# Crossing Matrices for $SU(2)$ and $SU(3)$*

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The $SU(2)$ crossing matrices for the scattering of $I=0, \frac{1}{2}, 1, \frac{3}{2}$ particles and antiparticles, and the $SU(3)$ crossing matrices for the scattering of singlets, octets, and decimets are listed. The $s-t$, $s-u$, and $t-u$ crossing matrices and their inverses are given for each case. The relative phases of the crossing matrix are discussed in detail.

## CONTENTS



## I. INTRODUCTION

Many calculations of amplitudes for high-energy processes require a knowledge of the crossed-channel processes. The idea that amplitudes related by crossing may be given by the same analytic function has met with success in both phenomenological applications and dynamical models. To use this idea in practice, one must be able to project the quantum numbers of a crossed channel onto the direct channel. The crossing matrix determines which linear superposition of crossed-channel amplitudes compose the direct-channel amplitudes. In this paper we consider the crossing matrices that are obtained when the amplitude is assumed to be $SU(2)$ or $SU(3)$ invariant.

Because of the importance of the crossing matrices, much good work has gone into the examination and ennumeration of their properties, and many explicit crossing matrices may be found in the literature.§ In fact, the problem has been completely solved for a number of years, at least for the two-body amplitude. However, to the best of our knowledge, no complete compilation of crossing matrices has appeared in the literature. The object of this paper is to present a compilation of $SU(2)$ crossing matrices in which all possible combinations of $I=0, \frac{1}{2}, 1, \frac{3}{2}$ particles and antiparticles may scatter off one another, and $SU(3)$ crossing matrices in which all possible combinations of singlets, octets, and decimets may scatter off one another. We have listed the crossing matrices between the $s$, $t$, and $u$ channels, along with their inverses. This list is complete, as long as the particles that are scattered are those of a quark model in which the mesons appear as $q\bar{q}$ and the baryons as $3q$ states.

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§ Mandelstam *et al.* (1962), Yang (1963), Foldy and Peierls (1963), Barut and Unal (1963), Carruthers and Krisch (1965), Lee (1967), de Swart (1964), Nieto (1965), Lin and Cutkosky (1965), Fairlie (1966), Sharp (1968), and Mani *et al.* (1966).

The $SU(2)$ and $SU(3)$ crossing matrices are derived in Sec. II, where a detailed analysis of the phases is given. We discuss the rules for transforming our crossing matrices to the crossing matrices for reactions in which the order of particles has been reversed, or in which particles have been replaced by their antiparticles. We also relate the isospin crossing matrix to the 6-$j$ symbol.

Section III contains the $SU(2)$ crossing matrices.

Section IV catalogs the $SU(3)$ crossing matrices.

## II. DERIVATION AND PHASES

The invariant amplitude is the $S$-matrix element with the energy–momentum delta function and the $1/(2E_i)^{1/2}$'s factored away. When spin is involved, certain kinematical singularities depending on the spin basis must also be removed. The invariant amplitude is assumed to be an analytic function of the Lorentz invariants, $s$, $t$, and $u$. We make the usual assumption that this amplitude, when continued to the values of $s$, $t$, and $u$ corresponding to the physical process in one of the cross channels, is just the amplitude for the crossed-channel process. Let us define the $s$, $t$, and $u$ channels as

$$\begin{aligned} &A+B\rightarrow C+D \qquad (s \text{ channel}),\\ &A+\bar{C}\rightarrow \bar{B}+D \qquad (t \text{ channel}),\\ &A+\bar{D}\rightarrow \bar{B}+C \qquad (u \text{ channel}).\end{aligned}$$

Then the crossing condition is

$$\langle CD \mid \mathfrak{M}(s,t,u) \mid AB\rangle = \langle \bar{B}D \mid \mathfrak{M}(t,s,u) \mid A\bar{C}\rangle = \langle \bar{B}C \mid \mathfrak{M}(u,s,t) \mid A\bar{D}\rangle, \quad (1)$$

where $|A\rangle$, $|B\rangle$, $|C\rangle$, and $|D\rangle$ are particle states and $|\bar{A}\rangle$, $|\bar{B}\rangle$, $|\bar{C}\rangle$, $|\bar{D}\rangle$ are antiparticle states. Incoming particle states of momentum $\mathbf{k}$ are transformed into outgoing antiparticle states of the same momentum by CPT. Thus, in Eq. (1) we have chosen the phase of the CPT operation to be $+1$. This is always possible because the phase of T is arbitrary.

If the $S$ matrix is invariant under an internal symmetry group, then we may expand the invariant amplitudes into eigenamplitudes of the group. We call these eigenamplitudes $A_s(I)$, $A_t(I)$, and $A_u(I)$, where the subscript labels the channel in which the expansion is performed, and $I$ labels the representation. The eigenamplitudes in one channel are linearly related to the eigenamplitudes of the crossed channels by Eq. (1).

The matrices of these equations are the "crossing matrices."

The expansion of the invariant amplitudes into isospin or $SU(3)$ eigenamplitudes involves the vector-coupling (V–C) coefficients. After expanding the amplitudes, it is straightforward to solve Eq. (1) for the crossing matrices.

Tables of V–C coefficients are available for $SU(2)$ and $SU(3)$.* However, the use of these tables requires some care since phase factors may be needed to relate the particle states to the isospin or $SU(3)$ states, i.e., to the vectors which are used in the construction of the V–C coefficients.

Suppose that the particle state $|C\rangle$ transforms according to the representation $\mathbf{R}_C$. Then, if $\langle C|$ is an outgoing state, it transforms according to the complex conjugate representation, $\mathbf{R}_C^*$. Moreover, to maintain Eq. (1) and the $SU(2)$ or $SU(3)$ invariance of the $S$ matrix, the $t$-channel incoming state $|A\bar{C}\rangle$ must transform according to $R_A\otimes R_C^*$.† In expanding the $t$-channel amplitudes we are then faced with the problem of reducing the direct product $R_A\otimes R_C^*$.

Let us first consider $SU(2)$, where all the representations are self-conjugate. Since $R$ and $R^*$ are equivalent (but not equal), the tables for the V–C coefficients display only the reduction of $R_A\otimes R_C$, and not $R_A\otimes R_C^*$. The basis for $R_C^*$ is related to the equivalent basis for $R_C$ by the operator, exp $(i\pi I_2)$. Let us denote the isospin state $|I_c, I_{3c}\rangle$ by $|c\rangle$ and the isospin state $|I_c, -I_{3c}\rangle$ by $|-c\rangle$. Then the vectors

$$|c^*\rangle = \exp(i\pi I_2)\,|c\rangle = (-1)^{I_c+I_{3c}}\,|-c\rangle \tag{2}$$

span the representation conjugate to $R_C$. We may identify the states $|c^*\rangle$ with the antiparticle states $|\bar{C}\rangle$.

The choice $|\bar{C}\rangle = (-1)^{I_c+I_{3c}}|-c\rangle$ is convenient for the half-integer-isospin states. However, when $I_c$ is an odd integer (as in the case of the $\pi$ multiplet), exp $(i\pi I_2)$ sends the neutral member of the multiplet into minus itself. Using the arbitrariness of the over-all phase between the antiparticle states and the isospin states, we may identify the antiparticle states with $|\bar{C}\rangle = (-1)^{I_{3c}}|-c\rangle$ [instead of $(-1)^{I_c+I_{3c}}|-c\rangle$] when $I_c$ is an integer. (At this point it is easy to recover the $G$-parity operation from the transformation that takes $|C\rangle$ to $|\bar{C}\rangle$. The extra phase we used in the integer-isospin case corresponds to the assignment $G=-1$ for the $\pi$ multiplet since the charge parity of the $\pi^0$ is $+1$.)

The situation is slightly more complicated for $SU(3)$. Not all of the representations of $SU(3)$ are self-conjugate. For the self-conjugate representations $(\mathbf{1}, \mathbf{8}, \mathbf{27}, \cdots)$, a basis for the conjugate representations is given by*

$$|c^*\rangle = (-1)^{I_{3c}+Y_c/2}\,|-c\rangle = (-1)^{Q_c}\,|-c\rangle,$$

where $|c\rangle = |N, I_c, I_{3c}, Y_c\rangle$ and $|-c\rangle = |N, I_c - I_{3c} - Y_c\rangle$. Moreover, by convention, this same factor has been retained in the construction of the V–C coefficients for the reduction of products in which non-self-conjugate representations appear, like $\mathbf{10}\otimes\mathbf{10}^*$ (de Swart, 1963; McNamee and Chilton, 1964). In other words, these tables do not list the V–C coefficients for the reduction of $\mathbf{R}_A\otimes\mathbf{R}_C^*$, but for the reduction of $\mathbf{R}_A\otimes\mathbf{R}_C'$, where $\mathbf{R}_C'$ is equivalent to $\mathbf{R}_C^*$. The basis for $\mathbf{R}_C'$ is given by $(-1)^{Q_c}|-c\rangle$.†

In summary, the antiparticle states in Eq. (1) are related to the isospin or $SU(3)$ states by the phase $\eta$, where

$$|\bar{A}\rangle = \eta_a\,|-a\rangle. \tag{3}$$

The phase $\eta_a$ is

$$\eta_a = (-1)^{I_{3a}} \tag{4}$$

for $SU(2)$, integer isospin;

$$\eta_a = (-1)^{I_a+I_{3a}} \tag{5}$$

for $SU(2)$, half-integer isospin; and

$$\eta_a = (-1)^{I_{3a}+Y_a/2} = (-1)^{Q_a} \tag{6}$$

for $SU(3)$.

Now that we have identified the particle and antiparticle states with basis vectors of the $SU(2)$ or $SU(3)$ representations, we may write Eq. (1) as

$$\begin{aligned}\langle c, d\,|\,\mathfrak{M}_s(s,t,u)\,|\,a,b\rangle &= \eta_b\eta_c\langle -b, d\,|\,\mathfrak{M}(t,s,u)\,|\,a,-c\rangle\\ &= \eta_b\eta_d\langle -b, c\,|\,\mathfrak{M}(u,s,t)\,|\,a,-d\rangle,\end{aligned} \tag{7}$$

where the matrix elements in Eq. (7) can be expanded into isospin or $SU(3)$ amplitudes with the tables of V–C coefficients. For example,

$$\langle -b, d\,|\,\mathfrak{M}(t,s,u)\,|\,a,-c\rangle = \sum_I C(a,-c;I)\,C(-b,d;I)\,A(I), \tag{8}$$

where

$$C(a,-c;I) = \langle I_aI_{3a}, I_c - I_{3c}\,|\,I_aI_c; I, I_{3a}-I_{3c}\rangle \tag{9}$$

for $SU(2)$, and

$$C(a,-c;I) = \begin{pmatrix} \mu_a & \mu_c^* & \mu_{I\gamma} \\ \nu_a & -\nu_c & \nu_I \end{pmatrix} \tag{10}$$

for $SU(3)$.

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* de Swart (1963), McNamee and Chilton (1964), Edmonds (1957), and Rotenberg *et al.* (1959).

† We also recover the known result for unitary groups that incoming particles and outgoing antiparticles must transform according to the same representation, whereas incoming antiparticles and outgoing particles transform according to the conjugate one.

* No operator in $SU(3)$ performs this operation. We wish to thank Dr. Jeffrey Mandula and Professor Yuval Ne'eman for a discussion of this point.

† This choice of phases is convenient for constructing the isoscalar factors for representations having zero triality. For other representations, such as the **3** or **3***, this phase convention gives complex isoscalar factors (de Swart, 1963; McNamee and Chilton, 1964).

TABLE I. The phases $\xi_{st}$, $\xi_{su}$, and $\xi_{tu}$ for Eq. (13). These phases depend on whether the particles that are crossed have integer or half-integer isospin. See Eqs. (4) and (5).

| Isospin | | |
|---|---|---|
| $I_b$ | $I_c$ | $\xi_{st}$ |
| Integer | Integer | $(-1)^{I+I'}$ |
| Half-integer | Integer | $(-1)^{I+I'+I_b}$ |
| Integer | Half-integer | $(-1)^{I+I'-I_c}$ |
| Half-integer | Half-integer | $(-1)^{I+I'+I_b-I_c}$ |
| $I_b$ | $I_d$ | $\xi_{su}$ |
| Integer | Integer | $(-1)^{I'+I_c+I_d}$ |
| Half-integer | Integer | $(-1)^{I'+I_c+I_d+I_b}$ |
| Integer | Half-integer | $(-1)^{I'+I_c}$ |
| Half-integer | Half-integer | $(-1)^{I'+I_c+I_b}$ |
| $I_c$ | $I_d$ | $\xi_{tu}$ |
| Integer | Integer | $(-1)^{2I_b-I_d+I_c}$ |
| Half-integer | Integer | $(-1)^{2I_b-I_d}$ |
| Integer | Half-integer | $(-1)^{2I_b+I_c}$ |
| Half-integer | Half-integer | $(-1)^{2I_b}$ |

Finally, Eq. (7) may be solved for $A_s(I)$ in terms of $A_t(I)$ or $A_u(I)$ to find the crossing matrices $X_{st}$ or $X_{su}$,

$$A_s(I)=\Sigma_{I'}(X_{st})_{I,I'}A_t(I'),$$

$$A_s(I)=\Sigma_{I'}(X_{su})_{I,I'}A_u(I'). \tag{11}$$

We relate the isospin crossing matrices to the 6-$j$ symbols. For example, one may solve Eq. (7) for $A_s(I)$ in terms of $A_t(I)$ using the orthogonality properties of the V–C coefficients. Then $(X_{st})_{I,I'}$ is given by

$$(X_{st})_{I,I'}=\sum_{abcd}\eta_b\eta_c C(a,b;I)C(c,d;I)\times C(a,-c;I')C(-b,d;I'). \tag{12}$$

The right-hand side of Eq. (12) is proportional to a 6-$j$ symbol. Some of the crossing matrices for $SU(2)$ in terms of the 6-$j$ symbols are*

$$(X_{st})_{I,I'}=\xi_{st}(2I'+1)\begin{Bmatrix} I_a & I_b & I \\ I_d & I_c & I' \end{Bmatrix},$$

$$(X_{su})_{I,I'}=\xi_{su}(2I'+1)\begin{Bmatrix} I_a & I_b & I \\ I_c & I_d & I' \end{Bmatrix},$$

$$(X_{tu})_{I,I'}=\xi_{tu}(2I'+1)\begin{Bmatrix} I_a & I_c & I \\ I_b & I_d & I' \end{Bmatrix}, \tag{13}$$

where $\xi_{st}$, $\xi_{su}$, and $\xi_{tu}$ are the phases given in Table I.

It may be necessary to relate the crossing matrices we give to others in which the $t$ and $u$ channels are defined differently or some particles have been replaced by their antiparticles. When the order of states is reversed, the crossing matrix may differ by some phase factor. This phase results from the symmetry of the V–C coefficients,

$$C(a,b;I)=\xi_1 C(b,a;I), \tag{14}$$

where $\xi_1$ is $(-1)^{I-I_a-I_b}$ for $SU(2)$, and is given in Table II for $SU(3)$. It follows that the crossing matrix for amplitudes in which the order of states is reversed is obtained simply by multiplying the corresponding amplitudes by the phase factor $\xi_1$.

Let us consider the crossing matrix for the reaction where a particle is replaced by its antiparticle, if the particle and antiparticle belong to equivalent representations. In deriving the crossing matrices, we may use the same isospin or $SU(3)$ state for the particle or the antiparticle, so that exactly the same V–C coefficients are needed. Compare the crossing condition

$$\langle AB\mid \mathfrak{M}_s\mid CD\rangle=\langle A\bar{C}\mid \mathfrak{M}_t\mid \bar{B}D\rangle$$

with

$$\langle A(-\bar{B})\mid \mathfrak{M}_s'\mid CD\rangle=\langle A\bar{C}\mid \mathfrak{M}_t'\mid (-B)D\rangle.$$

In terms of the isospin or $SU(3)$ basis, these equations are

$$\langle a,b\mid \mathfrak{M}_s\mid c,d\rangle=\eta_b\eta_c\langle a,-c\mid \mathfrak{M}_t\mid -b,d\rangle$$

and

$$\eta_{-b}\langle a,b\mid \mathfrak{M}_s'\mid c,d\rangle=\eta_c\langle a_1-c\mid \mathfrak{M}_t'\mid -b,d\rangle.$$

TABLE II. Phase factor for reversal of order of states, $\xi_1$, and "conjugation" of the V–C coefficient, $\xi_3$. See Eqs. (14) and (16).

| $\mu_1$ | $\mu_2$ | $\mu_3$ | $\xi_1$ | $\xi_3$ |
|---|---|---|---|---|
| 8 | 8 | 1 | 1 | 1 |
| | | $8_s$ | 1 | 1 |
| | | $8_A$ | $-1$ | $-1$ |
| | | 10 | $-1$ | 1 |
| | | $\overline{10}$ | $-1$ | 1 |
| | | 27 | 1 | 1 |
| 8 | 10 | 8 | 1 | $-1$ |
| | | 10 | $-1$ | $-1$ |
| | | 27 | $-1$ | 1 |
| | | 35 | 1 | 1 |
| 8 | $\overline{10}$ | 8 | 1 | $-1$ |
| | | $\overline{10}$ | $-1$ | $-1$ |
| | | 27 | $-1$ | 1 |
| | | $\overline{35}$ | 1 | 1 |
| 10 | 10 | $\overline{10}$ | $-1$ | 1 |
| | | 27 | 1 | 1 |
| | | 35 | $-1$ | 1 |
| | | 28 | 1 | 1 |
| 10 | $\overline{10}$ | 1 | $-1$ | $-1$ |
| | | 8 | 1 | 1 |
| | | 27 | $-1$ | $-1$ |
| | | 64 | 1 | 1 |

* The 6-$j$ symbol may be related to the Racah coefficient (Edmonds, 1957).

Comparing these equations, it is clear that

$$X_{st}'=\eta_{-b}\eta_b X_{st}. \qquad (15)$$

From Eqs. (4)–(6), we find that $\eta_{-b}\eta_b=(-1)^{2I_b}$ for $SU(2)$, and $\eta_{-b}\eta_b=1$ for $SU(3)$, as long as $B$ is in a **1**, **8**, **27**, $\cdots$. Thus, crossing matrices for reactions which differ by having a particle in a self-conjugate representation replaced by its antiparticle are related by $(-1)$ if the particle belongs to a half-integer isomultiplet and is crossed, and are equal otherwise.

If a particle which belongs to a non-self-conjugate representation like the **10** is replaced by its antiparticle, the new crossing matrix is, in general, not simply related to the old one. However, if all the particles in the reaction are changed into their antiparticles, then the relation is a phase coming from

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_{3\gamma} \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix} = \xi_3 \begin{pmatrix} \mu_1^* & \mu_2^* & \mu_{3\gamma}^* \\ -\nu_1 & -\nu_2 & -\nu_3 \end{pmatrix}. \qquad (16)$$

It follows that the crossing matrices are related by a product of $\xi_3$ factors. The $\xi_3$ are also listed in Table II.

The isospin crossing matrices in Eq. (13) may be immediately derived from one another using these prescriptions.

## III. ISOSPIN CROSSING MATRICES

| Isospin Structure | Example | Channel |
|---|---|---|
| $1/2+1/2'\to 0+0'$ | $\bar{K}N\to\Lambda\eta$ | $(s)$ |
| $1/2+0\to\overline{1/2}'+0'$ | $\bar{K}\bar{\Lambda}\to\bar{N}\eta$ | $(t)$ |
| $1/2+0'\to\overline{1/2}'+0$ | $\bar{K}\eta\to\bar{N}\Lambda$ | $(u)$ |

$$A_s(0)=(2^{1/2})A_t(1/2)=(2^{1/2})A_u(1/2).$$

| | | |
|---|---|---|
| $1/2+1/2'\to 1+0$ | $\bar{K}N\to\Sigma\eta$ | $(s)$ |
| $1/2+1\to\overline{1/2}'+0$ | $\bar{K}\bar{\Sigma}\to\bar{N}\eta$ | $(t)$ |
| $1/2+0\to\overline{1/2}'+1$ | $\bar{K}\eta\to\bar{N}\Sigma$ | $(u)$ |

$$A_s(1)=(1/3)(6^{1/2})A_t(1/2)=(1/3)(6^{1/2})A_u(1/2).$$

| | | |
|---|---|---|
| $1/2+1/2'\to 1/2''+1/2'''$ | $NK\to N'K'$ | $(s)$ |
| $1/2+\overline{1/2}''\to\overline{1/2}'+1/2'''$ | $N\bar{N}'\to\bar{K}K'$ | $(t)$ |
| $1/2+\overline{1/2}'''\to\overline{1/2}'+1/2''$ | $N\bar{K}'\to\bar{K}N'$ | $(u)$ |

$$\begin{bmatrix} A_s(0) \\ A_s(1) \end{bmatrix} = \begin{bmatrix} -1/2 & -3/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} A_t(0) \\ A_t(1) \end{bmatrix} = \begin{bmatrix} 1/2 & 3/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} A_u(0) \\ A_u(1) \end{bmatrix};$$

$$X_{ts}=X_{st};\qquad X_{tu}=X_{ut}=\begin{bmatrix} 1/2 & -3/2 \\ -1/2 & -1/2 \end{bmatrix};\qquad X_{us}=\begin{bmatrix} 1/2 & -3/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

| | | |
|---|---|---|
| $1/2+1/2'\to 1+1'$ | $\bar{K}N\to\Sigma\pi$ | $(s)$ |
| $1/2+1\to\overline{1/2}'+1'$ | $\bar{K}\bar{\Sigma}\to\bar{N}\pi$ | $(t)$ |
| $1/2+1'\to\overline{1/2}'+1$ | $\bar{K}\pi\to\bar{N}\Sigma$ | $(u)$ |

$$\begin{bmatrix} A_s(0) \\ A_s(1) \end{bmatrix} = \begin{bmatrix} -(1/3)(6^{1/2}) & -(2/3)(6^{1/2}) \\ -2/3 & 2/3 \end{bmatrix} \begin{bmatrix} A_t(1/2) \\ A_t(3/2) \end{bmatrix} = \begin{bmatrix} -(1/3)(6^{1/2}) & -(2/3)(6^{1/2}) \\ 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} A_u(1/2) \\ A_u(3/2) \end{bmatrix};$$

$$X_{ts}=\begin{bmatrix} -(1/6)(6^{1/2}) & -1 \\ -(1/6)(6^{1/2}) & 1/2 \end{bmatrix};\qquad X_{tu}=X_{ut}=\begin{bmatrix} -1/3 & 4/3 \\ 2/3 & 1/3 \end{bmatrix};\qquad X_{us}=\begin{bmatrix} -(1/6)(6^{1/2}) & 1 \\ -(1/6)(6^{1/2}) & -1/2 \end{bmatrix}.$$

$$\begin{array}{lll} 1/2+1/2'\to 3/2+1/2'' & NK\to \Delta K' & (s) \\ 1/2+\overline{3/2}\to \overline{1/2}'+1/2'' & N\bar{\Delta}\to \bar{K}K' & (t) \\ 1/2+\overline{1/2}''\to \overline{1/2}'+3/2 & N\bar{K}'\to \bar{K}\Delta & (u) \end{array}$$

$$A_s(1)=A_t(1)=A_u(1).$$

$$\begin{array}{lll} 1/2+1/2'\to 3/2+3/2' & NN'\to \Delta\Delta' & (s) \\ 1/2+\overline{3/2}\to \overline{1/2}'+3/2' & N\bar{\Delta}\to \bar{N}'\Delta' & (t) \\ 1/2+\overline{3/2}'\to \overline{1/2}'+3/2 & N\bar{\Delta}'\to \bar{N}'\Delta & (u) \end{array}$$

$$\begin{bmatrix} A_s(0) \\ A_s(1) \end{bmatrix} = \begin{bmatrix} -(3/4)(2^{1/2}) & -(5/4)(2^{1/2}) \\ -(1/4)(10^{1/2}) & (1/4)(10^{1/2}) \end{bmatrix} \begin{bmatrix} A_t(1) \\ A_t(2) \end{bmatrix} = \begin{bmatrix} (3/4)(2^{1/2}) & (5/4)(2^{1/2}) \\ -(1/4)(10^{1/2}) & (1/4)(10^{1/2}) \end{bmatrix} \begin{bmatrix} A_u(1) \\ A_u(2) \end{bmatrix};$$

$$X_{ts}=\begin{bmatrix} -(1/4)(2^{1/2}) & -(1/4)(10^{1/2}) \\ -(1/4)(2^{1/2}) & (3/20)(10^{1/2}) \end{bmatrix}; \quad X_{tu}=X_{ut}=\begin{bmatrix} 1/4 & -5/4 \\ -3/4 & -1/4 \end{bmatrix}; \quad X_{us}=\begin{bmatrix} (1/4)(2^{1/2}) & -(1/4)(10^{1/2}) \\ (1/4)(2^{1/2}) & (3/20)(10^{1/2}) \end{bmatrix}.$$

$$\begin{array}{lll} 1+0\to 1'+0' & \pi\Lambda\to \pi'\Lambda' & (s) \\ 1+1'\to 0+0' & \pi\pi'\to \bar{\Lambda}\Lambda' & (t) \\ 1+0'\to 0+1' & \pi\bar{\Lambda}'\to \bar{\Lambda}\pi' & (u) \end{array}$$

$$A_s(1)=-(1/3)(3^{1/2})A_t(0)=A_u(1).$$

$$\begin{array}{lll} 1+0\to 1'+1'' & \pi\Lambda\to \pi'\Sigma & (s) \\ 1+1'\to 0+1'' & \pi\pi'\to \bar{\Lambda}\Sigma & (t) \\ 1+1''\to 0+1' & \pi\bar{\Sigma}\to \bar{\Lambda}\pi' & (u) \end{array}$$

$$A_s(1)=-A_t(1)=A_u(1).$$

$$\begin{array}{lll} 1+0\to 3/2+1/2 & \Sigma\Lambda\to \Delta\Xi & (s) \\ 1+\overline{3/2}\to 0+1/2 & \Sigma\bar{\Delta}\to \bar{\Lambda}\Xi & (t) \\ 1+\overline{1/2}\to 0+3/2 & \Sigma\bar{\Xi}\to \bar{\Lambda}\Delta & (u) \end{array}$$

$$A_s(1)=-(1/3)(6^{1/2})A_t(1/2)=(2/3)(3^{1/2})A_u(3/2).$$

$$\begin{array}{lll} 0+3/2\to 0'+3/2' & \eta\Delta\to \eta'\Delta' & (s) \\ 0+0'\to \overline{3/2}+3/2' & \eta\eta'\to \bar{\Delta}\Delta' & (t) \\ 0+\overline{3/2}'\to \overline{3/2}+0' & \eta\bar{\Delta}'\to \bar{\Delta}\eta' & (u) \end{array}$$

$$A_s(3/2)=(1/2)A_t(0)=A_u(3/2).$$

$$\begin{array}{lll} 1+3/2\to 0+3/2' & \pi\Delta\to \eta\Delta' & (s) \\ 1+0\to \overline{3/2}+3/2' & \pi\eta\to \bar{\Delta}\Delta' & (t) \\ 1+\overline{3/2}'\to \overline{3/2}+0 & \pi\bar{\Delta}'\to \bar{\Delta}\eta & (u) \end{array}$$

$$A_s(3/2)=(1/2)(3^{1/2})A_t(1)=-A_u(3/2).$$

$$1+1/2\to 1'+1/2' \qquad \pi N\to\pi' N' \qquad (s)$$
$$1+1'\to\overline{1/2}+1/2' \qquad \pi\pi'\to\bar{N}N' \qquad (t)$$
$$1+\overline{1/2}'\to\overline{1/2}+1' \qquad \pi\bar{N}'\to\bar{N}\pi' \qquad (u)$$

$$\begin{bmatrix} A_s(1/2) \\ A_s(3/2) \end{bmatrix} = \begin{bmatrix} -(1/6)(6^{1/2}) & -1 \\ -(1/6)(6^{1/2}) & 1/2 \end{bmatrix} \begin{bmatrix} A_t(0) \\ A_t(1) \end{bmatrix} = \begin{bmatrix} 1/3 & 4/3 \\ -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} A_u(1/2) \\ A_u(3/2) \end{bmatrix};$$

$$X_{ts} = \begin{bmatrix} -(1/3)(6^{1/2}) & -(2/3)(6^{1/2}) \\ -2/3 & 2/3 \end{bmatrix}; \quad X_{tu} = \begin{bmatrix} (1/3)(6^{1/2}) & -(2/3)(6^{1/2}) \\ -2/3 & -2/3 \end{bmatrix};$$

$$X_{ut} = \begin{bmatrix} (1/6)(6^{1/2}) & -1 \\ -(1/6)(6^{1/2}) & -1/2 \end{bmatrix}; \quad X_{us} = \begin{bmatrix} 1/3 & -4/3 \\ 2/3 & 1/3 \end{bmatrix}.$$

---

$$1+1/2\to 1'+3/2 \qquad \pi N\to\pi'\Delta \qquad (s)$$
$$1+1'\to\overline{1/2}+3/2 \qquad \pi\pi'\to\bar{N}\Delta \qquad (t)$$
$$1+\overline{3/2}\to\overline{1/2}+1' \qquad \pi\bar{\Delta}\to\bar{N}\pi' \qquad (u)$$

$$\begin{bmatrix} A_s(1/2) \\ A_s(3/2) \end{bmatrix} = \begin{bmatrix} -1/2 & -(5/6)(3^{1/2}) \\ -(1/4)(10^{1/2}) & (1/12)(30^{1/2}) \end{bmatrix} \begin{bmatrix} A_t(1) \\ A_t(2) \end{bmatrix} = \begin{bmatrix} -2/3 & -(1/3)(10^{1/2}) \\ (1/6)(10^{1/2}) & -2/3 \end{bmatrix} \begin{bmatrix} A_u(1/2) \\ A_u(3/2) \end{bmatrix};$$

$$X_{ts} = \begin{bmatrix} -1/3 & -(1/3)(10^{1/2}) \\ -(1/3)(3^{1/2}) & (1/15)(30^{1/2}) \end{bmatrix}; \quad X_{tu} = \begin{bmatrix} -1/3 & (1/3)(10^{1/2}) \\ (1/3)(3^{1/2}) & (1/15)(30^{1/2}) \end{bmatrix};$$

$$X_{ut} = \begin{bmatrix} -1/2 & (5/6)(3^{1/2}) \\ (1/4)(10^{1/2}) & (1/12)(30^{1/2}) \end{bmatrix}; \quad X_{us} = \begin{bmatrix} -2/3 & (1/3)(10^{1/2}) \\ -(1/6)(10^{1/2}) & -2/3 \end{bmatrix}.$$

---

$$1+1'\to 1''+1''' \qquad \pi\pi'\to\pi''\pi''' \qquad (s)$$
$$1+1''\to 1'+1''' \qquad \pi\pi''\to\pi'\pi''' \qquad (t)$$
$$1+1'''\to 1'+1'' \qquad \pi\pi'''\to\pi'\pi'' \qquad (u)$$

$$\begin{bmatrix} A_s(0) \\ A_s(1) \\ A_s(2) \end{bmatrix} = \begin{bmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{bmatrix} \begin{bmatrix} A_t(0) \\ A_t(1) \\ A_t(2) \end{bmatrix} = \begin{bmatrix} 1/3 & 1 & 5/3 \\ -1/3 & -1/2 & 5/6 \\ 1/3 & -1/2 & 1/6 \end{bmatrix} \begin{bmatrix} A_u(0) \\ A_u(1) \\ A_u(2) \end{bmatrix};$$

$$X_{ts} = X_{st}; \quad X_{tu} = X_{ut} = \begin{bmatrix} 1/3 & -1 & 5/3 \\ -1/3 & 1/2 & 5/6 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}; \quad X_{us} = \begin{bmatrix} 1/3 & -1 & 5/3 \\ 1/3 & -1/2 & -5/6 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}.$$

---

$$1+1'\to 3/2+\overline{3/2}' \qquad \pi\pi'\to\Delta\bar{\Delta}' \qquad (s)$$
$$1+\overline{3/2}\to 1'+\overline{3/2}' \qquad \pi\bar{\Delta}\to\pi'\bar{\Delta}' \qquad (t)$$
$$1+3/2'\to 1'+3/2 \qquad \pi\Delta'\to\pi'\Delta \qquad (u)$$

$$\begin{bmatrix} A_s(0) \\ A_s(1) \\ A_s(2) \end{bmatrix} = \begin{bmatrix} (1/3)(3^{1/2}) & (2/3)(3^{1/2}) & 3^{1/2} \\ (1/6)(10^{1/2}) & (2/15)(10^{1/2}) & -(3/10)(10^{1/2}) \\ (1/6)(6^{1/2}) & -(4/15)(6^{1/2}) & (1/10)(6^{1/2}) \end{bmatrix} \begin{bmatrix} A_t(1/2) \\ A_t(3/2) \\ A_t(5/2) \end{bmatrix}$$

$$= \begin{bmatrix} (1/3)(3^{1/2}) & (2/3)(3^{1/2}) & 3^{1/2} \\ -(1/6)(10^{1/2}) & -(2/15)(10^{1/2}) & (3/10)(10^{1/2}) \\ (1/6)(6^{1/2}) & -(4/15)(6^{1/2}) & (1/10)(6^{1/2}) \end{bmatrix} \begin{bmatrix} A_u(1/2) \\ A_u(3/2) \\ A_u(5/2) \end{bmatrix};$$

$$X_{ts} = \begin{bmatrix} (1/6)(3^{1/2}) & (1/4)(10^{1/2}) & (5/12)(6^{1/2}) \\ (1/6)(3^{1/2}) & (1/10)(10^{1/2}) & -(1/3)(6^{1/2}) \\ (1/6)(3^{1/2}) & -(3/20)(10^{1/2}) & (1/12)(6^{1/2}) \end{bmatrix}; \qquad X_{tu} = X_{ut} = \begin{bmatrix} 1/6 & -2/3 & 3/2 \\ -1/3 & 11/15 & 3/5 \\ 1/2 & 2/5 & 1/10 \end{bmatrix};$$

$$X_{us} = \begin{bmatrix} (1/6)(3^{1/2}) & -(1/4)(10^{1/2}) & (5/12)(6^{1/2}) \\ (1/6)(3^{1/2}) & -(1/10)(10^{1/2}) & -(1/3)(6^{1/2}) \\ (1/6)(3^{1/2}) & (3/20)(10^{1/2}) & (1/12)(6^{1/2}) \end{bmatrix}.$$

---

$$\begin{array}{lll} 1/2+3/2 \to 3/2'+3/2'' & N\Delta \to \Delta'\Delta'' & (s) \\ 1/2+\overline{3/2}' \to \overline{3/2}+3/2'' & N\bar{\Delta}' \to \bar{\Delta}\Delta'' & (t) \\ 1/2+\overline{3/2}'' \to \overline{3/2}+3/2' & N\bar{\Delta}'' \to \bar{\Delta}\Delta' & (u) \end{array}$$

$$\begin{bmatrix} A_s(1) \\ A_s(2) \end{bmatrix} = \begin{bmatrix} 1/2 & (1/2)(5^{1/2}) \\ (3/10)(5^{1/2}) & -1/2 \end{bmatrix} \begin{bmatrix} A_t(1) \\ A_t(2) \end{bmatrix} = \begin{bmatrix} 1/2 & (1/2)(5^{1/2}) \\ -(3/10)(5^{1/2}) & 1/2 \end{bmatrix} \begin{bmatrix} A_u(1) \\ A_u(2) \end{bmatrix};$$

$$X_{ts} = X_{st}; \qquad X_{tu} = X_{ut} = \begin{bmatrix} -1/2 & (1/2)(5^{1/2}) \\ (3/10)(5^{1/2}) & 1/2 \end{bmatrix}; \qquad X_{us} = \begin{bmatrix} 1/2 & -(1/2)(5^{1/2}) \\ (3/10)(5^{1/2}) & 1/2 \end{bmatrix}.$$

---

$$\begin{array}{lll} 3/2+3/2' \to 3/2''+3/2''' & \Delta\Delta' \to \Delta''\Delta''' & (s) \\ 3/2+\overline{3/2}'' \to \overline{3/2}'+3/2''' & \Delta\bar{\Delta}'' \to \bar{\Delta}'\Delta''' & (t) \\ 3/2+3/2''' \to \overline{3/2}'+3/2'' & \Delta\bar{\Delta}''' \to \bar{\Delta}'\Delta'' & (u) \end{array}$$

$$\begin{bmatrix} A_s(0) \\ A_s(1) \\ A_s(2) \\ A_s(3) \end{bmatrix} = \begin{bmatrix} -1/4 & -3/4 & -5/4 & -7/4 \\ -1/4 & -11/20 & -1/4 & 21/20 \\ -1/4 & -3/20 & 3/4 & -7/20 \\ -1/4 & 9/20 & -1/4 & 1/20 \end{bmatrix} \begin{bmatrix} A_t(0) \\ A_t(1) \\ A_t(2) \\ A_t(3) \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 & 5/4 & 7/4 \\ -1/4 & -11/20 & -1/4 & 21/20 \\ 1/4 & 3/20 & -3/4 & 7/20 \\ -1/4 & 9/20 & -1/4 & 1/20 \end{bmatrix} \begin{bmatrix} A_u(0) \\ A_u(1) \\ A_u(2) \\ A_u(3) \end{bmatrix}$$

$$X_{ts} = X_{st}; \qquad X_{tu} = X_{ut} = \begin{bmatrix} 1/4 & -3/4 & 5/4 & -7/4 \\ -1/4 & 11/20 & -1/4 & -21/20 \\ 1/4 & -3/20 & -3/4 & -7/20 \\ -1/4 & -9/20 & -1/4 & -1/20 \end{bmatrix};$$

$$X_{us}=\begin{bmatrix} 1/4 & -3/4 & 5/4 & -7/4 \\ 1/4 & -11/20 & 1/4 & 21/20 \\ 1/4 & -3/20 & -3/4 & -7/20 \\ 1/4 & 9/20 & 1/4 & 1/20 \end{bmatrix}.$$

## IV. $SU(3)$ CROSSING MATRICES

| $SU(3)$ Structure | Example | Channel |
|---|---|---|
| $1+8\to1'+8'$ | $XB\to X'B'$ | $(s)$ |
| $1+1'\to8+8'$ | $X\bar{X}'\to\bar{B}B$ | $(t)$ |
| $1+8'\to8+1'$ | $X\bar{B}'\to\bar{B}X'$ | $(u)$ |

$$A_s(8)=-(1/8)(8^{1/2})A_t(1)=A_u(8).$$

| | | |
|---|---|---|
| $1+10\to1'+10'$ | $X\Delta\to X'\Delta'$ | $(s)$ |
| $1+1'\to\overline{10}+10$ | $X\bar{X}'\to\bar{\Delta}\Delta'$ | $(t)$ |
| $1+\overline{10}'\to\overline{10}+1'$ | $X\bar{\Delta}'\to\bar{\Delta}X'$ | $(u)$ |

$$A_s(10)=-(1/10)(10^{1/2})A_t(1)=A_u(\overline{10}).$$

| | | |
|---|---|---|
| $1+8\to8'+8''$ | $XB\to PB'$ | $(s)$ |
| $1+8'\to8+8''$ | $XP\to\bar{B}B'$ | $(t)$ |
| $1+8''\to8+8'$ | $X\bar{B}'\to\bar{B}P$ | $(u)$ |

$$\begin{bmatrix} A_s(8_s) \\ A_s(8_A) \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} A_t(8_s) \\ A_t(8_A) \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A_u(8_s) \\ A_u(8_A) \end{bmatrix};$$

$$X_{ts}=X_{st}; \qquad X_{tu}=X_{ut}=X_{st}; \qquad X_{us}=X_{su}.$$

| | | |
|---|---|---|
| $1+8\to8'+10$ | $XB\to P\Delta$ | $(s)$ |
| $1+8'\to8+10$ | $XP\to\bar{B}\Delta$ | $(t)$ |
| $1+\overline{10}\to8+8'$ | $X\bar{\Delta}\to\bar{B}P$ | $(u)$ |

$$A_s(8)=-A_t(8)=-(1/2)(5^{1/2})A_u(\overline{10}).$$

| | | |
|---|---|---|
| $1+10\to8+8'$ | $X\Delta\to PB$ | $(s)$ |
| $1+8\to\overline{10}+8'$ | $XP\to\bar{\Delta}B$ | $(t)$ |
| $1+8'\to\overline{10}+8$ | $X\bar{B}\to\bar{\Delta}P$ | $(u)$ |

$$A_s(10)=(2/5)(5^{1/2})A_t(8)=-(2/5)(5^{1/2})A_u(8).$$

| | | |
|---|---|---|
| $1+10\to8+10'$ | $X\Delta\to P\Delta$ | $(s)$ |
| $1+8\to\overline{10}+10'$ | $XP\to\bar{\Delta}\Delta$ | $(t)$ |
| $1+\overline{10}'\to\overline{10}+8$ | $X\bar{\Delta}\to\bar{\Delta}P$ | $(u)$ |

$$A_s(10)=-(2/5)(5^{1/2})A_t(8)=A_u(\overline{10}).$$

| | | |
|---|---|---|
| $8+8'\to8''+8'''$ | $PB\to P'B'$ | $(s)$ |
| $8+8''\to8'+8'''$ | $PP'\to\bar{B}B'$ | $(t)$ |
| $8+8'''\to8'+8''$ | $P\bar{B}'\to\bar{B}P'$ | $(u)$ |

$$
\begin{bmatrix} A_s(1) \\ A_s(8_{As}) \\ A_s(8_{sA}) \\ A_s(8_{ss}) \\ A_s(8_{AA}) \\ A_s(10) \\ A_s(\overline{10}) \\ A_s(27) \end{bmatrix}
=
\begin{bmatrix}
1/8 & 0 & 0 & 1 & 1 & 5/4 & 5/4 & 27/8 \\
0 & -1/2 & -1/2 & 0 & 0 & (1/4)(5^{1/2}) & -(1/4)(5^{1/2}) & 0 \\
0 & -1/2 & -1/2 & 0 & 0 & -(1/4)(5^{1/2}) & (1/4)(5^{1/2}) & 0 \\
1/8 & 0 & 0 & -3/10 & 1/2 & -1/2 & -1/2 & 27/40 \\
1/8 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & -9/8 \\
1/8 & (1/5)(5^{1/2}) & -(1/5)(5^{1/2}) & -2/5 & 0 & 1/4 & 1/4 & -9/40 \\
1/8 & -(1/5)(5^{1/2}) & (1/5)(5^{1/2}) & -2/5 & 0 & 1/4 & 1/4 & -9/40 \\
1/8 & 0 & 0 & 1/5 & -1/3 & -1/12 & -1/12 & 7/40
\end{bmatrix}
\begin{bmatrix} A_t(1) \\ A_t(8_{As}) \\ A_t(8_{sA}) \\ A_t(8_{ss}) \\ A_t(8_{AA}) \\ A_t(10) \\ A_t(\overline{10}) \\ A_t(27) \end{bmatrix}
$$

$$
=
\begin{bmatrix}
1/8 & 0 & 0 & 1 & 1 & 5/4 & 5/4 & 27/8 \\
0 & 1/2 & 1/2 & 0 & 0 & -(1/4)(5^{1/2}) & (1/4)(5^{1/2}) & 0 \\
0 & -1/2 & -1/2 & 0 & 0 & -(1/4)(5^{1/2}) & (1/4)(5^{1/2}) & 0 \\
1/8 & 0 & 0 & -3/10 & 1/2 & -1/2 & -1/2 & 27/40 \\
-1/8 & 0 & 0 & -1/2 & -1/2 & 0 & 0 & 9/8 \\
-1/8 & -(1/5)(5^{1/2}) & (1/5)(5^{1/2}) & 2/5 & 0 & -1/4 & -1/4 & 9/40 \\
-1/8 & (1/5)(5^{1/2}) & -(1/5)(5^{1/2}) & 2/5 & 0 & -1/4 & -1/4 & 9/40 \\
1/8 & 0 & 0 & 1/5 & -1/3 & -1/12 & -1/12 & 7/40
\end{bmatrix}
\begin{bmatrix} A_u(1) \\ A_u(8_{As}) \\ A_u(8_{sA}) \\ A_u(8_{ss}) \\ A_u(8_{AA}) \\ A_u(10) \\ A_u(\overline{10}) \\ A_u(27) \end{bmatrix},
$$

where $A_s(8_{As}) = \langle 8_{A\ \mathrm{out}} \mid \mathfrak{M}_s \mid 8_{s\ \mathrm{in}} \rangle$, etc. (This definition is crucial for computing the phases for reversing the order of particles in the initial *or* final state.)

$$X_{ts}=X_{st};$$

$$X_{tu}=X_{ut}=\begin{bmatrix} 1/8 & 0 & 0 & 1 & -1 & -5/4 & -5/4 & 27/8 \\ 0 & -1/2 & 1/2 & 0 & 0 & (1/4)(5^{1/2}) & -(1/4)(5^{1/2}) & 0 \\ 0 & 1/2 & -1/2 & 0 & 0 & (1/4)(5^{1/2}) & -(1/4)(5^{1/2}) & 0 \\ 1/8 & 0 & 0 & -3/10 & -1/2 & 1/2 & 1/2 & 27/40 \\ -1/8 & 0 & 0 & -1/2 & 1/2 & 0 & 0 & 9/8 \\ -1/8 & (1/5)(5^{1/2}) & (1/5)(5^{1/2}) & 2/5 & 0 & 1/4 & 1/4 & 9/40 \\ -1/8 & -(1/5)(5^{1/2}) & -(1/5)(5^{1/2}) & 2/5 & 0 & 1/4 & 1/4 & 9/40 \\ 1/8 & 0 & 0 & 1/5 & 1/3 & 1/12 & 1/12 & 7/40 \end{bmatrix};$$

$$X_{us}=\begin{bmatrix} 1/8 & 0 & 0 & 1 & -1 & -5/4 & -5/4 & 27/8 \\ 0 & 1/2 & -1/2 & 0 & 0 & -(1/4)(5^{1/2}) & (1/4)(5^{1/2}) & 0 \\ 0 & 1/2 & -1/2 & 0 & 0 & (1/4)(5^{1/2}) & -(1/4)(5^{1/2}) & 0 \\ 1/8 & 0 & 0 & -3/10 & -1/2 & 1/2 & 1/2 & 27/40 \\ 1/8 & 0 & 0 & 1/2 & -1/2 & 0 & 0 & -9/8 \\ 1/8 & -(1/5)(5^{1/2}) & -(1/5)(5^{1/2}) & -2/5 & 0 & -1/4 & -1/4 & -9/40 \\ 1/8 & (1/5)(5^{1/2}) & (1/5)(5^{1/2}) & -2/5 & 0 & -1/4 & -1/4 & -9/40 \\ 1/8 & 0 & 0 & 1/5 & 1/3 & 1/12 & 1/12 & 7/40 \end{bmatrix}.$$

---

| | | |
|---|---|---|
| $8+8'\to 8''+10$ | $PB\to P'\Delta$ | $(s)$ |
| $8+8''\to 8'+10$ | $PP'\to \bar{B}\Delta$ | $(t)$ |
| $8+\overline{10}\to 8'+8''$ | $P\bar{\Delta}\to \bar{B}P'$ | $(u)$ |

$$\begin{bmatrix} A_s(8_s) \\ A_s(8_A) \\ A_s(10) \\ A_s(27) \end{bmatrix} = \begin{bmatrix} 2/5 & (1/5)(5^{1/2}) & (1/4)(2^{1/2}) & 27/20 \\ (1/5)(5^{1/2}) & 0 & (1/4)(10^{1/2}) & -(9/20)(5^{1/2}) \\ (1/5)(2^{1/2}) & (1/5)(10^{1/2}) & -1/2 & -(9/20)(2^{1/2}) \\ 2/5 & -(2/15)(5^{1/2}) & -(1/6)(2^{1/2}) & 1/10 \end{bmatrix} \begin{bmatrix} A_t(8_s) \\ A_t(8_A) \\ A_t(10) \\ A_t(27) \end{bmatrix}$$

$$= \begin{bmatrix} -2/5 & (1/5)(5^{1/2}) & -(1/4)(2^{1/2}) & 27/20 \\ (1/5)(5^{1/2}) & 0 & (1/4)(10^{1/2}) & (9/20)(5^{1/2}) \\ (1/5)(2^{1/2}) & -(1/5)(10^{1/2}) & -1/2 & (9/20)(2^{1/2}) \\ -2/5 & -(2/15)(5^{1/2}) & (1/6)(2^{1/2}) & 1/10 \end{bmatrix} \begin{bmatrix} A_u(8_s) \\ A_u(8_A) \\ A_u(\overline{10}) \\ A_u(27) \end{bmatrix};$$

$$X_{ts}=X_{st};$$

$$X_{tu}=\begin{bmatrix} -2/5 & -(1/5)(5^{1/2}) & (1/4)(2^{1/2}) & 27/20 \\ (1/5)(5^{1/2}) & 0 & -(1/4)(10^{1/2}) & (9/20)(5^{1/2}) \\ (1/5)(2^{1/2}) & (1/5)(10^{1/2}) & 1/2 & (9/20)(2^{1/2}) \\ -2/5 & (2/15)(5^{1/2}) & -(1/6)(2^{1/2}) & 1/10 \end{bmatrix};$$

$$X_{ut}=\begin{bmatrix} -2/5 & (1/5)(5^{1/2}) & (1/4)(2^{1/2}) & -27/20 \\ -(1/5)(5^{1/2}) & 0 & (1/4)(10^{1/2}) & (9/20)(5^{1/2}) \\ (1/5)(2^{1/2}) & -(1/5)(10^{1/2}) & 1/2 & -(9/20)(2^{1/2}) \\ 2/5 & (2/15)(5^{1/2}) & (1/6)(2^{1/2}) & 1/10 \end{bmatrix};$$

$$X_{us}=\begin{bmatrix} -2/5 & (1/5)(5^{1/2}) & (1/4)(2^{1/2}) & -27/20 \\ (1/5)(5^{1/2}) & 0 & -(1/4)(10^{1/2}) & -(9/20)(5^{1/2}) \\ -(1/5)(2^{1/2}) & (1/5)(10^{1/2}) & -1/2 & (9/20)(2^{1/2}) \\ 2/5 & (2/15)(5^{1/2}) & (1/6)(2^{1/2}) & 1/10 \end{bmatrix}.$$

---

| | | |
|---|---|---|
| $8+8'\to 10+10'$ | $BB'\to\Delta\Delta'$ | $(s)$ |
| $8+\overline{10}\to 8'+10'$ | $B\bar{\Delta}\to\bar{B}'\Delta'$ | $(t)$ |
| $8+\overline{10}'\to 8'+10$ | $B\bar{\Delta}'\to\bar{B}'\Delta$ | $(u)$ |

$$\begin{bmatrix} A_s(\overline{10}) \\ A_s(27) \end{bmatrix} = \begin{bmatrix} (2/5)(2^{1/2}) & (9/10)(2^{1/2}) \\ (2/15)(10^{1/2}) & -(1/5)(10^{1/2}) \end{bmatrix} \begin{bmatrix} A_t(8) \\ A_t(27) \end{bmatrix} = \begin{bmatrix} -(2/5)(2^{1/2}) & -(9/10)(2^{1/2}) \\ (2/15)(10^{1/2}) & -(1/5)(10^{1/2}) \end{bmatrix} \begin{bmatrix} A_u(8) \\ A_u(27) \end{bmatrix};$$

$$X_{ts}=\begin{bmatrix} (1/2)(2^{1/2}) & (9/20)(10^{1/2}) \\ (1/3)(2^{1/2}) & -(1/5)(10^{1/2}) \end{bmatrix}; \qquad X_{tu}=X_{ut}=\begin{bmatrix} 1/5 & -9/5 \\ -8/15 & -1/5 \end{bmatrix};$$

$$X_{us}=\begin{bmatrix} -(1/2)(2^{1/2}) & (9/20)(10^{1/2}) \\ -(1/3)(2^{1/2}) & -(1/5)(10^{1/2}) \end{bmatrix}.$$

---

| | | |
|---|---|---|
| $8+10\to 8'+10'$ | $B\Delta\to B'\Delta'$ | $(s)$ |
| $8+8'\to\overline{10}+10'$ | $B\bar{B}'\to\bar{\Delta}\Delta'$ | $(t)$ |
| $8+\overline{10}'\to\overline{10}+8'$ | $B\bar{\Delta}'\to\bar{\Delta}B'$ | $(u)$ |

$$\begin{bmatrix} A_s(8) \\ A_s(10) \\ A_s(27) \\ A_s(35) \end{bmatrix} = \begin{bmatrix} (1/20)(5^{1/2}) & (1/5)(2^{1/2}) & (1/5)(10^{1/2}) & (9/20)(7^{1/2}) \\ (1/20)(5^{1/2}) & (3/10)(2^{1/2}) & (1/10)(10^{1/2}) & -(9/20)(7^{1/2}) \\ (1/20)(5^{1/2}) & -(3/10)(2^{1/2}) & (1/30)(10^{1/2}) & -(1/20)(7^{1/2}) \\ (1/20)(5^{1/2}) & (1/10)(2^{1/2}) & -(1/10)(10^{1/2}) & (9/140)(7^{1/2}) \end{bmatrix} \begin{bmatrix} A_t(1) \\ A_t(8_s) \\ A_t(8_A) \\ A_t(27) \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & 1/2 & 9/20 & 7/4 \\ -2/5 & -3/4 & 9/40 & 7/8 \\ -2/15 & 1/12 & -37/40 & 7/24 \\ 2/5 & -1/4 & -9/40 & 1/8 \end{bmatrix} \begin{bmatrix} A_u(8) \\ A_u(\overline{10}) \\ A_u(27) \\ A_u(\overline{35}) \end{bmatrix};$$

$$X_{ts}=\begin{bmatrix} (2/5)(5^{1/2}) & (1/2)(5^{1/2}) & (27/20)(5^{1/2}) & (7/4)(5^{1/2}) \\ (1/5)(2^{1/2}) & (3/8)(2^{1/2}) & -(81/80)(2^{1/2}) & (7/16)(2^{1/2}) \\ (1/5)(10^{1/2}) & (1/8)(10^{1/2}) & (9/80)(10^{1/2}) & -(7/16)(10^{1/2}) \\ (2/15)(7^{1/2}) & -(1/6)(7^{1/2}) & -(1/20)(7^{1/2}) & (1/12)(7^{1/2}) \end{bmatrix};$$

$$X_{tu}=\begin{bmatrix} (2/5)(5^{1/2}) & -(1/2)(5^{1/2}) & -(27/20)(5^{1/2}) & (7/4)(5^{1/2}) \\ (1/5)(2^{1/2}) & -(3/8)(2^{1/2}) & (81/80)(2^{1/2}) & (7/16)(2^{1/2}) \\ -(1/5)(10^{1/2}) & (1/8)(10^{1/2}) & (9/80)(10^{1/2}) & (7/16)(10^{1/2}) \\ (2/15)(7^{1/2}) & (1/6)(7^{1/2}) & (1/20)(7^{1/2}) & (1/12)(7^{1/2}) \end{bmatrix};$$

$$X_{ut}=\begin{bmatrix} (1/20)(5^{1/2}) & (1/5)(2^{1/2}) & -(1/5)(10^{1/2}) & (9/20)(7^{1/2}) \\ -(1/20)(5^{1/2}) & -(3/10)(2^{1/2}) & (1/10)(10^{1/2}) & (9/20)(7^{1/2}) \\ -(1/20)(5^{1/2}) & (3/10)(2^{1/2}) & (1/30)(10^{1/2}) & (1/20)(7^{1/2}) \\ (1/20)(5^{1/2}) & (1/10)(2^{1/2}) & (1/10)(10^{1/2}) & (9/140)(7^{1/2}) \end{bmatrix};$$

$$X_{us}=\begin{bmatrix} 1/5 & -1/2 & -9/20 & 7/4 \\ 2/5 & -3/4 & 9/40 & -7/8 \\ 2/15 & 1/12 & -37/40 & -7/24 \\ 2/5 & 1/4 & 9/40 & 1/8 \end{bmatrix}.$$

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$$8+10\rightarrow10'+10'' \qquad B\Delta\rightarrow\Delta'\Delta'' \qquad (s)$$

$$8+\overline{10}'\rightarrow\overline{10}+10'' \qquad B\bar{\Delta}'\rightarrow\bar{\Delta}\Delta'' \qquad (t)$$

$$8+\overline{10}''\rightarrow\overline{10}+10' \qquad B\bar{\Delta}''\rightarrow\bar{\Delta}\Delta' \qquad (u)$$

$$\begin{bmatrix} A_s(27) \\ A_s(35) \end{bmatrix} = \begin{bmatrix} (2/15)(5^{1/2}) & (1/10)(70^{1/2}) \\ 2/5 & -(9/70)(14^{1/2}) \end{bmatrix} \begin{bmatrix} A_t(8) \\ A_t(27) \end{bmatrix} = \begin{bmatrix} (2/15)(5^{1/2}) & (1/10)(70^{1/2}) \\ -2/5 & (9/70)(14^{1/2}) \end{bmatrix} \begin{bmatrix} A_u(8) \\ A_u(27) \end{bmatrix};$$

$$X_{ts}=\begin{bmatrix} (9/20)(5^{1/2}) & 7/4 \\ (1/10)(70^{1/2}) & -(1/6)(14^{1/2}) \end{bmatrix}; \qquad X_{tu}=X_{ut}=\begin{bmatrix} -2/5 & (9/20)(14^{1/2}) \\ (2/15)(14^{1/2}) & 2/5 \end{bmatrix};$$

$$X_{us}=\begin{bmatrix} (9/20)(5^{1/2}) & -7/4 \\ (1/10)(70^{1/2}) & (1/6)(14^{1/2}) \end{bmatrix}.$$

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$$10+10\rightarrow10''+10''' \qquad \Delta\Delta'\rightarrow\Delta''\Delta''' \qquad (s)$$

$$10+\overline{10}''\rightarrow\overline{10}'+10''' \qquad \Delta\bar{\Delta}''\rightarrow\bar{\Delta}'\Delta''' \qquad (t)$$

$$10+\overline{10}'''\rightarrow\overline{10}'+10'' \qquad \Delta\bar{\Delta}'''\rightarrow\bar{\Delta}'\Delta'' \qquad (u)$$

$$\begin{bmatrix} A_s(\overline{10}) \\ A_s(27) \\ A_s(35) \\ A_s(28) \end{bmatrix} = \begin{bmatrix} -1/10 & -2/5 & -9/10 & -8/5 \\ -1/10 & -4/15 & -1/10 & 16/15 \\ -1/10 & 0 & 9/14 & -16/35 \\ -1/10 & 2/5 & -27/70 & 4/35 \end{bmatrix} \begin{bmatrix} A_t(1) \\ A_t(8) \\ A_t(27) \\ A_t(64) \end{bmatrix}$$

$$= \begin{bmatrix} 1/10 & 2/5 & 9/10 & 8/5 \\ -1/10 & -4/15 & -1/10 & 16/15 \\ 1/10 & 0 & -9/14 & 16/35 \\ -1/10 & 2/5 & -27/70 & 4/35 \end{bmatrix} \begin{bmatrix} A_u(1) \\ A_u(8) \\ A_u(27) \\ A_u(64) \end{bmatrix};$$

$$X_{ts} = \begin{bmatrix} -1 & -27/10 & -7/2 & -14/5 \\ -1/2 & -9/10 & 0 & 7/5 \\ -1/3 & -1/10 & 5/6 & -2/5 \\ -1/4 & 9/20 & -1/4 & 1/20 \end{bmatrix};$$

$$X_{tu} = X_{ut} = \begin{bmatrix} 1/10 & -4/5 & 27/10 & -32/5 \\ -1/10 & 3/5 & -9/10 & -8/5 \\ 1/10 & -4/15 & -47/70 & -32/105 \\ -1/10 & -1/5 & -9/70 & -1/35 \end{bmatrix};$$

$$X_{us} = \begin{bmatrix} 1 & -27/10 & 7/2 & -14/5 \\ 1/2 & -9/10 & 0 & 7/5 \\ 1/3 & -1/10 & -5/6 & -2/5 \\ 1/4 & 9/20 & 1/4 & 1/20 \end{bmatrix}.$$

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## APPENDIX

The vector coupling coefficients for the group $SU(3)$ can be obtained from those of $SU(2)$ by means of the "isoscalar factors," according to the following relation (de Swart, 1963):

$$\begin{pmatrix} \mu_a & \mu_b & \mu_{c\gamma} \\ \nu_a & \nu_b & \nu_c \end{pmatrix} = (I_a I_{3a}, I_b I_{3b}; I_a I_b, I_c I_{3c}) \times \left(\begin{array}{cc|c} \mu_a & \mu_b & \mu_{c\gamma} \\ I_a Y_a & I_b Y_b & I_c Y_c \end{array}\right). \quad \text{(A1)}$$

The isoscalar factors

$$\left(\begin{array}{cc|c} \mu_a & \mu_b & \mu_{c\gamma} \\ I_a Y_a & I_b Y_b & I_c Y_c \end{array}\right)$$

depend upon the $SU(3)$ representations, the isospins and hypercharges of the particles involved, but not on the third components of their isospin.

The isoscalar factors can be used to derive the crossing matrices for $SU(3)$ directly, without using the $SU(3)$ V–C coefficients, once the $SU(2)$ crossing matrices are known.

TABLE III. Isoscalar factors for $8\otimes 8$, $8\otimes 10$, $10\otimes 10$, $10\otimes 10^*$.

Isoscalar factors for $\{8\}\otimes\{8\}$; given are the isoscalar factors

$$\begin{pmatrix} 8 & 8 & \mu_\gamma \\ I_1 Y_1 & I_2 Y_2 & IY \end{pmatrix}$$

for the CG series $\{8\}\otimes\{8\}=\{27\}\oplus\{10\}\oplus\{10^*\}\oplus\{8\}_1\oplus\{8\}_2\oplus\{1\}$

$Y=2 \quad I=1$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| $\frac{1}{2}$, | 1; | $\frac{1}{2}$, | 1 | 1 | |

$Y=1 \quad I=\frac{3}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 | 10 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|
| $\frac{1}{2}$, | 1; | 1, | 0 | $(2^{1/2})/2$ | $-(2^{1/2})/2$ | |
| 1, | 0; | $\frac{1}{2}$, | 1 | $(2^{1/2})/2$ | $(2^{1/2})/2$ | |

$Y=1 \quad I=\frac{1}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 | $8_1$ | $8_2$ | $10^*$ | $\mu_\gamma$ |
|---|---|---|---|---|---|---|---|---|
| $\frac{1}{2}$, | 1; | 1, | 0 | $(5^{1/2})/10$ | $3(5^{1/2})/10$ | 1/2 | $-1/2$ | |
| 1, | 0; | $\frac{1}{2}$, | 1 | $-(5^{1/2})/10$ | $-3(5^{1/2})/10$ | 1/2 | $-1/2$ | |
| $\frac{1}{2}$, | 1; | 0, | 0 | $3(5^{1/2})/10$ | $-(5^{1/2})/10$ | 1/2 | 1/2 | |
| 0, | 0; | $\frac{1}{2}$, | 1 | $3(5^{1/2})/10$ | $-(5^{1/2})/10$ | $-1/2$ | $-1/2$ | |

$Y=0 \quad I=2$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| 1, | 0; | 1, | 0 | 1 | |

$Y=0 \quad I=1$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 | $8_1$ | $8_2$ | 10 | $10^*$ | $\mu_\gamma$ |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{1}{2}$, | 1; | $\frac{1}{2}$, | $-1$ | $(5^{1/2})/5$ | $-(30^{1/2})/10$ | $(6^{1/2})/6$ | $-(6^{1/2})/6$ | $(6^{1/2})/6$ | |
| $\frac{1}{2}$, | $-1$; | $\frac{1}{2}$, | 1 | $(5^{1/2})/5$ | $-(30^{1/2})/10$ | $-(6^{1/2})/6$ | $(6^{1/2})/6$ | $-(6^{1/2})/6$ | |
| 1, | 0; | 1, | 0 | 0 | 0 | $(6^{1/2})/3$ | $(6^{1/2})/6$ | $-(6^{1/2})/6$ | |
| 1, | 0; | 0, | 0 | $(30^{1/2})/10$ | $(5^{1/2})/5$ | 0 | 1/2 | 1/2 | |
| 0, | 0; | 1, | 0 | $(30^{1/2})/10$ | $(5^{1/2})/5$ | 0 | $-1/2$ | $-1/2$ | |

$Y=2 \quad I=0$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | $10^*$ | $\mu_\gamma$ |
|---|---|---|---|---|---|
| $\frac{1}{2}$, | 1; | $\frac{1}{2}$, | 1 | $-1$ | |

$Y=0 \quad I=0$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 | $8_1$ | 1 | $8_2$ | $\mu_\gamma$ |
|---|---|---|---|---|---|---|---|---|
| $\frac{1}{2}$, | 1; | $\frac{1}{2}$, | $-1$ | $(15^{1/2})/10$ | $(10^{1/2})/10$ | 1/2 | $(2^{1/2})/2$ | |
| $\frac{1}{2}$, | $-1$; | $\frac{1}{2}$, | 1 | $-(15^{1/2})/10$ | $-(10^{1/2})/10$ | $-1/2$ | $(2^{1/2})/2$ | |
| 1, | 0; | 1, | 0 | $-(10^{1/2})/20$ | $-(15^{1/2})/5$ | $(6^{1/2})/4$ | 0 | |
| 0, | 0; | 0, | 0 | $3(30^{1/2})/20$ | $-(5^{1/2})/5$ | $-(2^{1/2})/4$ | 0 | |

$Y=-1 \quad I=\frac{3}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 | $10^*$ | $\mu_\gamma$ |
|---|---|---|---|---|---|---|
| $\frac{1}{2}$, | $-1$; | 1, | 0 | $(2^{1/2})/2$ | $-(2^{1/2})/2$ | |
| 1, | 0; | $\frac{1}{2}$, | $-1$ | $(2^{1/2})/2$ | $(2^{1/2})/2$ | |

$Y=-1 \quad I=\frac{1}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 | $8_1$ | $8_2$ | 11 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|---|---|
| $\frac{1}{2}$, | $-1$; | 1, | 0 | $-(5^{1/2})/10$ | $-3(5^{1/2})/10$ | 1/2 | 1/2 | |
| 1, | 0; | $\frac{1}{2}$, | $-1$ | $(5^{1/2})/10$ | $3(5^{1/2})/10$ | 1/2 | 1/2 | |
| $\frac{1}{2}$, | $-1$; | 0, | 0 | $3(5^{1/2})/10$ | $-(5^{1/2})/10$ | $-1/2$ | 1/2 | |
| 0, | 0; | $\frac{1}{2}$, | $-1$ | $3(5^{1/2})/10$ | $-(5^{1/2})/10$ | 1/2 | $-1/2$ | |

$Y=-2 \quad I=1$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| $\frac{1}{2}$, | $-1$; | $\frac{1}{2}$, | $-1$ | 1 | |

$Y=-2 \quad I=0$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 10 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| $\frac{1}{2}$, | $-1$; | $\frac{1}{2}$, | $-1$ | 1 | |

TABLE III (*Continued*)

Isoscalar factors for $\{8\}\otimes\{10\}$; given are the isoscalar factors

$$\left(\begin{array}{cc|c} 8 & 10 & \mu_\gamma \\ I_1Y_1 & I_2Y_2 & IY \end{array}\right)$$

for the CG series $\{8\}\otimes\{10\}=\{35\}\oplus\{27\}\oplus\{10\}\oplus\{8\}$

$Y=2 \quad I=2$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 $\mu_\gamma$ |
|---|---|---|---|---|
| $\frac{1}{2}$, | 1; | $\frac{3}{2}$, | 1 | 1 |

$Y=2 \quad I=1$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 $\mu_\gamma$ |
|---|---|---|---|---|
| $\frac{1}{2}$, | 1; | $\frac{3}{2}$, | 1 | $-1$ |

$Y=1 \quad I=\frac{5}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 $\mu_\gamma$ |
|---|---|---|---|---|
| 1, | 0; | $\frac{3}{2}$, | 1 | 1 |

$Y=1 \quad I=\frac{3}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 | 27 | 10 $\mu_\gamma$ |
|---|---|---|---|---|---|---|
| 1, | 0; | $\frac{3}{2}$, | 1 | $-1/4$ | $-(5^{1/2})/4$ | $(10^{1/2})/4$ |
| 0, | 0; | $\frac{3}{2}$, | 1 | $(5^{1/2})/4$ | $-3/4$ | $-(2^{1/2})/4$ |
| $\frac{1}{2}$, | 1; | 1, | 0 | $(10^{1/2})/4$ | $(2^{1/2})/4$ | $1/2$ |

$Y=0 \quad I=0$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 | 8 $\mu_\gamma$ |
|---|---|---|---|---|---|
| 1, | 0; | 1, | 0 | $(10^{1/2})/5$ | $-(15^{1/2})/5$ |
| $\frac{1}{2}$, | 1; | $\frac{1}{2}$, | $-1$ | $-(15^{1/2})/5$ | $-(10^{1/2})/5$ |

$Y=-1 \quad I=\frac{3}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 | 27 $\mu_\gamma$ |
|---|---|---|---|---|---|
| 1, | 0; | $\frac{1}{2}$, | $-1$ | $(2^{1/2})/2$ | $(2^{1/2})/2$ |
| $\frac{1}{2}$, | $-1$; | 1, | 0 | $(2^{1/2})/2$ | $-(2^{1/2})/2$ |

$Y=-1 \quad I=\frac{1}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 | 27 | 10 | 8 $\mu_\gamma$ |
|---|---|---|---|---|---|---|---|
| 1, | 0; | $\frac{1}{2}$, | $-1$ | $-1/4$ | $-7(5^{1/2})/20$ | $(2^{1/2})/4$ | $(5^{1/2})/5$ |
| 0, | 0; | $\frac{1}{2}$, | $-1$ | $3/4$ | $-3(5^{1/2})/20$ | $(2^{1/2})/4$ | $-(5^{1/2})/5$ |
| $\frac{1}{2}$, | 1; | 0, | $-2$ | $(2^{1/2})/4$ | $3(10^{1/2})/20$ | $1/2$ | $(10^{1/2})/5$ |
| $\frac{1}{2}$, | $-1$; | 1, | 0 | $-1/2$ | $(5^{1/2})/10$ | $(2^{1/2})/2$ | $-(5^{1/2})/5$ |

$Y=1 \quad I=\frac{1}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 | 8 $\mu_\gamma$ |
|---|---|---|---|---|---|
| 1, | 0; | $\frac{3}{2}$, | 1 | $(5^{1/2})/5$ | $-2(5^{1/2})/5$ |
| $\frac{1}{2}$, | 1; | 1, | 0 | $-2(5^{1/2})/5$ | $-(5^{1/2})/5$ |

$Y=0 \quad I=2$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 | 27 $\mu_\gamma$ |
|---|---|---|---|---|---|
| 1, | 0; | 1, | 0 | $(3^{1/2})/2$ | $1/2$ |
| $\frac{1}{2}$, | $-1$; | $\frac{3}{2}$, | 1 | $1/2$ | $-(3^{1/2})/2$ |

$Y=0 \quad I=1$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 | 27 | 10 | 8 $\mu_\gamma$ |
|---|---|---|---|---|---|---|---|
| 1, | 0; | 1, | 0 | $-(3^{1/2})/6$ | $-3(5^{1/2})/10$ | $(3^{1/2})/3$ | $(30^{1/2})/15$ |
| 0, | 0; | 1, | 0 | $(2^{1/2})/2$ | $-(30^{1/2})/10$ | 0 | $-(5^{1/2})/5$ |
| $\frac{1}{2}$, | 1; | $\frac{1}{2}$, | $-1$ | $(3^{1/2})/3$ | $(5^{1/2})/5$ | $(3^{1/2})/3$ | $(30^{1/2})/15$ |
| $\frac{1}{2}$, | $-1$; | $\frac{3}{2}$, | 1 | $-(3^{1/2})/6$ | $(5^{1/2})/10$ | $(3^{1/2})/3$ | $-2(30^{1/2})/15$ |

$Y=-2 \quad I=1$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 | 27 $\mu_\gamma$ |
|---|---|---|---|---|---|
| 1, | 0; | 0, | $-2$ | $1/2$ | $(3^{1/2})/2$ |
| $\frac{1}{2}$, | $-1$; | $\frac{1}{2}$, | $-1$ | $(3^{1/2})/2$ | $-1/2$ |

$Y=-2 \quad I=0$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 | 10 $\mu_\gamma$ |
|---|---|---|---|---|---|
| 0, | 0; | 0, | $-2$ | $(2^{1/2})/2$ | $(2^{1/2})/2$ |
| $\frac{1}{2}$, | $-1$; | $\frac{1}{2}$, | $-1$ | $-(2^{1/2})/2$ | $(2^{1/2})/2$ |

$Y=-3 \quad I=\frac{1}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 $\mu_\gamma$ |
|---|---|---|---|---|
| $\frac{1}{2}$, | $-1$; | 0, | $-2$ | 1 |

TABLE III (*Continued*)

Isoscalar factors for $\{10\} \otimes \{10\}$; given are the isoscalar factors

$$\left(\begin{array}{cc|c} 10 & 10 & \mu_\gamma \\ I_1 Y_1 & I_2 Y_2 & I Y \end{array}\right)$$

for the CG series $\{10\} \otimes \{10\} = \{35\} \oplus \{28\} \oplus \{27\} \oplus \{10^*\}$

$Y=2 \quad I=3$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 28 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | $\frac{3}{2}$, | 1 | 1 | |

$Y=2 \quad I=2$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | $\frac{3}{2}$, | 1 | $-1$ | |

$Y=2 \quad I=1$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | $\frac{3}{2}$, | 1 | 1 | |

$Y=2 \quad I=0$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 10* | $\mu_\gamma$ |
|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | $\frac{3}{2}$, | 1 | $-1$ | |

$Y=1 \quad I=\frac{5}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 28 | 35 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | 1, | 0 | $(2^{1/2})/2$ | $(2^{1/2})/2$ | |
| 1, | 0; | $\frac{3}{2}$, | 1 | $(2^{1/2})/2$ | $-(2^{1/2})/2$ | |

$Y=1 \quad I=\frac{3}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 | 27 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | 1, | 0 | $-(2^{1/2})/2$ | $-(2^{1/2})/2$ | |
| 1, | 0; | $\frac{3}{2}$, | 1 | $-(2^{1/2})/2$ | $(2^{1/2})/2$ | |

$Y=1 \quad I=\frac{1}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 | 10* | $\mu_\gamma$ |
|---|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | 1, | 0 | $(2^{1/2})/2$ | $(2^{1/2})/2$ | |
| 1, | 0; | $\frac{3}{2}$, | 1 | $(2^{1/2})/2$ | $-(2^{1/2})/2$ | |

$Y=0 \quad I=2$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 28 | 35 | 27 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | $\frac{1}{2}$, | $-1$ | $(5^{1/2})/5$ | $(2^{1/2})/2$ | $(30^{1/2})/10$ | |
| $\frac{1}{2}$, | $-1$; | $\frac{3}{2}$, | 1 | $(5^{1/2})/5$ | $-(2^{1/2})/2$ | $(30^{1/2})/10$ | |
| 1, | 0; | 1, | 0 | $(15^{1/2})/5$ | 0 | $-(10^{1/2})/5$ | |

$Y=0 \quad I=1$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 | 27 | 10* | $\mu_\gamma$ |
|---|---|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | $\frac{1}{2}$, | $-1$ | $-(6^{1/2})/6$ | $-(2^{1/2})/2$ | $-(3^{1/2})/3$ | |
| $\frac{1}{2}$, | $-1$; | $\frac{3}{2}$, | 1 | $-(6^{1/2})/6$ | $(2^{1/2})/2$ | $-(3^{1/2})/3$ | |
| 1, | 0; | 1, | 0 | $-(6^{1/2})/3$ | 0 | $(3^{1/2})/3$ | |

$Y=0 \quad I=0$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 27 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| 1, | 0; | 1, | 0 | 1 | |

$Y=-1 \quad I=\frac{3}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 28 | 35 | 27 | 10* | $\mu_\gamma$ |
|---|---|---|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | 0, | $-2$ | $(5^{1/2})/10$ | 1/2 | $3(5^{1/2})/10$ | 1/2 | |
| 0, | $-2$; | $\frac{3}{2}$, | 1 | $(5^{1/2})/10$ | $-1/2$ | $3(5^{1/2})/10$ | $-1/2$ | |
| 1, | 0; | $\frac{1}{2}$, | $-1$ | $3(5^{1/2})/10$ | 1/2 | $-(5^{1/2})/10$ | $-1/2$ | |
| $\frac{1}{2}$, | $-1$; | 1, | 0 | $3(5^{1/2})/10$ | $-1/2$ | $-(5^{1/2})/10$ | 1/2 | |

TABLE III (*Continued*)

Isoscalar factors for $\{10\}\otimes\{10\}$; given are the isoscalar factors

$$\left(\begin{array}{cc|c} 10 & 10 & \mu_\gamma \\ I_1Y_1 & I_2Y_2 & IY \end{array}\right)$$

for the CG series $\{10\}\otimes\{10\}=\{35\}\oplus\{28\}\oplus\{27\}\oplus\{10^*\}$

$Y=-1\quad I=\frac{1}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 | 27 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|
| 1, | 0; | $\frac{1}{2}$, | $-1$ | $-(2^{1/2})/2$ | $-(2^{1/2})/2$ | |
| $\frac{1}{2}$, | $-1$; | 1, | 0 | $-(2^{1/2})/2$ | $(2^{1/2})/2$ | |

$Y=-2\quad I=1$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 28 | 35 | 27 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|---|
| 1, | 0; | 0, | $-2$ | $(5^{1/2})/5$ | $(2^{1/2})/2$ | $(30^{1/2})/10$ | |
| 0, | $-2$; | 1, | 0 | $(5^{1/2})/5$ | $-(2^{1/2})/2$ | $(30^{1/2})/10$ | |
| $\frac{1}{2}$, | $-1$; | $\frac{1}{2}$, | $-1$ | $(15^{1/2})/5$ | 0 | $-(10^{1/2})/5$ | |

$Y=-2\quad I=0$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 35 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| $\frac{1}{2}$, | $-1$; | $\frac{1}{2}$, | $-1$ | $-1$ | |

$Y=-3\quad I=\frac{1}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 28 | 35 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|
| $\frac{1}{2}$, | $-1$; | 0, | $-2$ | $(2^{1/2})/2$ | $(2^{1/2})/2$ | |
| 0, | $-2$; | $\frac{1}{2}$, | $-1$ | $(2^{1/2})/2$ | $-(2^{1/2})/2$ | |

$Y=-4\quad I=0$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 28 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| 0, | $-2$; | 0, | $-2$ | 1 | |

Isoscalar factors for $\{10\}\otimes\{10^*\}$; given are the isoscalar factors

$$\left(\begin{array}{cc|c} 10 & 10^* & \mu_\gamma \\ I_1Y_1 & I_2Y_2 & IY \end{array}\right)$$

for the CG series $\{10\}\otimes\{10\}^*=\{64\}\oplus\{27\}\oplus\{8\}\oplus\{1\}$

$Y=3\quad I=\frac{3}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | 0, | 2 | 1 | |

$Y=2\quad I=2$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | $\frac{1}{2}$, | 1 | 1 | |

$Y=2\quad I=1$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | 27 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | $\frac{1}{2}$, | 1 | $(21^{1/2})/7$ | $2(7^{1/2})/7$ | |
| 1, | 0; | 0, | 2 | $2(7^{1/2})/7$ | $-(21^{1/2})/7$ | |

$Y=1\quad I=\frac{5}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | 1, | 0 | 1 | |

$Y=1\quad I=\frac{3}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | 27 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | 1, | 0 | $(14^{1/2})/7$ | $(35^{1/2})/7$ | |
| 1, | 0; | $\frac{1}{2}$, | 1 | $(35^{1/2})/7$ | $-(14^{1/2})/7$ | |

$Y=1\quad I=\frac{1}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | 27 | 8 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | 1, | 0 | $(7^{1/2})/7$ | $4(35^{1/2})/35$ | $(10^{1/2})/5$ | |
| 1, | 0; | $\frac{1}{2}$, | 1 | $2(7^{1/2})/7$ | $(35^{1/2})/35$ | $-(10^{1/2})/5$ | |
| $\frac{1}{2}$, | $-1$; | 0, | 2 | $(14^{1/2})/7$ | $-3(70^{1/2})/35$ | $(5^{1/2})/5$ | |

**Table III** (*Continued*)

Isoscalar factors for $\{10\}\otimes\{10^*\}$; given are the isoscalar factors

$$\left(\begin{array}{cc|c} 10 & 10^* & \mu_\gamma \\ I_1Y_1 & I_2Y_2 & IY \end{array}\right)$$

for the CG series $\{10\}\otimes\{10\}^*=\{64\}\oplus\{27\}\oplus\{8\}\oplus\{1\}$

$Y=0\quad I=3$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | $\frac{3}{2}$, | $-1$ | 1 | |

$Y=0\quad I=2$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | 27 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | $\frac{3}{2}$, | $-1$ | $(7^{1/2})/7$ | $(42^{1/2})/7$ | |
| 1, | 0; | 1, | 0 | $(42^{1/2})/7$ | $-(7^{1/2})/7$ | |

$Y=0\quad I=1$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | 27 | 8 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | $\frac{3}{2}$, | $-1$ | $(21^{1/2})/21$ | $(14^{1/2})/7$ | $(6^{1/2})/3$ | |
| 1, | 0; | 1, | 0 | $(210^{1/2})/21$ | $3(35^{1/2})/35$ | $-2(15^{1/2})/15$ | |
| $\frac{1}{2}$, | $-1$; | $\frac{1}{2}$, | 1 | $(210^{1/2})/21$ | $-4(35^{1/2})/35$ | $(15^{1/2})/15$ | |

$Y=0\quad I=0$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | 27 | 8 | 1 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|---|---|
| $\frac{3}{2}$, | 1; | $\frac{3}{2}$, | $-1$ | $(35^{1/2})/35$ | $(210^{1/2})/35$ | $(10^{1/2})/5$ | $(10^{1/2})/5$ | |
| 1, | 0; | 1, | 0 | $2(105^{1/2})/35$ | $(70^{1/2})/14$ | 0 | $-(30^{1/2})/10$ | |
| $\frac{1}{2}$ | $-1$; | $\frac{1}{2}$, | 1 | $3(70^{1/2})/35$ | $-(105^{1/2})/35$ | $-(5^{1/2})/5$ | $(5^{1/2})/5$ | |
| 0, | $-2$; | 0, | 2 | $2(35^{1/2})/35$ | $-3(210^{1/2})/70$ | $(10^{1/2})/5$ | $-(10^{1/2})/10$ | |

$Y=-1\quad I=\frac{5}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| 1, | 0; | $\frac{3}{2}$, | $-1$ | 1 | |

$Y=-1\quad I=\frac{3}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | 27 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|
| 1, | 0; | $\frac{3}{2}$, | $-1$ | $(14^{1/2})/7$ | $(35^{1/2})/7$ | |
| $\frac{1}{2}$, | $-1$; | 1, | 0 | $(35^{1/2})/7$ | $-(14^{1/2})/7$ | |

$Y=-1\quad I=\frac{1}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | 27 | 8 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|---|
| 1, | 0; | $\frac{3}{2}$, | $-1$ | $(7^{1/2})/7$ | $4(35^{1/2})/35$ | $(10^{1/2})/5$ | |
| $\frac{1}{2}$, | $-1$; | 1, | 0 | $2(7^{1/2})/7$ | $(35^{1/2})/35$ | $-(10^{1/2})/5$ | |
| 0, | $-2$; | $\frac{1}{2}$, | 1 | $(14^{1/2})/7$ | $-3(70^{1/2})/35$ | $(5^{1/2})/5$ | |

$Y=-2\quad I=2$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| $\frac{1}{2}$, | $-1$; | $\frac{3}{2}$, | $-1$ | 1 | |

$Y=-2\quad I=1$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | 27 | $\mu_\gamma$ |
|---|---|---|---|---|---|---|
| $\frac{1}{2}$, | $-1$; | $\frac{3}{2}$, | $-1$ | $(21^{1/2})/7$ | $2(7^{1/2})/7$ | |
| 0, | $-2$; | 1, | 0 | $2(7^{1/2})/7$ | $-(21^{1/2})/7$ | |

$Y=-3\quad I=\frac{3}{2}$

| $I_1$, | $Y_1$; | $I_2$, | $Y_2$ | 64 | $\mu_\gamma$ |
|---|---|---|---|---|---|
| 0, | $-2$; | $\frac{3}{2}$, | $-1$ | 1 | |

Indeed, take any reaction $AB \to CD$ ($s$ channel) with corresponding $t$ channel $A\bar{C} \to \bar{B}D$. Let the amplitudes for the reaction to occur in a state of total isospin $I$ be $A(I)$, and the $SU(3)$ eigenamplitudes be $A(\mu_\gamma)$. We have

$$A_s(I) = \sum_{\mu_\gamma} \left( \begin{array}{cc|c} \mu_A & \mu_B & \mu_\gamma \\ I_A Y_A & I_B Y_B & I Y_A + Y_B \end{array} \right) \times \left( \begin{array}{cc|c} \mu_C & \mu_D & \mu_\gamma \\ I_C Y_C & I_D Y_D & I Y_C + Y_D \end{array} \right) A_s(\mu_\gamma) \quad \text{(A2)}$$

with an analogous equation holding for the $t$-channel amplitudes.

On the other hand, $A_s(I)$ and $A_t(I)$ are related by Eq. (11):

$$A_s(I) = \sum_{I'} (X_{st})_{I,I'} A_t(I').$$

The various relations that can be obtained from Eqs. (A2) and (11) by a suitable choice of the external particles can finally be used to express the $SU(3)$ amplitudes $A_s(\mu_\gamma)$ by $A_t(\mu_\gamma)$, i.e., to derive the $SU(3)$ crossing matrix.

A word of caution, however: The $SU(2)$ phase factors $(-)^{I_3}$ (integer isospin) or $(-)^{I+I_3}$ (half-integer isospin) do not always coincide with the $SU(3)$ phase $(-)^Q = (-)^{I_3 + (Y/2)}$. Therefore, if the isoscalar coefficients are used to derive the $SU(3)$ crossing matrices, a phase $-1$ should be added wherever one of the following particles is crossed: $\Delta$, $\bar{N}$, $\Xi$, $\Omega$, $\bar{\Omega}$.

For convenience of the reader, we reproduce in Table III the isoscalar factors for $8\otimes 8$, $8\otimes 10$, $10\otimes 10$, $10\otimes \overline{10}$.* Other isoscalar factors can be obtained by using the identities:

$$\left( \begin{array}{cc|c} \mu_1 & \mu_2 & \mu_\gamma \\ I_1 Y_1 & I_2 Y_2 & I Y \end{array} \right) = \xi_1 (-)^{I_1+I_2-I} \left( \begin{array}{cc|c} \mu_2 & \mu_1 & \mu_\gamma \\ I_2 Y_2 & I_1 Y_1 & I Y \end{array} \right), \quad \text{(A3)}$$

$$\left( \begin{array}{cc|c} \mu_1 & \mu_2 & \mu_\gamma \\ I_1 Y_1 & I_2 Y_2 & I Y \end{array} \right) = \xi_3 (-)^{I_1+I_2-I} \left( \begin{array}{cc|c} \mu_1^* & \mu_2^* & \mu_\gamma^* \\ I_1 - Y_1 & I_2 - Y_2 & I_3 - Y_3 \end{array} \right). \quad \text{(A4)}$$

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