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Some Basic Definitions in Graph Theory

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A systematic list of definitions of some basic concepts in graph theory of application to physics is presented. An index, some illustrative theorems, and a brief bibliography are included.

CONTENTS

INTRODUCTION

This article presents a list of definitions of some of the basic concepts arising in the theory of linear graphs and their applications, particularly to problems in statistical mechanics and combinatorics. It was prepared by the authors mainly in an endeavor to systematize and specify precisely the terminology found in the literature. We have attempted to present—in sequence—complete, unambiguous definitions in which the "exceptional cases" (often a cause of confusion or imprecision) have not been overlooked. Doubtless we have not been completely successful in these tasks. Equally the choice of "basic" definitions must remain largely subjective; indeed the selection presented here is strongly influenced by our own interests. Nonetheless, the results of our labors may be of some value to others using graph theory—if only as a convenient reference point from which to depart!

Although our main aim has been to specify a consistent terminology, we have, at various places, stated without proof certain theorems (some trivial, some profound) which illustrate the significance of the definitions. The material has been organized into sections with, generally, the more elementary and widely applicable definitions placed near the beginning. We have frequently mentioned alternative terms (in parentheses); the order of these indicates our relative preferences although these are sometimes not very strong.

The reader's attention is drawn to the *index* of terms at the end of the list of definitions, which enables the list to be used as a dictionary of graph theory. We have consulted a range of references in assembling this list. Our main sources are listed in the bibliography which, needless to say, is in no way meant to indicate the scope of the literature on graph theory or its applications, even in the restricted fields with which the authors are familiar.

1. VERTICES, ARCS, EDGES, AND GRAPHS

1.1 A *vertex set* V is a set of objects $a, b, c, \cdots, i, j, \cdots$ called *vertices* (or *points*, or *sites*).[1] These are conventionally represented by labeled geometrical points in the plane. The number of vertices in V is denoted below by v.

1.2 An *arc* (or *directed* or *oriented*, *edge* or *line*) is an ordered pair of distinct vertices from a vertex set V. The arc (i,j) is said to be *incident* with the vertices i and j, being *incident out of* the *initial vertex* i (or *tail* of the arc) and *incident into* the *terminal vertex* (or *head* of the arc). The initial vertex is *joined* to the terminal vertex by the arc. The arc (i,j) is conventionally represented geometrically by a continuous directed line from points i to j.

1.3 A *multiarc* of *multiplicity* s is a set of s distinct arcs all incident out of one vertex and all incident into a second vertex. The members of such a set may be denoted $(i,j)_1$, $(i,j)_2, \cdots (i,j)_s$ and are said to be *strictly parallel arcs*. [The arcs (i,j) and (j,i) are *parallel* but not strictly parallel.]

1.4 An *arc set* A associated with a vertex set V is a set of arcs or multiarcs or both with vertices in V.

1.5 An *edge* (or *line*, *bond*[2] or *link* or, in electric network theory, *branch*) is an unordered pair of distinct vertices from a vertex set V. The edge $[i,j]$ is said to be *incident* with the vertices i and j and to *connect* them. An edge may be represented by a continuous line connecting the points i and j.

1.6 A *multiedge* of *multiplicity* t is a set of t edges incident with the same pair of vertices. The members of such a set may be denoted by $[i,j]_1$, $[i,j]_2, \cdots [i,j]_t$ and are said to be *parallel edges*.

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[1] In electric network theory the term *node* is sometimes used as equivalent to vertex. We feel it is useful to specialize the meaning of this word somewhat: see 1.23.

[2] We prefer to retain the term *bond* to denote the *graph* $K(2)$ of two vertices and one edge (2.12).

1.7 An *edge set* E associated with a vertex set V is a set of edges or multiedges, or both, with vertices in V. The number of edges in E is usually denoted by e in the definitions below.

1.8 A *loop* is the pair obtained by taking the same vertex twice from a vertex set V. The loop $[i, i]$ is said to be *incident* with the vertex i and may be represented by a continuous line beginning and ending on the point i. A loop is not an edge[3] and may be considered either ordered or unordered.

1.9 A *multiloop* of *multiplicity* u is a set of u loops incident with the same vertex. The members of such a set may be denoted by $[i, i]_1$, $[i, i]_2, \cdots [i, i]_u$.

1.10 A *loop set* O associated with a vertex set V is a set of loops or multiloops, or both, with vertices in V.

1.11 A *directed* (or *oriented*) *graph*[4] $G \equiv (V, A, E)$ is a vertex set V of $v(G)$ vertices having at least one member, together with an associated arc set A, of $a(G)$ arcs. Since it is often convenient to ignore the directions of the arcs, it is supposed that a directed graph has an edge set E of $e(G)$ edges which contains the edge $[i, j]$ if and only if the arc set contains the *associated arc* (i, j). If both the arcs (i, j) and (j, i) are contained in A, then E contains a multiedge, $[i, j]_1$, $[i, j]_2$, of multiplicity two.

1.12 An *undirected graph* $G = (V, E)$ is a vertex set V, having at least one member, together with an associated edge set E. The term *multigraph* may be used to emphasize that multiedges are allowed.

1.13 A *null graph* has an empty vertex set. It may be denoted by Ω.

1.14 A *graph with loops* (or *general graph*): If any graph G, directed or undirected, also has a loop set O, it will be called a graph with loops, i.e., $G \equiv (V, A, E, O)$ or $G \equiv (V, E, O)$, respectively. The term *abstract graph* may be used to emphasize the set-theoretic character of these definitions of a graph in contrast to its possible geometrical *representations drawn* on some surface, etc. [see (7.24) and (7.25)].

1.15 *A graph.* It is useful to retain this term for graphs directed or undirected which have no loops (empty loop set) since this frequently avoids tedious complications in the statements of definitions and theorems. Where it is wished to emphasize that a statement applies also to graphs with loops, the explicit phrase "graph with loops" will be used. In other cases the necessary extensions and conditions may generally be provided by the reader. A graph of any type is *finite* if both its vertex and edge sets are finite; otherwise it is *infinite*.

[3] In some applications it may be useful to regard a loop as a special kind of edge. In this case the term *link* may be used specifically to indicate an edge which is not a loop.

[4] The term *digraph* has been used as an abbreviation of "directed graph" but since the word already has a well-known technical meaning, especially in cryptanalysis, namely "pair of letters," we feel its use in graph theory should not be encouraged.

1.16 *Simple:* a *simple arc* or *edge* is a multiarc or multiedge of multiplicity one; a *simple directed graph* contains only simple arcs; a *simple graph* contains only simple edges.

1.17 An *s-graph* is an undirected graph in which no multiedge has a multiplicity exceeding s. (Evidently a one-graph is a simple graph.)

1.18 An *isolated vertex* of a graph is one having no incident arcs, edges, or loops. If every vertex of a graph is isolated, the graph is *degenerate* (and has no arcs, edges, or loops).

1.19 The *valence* or *degree* (or *coordination number*) *of a vertex* of a graph G is the number of edges of G incident with that vertex. Multiedges are counted with appropriate multiplicity. In a graph with loops, each loop is counted twice.

1.20 The *in-valence* (or *in-degree*) of a vertex of a directed graph G is the number of arcs of G incident *into* that vertex. In a graph with loops, each directed loop is counted once.

1.21 The *out-valence* (or *out-degree*) of a vertex of a directed graph G is the number of arcs of G incident *out* of that vertex. In a graph with loops, each directed loop is counted once.

1.22 A *pendant vertex* of a graph is a vertex of valence one.[5] (An isolated vertex has valence zero.)

1.23 A *node* (or *principal point*) of a graph is a vertex of valence three or more (but note that in electric network theory, *node* is sometimes used to mean a vertex of any valence.)

1.24 An *antinode* is a vertex of a graph of valence one or two.

1.25 A *closed graph:* a graph is *closed* if and only if it has no vertices of valence zero or one (i.e., no isolated or pendant vertices). A graph is *open* if and only if it has vertices of valence one, i.e., pendant vertices.

1.26 A graph (with loops) is *locally finite* if and only if the valence of each vertex is finite.

1.27 A *graph of valence* or *degree* d is a graph in which each vertex is of valence d. It is said to be *d-valent*, e.g., *trivalent*, *quadrivalent*, etc. (or *regular*, *d-regular*, etc.).

1.28 In a *closed directed graph* the out-valence of each vertex is equal to the in-valence of that vertex.

THEOREM 1.1. In a finite graph (with loops) the number of vertices of odd valence or degree, is even.

1.29 *Adjacent arcs:* two distinct arcs are said to be adjacent if they have a vertex in common (which may be either an initial or a terminal vertex of either arc).

1.30 *Adjacent edges:* two distinct edges are said to be adjacent if they have a vertex in common.

1.31 *Adjacent* (*or neighboring*) *vertices:* two distinct

[5] The term *terminal node* or *terminal vertex* is sometimes used in this sense, but see 1.2, 1.23, and 2.27.

vertices are said to be adjacent if they are incident with the same edge (or arc).

1.32 *Consecutive arcs:* two arcs are *consecutive* if the terminal vertex of one is the initial vertex of the other.

1.33 A *symmetric* (*directed*) *graph* is a simple directed graph (1.16) in which any two adjacent vertices i and j are connected by both the arcs (i, j) and (j, i). An *antisymmetric* (*directed*) *graph* never contains both the arcs (i, j) *and* (j, i).

1.34 A *reflexive* (*directed*) *graph* has a directed loop at each vertex.

1.35 *Transitively directed:* a directed graph is *transitive* (or *transitively directed*) if and only if it contains the arc (i, k) whenever it contains both the arcs (i, j) and (j, k), and it never contains both the arcs (l, m) and (m, l).

1.36 A *comparability graph* is an undirected graph the edges of which may be *oriented* {i.e., for each edge $[i, j]$ either the arc (i, j) or the arc (j, i) is selected to form an arc set} such that the resulting directed graph is transitive.

1.37 An *intersection graph* of a family of sets $\{S_i\}$ is a simple graph which has a vertex i for each set S_i and which contains the edge $[i, j]$ if and only if the intersection, $S_i \cap S_j$, of the sets S_i and S_j is not empty.

1.38 An *interval graph* is the intersection graph of a set of intervals on a line.

2. HOMEOMORPHIC, ASSOCIATED, AND SPECIAL GRAPHS

2.1 An *automorphism of a graph* is a one–one correspondence between the vertices of the graph which induces a one–one correspondence between its edges and between its arcs (if any).

2.2 The *graph group* $\mathcal{G}$ of a graph G is the abstract group formed from the set of automorphisms of G.

2.3 A *free* (or *unlabeled*) graph is best defined as an equivalence class of graphs G under the automorphisms of G constituting the graph group. In less abstract terms it is a graph in which the vertices are considered to be indistinguishable. (Notice that in the original definitions of a graph, the vertices of the vertex set V are distinguishable or, equivalently, *labeled* so that any associated graph, $G \equiv (V, \cdots)$, is implicitly labeled.)

2.4 The *symmetry number* $s(G)$ of a graph G is the order of the graph group. There are thus $[v(G)]!/s(G)$ (labeled) graphs corresponding to a given free graph of $v(G)$ vertices. In the case of a graph with multiedges (1.6) of multiplicities $t_k(G)$ $(k=1, 2, \cdots)$, it is useful to define the modified symmetry number $s'(G) = s(G) \prod_k [t_k(G)]!$.

2.5 *Isomorphic graphs:* two graphs (with loops) G_1 and G_2 are *isomorphic* if there is a one–one correspondence between the vertex sets V_1 and V_2 which induces a one–one correspondence between their edge sets E_1 and E_2 and their arc and loop sets (if any). The graph G_1 is said to be an *isomorph* of G_2.

2.6 The *insertion of a vertex* of valence two on an edge, $[i, j]$, loop (i, i) or arc (i, j) means (a) the removal of the edge, loop, or arc from its respective set, (b) the addition of a new vertex k to the vertex set, and (c) the addition of the new edges $[i, k]$ and $[k, j]$, or multiedge $[i, k]_1$, $[i, k]_2$ to the edge set or the new arcs (i, k) and (k, j) to the arc set, respectively. This process is also known as the *subdivision* or *decoration* of the edge $[i, j]$.

2.7 The *suppression of a vertex* of valence two: If a graph G_1 can be obtained from a graph G_2 by the insertion of a vertex k of valence two, one may, alternatively, say that G_2 is obtained from G_1 by the suppression of the vertex k. One may write $G_2 = G^{\sigma}{}_{1(k)}$.

2.8 *Homeomorphic graphs:* two graphs are said to be *homeomorphic* (and to be *homeomorphs* of one another) if they may be made isomorphic by the insertion of one or more vertices of valence two on either, both, or neither of the graphs (or equivalently by the suppression of vertices of valence two).

2.9 *Basic topology:* two homeomorphic graphs are said to have the same basic topology. [Note that all homeomorphic graphs have the same cyclomatic number (5.10) and weak k-weight (5.34).]

2.10 *Topological type:* a graph of a given basic topology which is homeomorphic to no graph with fewer vertices is said to *represent faithfully* the topological type of all its homeomorphs.

2.11 A *complete graph* $K(n)$ (or K_n) of n vertices is a graph the edge set of which contains each of the possible $\frac{1}{2}n(n-1)$ vertex pairs once and once only.

2.12 *The bond* or *dimer* (or *line* or *edge*) is the complete graph $K(2)$. A *directed bond* or *dimer* is a graph of two vertices, one arc, and one edge.

2.13 The *double bond* (or *digon* or *lune*) is a graph of two vertices and two edges (constituting a multiedge of multiplicity two). The *triple bond*, *quadruple bond*, $\cdots$, *multibond* is a graph of two vertices and a multiedge of multiplicity 3, 4, $\cdots$, t, respectively.

2.14 The *triangle* is the complete graph $K(3)$. The *tetrahedron* is the complete graph $K(4)$.

2.15 A *complete bichromatic graph* $K(m_1, m_2)$ [or K_{m_1,m_2}] is a graph of $n = m_1 + m_2$ vertices whose vertex set is partitioned into a subset V_1 of m_1 vertices, and a subset V_2 of m_2 vertices and whose edge set consists of all of the $m_1 m_2$ possible vertex pairs with one vertex in V_1 and one in V_2.

2.16 A *complete p-chromatic graph* $K(m_1, m_2, \cdots, m_p)$ is a graph of $n = \sum_k m_k$ vertices whose vertex set is partitioned into p subsets V_k of m_k vertices $(k=1, 2, \cdots, p)$, and whose edge set consists of all the $\sum_{k<l} m_k m_l$ possible vertex pairs in which the two vertices belong to different subsets.

2.17 A *polygon* is any graph of $n \geq 2$ vertices which is homeomorphic to a triangle [or to the double bond (2.13), or to the graph of one vertex and one loop]. An *n-gon* is a polygon of n vertices. A *quadrilateral* is a 4-gon, etc.

2.18 A *theta-graph* is any graph homeomorphic to the complete bichromatic graph $K(2, 3)$ (or to the graph of two vertices and three edges, i.e., the triple bond). A theta-graph has cyclomatic number two (5.10).

2.19 An *alpha-graph* (or *tetrahedral graph*) is any graph homeomorphic to the tetrahedron $K(4)$.

2.20 A *beta-graph* is any graph homeomorphic to the graph of four vertices with edges $[1, 2]_1$, $[1, 2]_2$, $[2, 3]$, $[3, 4]_1$, $[3, 4]_2$, $[4, 1]$.

2.21 A *gamma-graph* is any graph homeomorphic to the graph of three vertices with edges $[1, 2]_1$, $[1, 2]_2$, $[2, 3]$, $[3, 1]_1$, $[3, 1]_2$.

2.22 A *delta-graph* is any graph homeomorphic to the complete bichromatic graph $K(2, 4)$ (or to the graph of two vertices and four edges).

Note that polygons, θ-graphs, α-, β-, γ- and δ-graphs, exemplify all the possible topological types of multiply connected graphs, (5.19), of cyclomatic numbers, (5.10), three or less.

2.23 The *complementary graph* $\bar{G}$ of a simple graph G of n vertices (1.16) is obtained by removing from the edge set of the complete graph $K(n)$ all those edges contained in the edge set of G.

2.24 The *reverse graph* G^R of a directed graph G is the graph obtained by reversing the sense (i.e., the orders of the vertices) of each arc.

2.25 *The covering graph* G^C *of an undirected graph* G (or *interchange graph* of G) is constructed as follows: (a) with each edge of G is associated a new vertex; these new vertices constitute the vertex set of G^C; (b) any two distinct vertices of G^C corresponding to *adjacent edges* (1.30) of G, but not belonging to the same multiedge of G, are connected by a single edge of G^C; (c) any two distinct vertices of G^C corresponding to components of the same multiedge of G are connected by two edges of G^C (i.e., by a multiedge of multiplicity two). [Notice that the covering graph of a simple graph G is just the intersection graph (1.37) of the family of edges of G.] A graph which is a covering graph is sometimes called a *line graph*.

2.26 The *covering graph* G^C *of a directed graph* G is constructed as follows: (a) with each arc of G is associated a new vertex; these new vertices constitute the vertex set of G^C; (b) any two distinct vertices a and b of G^C corresponding, respectively, to the consecutive arcs (i, j) (j, k) of G are connected by an arc (a, b) of G^C. [Note if $k=i$, the arc (b, a) is also in G^C.]

2.27 The *terminal graph* G^T of an undirected graph G (with no loops) is constructed as follows: (a) with each vertex i of valence d_i in G is associated a *cluster* (or *city*) of d_i new vertices (*terminals*), one for each edge incident with i; these vertices or terminals form the vertex set of G^T; (b) two terminals of G^T corresponding to the same edge of G (and hence belonging to adjacent clusters) are connected by an edge (*external edge*) of G^T; (c) within a cluster each terminal is connected to every other terminal by an edge (*internal edge*) of G^T.

Note that the external edges of G^T correspond to the edges of G, while the internal edges correspond to the edges of the covering graph G^C.

2.28 An *expanded graph* G^E of an undirected graph G (with no loops) is constructed in the same way as the terminal graph, except that in step (c) only the (d_i-1) internal edges $[i_1, i_2]$, $[i_2, i_3]$, $\cdots$, $[i_{d_i-1}, i_{d_i}]$ are included in the edge set of G^E ($i_1, i_2, \cdots, i_{d_i}$ denotes *any* labeling of the d_i terminals within the cluster i). The degree of any vertex of an expanded graph does not exceed three.

2.29 An *n-dimensional lattice* (or *point lattice*)[6] is an array of points (or *sites*) in n-dimensional Euclidean space with position vectors $\mathbf{r}_\nu = \nu_1\mathbf{a}_1 + \cdots + \nu_n\mathbf{a}_n$, where $\mathbf{a}_1, \cdots, \mathbf{a}_n$ are n linearly independent *primitive vectors* and where $\nu_1, \cdots, \nu_n$ are integers which take a consecutive series of values. The lattice is a *finite* $N_1 \times N_2 \times \cdots \times N_n$ *lattice* if ν_k runs from 1 to N_k $(k=1, 2, \cdots, n)$. The lattice is *infinite* if all ν_k run from $-\infty$ to $+\infty$. (For a *lattice with bonds* see 2.35–2.39 below.)

2.30 The *linear chain*, the (*plane*) *square lattice*, the *simple cubic lattice*, and the *hypercubical lattices* are lattices of $n=1, 2, 3, \geq 4$ dimensions, respectively, in which the n primitive vectors $\mathbf{a}_k$ are orthogonal and of a uniform length a, called the *lattice spacing*.

2.31 The (*plane*) *triangular lattice* is generated by the primitive vectors $\mathbf{a}_1 = (a, 0)$, $\mathbf{a}_2 = (\frac{1}{2}a, \frac{1}{2}\sqrt{3}a)$; the *body-centered cubic lattice* is generated by the vectors $\mathbf{a}_1 = (a, 0, 0)$, $\mathbf{a}_2 = (0, a, 0)$, $\mathbf{a}_3 = (\frac{1}{2}a, \frac{1}{2}a, \frac{1}{2}a)$; the *face-centered* cubic lattice is generated by the vectors $\mathbf{a}_1 = (a, 0, 0)$, $\mathbf{a}_2 = (\frac{1}{2}a, \frac{1}{2}\sqrt{3}a, 0)$, $\mathbf{a}_3 = (\frac{1}{2}a, \frac{1}{6}\sqrt{3}a, \frac{1}{3}\sqrt{6}a)$. In each of these cases the (nearest-neighbor) lattice spacing is a.

2.32 A *generalized lattice* (or a *crystal lattice*) is derived from an n-dimensional lattice (2.29) of points with position vectors $\mathbf{r}_\nu$ by replacing the point at $\mathbf{r}_\nu$ by m points with position vectors $\mathbf{r}_\nu^{(l)} = \mathbf{r}_\nu + \mathbf{b}_l$ $(l=1, 2, \cdots, m)$. The *basis vectors* $\mathbf{b}_l$ specify the positions of the m points in a *cell* of the lattice described by the vectors $\mathbf{a}_k$ (2.29).

2.33 The *plane hexagonal* or *honeycomb lattice* of lattice spacing a is a generalized lattice with primitive vectors $\mathbf{a}_1 = (\sqrt{3}a, 0)$, $\mathbf{a}_2 = (\frac{1}{2}\sqrt{3}a, \frac{3}{2}a)$, and basis vectors $\mathbf{b}_1 = (0, 0)$, $\mathbf{b}_2 = (0, a)$. The *close-packed hexagonal lattice* and the *diamond* and *ice* (*tetrahedral*) *lattices* are three-dimensional generalized lattices.

2.34 In a (*generalized*) *lattice with periodic boundary conditions* the points $\mathbf{r}_\nu$ and $\mathbf{r}_\nu^{(l)}$, $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_n)$ are identified modulo $\mathbf{N} = (N_1, \cdots, N_n)$.

2.35 A *lattice graph* L is a graph whose vertices are the points of a lattice or a generalized lattice and whose edges (or *bonds*) correspond to lines between the points. (No regularity of the bonds is implied at this stage although one may require that L be a subgraph [partial

[6] Note that in abstract algebra the term *lattice* is used with a completely different meaning (namely, a partially ordered set in which any two elements have a greatest lower bound and a least upper bound). See G. Birkhoff and S. MacLane, *A Survey of Modern Algebra* (Macmillan Co., New York, 1953).

graph (3.6)] of a uniform infinite lattice graph, (2.36).)
2.36 A *uniform infinite lattice graph* (often abbreviated to *lattice*[6]) is a graph whose vertices are the points of an infinite lattice (2.29), or generalized lattice (2.32), and whose edges (or bonds) are assigned in a translationally invariant manner (i.e., if $[\mathbf{r}_\nu^{(l)}, \mathbf{r}_{\nu'}^{(l')}]$ is an edge, then so is $[\mathbf{r}_\nu^{(l)}+\mathbf{r}_\mu, \mathbf{r}_{\nu'}^{(l')}+\mathbf{r}_\mu]$ for all lattice vectors $\mathbf{r}_\mu$). In applications, edges are frequently assigned only to pairs of points which are geometrical *nearest neighbors*. The primitive vectors, the basis vectors, and the set of edges incident with the vertices in a typical cell specify the *structure* of the lattice.
2.37 A *regular infinite lattice graph of coordination number q* (often abbreviated to a *lattice*[6] *of coordination q*) is a uniform infinite lattice graph in which each vertex has the same valence, or *coordination number q*, and, furthermore, in which *all* points are equivalent under translations, reflections, and rotations of Euclidean space.

The *nearest-neighbor linear chain*, *square lattice*, *hexagonal lattice*, etc., (see 2.30, 2.31, 2.33) are regular lattice graphs.
2.38 A (*uniform* or *regular*) *torus* or *toroidal lattice* is a (uniform or regular) lattice graph defined on a point lattice with periodic boundary conditions (2.34).
2.39 A *ring* is a one-dimensional toroidal lattice. (It is isomorphic to a polygon.)
2.40 A *wheel* $W(n)$ [or W_n] of *order n* (≥ 3) is a graph constructed from a polygon of n vertices called its *rim* by adding one further vertex, called the *hub*, and n further edges, or *spokes*, connecting the hub to the n vertices of the rim.
2.41 A *vertex star* of n edges is a graph of $n+1$ vertices consisting of one vertex incident with n edges [the remaining n vertices are pendant (1.22)]. Alternatively, it is a complete bichromatic graph $K(1, n)$, (2.15). The term *star* is sometimes used in place of *vertex star*, but we reserve this term for a more important use (5.12).
The *vertex star of a vertex i in a graph G* is a subgraph of G (3.5), consisting of the vertex i and all edges and multiedges of G incident with i, and of all vertices of G adjacent to i.

3. SUBGRAPHS, EMBEDDINGS, AND LATTICE CONSTANTS

3.1 The *deletion of a vertex i* from any graph G (with loops) means the removal of the vertex i from the vertex set of G and the removal of all incident arcs, edges, and loops from the arc, edge, and loop sets of G. The resulting graph may be denoted $G^\delta{}_{(i)}$.
3.2 The *deletion of an arc* (i, j) from a directed graph G means the removal of the arc (i, j) from the arc set of G and the removal of the corresponding edge from the edge set of G; the resulting graph may be denoted $G^\delta{}_{(i,j)}$.
3.3 The *deletion of an edge* $[i, j]$ from a graph G means the removal of the edge $[i, j]$ from the edge set of G (and, in the case of a directed graph, the removal of the corresponding arc from the arc set); the resulting graph may be denoted $G^\delta{}_{[i,j]}$. The *deletion of a loop* is defined similarly.
3.4 The *contraction of an edge* $[i, j]$, *or* of a set of edges $\{[i, j]_k\}$ constituting *a multiedge* (1.6), means the deletion of the edge or multiedge and the *identification* of the vertices i and j (thereby reducing the number of vertices in the vertex set by one). The graph resulting on contracting $[i, j]$ may be denoted $G^\gamma{}_{[i,j]}$.

Note that the contraction of all the internal edges of a terminal graph G^T (2.27) transforms it into G, while the contraction of the external edges transforms G^T into the covering graph G^C (2.25).
3.5 A *subgraph* G' of a graph G (with loops) is a graph obtained from G by deleting subsets (which may be null sets) of its vertices, arcs, edges, and loops; G' is said to be *contained* in G which is a *supergraph* of G'. One may write $G'\subseteq G$.
3.6 A *partial graph* or *spanning subgraph* of a graph G (with loops) is a graph obtained from G by deleting a subset (which may be the null set) of its arcs, edges, and loops (but retaining all the vertices of G).
3.7 A *section graph* G^* of a graph G (with loops) is a graph obtained from G by deleting a subset (which may be the null set) of its vertices. (Note that edges and loops may *not* be deleted.) Both section graphs and partial graphs are subgraphs, but the converse is not necessarily true.
3.8 An *associated section graph* of a subgraph is the section graph having the same vertex set. Two subgraphs may have the same associated section graph. The associated section graph of a partial graph or spanning subgraph of a graph G is G itself, thus all partial graphs have the same associated section graph.
3.9 A *proper subgraph, section graph*, etc., of a graph G is one which is not G itself. One may write $G'\subset G$, $G^*\subset G$, etc.
3.10 A *maximal subgraph, section graph*, etc., possessing a given property is one which is contained in no other subgraph, section graph, etc., possessing that property. A *minimal subgraph, section graph*, etc., possessing a given property is one which contains no other subgraph, section graph, etc., possessing that property.
3.11 The *union* (or *sum graph*), $G'\cup G''$ (or $G'+G''$), *of two subgraphs* of a graph G is the minimal subgraph of G containing both G' and G''. Similarly, the *intersection*, $G'\cap G''$, is the maximal subgraph contained in both G' and G''. More generally, the *union*

$$\bigcup_{l=1}^{n} G_l, \qquad \text{or } \textit{intersection}, \qquad \bigcap_{l=1}^{n} G_l,$$

of a set $\{G_l\}$ *of n graphs* (not necessarily subgraphs) is the graph whose vertex, arc, edge, and loop sets (as the case may be) are the unions or intersections, respectively, of the vertex, arc, edge, and loop sets, respectively, of the G_l.

3.12 A set $\{G_l\}$ of graphs or subgraphs is *disjoint* (or *vertex disjoint*) if the intersection of any distinct pair, $G_k \cap G_l$, is the null graph. Thus two disjoint graphs have no common vertices. Two graphs are *edge-disjoint* if they have no common edges.

3.13 An *overlap partition* $\{G', G'', \cdots, G^{(p)}\}$ of a graph G is a set (unordered) of subgraphs whose sum graph $G' \cup G'' \cup \cdots \cup G^{(p)}$ is G itself.

3.14 The *difference graph*, $G^* - G^{**}$, *of two section graphs* of a graph G is the section graph of G obtained by deleting from G^* any vertices also contained in G^{**}.

3.15 *Weak embedding:* A subgraph G' (3.5) of G which is isomorphic with a graph G_1 is said to represent an embedding of G_1 in G in the weak sense.

3.16 *Strong embedding:* A section graph G^* (3.7) of G which is isomorphic with a graph G_1 is said to represent an embedding of G_1 in G in the strong sense. A strong embedding is also a weak embedding, but the converse need not be true.

3.17 The *weak lattice constant* $(G_1; G)$ of G_1 in G is the number of weak embeddings of G_1 in G, i.e., the number of subgraphs (3.5) of G isomorphic to G_1.

3.18 The *strong lattice constant* $[G_1; G]$ of G_1 in G is the number of strong embeddings of G_1 in G, i.e., the number of section graphs (3.7) of G isomorphic to G_1.

THEOREM 3.1. (Sykes, Essam, Heap, and Hiley). If G_1 has v_1 vertices and G is any graph,

$$(G_1; G) = \sum_{v_j = v_1} (G_1; G_j)[G_j; G],$$

where the sum runs over all graphs G_j of $v_j = v_1$ vertices.

3.19 The *overlap constant* $\{G_1 \cup G_2 \cup \cdots \cup G_p = G\}$ of the set $\{G_1, G_2, \cdots, G_p\}$ in a graph G is the number of overlap partitions (3.13) of G isomorphic to $\{G_1, G_2, \cdots, G_p\}$.

THEOREM 3.2. (Sykes, Essam, Heap, and Hiley). If G_1 and G_2 are two disjoint graphs with v_1 and v_2 vertices, respectively, and $G_1 \cup G_2$ is their union (5.9), then

$$(G_1 \cup G_2; G) = (G_1; G)(G_2; G) - \sum_{v_k < v_1 + v_2} \{G_1 \cup G_2 = G_k\}(G_k; G),$$

where the sum runs over all graphs G_k of $v_k < v_1 + v_2$ vertices, and where the first term must be multiplied by a factor $\frac{1}{2}$ when G_1 and G_2 are isomorphic.

THEOREM 3.3. The weak and strong lattice constants of any finite graph G in any sufficiently large, locally finite, uniform lattice graph with periodic boundary conditions, or toroidal lattice L (2.38), of fixed structure (2.36), are finite polynomials without constant term in the variable $N = v(L)$, the number of (distinct) vertices of L, with fixed coefficients depending only on the structure of the lattice.

3.20 The *weak-* and *strong-lattice-constant polynomials*, $(G; L \mid N)$ and $[G; L \mid N]$, of a graph G in a uniform toroidal lattice graph L are the polynomials specified in Theorem 3.3. Note that the degree $n(G)$ of these polynomials will be equal to the *number of* (connected) *components* (5.9) in G.

3.21 The *weak* and *strong lattice constants per site* of an infinite uniform lattice graph L are defined in terms of the corresponding lattice-constant polynomials for the associated toroidal lattices (with periodic boundary conditions) by

$$(G; L) = N^{-1}(G; L \mid N) \mid_{N=0},$$

$$[G; L] = N^{-1}[G; L \mid N] \mid_{N=0},$$

i.e., by the coefficient of N in the respective polynomial. [Note that this definition applies irrespective of the number of components (5.9), in G.]

3.22 A *clique*, or *maximal clique*, of a graph G is a subgraph, or maximal subgraph, respectively, of G consisting of a set of vertices of G any two of which are adjacent (1.31). (A clique has no edges, arcs, or loops but in a simple undirected graph the associated section graph of a clique is a complete graph.)

3.23 A *simplicial vertex* in a graph G is a vertex which, together with *all* its adjacent vertices in G, forms a clique.

3.24 The *clique number* of a graph G is the maximum number of vertices which can constitute a clique of G.

3.25 A *stable* (or *independent*) *set of vertices* or *of edges* in a graph G is a set consisting of vertices or edges of G, respectively, no two of which are adjacent (1.30, 1.31).

3.26 The *vertex stability number* (or *coefficient of internal stability*), $\alpha(G)$, of a graph is the maximum number of vertices of G which can constitute a stable set. *The edge stability number*, $\beta(G)$, is defined similarly.

3.27 A *dominating set of vertices*, V_D, in a graph G is a subset of vertices of G such that every vertex of G is either in V_D or is adjacent to a vertex in V_D.

3.28 A *covering set of vertices*, V_C, in a graph G is a subset of vertices of G such that every edge of G is incident (1.5) with some vertex in V_C.

3.29 A *vertex of attachment of a subgraph* G' in a graph G is a vertex of G' incident (1.5) with an edge not in G'. The set of vertices of attachment may be denoted $W(G', G)$.

3.30 The *complement*, $\bar{G}'$, *of a subgraph* G' *in a graph* G is the graph obtained from G by (i) deleting all edges of

G', and (ii) deleting all vertices of G' which are not vertices of attachment of G' in G. (Note these two operations commute.)

3.31 A *perimeter vertex of a subgraph G' in a graph G* is a vertex of G which is (i) not in G', but (ii) adjacent to a vertex (of attachment) of G'. The (*vertex*) *perimeter* of a subgraph is the set of its perimeter vertices.

3.32 An *edge of attachment* of a subgraph G' in a graph G is an edge of G incident both with a vertex of G' and with a vertex not in G'. (Thus one vertex of an edge of attachment is a vertex of attachment, while the other is a perimeter vertex.)

3.33 A *perimeter edge* of a subgraph G' in a graph G is an edge of G not belonging to G' but adjacent to a vertex (of attachment) of G'. The *edge perimeter* of a subgraph is the set of its perimeter edges. Note all edges of attachment are perimeter edges, but the converse is not generally true.

3.34 A subgraph G' of a graph G is said to have *full edge* (or *vertex*) *perimeter* if and only if all edges (or vertices) of G not in G' are in the edge (or vertex) perimeter of G', respectively.

Note the important topic of the connectivity of graphs may be approached via the concepts of the vertices and edges of attachment and, correspondingly, of *detachment* modulo a vertex, an edge, or a more general subgraph (Tutte, Ore). We have preferred a development which places emphasis on paths and chains (Berge, Busacker and Saaty). These and related terms are defined in the next section.

4. PATHS, WALKS, AND CHAINS

4.1 A *path* (or *arc progression*) *of n steps* on a directed graph is an alternating sequence of $n+1$ vertices and n arcs[7] (the *steps*) in which each vertex (except the first, or *initial*, and last, or *terminal*, vertex of the path) is the terminal vertex of the preceding arc and the initial vertex of the succeeding arc, i.e., the arcs are *consecutive*. (A single vertex may be regarded as a path.) A *spanning path* includes (*visits*) every vertex of the graph.

4.2 In a *closed path*[7] the initial and terminal vertices coincide; in an *open path* these vertices are distinct.

4.3 A *circuit* is a closed path[7] of at least one step in which the initial vertex is not distinguished (i.e., sequence is defined only relative to circular order).

4.4 A *walk* (or *edge progression*) *of n steps* on a graph is an alternating sequence of n edges[7] (the *steps*) and $n+1$ vertices ($n>0$) such that each vertex (except the first and last) is incident with the preceding, and with the succeeding, edge. A *spanning walk visits* every vertex of the graph. Note that paths and circuits refer to arcs[7] and hence have an *orientation*, whereas walks, and chains and cycles (see below) refer to edges[7] and do not have a specific orientation (although a walk does have a *sense*, namely, from initial to final vertex).

4.5 A step of a walk on a directed graph is *cooriented* if the initial vertex of the corresponding arc preceeds the terminal vertex in the walk. Conversely, it is *contraoriented* if the terminal vertex preceeds the initial vertex. If all the steps of a walk are cooriented, it is a path (4.1).

4.6 A *chain of n links* (or *length n*) is a walk of n steps in which the sequence of steps (now called *links*) and the reverse sequence are not distinguished i.e., a chain has no sense. It has two *terminal vertices* which it *connects* and $n-1$ *intermediate vertices*. (To each chain correspond two walks.)

4.7 In a *closed walk* or a *closed chain* the initial and terminal vertices coincide; *open walks* and *chains* are not closed.

4.8 A *cycle* of *length n* is a closed chain of $n>2$ links in which the terminal (or initial) vertex is not distinguished. (To each cycle correspond $2n$ distinct closed walks.)

4.9 A *simple path* or *circuit*, is one in which no arc occurs (is passed through) more than once.

4.10 A *simple walk* (or a *trail*) and a *simple, chain* or *cycle* is one in which no edge occurs more than once.

4.11 An *elementary* (or *self-avoiding*) *path, circuit, walk, chain, or cycle* is one in which no edge and no vertex occurs (or is *visited*) more than once, except in the case of closed paths, walks, or chains where the initial and final vertices coincide.

4.12 A *Hamilton path* (or a *Hamilton circuit, walk, chain, or cycle*) on a graph G is an elementary path (circuit, walk, chain, cycle) which visits each vertex of G, once and only once. (Note that a Hamilton path spans the graph.)

4.13 An *Euler path* (or *Euler circuit, walk, chain, or cycle*) on a graph G is a path (circuit, walk, chain, or cycle) which contains each arc or edge of G exactly once. (An Euler path or walk is in general *not* elementary.)

4.14 A *bridge* on a graph is a simple chain the terminal vertices of which are nodes (1.23), while the intermediate vertices (if any) are of valence two.

4.15 A *pendant chain* on a graph is a simple chain in which one terminal vertex is a node (1.23) and the other is a pendant vertex, i.e., of valence one (1.22), while the intermediate vertices (if any) are of valence two.

4.16 The *distance* $l(i,j)$ between two (ordered) vertices i and j in a directed or undirected graph is the number of steps in the shortest path or walk, respectively, with initial vertex i and terminal vertex j. The distance is conventionally taken as infinite if a path or walk of the specified class does not exist.

[7] In the case of a graph with loops, paths and circuits may include directed loops in place of arcs, while walks, chains, and cycles may include loops in place of edges. A single loop and its vertex may be regarded as a circuit or chain or unit length.

4.17 The *associated* (or *remoteness*[8]) *number* $\lambda(i \mid G)$ *of a vertex* i of a graph G is defined in terms of the distances $l(i,j)$ by

$$\lambda(i \mid G) = \max_{j\epsilon V} l(i,j).$$

Note $\lambda(i \mid G) = \infty$ if G is disconnected (5.5).

4.18 The *radius* $\rho(G)$ and *diameter* $\delta(G)$ of a graph G are defined in terms of the associated numbers $\lambda(i \mid G)$ by

$$\rho(G) = \min_{i\epsilon V} \lambda(i \mid G), \qquad \delta(G) = \max_{i\epsilon V} \lambda(i \mid G),$$

with the convention $\rho(G) = \infty$ if the minimum does not exist. For a disconnected graph, $\delta(G) = \infty$.

4.19 A *center* of a graph G is a vertex i_0 for which the associated number $\lambda(i_0 \mid G)$ equals the radius $\rho(G)$ of G. A graph may have one, several, or no centers.

4.20 A *chord* of a path or circuit, or of a walk, chain, or cycle with vertices $i_1, \cdots, i_k, \cdots, i_l, \cdots$ is an arc (i_k, i_l), or an edge $[i_k, i_l]$, respectively.

4.21 A *triangular chord* of a path or circuit, or of a walk, chain, or cycle with vertices $i_1, \cdots, i_{k-1}, i_k, i_{k+1}, \cdots$ is an arc or an edge of type (i_{k-1}, i_{k+1}) or $[i_{k-1}, i_{k+1}]$, respectively.

4.22 A *triangulated* (or *rigid circuit*) *graph* is a graph in which every cycle of length greater than three has a chord.

5. CONNECTIVITY AND COHESIVITY

5.1 Two vertices i and j in a directed graph are *weakly connected* (or *joined*) if *either* there is a path (4.1) from i to j, that is, with initial vertex i and terminal vertex j, *or* if there is a path from j to i. A *total* or *weakly connected* (*directed*) *graph* is one in which each pair of vertices is weakly connected.

5.2 Two vertices i and j in a directed graph are *strongly* (or *mutually*) *connected* if there is a path (4.1) *both* from i to j and from j to i. A *strongly connected* (*directed*) *graph* is one in which each pair of vertices is strongly connected. The graph consisting of a single vertex is defined to be strongly connected.

5.3 *Connected vertices.* Two vertices in a graph are connected if there is a chain (of edges) containing both of them (i.e., *connecting* them).

5.4 A *connected graph C* is one in which every pair of vertices are connected. A single vertex is defined to be connected. A strongly connected graph is connected, but the converse is not necessarily true.

5.5 A *disconnected graph* is a graph which is not connected. [It decomposes into two or more (connected) components (5.9).]

[8] We propose the new term *remoteness number* as a preferred alternative since it is more suggestive of the meaning.

5.6 A *tree* (*Cayley tree*) or *finite tree* of *n edges* $T(n)$ [or T_n] is a connected graph of n (simple) edges which contains no polygons (2.17). Clearly a tree $T(n)$ has $n+1$ vertices. Any tree is planar (7.10). (For a *planted tree* see 8.24.)

5.7 A *Bethe lattice of degree* (or *coordination number*) d is an infinite tree in which each vertex has the same valence d (≥ 2). (The name of this class of graphs arose from Bethe's approximation in statistical mechanics which is exact for such a structure.)

5.8 An *arborescence* (*of n arcs*) is a (weakly connected) directed graph of n arcs with no circuits in which every vertex except one, the *root*, is the terminal of a single arc. The root is not the terminal of any arc. Every arborescence is a tree.

5.9 A *component* (or a *connected component*) of a graph is a maximal connected subgraph (3.10). A disconnected graph is said to be the *union of all its components* and is denoted by $G = C_1 \cup C_2 \cup C_3 \cdots$, where $C_1, C_2, C_3 \cdots$ are the components of G.

5.10 The *cyclomatic number* (or *cycle rank*[9]) $c(G)$ of a graph G is defined by

$$c(G) = e(G) - v(G) + n(G),$$

where $e(G)$, $v(G)$, and $n(G)$ are the numbers of edges, vertices, and components of G, respectively; $n(G)$ and $c(G)$ are sometimes called the *Betti numbers* of order zero and one, respectively.

5.11 An *articulation point* or *cut-vertex* (or *separating vertex*) is a vertex of a connected graph, the deletion of which produces a graph which is not connected. The *multiplicity of an articulation point* is the number of components resulting upon its deletion. (Note that an isolated vertex has no articulation point.) Clearly a disconnected graph may also be said to have an articulation point if one of its components has an articulation point. A *separable graph* has a cut-vertex; a *nonseparable graph* does not.

5.12 A *star S* is a connected graph of *two or more* vertices having no articulation point.[10] [This includes the bond $K(2)$ (2.12).] A *star-subgraph* is a subgraph which is a star.

5.13 An *articulation set of order k* is a subset of k vertices of a connected graph the deletion of which produces a graph which is no longer connected.

5.14 A *k-irreducible* graph is a connected graph which has no articulation set of order k; conversely, if a graph has an articulation set of order k, it is *k-reducible.*

5.15 An *isolation set of order k* is a subset of k vertices of a connected graph the deletion of which leaves a

[9] H. Whitney in his pioneering papers on graph theory called $c(G)$ the *nullity*, and $r(G) = v(G) - n(G)$ the *rank* of the graph, although the latter may be referred to more specifically as the *cocycle rank* (see also Theorem 6.7).

[10] This concept of a *star* should not be confused with a *vertex star* (2.41).

single, isolated vertex. (This concept is needed for completeness in the following definitions.)

5.16 The *connection number* (or *connectivity*) $\omega(C)$ of a connected graph C is the order of the smallest articulation or isolation set of C.

5.17 A connected graph C is *h-connected* if and only if $1 \leq h \leq \omega(C)$. An isolated vertex is defined to be one-connected. [Note that a graph which is h-connected is also $(h-1)$-irreducible. The converse is also true except for the complete graphs $K(n)$ (2.11).]

5.18 *Independent chains* between two vertices on a graph G have no vertices of G in common other than the two terminal vertices. If G is a graph with multiedges, not more than one chain may have no intermediate vertex (4.6).

THEOREM 5.1. If a graph other than an isolated vertex is h-connected, any pair of vertices can be connected by h independent self-avoiding chains (4.11).

5.19 A *multiply connected* (*biconnected*) graph is one which is two-connected. (Note that the bond, the double bond, etc. (2.13), are *not* multiply connected.)

5.20 A *lobe* (or a *block*) of a graph G is a maximal star-subgraph of G (5.12). (Thus a lobe can be a bond or dimer.)

5.21 A *piece of a connected graph C, relative to an articulation point x*, is a section graph of C the vertex set of which is the union of x with the vertex set of some component isolated by the deletion of x. (A piece may be a bond or dimer.)

5.22 A *minimal* or *elementary piece* of a connected graph C is a piece of C which contains no articulation point and, hence, is a star (5.12). A minimal piece is a lobe, but the converse need not be true.

5.23 A *k-piece of a connected graph C, relative to an articulation set V_x of k vertices*, is a section graph of C the vertex set of which is the union of the vertex set V' of some component isolated by the deletion of V_x with those vertices of V_x adjacent to vertices in V'.

5.24 A *Husimi tree* is a connected graph whose lobes are all polygons (2.17) or bonds (2.12). Alternatively it is a simple (1.16) connected graph in which no edge belongs to more than one simple cycle (4.10). A Husimi tree is *pure* if all its lobes are isomorphic; it is *mixed* if it contains nonisomorphic lobes.

5.25 A *cactus* is a pure Husimi tree of triangles (2.14).

5.26 A *star tree* is a connected graph whose lobes are all stars (5.12) isomorphic to stars in some *star collection*. A star tree is *pure* if all its lobes are isomorphic; otherwise it is mixed. Note that a general star tree is merely a connected graph.

5.27 A *cut-edge* (or *isthmus* or *separating edge*) of a connected graph is an edge the deletion of which produces a disconnected graph. A disconnected graph has a cut-edge if one of its components has one.

5.28 A *cut-set of edges* is a set of edges of a connected graph the deletion of which produces a disconnected graph.

5.29 The *cohesion number* $\chi(C)$ of a connected graph C is the minimum number of edges which form a cut-set of edges for C.

5.30 A connected graph C is *h-coherent* if, and only if, $1 \leq h \leq \chi(C)$. An isolated vertex is defined to be one-coherent.

5.31 A *cocycle* is a minimal cut set of edges.

THEOREM 5.2. If a connected graph is h-coherent, any pair of vertices can be connected by h edge-disjoint chains (3.12).

5.32 A *forest* is a disconnected graph (5.5) whose components are all trees. Every edge of a forest is a cut-edge.

5.33 The *girth* $\gamma(G)$ of a graph G which is not a forest (and hence contains a polygon) is the least integer n such that G contains a polygon of n edges.

5.34 The *weak k-weight*[11] of a graph G is defined recursively by

$$k(G) = n(G) - \sum_{G'}{}' k(G'),$$

where $n(G)$ is the number of components (5.9) of G, and the sum runs over all proper subgraphs, G', of G. The definition may be re-expressed in terms of weak lattice constants (3.17) as

$$\sum_{G'}{}^{\dagger} (G'; G) k(G') = n(G),$$

where the sum now runs over all isomorphically inequivalent (2.5) subgraphs G' of G.

The weak k-weight of a single vertex is $+1$. Homeomorphic graphs have equal weak k-weights; for polygons $k = +1$; for theta-graphs $k = -1$; if G is separable (and thus not a star), $k(G) \equiv 0$ (Essam and Sykes.[11])

5.35 The *strong K-weight* $K(G)$ of a graph G is defined recursively by

$$K(G) = n(G) - \sum_{G*}{}' K(G^*),$$

where $n(G)$ is the number of components of G and the sum runs over all proper section graphs G^* of G, (3.7). In terms of strong lattice constants (3.18), the definition may be written

$$\sum_{G*}{}^{\dagger} [G^*; G] K(G^*) = n(G),$$

where the sum now runs over all isomorphically inequivalent section graphs of G.

The strong weight $K(G)$ vanishes whenever the

[11] See J. W. Essam and M. F. Sykes, J. Math. Phys. **7**, 1573 (1966). The magnitude $|k(G)|$ of a weak k-weight is also known as the *Crapo number* of G [H. H. Crapo, J. Comb. Theory **2**, 406 (1967)].

weak weight $k(G)$ does so; in addition, $K(G)\equiv 0$ whenever G has an articulation set (5.13) which forms a clique (3.22).

6. THE MATRICES OF A GRAPH

6.1 *The incidence matrix of the arcs* **S** is a rectangular matrix representation of a directed graph G, the (i, k) element of which is defined by

$s_{ik}=+1$ if the arc k is incident out of the vertex i,

$=-1$ if the arc k is incident into the vertex i,

$=0$ otherwise.

THEOREM 6.1. The determinant of any square submatrix of **S** has the value $+1$, -1, or 0.

6.2 The *incidence matrix of the edges* **R** of a graph G is defined by

$r_{ik}=+1$ if the edge k is incident with the vertex i,

$=0$ otherwise.

This is a faithful representation of an undirected graph but does not give full information about a directed graph.

6.3 The *adjacency matrix of a graph with loops* is a square matrix **A**, the rows and columns of which are labeled by the vertices of the graph, the elements being defined by

$a_{ii}=$ the number of loops incident with the vertex i,

$a_{ij}=$ the number of arcs incident out of i into $j\,(i\neq j)$ in the case of a directed graph, or the number of edges incident with both the vertices i and j in the case of an undirected graph.

6.4 The *graph matrix* **B** of a graph G is a symmetric matrix the rows and columns of which are labeled by the vertices of G, the elements being defined by

$b_{ii}=$ the degree of the ith vertex,

$b_{ij}=(-1)\times$ the number of edges incident with both the vertices i and j $(i\neq j)$.

Note that $\sum_i b_{ij}=\sum_j b_{ij}=0$.

THEOREM 6.2. $\mathbf{B}=\mathbf{S}\tilde{\mathbf{S}}$ (where $\tilde{\mathbf{S}}$ denotes the transpose of **S**). (In the case of an undirected graph, the arc set needed to define **S** may be obtained by giving each edge an arbitrary orientation.)

THEOREM 6.3. If $v(G)$ is the number of vertices in G, and $n(G)$ the number of components (5.9), then[9]

$$\text{rank}\,\{\mathbf{B}\}=v(G)-n(G)$$

(see Uhlenbeck and Ford).

6.5 The *complexity* $\Delta(C)$ of a connected graph C is the determinant of any principal minor of **B**.

THEOREM 6.4. The complexity of a connected graph C with articulation points is the product of the complexities of its lobes (5.20). (See Uhlenbeck and Ford.)

THEOREM 6.5. (Kirchhoff) The number of *spanning trees*, that is, trees which are partial graphs (3.6) of a graph C, is equal to the complexity of C.

6.6 The *orientation matrix* **D** of an antisymmetric directed graph G (1.34) is an antisymmetric matrix with rows and columns labeled by the vertices of G and elements

$d_{ij}=-d_{ji}=$ number of arcs incident out of vertex i into vertex j, *minus* the number of arcs incident out of j into i.

The determinant of the orientation matrix can be related to the number of perfect matchings or dimer coverings (7.2) of the graph G (Kasteleyn).

6.7 The *cycle matrix* **C** of a graph G is defined by

$c_{lk}=+1$ if the edge k belongs to the lth simple cycle (4.10) in G,

$=0$ otherwise.

The *cocycle matrix* $\mathbf{C}^{\times}$ is defined similarly

$c_{lk}^{\times}=+1$ if the edge k belongs to the lth cocycle (5.31),

$=0$ otherwise.

THEOREM 6.6. The incidence matrix of edges, **R**, and the transpose **C** of the cycle matrix are orthogonal, i.e., $\mathbf{R}\tilde{\mathbf{C}}=0$, under addition modulo 2.

THEOREM 6.7. Under arithmetic modulo 2 the ranks of the incidence, cocycle, and cycle matrices of a graph G are

$$\text{rank}_2\{\mathbf{R}\}=\text{rank}_2\{\mathbf{C}^{\times}\}=v(G)-n(G)$$

$$\text{rank}_2\{\mathbf{C}\}=c(G),$$

where $c(G)$ is the cyclomatic number (5.10) of G. (See Harary.)

7. MATCHINGS, COLORINGS, AND PLANAR AND TOPOLOGICAL GRAPHS

7.1 A *matching* of a graph G or a *dimer arrangement* on G is a graph G together with a subgraph G' in which each component is a bond or dimer (2.12) (or directed bond or dimer). Alternatively, a matching of G is the assignment of a color to *some* of the arcs or edges of G such that no two adjacent arcs or edges are colored.

7.2 A *perfect matching* of G or a *dimer covering* of G is a graph G together with a spanning subgraph (partial graph, 3.6) of G in which each component is a dimer (or directed dimer).

7.3 A *proper* (*vertex*) *coloring* of a graph G is the assignment of a color to each vertex of G such that no two adjacent vertices have the same color.
7.4 A *bicolored graph* is a graph with a proper vertex coloring of two colors. Not all graphs are *bicolorable* [or *bipartite* or *bichromatic*, see 7.6], i.e., admit such a coloring.

THEOREM 7.1. The determinant of any square submatrix of the incidence matrix of edges, **R**, of a graph G (6.2) takes the values $+1$, -1, or 0 if and only if G is bicolorable.

7.5 A *p-colorable graph* is a graph which admits a proper vertex coloring of p distinct colors. Equivalently, the vertex set admits a partition into p disjoint sets such that no two vertices from distinct sets are adjacent.
7.6 A *p-chromatic graph* is a graph which is p-colorable but not $(p-1)$-colorable. Evidently the complete p-chromatic graph $K(m_1\cdots m_p)$ (2.16), is p-chromatic.

THEOREM 7.2. (König) A graph is bichromatic if and only if it contains no cycles (4.8) of uneven length.

7.7 The *chromatic number* $\gamma(G)$ of a graph G is the least number p for which G is p-colorable. The *chromatic polynomial*,

$$P_v(p;G)=\sum_{r=1}^{v}\pi_r(G)p^r,$$

of a graph G of v vertices is the total number of distinct proper colorings given p distinct colors.
7.8 A *proper edge* (or *arc*) *coloring* is the assignment of a color to each edge (or arc) of a graph G such that no two adjacent edges (or arcs) have the same color.
7.9 The *chromatic index* of a graph G is the least number of distinct colors required for proper edge coloring of G. Note that the chromatic index of a graph is the chromatic number of its covering graph (2.26), namely $\gamma(G^C)$.
7.10 A *planar graph* is a graph (*with* or *without loops*) which can be *embedded* in the plane, that is, can be *faithfully represented* by points and lines drawn in the plane in such a way that no two lines meet except at a vertex. In the case of a directed planar graph with loops, the loops of the embedded graph are assigned an orientation (with respect to one side of the plane).

Note that a given planar graph G may be embeddable in the plane in several topologically distinct ways [e.g., consider a beta-graph (2.20)]. Each distinct embedding represents a *plane graph*.
7.11 A *finite* (or *inner*) *face* of a planar graph *embedded in the plane* is a connected domain of the plane bounded by lines representing the arcs, edges, and loops of the graph. The bounding edges form the *edges of the face*. The bounding edges (and loops) taken in sequence form the *contour* or *contour cycle* (4.8) of the face.
7.12 The *infinite* (or *outer*) *face* of a planar graph embedded in the plane, is the domain of the plane which is *not* bounded by lines representing the edges, etc., of the graph.
7.13 The term *face* means either a finite or an infinite face. (The word *region* is sometimes used in place of *face* but we do not recommend it.)
7.14 Distinct faces are *adjacent* if they have a common boundary edge or loop.
7.15 A *simple face* of $n=3, 4, 5, \cdots$ edges and vertices has no loops as any part of its boundary and contains no edges within its boundary. A *simple planar graph* (or *mosaic*) is a connected planar graph in which every face is simple.

THEOREM 7.3. (Euler's law of the edges) The number of finite faces $f(G)$ of a planar graph embedded in the plane is equal to the cyclomatic number $c(G)=e(G)-v(G)+n(G)$, (5.10). Thus the number of faces is independent of the embedding.

THEOREM 7.4. (Kuratowski's theorem) The necessary and sufficient condition for a graph to be planar is that it should contain no subgraphs homeomorphic to the complete graph $K(5)$ or to the complete bichromatic graph $K(3, 3)$ [see 2.8, 2.11, and 2.15].

Any tree is clearly planar.

THEOREM 7.5. Every planar graph is five-colorable. (The *four-color problem* is to prove that every planar graph is four-colorable).

7.16 The *dual* G^D of an *undirected connected planar graph* G embedded in the plane is constructed as follows: (a) a new vertex is placed within each face of G; these vertices constitute the vertex set of G^D; (b) for each edge or loop of G which separates two distinct faces, an edge of G^D is drawn which joins the vertices in the two faces and crosses no line except the edge or loop of G, which it crosses once only; (c) for each edge of G which does not separate two distinct faces (and which thus lies wholly within one face) a loop of G^D is drawn, from the vertex in the same face, which crosses no line except the edge of G, which it crosses once only.

Note that (i) a *bridge* (4.14) of G having two or more edges gives rise to a multiedge of G^D, and (ii) the dual of an isolated vertex is an isolated vertex.
7.17 The (*clockwise*) *dual* G^D of a *directed connected planar graph* embedded in the plane is constructed as for an undirected graph except that each edge and loop of G^D is assigned an orientation by the convention that the direction of the arc or loop of G at the crossing point is rotated clockwise (with respect to the front of the plane) to yield the sense of the corresponding arc (edge) or loop in G^D.

THEOREM 7.6. The dual of a directed or undirected connected planar graph is a directed or undirected connected planar graph, respectively.

7.18 The *dual* G^D of a general planar graph G is obtained by taking the union of the duals of the separate components of G (5.9).

THEOREM 7.7. The dual of the dual of an undirected graph G is G itself. If G is directed, the clockwise dual of the clockwise dual is the reverse graph G^R (2.24).

7.19 The *four-color problem for planar maps* is to prove that the faces of any (undirected) planar graph may be colored using only four distinct colors in such a way that no two adjacent faces are the same color. (This is equivalent to proving the dual graph is four-colorable).

7.20 The *decoration* or *completion of a simple face* (7.15) of $n>2$ vertices of an undirected plane graph is the addition of $\frac{1}{2}n(n-3)$ new edges constructed by drawing, within the face, all possible diagonal lines. This converts the face with its boundary edges to a complete graph $K(n)$ drawn with crossing lines (for $n>3$) which is termed a *multiface* in distinction to an *ordinary face*. (Note that a triangular face remains invariant under completion.)

7.21 A *semiplanar graph G* is derived from an *underlying graph* G_0, which is a planar graph embedded on the plane, by completing some or all of the simple faces to multifaces.

THEOREM 7.8. (Euler's extended theorem) If G is a semiplanar graph with $f_1(G)$ ordinary faces and with multifaces of $n_1, n_2, \cdots, n_k, \cdots$ vertices, respectively, then

$$f_1(G)=c(G)-\sum_k \tfrac{1}{2}(n_k-1)(n_k-2),$$

where $c(G)$ is the cyclomatic number of G (5.10).

7.22 A *simple semiplanar graph* (or a *decorated mosaic*) is a connected semiplanar graph having only simple faces and multifaces.

7.23 The *matching graph* G^M of a simple semiplanar graph G is obtained from the underlying graph G_0 by completing all those faces of G_0 not completed in G.

Evidently the matching graph of G^M is G itself. The significance of semiplanar graphs and their matching graphs is that the covering graph (2.25) of any (suitably restricted) planar graph G_1 can be represented as a semiplanar graph G. The matching graph of G is then isomorphic to the covering graph of the dual G_1^D.

7.24 A *topological graph* on a given surface is a graph whose vertices are geometrical points on the surface and whose edges are line segments on the surface ending at the corresponding points. Note that the line segments may cross or meet at interior points of one or both segments but such points of intersection are *not* identified as vertices of the graph. A topological graph is said to be *drawn* on the surface and to *represent* the graph (or, for emphasis, *abstract graph*) to which it is isomorphic.

7.25 A graph G is *embeddable* in a surface if and only if there exists a topological graph which represents G *faithfully*, that is, such that no line segments meet except at a vertex point. (Compare with 7.10; there are no crossing lines in an embedding.)

7.26 A graph G is of *genus g* if it is embeddable in a surface of genus g but cannot be embedded in a surface of lower genus. A planar graph is of genus zero. (Note that every closed two-sided surface is topologically equivalent, or homeomorphic, to a "normal surface" of genus g, namely a "sphere with g handles" or a "pretzel with g holes".)

7.27 A graph is called *polyhedral* (or *n-polyhedral*) if its vertices and edges may be identified with the vertices and edges of a convex polyhedron in a Euclidean space (of n dimensions). The complete graph $K(n)$ is $(n-1)$-polyhedral; the corresponding polyhedron is an $(n-1)$-*simplex*.

THEOREM 7.9. (See Busacker and Saaty). Every complete graph $K(n)$ on $n\geq 5$ vertices is 4-polyhedral.

8. ROOTED GRAPHS

8.1 A *rooted graph* of some type (directed, undirected with loops, etc.) has one or more vertices, called *roots* (or *root points* or *external* vertices), which are specially distinguished from the remaining, *ordinary* or *internal* vertices. (Root points may be represented as open circles drawn in the plane, the remaining vertices being represented by solid circles.)

8.2 A *two-rooted* or *birooted graph* has precisely two roots. An *r-rooted graph* has precisely r roots.

8.3 An *automorphism of a rooted graph* is a one–one correspondence between its root points and a one–one correspondence between its remaining vertices which induces a one–one correspondence between its edges and between its arcs (if any). The *graph group* of a rooted graph is the group of its automorphisms. The *symmetry number* $s(G^{(r)})$ *of a rooted graph* $G^{(r)}$ is the order of its graph group.

8.4 *Isomorphic rooted graphs:* two-rooted graphs are *isomorphic* if there is a one–one correspondence between their roots and a one–one correspondence between their remaining vertices which induces a one–one correspondence between their edge and arc sets (if any).

8.5 *Homeomorphic rooted graphs:* two-rooted graphs are *homeomorphic* if by the insertion of any number of ordinary vertices of valence two (2.6) they can be made isomorphic.

8.6 *Subgraphs*, *section graphs*, and *partial graphs* of a rooted graph G are defined as for unrooted graphs (see Sec. 3) and may have fewer roots than G.

8.7 The *weak* and *strong lattice constants*, $(G_1; G)$ and

$[G_1; G]$, are defined as for unrooted graphs, (3.17) and (3.18), and vanish identically if G_1 has more roots than G.

8.8 A *rooting* of an unrooted graph G (with loops) is a designation of some of the vertices of G as roots. An *r-rooting* is a rooting in which precisely r vertices are designated as roots.

8.9 The *number of rootings*, $((G^{(r)}; G))$, of an r-rooted graph $G^{(r)}$ in an unrooted graph G, is the number of distinct rootings of G yielding a graph isomorphic to $G^{(r)}$. In terms of symmetry numbers (2.4, 8.3) we have $((G^{(r)}; G)) = s(G)/s(G^{(r)})$.

8.10 The *product graph* $G_1 * G_2$ of two r-rooted graphs G_1 and G_2 with roots labeled 1, 2, 3, $\cdots r$ is constructed by taking the union of G_1 and G_2, and then identifying each root of G_1 with the root of G_2 bearing the same label. More generally, if G_1 and G_2 have different numbers of roots (including none) and the roots are labeled in some general way, the product graph is defined as above except that only pairs of roots bearing the *same* label in G_1 and G_2 are identified. (Roots with unmatched labels remain unaffected.) The graphs G_1 and G_2 are said to be *joined in parallel* in $G_1 * G_2$. (See Stell.)

8.11 A *linked graph*[12] is a rooted graph which has at least one root in each component (defined as for a graph, 5.9). (The idea is that the rooted components are linked to one another through the "ground.")

8.12 An *unlinked graph* is a rooted graph in which at least one component has no root.

8.13 The *derived graph* of a rooted graph G is the unrooted graph which has the same vertex set as G (the distinction between root points and other vertices being ignored), and the edge set of which is the union of the edge set of G with the edges formed by taking all pairs of root points of G once and once only. These additional edges are called *linking edges*.

THEOREM 8.1. (self-evident) A rooted graph is linked if and only if its derived graph is connected.

8.14 An *h-linked graph* is a linked graph the derived graph of which is h-connected (5.17).

8.15 An *articulation set of a rooted graph* is an articulation set for the derived graph (5.13).

8.16 A *k-irreducible rooted graph* is a rooted graph for which the derived graph is k-irreducible (5.14).

8.17 A *k-piece of a linked graph*, relative to an articulation set V_x of k vertices, is a section graph (3.7) the vertex set of which is the union of the vertex set V' of (a) some unrooted component, or (b) the maximal linked subgraph (3.10) isolated by the deletion of V_x, with those vertices of V_x adjacent to vertices in V'.

8.18 A *simple one-irreducible two-rooted graph*[13] is a two-rooted graph in which the root points are not adjacent (1.29), and for which the root points do not constitute an articulation pair (5.13).

8.19 A *ladder graph* is a one-irreducible two-rooted graph which is not simple in the sense of 8.18, and which is not a two-rooted bond. (A two-rooted multibond (2.13) is thus a ladder graph.)

8.20 A *composite one-irreducible two-rooted graph* is a ladder graph in which the root points are not adjacent.

8.21 A *nodal point* is a vertex on a simple one-irreducible two-rooted graph the deletion of which separates it into two components each of which contains a root point. [Note that a nodal point is distinct from a node as defined in 1.23.]

8.22 A *nodal one-irreducible two-rooted graph* is a simple one-irreducible two-rooted graph containing one or more nodal points.

8.23 An *elementary one-irreducible two-rooted graph* is a simple one-irreducible two-rooted graph which is nonnodal.

8.24 A *planted tree* is a tree with a single root at a vertex of valence one.

[12] See C. Bloch, *Studies in Statistical Mechanics*, J. de Boer and G. E. Uhlenbeck, Eds. (North-Holland Publ. Co., Amsterdam, (1965) Vol. III.

[13] The following classification of one-irreducible two-rooted graphs is used in the study of pair correlation functions. [See, for example, J. M. J. van Leeuwen, J. Groeneveld, and J. de Boer, Physica **25**, 792 (1959).]

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