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Charges and Generators of Symmetry Transformations in Quantum Field Theory*

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Within the Wightman approach to quantum field theory, we review and clarify the properties of formal charges, defined as space integrals for the fourth component of a local current. The conditions for a formal charge to determine an operator (generator) are discussed, in connection with the well-known theorems of Goldstone and of Coleman. The symmetry transformations generated by this operator—given its existence—are also studied in some detail. For generators in a scattering theory, we prove their additivity and thus provide a simple way to characterize them from their matrix elements between one-particle states. This characterization allows an immediate construction of the unitary operators implementing the symmetry transformations, and implies that all internal symmetry groups are necessarily compact. We also indicate how to construct interacting fields having definite internal quantum numbers. The present status of the proof of Noether's theorem and of its converse is discussed in the light of the rather delicate properties of formal charges.

CONTENTS

1. INTRODUCTION

Following the results of Goldstone *et al.* (Goldstone, 1961; Goldstone, Salam, and Weinberg, 1962) and of Coleman (1966), in recent years there has been a continual interest in the properties of formal charges. A formal charge Q is defined here as the space integral of the zeroth component of a local four-vector current:

$$Q(x_0)=\int d\mathbf{x}j_0(x). \tag{1.1}$$

Quantities of this kind appear in the discussion of symmetries and broken symmetries in quantum field theory, and are one of the basic tools in the modern "current-algebraic" approach to elementary particle physics (Gell-Mann, 1962; Adler and Dashen, 1968).

It has been repeatedly emphasized (Kastler, Robinson, and Swieca, 1966; Schroer and Stichel, 1966; Dell'Antonio, 1967; Swieca, 1966; Katz, 1966; Fabri and Picasso, 1966; Fabri, Picasso, and Strocchi, 1967; and De Mottoni, 1967) that equations of the type (1.1) have rather delicate convergence properties, and that a certain care has to be exercised when considering such expressions. This fact limits the extent to which Q can be thought of as a generator of symmetry or broken symmetry transformations.[1] The same convergence properties are at the basis of Goldstone's theorem (Goldstone, 1961; Goldstone, Salam and Weinberg, 1962; Kastler, Robinson and Swieca, 1966), and of Coleman's theorems (Coleman, 1965 and 1966; Pohlmeyer, 1966; Schroer and Stichel, 1966; and

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[1] The nomenclature as well as the mentioned restrictions will be clarified later on. For present purposes, a generator of symmetry transformations is to be identified with a self-adjoint operator which commutes with **P**, the momentum operator, and commutes with the S matrix. A conserved current leads to an exact symmetry if the associated charge is a generator of symmetry transformations. Spontaneously broken symmetries occur when current conservation does *not* imply the existence of a symmetry. Intrinsically broken symmetries arise when the current is not conserved.

Dell'Antonio, 1967), which impose severe restrictions on the conditions for having a spontaneously broken symmetry and, respectively, on the formulation of theories exhibiting intrinsically broken symmetries.

The main purpose of this paper is to review and clarify the properties of formal charges and to provide a unified treatment of the above-mentioned problems.

The content of the following sections can be summarized as follows.

In Sec. 2, we outline the main criticisms which can be held against the existing proofs of a quantum Noether's theorem, state the problems of our concern somewhat more precisely, and give a few basic definitions and theorems.

In Sec. 3 we state, comment on, and finally prove some relevant properties of formal charges. The most important outcome of this section is the emphasis on the fact that a formal charge Q is *always* a divergent quantity and determines an operator G if and only if the current is conserved. Even then, the connection between a formal charge and its associated operator is rather indirect, and one cannot, strictly speaking, identify Q with G. We also discuss the extent to which these results are independent of the particular definition one adopts for the formal charge.

In Sec. 4 we discuss a constructive definition of the operator associated with a formal charge. This definition is consistent provided no Goldstone-type phenomena are present. After a brief digression on the Goldstone theorem, we discuss the construction of the symmetry transformations generated by a formal charge.

In Sec. 5, we take a more practical attitude and describe a simple way to characterize a formal charge from its matrix elements between one-particle states in a scattering theory. This characterization corresponds to the usual textbook expression of a charge in terms of creation and annihilation operators, an expression which is thus derived in a rigorous way. This allows for a convenient way of characterizing the symmetry transformations generated by the formal charge. We then show how it is possible to construct interpolating fields having the same internal quantum numbers as those of the corresponding asymptotic free fields.

In the final section we summarize and discuss the main results.

Apart from the style of the presentation, several scattered remarks, and the above-mentioned construction of Sec. 5, not many new results are derived here. Thus, one of the main intents of the paper is to provide a pedagogically useful review of the problems outlined above.

2. PRELIMINARIES

A. Some Preliminary Remarks on the Quantal Noether Theorem

Let $j_\mu(x)$ be a local current; its associated "formal charge" $Q(x_0)$ is defined as the expression

$$Q(x_0)=\int j_0(x)\,d\mathbf{x}. \qquad (2.1)$$

(At this stage, we do not worry about the meaning of this definition.)

The importance of formal charges is most easily understood by considering the case of a Hermitian conserved current j_μ in a Lagrangian field theory. According to the usual arguments, Q is actually time independent, and the unitary transformations

$$U(\tau)=\exp[iQ\tau], \qquad \tau \text{ real} \qquad (2.2)$$

are symmetries of the theory. Thus, *the formal charge associated with a conserved current is the generator of a one-parameter continuous Abelian group of symmetry transformations*. In the following, this statement will be referred to as "*the converse of Noether's Theorem.*"

Noether's theorem itself states that, in a theory described by a given Lagrangian $\mathcal{L}$, if the action integral is invariant under a continuous group of point transformations in the field variables, then these transformations are symmetries of the theory—described by unitary operators—and the corresponding generators are of the form (2.1), with j_μ a conserved current. Furthermore, the explicit (formal) expression for j_μ can also be specified once one is given $\mathcal{L}$ and the infinitesimal transformation properties of the fields.

In summary, Noether's theorem and its converse provide the bridge between global invariance properties and local conservation laws. Since the idea of locality is one of the main ingredients of quantum field theory, it is of vital importance to have a solution to the problem of characterizing symmetries which are associated with local conservation laws and which act locally on the fields. In view of this, we need not emphasize the great importance of Noether-type theorems.

We assume that the reader is familiar with the usual (textbook) discussions of Noether's theorem,[2] and thus omit its standard proof. We do, however, make the following comments so as to emphasize why such standard proofs *cannot* be accepted as satisfactory:

(i) In quantum field theory, Lagrangians are no longer seen as fundamental objects. Rather, they are used as a tool to generate a perturbative expansion of the S matrix.

(ii) The usual heuristic forms of Lagrangians involve ill-defined products of quantized fields at the same space–time point. This, among other things, usually makes the action integral an ill-defined quantity.

(iii) The existing proofs of Noether's theorem involve the use of formal operations, such as functional differentiation of operators. These operations are particularly problematic when applied to already ill-defined expressions involving products of fields at the same space–time point.

Thus, a rigorous formulation of a quantal action principle is still lacking. Furthermore, the following additional difficulties undermine the usual treatments of charge operators.

[2] See, e.g., Hill (1951), Schroeder (1968), and Bogoliubov and Shirkov (1959) for more complete treatments.

(iv) Canonical commutation relations (CCR's) are usually invoked in order to determine the commutation relations (CR's) between a charge and a field (this is done in order to verify that the charge generates the right transformations on the field variables). CCR's are not necessarily valid for interacting fields. Furthermore, the formal charge involves nonlinear expressions in the fields at the same space–time point; such expressions might violate associative-type laws[3] for products appearing in commutators, thus preventing the use of CCR's in order to deduce the above-mentioned transformation properties.

(v) While a classical current density $\rho(x)$ is required to vanish as $|\mathbf{x}|\to\infty$, and thus a classical charge is usually well-defined, the same property breaks down for a quantal current density. Indeed, one of our tasks will be to prove and discuss the lack of convergence of $\int_V j_0(x)\,d\mathbf{x}$ as the volume V tends to ∞, for a general current operator j_μ.

A typical consequence of this state of affairs can be recognized in the modern formulation of renormalized perturbation theory in terms of local field equations (Wilson, 1965; Brandt, 1967; 1969; Zimmermann, 1967). In the example of quantum electrodynamics, the electromagnetic current is constructed from the basic (ill-defined) Noether current $\bar{\Psi}\gamma_\mu\Psi$ by adding suitable (infinite) counterterms which make the total expression meaningful. In this construction, the local conservation of the renormalized current is heavily used in order to determine the form of the counterterms. In other words, the original Lagrangian $\mathcal{L}$ and the associated Noether current $\bar{\Psi}\gamma_\mu\Psi$ are only used as a general guide in order to construct a meaningful current from the renormalized perturbation expansion dictated by $\mathcal{L}$ and $\bar{\Psi}\gamma_\mu\Psi$.

A more serious difficulty seems to arise in γ_5-invariant spinor electrodynamics, where one is unable, in perturbation theory, to construct a gauge-invariant local conserved current such that the associated charge generates the unitaries implementing the γ_5 invariance (Adler, 1969; Brandt, 1969a). To the extent that perturbation theory is a guide, this indicates the possible existence of counterexamples to Noether's theorem.

These examples and remarks (i)–(v) are mentioned only so as to support our belief that a rigorous and clear understanding of the Noether problem is lacking in a quantum field–theoretic framework. In rigorous quantum field theory, where formal Lagrangians and formal manipulations are excluded from the game, any attempt at seriously investigating the Noether problem faces prohibitive difficulties. While we shall return to this point, *we will devote most of our attention to the converse of Noether's theorem.* Thus, given a local current, we will study the properties of the associated formal charge: this is clearly a necessary first step in tackling the problem at hand.

[3] Thus, e.g., for a free spinor field Ψ, the electric charge is $Q\sim\int d\mathbf{x}:\Psi^\dagger(x)\Psi(x):$. One then calculates $[Q,\Psi]$ by using the associative-type law $[AB,C]=A[C,B]_+-[A,C]_+B$ and the canonical anticommutation relations. This associativity, while valid for ordinary products, may break down for quantities such as $:\Psi^\dagger(x)\Psi(x):$, which are not ordinary products but suitably defined limits.

B. Some Basic Definitions and Theorems

We now provide some basic definitions and theorems which will be used in the following sections. The content of this section is included only as a convenient reminder; it cannot be a substitute for the existing excellent treatments of the general theory of quantized fields (see e.g., Streater and Wightman, 1964; and Jost, 1965).

We will work in a Wightman field theory (Streater and Wightman, 1964) determined by a set $\{\phi_i(x)\}_{i=1}^J$ of local and relatively local fields. "Locality"[4] for a field ϕ_i means that

$$[\phi_i(x),\phi_i(y)]=0 \quad \text{for}^5\; x\sim y, \tag{2.3}$$

while "relative locality" of ϕ_i relative to ϕ_j means that

$$[\phi_i(x),\phi_j(y)]=0 \quad \text{for } x\sim y. \tag{2.4}$$

The fields ϕ_i are fields in the sense of Wightman. In particular, the ϕ_j's do not necessarily create one-particle states from the vacuum (e.g., for some i, ϕ_i might be a current operator). The vacuum $|0\rangle$ is defined as a Poincaré-invariant state and is assumed to be unique.

We recall that a (*test*) function f is of class $\mathcal{S}$ if it is C^∞ (i.e., continuous together with all its derivatives) and f as well as all its derivatives are fast decreasing;[6] f is of class $\mathcal{D}$ if it is C^∞ and it vanishes (together with all its derivatives) outside of a bounded set.

A Wightman field ϕ_i is an operator-valued distribution over $\mathcal{S}$; this means that, for each $f\in\mathcal{S}$, one is given an (unbounded) operator $\phi_i(f)$ acting in the Hilbert space $\mathcal{H}$ of all physical states. The smeared field $\phi_i(f)$ is indicated by the following (formal) integral:

$$\phi_i(f)\equiv\int\phi_i(x)f(x)\,dx. \tag{2.5}$$

[4] Also called "local commutativity".

[5] Notation: $x\sim y\leftrightarrow(x-y)^2<0$ in our metric, i.e., $g_{\mu\nu}=0$ for $\mu\neq\nu$, $g_{00}=1=-g_{kk}$, $k=1, 2, 3$. Thus, $x\sim y$ means "x spacelike relative to y." If, for given i, j, ϕ_i and ϕ_j are fields with half-integer spin, the anticommutator should be substituted for the commutator in Eqs. (2.3) and (2.4). We choose the natural units $c=\hbar=1$. We will indicate by $|x|$ the Euclidean norm $x_0^2+|\mathbf{x}|^2$ of x.

[6] More precisely, for a function f of n real variables, $f\in\mathcal{S}$ if it is C^∞ and it satisfies, for all $r, s<\infty$,

$$\|f\|_{r,s}\equiv\sum_{|k|,k\leq r}\;\sum_{|l|,l\leq s}\;\sup_x|x^{|k|}D^{|l|}f(x)|<\infty,$$

where

$$x\equiv(x_1,\cdots,x_n),\qquad x^{|k|}\equiv x_1^{k_1}\cdots x_1^{k_n},\qquad k\equiv k_1+\cdots+k_n,\qquad k_i\geq0,$$

and

$$D^{|l|}=\partial^l/(\partial x_1)^{l_1}\cdots(\partial x_n)^{l_n}].$$

When needed, we write $f\in\mathcal{S}(\mathbf{R}^n)$ for a function of class $\mathcal{S}$ depending on n real variables. $\mathbf{R}^n$ is the Euclidean space in n dimensions. If $f\in\mathcal{S}$, then also $\partial_\mu f\in\mathcal{S}$. This allows to define, for a distribution φ, its derivative $\partial_\mu\phi$ by $(\partial_u\phi)\;(f)\equiv-\varphi(\partial_u f)$, $\forall f\in\mathcal{S}$. Thus, distributions are always differentiable and their derivatives are also distributions. For a Wightman field, it is easily shown that its derivatives are also Wightman fields.

Let $\mathfrak{D}$ be a common domain of vectors in $\mathfrak{H}$ where the $\phi_i(f)$ are defined, for all $i=1, 2, \cdots, J$ and all $f\in \mathcal{S}$; $\mathfrak{D}$ is assumed to be dense in $\mathfrak{H}$, to contain the vacuum $|\,\mathbf{0}\rangle$ and to be stable upon application of all $\phi_i(f)$: $\phi_i(f)\mathfrak{D}\subseteq\mathfrak{D}$.

Concerning the vacuum $|\,\mathbf{0}\rangle$, it is assumed to be *cyclic* with respect to the given set of fields $\{\phi_i\}_{i=1}{}^J$; this means that the set of all states obtained by applying polynomials in the smeared fields on $|\,\mathbf{0}\rangle$, is *dense* in $\mathfrak{H}$.

In a theory with non-negative mass spectrum, it can be shown that, for a set of fields ϕ_i, $i=1, \cdots, J$, satisfying the usual properties, uniqueness and cyclicity of $|\,\mathbf{0}\rangle$ imply that the set $\{\phi_i\}_{i=1}{}^J$ is *irreducible*, i.e., that any operator which commutes with all bounded functions of the $\{\phi_i(f)\}$ (for all i, and $f\in\mathcal{S}$) is a constant multiple of the identity operator.[7]

If $f(x_1, \cdots, x_n)$ is of class $\mathcal{S}$ in all variables [i.e., $f\in\mathcal{S}(\mathbf{R}^{4n})$], one can show that f can be approximated arbitrarily well by sums of products

$$\prod_{i=1}^{n} f_i(x_i)$$

of functions in $\mathcal{S}(\mathbf{R}^4)$.[8] This allows the consideration of states of the form

$$|\,\mathbf{\Psi}\rangle = A_\Psi\,|\,\mathbf{0}\rangle, \tag{2.6a}$$

where

$$A_\Psi=\sum_{m=0}^{M}\int\prod_{j=1}^{m} dx_j g_m(x_1, \cdots, x_m)\phi_{i1}(x_1)\cdots\phi_{i_m}(x_m). \tag{2.6b}$$

Here the term $m=0$ is defined as proportional to the identity, and the g_m's, for $m=1, 2, \cdots, M$, are of class $\mathcal{S}$ or of class $\mathfrak{D}$. States (operators) of the form (2.6a) [respectively, (2.6b)], with g of class $\mathcal{S}$ and $\mathfrak{D}$ are called, respectively, *quasilocal states* (respectively, *quasilocal operators*) and *strictly localized* or *local states* (respectively, *localized operators*).[9]

The sets $\mathfrak{D}_{qL}$ and $\mathfrak{D}_L$ of all quasilocal and local states are *dense sets in* $\mathfrak{H}$.

Given a local or quasilocal operator A_Ψ and the corresponding state $|\,\mathbf{\Psi}\rangle$, we can translate them by using the translation operator $T(x)$:

$$A_\Psi\to A_\Psi(x)\equiv T(x)A_\Psi T^\dagger(x), \tag{2.7a}$$

$$|\,\mathbf{\Psi}\rangle\to T(x)\,|\,\mathbf{\Psi}\rangle = A_\Psi(x)\,|\,\mathbf{0}\rangle. \tag{2.7b}$$

We can also consider superpositions of these operators and states, such as

$$B_\Phi\equiv\int dxh(x)A_\Psi(x), \tag{2.8a}$$

$$|\,\mathbf{\Phi}\rangle = B_\Phi\,|\,\mathbf{0}\rangle, \tag{2.8b}$$

for some suitable averaging function $h(x)$.

We call the operator B_Φ and the corresponding $|\,\mathbf{\Phi}\rangle$ *quasilocal of order* N $(N\geq 0)$ if

$$\lim_{|\mathbf{x}|\to\infty}\,|\,x\,|^N\,|\,h(x)\,| = 0.^{10} \tag{2.9}$$

Besides these definitions, we need the following preparatory important theorems:[11]

THEOREM 2.1 (Reeh and Schlieder, 1961). The set $\mathfrak{D}_L(\mathcal{O})$ of all states localized in a bounded open space–time region $\mathcal{O}$ is dense in $\mathfrak{H}$.

This theorem has the following important corollary:

Corollary 2.1 (Schroer, 1958; Jost, 1959; and Federbush and Johnson, 1960). Let $\phi(x)$ be a local Wightman field. The following two conditions are equivalent:

(i) $\quad\phi(x)\,|\,\mathbf{0}\rangle = 0,$

(ii) $\quad\phi(x) = 0.$

Proof: Choose a space–time region $\mathcal{O}$ which is totally spacelike relative to the point x. If A is any operator localized in $\mathcal{O}$, by local commutativity we have $[A, \phi(x)]=0$. Upon applying this commutator to the vacuum, from (i) we obtain $\phi(x)A\,|\,\mathbf{0}\rangle=0$. But the set of states $\{A\,|\,\mathbf{0}\rangle\}$, with A localized in $\mathcal{O}$, is dense in $\mathfrak{H}$; thus, $\phi(x)$ annihilates a dense set of states, and it must be identically zero.

THEOREM 2.2[12] (Borchers, 1964). A Wightman field $\phi(x)$ is a distribution in the time variable, and a C^∞ function in the space variables. By this we mean that the vector

$$\phi(\mathbf{x}, g)\,|\,\mathbf{\Psi}\rangle\equiv\int dx_0\phi(\mathbf{x}, x_0)g(x_0)\,|\,\mathbf{\Psi}\rangle, \tag{2.10}$$

for $|\,\mathbf{\Psi}\rangle\in\mathfrak{D}_{qL}$, $g\in\mathcal{S}$ is a C^∞ function in $\mathbf{x}$, so that the matrix element

$$\langle\mathbf{\Phi}\,|\,\phi(\mathbf{x}, g)\,|\,\mathbf{\Psi}\rangle \tag{2.11}$$

is a C^∞ function in $\mathbf{x}$ for any $|\,\mathbf{\Phi}\rangle\in\mathfrak{H}$.

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[7] A standard definition of irreducibility for a set $\{A_i\}$ of bounded operators is: if B is any bounded operator commuting with all the A_i, then B is a constant multiple of the identity. Here, because the $\phi_i(f)$ are unbounded and given only on the domain $\mathfrak{D}$, we replace this definition with the one given by Ruelle: $\{\phi_i\}_{i=1}{}^J$ is *irreducible* if any bounded operator B satisfying

$$\langle B^\dagger\mathbf{\Phi}\,|\,\phi_i(f)\mathbf{\Psi}\rangle = \langle\phi_i{}^\dagger(f)\mathbf{\Phi}\,|\,B\mathbf{\Psi}\rangle,\quad\forall f\in\mathcal{S},\ \ \forall\,|\,\mathbf{\Phi}\rangle,\ \ |\,\mathbf{\Psi}\rangle\in\mathfrak{D},$$

is a constant multiple of the identity. The irreducibility of the fields gives a precise meaning to the somewhat vague idea that $\{\phi_i\}_{i=1}{}^J$ determines the theory.

[8] Convergence (and thus approximation) in $\mathcal{S}$ is defined as follows: $\{f_n\}_1{}^\infty$, $f_n\in\mathcal{S}$ is a Cauchy sequence if a finite $n(\epsilon)$ exists for any given $\epsilon>0$, such that, for $m>n>n(\epsilon)$, $\|f_n-f_m\|_{r,s}<\epsilon$ for all r, s. A sequence $\{f_n\}_1{}^\infty$ in $\mathcal{S}$ has a limit point f in $\mathcal{S}$, and we write $f_n\to f$ in $\mathcal{S}$, when $\|f_n-f\|_{r,s}\to_{n\to\infty}0$ for all r, s. If $f_n\to f$ in $\mathcal{S}$ and φ is a distribution, then $\varphi(f_n)\to\varphi(f)$. This implies that, given states of the form, for $|\,\chi\rangle\in\mathfrak{D}$,

$$|\,\mathbf{\Phi}_n\rangle = \prod_{i=1}^{m}\phi_{ji}(f_{n;i})|\,\chi\rangle,\ \text{if}\ \prod_{i=1}^{m} f_{n;i}\underset{n\to\infty}{\longrightarrow} g\in\mathcal{S}(\mathbf{R}^{4n}),$$

then $|\,\mathbf{\Phi}_n\rangle$ converges to a vector $|\,\mathbf{\Phi}\rangle\in\mathfrak{H}$.

[9] The reader should not confuse between "localization" and "local commutativity" [expressed by Eqs. (2.3)–(2.4)]. See also Footnote 11 and Sec. 4.

[10] Thus, one sees that quasilocal states are quasilocal of infinite order.

[11] One says that an operator of the type (2.6b) is localized in a region $\mathcal{O}$ if all g_m's vanish for $x_i\notin\mathcal{O}$, $i=1, 2, \cdots, m$. Operators which are localized in regions which are totally spacelike separated commute with each other as a consequence of Eqs. (2.3) and (2.4).

[12] This theorem is also valid for nontempered fields, i.e., fields which are operator-valued distributions over $\mathfrak{D}$. See, e.g., Dell'Antonio (1967).

We now introduce an assumption on the mass spectrum of our theory:

Assumption (*"Mass gap"*) Denoting by P_μ the energy–momentum operators, the spectrum of the operator $P_\mu P^\mu$ is in $V_m^+ \equiv \{p: p_0>0,\ p^\mu p_\mu \geq m^2>0\}$, with the exception of the point $p=0$, which corresponds to the projection onto the unique vacuum.

In other words, there are no states, excepting the vacuum, whose effective mass is less than m. Unless otherwise stated, *this "strong spectrum condition"*[13] or *"presence of a mass gap" will always be assumed*, and a field $\phi(x)$ will be adjusted so as to have

$$\langle 0 \mid \phi(x) \mid 0 \rangle = 0. \tag{2.12}$$

The usual cluster properties (Streater and Wightman, 1964) in a theory satisfying the mass-gap assumption can then be used to prove also the following property:

THEOREM 2.3 For $|\Phi\rangle$ and $|\Psi\rangle$ in $\mathfrak{D}_L$ or $\mathfrak{D}_{qL}$, as $|\mathbf{x}| \to \infty$,

$$|\mathbf{x}|^N \langle \Phi \mid \phi(\mathbf{x}, g) \mid \Psi \rangle \to 0 \tag{2.13}$$

for any N.

Combining Theorems 2.2 and 2.3, we see that the matrix element (2.11) is a function in $\mathcal{S}(\mathbf{R}^3)$ for $|\Psi\rangle$ and $|\Phi\rangle$ quasilocal.

3. PROPERTIES OF FORMAL CHARGES

A. Generalities

In this section we discuss the convergence properties of formal charges. As was emphasized under (vi) in Sec. 2.A, when defining the "charge"

$$Q(x_0) \equiv \int j_0(x)\, d\mathbf{x}, \tag{1.1}$$

we must proceed with care: the Expression (1.1) is formal and not necessarily meaningful since it corresponds to smearing $j_0(x)$ with the "function" $1 \times \delta(x_0' - x_0)$, which is certainly not expected to be a good test function for the field $j_0(x)$. This is actually the reason for our referring to (1.1) as to the "formal charge."

In order to obtain something similar to the formal charge, we consider the following expression (Kastler, Robinson, and Swieca, 1966):

$$j_0(f_R f_T) \equiv \int dx f_R(\mathbf{x}) f_T(x_0) j_0(x), \tag{3.1}$$

where

$$\begin{cases} f_R(\mathbf{x}) \in \mathcal{S}(\mathbf{R}^3), f_R(\mathbf{x}) = \begin{cases} 1 & \text{for } |\mathbf{x}| < R \\ f_R(|\mathbf{x}|) & \text{for } R \leq |\mathbf{x}| \leq R+\Lambda,\ \Lambda>0, \\ 0 & \text{for } |\mathbf{x}| > R+\Lambda \end{cases} \\ f_T(x_0) \in \mathcal{S}(\mathbf{R}^1), \begin{cases} \int f_T(x_0)\, dx_0 = 1, & f_T(-x_0) = f_T(x_0) \geq 0, \\ f_T(x_0) = 0 & \text{for } |x_0| > T. \end{cases} \end{cases} \tag{3.2}$$

In Eq. (3.1) we are "cutting" the tails of the charge for large spatial distances and averaging in time around the point $x_0=0$.[14] The formal expression (1.1) should then be understood as defined by the limit

$$\lim_{T\to 0} \lim_{R\to\infty} j_0(f_R f_T) \equiv Q \equiv Q(0), \tag{3.3}$$

where we are still proceeding formally since we are not specifying what kind of limit we are considering.[15] Our present problem is to arrive at the exact meaning of the limit (3.3): After this is understood, we shall explore the connection between formal charges and generators of groups of symmetry transformations.

[13] The *weak* spectrum condition requires only that the spectrum of $P_\mu P^\mu$ be nonnegative.

[14] For notational convenience, we are choosing $x_0=0$ in considering the analog of Eq. (1.1). Clearly, $f_R \to 1$ as $R\to\infty$, and $f_T \to \delta$ as $T\to 0$.

[15] For varying $f_R f_T$, $j_0(f_R f_T)$ provides a family of (unbounded) operators. Given a vector $|\Psi\rangle$ in the common domain of definition of these operators, one says that the given family converges *strongly* as $E\to\infty$, $T\to 0$ if $j_0(f_R f_T)|\Psi\rangle$ is a strongly convergent family of vectors (see below), while it converges *weakly* if $\langle\Phi | j_0(f_R f_T) | \Psi\rangle$ converges for any (fixed) vector $|\Phi\rangle$. We indicate strong convergence for vectors by an arrow: $|\Phi_n\rangle \to |\Phi\rangle$ as $n\to\infty$ if and only if $\|\Phi_n - \Phi\| \to 0$ as $n\to\infty$. $\|\Phi\|$ is the norm of the vector $|\Phi\rangle : \|\Phi\| \equiv \langle\Phi|\Phi\rangle^{1/2}$.

In this section, we only consider the problem of the large-R behavior of formal charges. The importance of the time smearing and related questions will be discussed in Sec. 4.

We leave momentarily unsettled the question of how Λ in Eq. (3.2) should be allowed to vary with R. Unless otherwise specified, we take Λ to be a fixed constant; the dependence of our statements on the choice of Λ as a function of R will be discussed in Sec. 3.E.

B. The Main Theorems

We now state some important theorems on the high-R behavior of the expression (3.1). We add only very short comments to the statements of the theorems: their interpretation and consequences are discussed in Sec. 3.C, while the relevant proofs are given in 3.D.

The current j_μ is assumed to be a local field,[16] local relative to the basic fields ϕ_i, $i=1,\cdots,J$, of our Wightman theory. We also assume that the domain of

[16] In particular, j_μ is assumed to transform like a field under translations: $j_\mu(x+a) = T(a) j_\mu(x) T(a)^\dagger$. Thus, we are excluding an *explicit* x dependence in $j_\mu(x)$. Cases in which this condition is violated require separate consideration and will not be discussed here.

$j_\mu(f)$, for all $f\in \mathcal{S}$, includes $\mathfrak{D}_{qL}$, and that $j_\mu(f)\mathfrak{D}_{qL}\subseteq\mathfrak{D}_{qL}$, $\forall f\in \mathcal{S}$. In short, j_μ is itself assumed to be a Wightman field.

We adjust j_μ as having vanishing vacuum expectation value:

$$\langle \mathbf{0} \mid j_\mu(x) \mid \mathbf{0}\rangle = 0, \tag{3.4}$$

and state a first theorem on the behavior of the norm of the state $j_0(f_R f_T) \mid \mathbf{0}\rangle$ for large R:

THEOREM 3.1 (Kastler, Robinson and Swieca, 1966; Schroer and Stichel, 1966). As $R\rightarrow\infty$,

$$\langle \mathbf{0} \mid j_0^\dagger(f_R f_T) j_0(f_R f_T) \mid \mathbf{0}\rangle \geq cR^2, \tag{3.5}$$

with $c\neq 0$ unless $j_\mu = 0$.

This theorem shows that the limit (3.3) cannot be taken in the sense of strong operator convergence since the norm of the state $j_0(f_R f_T) \mid \mathbf{0}\rangle$ blows up as R increases.[17]

Consider now a state of the form (2.8b), but obtained by taking purely spatial translations:

$$\mid \mathbf{\Phi}\rangle \equiv \int h(\mathbf{x}) A_\Psi(\mathbf{x}) \mid \mathbf{0}\rangle d\mathbf{x}, \tag{3.6}$$

where A_Ψ is a quasilocal operator, and $h(\mathbf{x})$ is a C^∞ function. Then we have the following theorem.

THEOREM 3.2 (Kastler, Robinson, and Swieca, 1966; Schroer and Stichel, 1966). If

$$\lim_{|\mathbf{x}|\rightarrow\infty} h(\mathbf{x}) \mid \mathbf{x} \mid^2 \neq 0,$$

then

$$\lim_{R\rightarrow\infty} \langle \mathbf{\Phi} \mid j_0(f_R f_T) \mid \mathbf{0}\rangle \tag{3.7}$$

is finite for all quasilocal operators A_Ψ if and only if $j_\mu(x)\equiv 0$.

In other words, if $j_\mu(x)\neq 0$, one can always find normalizable states $\mid \mathbf{\Phi}\rangle$, quasilocal of order less than two, such that $\langle \mathbf{\Phi} \mid j_0(f_R f_T) \mid \mathbf{0}\rangle$ becomes arbitrarily large as R increases. Theorem 3.2 eliminates the possibility of interpreting the limit (3.3) in the sense of weak operator convergence. Indeed, the sequence of vectors $j_0(f_R f_T) \mid \mathbf{0}\rangle$ does not converge weakly, since there are normalizable states $\mid \mathbf{\Phi}\rangle$ in $\mathcal{H}$ for which the limit (3.7) diverges.

Theorems 3.1 and 3.2 do not depend on the mass-gap hypothesis. In the presence of a mass gap, for a conserved current, we also have the following positive statement:

THEOREM 3.3 (Kastler, Robinson, and Swieca, 1966; Schroer and Stichel, 1966). If $\partial^\mu j_\mu = 0$, then

$$\lim_{R\rightarrow\infty} \langle \mathbf{\Phi} \mid j_0(f_R f_T) \mid \mathbf{0}\rangle = 0 \tag{3.8}$$

for any state $\mid \mathbf{\Phi}\rangle$ of the form (3.6), satisfying

$$\lim_{|\mathbf{x}|\rightarrow\infty} \mid \mathbf{x} \mid^2 h(\mathbf{x}) = 0. \tag{3.9}$$

We now recall that a *sesquilinear*[18] *form* (*s.f.*) $F(\mathbf{\Psi}, \chi)$, over $\mathcal{H}$ is a map from ordered pairs $(\mid \mathbf{\Psi}\rangle, \mid \chi\rangle)$ of vectors in certain linear manifolds[19] in $\mathcal{H}$ to the complex numbers and satisfying the usual linear properties, i.e., given any complex numbers α and β, F satisfies

$$F(\alpha\mathbf{\Psi}_1 + \beta\mathbf{\Psi}_2, \chi) = \alpha^* F(\mathbf{\Psi}_1, \chi) + \beta^* F(\mathbf{\Psi}_2, \chi),$$
$$F(\mathbf{\Psi}, \alpha\chi_1 + \beta\chi_2) = \alpha F(\mathbf{\Psi}, \chi_1) + \beta F(\mathbf{\Psi}, \chi_2). \tag{3.10}$$

If a *s.f.* $F(\mathbf{\Psi}, \chi)$ is defined for all $\mid \mathbf{\Psi}\rangle\in\mathfrak{M}_1$, $\mid \chi\rangle\in\mathfrak{M}_2$, where $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are linear manifolds in $\mathcal{H}$, then $(\mathfrak{M}_1, \mathfrak{M}_2)$ is called the *domain* of definition of F. In other terms, a *s.f.* is something like a *matrix*.

A smeared field defines an operator, and thus *a fortiori* it defines a *s.f.* via its matrix elements. Thus, the "partial charges" $j_0(f_R f_T)$ also define *s.f.*'s which vary with $f_R f_T$:

$$Q_R(\mathbf{\Psi}, \chi) \equiv \langle \mathbf{\Psi} \mid j_0(f_R f_T) \mid \chi\rangle, \tag{3.11}$$

for $\mid \mathbf{\Psi}\rangle$ and $\mid \chi\rangle$ in $\mathfrak{D}_{qL}$, defines a *s.f.* If, for states $\mid \mathbf{\Psi}\rangle$ and $\mid \chi\rangle$ in given linear manifolds, the limit

$$Q(\mathbf{\Psi}, \chi) \equiv \lim_{R\rightarrow\infty} Q_R(\mathbf{\Psi}, \chi) \tag{3.12}$$

exists, it will define a *s.f.* associated with the formal charge. As we have seen (Theorem 3.2), we do not expect the limit (3.12) to exist for arbitrary states; however, in the presence of a mass gap, we have:

THEOREM 3.4 (Kastler, Robinson, and Swieca, 1966; Schroer and Stichel, 1966; Dell'Antonio, 1967). For $\mid \mathbf{\Psi}\rangle$ and $\mid \chi\rangle$ quasilocal, the limit

$$Q(\mathbf{\Psi}, \chi) = \lim_{R\rightarrow\infty} \langle \mathbf{\Psi} \mid j_0(f_R f_T) \mid \chi\rangle \tag{3.13}$$

exists and defines a *s.f.*

In particular,

$$Q(\mathbf{\Psi}, \mathbf{0}) \equiv \lim_{R\rightarrow\infty} \langle \mathbf{\Psi} \mid j_0(f_R f_T) \mid \mathbf{0}\rangle \tag{3.14}$$

exists for $\mid \mathbf{\Psi}\rangle\in\mathfrak{D}_{qL}$.

Thus we see that a formal charge defines *at least* a *s.f.* with domain $(\mathfrak{D}_{qL}, \mathfrak{D}_{qL})$. The problem is now to determine some more properties of this *s.f.* It will turn out that the crucial point is whether or not $Q(\mathbf{\Psi}, \chi)$ is *bounded* in $\mid \mathbf{\Psi}\rangle$, i.e., if

$$Q(\mathbf{\Psi}, \chi) \leq K(\chi) \parallel \mathbf{\Psi} \parallel \tag{3.15}$$

for any $\mid \mathbf{\Psi}\rangle\in\mathfrak{D}_{qL}$, where the nonnegative number $K(\chi)$ is independent of $\mid \mathbf{\Psi}\rangle$.

It is in this respect that current conservation plays the vital role[20]:

THEOREM 3.5 For a conserved current, and any fixed $\mid \chi\rangle\in\mathfrak{D}_{qL}$, $Q(\mathbf{\Psi}, \chi)$ satisfies the boundedness condition (3.15).

[17] The exact R^2 behavior depends on the choice of the sequence of functions f_R. See Secs. 3.D.c and 3.E.

[18] Also (improperly) called *bilinear*.

[19] A linear manifold $\mathfrak{M}$ is a set of vectors in $\mathcal{H}$ such that $\alpha \mid \mathbf{\Psi}_1\rangle + \beta \mid \mathbf{\Psi}_2\rangle \in\mathfrak{M}$ if $\mid \mathbf{\Psi}_1\rangle$ and $\mid \mathbf{\Psi}_2\rangle$ are in $\mathfrak{M}$, and α, β are any complex numbers.

[20] The mass-gap hypothesis is assumed. The importance of Theorems 3.5 and 3.6 is particularly emphasized in Schroer and Stichel (1966) and Robinson (1966).

THEOREM 3.6 For a nonconserved current, and any fixed $|\chi\rangle\in\mathfrak{D}_{qL}$, $Q(\Psi,\chi)$ does *not* satisfy the boundedness condition (3.15).

Thus, we see that current conservation is a *necessary and sufficient* condition for the validity of Eq. (3.15). The importance of this fact will be made clear in the next subsection.

C. What It All Means

Our problem was to determine the meaning of the formal expression (1.1) or, equivalently,[21] of the limit (3.3). The ultimate motivation for investigating this point is that we want to understand the connection between a formal charge and the generator of a (broken) symmetry group. A symmetry group has a generator G if the (unitary) symmetry transformations $U(\tau)$ are of the form $\exp[iG\tau]$; for the latter to be unitaries, G must be a self-adjoint operator.[22] Our problem can then be summarized in the following two questions:

(a) Is a formal charge an operator in $\mathfrak{H}$?

(b) When and to what extent is it possible to identify a formal charge with a generator?

To question (a) we found a *negative answer*: a *formal charge is never an operator*, since we saw that, as a consequence of Theorems 3.1 and 3.2, the limit (3.3) does not exist in the sense of strong or weak operator convergence.[23] Nevertheless, we saw that a charge can be identified with a *s.f.* $Q(\Psi,\chi)$, defined for $|\Psi\rangle$ and $|\chi\rangle$ in $\mathfrak{D}_{qL}$. Leaving aside until Sec. 4 the problem of self-adjointness, which is required of a generator, question (b) asks whether or not $Q(\Psi,\chi)$ determines an operator or, as we now explain, is the form of an operator.

Suppose that there exists an unbounded operator G with domain $\mathfrak{D}_{qL}$, and satisfying

$$\langle\Psi\mid G\chi\rangle=Q(\Psi,\chi),\ \forall\mid\Psi\rangle,\mid\chi\rangle\in\mathfrak{D}_{qL}. \quad (3.16)$$

If such an operator exists, it is called an *operator extension* of $Q(\Psi,\chi)$, and $Q(\Psi,\chi)$ is called *the form of the operator* G. Since $\mathfrak{D}_{qL}$ is dense in $\mathfrak{H}$, if an operator G satisfying (3.16) exists, it is easily shown that G is uniquely determined by $Q(\Psi,\chi)$.

In order to conveniently summarize our analysis so far, we introduce a simplifying notion of convergence of *densely defined sesquilinear forms* (*d.d.s.f.*'s). A *d.d.s.f.* F is a *s.f.* with domain $(\mathfrak{D}_1,\mathfrak{D}_2)$, such that $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are dense in $\mathfrak{H}$. In the following, we only need to consider the case $\mathfrak{D}_1=\mathfrak{D}_2=\mathfrak{D}$, and thus simply write $\mathfrak{D}$ for the domain of F.

Given a sequence $\{F_n\}_1^\infty$, of *d.d.s.f.*'s, all defined on a common dense domain $\mathfrak{D}$, if for any $|\Psi\rangle$, $|\chi\rangle\in\mathfrak{D}$, $F_n(\Psi,\chi)\rightarrow_{n\to\infty}F(\Psi,\chi)$, F a *d.d.s.f.* with domain $\mathfrak{D}$, we say that $F_n\rightarrow F$ in the sense of *d.d.s.f.*'s. If, in addition, F is the form of an operator G, we say that F_n converges to G *in the sense of d.d.s.f.*'s. As we shall illustrate below [see Eqs. (3.20)–(3.21)], convergence to an operator in the sense of *d.d.s.f.*'s is a weaker notion than strong or weak operator convergence.

Given that strong or weak operator convergence of $j_0(f_Rf_T)$ is excluded as $R\rightarrow\infty$, we are asking if the $Q_R(\Psi,\chi)$ of Eq. (3.11) converge to an operator in the (weaker) sense of *d.d.s.f.*'s. Theorem 3.4 guarantees that $Q_R(\Psi,\chi)$ converges, in the sense of *d.d.s.f.*'s, to the *d.d.s.f.* $Q(\Psi,\chi)$, for $|\Psi\rangle,|\chi\rangle\in\mathfrak{D}_{qL}$. Thus, we only have to find out whether or not $Q(\Psi,\chi)$ is the form of an operator. In order to answer this question, we first establish the following lemma:

Lemma 3.1 Let $F(\Psi,\chi)$ be a *d.d.s.f.* with domain $(\mathfrak{D}_1,\mathfrak{D}_2)$. Then

(i) F is the form of an operator G with domain $\mathfrak{D}_2$ if and only if for each fixed $|\chi\rangle\in\mathfrak{D}_2$ there exists a constant $K(\chi)$, independent of $|\Psi\rangle$, such that

$$|F(\Psi,\chi)|\leq K(\chi)\,\|\Psi\|,\ \forall\,|\Psi\rangle\in\mathfrak{D}_1; \quad (3.17)$$

(ii) for $\mathfrak{D}_1=\mathfrak{D}_2=\mathfrak{D}$, if (3.17) holds and in addition F is Hermitian in the sense that

$$F(\Psi,\chi)^*=F(\chi,\Psi),\ \forall\,|\Psi\rangle,|\chi\rangle\in\mathfrak{D}, \quad (3.18)$$

then G is also Hermitian

$$\langle\Psi\mid G\chi\rangle=\langle G\Psi\mid\chi\rangle,\ \forall\,|\Psi\rangle,|\chi\rangle\in\mathfrak{D}.$$

Proof (i) Since $F(\Psi,\chi)$ is antilinear in $|\Psi\rangle$ [see Eq. (3.10)], $F(\Psi,\chi)^*$ is[24] a linear function in $|\Psi\rangle$ for any fixed $|\chi\rangle\in\mathfrak{D}_2$. If Eq. (3.17) holds, $F(\Psi,\chi)^*$ is continuous in $|\Psi\rangle$ for $|\Psi\rangle\in\mathfrak{D}_1$, and can therefore be defined on all of $\mathfrak{H}$ by continuity. By the Riesz representation theorem (Akhiezer and Glazman, 1961; Sec. 16), there exists a unique vector $|\bar{\chi}\rangle$ such that

$$F(\Psi,\bar{\chi})^*=\langle\bar{\chi}\,|\Psi\rangle,$$

and thus

$$F(\Psi,\chi)=\langle\Psi\mid\bar{\chi}\rangle.$$

Define now an operator G by

$$|\bar{\chi}\rangle=G\,|\chi\rangle,\ |\chi\rangle\in\mathfrak{D}_2;$$

G is clearly a linear operator having $\mathfrak{D}_2$ in its domain, and it satisfies

$$F(\Psi,\chi)=\langle\Psi\mid G\chi\rangle \quad (3.19)$$

for all $|\Psi\rangle\in\mathfrak{D}_1$, $|\chi\rangle\in\mathfrak{D}_2$. Thus, F is the form of G.

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21 Apart from the problem of time smearing, i.e., the process of taking the limit $T\rightarrow0$ in Eq. (3.3). This problem is irrelevant at this stage, and will be reconsidered in Sec. 4.

22 Rather, G should be essentially self-adjoint and the unitary transformations should be expressed as $\exp[iG^\dagger\tau]$. This point will be discussed in Sec. 4. For the moment, we choose not to delve into such mathematical complications.

23 Strictly speaking, we proved that the formal charge is not an operator limit having quasilocal states in its domain. The limit (3.3) might still converge for states which are not quasilocal. However, in this case the practical usefulness of the charge would be highly impaired. For the sake of brevity, we maintain our approximate terminology and simply say "the formal charge is not an operator" rather than "the formal charge is not an operator whose domain includes quasilocal states."

24 This extension, by continuity, is illustrated below, following Eq. (3.20), in the special case $|\chi\rangle=|0\rangle$.

Vice versa, if F is the form of an operator G, the Cauchy–Schwartz inequality gives

$$|F(\Psi,\chi)| = |\langle\Psi|G\chi\rangle| \leq ||\Psi|| \cdot ||G|\chi\rangle||,$$

and (3.17) is satisfied with $K(\chi) = ||G|\chi\rangle||$.

(ii) Equation (3.18) and the fact that F is the form of G imply, for $|\Psi\rangle, |\chi\rangle \in \mathfrak{D}$,

$$\langle\Psi|G\chi\rangle = \langle\chi|G\Psi\rangle^* = \langle G\Psi|\chi\rangle$$

and thus G is Hermitian.

Returning to the formal charge, we see that $Q(\Psi,\chi)$ is the form of an operator if and only if the boundedness condition (3.15) is satisfied.

For a conserved current, Theorem 3.5 states that $Q(\Psi,\chi)$ satisfies (3.15) for $|\chi\rangle \in \mathfrak{D}_{qL}$. Thus, $Q(\Psi,\chi)$ is the form of an operator G having $\mathfrak{D}_{qL}$ as its domain and whose matrix elements satisfy (3.16) for $|\Phi\rangle$, $|\chi\rangle \in \mathfrak{D}_{qL}$. G is uniquely determined by $Q(\Psi,\chi)$ and, for a *Hermitian* current, G is a *Hermitian operator.*[25]

In the special case $|\chi\rangle = |0\rangle$, by Theorem (3.3) we have

$$Q(\Phi, 0) = 0 \tag{3.20}$$

for any state $|\Phi\rangle$ of the form (3.6) satisfying (3.9), and thus *a fortiori* for $|\Phi\rangle \in \mathfrak{D}_{qL}$. Since $|0\rangle$ is in the domain of G, G is continuous on $|0\rangle$, and one has[26]

$$\langle\Phi|G|0\rangle = 0 \tag{3.21}$$

for *any* $|\Phi\rangle \in \mathfrak{H}$. This is to be contrasted with the fact that $Q(\Phi, 0)$ might diverge for $|\Phi\rangle \notin \mathfrak{D}_{qL}$. The reader can easily convince himself that the reason there is no contradiction is that $\langle\Phi|G|0\rangle$ is the limit of $Q_R(\Phi, 0)$ *only in the sense of d.d.s.f.*'s. Thus, the equality of $\langle\Phi|G|0\rangle$ and $\lim_{R\to\infty} Q_R(\Phi, 0)$ need hold for $|\Phi\rangle \in \mathfrak{D}_{qL}$ *only.* This point illustrates the difference between weak operator convergence and convergence in the sense of *d.d.s.f.*'s.

To summarize: *For a conserved current, the densely defined sesquilinear form defined by the formal charge is the form of a uniquely determined operator G; G is Hermitean if the current is Hermitean.*

For nonconserved currents, the boundedness condition (3.15) *is violated in* $\mathfrak{D}_{qL}$, *and the d.d.s.f.* $Q(\Phi,\chi)$ *is not the form of an operator.* Thus, the charge cannot be defined as a limit of *d.d.s.f.*'s. As no time-dependent generator is associated with the charge, we cannot exponentiate this nonexisting generator to obtain the transformations forming the broken-symmetry group. If we try to postulate that a generator exists as an operator associated with the *s.f.* $Q(\Psi, \chi)$, we find[27] that (3.15) must be satisfied and we must have a conserved current, i.e., an exact symmetry.[28] Thus, it is impossible to construct the analog of the formal expression $\exp[iQ_T]$ for a broken symmetry, where Q would be the generator of broken-symmetry transformations. This should perhaps not be too surprising.[29] What is surprising is the fact that, if the symmetry is broken, it must be *globally* broken in the sense that no quasilocal state is left strictly invariant upon broken symmetry transformations. In Sec. 6 we shall discuss to a greater extent the effect of all this on the formulation of broken symmetries in quantum field theory.

We conclude this discussion of properties of formal charges by stating the converse of Theorem 3.3:

THEOREM 3.7 (Coleman, 1966; Schroer and Stichel, 1966; and Dell'Antonio, 1967). If the charge annihilates the vacuum *weakly,* then the current is conserved.

This means if $Q(\Psi, 0) = 0$ for all $|\Psi\rangle \in \mathfrak{D}_{qL}$, then $\partial^\mu j_\mu = 0$.

This theorem is a refined version of Coleman's original theorem (Coleman, 1966), and was proved by Schroer and Stichel (1966) and Dell'Antonio (1967). In the original version, Coleman treated the formal charge as an operator having $|0\rangle$ in its domain, and annihilating $|0\rangle$. We now know that this hypothesis cannot be accepted, and Theorem 3.7 only assumes that $Q(\Psi, 0)$, for $|\Psi\rangle \in \mathfrak{D}_{qL}$, vanishes.

D. The Proofs

We now outline the proofs of the main theorems stated in Secs. 3.B and 3.C. Most of the proofs are exercises in the use of the integral representation of the two-point functions of a local field. A knowledge of this representation is thus required on the part of the reader (see e.g., Umezawa and Kamefuchi, 1952; Källen, 1952; and Lehmann, 1954).

Proof of Theorem 3.1 (Schroer and Stichel, 1966, p. 260; Reeh, 1968, p. 692). We only consider the special case of a conserved vector current and prove that, as $R\to\infty$,

$$||j_0(f_Rf_T)|0\rangle||^2 \equiv \langle 0|j_0^\dagger(f_Rf_T)j_0(f_Rf_T)|0\rangle \sim cR^2 \tag{3.22}$$

with $c\neq 0$ unless $j_\mu(x)\equiv 0$.

[25] As we shall see in Sec. 4, Hermiticity is not sufficient for G to be the generator of a continuous symmetry group: the stronger condition that G be self-adjoint is needed.

[26] In order to illustrate this "extension by continuity," also used in the proof of Lemma 3.1, we define $\langle\Phi|G|\chi\rangle$ for $|\Phi\rangle \in \mathfrak{D}_{qL}$. Since $\mathfrak{D}_{qL}$ is dense in $\mathfrak{H}$, we can find a sequence $\{|\Phi_n\rangle\}_1^\infty$, with $|\Phi^n\rangle \in \mathfrak{D}_{qL}$ and $|\Phi_n\rangle \to |\Phi\rangle$. Then

$$\langle\Phi|G|\chi\rangle \equiv \lim_{n\to\infty} \langle\Phi_n|G|\chi\rangle = \lim_{n\to\infty} Q(\Phi_n, \chi).$$

One easily shows that, by virtue of (3.15), the above limit is always finite and independent of the choice of the sequence $\{|\Phi_n\rangle\}_1^\infty$, $|\Phi_n\rangle \in \mathfrak{D}_{qL}$, converging to $|\Phi\rangle$.

[27] Barring difficulties associated with the differences between "Hermitian" and "self-adjoint."

[28] We recall that we are considering only currents j_μ which are Wightman fields, and which thus transform as fields under translations:

$$T(a)j_\mu(x)T(a)^+ = j_\mu(x+a).$$

This excludes an *explicit* dependence on x in j_μ. The case of, e.g., Lorentz boosts, which show an explicit time dependence, ought to be separately discussed.

[29] In nonrelativistic quantum mechanics a nonconserved current may lead to a well-defined charge operator. However, when the theory is required to be causal, to be Lorentz-invariant, and to satisfy the spectrum condition, lack of current conservation and symmetry breaking are tied together to the extent of preventing the existence of the charge.

The two-point function of a conserved vector current has the following representation:

$$\langle \mathbf{0} \mid j_\mu^\dagger(x) j_\nu(y) \mid \mathbf{0} \rangle$$
$$= -i \int dm^2 [g_{\mu\nu} + (\partial_\mu \partial_\nu / m^2)] \Delta^+(x-y; m^2) \rho(m^2), \quad (3.23)$$

where $\rho(m^2)$ is a tempered measure.

If one defines $g(p)$ as follows:

$$g(p) = (2\pi)^{-1} \int dm^2 [\rho(m^2)/2m^2(m^2+p^2)^{1/2}] \times |\tilde{f}_T[(p^2+m^2)^{1/2}]|^2, \quad (3.24)$$

where a tilde denotes the Fourier transform, a straightforward calculation leads to the result:

$$\| j_0(f_R f_T) \mid \mathbf{0} \rangle \|^2 = \int d\mathbf{p} \mid p \tilde{f}_R(\mathbf{p}) \mid^2 g(p). \quad (3.25)$$

One also easily finds the following expression for $p\tilde{f}_R(\mathbf{p})$:

$$p\tilde{f}_R(\mathbf{p}) = \frac{2}{(2\pi)^{1/2}} \int_0^\infty dr r f_R(r) \sin pr$$
$$= \frac{2}{(2\pi)^{1/2}} \int_0^\infty dr \frac{r}{p} \left[\frac{d}{dr} f_R(r)\right] \left(\cos pr - \frac{\sin pr}{pr}\right). \quad (3.26)$$

We now make the following choice for the sequence of smearing functions $f_R(r)$:

$$f_R(r) = 1 \quad \text{for } r \leq R$$
$$= f(r-R) \quad \text{for } R \leq r \leq R+\Lambda$$
$$= 0 \quad \text{for } r \geq R+\Lambda, \quad (3.27)$$

where Λ is kept constant, and $f(r')$ is C^∞ and equals 1 for $r'=0$, while it vanishes for $r' \geq \Lambda$. The choice (3.27) is made mainly so as to simplify things as much as possible: as R varies, the function $f_R(r)$ is displaced in a parallel fashion in the region $(R, R+\Lambda)$; its derivative satisfies

$$df_R(r)/dr = df(r')/dr' \mid_{r'=r-R}. \quad (3.28)$$

Next, we change the variable of integration, use the addition formulae for $\sin p(r+R)$ and $\cos p(r+R)$, and perform some partial integrations, so as to bring Eq. (3.26) to the following form:

$$p\tilde{f}_R(\mathbf{p}) = [2/(2\pi)^{1/2}](p^{-2}[f_c(p) \sin pR + f_s(p) \cos pR]$$
$$- (p)^{-1}\{[(d/dp) f_c(p)] \sin pr + [(d/dp) f_s(p)] \cos pR\}$$
$$- R(p^{-1})[f_c(p) \cos pR - f_s(p) \sin pR]), \quad (3.29)$$

where

$$f_s(p) \equiv \int_0^\Lambda dr' \left[\frac{d}{dr'} f(r')\right] \sin pr',$$

$$f_c(p) \equiv \int_0^\Lambda dr' \left[\frac{d}{dr'} f(r')\right] \cos pr'. \quad (3.30)$$

After insertion of Eq. (3.29) in (3.25), the large-R behavior is determined as follows: The resulting bilinear terms in $\cos pR$ and $\sin pR$ are reduced to sums of linear terms in $\cos 2pR$ and $\sin 2pR$ and terms containing no oscillating factor. By the Riemann–Lebesgue lemma,[30] the factors of R and R^2 containing $\cos 2pR$ or $\sin 2pR$ vanish as $R \rightarrow \infty$. The remaining highest-order contribution gives, for large R, the following behavior:

$$\| j_0(f_R f_T) \mid \mathbf{0} \rangle \|^2 \geq \pi^{-1} R^2 \int_\delta^\infty [f_c^2(p) + f_s^2(p)] \bar{g}(p)\, dp, \quad (3.31)$$

where[31] $\delta > 0$ and

$$\bar{g}(p) = \int d\Omega_{\mathbf{p}} g(p), \qquad d\mathbf{p} = p^2 dp d\Omega_{\mathbf{p}}. \quad (3.32)$$

Since f_c^2 and f_s^2 are chosen to be positive definite,[32] while $\bar{g}(p)$ is nonnegative, the coefficient of R^2 in Eq. (3.31) is seen to be nonvanishing unless $\rho(m^2)=0$, as follows from Eqs. (3.32) and (3.24). Equation (3.23) then implies $\| j_\mu(f) \mid 0 \rangle \| = 0$ for any $f \in \mathcal{S}$ or, equivalently,

$$j_\mu(x) \mid 0 \rangle = 0. \quad (3.33)$$

This and Corollary 2.1 imply that if $\rho(m^2)$ vanishes, $j_\mu(\mathbf{x}) = 0$, and the behavior claimed in Eq. (3.22) is thus proved.

Remarks: (i) From the mass-gap hypothesis and Theorems 2.2 and 2.3, one can see that

$$\langle \mathbf{0} \mid j_0^\dagger(\mathbf{x}, f_T) j_0(\mathbf{y}, f_T) \mid \mathbf{0} \rangle$$
$$= \int dx_0 dy_0 f_T(x_0) f_T(y_0) \langle \mathbf{0} \mid j_0^\dagger(x) j_0(y) \mid \mathbf{0} \rangle \quad (3.34)$$

is a fast-decreasing C^∞ function in $\mathbf{x}-\mathbf{y}$. This in turn can be seen to imply that $\bar{g}(p)$ is C^∞ and fast decreasing in p. Since we can always choose $f(r)$ to be C^∞ and fast decreasing, the integrand in Eq. (3.25) will be C^∞ and fast decreasing. This is much more than is needed for applying the Riemann–Lebesgue lemma. As a matter of fact, it turns out that *the mass-gap hypothesis is superfluous*: when massless particles are present, the expression in (3.34) will vanish as $\mathbf{x}-\mathbf{y} \rightarrow \infty$ (like some inverse power of $\mathbf{x}-\mathbf{y}$), and this can be shown to be sufficient for deriving the behavior (3.31). However, the contributions from $\mathbf{p}=0$ could now play a more significant role, due to the singular behavior of $\rho(m^2)$ for $m^2=0$. Thus, when massless particles are present, $\| j_0(f_R f_T) \mid \mathbf{0} \rangle \|^2$ might behave worse than R^2 for large R. These points are discussed in greater detail in Reeh (1968) where it is shown that $\| j_0(f_R f_T) \mid \mathbf{0} \rangle \|^2$ is always bounded by cR^3 for large R.

(ii) The dependence of our results on the choice of the sequence of function f_R will be discussed in Sec. 3.E.

(iii) In the case of a nonconserved current, one follows roughly the same steps performed in the proof given above. The main difference is that, in Eq. (3.23), one must use a more general integral representation, since the form (3.23) takes current conservation into account. We refer to Reek (1968), p. 692 for additional details.

[30] See, e.g., Titchmarsh (1939). See also our remark (i) below.
[31] The cutoff $\delta > 0$ is introduced to avoid inessential complications arising from the neighborhood of $\mathbf{p}=0$.
[32] We recall that f_R is taken to be real.

Proof of Theorem 3.2 [Schroer and Stichel (1966), p. 262] Theorem 3.2 states that the limit

$$\lim_{R\to\infty} \langle \mathbf{\Phi} \mid j_0(f_R f_T) \mid \mathbf{0}\rangle \qquad (3.7)$$

cannot be finite for all states $|\mathbf{\Phi}\rangle$ of the form (3.6) with $h(\mathbf{x})$ satisfying

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|^2 h(\mathbf{x})\neq 0. \qquad (3.35)$$

The theorem can be proved by performing a calculation similar to the one given for the proof of Theorem 3.1 and by choosing

$$|\mathbf{\Phi}\rangle = \int d\mathbf{x} h(\mathbf{x}) T(\mathbf{x}) \int dx_0 j_0(\mathbf{0}, x_0) f_T(x_0) \mid \mathbf{0}\rangle \qquad (3.36)$$

with $h(\mathbf{x})$ satisfying (3.35).

Proof of Theorem 3.3 (Kastler, Robinson, and Swieca, 1966, Lemma II; Schroer and Stichel, 1966, p. 261.) Theorem 3.3 states that the formal charge associated with a conserved current annihilates the vacuum *weakly*, i.e., the limit $Q(\mathbf{\Phi}, \mathbf{0})$ vanishes for $|\mathbf{\Phi}\rangle$ in the dense set of all states quasilocal of order[33] $N\geq 2$. This result is intuitively obvious, but one should show that the pathologies expressed by Theorem 3.2 do not occur for states quasilocal of order $N\geq 2$.

In the heuristic proof of Theorem 3.3, one would proceed as follows: since $\langle \mathbf{0} \mid j_0(x) \mid \mathbf{0}\rangle$ vanishes, we can choose $|\mathbf{\Phi}\rangle$ to be orthogonal to the vacuum and expand $|\mathbf{\Phi}\rangle$ in a complete set of (continuum) eigenstates of the energy operator:

$$|\mathbf{\Phi}\rangle = \int dp_0 \langle \mathbf{\Psi}(p_0) \mid \mathbf{\Phi}\rangle \mid \mathbf{\Psi}(p_0)\rangle; \qquad (3.37)$$

by virtue of the mass-gap hypothesis, the integration starts at $p^0\geq m_{\min}>0$, where $m_{\min}$ is the lowest mass in the theory. Thus, the energy operator has an inverse on states of this kind, and we obtain

$$\langle \mathbf{\Phi} \mid j_0(f_R f_T) \mid \mathbf{0}\rangle = \langle \mathbf{\Phi}' \mid [P_0, j_0(f_R f_T)] \mid \mathbf{0}\rangle, \qquad (3.38)$$

where

$$|\mathbf{\Phi}'\rangle \equiv (P_0)^{-1} \mid \mathbf{\Phi}\rangle. \qquad (3.39)$$

Since $[P_0, j_0(x)] = -i\partial_0 j_0(x)$, and, by current conservation, $\partial^0 j_0 = \mathbf{\nabla}\cdot\mathbf{j}$, we can then use Gauss' theorem to show that (3.38) vanishes as $R\to\infty$.

In a polished version of the proof, the operation of "dividing" by P_0 is still allowed: $|\mathbf{\Phi}\rangle$ can be expressed as follows:

$$|\mathbf{\Phi}\rangle = \int dE(p_0) \mid \mathbf{\Phi}\rangle, \qquad (3.40)$$

where the "spectral measure" $dE(p_0)$ corresponds to $|\mathbf{\Psi}(p_0)\rangle\langle\mathbf{\Psi}(p_0)| \, dp_0$. By the mass-gap condition, $dE(p_0)$ vanishes for[34] $p_0\leq m_{\min}$, and we can consider a state

$$|\mathbf{\Phi}'\rangle \equiv \int dE(p_0) [(p_0)^{-1} \mid \mathbf{\Phi}\rangle], \qquad (3.41)$$

which clearly corresponds to $(P_0)^{-1}|\mathbf{\Phi}\rangle$, since

$$P_0 \mid \mathbf{\Phi}'\rangle = \mid \mathbf{\Phi}\rangle. \qquad (3.42)$$

[33] Compare Sec. 2.B for the definition of quasilocal state of order N.

[34] We take $\langle \mathbf{\Phi} \mid \mathbf{0}\rangle = 0$; see also below.

The only open question at this point is whether or not dividing by P_0 can destroy the localization properties of $|\mathbf{\Phi}\rangle$. *A priori* this might happen, since division by the energy corresponds to an integration in the time variable. However, it turns out that $|\mathbf{\Phi}'\rangle$ can be chosen as localized as the original $|\mathbf{\Phi}\rangle$. We do not prove this result,[35] but we do state it in the following generalized form.

Lemma 3.2 Let $K\geq 1$ be a fixed integer, and $|\mathbf{\Phi}\rangle$ be a quasilocal state of order $N\geq 2$. *Under the mass-gap hypothesis*, there exists another state $|\mathbf{\Phi}'^{(K)}\rangle$, also quasilocal of order N, such that

$$P_0^K \mid \mathbf{\Phi}'^{(K)}\rangle = \mid \mathbf{\Phi}\rangle - \langle 0 \mid \mathbf{\Phi}\rangle \mid \mathbf{0}\rangle. \qquad (3.43)$$

We now outline the rigorous proof of Theorem 3.3. By the definition of derivative of a distribution, one has

$$\partial^\nu j_\mu(f_R f_T) = -j_\mu(\partial^\nu f_R f_T) \qquad (3.44)$$

and, by current conservation,

$$[P_0, j_0(f_R f_T)] = ij_0[f_R(df_T/dx_0)] = -\sum_{k=1}^{3} ij^k(\partial_k f_R f_T) \equiv i\mathbf{j}(\cdot\mathbf{\nabla} f_R f_T). \qquad (3.45)$$

From Eqs. (3.38) and (3.45) we obtain

$$\langle \mathbf{\Phi} \mid j_0(f_R f_T) \mid \mathbf{0}\rangle = i\langle \mathbf{\Phi}' \mid \mathbf{j}(\cdot\mathbf{\nabla} f_R f_T) \mid \mathbf{0}\rangle, \qquad (3.46)$$

where, by Lemma 3.2, $|\mathbf{\Phi}'\rangle$ is of the form[36]

$$|\mathbf{\Phi}'\rangle = \int d\mathbf{x} \bar{h}(\mathbf{x}) T(\mathbf{x}) \mid \mathbf{\Psi}'\rangle, \qquad (3.47)$$

with $\bar{h}(\mathbf{x})$ satisfying Eq. (3.35), and $|\mathbf{\Psi}'\rangle$ a quasilocal state. Since $\mathbf{\nabla} f_R f_T$ is nonvanishing only for

$$x\in D_R \equiv \{x: R\leq |\mathbf{x}| \leq R+\Lambda, |x_0| \leq T\}, \qquad (3.48)$$

we see that the state

$$\mathbf{j}(\cdot\mathbf{\nabla} f_R f_T) \mid \mathbf{0}\rangle$$

is localized in the region D_R. If one decomposes $|\mathbf{\Phi}'\rangle$ as a sum of two states:

$$|\mathbf{\Phi}'\rangle = |\mathbf{\Phi}'_1\rangle + |\mathbf{\Phi}'_2\rangle = \int_0^{|\mathbf{x}|\leq R/2} d\mathbf{x} \bar{h}(\mathbf{x}) T(\mathbf{x}) \mid \mathbf{\Psi}'\rangle + \int_{|\mathbf{x}|\geq R/2}^{\infty} d\mathbf{x} \bar{h}(\mathbf{x}) T(\mathbf{x}) \mid \mathbf{\Psi}'\rangle, \qquad (3.49)$$

we see that the first state is quasilocal and "effectively localized" around $|\mathbf{x}|\leq R/2$, since outside of this region it is quasilocal.

As $R\to\infty$, D_R is shifted to infinity, and one can use the cluster theorem to conclude that

$$\langle \mathbf{\Phi}_1' \mid \mathbf{j}(\cdot\mathbf{\nabla} f_R f_T) \mid \mathbf{0}\rangle \xrightarrow[R\to\infty]{} 0, \qquad (3.50)$$

the limit being approached faster than any inverse

[35] See Kastler, Robinson, and Swieca (1966), Lemma III.

[36] Since $\langle \mathbf{0} \mid j_\mu \mid \mathbf{0}\rangle = 0$, $|\mathbf{\Phi}\rangle$ can be taken to be orthogonal to $|0\rangle$ and there is no contribution from the last term in Eq. (4.43).

power of R. For the second matrix element, we use the Cauchy–Schwartz inequality and obtain:

$$\langle \mathbf{\Phi}'_2 \mid \mathbf{j}(\cdot \nabla f_R f_T) \mid \mathbf{0}\rangle \leq \|\mathbf{\Phi}'_2\| \cdot \|\mathbf{j}(\cdot \nabla f_R f_T) \mid \mathbf{0}\rangle\|$$

$$= \left[\int_{|\mathbf{x}|\geq R/2}^{\infty} d\mathbf{x} \int_{|\mathbf{x}|\geq R/2}^{\infty} d\mathbf{y} \bar{h}(\mathbf{x}) \bar{h}(\mathbf{y}) \langle \mathbf{\Psi}' \mid T(\mathbf{x}-\mathbf{y}) \mid \mathbf{\Psi}' \rangle \right.$$

$$\left. \cdot \langle \mathbf{0} \mid \mathbf{j}^\dagger(\cdot \nabla f_R f_T) \mathbf{j}(\cdot \nabla f_R f_T) \mid \mathbf{0} \rangle \right]^{1/2}. \quad (3.51)$$

By the cluster theorem, $\langle \mathbf{\Psi}' \mid T(\mathbf{x}-\mathbf{y}) \mid \mathbf{\Psi}' \rangle$ is fast decreasing[37] in $\mathbf{x}-\mathbf{y}$. This, and the assumed behavior of $\bar{h}(\mathbf{x})$, can be shown to imply that the first factor decreases *faster* than $R^{-1/2}$ as $R \rightarrow \infty$.

For the second factor, we can proceed as in the proof of Theorem 3.1 and evaluate the high-R behavior. Rather than choosing the usual f_R, it is convenient to choose

$$f_R(\mathbf{x}) = f(r/R), \quad (3.52)$$

where $f(\mathbf{0}) = 1$, and f is a smooth C^∞ function.[38]

Now, since $\langle 0 \mid j_\mu(x) j_\nu(y) \mid 0\rangle$, for $(x-y)^2 \rightarrow -\infty$, is fast decreasing by the cluster theorem, while $df/dr \leq$ const $1/R$, one can easily derive the majorization

$$\| \mathbf{j}(\cdot \nabla f_R f_T) \mid 0 \rangle \| \leq \text{const } R^{1/2}$$

(Kastler, Robinson, and Swieca, 1966, p. 114; Schroer and Stichel, 1966, p. 262). This, combined with the faster-than-$R^{-1/2}$ behavior of the first factor, leads to the conclusion stated in Theorem 3.3.

Proofs of Theorems 3.4–3.7 We must prove that the limits

$$Q(\mathbf{\Psi}, \chi) = \lim_{R\rightarrow\infty} \langle \mathbf{\Psi} \mid j_0(f_R f_T) \mid \chi \rangle, \quad (3.13)$$

$$Q(\mathbf{\Psi}, \mathbf{0}) = \lim_{R\rightarrow\infty} \langle \mathbf{\Psi} \mid j_0(f_R f_T) \mid \mathbf{0} \rangle, \quad (3.14)$$

for $\mid \mathbf{\Psi}\rangle$ and $\mid \chi \rangle$ quasilocal,

(i) exist and define bilinear forms;

(ii) satisfy the boundedness condition (3.15), if and only if, the current is conserved.

Property (i) is an immediate consequence of Theorems 2.2 and 2.3,[39] and of the fact that the set of all quasilocal states is a linear manifold.

In order to prove (ii), we begin by considering a quasilocal operator A_χ such that

$$A_\chi \mid \mathbf{0} \rangle = \mid \chi \rangle. \quad (3.53)$$

Then we have

$$\langle \mathbf{\Psi} \mid j_0(f_R f_T) \mid \chi \rangle = \langle \mathbf{\Psi} \mid [j_0(f_R f_T), A_\chi] \mid \mathbf{0} \rangle$$
$$+ \langle \mathbf{\Psi}' \mid j_0(f_R f_T) \mid \mathbf{0} \rangle, \quad (3.54)$$

where

$$\mid \mathbf{\Psi}' \rangle \equiv A_\chi^\dagger \mid \mathbf{\Psi} \rangle \quad (3.55)$$

is a quasilocal state.

The commutator $[j_0(\mathbf{x}, f_T), A_\chi]$ is fast decreasing in $\mathbf{x}$, as follows from local commutativity. [This can be most easily understood by considering the special case of an A_χ which is strictly localized in some region $\mathcal{O}$. For $\mid \mathbf{x} \mid$ sufficiently large, the segment $(\mathbf{x}, \mid x_0 \mid \leq T)$ becomes spacelike with respect to $\mathcal{O}$, and thus $j_0(\mathbf{x}, f_T)$ commutes with A_χ].[40] On the other hand, we can use the Cauchy–Schwartz inequality and obtain

$$\mid \langle \mathbf{\Psi} \mid [j_0(f_R f_T), A_\chi] \mid \mathbf{0} \rangle$$
$$\leq \| \mathbf{\Psi} \| \cdot \| [j_0(f_R f_T), A_\chi] \mid \mathbf{0} \rangle \|. \quad (3.56)$$

Since $[j_0(\mathbf{x}, f_T), A_\chi]$ is fast decreasing in $\mathbf{x}$, the rhs has a limit as $R \rightarrow \infty$, and we conclude that the first term in the rhs of Eq. (3.54) satisfies the boundedness criterion (3.15).

Thus everything depends on whether or not $Q(\mathbf{\Psi}, \mathbf{0})$ satisfies, for all quasilocal states $\mid \mathbf{\Psi} \rangle$,

$$Q(\mathbf{\Psi}, \mathbf{0}) \leq K \| \mathbf{\Psi} \| \quad (3.57)$$

for some constant K.

If the current is conserved, we already known (Theorem 3.3) that (3.57) holds with $K = 0$.

Vice versa, let us assume that (3.57) holds for some finite K. If we can prove that K must necessarily vanish, we will fall within the conditions of Theorem 3.7.

The proof of the fact that (3.57) implies $K = 0$ is as follows (Robinson, 1966): we know from Sec. 3.C that, if (3.57) is satisfied, there exists an operator G such that

$$Q(\mathbf{\Psi}, \mathbf{0}) = \langle \mathbf{\Psi} \mid G \mid \mathbf{0} \rangle \quad (3.58)$$

for any quasilocal $\mid \mathbf{\Psi} \rangle$. On the other hand, if $\mid \mathbf{\Psi}(\mathbf{x}) \rangle = T(\mathbf{x}) \mid \mathbf{\Psi} \rangle$ is "translated" by $\mathbf{x}$ of $\mid \mathbf{\Psi} \rangle$, it is not difficult to show that

$$Q(\mathbf{\Psi}(\mathbf{x}), \mathbf{0}) = Q(\mathbf{\Psi}, \mathbf{0}). \quad (3.59)$$

We now take $\mid \mathbf{x} \mid \rightarrow \infty$ and use the cluster theorem, which implies

$$\lim_{|\mathbf{x}|\rightarrow\infty} \langle \mathbf{\Psi} \mid T(\mathbf{x}) G \mid \mathbf{0} \rangle \rightarrow \langle \mathbf{\Psi} \mid \mathbf{0} \rangle \langle \mathbf{0} \mid G \mid \mathbf{0} \rangle. \quad (3.60)$$

Thus, since $\mid \mathbf{\Psi} \rangle$ is any quasilocal state, and quasilocal states from a dense set,

$$G \mid \mathbf{0} \rangle = \langle \mathbf{0} \mid G \mid \mathbf{0} \rangle \mid \mathbf{0} \rangle, \quad (3.61)$$

and $\langle \mathbf{0} \mid G \mid \mathbf{0} \rangle$ must vanish in view of $\langle \mathbf{0} \mid j_0(x) \mid \mathbf{0} \rangle = 0$.

From this and Eq. (3.58), we conclude that

$$Q(\mathbf{\Psi}, \mathbf{0}) = 0 \quad (3.62)$$

is implied by Eq. (3.57).

Equation (3.62) says that the charge annihilates the vacuum weakly and is exactly the condition for the validity of Theorem 3.7. Thus, we have reduced the

[37] The mass-gap hypothesis is used here for the second time. It was also crucial for Lemma 3.2.

[38] See the next subsection for a discussion on the dependence of our results on the choice of the sequence f_R.

[39] $\int dx_0 f_T(x_0) \langle \mathbf{\Psi} \mid j_0(\mathbf{x}, x_0) \mid 0 \rangle$ and $\int dx_0 f_T(x_0) \langle \mathbf{\Psi} \mid j_0(\mathbf{x}, x_0) \mid \chi \rangle$ are C^∞ and fast-decreasing functions of $\mathbf{x}$; thus, they are integrable functions of $\mathbf{x}$.

[40] See Theorem 4.1 below for a precise statement and proof of the properties of the commutator appearing in Eq. (3.54).

problem of proving the equivalence of current conservation and boundedness of $Q(\Psi, \chi)$, for an arbitrary $|\chi\rangle \in \mathfrak{D}_{qL'}$ to the same problem for $Q(\Psi, \mathbf{0})$. We already know that $Q(\Psi, \mathbf{0})$, $|\Psi\rangle \in \mathfrak{D}_{qL'}$ is bounded for a conserved current. Thus, we need only to prove the following: If the current is not conserved, then $Q(\Psi, \mathbf{0})$ is unbounded as $|\Psi\rangle$ varies in $\mathfrak{D}_{qL}$. For this, we follow Schroer and Stichel, 1966, p. 263 and look for a sequence $\{|\Psi_N\rangle\}_1^\infty$ of *quasilocal* states all having unit norm and such that $Q(\Psi_N, \mathbf{0})$ diverges as $N\rightarrow\infty$. Proceeding as in the proof of Theorem 3.3, we use Lemma 3.2 to divide $|\Psi_N\rangle$ by the energy operator, and obtain:

$$Q(\Psi_N, \mathbf{0}) = \lim_{R\rightarrow\infty} \langle \Psi_N' \mid \mathbf{j}(\cdot\nabla f_R f_T) \mid \mathbf{0}\rangle + \lim_{R\rightarrow\infty} \langle \Psi_N' \mid j^\mu(\partial_\mu f_R f_T) \mid \mathbf{0}\rangle, \quad (3.63)$$

where $|\Psi_N'\rangle$ is a quasilocal state of the form $(P_0)^{-1}|\Psi_N\rangle$.

The first term in the right-hand side of Eq. (3.63) is seen to vanish by an argument similar to the one given in the proof of Theorem 3.3. Now, if $\partial^\mu j_\mu \neq 0$, we make the choice (Schroer and Stichel, 1966)[41]

$$|\Psi_N'\rangle = \frac{1}{\| j^\mu(\partial_\mu f_N f_T) \mid \mathbf{0}\rangle \|} \cdot j^\mu(\partial_\mu f_N f_T) \mid \mathbf{0}\rangle; \quad (3.64)$$

then by a calculation similar to the one performed in the proof of Theorem 3.1, one can conclude that $Q(\Psi_N, \mathbf{0})$ behaves as $N^{3/2}$ as $N\rightarrow\infty$.

E. Additional Remarks

As we have shown in the preceding subsection, the study of the large-R behavior of a formal charge is carried out by evaluating the behavior of $\| j_0(f_R f_T) \mid \mathbf{0}\rangle \|$ or of closely related expressions. The question arises as to how our results depend upon the choice of the sequence of functions f_R considered in Eqs. (3.1)–(3.3).

With the choice (3.2), where Λ is kept fixed, we found the behavior (3.22), i.e.,

$$\| j_0(f_R f_T) \mid \mathbf{0}\rangle \| \sim cR. \quad (3.65)$$

However, the choice

$$f_R(\mathbf{x}) = f(|\mathbf{x}|/R), \qquad f_R(\mathbf{0}) = 1, \quad (3.66)$$

leads to a better behavior:[42] by a calculation parallel to the one carried out in Sec. 3.D (proof of Theorem 3.2), one can easily show that (3.66) leads to, for a conserved current,

$$\| j_0(f_R f_T) \mid \mathbf{0}\rangle \| \sim c' R^{1/2}. \quad (3.67)$$

The problem of determining the optimal choice for the sequence $\{f_R\}$ has no general answer; in special cases one can conclude that the least divergent behavior is as expressed in Eq. (3.67),[43] but in general the question cannot be settled without further knowledge of $g(p)$ [defined in Eq. (3.24)]. This fact can perhaps be qualitatively recognized as a consequence of the Reeh–Schlieder Theorem (see Theorem 2.1).

Indeed, by this theorem, given an $f_R(\mathbf{x})$, we can construct $\bar{f}_R(\mathbf{x})$ such that

$$\bar{f}_R(\mathbf{x}) = f_R(\mathbf{x}) \qquad \text{for} \quad |\mathbf{x}| \leq R,$$

and find some operator A, localized in the region where $\bar{f}_R(\mathbf{x}) \neq f_R(\mathbf{x})$ and in $-T \leq x_0 \leq T$, such that $A|\mathbf{0}\rangle$ approximates arbitrarily well the state $j_0(f_R f_T)|\mathbf{0}\rangle$. If in turn $A|\mathbf{0}\rangle$ could be well approximated by $j_0[(\bar{f}_R - f_R)f_T]|\mathbf{0}\rangle$, we see that, by changing f_R outside of the region $0 \leq |\mathbf{x}| \leq R$, the value of $\| j_0(f_R f_T) \mid \mathbf{0}\rangle \|$ could be changed at will.

In view of these qualitative remarks, one might wonder if, by a suitable choice of functions f_R such that $f_R(\mathbf{x})\rightarrow 1$ as $R\rightarrow\infty$, one could make the limit as $R\rightarrow\infty$ of $\| j_0(f_R f_T) \mid \mathbf{0}\rangle \|$ finite. However, this possibility is excluded and we now show that, *independently of how $f_R \rightarrow 1$ as $R\rightarrow\infty$, $\| j_0(f_R f_T) \mid \mathbf{0}\rangle \|$ cannot remain bounded as $R\rightarrow\infty$.*[44]

In order to prove this property, we first remark that, for a general current operator j_μ, we have the integral representation (Umezawa and Kamefuchi, 1952; Källèn, 1952; and Lehmann, 1954)

$$\langle \mathbf{0} \mid j_\mu(x), j_\nu(y) \mid \mathbf{0}\rangle = \int dp e^{ip(x-y)} \rho_{\mu\nu}(p), \quad (3.68)$$

with $\rho_{\mu\nu}$ the derivative of a measure having support in V_m^+. With the notation

$$\hat{g}(\mathbf{p}) = \int dp_0 |\tilde{f}_T(p_0)|^2 \rho_{00}(p),$$
$$\bar{g}(|\mathbf{p}|) = \int d\Omega_p \hat{g}(\mathbf{p}), \quad (3.69)$$

one finds (Reeh, 1968, p. 692) that $\hat{g}(\mathbf{p})$ and $\bar{g}(|\mathbf{p}|)$ are a C^∞ and fast decreasing as $|\mathbf{p}|\rightarrow\infty$.

Consider a state of the form

$$|\Psi_\epsilon\rangle = \int dx j_0(x) h_\epsilon(\mathbf{x}) f_T(x_0) \mid \mathbf{0}\rangle, \quad (3.70)$$

with

$$\tilde{h}_\epsilon(\mathbf{p}) \sim [(|\mathbf{p}|^2+\epsilon)^{k+1}]^{-1}, \ \epsilon > 0, \ \tfrac{1}{4} > k > 0. \quad (3.71)$$

$|\Psi_\epsilon\rangle$ is not quasilocal but is normalizable, since $\bar{g}$ is integrable and

$$\langle \Psi_\epsilon \mid \Psi_\epsilon\rangle \sim \int d|\mathbf{p}|\, \bar{g}(|\mathbf{p}|)[|\mathbf{p}|^2/(|\mathbf{p}|^2+\epsilon)^{2(k+1)}] \leq \langle \Psi_\epsilon \mid \Psi_\epsilon\rangle|_{\epsilon=0} \equiv C^2.$$

One also finds

$$\langle \Psi_\epsilon \mid j_0(f_R f_T) \mid \mathbf{0}\rangle \sim \int d\mathbf{p} \tilde{f}_R(\mathbf{p}) \hat{g}(\mathbf{p}) [(|\mathbf{p}|^2+\epsilon)^{k+1}]^{-1}.$$

As $R\rightarrow\infty$, $\tilde{f}_R(\mathbf{p}) \rightarrow (2\pi)^3 \delta(\mathbf{p})$ in the sense of distributions. Since $\hat{p}(\mathbf{p})[(|\mathbf{p}|^2+\epsilon)^{k+1}]^{-1}$ is a good test function in $\mathbf{p}$, we can take the limit with impunity and obtain

$$\langle \Psi_\epsilon \mid j_0(f_R f_T) \mid \mathbf{0}\rangle \sim \text{const } (\epsilon)^{-k-1}.$$

[41] States $|\Psi_N\rangle$ which are quasilocal and which satisfy $P_0|\Psi_N'\rangle = |\Psi_N\rangle$ with $|\Psi_N\rangle$ given below are easily constructed.

[42] Alternatively, the choice $\Lambda \propto R$ leads to the same behavior.

[43] This is the case for free fields and is probably the best possible behavior in the general case.

[44] If $\| j_0(f_R f_T)|\mathbf{0}\rangle \|$ were bounded in R but did not have a limit as $R\rightarrow\infty$, we could still conclude that for the given sequence $\{f_R\}$ there exists a subsequence $\{\bar{f}_R\}$ such that $j_0(\bar{f}_R f_T)|\mathbf{0}\rangle$ converged weakly. (This is so because any bounded sequence of vectors in $\mathfrak{H}$ contains a weakly convergent subsequence) (Akhiezer and Glazman, 1961). This would contradict Theorem 3.2.

By virtue of the Cauchy–Schwartz inequality, we have

$$\| j_0(f_R f_T) \mid \mathbf{0}\rangle \| \geq (\| \Psi_\epsilon \|)^{-1} \mid \langle \Psi_\epsilon \mid j_0(f_R f_T) \mid \mathbf{0}\rangle \|. \tag{3.72}$$

Thus, as $R \to \infty$, the left-hand side of (3.72) becomes bigger than $(\epsilon)^{-k-1}$ times a constant independent of ϵ. Since ϵ can be chosen arbitrarily small, we conclude that $\| j_0(f_R f_T) \mid \mathbf{0}\rangle \|$ cannot be bounded as $R \to \infty$, whatever choice is made for the sequence $\{ f_R \}$ (provided, of course, that $f_R \to 1$ as $R \to \infty$).

For a conserved current, it is instructive to look for more explicit examples of states $\mid \Psi \rangle$ for which

$$\lim_{R\to\infty} \langle \Psi \mid j_0(f_R f_T) \mid \mathbf{0}\rangle \neq 0. \tag{3.73}$$

Clearly, $\mid \Psi \rangle$ cannot be quasilocal and thus must describe long-range correlations. Furthermore, the quantum numbers of $\mid \Psi \rangle$ must be the same as those of $j_0(f_R f_T) \mid \mathbf{0}\rangle$. For a Hermitian current, this implies that $\mid \Psi \rangle$ can only contain particle–antiparticle pairs. In order to further illustrate this point, we consider the simple example of a free charged scalar field $\varphi(x)$, with the usual current $j_\mu = i : \varphi^* \overleftrightarrow{\partial}_\mu \varphi :$. With $a^+(\mathbf{k})$, $b^+(\mathbf{k})$ the creation operators for particles and antiparticles of momentum $\mathbf{k}$, consider the state (Swieca, 1966)

$$\mid \mathbf{2}\rangle \equiv \int f(\mathbf{k}_1, \mathbf{k}_2) \{ [(\mathbf{k}_1+\mathbf{k}_2)\cdot(\mathbf{k}_1-\mathbf{k}_2)]/(\mathbf{k}_1+\mathbf{k}_2)^2 \} \times a^+(\mathbf{k}_1) b^+(\mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2 \mid \mathbf{0}\rangle, \tag{3.74}$$

with a wave function $f(\mathbf{k}_1, \mathbf{k}_2) > 0$ and such that $\langle \mathbf{2} \mid \mathbf{2}\rangle = 1$. It is then easily verified that, with the choice $\mid \Psi \rangle = \mid \mathbf{2}\rangle$, Eq. (3.73) is satisfied in this model, although $\mid \mathbf{2}\rangle$ is a neutral state (and thus $G \mid \mathbf{2}\rangle = 0$). Thus, we see that the origin of our troubles lies in the possibility of having pairs produced from the vacuum by the current j_μ. This and the translation invariance of the "formal" state $Q \mid \mathbf{0}\rangle$ provide an intuitive explanation of the origin of the divergence of $\| Q \mid \mathbf{0}\rangle \|$. In fact, translation invariance implies that these vacuum fluctuations must appear "everywhere in the same way" and add up to an infinite value for $\| Q \mid \mathbf{0}\rangle \|$. These considerations also indicate the reason why similar difficulties do not appear in nonrelativistic quantum mechanics.

An interesting problem, and one which is still totally open, is the one of finding a relation between the "deviation" of the formal charge from an operator, and the lack of current conservation. So far, all we know is that the formal charge coincides on a dense set with an operator if and only if the current is conserved. However, no relation is known between the "amount" of current nonconservation (symmetry breaking) and properties of the charge.

For the sake of completeness, we also mention that a formal charge can be defined as the limit of a sequence of operators, *provided* one takes the limit in a "superweak" topology. This amounts to a further enrichment in the structure of the Hilbert space in the sense of embedding it in a larger space. Improper vectors (i.e., vectors having infinite ordinary norms) can then be treated as ordinary vectors, with the proviso that their scalar product is defined only with vectors in a subset of the original space.[45] Although one might believe that an approach along these lines could be helpful for answering the mentioned problem of characterizing nonconserved charges, the existing work on the subject (Katz, 1966; De Mottoni, 1967) does not shed much additional light on the properties of formal charges. Thus, we feel justified in not giving an account of this approach.

Finally, we recall that Theorems 3.1 and 3.2 do not depend on the mass-gap hypothesis, while we made use of this assumption in the proof of Theorems 3.3–3.7. However, Theorems 3.4–3.7 can also be proved under the weaker assumption that no massless particles are present (Ezawa and Swieca, 1967; Swieca, 1970). Thus, the mass spectrum could be allowed to come arbitrarily close to the point zero, provided no discrete eigenstates belonging to the eigenvalue zero of P^2 exist that are different from the vacuum.

4. CHARGES AS GENERATORS OF SYMMETRY TRANSFORMATIONS[46]

At the heuristic level, a charge Q is identified with the generator of the one-parameter continuous Abelian group of unitary operators $U(\tau)$, τ real, defined as[47]

$$U(\tau) \equiv \exp[iQ\tau]. \tag{4.1}$$

When acting on operators A, these unitaries induce the transformations

$$A \to A_\tau \equiv \exp[iQ\tau] A \exp[-iQ\tau]. \tag{4.2}$$

If this map is C^∞ in τ, one finds

$$A_{(0)} = A, \qquad dA_\tau/d\tau \mid_{\tau=0} = i[Q, A], \tag{4.3}$$

$$d^n A_\tau / d\tau^n \mid_{\tau=0} = i^n [_n[Q, A]], \qquad n = 0, 1, 2, \cdots, \tag{4.4}$$

where $[_n[\cdot, A]]$ is inductively defined as follows:

$$[_0[\cdot, A]] = A; \quad [_1[\cdot, A]] = [\cdot, A],$$
$$[_n[\cdot, A]] = [\cdot, [_{n-1}[\cdot, A]]. \tag{4.5}$$

Thus, one obtains *formally*

$$A_\tau = \sum_{n=0}^{\infty} \frac{(i\tau)^n}{n!} [_n[Q, A]]. \tag{4.6}$$

Vice versa, given a continuous one-parameter group of unitary transformations $U(\tau)$, $-\infty < \tau < +\infty$, there exists a self-adjoint operator that generates it. (This is Stone's Theorem, see the Appendix)

In the case of symmetries associated with a Hermitian conserved local current, at a more rigorous level the identification of a charge with a generator has to be

[45] To be precise, one constructs from the original space a "rigged" Hilbert space (Gel'fand and Vilenkin, 1964).

[46] Throughout this section, we only consider Hermitian conserved currents. The mass-gap hypothesis will be explicitly stated whenever needed.

[47] See the Appendix.

reconsidered. In fact, by Lemma 3.1, we know that, when it exists, the operator G, associated with the formal charge, is Hermitian. However, self-adjointness is a stronger condition than Hermiticity, and G must be self-adjoint in order to be a generator of symmetry transformations (see the Appendix). Furthermore, G is only defined on a dense set, and there are domain problems which restrict the validity of Eqs. (4.2)–(4.6).

We shall proceed as follows: we first consider the commutator between the "partial charge" $j_0(f_R f_T)$ and a localized operator A, and show that the limit

$$\lim_{T\to 0}\lim_{R\to\infty}[j_0(f_R f_T), A]\equiv[Q, A] \tag{4.7}$$

exists and is independent of how $f_R\to 1$, $f_T\to\delta$. This will provide a tool for constructing the multiple commutators $[_n[Q, A]]$. It will also allow us to complete our discussion of the properties of the operator G associated with a formal charge. Indeed, (4.7) leads to a constructive definition of $G|\Psi\rangle$ for any localized state $|\Psi\rangle$. We check that this definition coincides with the less direct one discussed in Sec. 3.E. For the definition to consistently define an operator, we find that Goldstone phenomena must be absent.

In order to make our discussion fairly self-contained, in Sec. 4.B we briefly digress on the Goldstone Theorem for spontaneously broken symmetries. We emphasize there that the Goldstone Theorem can be stated as the impossibility of consistently defining an operator for the formal charge when spontaneously broken symmetries are present, a property which (as we saw in Sec. 3) is shared by intrinsically broken symmetries (non-conserved currents).

In Sec. 4.C we briefly consider the more mathematical questions of the self-adjointness of G, the domain problems, etc., that were mentioned above.

A. Commutators Involving Charges and Definition of Generators

We first prove the following theorem.

THEOREM 4.1 (Kastler, Robinson, and Swieca, 1966) Let A be a localized operator, and j_μ a local, locally conserved current. The commutator

$$C(A)=[j_0(f_R f_T), A] \tag{4.8}$$

is independent of f_R and of f_T for sufficiently large R.

Since $j_0(f_R f_T)$ is only defined on the Wightman domain $\mathfrak{D}$ (see Sec. 2.B), Eq. (4.8) and all the operator relations to be written below should be understood as relations valid where each side is applied to a vector in $\mathfrak{D}$. Theorem 4.1 is valid independently of the mass-gap hypothesis. We remark that this theorem is of crucial importance if the identification (4.3) is to be preserved. In fact, Theorem 4.1 shows that the limit (4.7), which corresponds to $[Q, A]$, for A localized exists and is independent of how one approaches the formal limit Q.

Proof. Let A be localized in a (bounded) region $\mathcal{O}$. For sufficiently large L, we can construct two light cones having apexes at $(L, \mathbf{0})$, $(-L, \mathbf{0})$, and such that their intersection, the *diamond* **O**, contains all of $\mathcal{O}$ (see Fig. 1).

For $|\mathbf{x}|$ greater than $L+T$, the segment: $\mathbf{x}$ fixed, $0\leq|x_0|\leq T$ is totally spacelike relative to **O**. Thus, for $|\mathbf{x}|>L+T$, $j_0(\mathbf{x}, f_T)$ commutes with A by local commutativity. Since $f_R(\mathbf{x})=1$ for $|\mathbf{x}|\leq R$, one easily sees that $C(A)$ does not depend on the choice of f_R for $R>L+T$.

[We remark here that this part of the proof is independent of current conservation as well as of how Λ in Eq. (3.2) is allowed to depend on R].

In order to prove that $C(A)$ is independent of f_T, we define $\hat{f}$ as follows:

$$\hat{f}(x_0)=\int_{-T}^{x_0}[f_{T_1}{}^{(1)}(x_0')-f_{T_2}{}^{(2)}(x_0')]\,dx_0, \tag{4.9}$$

where, say, $T\geq T_1\geq T_2$, and $f_{T_1}{}^{(1)}$, $f_{T_2}{}^{(2)}$ are functions of the type considered in Eq. (3.2). Here $\hat{f}$ is a good test function since it is C^∞ and by (3.2) it vanishes outside of $(-T, T)$. Clearly,

$$j_0[f_R(df/dx_0)]=j_0(f_R f_{T_1}{}^{(1)})-j_0(f_R f_{T_2}{}^{(2)}) \tag{4.10}$$

and, by current conservation, we obtain

$$\begin{aligned} j_0[f_R(d\hat{f}/dx_0)]&\equiv-(\partial^0 j_0)(f_R\hat{f})\\ &=+\sum_{k=1}^{3}(\partial^k j_k)(f_R\hat{f})\\ &\equiv-\sum_{k=1}^{3}j_k\left(\frac{\partial f_R}{\partial x_k}\hat{f}\right). \end{aligned} \tag{4.11}$$

Since $\hat{f}(x_0)$ vanishes outside of $(-T, T)$ and since $\partial^k f_R(\mathbf{x})$ vanishes for $|\mathbf{x}|\leq R$, we see that, for $R>L+T$, $j_k(\partial^k f_R\hat{f})$ is localized in a region which is totally spacelike relative to **O**. Thus, $j_k(\partial^k f_R\hat{f})$ commutes with A for all $R>L+T$. From this and Eqs. (4.9)–(4.11), the second part of the theorem is immediately proved.

One easily shows that $C(A)$ is a strictly localized operator. By applying Theorem 4.1 to the commutator between the partial charge and $C(A)$, one also shows

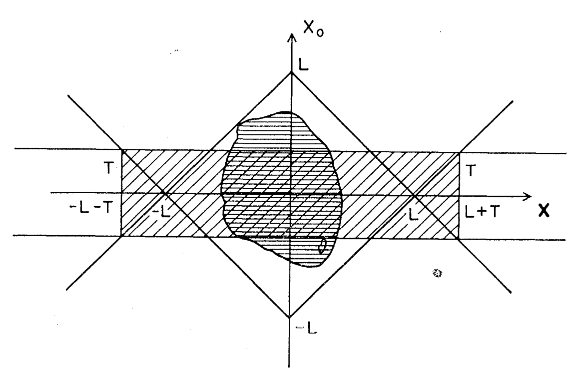

FIG. 1. The operator A is localized in the space–time region $\mathcal{O}$; $j_0(f_R f_T)$ is localized in the strip having width $2T$ for $|\mathbf{x}|>L+T$; all points in the strip are spacelike relative to $\mathcal{O}$.

the existence of all multiple commutators

$$[_n[Q, A]] \equiv \lim_{T\to 0} \lim_{R\to\infty} [_n[j_0(f_R f_T), A]] \quad (4.12)$$

for any local A. The commutator $[_n[j_0(f_R f_T), A]]$, for A localized, is independent of $f_R f_T$ for R greater than some finite R_n.[48]

We now proceed with the definition of the generator G associated with the formal charge Q. We define it as follows: G has all localized states in its domain and, for any localized operator A,

$$GA \mid \mathbf{0}\rangle \overset{\text{def}}{\equiv} [Q, A] \mid \mathbf{0}\rangle, \quad (4.13)$$

with $[Q, A]$ defined as in Eq. (4.12) for $n=1$. By taking A for the identity operator, we see that

$$G \mid \mathbf{0}\rangle = 0. \quad (4.14)$$

Of course, we must check that Eq. (4.13) does indeed consistently define an operator. If $\mid \Psi\rangle$ is a quasilocal state and *if the mass-gap hypothesis is satisfied*, G is the operator determined by Q in the sense of Sec. 3. In fact, Theorems 3.3 and 4.1 imply that, for $\mid \Psi\rangle$ a quasilocal state, and A a local operator,

$$\langle \Psi \mid GA \mid \mathbf{0}\rangle = \lim_{T\to 0} \lim_{R\to\infty} \langle \Psi \mid j_0(f_R f_T) A \mid \mathbf{0}\rangle$$
$$= \lim_{T\to 0} \lim_{R\to\infty} \langle \Psi \mid [j_0(f_R f_T), A] \mid \mathbf{0}\rangle, \quad (4.15)$$

and the existence of G is guaranteed by Theorem 3.5, while property (4.14) is satisfied in view of Eq. (3.21).[49] Thus, when no massless particles are present, Eq. (4.13) consistently defines an operator G on any localized state $\mid \Psi\rangle$.

Eq. (4.14) is a consistency check for the definition (4.13), and for it to be violated, massless particles must be present. The connection between this and the occurrence of spontaneously broken symmetries will be considered next. Then following this digression, we will return to our study of the finite transformations generated by G.

B. Symmetries and Spontaneously Broken Symmetries

We now investigate the consistency of the definition (4.13) of the generator G.

From (4.13) and (4.14), we see that

$$\langle \mathbf{0} \mid [Q, A] \mid \mathbf{0}\rangle = 0, \qquad \forall A \text{ localized}, \quad (4.16)$$

is a necessary condition for (4.13) to be a consistent definition. We have already proved (Theorem 3.3) that Eq. (4.16) is satisfied in the presence of a mass gap. It can be shown (Ezawa and Swieca, 1967; Swieca, 1970) that Eq. (4.16) also holds under the weaker assumption that no massless particles are present. (The mass spectrum can come arbitrarily close to zero, provided the vacuum is the only discrete eigenstate of the mass operator belonging to the eigenvalue zero).

Thus, in the absence of zero-mass particles,

$$\lim_{R\to\infty} \langle \mathbf{0} \mid [j_0(f_R f_T), A] \mid \mathbf{0}\rangle = 0 \quad (4.17)$$

for all localized operators A.[50]

This property is intimately connected with the impossibility of having spontaneously broken symmetries[51] in the absence of zero-mass particles. In fact, Eq. (4.17) can be taken as a statement of the Goldstone theorem (Goldstone, 1961; Goldstone, Salam, and Weinberg, 1962) (see below), a point which we now briefly discuss.

Spontaneously broken symmetries arise when one is given a conserved local current, but the formal charge is *not* the generator of symmetry transformations (i.e., unitary operators). As we shall illustrate in Sec. 4.C, the transition from the formal charge to the finite transformations corresponding to (4.1) is rather delicate from the mathematical point of view. In view of this fact, we will assume that such transformations have been defined in some way and that they are "generated" by the formal charge in a very weak sense to be specified below.

We specify our assumptions as follows[52]: for all localized operators A in the Hilbert space $\mathfrak{H}$ of physical states, and for all real τ, a map

$$A \to A_\tau \quad (4.18)$$

is given. This map is assumed to satisfy the following conditions:

(i) It is "internal": If A is localized in a region $\mathcal{O}$, A_τ for all τ, is also localized in the same region.

(ii) $A\to A_\tau$ is compatible with the usual operations with operators:

$$\mathfrak{D}_A \subseteq \mathfrak{D}_{A_\tau} \quad \text{for the domains of } A \text{ and } A_\tau;$$
$$(AB)_\tau = A_\tau B_\tau,$$
$$(A+B)_\tau = A_\tau + B_\tau,$$
$$(A^\dagger)_\tau = (A_\tau)^\dagger. \quad (4.19)$$

(iii) $A\to A_\tau$ is a continuous map:
If

$$\| (A - B_n) \mid \Psi\rangle \| \underset{n\to\infty}{\to} 0, \quad (4.20)$$

then

$$\| (A_\tau - (B_n)_\tau) \mid \Psi\rangle \| \underset{n\to\infty}{\to} 0; \quad (4.21)$$

$A\to A_\tau$ is also continuous in τ:

$$\| (A_\tau - A) \mid \Psi\rangle \| \underset{\tau\to 0}{\to} 0, \quad (4.22)$$

[48] We shall return to this point in Sec. 4.C.

[49] Equation (4.14) is also satisfied under the weaker assumption that there are no massless particles in theory. (See Sec. 4.B.)

[50] In view of Theorem 4.1, we need not take the limit $T\to 0$ since the commutator (4.8) is independent of f_T for large R.

[51] A precise definition of a spontaneously broken symmetry will be given below.

[52] See Kastler, Robinson, and Swieca (1966), where an analogous formulation is given within the algebraic approach to quantum field theory of Haag and Kastler (1964).

for

$$\forall \mid \Psi\rangle \in \bigcap_{n=1}^{\infty} \mathfrak{D}_{B_n} \cap \mathfrak{D}_A$$

in (4.20), (4.21), and $\forall \mid \Psi\rangle \in \mathfrak{D}_A$ in (4.22).

(iv) The maps $A \rightarrow A_\tau$ form an Abelian group: With additive notation

$$(A_\tau)_{\tau'} = A_{\tau+\tau'}; \tag{4.23}$$

this Abelian group is continuous in view of (iii).

(v) The formal charge associated with a given local, locally conserved current j_μ is the generator of the transformations $A \rightarrow A_\tau$ in the following sense: For all strictly localized A, $\langle 0 \mid A_\tau \mid 0 \rangle$ is differentiable in τ at $\tau = 0$ and

$$(d/d\tau)\langle 0 \mid A_\tau \mid 0 \rangle \mid_{\tau=0} = \lim_{R\to\infty} i\langle 0 \mid [j_0(f_R f_T), A] \mid 0 \rangle. \tag{4.24}$$

This is clearly a minimal requirement for the map $A \rightarrow A_\tau$ to be in some way generated by the formal charge.

Goldstone's theorem can now be stated as follows (Goldstone, 1961; Goldstone, Salam, and Weinberg, 1962; Kastler, Robinson, and Swieca, 1966; Ezawa and Swieca, 1967; Swieca, 1970; Streater, 1965 and 1965a).

THEOREM 4.2 The map (4.18), satisfying (i)–(v) is a symmetry of the theory in the absence of zero-mass particles. Thus, if there are no massless particles, there exists a family $\{U(\tau)\}$ of unitary operators implementing the map (4.18):

$$A_\tau = U(\tau) A U(\tau)^\dagger. \tag{4.25}$$

Furthermore, these unitary operators satisfy the following conditions:

$$U(\tau)U(\tau') = U(\tau+\tau'),$$
$$U(\tau) \mid 0 \rangle = \mid 0 \rangle, \tag{4.26}$$

and $U(\tau)$ is continuous in τ.

Proof of Theorem 4.2 In the absence of massless particles Eq. (4.17) holds, so that Eq. (4.24) implies

$$(d/d\tau)\langle 0 \mid A_\tau \mid 0 \rangle \mid_{\tau=0} = 0 \tag{4.27}$$

for all localized A. From (i), $A_{\tau'}$ has the same localization properties as A, and, by choosing $A_{\tau'}$ instead of A, Eq. (4.27) gives

$$(d/d\tau)\langle 0 \mid A_\tau \mid 0 \rangle \mid_{\tau=\tau'} = 0; \tag{4.28}$$

thus, for all τ,

$$(d/d\tau)\langle 0 \mid A_\tau \mid 0 \rangle = 0, \tag{4.29}$$

or, equivalently,

$$\langle 0 \mid A_\tau \mid 0 \rangle = \langle 0 \mid A \mid 0 \rangle. \tag{4.30}$$

Given any two localized operators B and C, we can apply Eq. (4.30) to $B^\dagger C$, and from (ii) we obtain

$$\langle 0 \mid B_\tau^\dagger C_\tau \mid 0 \rangle = \langle 0 \mid B^\dagger C \mid 0 \rangle. \tag{4.31}$$

Thus, the map (4.12) leaves invariant the scalar product of any two localized states. Since the set of all localized states is dense in $\mathfrak{H}$, one easily sees that the continuity requirement (4.21) implies that the scalar product of any two states in $\mathfrak{H}$ is left invariant under the transformation (4.18),[53] and the transformation must be a unitary operator $U(\tau)$. The invariance of the vacuum is a consequence of the so-called *reconstruction Theorem* (Streater and Wightman, 1964, Sec. 3–4), which states that the knowledge of the vacuum expectation values of all the quasilocal operators or of all localized operators is sufficient to completely and uniquely reconstruct the corresponding field theory. Equation (4.30) states that the expectation value of any localized operator in the transformed vacuum

$$\mid 0 \rangle_\tau = U(\tau) \mid 0 \rangle \tag{4.32}$$

equals the one in the original vacuum $\mid 0 \rangle$. From this to conclude that $\mid 0 \rangle_\tau = \mid 0 \rangle$ requires some technical arguments which the reader can easily reconstruct by adapting those in Kastler, Robinson, and Swieca (1966), p. 117. There, the reconstruction theorem is replaced by its analog for the Haag–Kastler algebraic approach (called the Gelfand–Naimark–Segal construction).

Goldstone's phenomena will not be discussed here in detail; excellent reviews can be found in Robinson (1966), Reeh (1968), Swieca (1970), Kastler (1967), Kibble (1967), Guralnik, Hagen, and Kibble (1968), and Katz and Frishman (1967). The Goldstone theorem states that, in a theory having the usual local structure, the occurrence of massless particles is a necessary condition for having a "spontaneously broken symmetry" (SBS). In our non-Lagrangian framework, a spontaneously broken symmetry can be precisely defined as a map (4.12) satisfying (i)–(iv) and associated with a conserved current in the sense of (v), which is not a symmetry of the theory (i.e., which cannot be unitarily implemented).[54]

From the proof of Theorem 4.2, we see that a necessary and sufficient condition for having a SBS is that there exists at least one localized operator A such that

$$\langle 0 \mid A_\tau \mid 0 \rangle \neq \langle 0 \mid A \mid 0 \rangle. \tag{4.33}$$

By (v), Eq. (4.32) implies

$$\langle 0 \mid [Q, A] \mid 0 \rangle \neq 0, \tag{4.34}$$

and a necessary condition for this is the presence of massless particles.

[53] Since the localized states form a dense set, any vector $\mid \Psi \rangle$ can be arbitrarily well approximated by $A \mid 0 \rangle$ for some suitably chosen *local* operator A. The transformed vector $\mid \Psi \rangle_\tau$ is then defined by continuity from $A_\tau \mid 0 \rangle$.

[54] This situation is to be contrasted with the case of an *intrinsically broken* symmetry, which arises when the current is not exactly conserved.

If Eq. (4.34) holds, according to our discussion in Sec. 4.A it is impossible to consistently define an operator G associated with the formal charge and having the vacuum in its domain.

A situation that seems to be relevant in practice is the one in which a SBS is an approximate symmetry in the following sense (Reeh, 1968): The transformations (4.18) are *locally* unitarily implementable but cannot be *globally* implemented by unitary transformations.

Local unitary implementability for (4.18) means that, for each region $\mathcal{O}$ in space time and for all A localized in $\mathcal{O}$, the map $A \rightarrow A_\tau$ is unitarily implementable:

$$A \rightarrow A_\tau = U(\mathcal{O}, \tau) A U(\mathcal{O}, \tau)^\dagger. \qquad (4.35)$$

If there is no unitary operator $U(\tau)$ which performs the transformation (4.35) and is *independent* of the region of localization for A, the map $A \rightarrow A_\tau$ is not *globally* unitarily implementable.

Transformations of this kind will not leave global entities invariant, a situation mirrored by Eq. (4.32), which reflects the noninvariance of the vacuum.

A thorough discussion of these points can be found in the quoted literature, especially Reeh (1968) where they are discussed within the Haag–Kastler algebraic approach.

We remark that, in this section, the transformations (4.18) were assumed as being given *a priori*. The problem of the actual construction of transformations generated by a formal charge will be discussed in outline in the next subsection.

C. Symmetry Transformations Generated by a Charge[55]

Given a conserved current in a local theory without massless particles, we know that the formal charge Q defines an operator G. For a Hermitian current, by Lemma 3.1, G is a Hermitian operator. This however is not sufficient for $\exp[iG\tau]$ to be a unitary operator depending on τ in a continuous fashion. If G can be closed, and if its closure is a self-adjoint operator, then $\exp[iG^\dagger\tau]$ is a unitary operator, since $G^\dagger = G^{\dagger\dagger}$, and $G^{\dagger\dagger}$ is the closure of G.

Equivalently[56] we can say that if G is essentially self-adjoint, its adjoint $G^\dagger$ is the generator of unitary transformations

$$U(\tau) = \exp[iG^\dagger\tau]. \qquad (4.36)$$

Furthermore, by Eq. (4.14), we have

$$U(\tau) \mid 0\rangle = \mid 0\rangle, \qquad (4.37)$$

while the converse of Stone's theorem[57] implies that the transformations (4.36) form a continuous Abelian group:

$$U(\tau)U(\tau') = U(\tau+\tau'), \qquad (4.38)$$

$$\| [U(\tau) - U(\tau')] \mid \Psi\rangle \| \xrightarrow[\tau\to\tau']{} 0. \qquad (4.39)$$

At this point one ought to look for conditions to be satisfied by the current j_0 that are equivalent to the essential self-adjointness of G.

No general answer to this highly mathematical question is presently available, and probably the problem can only be solved in some concrete examples.

Let us make the technical assumption[58] that the partial charges $j_0(f_R f_T)$ are essentially self-adjoint operators; this allows us to consider the unitary operators

$$U_{RT}(\tau) \equiv \exp[ij_0^\dagger(f_R f_T)\tau] \qquad (4.40)$$

and to make some comments on the connection between $U(\tau)$ and the limit

$$\lim_{R\to\infty, T\to 0} \exp[ij_0^\dagger(f_R f_T)\tau]. \qquad (4.41)$$

We remark that this connection cannot be very simple. Indeed, the vector

$$\mid 0\rangle_{RT(\tau)} \equiv \exp[ij_0^\dagger(f_R f_T)\tau] \mid 0\rangle \qquad (4.42)$$

does not converge in any sense to $\mid 0\rangle$ *as* $R\rightarrow\infty$.[59] Recalling Eq. (4.37), we see that

$$\mid 0\rangle = U(\tau) \mid 0\rangle \neq \lim_{R\to\infty, T\to 0} U_{RT}(\tau) \mid 0\rangle. \qquad (4.43)$$

For $\mid \Psi\rangle$ a localized state, we have[60] [cf. Eqs. (4.13–4.14)]:

$$\sum_{n=0}^{N} \frac{(i\tau)^n}{n!} G^n \mid \Psi\rangle \equiv \sum_{n=0}^{N} \frac{(i\tau)^n}{n!} [_n[j_0(f_R f_T), A_\Psi]] \mid 0\rangle, \qquad (4.44)$$

with R greater than a suitably chosen R_0. We remark that, from Theorem 4.1, one easily proves that the multiple commutators $[_n[j_0(f_R f_T), A_\Psi]]$ exist, are localized in a region which does not depend on n, and become independent of $f_R f_T$ for all R greater than a suitably chosen R_0, independent of n.

---

[55] We collect in the Appendix some mathematical definitions and theorems used in this subsection.

[56] The term $\exp[iG^\dagger\tau]$ is not necessarily given by its power series on all vectors in the domain of $G^\dagger$. See the Appendix, in particular the discussion on Nelson's theorem.

[57] See the Appendix.

[58] If the vacuum is an analytic vector for a (Hermitian) Wightman field smeared with (real) test functions of compact support, one says that "the vacuum is analytic." Borchers and Zimmermann (1963) proposed to adopt the analyticity of the vacuum as an extra postulate for quantum fields. They have shown that, if the vacuum is analytic, the smeared field has a dense set of localized analytic vectors and thus, by Nelson's Theorem (see Appendix), it is essentially self-adjoint. Furthermore, analyticity of the vacuum was shown by Borchers and Zimmermann to be the condition to be satisfied in order for a functional formulation of the theory to be possible. However, it must be emphasized that the physical significance of the analyticity of the vacuum is unclear and there are Wightman fields for which it is not satisfied.

[59] See Kastler, Robinson, and Swieca (1966) for a proof of this statement. We remark that $U_{RT}(\tau) \mid 0\rangle$ might still converge weakly to a vector $\lambda \mid 0\rangle$, with $|\lambda| \neq 1$. See Reeh (1968) for a discussion of this point.

[60] In the following, we omit taking $G^\dagger$ and $j_0^\dagger$ instead of G and j_0.

If a vector $|\Psi\rangle$ is analytic for G, then

$$\exp[iG\tau]|\Psi\rangle = \sum_{n=0}^{\infty} \frac{(i\tau)^n}{n!} G^n |\Psi\rangle. \tag{4.45}$$

If, in addition, $|\Psi\rangle$ is strictly localized, we see from Eq. (4.44) that

$$\exp[iG\tau]|\Psi\rangle = \sum_{n=0}^{\infty} \frac{(i\tau)^n}{n!} [_n[j_0(f_R f_T), A_\Psi]]|0\rangle|_{R>R_0}. \tag{4.46}$$

At this point, we need to make a distinction between internal and external symmetries. An internal symmetry is defined here as a symmetry which maps all operators localized in a region $\mathcal{O}$ into operators localized in the same region $\mathcal{O}$. An external symmetry is a symmetry which is not internal.

For an internal symmetry generated by G, it is at least plausible that there exists a dense subset of localized vectors which are analytic for G. This is probably the case when the theory is described by a finite set $\{\phi_i\}$ of Wightman fields having definite internal quantum numbers (see Sec. 5.C). Thus, the series (4.45) is expected to be convergent for internal symmetries and $|\Psi\rangle$ in a dense subset of all localized states. If one could actually prove this property, by Nelson's Theorem[61] the essential self-adjointness of G would follow. Furthermore, Eq. (4.46) would allow one to explicitly calculate the effect of $U(\tau)$ on localized states.

For an external symmetry, the constructive definition of G given in Sec. 4.A becomes clumsier for constructing $\exp[iG\tau]$. Indeed, although Eq. (4.44) retains its validity, one does not expect the series in Eq. (4.45) to be convergent for localized states. The intuitive reason for this can be most easily understood in the case in which G is one of the generators P_μ of space–time translations. If $|\Psi\rangle$ is strictly localized, it will contain components belonging to arbitrarily large eigenvalues of the energy and momenta. Thus, the series (4.45) might not have a finite radius of convergence.[62]

The above qualitative remarks are not compelling, since the convergence of (4.46) for $|\Psi\rangle$ localized might still hold. However, we conjecture that if G had a dense set of localized analytic vectors, the symmetry generated by G would necessarily be internal. Thus, in the case of an internal symmetry we suggest that one ought to look for an extension of the definition of G given in Sec. 4.A so as to include, say, quasilocal states which are not strictly localized.

Related to the above problems is the problem of finding the operators A for which

$$A_\tau \equiv U(\tau) A U(\tau)^\dagger \tag{4.47}$$

equals the following limit:

$$\lim_{R\to\infty, T\to 0} A_{R,T,\tau}, \tag{4.48}$$

with

$$A_{R,T,\tau} \equiv \exp[ij_0(f_R f_T)\tau] A \exp[-ij_0(f_R f_T)\tau]. \tag{4.49}$$

If the limit (4.48) exists for a localized A, we can explicitly compute the transformation (4.47) for this particular A. If the limit (4.48) exists for an irreducible set of localized operators, Eq. (4.46) provides a tool for explicitly calculating the transformed operators.

The mathematical problems mentioned in this section (essential self-adjointness of G, convergence of $U_{RT}(\tau)$ to $U(\tau)$, etc.) are to a large extent unsolved at present. One of our main purposes in mentioning them is to give the reader some idea of what remains to be done in order to completely clarify the mathematical content and the practical usefulness of the converse of Noether's Theorem.

We refer to Reeh (1968) and Maison (1969) for a more detailed discussion of some of the points raised in this subsection.

5. CHARACTERIZATION OF CHARGES AND SYMMETRIES IN SCATTERING THEORY

We now study the concrete problem of characterizing charges and internal symmetries in scattering theory. We shall find a simple expression for a charge in terms of the asymptotic free fields of the theory.[63] This will provide us with a useful practical tool for dealing with charges and symmetries: As typical applications, we consider the cases of the generators of Poincaré transformations and of internal symmetries. The mentioned characterization also provides a simple way to prove that internal symmetry groups must be compact in a theory in which the multiplicity of each mass multiplet is finite.[64]

We then obtain the same characterization of internal symmetry transformations from the action of the symmetry transformations on the interpolating fields, following a method due to Lopuszansky (1969).

We suggest that the combined use of these methods should provide a useful shortcut for bypassing some of the problems mentioned in Sec. 4.C.

Finally, we indicate a method for constructing interpolating fields having definite internal quantum numbers. The conceptual importance of this property for interpolating fields is briefly discussed.

Throughout this section, currents are taken as Hermitian and conserved, and the mass-gap hypothesis

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[61] See Appendix.

[62] In such cases, $\exp[iG\tau]|\Psi\rangle$ is *not* defined by (4.45), but by continuity from its values for $|\Psi_N\rangle$'s satisfying $|\Psi_N\rangle\to|\Psi\rangle$ and such that the series

$$\sum_{n=0}^{\infty} \frac{(i\tau)^n}{n!} G^n |\Psi_N\rangle$$

converges. Alternative definitions of $\exp[iG\tau]|\Psi\rangle$ can be found in the literature quoted in the Appendix.

[63] This problem was discussed in Orzalesi, Sucher, and Woo (1968).

[64] This statement needs the qualification that the symmetry group be semisimple and faithfully represented by unitary transformations.

is made.[65] We consider only theories satisfying *asymptotic completeness*, i.e., theories in which the asymptotic free fields form an irreducible set.

We assume that the reader is familiar with the usual asymptotic conditions in quantum field theory, at least at the level of the classic LSZ papers (1955 and 1955a). In order not to obscure the practical value of our discussion, we shall avoid the most technical points, and occasionally make use of semirigorous arguments. We do, however, indicate how these arguments ought to be modified in order to make them more rigorous.

A. Characterization of Charges

Perhaps the most important property of the generators of symmetry transformations is that they can be used without any detailed knowledge of the dynamics involved. This, in a sense, is the main motivation for considering symmetries at all: to obtain restrictions on physical quantities without having to study the dynamical details of the processes involved.

In scattering theory, the kinematics is fully described by the asymptotic configurations, whereby each particle, being well separated from all other particles, can be thought of as free. A symmetry implements the fact that certain properties are preserved by the dynamics of the system; such properties can thus be completely characterized by looking at the asymptotic configurations. All this is best described by the property of *additivity* that ought to be satisfied by the generator of symmetry transformations.[66] Thus, e.g., the total electric charge of a state describing N protons and M antiprotons is $(N-M)e$, irrespective of the configuration properties of this $(N+M)$-particle state.

In the introductory textbooks on quantum field theory, the additivity of generators is made apparent by writing expressions for them such as

$$\int d\mathbf{k} f(\mathbf{k}) a^{\dagger}(\mathbf{k}) a(\mathbf{k}), \tag{5.1}$$

with $f(\mathbf{k})$ a c-number function, and $a^{\dagger}(\mathbf{k})$, $a(\mathbf{k})$ the usual creation and annihilation operators {we consider the simplest case of a single scalar Hermitian field ϕ:

$$\phi(x) = (1/(2\pi)^{3/2}) \int d\mathbf{k} (1/(2\omega_{\mathbf{k}})^{1/2}) \times [e^{ikx} a^{\dagger}(\mathbf{k}) + e^{-ikx} a(\mathbf{k})] \tag{5.2}$$

with $\omega_{\mathbf{k}} = k_0 = (m^2 + \mathbf{k}^2)^{1/2}$, m being the mass of the particles associated with the field ϕ}.

Now, while expressions such as (5.1) are meaningful for free fields, the presence of interaction makes the creation and annihilation operators time dependent in a possibly discontinuous way, and this might make (5.1) a meaningless expression for interacting fields. Furthermore, an interacting field effectively describes infinitely many (virtual) particles, and (5.1), even when it is meaningful, is not guaranteed to be additive.

Our problem is to show that the generator associated with a formal charge, for a conserved current, does indeed satisfy the property of being additive on the asymptotic configurations. In order to prove this property, we ought to look for a simple expression of the generator in terms of the asymptotic fields.

In the simplest case of a theory describing only one type of scalar, neutral stable particle with mass m, we will find that a generator G can indeed be written as

$$G = \int d\mathbf{k} f(\mathbf{k}) a^{\dagger}{}_{\text{in,out}}(\mathbf{k}) a_{\text{in,out}}(\mathbf{k}) \tag{5.3}$$

in terms of the asymptotic "*in*" and "*out*" creation and annihilation operators. We already mentioned that the expression (5.3) is meaningful since it involves only time-independent free operators.[67] In order to make it more transparent that the expression (5.3) is indeed additive, we use the CCR's to write

$$[G, a^{\dagger}{}_{\text{in,out}}(\mathbf{k})] = f(\mathbf{k}) a^{\dagger}{}_{\text{in,out}}(\mathbf{k}),$$
$$[G, a_{\text{in,out}}(\mathbf{k})] = -f(\mathbf{k}) a_{\text{in,out}}(\mathbf{k}), \tag{5.4}$$

and remark that, since the integrand in (5.3) is normal ordered,

$$\langle \mathbf{0} \mid G \mid \mathbf{0} \rangle = 0 \tag{5.5}$$

Thus, the matrix elements

$$\langle \mathbf{k} \mid G \mid \mathbf{k}' \rangle = f(\mathbf{k}) \delta(\mathbf{k} - \mathbf{k}') \tag{5.6}$$

of G between one-particle states, together with its property of being additive, completely characterize G.

We now observe that, conversely, Eqs. (5.4) and (5.5) completely characterize G. Indeed, by asymptotic completeness, any operator that commutes with the "in" or "out" fields is a c number. Thus, Eq. (5.4) determines G up to a c number, which is fixed by Eq. (5.5). Hence, if we want to prove that a given operator L equals G, we only have to check that L satisfies the same *CR*'s as G with the "in" and "out" fields, and that the vacuum expectation values of L and G coincide.

We now return to our problem of proving additivity for the generator associated with a formal conserved charge.

We know that the formal charge determines an operator, defined as in Sec. 4.A. Let us call L this operator associated with the formal charge having the density j_0. One easily sees that L is translationally invariant and time independent. Thus, its matrix elements between one-particle states[68] will be of the following form:

$$\langle \mathbf{k} \mid L \mid \mathbf{k}' \rangle = f(\mathbf{k}) \delta(\mathbf{k} - \mathbf{k}') \tag{5.7}$$

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[65] Thus, we want to avoid the occurrence of broken symmetries, whether arising from lack of current conservation or from Goldstone-type phenomena. The mass-gap hypothesis is also made because no rigorous scattering theory is presently available for massless fields.

[66] An operator G is additive if g_1+g_2 is in the spectrum of G whenever g_1 and g_2 are in the spectrum of G. For the group-theoretical notion of additivity and a more complete and rigorous analysis of it, see Doplicher, Haag, and Roberts (1969, 1970).

[67] The function $f(\mathbf{k})$ in (5.3) is the same whether one expresses G in terms of the "in" or "out" operators. In fact, G on the one-particle states determines f, and the "in" and "out" one-particle states coincide.

[68] We momentarily neglect the problem of whether or not L has such states in its domain. See remark (iii) below.

for some function $f(\mathbf{k})$. We take Eq. (5.7) as *defining* $f(\mathbf{k})$ and then *construct* an operator G from Eq. (5.3) or, equivalently, from Eqs. (5.4–5.5). Thus, G is defined so as to be additive and to agree with L on the subspace of one-particle states. We want to prove the following theorem:

THEOREM 5.1 (Orzalesi, Sucher, and Woo, 1968) The generator L associated with the formal charge for a local, locally conserved current j_μ satisfies the *CR*'s[69]

$$[L, a^\dagger_{\text{in,out}}(\mathbf{k})]=f(\mathbf{k})a^\dagger_{\text{in,out}}(\mathbf{k}),$$
$$[L, a_{\text{in,out}}(\mathbf{k})]=-f(\mathbf{k})a_{\text{in,out}}(\mathbf{k}). \quad (5.8)$$

Equivalently, since $\langle \mathbf{0} \mid L \mid \mathbf{0}\rangle=0$, we must prove that

$$L=G, \quad (5.9)$$

with G defined by (5.4) and (5.5), and with $f(\mathbf{k})$ given by (5.7).

In order to prove Theorem 5.1, we first prove the following lemma.

Lemma 5.1 Let ϕ be a local field, local relative to the local conserved current j_μ. The commutator

$$C(x)\equiv[L, \phi(x)]\equiv \lim_{R\to\infty, T\to 0} [j_0(f_R f_T), \phi(x)] \quad (5.10)$$

is a local field. Thus, $C(x)$:

(i) transforms like a field under translations:

$$T(a)C(x)T(a)^\dagger=C(x+a); \quad (5.11)$$

(ii) commutes with itself for spacelike separations:

$$[C(x), C(y)]=0 \quad \text{for} \quad x\sim y; \text{ and} \quad (5.12)$$

(iii) commutes for spacelike separations with any local field ψ which is local relative to ϕ and j_0:

$$[C(x), \psi(y)]=0 \quad \text{for} \quad x\sim y. \quad (5.13)$$

Property (i) follows from the fact that $\phi(x)$ is a local field and L commutes with the space–time translation operators. We now prove (iii); since the proof of (ii) is completely analogous but slightly more involved.[70]

For $x\sim y$ and x, y, z all different from each other, we have

$$[[j_0(z), \phi(x)], \psi(y)]=[[j_0(z), \psi(y)], \phi(x)], \quad (5.14)$$

where we used the Jacobi identity[71] and dropped the term containing $[\phi(x), \psi(y)]$ which vanishes by local commutativity.

For $x\sim y$, we can choose z_0 in an open interval Z_0 in such a way that, for all $\mathbf{z}$, the four-vector z is spacelike separated from at least one of the two points x, y (see Fig. 2). From (5.14), it follows that, for $x\sim y$,

$$[[j_0(z), \phi(x)], \psi(y)]=0 \quad \text{for} \quad z_0\in Z_0. \quad (5.15)$$

By taking $f_T(z_0)$ concentrated in Z_0 and by integrating over z, we obtain

$$[[j_0(f_R f_T), \phi(x)], \psi(y)]=0 \quad \text{for} \quad x\sim y. \quad (5.16)$$

By Theorem 4.1, $C(x)$ is independent of f_T, and Lemma 5.1 is proved.

We make the following remarks:

(a) Often in the literature, locality is invoked in order to conclude that, for equal times,

$$[j_0(\mathbf{z}, x_0), \phi(x)]=F(x)\delta(\mathbf{x}-\mathbf{z})$$
$$+\mathbf{F}_1(x)\cdot\nabla\delta(\mathbf{z}-\mathbf{x})+\cdots, \quad (5.17)$$

where the dots indicate a finite sum of terms involving higher derivatives of $\delta(\mathbf{z}-\mathbf{x})$. Equation (5.17) expresses the fact that, for equal times, only the point $\mathbf{x}=\mathbf{z}$ is not spacelike relative to $\mathbf{z}$, so that the commutator can be nonvanishing only at that point. From (5.12), the locality of $C(x)$ formally follows upon integration. We emphasize that in our argument *we did not need to use equal-time commutators*. The formal proof indicated in (5.17) is faulty in many respects, chiefly because equal-time commutation need not be well defined on general principles.[72]

(b) In all practical circumstances, one is faced with basic local and relatively local fields $\{\phi_i\}$ which form an irreducible set, and with currents $j_\mu^{(i)}$ which are related to the basic fields by local field equations, and thus are local and local relative to the basic fields. Furthermore, the currents are used to describe the dynamics of the system, and thus are usually assumed to be themselves elements of an irreducible set of local and relatively local fields. In such cases, the following two theorems, due to

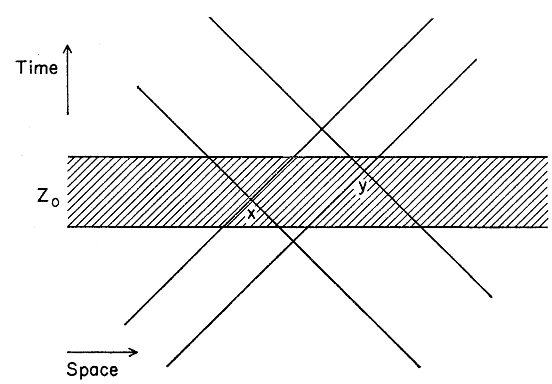

Fig. 2. The points x and y are spacelike separated. All z in the strip having width Z_0 are spacelike separated with respect to at least one of the two points x, y.

[69] Equation (5.8) and Eqs. (5.10)–(5.16) below are to be understood as relations valid when both sides are applied on a quasilocal state. As a general rule, in this section we will not enter into any details concerning domain problems. We understand that such problems ought to be taken care of by consistently working with limits of *d.d.s.f.*'s and making all the needed assumptions.

[70] To be more precise: when Theorem 5.2 below applies, (ii) follows immediately from (iii) [see remarks (a)–(c) below]. Otherwise, a direct proof of (ii) can be given by iterating the proof of (iii).

[71] We emphasize that we are taking x, y, z all different from each other. Thus, we are using the Jacobi identity for the *smeared-out* fields only.

[72] See, e.g., Orzalesi (1968) for additional details on this point.

Borchers (1960), are of the greatest importance:

THEOREM 5.2 Relative locality is a transitive property.

More precisely, let there be given two irreducible sets of local and relatively local fields $\{\phi_i\}$, $\{\chi_i\}$, and a third set of fields $\{\psi_i\}$ which are local relative to the ϕ_i's, and let all three sets of fields have a common Wightman dense domain $\mathfrak{D}$. Then, the ψ_i's are local and local relative to the χ_i's. Thus, locality is an equivalence relation, and we can consider the equivalence class of all local fields which are local relative to the fields of a given irreducible set (*Borchers class* for the given set of fields).

THEOREM 5.3 Two irreducible sets of local fields in the same Borchers class have the same S-matrix.

(c) As a consequence of Theorem 5.2, we see that in all cases of practical interest, $\psi(x)$ is local relative to j_μ, if it is local relative to ϕ. Thus, we might drop the "and j_μ" in our statement of property (iii). Furthermore, (5.12) becomes an immediate consequence of (5.13) and Theorem 5.2.

(d) Since we are assuming that the current j_μ is local relative to the interacting field ϕ, we cannot assume that j_μ is also local relative to the asymptotic field ϕ_{in}. Indeed, in the spirit of remark (b), by Theorem 5.2 we would have that ϕ and ϕ_{in} are relatively local. But then, by Theorem 5.3, ϕ would have S matrix equal to one, since ϕ_{in} is a free field.

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1 Since $C(x)$ is a local field, we can apply the LSZ asymptotic condition and write, in the sense of (weak) asymptotic convergence:

$$\begin{aligned}[L, \phi_{\text{in,out}}(x)] &= \lim_{x_0\to\mp\infty} C(x) \\ &= \lim_{x_0\to\mp\infty} \int d\mathbf{k}\{\langle \mathbf{k} \,||\, [L, \phi(x)] \,|\, \mathbf{0}\rangle a^\dagger{}_{\text{in,out}}(\mathbf{k}) \\ &\qquad + \langle \mathbf{0} \,|\, [L, \phi(x)] \,|\, \mathbf{k}\rangle a_{\text{in,out}}(\mathbf{k})\} \\ &= \int d\mathbf{k}\{\langle \mathbf{k} \,|\, [L, \phi_{\text{in,out}}(x)] \,|\, \mathbf{0}\rangle a^\dagger{}_{\text{in,out}}(\mathbf{k}) \\ &\qquad + \langle \mathbf{0} \,|\, [L, \phi_{\text{in,out}}(x)] \,|\, \mathbf{k}\rangle a_{\text{in,out}}(\mathbf{k})\} \\ &= \int d\mathbf{k}\{\langle \mathbf{k} \,|\, [G, \phi_{\text{in,out}}(x)] \,|\, \mathbf{0}\rangle a^\dagger{}_{\text{in,out}}(\mathbf{k}) \\ &\qquad + \langle \mathbf{0} \,|\, [G, \phi_{\text{in,out}}(x)] \,|\, \mathbf{k}\rangle a_{\text{in,out}}(\mathbf{k})\} \\ &= [G, \phi_{\text{in,out}}(x)]. \end{aligned} \tag{5.18}$$

In deriving Eq. (5.18), we made use of the fact that L is time independent, and satisfies $L\,|\,\mathbf{0}\rangle = 0$ [cf. Eq. (4.14)]. We then used the fact that $\phi_{\text{in,out}}$ only creates and destroys one particle, and L coincides with G on the subspace of one-particle states. Finally, in the last step we used the fact that G is defined to be additive.

By asymptotic completeness, Eq. (5.18) and the equality of the vacuum expectation values of G and L imply Eq. (5.9), so that Theorem 5.1 is proved.

Remarks: (i) In the following, we return to our usual notation and write G for the generator associated with a conserved formal charge. In view of Theorem 5.1, no confusion between L and the G defined in Eqs. (5.4–5.6) could possibly arise, since $G = L$.

(ii) According to the discussion of Sec. 3, the domain of definition of G is the set of all quasilocal states. In Sec. 4, we provided a constructive definition for the action of G on the localized states. We remark that G not only commutes with the energy–momentum operator on the localized states, but also on the quasilocal states. This property and the corresponding generalization of Theorem 4.1 for quasilocal operators A can be easily seen to be a consequence of Theorems 3.3 and 3.5, and of Theorem 2.4.

Thus, although the constructive definition indicated in 4.A becomes clumsier for quasilocal states, we are not restricted in applying G to a quasilocal state.

(iii) The above remark is relevant to the proof of Theorem 5.1. In fact, the smeared-out field $\phi(f)$ will, in general, create many-particle states from the vacuum. However, if f is suitably chosen[73] in the class $\mathcal{S}$, then $\phi(f)\,|\,\mathbf{0}\rangle$ is a one-particle state. Furthermore, the set of states obtained in such way is dense in the subspace of one-particle states, which are thereby guaranteed to be in the domain of the generator. If the domain of G only contained the strictly localized states, for f strictly localized, f would not vanish for large p^2 and the state $\phi(f)\,|\,0\rangle$ could not be a one-particle state.

(iv) It is not in the spirit of this section to seek absolute rigor. However, we remark that our Theorem 5.1 can be proved in a completely rigorous way by working within the Haag–Ruelle scattering theory (Haag, 1958; Ruelle, 1962; Hepp, 1965; and Araki and Haag, 1967). Once Lemma 5.1 is proved, since

$$\langle \mathbf{k} \,|\, C(x) \,|\, \mathbf{0}\rangle \neq 0, \tag{5.19}$$

one can directly apply the Haag–Nishijima–Zimmermann (1958) theorem to conclude that $C_{\text{in}}(x)$ and $\phi_{\text{in}}(x)$ must differ at most by a polynomial in the derivatives $\partial/\partial x_\mu$. This proof expressed by Eq. (5.18) ought to be regarded as an application of this theorem, and thus is also subject to the same limitations as most theorems proved within the Haag–Ruelle scattering theory; see especially Hepp (1965) and Araki and Haag (1967).

(v) The CR's (5.8) have a simple intuitive meaning: since L commutes with energy and momentum, it cannot change the support properties of a field, in momentum space. Thus, $[L, \phi_{\text{in,out}}(x)]$ must have the same energy–momentum support properties as $\phi_{\text{in}}(x)$. By asymptotic completeness, the only fields[74] satisfying

[73] The Fourier transform $\tilde{f}$ of f ought to have support concentrated around the one-particle mass m hyperboloid. $\tilde{f}(p)$ should vanish for $p^2 > \bar{m}^2$, with $\bar{m} < 2m$. See Haag (1958), Ruelle (1962), Hepp (1965), and Araki and Haag (1967) for additional details.

[74] In arriving at this conclusion, Lemma 5.1 is of crucial importance. In fact, e.g., $a_{\text{in}}(\mathbf{k}) + a_{\text{out}}(\mathbf{k})$ has the same support properties as $a_{\text{in}}(\mathbf{k})$, but it cannot be the asymptotic limit of a field local relative to ϕ. Thus, it is the locality of $[L, \phi(x)]$ relative to $\phi(x)$ that eliminates possibilities like this.

this property should have creation and annihilation operators of the form $f(\mathbf{k})a^\dagger{}_{\text{in,out}}(\mathbf{k})$, etc. In what follows, we will often use this more intuitive picture rather than its rigorous version, expressed by the use of the asymptotic condition.

B. Generalizations and Applications

1. Generalizations

If more than one type of particles is present, we can group the stable particles in mass multiplets. Thus, there will be N_i particles for each value m_i of the mass that corresponds to stable one-particle states. N_i will be assumed to be finite for each i, but we will impose no restrictions on how N_i might vary as the mass m_i increases. This situation appears to be realized in nature by the existing particles and resonances.

By energy–momentum conservation, the matrix elements of a conserved generator G_l between one-particle states will vanish unless the particles belong to the same mass multiplet. Following arguments parallel to those given in Sec. 5.A, one easily sees that Theorem 5.1 admits the following generalization:

THEOREM 5.4 In a theory which is asymptotically complete, satisfies the mass-gap hypothesis, and describes the interaction between N_1 types[75] of stable particles of mass m_1, N_2 of mass m_2, etc., a TCP-invariant generator G_l associated with a locally conserved Hermitian local current $j_\mu{}^l$ satisfies

$$[G_l, a^\dagger{}_{i;\,\text{in,out}}(\mathbf{k})] = \sum_{j=1,m_j=m_i}^{N_i} f_{lij}(\mathbf{k})a^\dagger{}_{j;\,\text{in,out}}(\mathbf{k}),$$

$$[G_l, a_{i;\,\text{in,out}}(\mathbf{k})] = -\sum_{j=1,m_j=m_i}^{N_i} f_{lji}{}^*(\mathbf{k})a_{j;\,\text{in,out}}(\mathbf{k}), \qquad (5.20)$$

$$\langle 0 \mid G_l \mid 0\rangle = 0, \qquad (5.21)$$

with

$$\delta(\mathbf{k}-\mathbf{k}')f_{lij}(\mathbf{k}) \equiv \langle j, \mathbf{k} \mid G_l \mid i, \mathbf{k}'\rangle, \qquad (5.22)$$

$| i, \mathbf{k}\rangle$ a one-particle state of type i and momentum $\mathbf{k}$, and $a^\dagger{}_{i;\,\text{in,out}}(\mathbf{k})$, $a_{i;\,\text{in,out}}(\mathbf{k})$ the creation and annihilation operators for the corresponding asymptotic free field.

2. The Generators of the Poincaré Group

We first discuss the case of energy–momentum operators. Clearly, our theorem finds application in all cases where one is given a local, conserved energy–momentum density $T_{\mu\nu}(x)$, satisfying $\partial^\mu T_{\mu\nu}=0$. The arbitrariness of $T_{\mu\nu}$ is best expressed by the following theorem which specifies sufficient conditions for a local field $T_{\mu\nu}$ to be the density of the energy-momentum operator.

THEOREM 5.5 (Orzalesi, Sucher, and Woo, 1968) Given: (i) $T_{\mu\nu}(x)$ is an (essentially) self-adjoint local field, transforming like a symmetric tensor and satisfying $\partial^\mu T_{\mu\nu}(x)=0$; (ii) $T_{\mu\nu}$ is local relative to all elements of an irreducible set of local fields, in an asymptotically complete theory satisfying the mass-gap hypothesis; (iii) the generator A_ν associated with the space integral of $T_{0\nu}$ commutes[76] with all operators whose eigenvalues serve to distinguish single-particle states degenerate in mass. Then: A_ν is proportional to the energy–momentum operator P_ν.

In view of Theorem 5.4, all that is needed in order to prove Theorem 5.5 is to show that A_ν and cP_ν, with c a constant, coincide in the subspace of one-particle states. We refer to Orzalesi, Sucher, and Woo (1968) for a proof of this property.

Under minor additional assumptions, it can also be shown (Orzalesi, Sucher, and Woo, 1968; Divgi and Woo, 1970) that the generators associated with $\int (x_\mu T_{0\nu} - x_\nu T_{0\mu})\, d\mathbf{x}$ coincide with $cM_{\mu\nu}$, where $M_{\mu\nu}$ are the generators of the homogeneous Lorentz transformations.[77]

3. Internal Symmetries

We now apply Theorem 5.4 to the study of internal symmetries. We adopt here a notion of "internal" that is probably not stronger than the one defined in Sec. 4.C: an internal symmetry transformation is defined here as one which commutes with all Poincaré transformations.[78]

If a generator G_l commutes with all Poincaré transformations, the $f_{lij}(\mathbf{k})$ appearing in Eq. (5.20) are actually independent of $\mathbf{k}$, their dependence on m_i^2 being absorbed into the index i. Thus, a generator G_l of internal symmetry transformations satisfies

$$[G_l, a^\dagger{}_{i;\,\text{in,out}}(\mathbf{k})] = f_{lij}a^\dagger{}_{j;\,\text{in,out}}(\mathbf{k}), \text{ etc.,} \qquad (5.23)$$

where j is summed from 1 to N_i.

Some additional properties of f_{lij} can be inferred if one specifies the properties of the fields having $a^\dagger{}_{j;\,\text{in,out}}$,

[75] The particle type is here determined by the internal quantum numbers and by the spin.

[76] For unbounded operators A and B, $[A, B]=0$ is to be understood as meaning $\langle A^\dagger\Phi \mid B\Psi\rangle = \langle B^\dagger\Phi \mid A\Psi\rangle$, $\forall \mid \Phi\rangle$, $\mid \Psi\rangle \in \mathfrak{D}(A)\cap\mathfrak{D}(B)$ so as to take care of a domain problems.

[77] See Divgi and Woo (1970) for a detailed analysis and proof of this property. The explicit coordinate dependence in the density of $M_{\mu\nu}$ does not create additional difficulties since it is particularly simple and known.

[78] The notion of "internal" defined in Sec. 4.C is not weaker than the one introduced here, at least for symmetries associated with conserved vector currents and for symmetries which act locally or almost locally on the interpolating fields (see Sec. 5.C, Theorem 5.6 for the characterization of such symmetries). In fact, in the case of a conserved vector current, it is not difficult to show that the corresponding generator is Poincaré invariant. For the case of an internal symmetry V acting locally or almost locally on the fields, one can show that V must commute with space–time translations as a consequence of properties of the two-point function of the transformed and original fields. For a Lorentz transformation Λ, one observes that $U(\Lambda)VU(\Lambda)^\dagger \equiv V_\Lambda$ is an internal symmetry acting locally on the fields if V is such. According to Theorem 5.6 below, $Va_{\text{in,out}}(\mathbf{k})V^\dagger = g(\mathbf{k})a_{\text{in,out}}(\mathbf{k})$, $g(\mathbf{k})$ a c-number function, for symmetries of this kind. It can then be shown that $V_\Lambda a_{\text{in,out}}(\mathbf{k})V_\Lambda{}^\dagger = g(\mathbf{k})a_{\text{in,out}}(\mathbf{k})$, with the same $g(\mathbf{k})$, so that g is actually independent of $\mathbf{k}$, so that $V_\Lambda = V$, and V commutes with Poincaré transformations. [The details can be found in Landan and Wichmann (1970) and Landan (1970).]

$a_{i;\,\mathrm{in\,out}}$ as their creation and annihilation operators, but we shall not provide a detailed analysis of such properties.[79]

From Eq. (5.23), since N_i is finite, we see that $\{f_l\}$ is a square finite-dimensional matrix.[80] Thus, we can easily iterate Eq. (5.23) and find the effect of the finite symmetry transformations on the asymptotic fields. In configuration space, we obtain, using a matrix notation,

$$\exp[iG_l\tau^l]\phi_{i;\,\mathrm{in,out}}(x)\exp[-iG_l\tau^l]\equiv\phi_{i;\,\mathrm{in,out}}(x)_{(\tau)}$$
$$=\exp[if_l\tau^l]\phi_{i;\,\mathrm{in,out}}(x),\quad \tau \text{ real.} \qquad (5.24)$$

If G_l is (essentially) self-adjoint, the matrices $\exp[if_l\tau^l]$ will be unitary. If furthermore these matrices form a unitary representation of a continuous group, this representation is finite dimensional, and one can conclude that *internal continuous symmetry groups are compact.*[81] This result follows only from current conservation in any local theory satisfying: (i) the mass-gap hypothesis, (ii) asymptotic completeness, and (iii) finite multiplicity for each mass multiplet. That internal continuous symmetry groups are necessarily compact under assumptions (i)–(iii) has been known for some time in a different context (Lopuszanski, 1969; and Doplicher, Haag, and Roberts, 1969). Our proof presents the advantage (over the other existing proofs) that we need no additional assumptions regarding the action of $\exp[iG_l\tau^l]$ on the interpolating fields. (We shall return to this point in the next subsection).

Remark: By virtue of Eq. (5.23), for an internal symmetry, G leaves invariant the subspace of all one-particle states having fixed energy and momentum. The converse of this statement is also true and can be heuristically formulated as follows:

Statement 5.1 [Coleman (1965)] Let j_μ be a local current in a theory satisfying asymptotic completeness and the mass-gap hypothesis. If the formal charge associated with j_μ leaves invariant the subspace of one-particle states with fixed energy and momentum, then the current j_μ must be conserved.

Thus, the symmetry associated with j_μ must be exact. We leave it to the reader to reformulate Statement 5.1 in rigorous terms by using the charge "matrix" rather than the formal charge.[82]

C. Characterization of Internal Symmetries

In the preceding subsection, we described a simple characterization of generators in a scattering theory. From this, we indicated how one can construct the action of the unitary operators implementing the symmetry on the asymptotic states.

In this process of "geometrization" of the charges and associated symmetries, nothing was said of the action of the unitary operators $\exp[iG\tau]$ on the interpolating fields, and the discussion of Sec. 4.C indicates that it might be very difficult to specify such action.

Here, we consider the case in which a certain symmetry transformation is defined for the interpolating fields and satisfies a rather weak assumption that is expected to hold for internal symmetries associated with a conserved current.

We then prove, following the arguments given by Lopuszanski (1969) that such a symmetry transformation acts on the asymptotic states in very much the same way as described in the preceding subsection.

We assume—throughout this section—that asymptotic completeness and the mass-gap assumption are fulfilled by our theory. We also assume that the theory describes the interaction of only J scalar stable particles, all having the same mass $m\neq 0$. This assumption is made mainly in order to simplify the discussion, and it could be weakened to cover a case as general as the one considered in Sec. 5.B.

Thus, let the theory be specified by an irreducible set of J local and relatively local interpolating fields $\phi_i(x)$, $i=1,\cdots,J$, with

$$\langle \mathbf{k}, i \mid \phi_j(x) \mid \mathbf{0}\rangle \sim \delta_{ij}\,\text{const}\, e^{ikx} \qquad (5.25)$$

Suppose now that we are given a unitary operator V such that

$$\langle \mathbf{0} \mid V \mid \mathbf{0}\rangle \neq 0 \qquad (5.26)$$

and such that V commutes with the Poincaré transformations[83] and the TCP operator.

We define new fields ψ_j as follows:

$$\psi_j(x)\equiv V\phi_j(x)V^\dagger \qquad (5.27)$$

and assume that V is such that ψ_j is local or almost local (see below) relative to the fields ϕ_i. "Almost local" in this context means that, for any N,

$$(x-y)^{2N}[\phi_i(x),\psi_j(y)]\xrightarrow[(x-y)^2\to-\infty]{}0. \qquad (5.28)$$

Then, we have the following property:

THEOREM 5.6 (Lopuszanski, 1969) Under the above assumptions, if the fields ψ_i form an irreducible set, V commutes with the S-matrix, and

$$\psi_{i;\,\mathrm{in,out}}(x)=V\phi_{i;\,\mathrm{in,out}}(x)V^\dagger=\sum_{j=1}^{J}\alpha_{ij}\phi_{j;\,\mathrm{in,out}}(x), \qquad (5.29)$$

where $\{\alpha\}$ is a unitary matrix.

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[79] See Lopuszanski (1969) for the details in a special case.

[80] This property considerably simplifies the problems connected with the fact that G is only defined as a limit of *d.d.s.f.*'s. In fact, it allows for an explicity calculation of the action of G on scattering states. However, not all is well since our results depend on the validity of Lemma 5.1, the proof of which did not take domain problems into due consideration.

[81] We are assuming that the symmetry group is a semisimple Lie group and is faithfully represented. Our assertion then follows from the fact that for a semisimple Lie group to have finite-dimensional faithful unitary representations, the group must be compact.

[82] Statement 5.1 has been rigorously proved in Pohlmeyer (1966) and, in a generalized form, in Dell'Antonio (1967).

[83] For an internal symmetry V satisfying the condition of "locality" or "almost locality" defined below, V commutes with the Poincaré group [see Landau (1970)]. Here, we list this property as a hypothesis in order to simplify the proof of Theorem 5.6.

This theorem can be generalized to the case in which one is given unitary operators $V(\tau_1, \cdots, \tau_n)$ which form a unitary representation of a continuous group. In this case, the corresponding matrices $\alpha(\tau_1, \cdots, \tau_n)$ will form a finite-dimensional unitary representation of the given group, which confirms the already stated result that internal continuous symmetry groups must be compact when only finite mass multiplets are present.

Proof of Theorem 5.6 From the fact that V commutes with all Poincaré transformations, and from the Poincaré invariance of the unique vacuum, Eq. (5.26) implies

$$V\mid 0\rangle = e^{i\beta}\mid 0\rangle, \tag{5.30}$$

with β a real number. We can absorb the phase factor $\exp[i\beta]$ in the definition of V and limit our discussion to a V satisfying

$$V\mid 0\rangle = \mid 0\rangle. \tag{5.31}$$

At this point, one uses a generalization of Theorem 4.3 [Araki, Haag, and Schroer (1961)] that applies under the weaker condition of relative almost locality. This theorem implies that the asymptotic fields $\phi_{i,\mathrm{in,out}}$ are proportional to the asymptotic fields $\psi_{i,\mathrm{in,out}}$. The proportionality constants can be obtained, by a procedure similar to the one used in Sec. 4.A, from the matrix elements of V between one-particle states.

We omit all additional details and applications, which can be found in Lopuszanski (1969), where the reader can also find a generalization of the Carruthers (1967) theorem.

Theorem 5.6 generalizes the applications of Theorem 5.4 discussed in Sec. 5.B in that V need not be of the form $\exp[iG_l\tau^l]$, with G_l the generator associated with the charge for a conserved current. On the other hand, we had to make the extra assumption—not needed for Theorem 5.4—that ψ_j as defined in Eq. (5.27) is almost local relative to ϕ_j. According to Lemma 5.1, and to the discussion of Sec. 4.C, it is plausible that

$$\exp[iG_l\tau^l]\phi_i \exp[-iG_l\tau^l] \tag{5.32}$$

is local or almost local relative to ϕ_i for the case of internal symmetries.

However, from the discussion in Sec. 4.C, we also know that very little can be said about the locality of (5.32) relative to ϕ_i.

From Eq. (5.29), one sees that the knowledge of V on the one-particle states determines V on the many-particle states. The relation between V defined on the one-particle subspace, and the unique operator induced by it on all of $\mathcal{H}$ by equations of the type (5.29), has been investigated in great generality by Segal (1959). Here, no domain problems arise since V is unitary and thus bounded.

D. Interpolating Fields With Definite Internal Quantum Numbers

In the ordinary introductory treatments of quantum field theory, it is implicitly assumed that definite internal quantum numbers can be ascribed to interpolating fields. The motivation for this assumption is, once more, the fact that, when the theory admits a continuous internal symmetry, certain properties will be unaffected by the dynamic changes that the system might undergo.

However, in the spirit of the converse of Noether's theorem, one ought to *prove* that interpolating fields can indeed be taken as having definite internal quantum numbers. Thus, for example, given a conserved current and given that, according to Sec. 4, the associated generator G satisfies[84]

$$[G, \phi_{\mathrm{in,out}}(x)] = i\phi_{\mathrm{in,out}}(x), \tag{5.33}$$

one ought to prove that the corresponding interpolating field $\phi(x)$ can always be chosen so as to satisfy

$$[G, \phi(x)] = i\phi(x), \tag{5.34}$$

i.e., it is a field having G charge one.

We now prove the above statement in a weakened form. Thus, suppose that there exists a local interpolating field ψ, having asymptotic fields $\phi_{\mathrm{in,out}}$. Let us assume that[85]

$$\psi_\tau(x) \equiv \exp[iG\tau]\psi(x)\exp[-iG\tau], \qquad 0\leq\tau\leq 2\pi \tag{5.35}$$

is a local field, which is local or almost local relative to ψ. Furthermore, we assume that the maps $\psi\rightarrow\psi_\tau$ form a continuous, differentiable Abelian group, and that the domain of definition of $\psi_\tau(f)$ is not smaller than the domain of $\psi(f)$, for $\forall f\in\mathcal{S}$.

Under the above assumptions,

$$\phi(x) \equiv \int_0^{2\pi} e^{i\tau}\psi_\tau(x)\, d\tau \tag{5.36}$$

is the desired field satisfying Eq. (5.34). Indeed, since

$$(d/d\tau)\psi_\tau = i[G, \psi_\tau], \qquad \psi_{2\pi} = \psi_0 = \psi, \tag{5.37}$$

Eq. (5.34) is clearly satisfied. Furthermore, ϕ is a local field since the ψ_τ, $0\leq\tau\leq 2\pi$ are local fields, while ϕ and ψ have the same S matrix since they are relatively (almost) local and have equal asymptotic fields (Araki, Haag, and Schroer, 1961).

The construction of ϕ indicated above is clearly independent of G being the charge for a conserved current. The essential ingredients are the group properties of the map $\psi\rightarrow\psi_\tau$ and the relative (almost) locality of ψ and ψ_τ.

Doplicher, Haag, and Roberts (1969, 1970) used constructions of the type indicated above to study theories with internal symmetries within an algebraic approach to quantum field theory (Haag and Kastler, 1964). There, the main problem was to study the

[84] For simplicity, we restrict our discussion to the case of a single generator and a single scalar field. We only consider generators which commute with Poincaré transformations. See Doplicher, Haag, and Roberts (1969, 1970) for a study of the case in which non-Abelian gauge groups are allowed.

[85] We normalize G so as to have $\exp[iG2n\pi]=1$ for all integers n.

interrelation between formulations of such theories in terms of gauge-invariant quantities only or in terms of charge-carrying fields.

6. SUMMARY AND CONCLUSIONS

According to the discussion in Sec. 3, the formal charge associated with a local current always has rather delicate convergence properties. The partial charges corresponding to $\int_V j_0(x)\,d\mathbf{x}$ do not converge in any simple operator mathematical sense as the volume V tends to infinity. At least for states $|\,\Psi\rangle$ of physical interest (states which are localized or quasilocalized of finite or infinite order), the convergence as $V\rightarrow\infty$ of $\int_V j_0(x)\,d\mathbf{x}\,|\,\Psi\rangle$ cannot be taken in the natural strong and weak topologies in the Hilbert space $\mathcal{H}$ of physical states.

We mentioned in Sec. 3.E that a precise mathematical meaning can be given to the limit as $V\rightarrow\infty$ of the above vectors. This can be done by introducing a "rigged" Hilbert space. However, the physical meaning of the additional structure thus introduced is unclear, and the existing results in this approach do not appear to clarify the issue in any significant way.

On the other hand, we found a convenient and mathematically precise way of dealing with formal charges by considering them as limits of sesquilinear forms defined on a dense set. We proved that the limit $Q(\Phi,\Psi)$ of $\langle\Phi\,|\,j_0(f_R f_T)\,|\,\Psi\rangle$ as $R\rightarrow\infty$ exists for $|\Phi\rangle$ and $|\,\Psi\rangle$ quasilocal and defines a sesquilinear form. We then reduced the problem of the infinite-volume behavior of the formal charge to the question of whether or not $Q(\Phi,\Psi)$ is the form of an operator.

This infinite-volume problem is probably the simplest example of divergences that can occur in quantum field theory. In spite of its simplicity, this problem creates a number of technical difficulties in the study of the converse of Noether's theorem:

(a) *In the case of an exactly conserved current, and when massless particles are absent,* we found it possible to construct an operator corresponding to the formal charge. This is somewhat gratifying, since it excludes the occurrence of very pathological situations. However, the mere existence of an operator G which "extends" the formal charge is not enough to answer the problem of constructing the symmetries associated with local current conservation. For this, one ought to prove that G is essentially self-adjoint, and that it does satisfy all the properties expected of a generator of continuous symmetry transformations.

The question of the essential self-adjointness of G was merely touched upon in Sec. 4.C. In all known relevant examples, questions of this kind appear to be extremely difficult to solve,[86] and we fear that little progress will be made in the near future in the case of general current operators.

[86] See Jaffe (1969) for a discussion of problems of a similar kind in the case of Hamiltonians in "constructive" quantum field theory.

In a scattering theory, and neglecting the mentioned delicate mathematical problems, we found that the operator G satisfies all the properties expected of a generator, summarized in the notion of "additivity" on the asymptotic states. This confirms our expectations from the Lagrangian formalism, and provides a simple way to characterize charges and symmetries associated with conserved currents.

We did not at all solve the problem of proving relative (almost) locality between a field and its transform under the symmetry generated by G. This problem also appears to be of the highest difficulty, as are most questions in the study of the structure of the Borchers class of a given field.

Leaving this last mentioned problem aside, a large class of symmetries which transform local fields into fields which are almost local relative to the untransformed ones were characterized by their simple action on asymptotic configurations. For continuous internal symmetries of this kind, we also found that the symmetry can be "dynamically" implemented, in the sense that the interpolating field can always be chosen as having a definite value for the internal quantum numbers.

We could not prove that the symmetry transformations associated with a conserved current satisfy the requirement of transforming the interpolating fields in a "local" or "almost local" fashion, although Lemma 5.1 indicates that this is likely to be true, since the commutator $[G,\phi]$ of the generator G with the field ϕ is local and local relative to ϕ. Thus, in order to apply the construction of Sec. 5.D to symmetries associated with conserved currents, one presently needs the additional technical assumption that the transformed field be almost local relative to the original one. However, we can still conclude that our treatment and construction of interpolating fields having definite internal quantum numbers is to be considered as a remarkable improvement over the standard treatments, which are based on much more questionable and stronger assumptions (e.g., the validity of CCR's, the use of formal expressions for the charge density, etc.)

Once the existence and properties of the generator of symmetry transformations are proved, the converse of Noether's theorem is to a large extent established, in the sense at least that (barring pathological domain problems) the characterization of charges given in Sec. 5 is sufficiently general for all practical applications.

(b) *In the case of nonconserved currents,* whereby the symmetry is *intrinsically broken,* we found that the divergence properties get worse. The formal charge no longer admits a physically interesting operator extension (having quasilocal states in its domain). We can restate Coleman's theorem in a strengthened form by saying that *whenever the formal charge determines an operator having the vacuum in its domain, the current must be conserved.*

Thus, we find a confirmation of the fact (Fabri and

Picasso, 1966; Fabri, Picasso, and Strocchi, 1967) that a unitary group describing an internal symmetry which does not commute with the Hamiltonian cannot be used for describing broken symmetries. To be more precise, we found that the generators of such one parameter unitary groups would either not be related to a formal charge or would have no quasilocal states in their domain of definition. In both cases, the practical usefulness of broken symmetries in this formulation would be highly impaired. It seems quite remarkable that we needed not recur to the study of many-point functions in order to draw our conclusions. Indeed, only two-point functions were used. This can be seen as one more instance of the power of the Reeh–Schlieder theorem (1961) [see also Jost (1969); Schroer (1958); Federbush and Johnson (1960); and Lopuszanski (1961)].

We remark that the additional difficulties inherent to the study of nonconserved charges do not appear to be relevant for the present current-algebraic formulation of broken symmetries (Gell-Mann, 1962; Adler and Dashen, 1968). Indeed, this approach, based on equal-time commutators and regularity conditions such as "partial axial-vector current conservation", can be formulated in rigorous terms in a way that takes care of the difficulties discussed here as well as of other difficulties that could be met in using equal-time commutators (Schroer and Stichel, 1966; Orzalesi, 1968).

In recent times, integrals of local densities over infinitesimally thin lightlike slabs have been the center of increasing attention. Their commutators "on the light cone" have been fruitfully investigated by several authors (see, e.g., Klauder, Leutwyler and Streit, 1970; Brandt, 1970; Leutwyler, 1968; and Jersak and Stern, 1969), as they are an important tool for studying the behavior of scattering amplitudes in certain high-energy domains. We emphasize that our results *do not* apply to such generalized "lightlike" charges, since we only considered integrals of local densities over spacelike slabs (and, in the limit $T\rightarrow 0$, over spacelike surfaces). In particular, Coleman's theorems are *not* expected to hold for lightlike charges. To the contrary, there are indications that an opposite theorem takes place, in the sense that a lightlike charge is always expected to annihilate the vacuum, irrespective of current conservation.

(c) *In the case of spontaneously broken symmetries*, whereby the current is conserved but there is no associated exact symmetry, we found that a generator cannot be consistently defined from the formal charge. Again, the fact that no charge operator exists was recognized as a consequence of the large-volume bepavior of formal charges in the presence of massless harticles: these are such as to make it impossible to define an operator extension of the formal charge having the vacuum (assumed to be unique) in its domain. Thus, Goldstone's Theorem is recognized as following from the fact that, in the presence of zero-mass particles, a formal charge can become, in a precise sense, "more divergent" than in the absence of zero-mass particles.

In a sense, spontaneously broken symmetries provide counterexamples to the converse of Noether's Theorem. On the other hand, some salient features of these counterexamples can be understood within the Lagrangian framework as well as in a more axiomatic setting (Kastler, Robinson, and Swieca, 1966; Swieca, 1966; Streater, 1965, 1965a). Of course, this does not imply that we possess a clearcut and complete understanding of Goldstone phenomena.

We can conclude that the converse of Noether's Theorem is to a large extent understood and proved in quantum field theory. Still, a great deal remains to be done: we lack, on the one hand, a satisfactory study of the mathematical properties of the operator extensions of charges (essential self-adjointness, etc.) and, on the other hand, a deeper and more complete study of the exceptions to the theorem (spontaneously broken symmetries).

Concerning Noether's theorem itself, little can be said: What emerges from our analysis of its converse is that a satisfactory study of the complete Noether problem should face the highest difficulties. Given that a certain continuous symmetry exists, one ought to prove the existence of a local conserved current whose associated charge operator generates the symmetry transformations. This statement is probably false as it stands, and a more promising attitude is to look for a classification of symmetries for which the statement is true. In any case, the fact that generators and formal charges only coincide on a dense set and cannot be strictly identified with each other, is expected to cause serious difficulties. Most likely, the proof of existence and the analysis of the properties of conserved local currents for given symmetries will face difficulties comparable to those—not yet solved!—that are met in the construction of a nontrivial Wightman quantum field.

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APPENDIX

We collect here some mathematical definitions and theorems that might prove useful especially for the understanding of Sec. 4.C. In the spirit of that section, we provide only the minimal amount of information necessary to get across some idea of the mathematical problems involved.

We only consider *linear* operators in a Hilbert space $\mathcal{H}$. Such an operator, A, is defined on a domain, $\mathfrak{D}_A$, a linear manifold of $\mathcal{H}$, and satisfies

$$A(\alpha\,|\,\Phi\rangle+\beta\,|\,\Psi\rangle)=\alpha A\,|\,\Phi\rangle+\beta A\,|\,\Psi\rangle \qquad \text{(A1)}$$

for all complex numbers α and β, and all $|\mathbf{\Phi}\rangle$ and $|\mathbf{\Psi}\rangle$ in $\mathfrak{D}_A$. To visualize the relations between linear operators it is helpful to use the notion of *graph*. The graph, $\Gamma(A)$, of a transformation A is the set of ordered pairs $\{|\mathbf{\Phi}\rangle, A|\mathbf{\Phi}\rangle\}$, where $|\mathbf{\Phi}\rangle$ runs over $\mathfrak{D}_A$. It is therefore a linear subset of $\mathcal{H}\oplus\mathcal{H}$. Then (A1) can be stated geometrically as: the graph of a linear transformation A, is a linear subset of $\mathcal{H}\oplus\mathcal{H}$. Not every linear subset of $\mathcal{H}\oplus\mathcal{H}$ is the graph of a linear transformation. The crucial point is that if $\{|\mathbf{\Phi}\rangle, 0\}$ is a point of the graph of a linear transformation then $|\mathbf{\Phi}\rangle=0$; this is a condition that guarantees that A is *single valued*.

An operator, B, is an *extension* of an operator, A, if the domain of B contains that of A: $\mathfrak{D}_B\supset\mathfrak{D}_A$, and $B|\mathbf{\Phi}\rangle=A|\mathbf{\Phi}\rangle$ for all vectors $|\mathbf{\Phi}\rangle$ in $\mathfrak{D}_A$. Under these circumstances we write $A\subset B$ or $B\supset A$. In terms of the graphs of A and B this relation is equivalent to $\Gamma(A)\subset\Gamma(B)$, where here the relation $\subset$ is ordinary set-theoretical "contained in."

An operator, A, is called *closed* if its graph $\Gamma(A)$ is a closed linear subset of $\mathcal{H}\oplus\mathcal{H}$. This can be seen to be equivalent to the condition that

$$|\mathbf{\Psi}_n\rangle\in\mathfrak{D}_A,\quad |\mathbf{\Psi}_n\rangle\to|\mathbf{\Psi}\rangle\in\mathcal{H},\quad A|\mathbf{\Psi}_n\rangle\to|\mathbf{\Phi}\rangle\in\mathcal{H} \tag{A2}$$

imply

$$|\mathbf{\Psi}\rangle\in\mathfrak{D}_A,\quad A|\mathbf{\Psi}\rangle=|\mathbf{\Phi}\rangle. \tag{A3}$$

If A is not closed, then one can look for its closed extensions. If it has any, there will be a smallest, whose graph is the closure $\bar{\Gamma}(A)$ of the graph, $\Gamma(A)$, of A. However, it can happen that $\bar{\Gamma}(A)$ is not the graph of a linear transformation, because it violates the above single valuedness condition. If $\bar{\Gamma}(A)$ is the graph of a linear transformation $\bar{A}$, $\bar{A}$ is called the *closure* of A.

Next we introduce the notion of the adjoint of an operator. This will be done in a geometrical way and then it will be verified afterwards that it coincides with the usual definition. Consider the orthogonal complement, $\Gamma(A)^\perp$ of the graph $\Gamma(A)$ of a linear transformation A. It consists of all pairs of vectors $\{\mathbf{\Psi},\chi\}$ such that the scalar product in $\mathcal{H}\oplus\mathcal{H}$ satisfies

$$(\{\mathbf{\Psi},\chi\},\{\mathbf{\Phi},A\mathbf{\Phi}\})\equiv\langle\mathbf{\Psi}|\mathbf{\Phi}\rangle+\langle\chi|A\mathbf{\Phi}\rangle=0 \tag{A4}$$

for all $\mathbf{\Phi}\in\mathfrak{D}_A$. We ask whether $\Gamma(A)^\perp$ is the graph of a transformation from the second $\mathcal{H}$ in $\mathcal{H}\oplus\mathcal{H}$ back to the first, i.e., whether there is a linear transformation $A^\perp$ such that the $\{\mathbf{\Phi},\chi\}$ satisfying (A4) are of the form $\{A^\perp\chi,\chi\}$. Applying the above-mentioned single-valuedness criterion we see that $A^\perp$ will indeed exist if $\langle\mathbf{\Psi}|\mathbf{\Phi}\rangle=0$ for all $\mathbf{\Phi}\in\mathfrak{D}_A$ implies $|\mathbf{\Phi}\rangle=0$. Thus, $A^\perp$ exists if A is defined on a dense set. Notice that the usual definition of the adjoint for an everywhere defined operator A is just given by $A^\dagger=-A^\perp$. More generally, if A is a densely defined linear operator, $\chi\in\mathfrak{D}_{A^\dagger}$ if there exists a vector $\mathbf{\Psi}$ such that

$$\langle\mathbf{\Psi}|\mathbf{\Phi}\rangle=\langle\chi|A\mathbf{\Phi}\rangle$$

for all $\mathbf{\Phi}\in\mathfrak{D}_A$. Then $\mathbf{\Psi}=A^\dagger\chi$ by definition. $A^\perp$ is called the *adjoint* of A. Since $\Gamma(A)\subset\Gamma(B)$ implies $\Gamma(A)^\perp\supset\Gamma(B)^\perp$ we have immediately that

$$A\subset B\quad\text{implies}\quad A^\dagger\supset B^\dagger. \tag{A5}$$

Clearly, since $(\Gamma(A)^\perp)^\perp=\bar{\Gamma}(A)$, $(A^\dagger)^\dagger$ exists for a densely defined A if and only if $\mathfrak{D}_{A^\dagger}$ is dense, and $(A^\dagger)^\dagger$ is the closure of A

$$\bar{A}=(A^\dagger)^\dagger. \tag{A6}$$

An operator, A, is *Hermitian* if

$$A\subset A^\dagger, \tag{A7}$$

or in words, if for every $\mathbf{\Phi}$ and $\mathbf{\Psi}\in\mathfrak{D}_A$, a dense linear set

$$\langle A\mathbf{\Psi}|\mathbf{\Phi}\rangle=\langle\mathbf{\Psi}|A\mathbf{\Phi}\rangle. \tag{A8}$$

Clearly, this implies immediately that $A^\dagger$ is densely defined, (because A must be so for $A^\dagger$ to exist) and

$$A\subset A^{\dagger\dagger}=\bar{A}\subset A^\dagger. \tag{A9}$$

An operator, A, is *self-adjoint* if

$$A=A^\dagger. \tag{A10}$$

It is *essentially self-adjoint if*

$$A^{\dagger\dagger}=A^\dagger. \tag{A11}$$

An essentially self-adjoint operator is one whose closure is self-adjoint. Every essentially self-adjoint operator is Hermitian, but not conversely [for examples, see Wightman (1966)].

The importance of self-adjoint operators in quantum mechanics arises from Stone's theorem. To state it we need the notion of a *continuous one-parameter Abelian group of unitary operators*. That is, a family of unitary operators $U(\tau)$ defined for $-\infty<\tau<\infty$ and satisfying

$$U(\tau_1)U(\tau_2)=U(\tau_1+\tau_2) \tag{A12}$$

and

$$\langle\mathbf{\Phi}|U(\tau)|\mathbf{\Psi}\rangle$$

is continuous for all τ for each fixed pair of vectors $|\mathbf{\Phi}\rangle$, $|\mathbf{\Psi}\rangle$. Stone's theorem says that every such continuous one-parameter group is of the form

$$U(\tau)=\exp[iA\tau], \tag{A13}$$

where A is a self-adjoint operator. Since there is a standard operator calculus for bounded functions of self-adjoint operators that says that for every self-adjoint A, (A.13) defines a continuous one parameter group of unitary operators, the significance of A being self-adjoint is clear [see Wightman (1966) for some examples that are relevant in physics].

In the operational calculus,[87] $\exp[iA\tau]$ is not defined by its power series, but there are useful things to be learned from a study of the power series. To state one of them, we introduce the notion of an *analytic vector* for an operator A. $\mathbf{\Phi}$ is an analytic vector for A

[87] See, e.g., Kato (1966).

if $A^n\Phi$ is defined and if the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} A^n \Phi$$

converges in some disk $|z| < R$. By the usual arguments about power series, this criterion is equivalent to the absolute convergence

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} \| A^n \Phi \|$$

in all smaller disks, $|z| < R-\epsilon$, $\epsilon < 0$. Now we can state *Nelson's theorem*: *a Hermitian operator is essentially self-adjoint if and only if it possesses a dense set of analytic vectors.*

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