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Landau, Brueckner–Bethe, and Migdal Theories of Fermi Systems

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The Landau theory for infinite Fermi systems is discussed, and it is then shown how one can apply the Landau technique of functional differentiation of the energy with respect to quasiparticle occupation numbers to obtain the effective interaction, beginning from the Brueckner–Bethe approximation to the energy. Calculations of this effective interaction by Bäckman from a nucleon–nucleon potential are discussed and criticized. Results of Bäckman's calculation are compared with the parameters in Migdal's effective force, evaluated at the center of the nucleus, the latter parameters having been derived phenomenologically by fitting various nuclear phenomena. There is a large discrepancy between the calculated value of the compressibility of nuclear matter and that obtained phenomenologically. The advantages of introducing a model space in order to improve the accuracy of the calculated results is discussed, and it is pointed out that such an introduction would also be advantageous in recent calculations by Barrett and Kirson of third-order terms in the effective interaction; also, that a model space is effectively introduced in recent calculations with the Midgal theory when it comes to deriving the quadrupole–quadrupole interaction. Calculation of the effective force between two atoms in ^{3}He is discussed briefly, and it is pointed out that this is a much more ambitious task than in nuclear matter. Calculations carried out within the Brueckner–Bethe formalism are reviewed.

CONTENTS

1. INTRODUCTION

The three theories in the title all claim validity in the description of strongly interacting, or moderately strongly interacting, fermions. We shall discuss here the application of all three to nuclear matter and to finite nuclei. Here, Migdal's theory of finite Fermi systems is a derivative of the Landau theory, with certain (major) added assumptions. We also discuss briefly the application of both the Landau and Brueckner–Bethe theory to liquid ^{3}He, noting that the latter must be supplemented by adding spin fluctuations before one can hope to achieve a successful description of this system.

The Landau theory gives a phenomenological description; the Brueckner–Bethe theory, on the other hand, starts from the force between two isolated particles and provides a framework within which one may be able to calculate the effective interactions between quasiparticles in the many-body system. Thus, one can try to calculate the parameters entering into the Landau theory within this latter framework. As will be shown, at the present time this can at best be done semiquantitatively.

My particular goal is to show the connections between the phenomenological Landau and Migdal theories, on the one hand, and the calculational Brueckner–Bethe theory on the other hand. I hope, in this way, to overcome the psychological block of the adherents of the phenomenological theories to consideration of the calculational one. It seems completely clear to me that one must at least *begin* within the Brueckner–Bethe framework, if one has any hope of making a microscopic description. This is because in the first step one must, in some way, get from the bare forces with hard cores to effective interactions (pseudopotentials) which are well behaved at short distances. The Brueckner–Bethe theory shows how to combine the strong short-range repulsion with correla-

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tions in the wave function which keep the particles apart over the range of this repulsive interaction, so as to give a well-behaved effective interaction. Of course, this effective interaction must then be used to include necessary many-body effects, as shall be indicated.

The main reason for the lack of interest among many-body physicists in the Brueckner–Bethe theory seems to me to be a philosophical one. Landau always wanted a small parameter; however, since he did not see one in, say, liquid ^{3}He, rather than expanding in powers of something not small, he preferred to parameterize the quantities determining physical phenomena, taking values from a given set of experiments, and using these to predict other phenomena. In this way he predicted zero sound, and his theory has great aesthetic advantages in unifying phenomena; in fact, it can even be extended to describe transport processes as shown by Pethick (1969).

The Brueckner–Bethe theory was applied in breadth and depth to nuclear matter, largely because one can actually calculate quantities starting from the "elementary" two-body force. Within the theory, as modified by Bethe (1965), one has a parameter κ which is small in the case of nuclear matter ($\kappa \cong 0.15$), but is admittedly not very small in liquid ^{3}He ($\kappa \cong 0.55$). This parameter, which is the one in which the expansion is carried out, is discussed in detail by Brandow (1967). We can understand it roughly in the following way.

On the average, each nucleon occupies a volume the size of a sphere of radius r_0, given by

$$\tfrac{4}{3}\pi r_0^3 = V/N = v, \tag{1}$$

where V is the volume and N the number of particles. Because of interactions with other particles which trespass into this domain, the particle is, on the average, excluded from part of this sphere, from a volume, say, of $\frac{4}{3}\pi d^3$, $d < r_0$. Think, for example, of a gas of hard spheres, each of radius c. Then it is clear, if we consider particles interacting pairwise, that the ratio of volume excluded to each particle by the presence of other particles to the average volume $\frac{4}{3}\pi r_0^3$ is $\sim c^3/r_0^3$ since the average distance between pairs is $\sim r_0$. In fact, quantum mechanically the ratio is somewhat larger, $\sim$4–8 times c^3/r_0^3, since the wave function of the interacting pair not only must be zero inside the region of hard core, but needs take some time to "heal" outside. An expression for κ will be given later, but we note here that κ, in the Brueckner–Bethe theory, is essentially $1-a$, where the a of the Landau theory is the residue at the quasiparticle pole, describing the probability that the excitation is a quasiparticle. It seems completely reasonable that if an excitation is a quasiparticle most of the time, more complicated configurations being admixed with only small probability, then it should be possible to make a microscopic theory in terms of quasiparticles.

2. THE LANDAU THEORY

Landau (1952, 1957, 1958) began from the idea of a system of quasiparticles—excitations which behave like particles, with properties modified by the interactions. He wrote down a Boltzmann equation for these:

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{x}}\frac{\partial \epsilon}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}}\frac{\partial \epsilon}{\partial \mathbf{x}} = I(n), \tag{2}$$

where n is the quasiparticle number, ϵ and p are the energy and momentum of the quasiparticle, respectively, and I is the collision term. He then tried to construct a momentum flux, to obtain conservation of total quasiparticle momentum, by multiplying by p_i and integrating,

$$\frac{\partial}{\partial t}\int p_i n d\tau + \int p_i \left(\frac{\partial n}{\partial \mathbf{x}}\frac{\partial \epsilon}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}}\frac{\partial \epsilon}{\partial \mathbf{x}}\right) d\tau = 0, \tag{3}$$

where

$$d\tau = g[d^3p/(2\pi)^3], \tag{3.1}$$

with g the degeneracy of each state (equal to 2 for liquid ^{3}He). As a result of conservation of momentum in the collisions, the integral $\int p_i I(n) d\tau$ is zero. By algebraic manipulation and integration by parts, one can bring Eq. (3) to the form

$$\frac{\partial}{\partial t}\int p_i n d\tau + \frac{\partial}{\partial \mathbf{x}}\int p_i \frac{\partial \epsilon}{\partial \mathbf{p}} n d\tau + \frac{\partial}{\partial x_i}\int n\epsilon d\tau - \int \epsilon \frac{\partial n}{\partial x_i} d\tau = 0. \tag{4}$$

We would like to be able to write this as

$$(\partial/\partial t)\int p_i n d\tau + (2\pi_{ik}/\partial x_k) = 0 \tag{4.1}$$

in terms of a momentum flux tensor π_{ik}. If this can be done, then it is trivial to obtain the conservation of total momentum

$$(\partial/\partial t)\int d^3x \int p_i n d\tau = 0, \tag{4.2}$$

assuming π_{ik} to be zero on the surfaces of the spatial volume integrated over. The term

$$\int \epsilon(\partial n/\partial x_i) d\tau$$

on the left-hand side of Eq. (4) is of the requisite form only if the $\partial/\partial x_i$ can be taken outside the integral, in other words, if $\epsilon(p)\delta n(p)$ is the differential of some quantity E. If we have

$$\delta E = \int \epsilon(p)\delta n(p) d\tau, \tag{5}$$

then

$$\int \epsilon(\partial n/\partial x_i) d\tau = (\partial/\partial x_i) E. \tag{5.1}$$

In a system described by the Hartree–Fock theory, it is clear that E is just the energy (per unit volume), and Landau then made the assumption that, quite generally,

$$\delta E/\delta n(p) = \epsilon(p), \tag{6}$$

i.e., that the quasiparticle energy $\epsilon(p)$ is obtained by varying the energy with respect to quasiparticle number. We see that Landau was led to this assumption by his construction of conservation laws for the quasiparticle momentum. (I am indebted to Professor I. M. Khalatnikov for explaining this to me.)

Now the energy E is a functional of occupation number $n(p)$ of all of the quasiparticles,

$$E=E\{n(p_1), n(p_2), \cdots\}, \tag{7}$$

and if one varies many of the $n(p)$ away from their equilibrium value $n_0(p)$ by, e.g., exciting a collective excitation, then the resulting E' is a functional

$$E'=E'\{\delta n(p_1), \delta n(p_2), \cdots\}. \tag{7.1}$$

Thus, for our present considerations, the $\epsilon(p)$ in Eq. (6) is, in general, a functional of the $\delta n(p)$'s. To describe collective phenomena, where many of the δn's enter, one needs to carry the variation of E to second order,

$$\delta E=E'-E$$
$$=\sum \epsilon^{(0)}(\mathbf{p})\delta n(\mathbf{p})+\tfrac{1}{2}\sum_{\mathbf{p}}\sum_{\mathbf{p}'} f(\mathbf{p},\mathbf{p}')\delta n(\mathbf{p})\delta n(\mathbf{p}'), \tag{7.2}$$

defining $f(\mathbf{p},\mathbf{p}')$. Here we have had to indicate vectors explicitly, since the dependence of f on angle between p and p' is crucial. Spin variables have been suppressed here, as elsewhere, for simplicity. It follows from Eq. (7.2) that

$$\epsilon(\mathbf{p})=\delta E/\delta n(\mathbf{p})=\epsilon^{(0)}(\mathbf{p})+\sum_{\mathbf{p}'} f(\mathbf{p},\mathbf{p}')\delta n(\mathbf{p}'). \tag{7.3}$$

The quantity $f(\mathbf{p},\mathbf{p}')$ turns out to be the interaction between quasiparticles, as might be deduced from Eq. (7.2), since it is the change in energy with removal of quasiparticles in states $\mathbf{p}$ and $\mathbf{p}'$. We shall see later that it is really a quasiparticle–quasihole interaction.

The "theory" part of the Landau theory consists in relating various physical phenomena to $f(\mathbf{p},\mathbf{p}')$, which, once determined, predicts other phenomena. To show this we repeat here Landau's famous argument, based on Galilean invariance, which gives the relation between the quasiparticle effective mass m^* and $f(\mathbf{p},\mathbf{p}')$.

Consider the Fermi sphere, shown in Fig. 1. Variations $\delta n(p)$ in the system are then made by displacing the Fermi sphere by a small momentum q (giving the system a uniform velocity) without changing the size or shape of the sphere. Then the energy per unit volume is increased by $\rho q^2/2m$, i.e.,

$$\delta E=\rho q^2/2m, \tag{8}$$

where ρ is the density. On the other hand, we can compute δE from Eq. (7.2). If a quasiparticle is assumed to have an effective mass expansion near the Fermi surface, then $\epsilon_p^{(0)}\cong p^2/2m^*$ for $p\simeq p_F$; in other words, the quasiparticles behave like particles with effective mass m^*, changed from m by the interactions with other particles. Changing the sum to an integral, we have

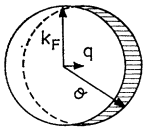

FIG. 1. Displacement of the Fermi sphere corresponding to a uniform velocity of the system as a whole.

$$\sum_{\mathbf{p}} \epsilon_p^{(0)}\delta n(\mathbf{p})\Rightarrow \frac{1}{(2\pi)^3}\int_{-1}^{1} d(\cos\theta)\,2\pi p_F^2 \times\int_0^{q\cos\theta}\frac{(p_F+\delta p)^2}{2m^*}d(\delta p)=\rho q^2/2m^*, \tag{8.1}$$

where we have used

$$\rho=p_F^3/6\pi^2, \tag{8.2}$$

appropriate for spinless fermions. The limit $q\cos\theta$ on δp is understandable if one looks at Fig. 1. The $\delta n(p)$ are equal to $+1$ for $0<\theta<\frac{1}{2}\pi$, and $0<\delta p<q\cos\theta$. Also, $\delta n(p)$ is -1 for $\frac{1}{2}\pi<\theta<\pi$.

Because of the quasiparticle–quasiparticle interaction $f(\mathbf{p},\mathbf{p}')$, there is a second term in δE [see Eq. (7.2)], and this is

$$\tfrac{1}{2}\sum_{\mathbf{p}}\sum_{\mathbf{p}'} f(\mathbf{p},\mathbf{p}')\delta n(\mathbf{p})\delta n(\mathbf{p}')$$
$$\Rightarrow\frac{1}{2}\frac{(2\pi p_F^2)^2}{(2\pi)^6}\int_{-1}^{1}d(\cos\theta)\int_{-1}^{1}d(\cos\theta')\times\int_0^{q\cos\theta}d(\delta p)\int_0^{q\cos\theta'}d(\delta p')f(\mathbf{p},\mathbf{p}'),\quad |\mathbf{p}|=|\mathbf{p}'|=p_F. \tag{8.3}$$

We have here neglected the dependence of $f(\mathbf{p},\mathbf{p}')$ on δp and $\delta p'$, which would lead to terms of order q^3 and higher. Thus, the magnitudes of $\mathbf{p}$ and $\mathbf{p}'$ are equal to p_F.

If we make the expansion

$$f(\mathbf{p},\mathbf{p}')=\sum_L f_L P_L(\chi), \tag{9}$$

where χ is the cosine of the angle between the two vectors $\mathbf{p}$ and $\mathbf{p}'$, we find, carrying out the elementary integrals in (8.3),

$$\rho q^2/2m=(\rho q^2/2m^*)+\tfrac{1}{2}(\rho p_F/6\pi^2)f_1 q^2. \tag{9.1}$$

Defining

$$F_1=(p_F m^*/2\pi^2)f_1, \tag{9.2}$$

we find

$$m^*=m(1+\tfrac{1}{3}F_1). \tag{10}$$

Phenomena described by the Landau theory require a knowledge of $f(\mathbf{p},\mathbf{p}')$ only for $|\mathbf{p}|=|\mathbf{p}'|=p_F$. Thus,

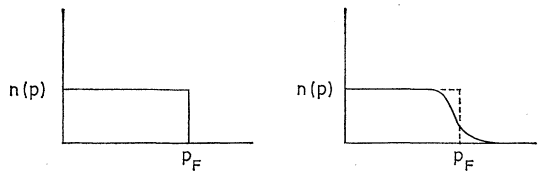

FIG. 2. Variation in $n(p)$ introduced by a small temperature.

just as m^*, they depend only on the numbers f_L or, equivalently, F_L.

Now f_0 can be determined from the compression modulus, which can be evaluated, given the velocity of ordinary sound. This is very reasonable, since sound propagation involves alternate local compressions and rarefactions, and these are spherically symmetrical distortions of the Fermi sea. The compression modulus also depends on the effective mass m^*, which must be determined before f_0 can be obtained.

The specific heat c_v depends upon the number of states per unit energy just at the Fermi surface, and can be used to determine m^*, so that F_1 can be obtained from Eq. (10). We repeat briefly this standard argument. By introducing an infinitesimal temperature $T=\delta T$, we vary the quasiparticle distribution from the one at zero temperature, as shown in Fig. 2. Then the δn introduced by this small temperature variation is proportional to

$$F(\epsilon)=(e^{(\epsilon-\mu)/kT}+1)^{-1}-1, \quad \epsilon<\mu$$
$$=(e^{(\epsilon-\mu)/kT}+1)^{-1}, \quad \epsilon>\mu \qquad (11)$$

expressed in terms of the Fermi distribution for quasiparticles; k is Boltzmann's constant. Here $\epsilon(p)$ can be replaced by $\epsilon^{(0)}(p)$, since the correction term gives only terms of higher order in $T=\delta T$. We then find

$$\delta E=\Omega c_v\delta T=\sum\epsilon(p)\delta n(p)$$
$$=[\Omega/(2\pi)^3]4\pi m^*p_F\int\epsilon d\epsilon F(\epsilon)$$
$$=\Omega[m^*p_F/2\pi^2](kT)^2\int xdxF(x), \qquad (11.1)$$

where Ω is the volume of the system, and

$$F(x)=(e^{x-x_0}+1)^{-1}-1, \quad x<x_0$$
$$=(e^{x-x_0}+1)^{-1}, \quad x>x_0. \qquad (11.2)$$

The value of the integral is

$$\int xdxF(x)=\pi^2/3, \qquad (11.3)$$

so that

$$c_v=m^*p_Fk^2T/6, \qquad (11.4)$$

and depends directly on m^*. Hence, knowing c_v, one can determine m^*.

The quantity $p_Fm^*/(2\pi^2)$ which appears in Eq. (9.2), and in many other places, is just the density of states per unit energy per unit volume at the Fermi surface, coming from

$$dN/d\epsilon\,|_{\epsilon_F}=(dN/dp)\,(dp/d\epsilon)\,|_{p_F}$$
$$=[\Omega/(2\pi)^3]4\pi p_F^2(m^*/p_F)$$
$$=\Omega(m^*p_F/2\pi^2)=\Omega N(0). \qquad (12)$$

With introduction of spin, one must multiply this density [and also the right-hand side of Eq. (11.4)] by 2. Isospin introduces another factor of 2 in nuclear matter.

In liquid ^{3}He, the value of F_0 determined from the velocity of sound is equal to 10.77, the F_1 determined from the specific heat, is 6.25, both at a pressure of $p=0.28$ atm. Knowing these, one can calculate the velocity of zero sound, if one assumes only F_0 and F_1 to be nonzero.

Zero sound is the phenomenon predicted by Landau to exist at low temperatures, where the collision term on the right-hand side of the Boltzmann equation [Eq. (2)] is negligible. (At low temperatures, few quasiparticles are present, and the Pauli principle inhibits scattering processes; this is just a consequence of the small phase space.)

Given

$$\frac{\partial n}{\partial t}+\frac{\partial n}{\partial\mathbf{x}}\frac{\partial\epsilon}{\partial\mathbf{p}}-\frac{\partial n}{\partial\mathbf{p}}\frac{\partial\epsilon}{\partial\mathbf{x}}=0, \qquad (13)$$

one then sees whether excitations of the form

$$n=n_0+\delta n; \quad \delta n=\delta n(\mathbf{p})\exp(i\mathbf{kx}-i\omega t), \qquad (13.1)$$

where δn is now a function of $\mathbf{p}$, $\mathbf{x}$, and t, will propagate. The quasiparticle energy ϵ varies with $\mathbf{x}$ only through its dependence on δn, since there is no driving force—such as, e.g., temperature difference—as is usual in transport processes. Here we are investigating whether there are self-sustaining oscillations in δn. Using

$$\partial n/\partial t=-i\omega\delta n(\mathbf{p})\exp(i\mathbf{kx}-i\omega t),$$
$$\partial n/\partial\mathbf{x}=i\mathbf{k}\delta n(\mathbf{p})\exp(i\mathbf{kx}-i\omega t),$$
$$\partial\epsilon/\partial\mathbf{p}=\mathbf{v},$$
$$\partial n/\partial\mathbf{p}\cong\partial n_0/\partial\mathbf{p}=(\partial n_0/\partial\epsilon)\,(\partial\epsilon/\partial\mathbf{p})=\mathbf{v}(\partial n_0/\partial\epsilon),$$
$$\partial\epsilon/\partial\mathbf{x}=\{i\mathbf{k}\int f(\mathbf{p},\mathbf{p}')\delta n(\mathbf{p}')2[d^3p'/(2\pi)^3]\}$$
$$\times\exp(i\mathbf{kx}-i\omega t), \qquad (13.2)$$

the linearized Boltzmann equation is

$$-i\omega\delta n(\mathbf{p})+i\mathbf{k}\cdot\mathbf{v}\delta n(\mathbf{p})=i\mathbf{k}\cdot\mathbf{v}(\partial n_0/\partial\epsilon)$$
$$\times\int f(\mathbf{p},\mathbf{p}')\delta n(\mathbf{p}')[2d^3p/(2\pi)^3]. \qquad (13.3)$$

To solve this, set $\delta n=\nu(\partial n_0/\partial\epsilon)$, where

$$\partial n_0/\partial\epsilon=(\partial n_0/\partial p)\,(m^*/p)=-\delta(p-p_F)\,(m^*/p_F) \qquad (13.4)$$

at zero temperature. Because of this δ function, we need know δn and $f(\mathbf{p},\mathbf{p}')$ only for $|\mathbf{p}|=|\mathbf{p}'|=p_F$, again. In

terms of ν,

$$(\omega-\mathbf{k}\cdot\mathbf{v})\nu=\mathbf{k}\cdot\mathbf{v}\int F(\mathbf{p},\mathbf{p}')\nu'(d\Omega'/4\pi),$$
$$|\mathbf{p}|=|\mathbf{p}'|=p_F \quad (13.5)$$

where $F(\mathbf{p},\mathbf{p}')=(m^*p_F/\pi^2)f(\mathbf{p},\mathbf{p}')$, the factor of 2 for spins now being included. Now, Eq. (13.5) is an integral equation for ν, involving, in principle, all multipoles of f in the expansion, Eq. (9). However, if one assumes that two terms in this expansion are sufficient, one predicts for a pressure of 0.28 atm

$$(c_0-c_1)/c_1=0.034,$$

where c_0 is the velocity of zero sound and c_1 is the velocity of ordinary sound, and it was found experimentally (Abel, Anderson, and Wheatley, 1960) that

$$(c_0-c_1)/c_1=0.035\pm0.003.$$

The predictive power of Landau theory exemplified here was a great success. Later, Pethick (1969) showed that the f_L determined as discussed above could be used to describe quantitatively transport properties.

3. THE BRUECKNER-BETHE THEORY

In the Brueckner theory it is realized that, because of the strong short-ranged interactions between nucleons or between ^{3}He molecules, ordinary perturbation theory cannot be used, and the pair interactions must be summed to all orders. One defines a G matrix (also called K matrix) by

$$G=V-V(Q/e)G, \quad (14)$$

where Q is the Pauli projection operator which excludes any states occupied by other particles, and e is the energy denominator. The diagonal matrix elements of G are given by

$$(\mathbf{k}_1\mathbf{k}_2\,|\,G\,|\,\mathbf{k}_1\mathbf{k}_2)=(\mathbf{k}_1\mathbf{k}_2\,|\,V\,|\,\mathbf{k}_1\mathbf{k}_2)$$
$$-\tfrac{1}{2}\sum_{k_3,k_4>k_F}\frac{(\mathbf{k}_1\mathbf{k}_2\,|\,V\,|\,\mathbf{k}_3\mathbf{k}_4)(\mathbf{k}_3\mathbf{k}_4\,|\,G\,|\,\mathbf{k}_1\mathbf{k}_2)}{(k_3^2/2m)+(k_4^2/2m)-\epsilon_1-\epsilon_2}, \quad (15)$$

where the Pauli principle has been put in explicitly. Exchange is assumed to be included, i.e., the matrix elements of V are direct minus exchange terms. Here ϵ_1 and ϵ_2 are the hole energies, inclusive of self-energy insertions, i.e.,

$$\epsilon(k)=(k^2/2m)+\sum_{k_2<k_F}(\mathbf{k}\mathbf{k}_2\,|\,G\,|\,\mathbf{k}\mathbf{k}_2). \quad (15.1)$$

It is usually convenient not to put self-energy insertions into particle lines, but rather to group them with other three-body clusters and to evaluate the entire sum by the use of the Fadeev equation (Bethe, 1965). Thus, there is a gap between particle and hole energies.

We now explain briefly what the parameter of convergence is in the Bethe–Brueckner theory. Let us consider a potential, as shown in Fig. 3, which has both a hard core and an external attractive region. The unperturbed wave function ϕ of relative motion can be expressed as $(kr)^{-1}\sin(kr)$ in the case of S waves. A correlated function ψ can be defined through

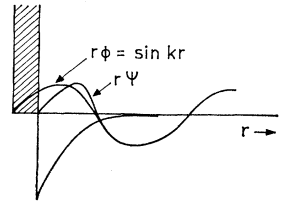

FIG. 3. Correlated and uncorrelated wave functions in nuclear matter. Here sin (kr) is (kr) times the S wave function $j_0(kr)$.

$$V\psi=G\phi. \quad (15.2)$$

It must, of course, be zero inside the hard core. We can then define a "defect" wave function χ by

$$\chi=\phi-\psi, \quad (15.3)$$

and $\int\chi^2d\tau$ is then just the "hole" in the wave function produced by the interaction. With inclusion of other partial waves in ϕ, ψ, and χ, the parameter κ mentioned in the Introduction is

$$\kappa=\rho\int\chi^2d\tau, \quad (15.4)$$

where ρ is the density. Thus, κ is just the ratio of the "hole" to the average volume occupied per particle. Of course, P and higher partial waves must be included in χ. The quantity κ will turn out to be what one calls the "renormalization at the quasiparticle pole" in the Landau theory.

In finite Fermi systems; e.g., nuclei, the G matrix is defined similarly to Eq. (15), but with k_1 and k_2 replaced by the relevant shell-model states. That is, for finite systems, we have

$$(\phi_1,\phi_2\,|\,G\,|\,\phi_1,\phi_2)=(\phi_1,\phi_2\,|\,V\,|\,\phi_1,\phi_2)$$
$$-\tfrac{1}{2}\sum_{k_3,k_4}{}'\frac{(\phi_1,\phi_2\,|\,V\,|\,\mathbf{k}_3,\mathbf{k}_4)(\mathbf{k}_3,\mathbf{k}_4\,|\,G\,|\,\phi_1,\phi_2)}{(k_3^2/2m)+(k_4^2/2m)-\epsilon_1-\epsilon_2}, \quad (16)$$

and off-diagonal matrix elements are also easily found by appropriate generalization. The sum

$$\sum_{k_3,k_4}{}'$$

is to be carried out over intermediate plane-wave states, but only after they are orthogonalized to the occupied shell-model states, so that the Pauli principle is properly taken into account. This is a difficult, but workable (Wong, 1967), procedure.

The use of plane-wave intermediate states is not meant as an approximation, but, rather, is a way of taking into account the fact that the self-energies of intermediate particles are off the energy shell. Consider the case which we shall use in the next section, where V is well behaved, and it is adequate to compute G to second order in V, i.e., we can replace the G on the right-hand side of Eqs. (15) or (16) by V. In this case, the

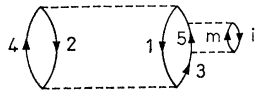

FIG. 4. Second-order self-energy insertion in a graph contributing to the second-order energy. Up-going solid lines depict particles; down-going solid lines depict holes. The dashed lines stand for the potential V.

second-order contribution to G is as shown in Fig. 4, where we have also included a second-order self-energy insertion in one of the particle lines "3." One can talk about particle 3 going to a two-particle, one-hole intermediate state, as shown. The energy denominator in this intermediate state is, however,

$$E_i - E_0 = \epsilon_4 + \epsilon_5 + \epsilon_m - \epsilon_1 - \epsilon_2 - \epsilon_i, \tag{16.1}$$

where the ϵ's are the various single-particle energies, since it must be reckoned from the ground state of the system, which is the state with no particles or holes. On the other hand, if only the single particle 3 were present to begin with, its self-energy insertion would look as shown in Fig. 5. The energy denominator here would be

$$E_i - E_0 = \epsilon_5 + \epsilon_m - \epsilon_3 - \epsilon_i, \tag{16.2}$$

particle 3 being present in the initial state. Thus, the energy denominator (16.1) has an additional

$$\epsilon_3 + \epsilon_4 - \epsilon_1 - \epsilon_2$$

as compared with (16.2), and this will affect the self-energy of the particle 3 in the process, Fig. 4. This is what we mean by saying that the self-energy is off the energy shell.

One can easily understand the nature of the effect by thinking about the uncertainty principle. In making the virtual excitation leading to particles in states 3 and 4, and holes in 1 and 2 at the bottom of Fig. 4, one is already an amount of energy $\Delta E = \epsilon_3 + \epsilon_4 - \epsilon_1 - \epsilon_2$ away from energy conservation. If one "borrows" an additional amount of energy $\epsilon_5 + \epsilon_m - \epsilon_3 - \epsilon_i$, to make the virtual excitation entering into the self-energy insertion in Fig. 4, then the time that this virtual excitation can last is

$$\tau^{-1}{}_{\text{off shell}} = \epsilon_5 + \epsilon_m - \epsilon_3 - \epsilon_i + \Delta E,$$

determined by the total amount of energy which is not conserved. Thus, the time of such virtual excitations is considerably smaller than the on-shell ones, shown in Fig. 5.

It is a complicated matter to calculate the off-shell energies of intermediate particles in all detail because of their dependence on the virtual excitations accompanying them, but estimates show (Brown, 1967) that, about as well as one can calculate the self-energies, they are zero, and thus, the replacement by plane waves, as shown in Eqs. (15) and (16) is justified. Such a replacement cannot be accurate just in the neighborhood of the Fermi surface, and we shall discuss later, with the introduction of a model space, how one can augment the accuracy of the treatment there.

Finally, we note that the hole energies ϵ_1 and ϵ_2 should be taken on the energy shell. The proof of this is given by Bethe, Brandow, and Petschek (1963), and repeated in Brown (1967). This asymmetrical treatment of particle and hole self-energies in the Brueckner theory arises from the fact that only particle–particle ladders are taken into account in calculating the G matrix. The rationale for this is that the phase space available for particles is much larger than that for holes, and the strong short-range interaction tends to make use of this large phase space.

4. A SIMPLE MODEL OF NUCLEAR MATTER

Our simple model to show the connection between the Landau and Brueckner–Bethe theories in the case of nuclear matter is to begin with the Brueckner expression for the energy

$$E = \sum_{k_1} (k_1^2/2m) n(k_1) + \tfrac{1}{2} \sum (\mathbf{k}_1\mathbf{k}_2 \mid G \mid \mathbf{k}_1\mathbf{k}_2) n(\mathbf{k}_1) n(\mathbf{k}_2) \tag{17}$$

in which we explicitly write in the dependence on particle occupation number, and then carry out the variations to obtain $\epsilon(p)$ and $f(\mathbf{p}, \mathbf{p}')$. Equations (15) and (15.1) are then rewritten

$$(\mathbf{k}_1\mathbf{k}_2 \mid G \mid \mathbf{k}_1\mathbf{k}_2) = (\mathbf{k}_1\mathbf{k}_2 \mid V \mid \mathbf{k}_1\mathbf{k}_2) - \sum \frac{(\mathbf{k}_1\mathbf{k}_2 \mid V \mid \mathbf{k}_3\mathbf{k}_4)\,(1-n(\mathbf{k}_3))(1-n(\mathbf{k}_4))(\mathbf{k}_3\mathbf{k}_4 \mid G \mid \mathbf{k}_1\mathbf{k}_2)}{(k_3^2/2m) + (k_4^2/2m) - \epsilon_1 - \epsilon_2}, \tag{15'}$$

$$\epsilon(k) = \delta E/\delta n(k) = (k^2/2m) + \sum_{\mathbf{k}_2} (\mathbf{k}\mathbf{k}_2 \mid G \mid \mathbf{k}\mathbf{k}_2) n(\mathbf{k}_2) + \tfrac{1}{2} \sum_{\mathbf{k}_1\mathbf{k}_2} (\mathbf{k}_1\mathbf{k}_2 \mid \delta G/\delta n(k) \mid \mathbf{k}_1\mathbf{k}_2) n(\mathbf{k}_1) n(\mathbf{k}_2). \tag{15.1'}$$

The $\epsilon(k)$ above in Eq. (15.1′) is distinguished from that in Eq. (15.1) and from those to be used in the denominator of Eq. (15′) by the last term involving $\delta G/\delta n(k)$. This is often called a rearrangement term, and enters into the Landau $\epsilon(k)$, so that the Landau $\epsilon(k_F)$ corresponds to the actual removal energy of the quasiparticle, whereas the Brueckner $\epsilon(k_F)$ in Eq. (15.1) does not.

Now the variations can be carried out to obtain $f(\mathbf{k}, \mathbf{k}')$. Whereas the $n(k)$ refer to be occupation numbers in the noninteracting, reference state (think of Rayleigh–Schrödinger perturbation theory), removal of a particle

in this state leads to the removal of a quasiparticle, since particles are assumed to transform smoothly into quasiparticles as the interaction is included.

Note from Eq. (15.1′) that the variation

$$\delta\epsilon(\mathbf{k})/\delta n(\mathbf{k}')$$

has to be carried out with respect to

(i) the $n(\mathbf{k}_2)$,
(ii) the $n(\mathbf{k}_3)$ and $n(\mathbf{k}_4)$ in Eq. (15′) for G,
(iii) the $n(\mathbf{k})$ appearing in ϵ_1 and ϵ_2 in the denominator of Eq. (15′).

Terms arising from (ii) and (iii), which are, respectively, corrections to the Pauli operator, and self-energy coming from removal of a particle in state $\mathbf{k}'$, are often called rearrangement terms.

Let us carry out these variations, replacing, for simplicity, G by V on the right-hand sides of Eqs. (15′) and (15.1′). This gives a theory which is essentially second-order Rayleigh–Schrödinger perturbation theory, but with the Hartree–Fock single-particle energies for the holes, i.e., we simplify Eq. (15.1′) to

$$\epsilon(k)=(k^2/2m)+\sum_{\mathbf{k}_2}(\mathbf{k}\mathbf{k}_2\mid G\mid\mathbf{k}\mathbf{k}_2)n(\mathbf{k}_2). \tag{15.1″}$$

We obtain the following contributions to $f(\mathbf{k},\mathbf{k}')$ from the variations listed as (i), (ii), (iii):

$$f_{(i)}(\mathbf{k},\mathbf{k}')=(\mathbf{k}\mathbf{k}'\mid V\mid\mathbf{k}\mathbf{k}')+\tfrac{1}{2}\sum_{\mathbf{k}_3,\mathbf{k}_4}\frac{(\mathbf{k}\mathbf{k}'\mid V\mid\mathbf{k}_3\mathbf{k}_4)(\mathbf{k}_3\mathbf{k}_4\mid V\mid\mathbf{k}\mathbf{k}')}{\epsilon_k+\epsilon_{k'}-(k_3^2/2m)-(k_4^2/2m)}(1-n_3)(1-n_4),$$

$$f_{(ii)}(\mathbf{k},\mathbf{k}')=-\sum_{\mathbf{k}_2,\mathbf{k}_4}\frac{(\mathbf{k}\mathbf{k}_2\mid V\mid\mathbf{k}'\mathbf{k}_4)(\mathbf{k}'\mathbf{k}_4\mid V\mid\mathbf{k}\mathbf{k}_2)}{\epsilon_k+\epsilon_2-(k'^2/2m)-(k_4^2/2m)}n_2(1-n_4)$$

$$-\sum_{\mathbf{k}_2,\mathbf{k}_4}\frac{(\mathbf{k}_2\mathbf{k}'\mid V\mid\mathbf{k}\mathbf{k}_4)(\mathbf{k}_4\mid V\mid\mathbf{k}_2\mathbf{k}')}{\epsilon_2+\epsilon_{k'}-(k^2/2m)-(k_4^2/2m)}n_2(1-n_4)+\tfrac{1}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\frac{(\mathbf{k}_1\mathbf{k}_2\mid V\mid\mathbf{k}\mathbf{k}')(\mathbf{k}\mathbf{k}'\mid V\mid\mathbf{k}_1\mathbf{k}_2)}{\epsilon_1+\epsilon_2-(k^2/2m)-(k'^2/2m)}n_1n_2,$$

$$f_{(iii)}(\mathbf{k},\mathbf{k}')=-(\mathbf{k}\mathbf{k}'\mid V\mid\mathbf{k}\mathbf{k}')\sum_{\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}\frac{(\mathbf{k}\mathbf{k}_2\mid V\mid\mathbf{k}_3\mathbf{k}_4)(\mathbf{k}_3\mathbf{k}_4\mid V\mid\mathbf{k}\mathbf{k}_2)}{[\epsilon_k+\epsilon_2-(k_3^2/2m)-(k_4^2/2m)]^2}n_2(1-n_3)(1-n_4)$$

$$-\sum_{\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}\frac{(\mathbf{k}_2\mathbf{k}'\mid V\mid\mathbf{k}_2\mathbf{k}')(\mathbf{k}\mathbf{k}_2\mid V\mid\mathbf{k}_3\mathbf{k}_4)(\mathbf{k}_3\mathbf{k}_4\mid V\mid\mathbf{k}\mathbf{k}_2)}{[\epsilon_k+\epsilon_2-(k_3^2/2m)-(k_4^2/2m)]^2}n_2(1-n_3)(1-n_4). \tag{18}$$

We draw these terms graphically as shown in Fig. 6. Terms with two internal hole lines, except for Fig. 6(ii)c, have been consistently dropped since they are of higher order in κ. The graphs, Figs. 6(i), would both be included in the G matrix $(\mathbf{k}\mathbf{k}'\mid G\mid\mathbf{k}\mathbf{k}')$, but otherwise one can go directly to the G-matrix case with the full G in the right-hand side of Eq. (15), simply by changing all matrix elements of V into those of G. One also has further graphs in the G-matrix case, but these are of higher order in κ than those here, and we shall not consider them here, although they may be numerically important in the case of liquid ^{3}He.

Since $f(\mathbf{k},\mathbf{k}')$ enters into the description of phenomena like zero sound, just the way that the particle–hole interaction enters into the random-phase approximation, it is clear that $f(\mathbf{k},\mathbf{k}')$ is just the complete particle–hole interaction in the long-wavelength limit. We have drawn it that way in Fig. 6, both the particle and hole having momentum $\mathbf{k}$ (or $\mathbf{k}'$), so that the total momentum is zero. The magnitude of $\mathbf{k}$ (or $\mathbf{k}'$) must be k_F, since only on the Fermi surface can a particle and hole have equal momentum. One can think of f as the limit of a particle–hole interaction between excitations of momentum $\mathbf{q}$, as $\mid\mathbf{q}\mid\to 0$.

The fact that $f(\mathbf{k},\mathbf{k}')$ is the particle–hole interaction in the long-wavelength limit can be made clearer by the following argument in which we think of a problem involving a weak interaction V which need be handled only to first order. Consider first the particle–hole interaction connecting particle–hole states of total momentum $\mathbf{q}$ as shown in Fig. 7. One can think of this interaction as

$$\Gamma=\sum_{\mathbf{k}'\mathbf{k}}(\mathbf{k}'+\mathbf{q},\mathbf{k}\mid V\mid\mathbf{k}',\mathbf{k}+\mathbf{q})\times(a_{\mathbf{k}'}{}^{\dagger}a_{\mathbf{k}'+\mathbf{q}})(a_{\mathbf{k}}{}^{\dagger}a_{\mathbf{k}+\mathbf{q}}), \tag{18.1}$$

that is, the matrix element of Γ between particle–hole

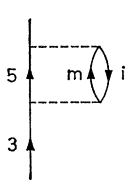

FIG. 5. Second-order self-energy insertion for a single particle.

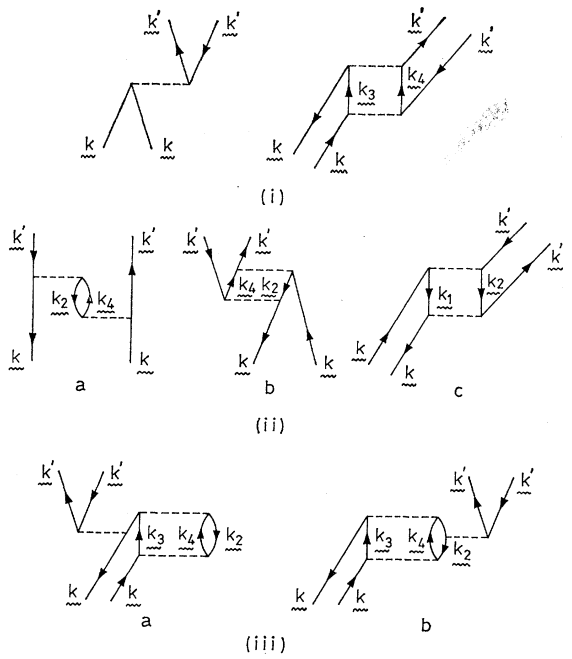

FIG. 6. Graphical representation of the terms entering into $f(\mathbf{k}, \mathbf{k}')$, Eq. (18). Exchange terms have not been explicitly drawn.

states $a_{\mathbf{k}_1+\mathbf{q}}{}^\dagger a_{\mathbf{k}_1}|0)$ and $a_{\mathbf{k}_2+\mathbf{q}}{}^\dagger a_{\mathbf{k}_2}|0)$, where $|0)$ is the vacuum, will give the interaction

$$(\mathbf{k}_2+\mathbf{q}, \mathbf{k}_1 \mid V \mid \mathbf{k}_2, \mathbf{k}_1+\mathbf{q})$$

shown in Fig. 7.

Such matrix elements enter into the description of collective excitations, the collective excitation being a linear combination of particle–hole states with coherent phases. For example, plasma oscillations are described in the random-phase approximation in such a way, where the interparticle Coulomb interaction is V. The momentum of the excitation is $\mathbf{q}$, so that in the long-wavelength limit, $\mathbf{q}\to 0$. In this limit, the operator pair $(a_{\mathbf{k}}{}^\dagger a_{\mathbf{k}+\mathbf{q}})$ in Eq. (18.1) becomes $(a_{\mathbf{k}}{}^\dagger a_{\mathbf{k}}) = n_{\mathbf{k}}$; similarly, $(a_{\mathbf{k}'}{}^\dagger a_{\mathbf{k}'+\mathbf{q}})$ becomes $n_{\mathbf{k}'}$. Consequently, the expectation value of Γ becomes

$$(0 \mid \Gamma \mid 0) = \sum_{\mathbf{k}_1\mathbf{k}_2} (\mathbf{k}_2, \mathbf{k}_1 \mid V \mid \mathbf{k}_2, \mathbf{k}_1) n(\mathbf{k}_2) n(\mathbf{k}_1), \quad (18.2)$$

i.e., it has the same form as the potential energy term in Eq. (17). Thus, removing the n's by functional differentiation gives the matrix element $(\mathbf{k}_2, \mathbf{k}_1 \mid V \mid \mathbf{k}_2, \mathbf{k}_1)$.

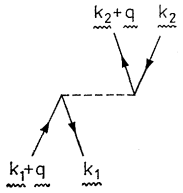

FIG. 7. Particle–hole interaction connecting two states of total momentum q.

To this order, we could have carried out the same considerations for the particle–particle interaction, with the same conclusions. However, the second-order particle–particle interaction will have terms like those shown in Fig. 8. Whereas the matrix element corresponding to the process Fig. 8(b) is just that of Fig. 6(ii)a, with the left-hand line pointing up instead of down, the matrix element corresponding to the process Fig. 8(a) is missing in Eq. (18). The reason is simple. If we redraw Fig. 8(a) as a particle–hole interaction as is done in Fig. 9, it is reducible; that is, it can be obtained by putting together two first-order interactions of the type shown in Fig. 7. But when the interaction, Fig. 7, is used as the kernel in an integral equation such as occurs in the random-phase approximation, or in Eq. (13.5), it is automatically iterated to all orders. Thus, reducible diagrams should not (and do not) occur in f.

The nucleon–nucleon potential which is used to fit scattering data is complicated, especially because it includes a large tensor force. It has been found (Bethe, 1966; Green, 1967), however, that the main effects of the latter—at least as far as the binding

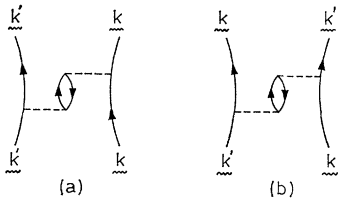

FIG. 8. Typical terms in the second-order particle–particle interactions.

energy is concerned—can be taken into account by giving the potential a density dependence.

We describe now calculations of Bäckman (1968, 1969), who used the reaction matrix

$$G=0.91[\tfrac{1}{4}(3+\sigma_1\cdot\sigma_2)](1-1.16\rho^{2/3})G_t + 0.90[\tfrac{1}{4}(1-\sigma_1\cdot\sigma_2)](1+0.29\rho^{2/3})G_s, \quad (19)$$

where G_s and G_t are the reaction matrices calculated, respectively, from the singlet and triplet parts of the Kallio–Kolltveit (1964) potential

$$V_i(r) = \infty \qquad \text{for } r<c$$
$$= -A_i \exp[-\alpha_i(r-c)] \qquad \text{for } r>c, \quad (19.1)$$

with parameters

$$c=0.4, \quad A_t=475 \text{ MeV}, \quad \alpha_t=2.52 \text{ fm}^{-1},$$
$$A_s=331 \text{ MeV}; \quad \alpha_s=2.40 \text{ fm}^{-1}. \quad (19.2)$$

The parameters of the Kallio–Kolltveit potential were chosen so as to fit low-energy nucleon–nucleon scattering, as well as average properties of the S wave phase shifts at higher energies. Although it was designed to be used only in relative S states, Bäckman uses the

singlet potential in both relative S and D states, and the triplet potential in only relative S states since the density-dependent coefficient is supposed to mock up the effect of tensor coupling to the D channel.

In nuclear matter, the expression for $f(\mathbf{k}, \mathbf{k}')$ must be generalized to a matrix in spin and isospin space because of the two spins and two kinds of particles. In general dimensionless form, it can be written[1]

$$\mathfrak{F}(\mathbf{k}, \mathbf{k}') = (\pi^2/m^* k_F) \times \{F + F' \tau_1 \cdot \tau_2 + (G + G' \tau_1 \cdot \tau_2) \sigma_1 \cdot \sigma_2\}, \quad (20)$$

where F, F', G, G' are all functions of the angle between $\mathbf{k}$ and $\mathbf{k}'$, and $m^* k_F/\pi^2$ is half the density of states on the Fermi surface.[2] Each is expanded in a Legendre series

$$F = \sum F_L P_L(\cos\theta), \quad \text{etc.} \qquad (20.1)$$

Bäckman calculates these various quantities by solving Eq. (15) for the G matrix, then using this G in place of V in Eqs. (18). Note, however, that $f_{(i)}(\mathbf{k}, \mathbf{k}')$ is replaced by $(\mathbf{kk}' \mid G \mid \mathbf{kk}')$, both terms in $f_{(i)}$ (and all of those higher order in the ladder sum) being included in G, Eq. (15).

Bäckman finds[3]

$$F_0 = -0.30 \qquad F_0' = 0.14$$
$$F_1' = 0.15$$
$$F_2' = 0.03$$
$$F_1 = -0.40 \qquad G_0 = 0.27$$
$$G_1 = 0.12$$
$$G_2 = 0.07$$
$$F_2 = -0.15 \qquad G_0' = 0.20$$
$$G_1' = 0.13$$
$$G_2' = 0.05.$$

Coefficients of the higher Legendre polynomials have not yet been calculated for F', G, and G'.

We shall discuss Migdal's theory later, but note here

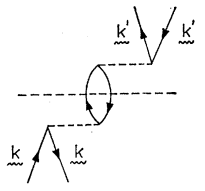

FIG. 9. Figure 8a redrawn as a particle–hole interaction.

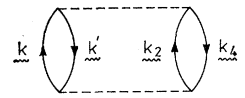

FIG. 10. Contribution to the energy which, when the lines labeled k and k' are broken, gives 6a in Fig. 6(ii).

that these should be compared with his values (Migdal, 1967) of

$$F_0 = 1 \pm 0.2 \qquad F_0' = 0.3 \pm 0.1$$
$$|F_1| < 0.1\text{–}0.2 \qquad |F_1'| < 0.1$$
$$G_0 \cong 0.5$$
$$G_0' \cong 0.5.$$

Since Migdal discusses finite Fermi systems, i.e., nuclei, his effective interaction varies in strength from the center of the nucleus to the surface, more or less as the nuclear density varies [see our later Eq. (23)]. We assume here that the values at the center of the nucleus are the ones that should be equated to the values for nuclear matter at the same density as the central density of the nucleus.

Before discussing the comparison with the results of Migdal, we make some remarks on Bäckman's calculation. The main contributions in his theory come from the processes shown in Fig. 6(i) and 6(iii). (Remember that we replace the potential V there by the G matrix.)

The contribution to $f(k, k')$ from Figs. 6(i) is just

$$f_{(i)}(\mathbf{k}, \mathbf{k}') = (\mathbf{kk}' \mid G \mid \mathbf{kk}'), \qquad (20.2)$$

i.e., just the bare G matrix. Inclusion of the process (a) in Fig. 6(iii) simply modifies this by multiplying by $(1-2\kappa)$, i.e., it expresses the renormalization at the quasiparticle pole in this theory to the order that this is treated. The process (b) in Fig. 6(iii) similarly has the effect of cutting f down. The value of κ for the potential Eq. (19.1) is found to be

$$\kappa = 0.07.$$

For potentials with tensor forces it is considerably larger, ~ 0.15.

The processes, Fig. 6(ii) contribute very little in Bäckman's calculations, but this is probably incorrect. Let us consider process (a) under Fig. 6(ii). Define the momentum transfer $\mathbf{q}$ by $\mathbf{q} = \mathbf{k}' - \mathbf{k}$. Now the energy denominator in this second-order process is

$$\Delta E = (k_4^2/2m) + (k_F^2/2m) - \epsilon_k - \epsilon_{k_2}, \qquad \mathbf{k}_4 = \mathbf{k}_2 + \mathbf{q}, \qquad (20.3)$$

and this does not go to zero as $\mathbf{q} \to 0$ ($\mathbf{k} \to \mathbf{k}'$ or $\chi \to 0$), because of the gap between particle and hole spectra. However, the phase space in the summation does go to zero, so that this graph does not contribute for $\chi = 0$. Had one been more careful, however, and not used an average gap—which is good enough for energy calculations but not for the effective interaction—one would have found that just for processes like those shown in Fig. 10, when $|\mathbf{k}| = |\mathbf{k}'| = k_F$, there is no gap

[1] Migdal uses the symbols f, g for the dimensionless quantities we call F, G. Our notation is more usual in other applications of the Landau theory.

[2] Note that the neutron–neutron and neutron–proton interactions are related to those reported here by $F_0 = \frac{1}{2}(F_0^{nn} + F_0^{np})$, $F_0' = \frac{1}{2}(F_0^{nn} - F_0^{np})$.

[3] Bäckman uses the density of states $2m^* k_F/\pi^2$ natural for nuclear matter in defining his F, G, etc. Here we use $m^* k_F/\pi^2$ as in Eq. (20), so as to be able to effect a direct comparison with Migdal's results later.

TABLE I. Values of the various quantities using $k_F = 1.39$ fm^{-1}.

Quantity	Bäckman	Migdal
K	128 MeV	700 MeV
β	22.5 MeV	20 MeV
m^*	0.74	~ 1.0

between particle and hole spectra, and the energy denominator goes to zero at the same rate as the phase space.

To improve the treatment, one should go over to a "model space" in which the energies of particles and holes are continuous, as we discuss in Sec. 6.

We have given no justification here of the validity of the procedure of starting from the particular approximate expression of Brueckner for the energy, and then carrying out the functional differentiations. This, of course, gives us only a few graphs of the totality envisaged in the complete theory. In particular, our procedure can be dangerous in cases where the effective interaction is especially sensitive to excitations just in the neighborhood of the Fermi surface, which might not be very important for the total energy of the system, and are neglected in the Brueckner expression for the energy. We shall see such an example in the case of the paramagnons in liquid ^{3}He.

5. DESCRIPTION OF PHYSICAL PHENOMENA IN THE MIGDAL THEORY

If one had systems of infinite nuclear matter, then the various constants F_0, F_1, F_0', etc., would be directly measurable. The derivation of the following formula is completely standard [see the book by Migdal (1967)] and we quote only the results:

(i) Compression modulus[4]:

$$K = r_0^2(d^2E/dr_0^2) = 6(k_F^2/2m^*)(1+2F_0), \qquad (21)$$

where E is the energy per particle.

(ii) Symmetry energy: If one has an excess of neutrons, a term appears in the energy per particle

$$\beta \ [(N-Z)/A]^2,$$

and β is given by

$$\beta = \tfrac{1}{3}(k_F^2/2m^*)(1+2F_0'). \qquad (22)$$

(iii) Effective mass:

$$m^* = m(1+\tfrac{2}{3}F_1) \qquad (22.1)$$

as derived in Eq. (10), but with the additional factor 2 for the two kinds of particles. Thus, as can be seen, the compression modulus, symmetry energy, and susceptibility all depend upon F_1, through the effective mass.

The situation with the susceptibility is rather complicated, because of the many corrections (exchange currents, etc.), which must be made in order to obtain magnetic moments in nuclei.

We show in Table I the values that Migdal and Bäckman would obtain for the three quantities K, β, and m^*.

Migdal's value for K is clearly much too large. The compression modulus is rather directly related to the surface energy, and Bethe (1968) obtains about the right value for this with the K of 157 MeV calculated by Dahlblom with the full nucleon–nucleon potential. Later work by Krainov (1968) in fitting the properties of the three state in ^{208}Pb gives $F_0 \cong 0$, and this would give a much more reasonable value for the compressibility. The original value of $F_0 \cong 1$ came chiefly from the fit to the isotope shift (Bunatyan and Mikulinskii, 1965). It is clear that a strongly repulsive interaction in the interior of the nucleus helps in obtaining isotope shifts of the right magnitude (Barrett, 1966; Lande, Molinari, and Brown, 1968).

The symmetry energy obtained by both authors is reasonable, in the range of the experimental value of 20–25 MeV.

As far as the effective mass is concerned there is some indication that in heavy nuclei, $m^* \cong 1$ (Brown, Gunn, and Gould, 1963). The fact that this value is larger than that normally calculated in Brueckner theory has been explained (Bertsch and Kuo, 1968) as coming from a rearrangement effect. Bäckman effectively neglects this by his essentially dropping the contribution from 6a in Fig. 6(ii), as discussed earlier. A better treatment of this term will increase both his m^* and F_0.

In the application of Migdal's theory to finite nuclei, one needs further assumptions about the variation of the effective interaction with density. The assumption made (Migdal, 1967) is that F_0 is not constant, but varies as a function of r as

$$F_0(r) = F_0 + [(F_0)_{\text{ext}} - F_0]/\{1+\exp[-\alpha'(r-R')]\}, \qquad (23)$$

where r is the center of mass of the two colliding particles. The quantities α' and R' differ little from the corresponding values α and R for the density variation. The F_0', G_0, G_0' seem to need no variation with density.

In fact, no empirical data determines the quasiparticle interaction in the center of the nucleus; all phenomena depending on either some interaction averaged over center and surface or, in the case of the isotope shift, the difference of interaction in the surface and in the center. Thus, F_0, the value of $F_0(r)$ in the interior, depends upon assumptions about the behavior of $F_0(r)$ as a function of r. The Fermi liquid theory, however, gives no guide to the behavior of $F_0(r)$ in the

[4] This K is the one defined by Brueckner; it is a factor of 9 larger than in Migdal's formula [which corresponds to defining K as $\rho_0^2(d^2E/d\rho_0^2)$].

region of varying density. Thus, the value of F_0 that Migdal finds depends upon his assumption, Eq. (23).

It should be emphasized that the r in Eq. (23) is that of the center of mass of the two particles. The truncation of the expansion (20.1) of F, etc., is equivalent to using a δ function for the dependence on interparticle distance; i.e., an effective interaction

$$V_{\text{eff}}=(\pi^2/m^*k_F)\{F_0(r_{\text{c.m.}})+F_0'\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + (G_0+G_0'\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\}\delta(\mathbf{r}_1-\mathbf{r}_2), \quad (23.1)$$

where we have put the lower suffix "c.m." on r in the argument of F_0 to emphasize that this is the center-of-mass coordinate.

Our main concern here is with the relation of different theories for infinite systems, so we shall not go into this further.

6. INTRODUCTION OF A MODEL SPACE

The concept of model space has been formulated in detail by Brandow (1967), following the early idea of Brueckner, Eden, and Francis (1955), and the work of Bloch and Horowitz (1958). We follow Brandow's work here.

The complete many-body wave function Ψ, satisfying

$$H\Psi=E\Psi, \quad (24)$$

can be expanded in states Φ_i, which satisfy

$$H_0\Phi_i=E_i\Phi_i. \quad (25)$$

From this, one immediately obtains

$$(E-E_i)a_i=\langle\Phi_i, V\Psi\rangle, \quad (26)$$

where

$$V=H-H_0. \quad (27)$$

A certain number of the Φ_i's, spanning a model space, which Brandow calls D, are selected. With this choice of D, one has a model wave function

$$\Psi_D=\sum_{i\epsilon D}a_i\Phi_i=P\Psi, \quad (28)$$

and a Green's function

$$g=\sum_{i\notin D}[|\Phi_i\rangle\langle\Phi_i|/(E-E_i)]=Q/(E-H_0), \quad (29)$$

where Q is defined so that (24) and (26) can be combined in the form

$$\Psi=\Psi_D+gV\Psi. \quad (30)$$

It is convenient to define a wave operator Ω by

$$\Psi=\Omega\Psi_D, \quad (31)$$

and a reaction matrix or "effective interaction"

$$G=V\Omega. \quad (32)$$

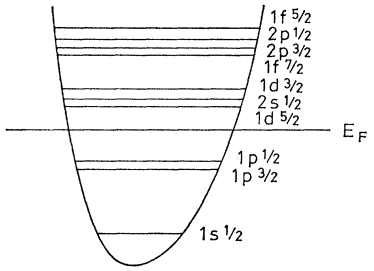

FIG. 11. Single-particle levels in ^{16}O.

Substitution of these in (30) gives

$$\Omega=1+gV\Omega \quad (33)$$

and

$$G=V+VgG. \quad (34)$$

In practice, the use of a model space is extremely useful in gaining an increased accuracy, for reasons which we shall discuss. But first, let us give an example of a useful type of model space. Consider calculations for nuclei in the region of ^{16}O, with single-particle levels as shown in Fig. 11, and Fermi energy as drawn. Here we have drawn the levels as if all were found, whereas the $2p$, $1f$ levels certainly are not; i.e., we pretend, however, that they are.

We consider our model space to be the space of levels shown in Fig. 11. More precisely, if we take the Φ_i to be of the form of antisymmetrized products of single-particle functions, then Φ_i lies within the model space if the states of all particles in Φ_i lie within the space of levels shown in Fig. 11. Thus, the model space for the many-body functions Φ_i can be discussed in terms of the model space for the single-particle functions.

One could think of these single-particle functions as the single-particle Hartree–Fock eigenfunctions, but in practice in nuclear physics, harmonic oscillator functions are usually used.

An evaluation of the matrix elements of G has been carried out in the above scheme (Brown and Wong, 1967). Although this is tedious, most accurate results for the binding energy and other quantities can be obtained by using such matrix elements as effective interactions within the model space, and then solving the problem within the model space as best possible.

We note that (Krainov and Malov, 1968) one must often effectively employ a model space within the Migdal theory if one wishes to begin from a δ-function interaction of the type (23.1), and obtain the type of effective forces observed in nuclei. For example, the quadrupole–quadrupole interaction (P_2 force) observed in nuclei comes mainly from the core-polarization process, Fig. 12. The same situation was found within the framework of the G-matrix theory (Brown and Kuo, 1967), except that there the wavy lines in Fig. 12 represent the G matrix.

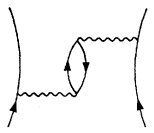

FIG. 12. Core-polarization process which gives rise to most of the quadrupole–quadrupole interactions in nuclei. In the Migdal theory (Krainov and Malov, 1968), the wavy line represents the interaction, Eq. (23.1). All particles and the hole are assumed to lie within the model space.

In the case of infinite systems, the model space should be taken to be somewhat larger than the Fermi sphere, as shown in Fig. 13.

It should be clear by now, but we emphasize the fact, that the particular G one calculates depends upon the choice of model space. This dependence comes from the dependence of the Pauli operator Q [Eq. (29)] on the model space.

There is one unaesthetic feature of working with a model space. In a two-body collision, one of the particles may go to a state outside of the set of single-particle states defined to be within the model space, the other particle staying within. According to our above definition, we have to define the many-body intermediate state Φ_i as being outside of the model space in such a case.

Now, if one wants to go back and make a connection with the Migdal theory by functionally differentiating the energy with respect to quasiparticle occupation number, one must write down an expression for the energy in terms of the occupation numbers within this model space. Of course, in principle, one would like to diagonalize the effective interaction G within this model space, but aside from certain formal troubles, in practice the amount of work is prohibitive.

The approximation employed by Brown and Wong (1967) was to take the bubble sum, Fig. 14, for the energy. Functional differentiation of the process, Fig. 14b, would give graphs which look like those of Fig. 6(ii), but with the difference that the particle and hole to be summed over [e.g., those corresponding to k_2 and k_4 in Fig. 6(ii)b] must now lie within the model space. Thus, one obtains certain terms essentially left out by Bäckman. (See the discussion at the end of Sec. 3.)

It is clear that the scheme advocated by Brandow, and described briefly here, can lead to a more accurate evaluation of the various quantities than the scheme employed by Bäckman. Using the model space, one can first calculate the G matrix in a way which involves only rather high-lying intermediate states, which should be adequately represented by plane waves. Then one uses G as an effective interaction to solve the problem to whatever accuracy is needed, within the model space.

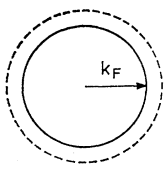

FIG. 13. Model space in the case of an infinite system.

We return now to a discussion of single-particle energies to be used for the particles within the model space.[5] As explained in Sec. 3, single-particle energies in the Brueckner–Bethe theory are, in general, to be calculated off the energy shell. Suppose we consider a virtual excitation of the type shown in Fig. 4, where the two particles 3 and 4, and the two holes 1 and 2 all lie within the model space. The amount particle 3 is off the energy shell is $\Delta E=\epsilon_3+\epsilon_4-\epsilon_1-\epsilon_2$, as explained following Eq. (16.2). Now, one can show[6] that the correction to the single-particle energy is then $\kappa\Delta E$, where κ is the quantity defined in Eq. (15.4). In the case of excitations in an infinite system from just below to just above the Fermi sea, as shown in Fig. 15, we find

$$\Delta E=\frac{(\mathbf{p}'-\mathbf{q})^2}{2m^*}+\frac{(\mathbf{p}+\mathbf{q})^2}{2m^*}-\frac{\mathbf{p}'^2}{2m^*}-\frac{p^2}{2m^*}$$
$$\cong(qp_F/m^*)(\cos\hat{p}\cdot\hat{q}-\cos\hat{p}'\cdot\hat{q}).$$

Thus, the off-energy-shell correction to the single-particle energy goes to zero with $\mathbf{q}$. This is highly relevant to the discussion of the process (a) of Fig. 6(ii) carried out at the end of Sec. 4.

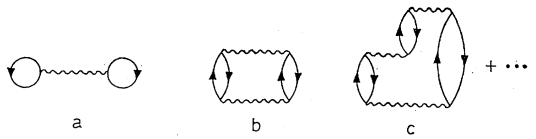

FIG. 14. "Bubble" sum for the binding energy.

In general, it is probably sufficiently accurate to take the single-particle energies in the model space on the energy shell, but if this is not so, the above provides a framework for putting in the off-energy-shell effect.

The convergence of the perturbation expansion for the effective interaction in mass-18 nuclei has been investigated by Barrett and Kirson (1968, 1968a, 1970). They use the Kuo–Brown (1966) G matrix in a more or less straightforward G-matrix expansion of the effective interaction, some terms of which are shown in Fig. 16. Like Bäckman, they are not working in a model space, and consequently, they do not include terms such as that shown in Fig. 17, since, in the calculation of the G matrix, the ladder sum has already been carried out, and this term is supposed to be included in the process,

[5] We follow here an unpublished treatment by H. A. Bethe.

[6] The relevant development in Bethe, Brandow, and Petschek (1963) is briefly sketched here. If one has two G matrices $G_A=V-V(Q/e_A)G_A$ and $G_B=V-V(Q/e_B)G_B$ differing only in energy denominator, then it is shown that $G_A-G_B=G_B(Q/e_B-Q/e_A)G_A$. One can take G_A to be the on-shell G matrix, and G_B to be the off-shell G matrix, so that $e_B-e_A=\Delta E=\epsilon_3+\epsilon_4-\epsilon_1-\epsilon_2$ for the case in hand. Thus, the diagonal matrix elements of G_B-G_A in states 3 and i, summed over i, will give the desired correction for being off-energy shell to the single-particle energy, which is found to be $\kappa\Delta E$.

Figs. 16(a) and 16(b), respectively. However, it is clear that if one uses the model-space approach, such terms—where the particles and holes in intermediate states lie within the model space—should be included. In the Kuo–Brown G-matrix, low-lying intermediate states are handled inaccurately by the approximation of plane-wave intermediate states, so that only a small part of the processes, Fig. 17, are included in the Barrett–Kirson treatment.

In summary, in both the infinite and finite cases, the accuracy of the treatment could be increased by going over to the use of the model-space approach.

7. THEORIES OF LIQUID ^{3}He

The Landau theory has been applied widely to liquid ^{3}He, in the way outlined in Sec. 2. Results of the Brueckner–Bethe theory are not so well known, and we review them here.

First of all, it is a much less ambitious task to calculate the energy of the system than to calculate the Landau parameters giving the effective interaction. The latter are expressed as the second functional derivative of the energy, and it is well known that differentiation of a series worsens the convergence. In addition, long-wavelength excitations of particle–hole type—and we shall discuss the case of paramagnons later—may play quite a role in the effective interaction on the Fermi surface, but may not contribute much to the total energy of the system if important only over a small range of momenta.

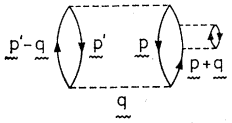

FIG. 15. Excitation from just below to just above the Fermi sea, with small momentum transfer **q**.

In a series of papers, Østgaard (1968, 1968a, 1968b, 1969) has worked out various bulk properties of liquid ^{3}He within the Brueckner–Bethe formalism, beginning from various forces of the van der Waals type between two free ^{3}He atoms. He does not find a particularly good convergence, even for the energy of the system. In fact, with the Frost–Musulin potential, he finds the contribution of the two-body clusters to be -3.05°K per particle, whereas that from the three-body clusters is -1.95°K per particle. He obtains a binding energy of -2.00°K per particle at a k_F of 0.78 Å, to be compared with the experimental binding energy of -2.5°K per particle at a k_F of 0.79 Å. Østgaard finds a compressibility of 4.3% per atmosphere, compared with the experimental value of 3.8% per atmosphere.

Bertsch (1969) has used Østgaard's G-matrix elements to calculate the effective interaction on the Fermi surface, evaluating processes such as those shown in Figs. 16(a) and 16(b), with the wavy lines representing G-matrix elements. Bertsch finds large differences

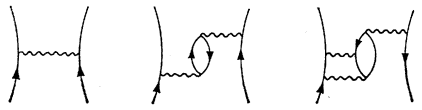

FIG. 16. Examples of a first-, second-, and third-order term, shown by a, b, and c, respectively, in the Barrett–Kirson calculation. The wavy line represents the G matrix.

in the contribution from the process Fig. 16(b) depending on whether or not he uses a gap in the single-particle spectrum (no gap corresponding roughly to the model-space calculation). In any case, the second-order processes are tremendous, of sufficient magnitude to change the signs of the F's from the lowest-order calculation. It is clear, therefore, that one must go beyond this theory.

It is tempting to try to make a theory of quasiparticles and collective excitations. A zero-sound excitation can, for example, be viewed roughly as a phonon. (It is actually made up out of fermion degrees of freedom.) Pictorially, the exchange of one zero-sound excitation will give the contribution to the effective particle–hole interaction shown in Fig. 18. Now, in the long-wavelength limit, zero sound can be viewed as a fluctuation in quasiparticle occupation $\delta n(\mathbf{p})$. If we expand

$$\delta n(\mathbf{p}) = \delta n_0(p) + \delta n_1(p) P_1(\hat{p}\cdot\hat{k}) + \cdots,$$

where $\hat{k}$ is the direction of the (vanishing) zero-sound wave vector, then $\delta n = \sum_p \delta n_0(p)$ is just a density fluctuation, and the coupling of zero sound to the quasiparticle or quasihole is just $f_0\delta n$ at each vertex. The $\delta n_1(p) P_1(\hat{p}\cdot\hat{k})$ represents a velocity field which could, in principle, couple to the quasiparticle velocity, but does not, in this case, because the directions of velocity field and quasiparticle momentum are perpendicular, so that the dot product is zero.

One could hope, then, the total effective interaction to be composed of a short-range part, which we would propose to represent by the G-matrix interaction, and a long-range part consisting of the interaction induced by phonon exchange. The coupling at the vertices between phonon and quasiparticle is given essentially by F_0, which is equal to 10.77 at a pressure of $P=0.28$ atm, so that the phonon-induced reaction is certainly not negligible.

However, it is clear that at least one more ingredient must be added, the persistent spin fluctuations, called "paramagnons." These have been found to give very important contributions to the effective mass of the ^{3}He

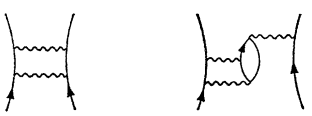

FIG. 17. Terms which do not enter into the Barrett–Kirson evaluation.

FIG. 18. Contribution of a virtual zero-sound excitation (shown by the wiggly line) to the effective particle–hole interaction.

atoms (Berk and Schrieffer, 1966). Exchange of such objects can also be expected to contribute importantly to the effective interaction. In fact, in recent calculations by S. Babu and the author, most of the effective force came from such a mechanism. However, the model used was speculative, and those calculations are certainly not sufficiently well founded to be included in an article which is primarily a review. Also, our main object is to discuss connections of the Landau–Migdal and Bethe–Brueckner theories, and here we are pointing out the case of liquid ^{3}He that one has to go beyond the latter theory to obtain the effective interaction.

Another interesting Fermi system is the liquid ^{3}He in ^{3}He–^{4}He mixtures, in which much of the effective force between ^{3}He atoms comes from exchange of ^{4}He excitations. Again, not much has been done here within the framework of the Brueckner–Bethe theory and we won't go into these matters.

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