower Bounds

References: Jin and Martin (1964), Sugawara (1965), and Wit (1965a, 1965b).

The bound on the forward amplitude given in Sec. 3.4 can be improved if assumptions are made about the low energy behavior of $F(W, 0)$, where $W=\frac{1}{2}(s-u)$. These conditions for pion–nucleon scattering are expressed by Sugawara (1965) in the form

$$\frac{1}{2\pi^2}\int_0^\infty \sigma(W)\,dq > \frac{1}{3}\left(\frac{M+m}{M}\right)(a_1+2a_3) + \frac{2g^2W_0}{(m^2-W_0^2)}, \quad (4.5.1)$$

where $\sigma(W)$ is the average of the π^+p and π^-p total cross sections, W and q denote the laboratory pion energy and momentum, $W_0=m^2/2M$, and $g^2=0.08$; a_1 and a_3 denote the s-wave scattering lengths for isospin $\frac{1}{2}$ and $\frac{3}{2}$, respectively.

The condition (4.5.1) ensures that the symmetric forward amplitude has at least two zeros in the W plane. From this, using the phase representation of Sugawara, or factoring out zeros to get a Herglotz function, one obtains the bound

$$|F(W, 0)| > C/(\log W)^{1/2}, \quad \text{as} \quad W\to\infty, \quad (4.5.2)$$

provided one excludes oscillations (or takes some average). In practice, the (3, 3) resonance alone is sufficient to satisfy the inequality (4.5.1) so the bound (4.5.2) follows for pion–nucleon scattering.

For nonforward scattering, lower bounds can be obtained only by assuming a limit on the allowed oscillations of Im F (for example, see Jin and Martin, 1964). The problem of oscillations of an amplitude has also been considered by Cornille (1970a, 1970b), who seeks suitable averaged amplitudes whose behavior is less oscillatory (see also Sec. 7 of this review, and Eden and Łukaszuk, 1967; and Gervais and Yndurain, 1968).

4.6 Some Related Topics

In most of this review we are concerned with high energy theorems for strongly interacting particles. Some of these theorems are closely related to other topics either by the mathematical techniques involved or through physical associations. A few related topics will be listed very briefly in this section, mainly to provide some useful references.

a. Sum Rules and Superconvergence Relations

Rigorous proofs of sum rules depend only on analyticity and asymptotic behavior of an appropriately chosen amplitude. References relating to rigorous and heuristic results have been given in Sec. 3.6.

b. Sum Rules and Bounds for Forward Compton Scattering

Reference: Truong (1970).

The zero mass of the photon prevents the use of many of the techniques discussed in this review. With some reasonable extra assumptions, Truong is able to obtain bounds from which sum rules can be derived.

c. Bounds on the Two-Pion Scattering Amplitude

References: Martin (1969), Lukaszuk (1966), Lukaszuk and Martin (1967), Wit (1970b), Bonnier and Vinh Mau (1967), and Common (1969).

Using the experimental fact that there is no stable bound state in the $\pi\pi$ system, it is possible to obtain bounds and inequalities on the $\pi\pi$ scattering amplitude. These bounds are rather weak unless assumptions are made additional to those of QFT. For example, at the symmetry point $s=t=u=(4m^2/3)$, for $\pi^0\pi^0$ scattering it is found that

$$-50 \leq F(4m^2/3, 4m^2/3) \leq 8, \qquad (4.6.1)$$

the bounds permit inequalities for certain scattering lengths. For example,

$$a_0(\pi^0\pi^0) = \tfrac{1}{3}a_0(I=0) + \tfrac{2}{3}a_0(I=2) > -4/m. \qquad (4.6.2)$$

d. Bounds for Special Models

The inequality $\sigma(\text{elastic}) < \sigma(\text{total})$ can be used in a variety of special situations to limit the rate of growth of the amplitude in the forward direction. Qualitatively, if $\Delta(s)$ denotes the width of the elastic forward peak, we will have

$$\frac{|F(s,0)|^2 \Delta(s)}{s^2} < \sigma(\text{total}). \qquad (4.6.3)$$

In the general case this inequality will lead only to bounds of the type described in Sec. 3.2, but for a Regge theory in which one uses only simple Regge poles, it leads to the result (Oehme, 1963; Leader, 1963; and Eden, 1967)

$$\alpha(0) \leq 1 \qquad (4.6.4)$$

for the leading trajectory.

This method has also been used by H. Sugawara (1963) in Regge pole theory. We will return to it also in connection with a more complicated form of Regge theory in Sec. 7.2.

5. HIGH ENERGY THEOREMS REQUIRING SPECIAL ASSUMPTIONS

General references: Greenberg (1964), Van Hove (1964), Kinoshita, and Eden (1967).

In this section we will begin by considering relations between the phase and the asymptotic rate of growth of scattering amplitudes. These relations provide the mathematical basis for results like the Pomeranchuk theorem. We also discuss situations where the Pomeranchuk theorem does not hold.

Notation: In this section and later sections it is more convenient to use the symmetric energy variable $W=\frac{1}{2}(s-u)$ instead of the variable s.

5.1 Asymptotic Phase and Growth Rate

References: Sugawara and Tubis (1963), Logunov *et al.* (1963), Van Hove (1964), Jin and MacDowell (1965), Khuri and Kinoshita (1965), and Eden (1967).

Relations between the asymptotic phase and growth rate follow from the symmetry between the right and left-hand cuts in the W plane, where $W=\frac{1}{2}(s-u)$ at fixed t. For a symmetric amplitude F_S, with W real,

$$\text{Im}\, F_S(-W+i0, t) = -\text{Im}\, F_S(W+i0, t). \qquad (5.1.1)$$

If the asymptotic behavior is given by

$$|F_S(W,t)| \sim C|W|^{\alpha}, \qquad (5.1.2)$$

it follows from (5.1.1) that

$$F_S(W,t) \sim \pm CW^{\alpha} \exp[i\pi(1-\tfrac{1}{2}\alpha)]. \qquad (5.1.3)$$

Similarly, for an antisymmetric amplitude

$$\text{Im}\, F_A(-W+i0, t) = +\text{Im}\, F_A(W+i0, t), \qquad (5.1.4)$$

giving

$$F_A(W,t) \sim \pm CW^{\alpha} \exp[i\pi(\tfrac{1}{2}-\tfrac{1}{2}\alpha)]. \qquad (5.1.5)$$

The above results are familiar from the phases associated with even and odd signature Regge poles. They can be generalized to include functions having more subtle asymptotic behavior in a variety of ways including

(a) the direct use of dispersion relations;

(b) the use of the Phragmèn–Lindelöff theorem (see Meiman, 1962; Logunov *et al.*, 1963; and Van Hove, 1964);

(c) the use of the phase representation of Sugawara and Kanezawa (1961, 1962) (see Sugawara and Tubis, 1963; and Jin and MacDowell, 1965);

(d) the use of results from the theory of univalent functions (see Khuri and Kinoshita, 1965; Khuri, 1969.

The last method, in particular, can be used to set bounds on the rate of growth even when there are severe oscillations. Some general classes of oscillating amplitudes have also been studied by Gervais and Yndurain (1968) with special reference to high energy results including the Pomeranchuk theorem.

5.2 Pomeranchuk Theorem for Differential Cross Sections

References: Pomeranchuk (1956), Logunov, Nguyen van Hieu, Todorov, and Krustalev (1963), and Van Hove (1964).

Let W denote the symmetric energy variable

$$W = \tfrac{1}{2}(s-u) = s + \tfrac{1}{2}t - \tfrac{1}{2}\sum m^2. \qquad (5.2.1)$$

Let F_1 and F_2 denote amplitudes for particle–target and antiparticle–target scattering

$$F_1: \bar{A}+B \rightarrow \bar{A}+B,$$

$$F_2: A+B \rightarrow A+B. \qquad (5.2.2)$$

Crossing and Hermitian analyticity gives the relations

$$F_1(-W-i0, t) = F_2(W+i0, t), \qquad (5.2.3)$$

$$F_1^*(-W+i0, t) = F_1(-W-i0, t). \qquad (5.2.4)$$

Hence

$$F_1(W \exp(i\pi), t) = F_2^*(W+i0, t), \qquad (5.2.5)$$

where W is real.

Assumption: For fixed real t, as $W(\text{real}) \to \infty$,

$$F_r(W+i0, t)/W^\alpha (\log W - \tfrac{1}{2}i\pi)^\beta \to C_r(t), \quad r=1, 2. \qquad (5.2.6)$$

Using analyticity and polynomial bounds, one obtains via the Phragmèn–Lindelöff theorem, as $W \to \infty$,

$$F_2^*(W+i0, t)/F_1(W+i0, t)$$
$$\to e^{i\pi\alpha}[(\log W + \tfrac{1}{2}i\pi)/(\log W - \tfrac{1}{2}i\pi)]^\beta \to e^{i\pi\alpha}. \qquad (5.2.7)$$

This gives the Pomeranchuk theorem for differential cross sections, at fixed t,

$$\frac{(d\sigma/dt)(A+B\to A+B)}{(d\sigma/dt)(A+B\to A+B)} \to 1, \quad \text{as} \quad W\to\infty, \qquad (5.2.8)$$

Note: (i) The assumption (5.2.6) excludes oscillations of the phase when W is sufficiently large. In Sec. 7 we give an example that violates this assumption. (ii) Specializing to $t=0$ we obtain

$$|F_2(W+i0, 0)/F_1(W+i0, 0)| \to 1. \qquad (5.2.9)$$

The result (5.2.9) gives no information about $\sigma_1(\text{total})$ and $\sigma_2(\text{total})$ unless F_1 and F_2 are dominantly pure imaginary in the forward direction.

5.3 Pomeranchuk Theorem for Total Cross Sections

Reference: Pomeranchuk (1958).

a. Statement of the Theorem

Let $\sigma_{\text{tot}}(A+B)$ and $\sigma_{\text{tot}}(A+B)$ denote total cross sections for particle–target and for antiparticle–target collisions. Then

$$\sigma_{\text{tot}}(A+B)/\sigma_{\text{tot}}(A+B) \to 1, \quad \text{as} \quad W\to\infty. \qquad (5.3.1)$$

b. Proof When Cross Sections Are Asymptotically Constant

References: Pomeranchuk (1958) and Martin (1965).

Assumption (i)

$$\sigma_{\text{tot}}(A+B) \to C_1; \qquad \sigma_{\text{tot}}(A+B) \to C_2. \qquad (5.3.2)$$

Assumption (ii)

$$\operatorname{Re} F_r(W, 0)/(\log W) \operatorname{Im} F_r(W, 0) \to 0 \qquad (r=1, 2) \qquad (5.3.3)$$

where F_1 and F_2 are defined in (5.2.2). From the dispersion relations and the optical theorem, it follows from (5.3.2) that

$$\operatorname{Re}(F_1-F_2) \sim -(2/\pi)(C_1-C_2) W \log W$$
$$\operatorname{Im}(F_1-F_2) \sim (C_1-C_2) W. \qquad (5.3.4)$$

Using also the dispersion relation for (F_1+F_2), one finds that assumption (ii), Eq. (5.3.3), is violated unless $C_1=C_2$ and the Pomeranchuk theorem is satisfied.

Note: Assumption (ii) has not been derived from QFT but is introduced as an *ad hoc* hypothesis. Intuitive arguments suggesting that $\operatorname{Im} F \gg \operatorname{Re} F$ as $W\to\infty$ appear to be based on models in which only a number $L'=cW^{1/2}$ of partial waves are important in the series (3.1.3), whereas in the simplest Regge picture, $L''=cW^{1/2}(\log W)^{1/2}$ partial waves are important, and rigorously from QFT, $L=cW^{1/2}\log W$ may be important.

c. Proof When Cross Sections Diverge Asymptotically

References: Eden (1966b) and Kinoshita (1966).

In this case the special assumption Eq. (5.3.3) is not an additional requirement since it follows rigorously from unitarity and QFT since, for example, Eq. (3.2.5) gives

$$\operatorname{Re} F_r(W, 0)/(\log W) \operatorname{Im} F_r(W, 0)$$
$$\leq (\pi/m^2)[\sigma_r(\text{total})]^{-1/2}. \qquad (5.3.5)$$

For example, let us assume that total cross sections behave asymptotically like

$$\sigma_1(W) \sim C_1(\log W)^\alpha, \qquad \sigma_2(W) \sim C_2(\log W)^\alpha, \qquad (5.3.6)$$

where α is strictly positive. It follows from the dispersion relations that the QFT result (5.3.5) is violated unless $C_1=C_2$. This proves the Pomeranchuk theorem (5.3.1).

Limited oscillations of $\sigma(\text{total})$ could be introduced without invalidating this result (for example, see Cornille, 1970b).

d. Bounded Growth Rate for $(\sigma_1-\sigma_2)$

We replace Eq. (5.3.6) by a form consistent with the result (5.3.5), but allowing a nonzero difference between σ_1 and σ_2, namely,

$$\sigma_1(W) \sim C(\log W)^\alpha + D(\log W)^\beta, \quad 0<\beta<\alpha,$$

$$\sigma_2(W) \sim C(\log W)^\alpha - D(\log W)^\beta, \quad 0<\beta<\alpha. \qquad (5.3.7)$$

From the dispersion relations, we can obtain $\operatorname{Re} F/\operatorname{Im} F$ for each amplitude. In particular, the dispersion relation for the antisymmetric amplitude will lead to a term in $\operatorname{Re} F(W, 0)$ involving $W(\log W)^{\beta+1}$. Substituting in Eq. (5.3.5), one obtains the consistancy requirement

$$D(\log W)^\beta \leq (\pi^2/2m^2)(\log W)^{\frac{1}{2}\alpha}, \qquad (5.3.8)$$

from which we deduce that $\beta \leq \frac{1}{2}\alpha$. Hence as $W\to\infty$,

$$|[\sigma_{\text{tot}}(A+B)/\sigma_{\text{tot}}(A+B)]-1|$$
$$\leq C/(\log W)^{\frac{1}{2}\alpha}. \qquad (5.3.9)$$

This may be written in the form

$$|(\sigma_1-\sigma_2)| \leq C(\sigma_r)^{1/2}, \qquad r=1, 2. \qquad (5.3.10)$$

e. Countertheorem

In Sec. 6, we will see that the situation,

$$\sigma_1(\text{total})\to C_1, \qquad \sigma_2(\text{total})\to C_2, \qquad C_1\neq C_2, \quad (5.3.11)$$

may be important. In this case the Pomeranchuk theorem is violated and from the dispersion relations one obtains the countertheorem

$$\text{Re } F_r(W,0)/\text{Im } F_r(W,0)$$
$$\sim[(C_2-C_1)/C_r](1/\pi)\log W, \qquad r=1,2. \quad (5.3.12)$$

f. Proof of a Pomeranchuk-like Theorem from QFT

References: Kinoshita (1970).

Using only results that have been established from QFT, Kinoshita has proved the following theorem:

If $\sigma_1(\text{total})\to C_1$, $\sigma_2(\text{total})\to C_2$, as $W\to\infty$, then either one of the following relations, or both of these relations, must be satisfied as $W\to\infty$,

$$\text{(i)} \quad \sigma_1(\text{total})/\sigma_2(\text{total})\to 1, \quad (5.3.13)$$

$$\text{(ii)} \quad d\sigma_1(W,t(W))/d\sigma_2(W,t(W))\to 1, \quad (5.3.14)$$

where $d\sigma$ denotes $d\sigma/dt$, and

$$t(W)=-[C/(\log W)^2]. \quad (5.3.15)$$

The proof can be achieved by the following steps: If (5.3.13) does not hold, then Re (F_1-F_2) given by (5.3.4) will dominate both forward amplitudes. The t derivative of the forward amplitudes satisfies the rigorous inequality (3.2.3), which in view of (5.3.4) becomes, in this case,

$$(d/dt)\log|F(W,t=0)|\leq C(\log W)^2. \quad (5.3.16)$$

This can be generalized to give a bound on the nth derivative and sets a bound on the Taylor series for $F(W,t)$. From this bound one can see that if C in (5.3.15) is chosen sufficiently small, $F_r(W,t(W))$ will still be dominated by Re F_r coming from the antisymmetric combination of F_1 and F_2. This proves that (5.3.14) must hold if (5.3.13) is violated. If (5.3.13) holds, it is not known whether any general statement can be made about the validity of (5.3.14). The foregoing outline proof is based on a method of Eden and Kaiser (1970a, 1970b) and differs from the elegant but more complicated proof of Kinoshita (1970).

5.4 Invariance Properties and Exchange Cross Sections

References: Okun and Pomeranchuk (1956) and Roy and Singh (1969).

For illustration we consider pion–nucleon scattering:

Assumption: (i) Isospin invariance is asymptotically exact. This gives, as $W\to\infty$,

$$F_1=F_1(\pi^+p\to\pi^+p)\sim F(\tfrac{3}{2}), \quad (5.4.1)$$

$$F_2=F_2(\pi^-p\to\pi^-p)\sim\tfrac{1}{3}[F(\tfrac{3}{2})+2F(\tfrac{1}{2})], \quad (5.4.2)$$

$$F_3=F_3(\pi^-p\to\pi^0n)\sim\tfrac{1}{3}\sqrt{2}[F(\tfrac{3}{2})-F(\tfrac{1}{2})]. \quad (5.4.3)$$

Apply the inelastic bound (3.5.3) to the exchange process, giving

$$|F_3(W,0)|^2\leq CW^2(\log W)^2\sigma_3(\pi^-p\to\pi^0n). \quad (5.4.4)$$

Hence, from isospin invariance,

$$|F_1-F_2|/W\log W\leq C(\sigma_3)^{1/2}. \quad (5.4.5)$$

Assumption: (ii) For total cross sections,

$$\sigma_1\to C_1, \qquad \sigma_2\to C_2. \quad (5.4.6)$$

Using the dispersion relations for (F_1-F_2) and substituting in (5.2.14) we obtain

$$|(C_1-C_2)|\leq C(\sigma_3)^{1/2}. \quad (5.4.7)$$

In their derivation of this result, Roy and Singh evaluate the constant C to give, as $W\to\infty$,

$$|\sigma_{\text{total}}(\pi^-p)-\sigma_{\text{total}}(\pi^+p)|$$
$$\leq[(2\pi)^{3/2}/2m][\sigma_{\text{exch}}(\pi^-p\to\pi^0n)]^{1/2}. \quad (5.4.8)$$

The Pomeranchuk theorem (5.3.1) can be deduced from (5.4.8) if one makes the following extra assumption:

Assumption: (iii) The exchange cross section σ_3 tends to zero as the energy tends to infinity. Then, if the previous assumptions (i) and (ii) are also valid, the result (5.3.1) follows from (5.4.8).

If the Pomeranchuk theorem (5.3.1) does not hold, the inequality (5.4.8) provides a possible experimental test for the validity of isospin invariance at asymptotic energies (see Sec. 6.3.c).

6. EXPERIMENTAL RESULTS AND HIGH ENERGY THEOREMS

6.1 Total Cross Sections and Dispersion Relations

References: Lindenbaum (1968, 1967a, 1967b), Foley *et al.* (1967), Allaby *et al.* (1969), and West and Yennie (1968).

Dispersion relations may be used to determine the phase of a forward amplitude from measurements of total cross sections only if assumptions are made about cross sections at energies beyond those obtained experimentally. Asymptotic smoothness together with bounds on asymptotic cross sections and the use of simple functions of energy will set limits on the phase (or Re F/Im F) in the forward direction. These limits can be tested experimentally by directly measuring the phase by its effect on Coulomb interference at small angles of scattering. This provides a valuable check on the reasonable nature of the asymptotic assumptions (however, see the comments in Sec. 6.4.g).

6.2 Tests of High Energy Bounds

References: MacDowell and Martin (1964) and Singh and Roy (1970b, 1970c).

The phase measurements by Foley *et al.* (1967) indicate that from 8 to 30 GeV, forward amplitudes are dominantly imaginary. The bounds on Im $F(s, t)$ may therefore be relevant near the forward direction, so long as Im $F \gg$ Re F.

MacDowell and Martin give the bound (3.2.6), namely,

$$(d/dt) \log [\operatorname{Im} F(s,t)]_{t=0} \geq \tfrac{1}{9}\{[(\sigma_{\text{tot}})^2/4\pi\sigma_{\text{el}}]-(1/k^2)\}. \quad (6.2.1)$$

On the assumption that both F and dF/dt are pure imaginary at $t=0$, it appears that this bound is nearly reached for the case of pion–nucleon scattering (assuming spin independence of the unpolarized cross sections near $t=0$).

Singh and Roy have made a comparison with experiment of their upper bound on Im $F(s, t)$ given in Eq. (3.3.6), assuming also that $d\sigma/dt$ is dominated by Im F. In the near-forward direction {where $d\sigma/dt > [\frac{1}{2}(d\sigma/dt)$ with $t=0]$} they find the bound differs by only about 10% from experimental values for pion–proton scattering between 7 and 13 GeV energies.

6.3 The Serpukhov Data on Total Cross Sections

References: Allaby *et al.* (1969).

a. The Experimental Results

The IHEP–CERN collaboration at Serpukhov (Allaby *et al.*, 1969) gives results as follows (σ denotes σ_{total}):

(i) $\sigma(\bar{p}p)$ continues to decrease at Serpukhov energies and may be asymptotically equal to $\sigma(pp)$.

(ii) $\sigma(\pi^- p)$ is nearly constant from 30 to 60 GeV/c at about 24.7 mb. This value is about 1 mb greater than $\sigma(\pi^- n)$, which (assuming charge symmetry) should be equal to $\sigma(\pi^+ p)$. In view of the experimental errors and the uncertainty about allowing for the Glauber effect, this difference may not be significant (see, for example, Wit, 1970a).

(iii) $\sigma(K^- p)$ is nearly constant from 20 to 60 GeV/c at about 21 mb. This compares with $\sigma(K^+ p)$, which has a constant value of about 17 mb from 5 to 25 GeV/c.

b. Theoretical Discussion

At first sight it appears that the Serpukhov data on σ(total) for $K^- p$ taken with previous data for $K^+ p$ contradict the Pomeranchuk theorem (5.3.1). This naive interpretation supposes that cross sections that appear to be experimentally constant over a large energy range remain constant out to asymptotic energies. Many alternative interpretations are possible depending in part on the value of the energy that one chooses for the onset of "asymptopia." Since most of the interpretations are likely to be ephemeral in character (or not susceptible to an experimental test), they will be mentioned only briefly. They include the following possibilities:

1. Simple violation of the Pomeranchuk theorem:

$$\sigma(K^- p) \to C_1, \qquad \sigma(K^+ p) \to C_2, \qquad C_1 \neq C_2. \quad (6.3.1)$$

In this case the countertheorem (5.3.12) applies (see Eden, 1970), giving

$$F_A(W, 0) = F(K^- p) - F(K^+ p), \quad (6.3.2)$$

$$F_A(W, 0) \sim -(2/\pi)(C_1 - C_2)(\log W - \tfrac{1}{2}i\pi). \quad (6.3.3)$$

From (6.3.1) and the rigorous inequality (3.2.2), we see that

$$C_r > \sigma_r(\text{elastic}) \geq 4m^2/\pi^3 (C_1 - C_2)^2, \qquad r = 1, 2. \quad (6.3.4)$$

Therefore, in this case, the elastic cross section will be bounded above and below by constant values.

From the rigorous inequality (3.2.3), we see that in this case

$$(d/dt) \log [F(W, t)]_{t=0} \leq C' (\log W)^2. \quad (6.3.5)$$

We will see from an example in Sec. 7.2 that this last inequality does not need to be saturated. However, if one also takes account of higher derivatives, the "effective" width of the forward peak must shrink like $(\log W)^{-2}$ so that σ(elastic) does not exceed σ(total). Further theoretical and experimental consequences of (6.3.1) have been discussed by Eden and Kaiser (1970a, b).

2. A minimum in σ(total). References: Frautschi and Margolis (1968), Barger and Phillips (1970), and Rarita (1970). In Regge theory it is possible for minima to arise in total cross sections due to interference between Regge pole terms and cut contributions of opposite sign. Taking advantage of this, Barger and Phillips and Rarita have shown that existing data can be fitted by a suitable choice of parameters in Regge theory. Their choice leads to flat minima, which agree with the nearly constant experimental values in $K^- p$ and $K^+ p$ collisions, but make asymptopia rather distant.

3. Oscillations in σ(total). Reference: Arnowitt and Rotelli (1970). There is some theoretical evidence that Regge trajectories may be complex even below threshold (Chew, 1969; and Ball and Zachariasen, 1969). This could lead to oscillations in the total cross sections due to interference between different terms in the Regge series for a forward scattering amplitude. Making use of the uncertainty in fitting Regge theory to data at low energies, Arnowitt and Rotelli have obtained a reasonable fit to the Serpukhov data.

6.4 Comments on Experimental Tests

The following aspects of high energy theorems appear to offer the most favorable possibilities for experimental tests. However, it should be emphasized that no amount of accurate experimental information will prevent a theorist from avoiding the issue *as far as general theory is concerned* by postulating ever increasing values for the energy where asymptopia commences. In terms of particular models, or real theories which actually make predictions (without having an indefinite number of variable parameters in reserve), these experimental tests may be much more decisive.

a. The Phase of Forward Amplitudes

If the Pomeranchuk theorem is violated as suggested in (6.3.1), then $\eta(W)=(\text{Re}\,F/\text{Im}\,F)$ should grow logarithmically as in (5.3.12). Qualitatively one would expect a change in the slope of $\eta(W)$ in about the same region that σ_1(total) and σ_2(total) take on nearly constant (unequal) values. Different results hold if alternative explanations of the Serpukhov data are assumed, such as Regge-cut-pole interference minima in total cross sections. But, in any event, the experimental values of the phases would give useful theoretical restrictions.

b. Differential Cross Sections

In the near-forward direction the axiomatic "Pomeranchuk-like" theorem discussed in Sec. 5.3.f can be tested in principle. In practice, it seems most unlikely that $\text{Re}\,F$ will dominate over $\text{Im}\,F$ at accessible energies sufficiently to make (5.3.14) even approximately correct on that account. One is more likely to see the effect of $\text{Re}\,F$ through an increase in the separate differential cross sections very-near-the-forward direction.

Away from the very-near-to-forward direction it is possible that $|F(W,t)|$ will develop oscillations if the Pomeranchuk theorem is violated as in (6.3.1). These oscillations would arise from zeros in $F(W,t)$ near to $t=0$ and may be detectable in the differential cross section (see Sec. 7).

If the Pomeranchuk theorem is violated as indicated in (6.3.1) the width of the forward peak should shrink like $(\log W)^{-2}$. This will be discussed further in Sec. 7.2 (see also Eden and Kaiser, 1970a, 1970b).

c. Tests of a Lower Bound on the Forward Peak

The rigorous bound (3.4.1) has already been discussed in relation to experiment in Sec. 6.2.

d. Comment on Asymptotic Isospin or SU3 Invariance

Reference: Roy and Singh (1969).

Asymptotic isospin invariance gives the inequality (5.4.8). If $\sigma_{\text{tot}}(\pi^-p)-\sigma_{\text{tot}}(\pi^+p)$ does *not* tend to zero, then *either* $\sigma_{\text{exch}}(\pi^-p\rightarrow\pi^0n)$ is also nonzero asymptotically, *or* isospin invariance does not become exact at asymptotic energies. The corresponding situation for kaons with asymptotic SU3 invariance is less simple. One obtains, as $W\rightarrow\infty$,

$$|\sigma(K^+p)-\sigma(K^-p)-\sigma(K^0p)+\sigma(\bar{K}^0p)|<(\pi^{3/2}/m)\times\min\{[\sigma_{\text{exch}}(K^0p\rightarrow K^+n)]^{1/2},[\sigma_{\text{exch}}(K^-p\rightarrow\bar{K}^0n)]^{1/2}\},\tag{6.3.6}$$

where m denotes the pion mass.

e. K^0 Regeneration

K^0 regeneration experiments determine the phase of the amplitude

$$F(K_1^0p\rightarrow K_2^0p)=\tfrac{1}{2}F(K^+n\rightarrow K^+n)-\tfrac{1}{2}F(K^-n\rightarrow K^-n),\tag{6.3.7}$$

in the forward direction. If the total cross sections for K^-n and K^+n are asymptotically unequal constants, the amplitude (6.3.7) will be dominated by the part that leads to violation of the Pomeranchuk theorem,

$$F(K_1^0p\rightarrow K_2^0p)\sim(2s/\pi)(\Delta\sigma)[\log(s)-\tfrac{1}{2}i\pi].\tag{6.3.8}$$

Thus the phase will tend to zero as $s\rightarrow\infty$. The way in which the phase approaches zero has been estimated in various Regge type models by Barger and Phillips (1970).

f. Charge Exchange

If $\sigma_{\text{total}}(\pi^-p)-\sigma_{\text{total}}(\pi^+p)$ does *not* tend to zero, and if isospin invariance becomes asymptotically exact, then the charge exchange amplitude $F(\pi^-p\rightarrow\pi^0n)$ will contain a term $F_A(W,0)$ like (6.3.3) which is responsible for violating the Pomeranchuk theorem. This term means that the charge exchange cross section should grow like $(\log W)^2$ at $t=0$ and it should have a sharp forward peak that shrinks like $(\log W)^{-2}$ as $W\rightarrow\infty$. Since this peak is to be compared in this case with ω, ρ contributions (instead of P, P' contributions as in the elastic case), there should be a better chance of observing it.

g. Coulomb Interference

References: Eden and Kaiser (1970a, 1970b) and West and Yennie (1968).

In Sec. 6.2 we remarked that the phase of a forward amplitude $F(W,0)$ could be measured by Coulomb interference. This deduction requires an extrapolation from measurements of $(d\sigma/dt)$ at small t in the region $t\approx-0.003$ (GeV)^2, where Coulomb scattering and strong (nuclear) scattering are comparable in magnitude. The extrapolation requires an assumption about the smooth behavior of the nuclear scattering in the Coulomb interference region. If the Pomeranchuk theorem is violated, the nuclear scattering contains a $(\log W)^2$ forward peak in addition to the usual well-known peak. This may induce a rapid change in the part of $(d\sigma/dt)$ due to strong interactions as t varies within the Coulomb interference region. It therefore becomes very important to measure $(d\sigma/dt)$ precisely in the Coulomb-dominant region $|t|<(0.003)$ in order to investigate whether there is a narrow nuclear peak in this region. This possibility also creates some doubt about the experimental estimates of total cross sections that also involve an extrapolation through the Coulomb region to zero scattering angle.

7. MODELS AND EXAMPLES OR COUNTER EXAMPLES

7.1 An Amplitude for Violating the Pomeranchuk Theorem

Reference: Finkelstein (1970).

We will assume that total cross sections are asymptotically constant:

$$\sigma_{\text{total}}(K^-p)\to C_1; \qquad \sigma_{\text{total}}(K^+p)\to C_2; \qquad (7.1.1)$$

with $C_1\neq C_2$. Let F_S and F_A denote the symmetric and antisymmetric amplitudes,

$$F_S=\tfrac{1}{2}[F(K^-p)+F(K^+p)], \qquad (7.1.2)$$

$$F_A=\tfrac{1}{2}[F(K^-p)-F(K^+p)]. \qquad (7.1.3)$$

F_S presents no problem since it has $\text{Re}\,F_S<\text{Im}\,F_S$ in the forward direction. However $F_A(W,t)$ must satisfy conditions which include

(i) analyticity for $|t|<t_0$, and the Froissart bound as $W\to\infty$,
(ii) the polynomial bound (3.3.3) for positive real t,
(iii) the bound (6.3.5) on the slope of the forward peak,
(iv) unitarity, $\sigma(\text{elastic})\leq\sigma(\text{total})$,
(v) $\text{Re}\,F_A(W,0)=CW(\log W)$
$\text{Im}\,F_A(W,0)=C'W$.

The Finkelstein amplitude satisfying these conditions has the form

$$F_A(W,t)\sim\frac{CW^{(1+t)}\{\sin[(-t)^{1/2}(\log W-\frac{1}{2}i\pi)]\}^2}{t(\log W-\frac{1}{2}i\pi)}. \qquad (7.1.4)$$

This amplitude has several interesting properties and consequences:

(i) It can be derived in Regge theory from a partial wave amplitude having two branch cuts running from $l=1+t$ to $l=1+t\pm i(-t)^{1/2}$.

(ii) It has zeros near $(-t)=n^2\pi^2/(\log W)^2$. Thus the differential cross sections will have oscillations within the forward peak since F_A dominates over F_S near to $t=0$.

7.2 Zeros near the Forward Direction

References: Bessis (1966), Eden and Lukaszuk (1967), Casella (1970), and Eden and Kaiser (1970a, 1970b).

The Finkelstein amplitude (7.1.4) can clearly be generalized, for example, to

$$F_A(W,t)\sim\frac{CW^{1+t}\{\sum a_n\sin[b_n(-t)^{1/2}(\log W-\frac{1}{2}i\pi)]\}^2}{t(\log W-\frac{1}{2}i\pi)}. \qquad (7.2.1)$$

By choice of constants we could arrange that

$$(d/dt)[\log|F_A(W,t)|]_{t=0}=0. \qquad (7.2.3)$$

Thus the inequality (6.3.5) need not be saturated. However, if the conditions (7.1.1) hold, we can deduce some rigorous conclusions about $F(W,t)$ near the forward direction, where F corresponds to either K^+p, or to K^-p, scattering:

(a) $F(W,0)$ is dominated by $CW\log W$ as $W\to\infty$.

(b) Since $\sigma(\text{elastic})\leq\sigma(\text{total})\leq\text{const}$,

we can deduce from

$$\sigma(\text{elastic})\geq\int_{-\epsilon}^{0}dt\left|\frac{F(s,t)}{s}\right|^2, \qquad (7.2.4)$$

that $(d\sigma/dt)$ must decrease so that the effective width of the forward peak $\Delta(\text{effective})$ satisfies

$$\Delta(\text{effective})\leq C_1/(\log W)^2. \qquad (7.2.5)$$

The constant C_1 has been bounded by Eden and Kaiser (1970a, 1970b), giving $C_1\approx10^2$ (GeV)^2. They also obtain a lower bound

$$C_2/(\log W)^2\leq\Delta(\text{effective}), \qquad (7.2.6)$$

where $C_2\approx10^{-4}$ (GeV)^2.

(c) By studying the rate of growth of the amplitude (3.1.3) for small positive t as $s\to\infty$, Eden and Kaiser also obtain lower bounds on $|F(s,t)|$ for $t<0$. Combining their results with (6.3.3), which follows from the (assumed) violation of the Pomeranchuk theorem, and with the unitarity condition (7.2.4), they show that $F(s,t)$ must have zeros as $s\to\infty$ which lie in the range

$$C_3/(\log W)^2<|t|<C_4/(\log W)^2, \qquad (7.2.7)$$

where $C_3\approx10^{-3}$ (GeV)$^{-2}$ and $C_4\approx10^8$ (GeV)^2. There cannot be any zeros closer to $t=0$ than is allowed by the lower bound in (7.2.7).

(d) The possibility of observing the above behavior of the forward peak in, for example,

$$(d\sigma/dt)(K^-p\to K^-p)$$

depends on the value of $\Delta(\text{effective})$ within the allowed range (7.2.5) and (7.2.6). With favorable values the peak could lie outside the Coulomb-dominant region $|t|>(0.003)$ (GeV)^2 (see the discussion in Sec. 6.4.g). It would be relatively more important in the charge exchange process,

$$K^- p \rightarrow \bar{K}^0 n,$$

for which one might also be able to see the effects of oscillations due to zeros of $F(W, t)$ if these were at experimentally favorable values in the range (7.2.7). In this situation a positive observation of oscillations of $(d\sigma/dt)$ would be most valuable, but failure to observe them would be indecisive.

7.3 Saturation of the Froissart Bound

The Finkelstein amplitude (7.1.4) can be adapted to give a symmetric amplitude that saturates the Froissart bound (3.1.5) and also satisfies all the conditions that we have noted in Sec. 3 as derivable from QFT. A suitable amplitude is given by

$$F_S(W, t) \sim iCW(\log W - \tfrac{1}{2}i\pi)^2 \times \left(\frac{\sin[(-t)^{1/2}(\log W - \frac{1}{2}i\pi)]}{(-t)^{1/2}(\log W - \frac{1}{2}i\pi)}\right)^n \quad (7.3.1)$$

for any integer $n > 2$. This amplitude can be further generalized as in (7.2.1) if this relatively simple form is found to contradict new conditions that may be derived in the future from QFT. The most likely source of such conditions lies in the combination of unitarity in the t channel with crossing symmetry. However, Regge theory already takes some (nonrigorous) account of t-channel unitarity and crossing; the reader is therefore invitėd to verify that the expression (7.3.1) has a representation in the complex l plane analogous to that for (7.1.4).

8. SUMMARY OF AXIOMATIC RESULTS

The following results as $s \rightarrow \infty$ have been proved from axiomatic QFT.

Statement of Bound	Section of Review giving References
Forward Scattering	
$\sigma(\text{total}) \leq (\pi/m^2)[\log(s/s_0)]^2$ m = pion mass	3.1
General spin	3.6
$\lvert F(s,0)\rvert \leq (\pi/m^2)[\log(s/s_0)]^2$ Normalized with Im $F \sim s\sigma(\text{total})$.	3.1
$[d\sigma(\text{elastic})/dt]_{t=0} \leq (1/16m^2)(\log s/s_0)^2\sigma(\text{elastic})$	3.2
$[\sigma(\text{total})]^2 \leq (\pi/m^2)(\log s/s_0)^2\sigma(\text{elastic})$	3.2
$\lvert \text{Re}\,F(s,0)/\text{Im}\,F(s,0)\rvert \leq (\pi/m^2)(\log s/s_0)[\sigma(\text{elastic})]^{1/2}/\sigma(\text{total})$	3.2
$\lvert \text{Re}\,F(s,0)/\text{Im}\,F(s,0)\rvert \leq (\pi/m^2)(\log s/s_0)[\sigma(\text{total})]^{-1/2}$	3.2
Diffraction Peak	
$\left[\frac{d}{dt}\log\lvert F(s,t)\rvert\right]_{t=0} \leq C[\sigma(\text{elastic})]^{1/2}\left(\frac{s[\log(s/s_0)]^3}{\lvert F(s,0)\rvert}\right)$	3.2
$[(d/dt)\log\lvert \text{Im}\,F(s,t)\rvert]_{t=0} \geq \frac{1}{9}\{[(\sigma(\text{total}))^2/4\pi\sigma(\text{elastic})] - (1/k^2)\}$	3.4
Fixed Transfer $(t < -\epsilon < 0)$	
$\frac{d\sigma(\text{elastic})}{dt} \leq \left[\frac{[\log(s/s_0)]\sigma(\text{elastic})}{4\pi m(-t)^{1/2}}\right]$	3.3
$\lvert F(s,t)\rvert \leq \left[\frac{s[\log(s/s_0)]^{3/2}}{2\pi^{1/2}m^{3/2}(-t)^{1/4}}\right]$	3.3
$\lvert \text{Im}\,F(s,t)/\text{Im}\,F(s,0)\rvert \leq [1 - \frac{1}{9}x + \frac{3}{8}(\frac{1}{9}x)^2 - \frac{21}{320}(\frac{1}{9}x)^3 + \cdots]$ if $2.5 > x \equiv (-t)[\sigma(\text{total})]^2/4\pi\sigma(\text{elastic})$	3.3
Fixed Transfer $(0 < t < 4m^2)$	
$\lvert F(s,t)\rvert \leq C(s)^{2-\epsilon}$	3.3

Fixed Angle $0<\theta<\pi$

$$|F(s,t(\cos\theta))|\leq\left[\frac{s^{3/4}[\log(s/s_0)]^{3/2}}{\pi^{1/2}2^{3/2}m^{3/2}(\sin\theta)^{1/2}}\right] \qquad 3.3$$

$$\frac{d\sigma(\text{elastic})}{d\Omega}\leq\left[\frac{s^{1/2}[\log(s/s_0)]\sigma(\text{elastic})}{8m\pi^2(\sin\theta)}\right] \qquad 3.3$$

Lower Bounds

Exact for complex s, average for real s:

$$|F(s,0)|\geq C/s^2 \qquad 3.4$$

$$\sigma(\text{total})\geq\frac{C'}{s^6[\log(s/s_0)]^2} \qquad 3.4$$

Statement of Bound	Section of Review giving References

Inelastic Two-Body Reactions

$$\left[\frac{d\sigma(ab\rightarrow cd)}{dt}\right]_{t=0}\leq\left[\frac{\sigma(ab\rightarrow cd)[\log(s/s_0)]^2}{16m^2}\right] \qquad 3.5$$

At fixed angle:

$$d\sigma(ab\rightarrow cd)/d(\cos\theta)\leq\{Cs^{1/2}(\log s/s_0)[\sigma(ab\rightarrow cd)/\sin\theta]\} \qquad 3.5$$

Form Factors

$$|F(t)|\geqslant A\exp(-b|t|^{1/2})\text{ as }t\rightarrow\infty \qquad 3.7$$

Zeros of $F(s,t)$

$$N(s)\leq C\log(s),\qquad\text{in }|t|<t_0 \qquad 3.8$$

Concluding Remarks

I wish to apologize to those authors whose work has been mentioned only briefly or has not been listed fully in the references. The various review articles noted in the text give a wider list of references to which the reader may refer for the many interesting methods and results that could not be contained in this short survey.

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