Whither Axiomatic Field Theory?*

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We survey some recent progress in constructing models of local, relativistic quantum mechanics. These field-theory models have nontrivial scattering.

I. CONSTRUCTIVE QUANTUM FIELD THEORY

In this paper we review some recent progress in quantum dynamics—progress made possible by a closer tie between mathematics and physics.

Relativistic quantum mechanics provides a general framework for dynamics and leads naturally to quantum field theory. The basic assumption of quantum field theory is that each field—the electric-current density, for example—is an operator on the Hilbert space of physical states. Field-theory dynamics may be specified by a Lagrangian or by some other scheme.

Despite the simplicity of this starting point, examples of interacting fields are very scarce. Even quantum electrodynamics, which gives incredible agreement between calculations and experiments, has not been shown to be internally consistent. Thus the fundamental question field theory faces today is the same one it faced many years ago: namely, "Is quantum field theory consistent?" A more concrete form of the same question is "Can we construct model field theories?"

Constructive field theorists have approached this problem with Lagrangian field theories and attempted to solve particular model Lagrangians. Because of the difficulty of the problem, we have felt justified in beginning with the simplest possible interaction Lagrangian. Once a simple model has been solved we can proceed to a more realistic and technically more difficult Lagrangian.

With the Lagrangian chosen, we can proceed with the construction of the dynamics. First we delve into the details of the particular Hamiltonian in order to get quantitative control over the interaction. We derive estimates to prove, for instance, that the renormalized Hamiltonian is positive. Second we combine these specific estimates with modern mathematical analysis in order to complete the construction. Thus, recent progress on constructing models has been the result of using modern functional analysis to control these physics problems. Among physicists, Wightman[1] has continually encouraged this point of view. Some mathematicians have encouraged this view as well.[2,3]

The program just described is, however, unrealistically simple. Neither the specific estimates nor the mathematical theorems appropriate for quantum field theory can be found in books; we must find and prove them before we can use them. We have found it necessary, then, both to develop the mathematical tools and to apply these tools to specific problems of quantum field theory. This procedure has led to productive crossing of boundaries between mathematics and physics. In fact, much of the work that I shall now describe has been the result of my collaboration with J. Glimm, a mathematician.[4–6]

We have constructed a simple field-theory model—the theory of a scalar field ϕ with a ϕ^4 self-interaction in space–time of two dimensions. We have already established for this theory many of the properties desired in a model. The remaining properties are valid on the level of perturbation theory. In particular, this $(\phi^4)_2$ theory provides an example of a local quantum field theory with interaction. It is the first step toward constructing a more realistic field theory.

II. RESULTS FOR $(\phi^4)_2$

I will now describe this $(\phi^4)_2$ model.

The field ϕ in the $(\phi^4)_2$ model satisfies a nonlinear equation of motion

$$(\square+m^2)\phi(x,t)=-4\lambda\phi^3(x,t).$$

Here $\square$ is the two-dimensional wave operator

$$\square=(\partial^2/\partial t^2)-(\partial^2/\partial x^2).$$

Furthermore, the field ϕ is quite singular, so that the nonlinear term ϕ^3 has no *a priori* meaning. Part of the problem of constructing the theory is to *define* the ϕ^3 which appears in the equation of motion.

For orientation, I shall now give a list of the results[4–6]; afterwards I shall discuss them in detail.

(1) The Heisenberg picture field $\phi(x,t)$ is known to exist. Expectation values of the field $(\psi, \phi(x,t)\psi)$ are defined for a dense set of vectors ψ in Fock space. Mathematically, $\phi(x,t)$ is a densely defined bilinear form.

(2) The average field

$$\phi(f)=\int\phi(x,t)f(x,t)\,dx\,dt$$

is an operator on Fock space. For a real, differentiable

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test function $f(x, t)$ vanishing outside a bounded region, $\phi(f)$ is a self-adjoint operator.

(3) The field is local:

$$[\phi(x, t), \phi(x', t')] = 0, \qquad \text{if } |x-x'| > |t-t'|.$$

In terms of the average fields, $\phi(f)$ commutes with $\phi(g)$ if f and g have space-like separated supports.

(4) By a general result, called Haag's theorem, the Hamiltonian for ϕ cannot exist on Fock space. This result applies to the Hamiltonian, but not to the field. In order to write down the Hamiltonian for ϕ, we must construct the space of physical states. We call it $\mathcal{H}_{\text{ren}}$ to distinguish it from Fock space. The construction of $\mathcal{H}_{\text{ren}}$ has been carried out.

(5) On $\mathcal{H}_{\text{ren}}$ there is a renormalized Hamiltonian operator H, which is self-adjoint and gives rise to the Schrödinger picture time development

$$\psi(t) = \exp(-iHt)\psi.$$

(6) On $\mathcal{H}_{\text{ren}}$ we also construct a self-adjoint momentum operator P, which commutes with H. The unitary operator

$$U(x, t) = \exp(-iHt + iPx)$$

gives the correct space–time translation for the theory.

(7) There is a vacuum vector Ω in $\mathcal{H}_{\text{ren}}$. The renormalized Hamiltonian H is positive, $H \geq 0$, and

$$H\Omega = 0, \qquad P\Omega = 0, \qquad \|\Omega\| = 1.$$

III. DISCUSSION OF THE RESULTS[4–7]

First we write the formal solution of the $(\phi^4)_2$ theory in terms of a canonical time-zero field. Then we shall see what modifications are necessary to make the formal procedure mathematically correct.

The time-zero canonical fields $\phi(x)$ and $\pi(x)$ satisfy the usual equal-time commutation relations

$$[\pi(x), \phi(y)] = -i\delta(x-y),$$

$$[\phi(x), \phi(y)] = 0 = [\pi(x), \pi(y)].$$

We choose to realize the time-zero field on Fock space—a convenience for computations. The correct formal Hamiltonian can be written in terms of the canonical fields. It is

$$H = H_0 + \lambda \int :\phi^4(x):\, dx - E.$$

Here H_0 is the free Hamiltonian, the interaction is Wick ordered, and E is a constant chosen to adjust the ground-state energy of H so that it is zero. Since the mass renormalization constant and field-strength renormalization constant for this model are finite in perturbation theory, we have chosen to omit them from H.

Writing $\phi(x, t)$ in terms of its time-zero value

$$\phi(x, t) = \exp(iHt)\phi(x)\exp(-iHt),$$

we see that $\phi(x, t)$ formally satisfies the correct equation of motion. To verify this we compute

$$(\partial^2/\partial t^2)\phi(x, t) = [iH, [iH, \phi(x, t)]] = [iH, \pi(x, t)]$$

$$= (\partial^2/\partial x^2 - m^2)\phi(x, t) - 4\lambda\phi^3(x, t),$$

where $\phi^3(x, t)$ is defined by

$$\phi^3(x, t) = \exp(iHt) :\phi^3(x):\, \exp(-iHt).$$

This formal procedure is not mathematically correct since the Hamiltonian H is not an operator. Likewise, $\exp(iHt)$ is not an operator. In fact, there is no nonzero vector ψ in Fock space such that $H\psi$ is again in Fock space. Since the ultraviolet behavior of H is good after Wick ordering, the difficulty with H is an infinite-volume problem.

By using a spatial cutoff in the $(\phi^4)_2$ Hamiltonian, we can avoid this difficulty. Let $g(x) \geq 0$ be a smooth function that equals 1 for $|x| \leq 1$ and vanishes for $|x|$ large. Then we define the cutoff Hamiltonian

$$H_n = H_0 + \lambda \int :\phi(x)^4: g(x/n)\, dx - E_n$$

$$= H_0 + H_{I,n}.$$

The constant E_n is chosen so that the lower bound of H_n is zero, and E_n is known to be finite[8,9] for all positive values of the coupling constant λ.

Clearly the Hamiltonian H_n agrees with the correct Hamiltonian in the region of space $|x| < n$. Noting that influence propagates in the Heisenberg picture at the speed of light, we conclude that the spatial cutoff n does not affect the field. In fact the well-defined field

$$\phi(x, t) = \exp(iH_n t)\phi(x)\exp(-iH_n t)$$

is independent of the spatial cutoff n, provided that

$$n > |x| + |t|.$$

Figure 1 illustrates this phenomenon, for the effect of $g(x) \neq 1$, which propagates at the speed of light, cannot be detected inside the diamond $n > |x| + |t|$. Hence if (x, t) lies inside the diamond, the field $\phi(x, t)$ must be independent of any cutoff. Thus $\phi(x, t)$ defined with H_n actually is a solution to the equation of motion. A similar argument based on the propagation of influence at the speed of light, shows that the field $\phi(x, t)$ is local. This method of removing the spatial cutoff was first proposed by Guenin. In giving a proof, we use a theorem of Segal that relies on a theorem of Trotter. See Ref. 4 for a discussion of the literature on that point.

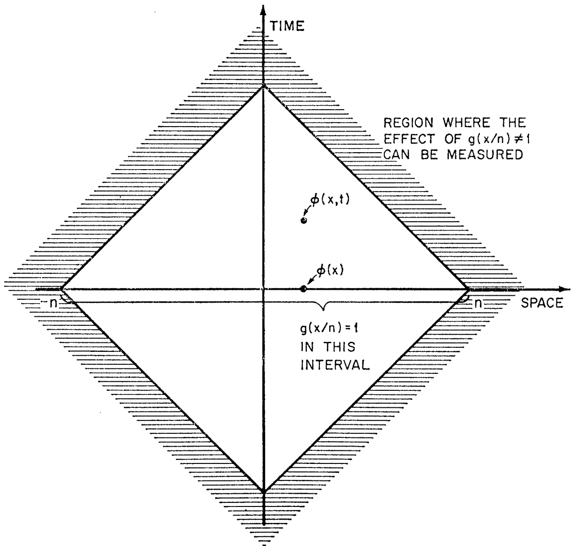


FIG. 1. Influence propagates at the speed of light for a local Hamiltonian.

In order to make these arguments mathematically precise, we must prove that H_n is self-adjoint. At this point I want to warn you that self-adjointness is a subtle mathematical property of an operator. Formal self-adjointness is insufficient, and a proof requires some detailed knowledge of those vectors on which the operator is defined. I will give you a simple instance of the pathology that may occur with just one degree of freedom. The formally self-adjoint operator

$$T=pq^3+q^3p;$$

it is not self-adjoint since

$$T\psi=-i\psi,$$

where ψ is the square integrable function

$$\psi(q)=q^{-3/2}\exp(-1/4q^2).$$

Detailed investigations show that such pathologies do not occur in $(\phi^4)_2$. We have proved that H_n is self-adjoint.[4,5] An essential new point of the proof is the quadratic estimate[4]

$$H_0^2+H_{I,n}^2\leq a(H_n+b)^2.$$

This estimate allows us to approximate H_n by a self-adjoint operator[10] $H_{n,\kappa,V}$ and then to remove the approximation while preserving self-adjointness.[5] We show that the resolvents of the $H_{n,\kappa,V}$ converge to the resolvent of a self-adjoint operator. Here κ denotes an ultraviolet cutoff and V denotes a periodic box.

In fact, we use this inequality to prove convergence of the resolvents $R_n=(H_n-z)^{-1}$ in norm,

$$\|R_{\kappa,V}-R_{\kappa',V'}\| = \|R_{\kappa,V}(H_{n,\kappa,V}-H_{n,\kappa',V'})R_{\kappa',V'}\|$$
$$\leq \text{const}\,\|(I+H_0)^{-1}(H_{n,\kappa,V}-H_{n,\kappa',V'})(1+H_0)^{-1}\|\rightarrow 0.$$

Having established the self-adjointness of H_n, we can define the field ϕ. As mentioned before, we have also proved that the field $\phi(f)$ is self-adjoint.

The corresponding self-adjointness for the localized field operator[6] $\phi(f)$ is a consequence of the estimate

$$\phi(f)^2\leq |f|^2(H_n+b),$$

where $|f|$ denotes a norm of f, and b is sufficiently large.

The self-adjointness of H_n and the self-adjointness proof for the field $\phi(f)$ are the main results valid on Fock space. We now describe a second class of results—those that deal with the physical Hilbert space and the Hamiltonian.[6] We saw that H_n has no limit in Fock space, and the necessity of changing Hilbert spaces is a usual renormalization phenomenon in quantum field theories. Since

$$\lim_{n\rightarrow\infty} g(x/n)=1,$$

we have the formal identity

$$\lim_{n\rightarrow\infty} H_n=H.$$

However, this limit does not exist in Fock space since H is ill defined. To define H, we must at the same time construct the physical Hilbert space.

The first step toward constructing $\mathcal{H}_{\text{ren}}$ is to prove that H_n has a vacuum vector Ω_n:

$$\|\Omega_n\|=1, \qquad H_n\Omega_n=0.$$

We have shown that this is the case, and in fact that Ω_n is unique up to a phase.

Perturbation theory predicts that as $n\rightarrow\infty$, $E_n\rightarrow-\infty$ and Ω_n converges weakly to zero. Although perturbation theory is inapplicable,[11] the exact vacuum energy diverges no faster than the perturbation-theory bound[6],

$$E_n\geq -Mn.$$

Hence the average energy density E_n/n is bounded below.

Though Ω_n does not converge as $n\rightarrow\infty$, we can construct the physical representation by taking limits of vacuum expectation values[6,12] (Wightman functions). We define the expectation values

$$\omega_n(A)=(\Omega_n, A\Omega_n),$$

where A is a local function of the field. Technically, A belongs to a C^* algebra $\mathfrak{A}$.

As $n\rightarrow\infty$, a generalized sequence of the $\omega_n(A)$ converges to a limit $\omega(A)$ for every A. These limiting

Wightman functions determine a Hilbert space $\mathcal{H}_{\text{ren}}$. The relation between the $\omega(A)$ and $\mathcal{H}_{\text{ren}}$ is summarized by the Gelfand–Segal construction, and this relation is similar to the Wightman reconstruction of a field theory from its vacuum expectation values. According to this construction, there exists a vector $\Omega \in \mathcal{H}_{\text{ren}}$ such that

$$\omega(A) = (\Omega, A\Omega);$$

and so we call Ω the vacuum vector.

I will show you now how vacuum expectation values will converge properly while vectors give the wrong answers. Consider the linear interaction

$$H_n = H_0 + \int_{-n}^{n} \pi(x)\, dx - E_n.$$

The vacuum for H_n is

$$\Omega_n = \exp\left[i \int_{-n}^{n} \phi(x)\, dx \right] \Omega_0, \qquad \| \Omega_n \| = 1.$$

For ψ in Fock space the scalar products (ψ, Ω_n) converge to zero. For instance, if $\psi = \Omega_0$, the no-particle state, then

$$(\Omega_0, \Omega_n) = \exp\left[-\frac{1}{2} \int_{-n}^{n} dx \int_{-n}^{n} dy i^{-1} \Delta^{(+)}(x-y) \right] \leq \exp(-Mn)$$

converges exponentially to zero. It is in this manner that the vacuum for the $(\phi^4)_2$ theory converges to zero. We cannot use this answer. However, Wightman functions do converge. For example,

$$\begin{aligned} \omega_n(\pi(f)) &= (\Omega_n, \pi(f)\Omega_n) \\ &= \left(\Omega_0, \left\{ \pi(f) + \int_{-n}^{n} f(x)\, dx \right\} \Omega_0 \right) \\ &= \int_{-n}^{n} f(x)\, dx \\ &\rightarrow f(0) = \omega(\pi(f)). \end{aligned}$$

Using the convergence of Wightman functions, we construct the physical Hilbert space $\mathcal{H}_{\text{ren}}$ for the $(\phi^4)_2$ theory.

On the physical Hilbert space $\mathcal{H}_{\text{ren}}$, we construct a global Hamiltonian H. We arrive at H, a limit of the approximate H_n's, by studying a limit of Wightman functions.

Recall that

$$A(t) = \exp(iH_n t) A(0) \exp(-iH_n t),$$

where A is a local observable associated with a bounded space–time region, and n is large; also

$$\exp(iH_n t)\Omega_n = \Omega_n.$$

Thus the approximate expectation value $(\Omega_n, A(t)B\Omega_n)$ equals

$$\begin{aligned} (\Omega_n, A(t)B\Omega_n) &= (\Omega_n, \exp(iH_n t) A \exp(-iH_n t) B\Omega_n) \\ &= (\Omega_n, A \exp(-iH_n t) B\Omega_n). \end{aligned}$$

We can write the Wightman function

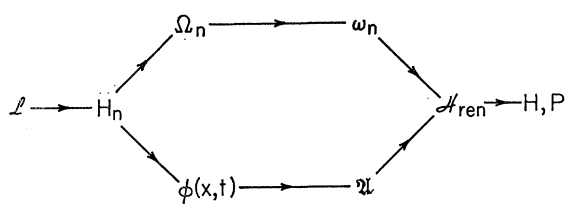
$$\begin{aligned} (\Omega, A(t)B\Omega_n) &= \lim_{n\to\infty} (\Omega_n, A(t)B\Omega_n) \\ &= \lim_{n\to\infty} (\Omega_n, A \exp(-iH_n t) B\Omega_n) \\ &= (\Omega, A \exp(-iHt) B\Omega). \end{aligned}$$

In this sense H_n converges to a limit H:

$$A(t) = \exp(iHt) A \exp(-iHt).$$

Since $H_n \geq 0$, H is positive too, and $H\Omega = 0$. The operator H is the renormalized Hamiltonian. A modification of this construction yields the momentum operator P. A suitable choice of $\mathfrak{A}$ ensures that both H and P are self adjoint.

In summary, the construction has proceeded schematically as follows:



We start from a formal Lagrangian $\mathfrak{L}$ and construct a cutoff, but locally correct, Hamiltonian H_n. Using H_n, we branch out in two directions. On the one hand, we construct the correct local field $\phi(x, t)$ and the bounded functions of the field $\mathfrak{A}$. On the other hand we prove that H_n has a (unique) vacuum vector Ω_n which give rise to vacuum expectation values ω_n. Combining the local functions $\mathfrak{A}$ and the vacuum expectation values ω_n, we pass to the limit $n \rightarrow \infty$ in order to construct the physical Hilbert space $\mathcal{H}_{\text{ren}}$. On $\mathcal{H}_{\text{ren}}$ we have the self-adjoint operators H and P.

We would like to determine more about the spectrum of H and the S matrix in the $(\phi^4)_2$ theory.

IV. THE FUTURE

The $(\phi^4)_2$ theory described fits into the spectrum of models illustrated by the "field-theory tree," Fig. 2. The difficulties of constructive field theory increase with the order of divergence. We chose $(\phi^4)_2$ because it is the

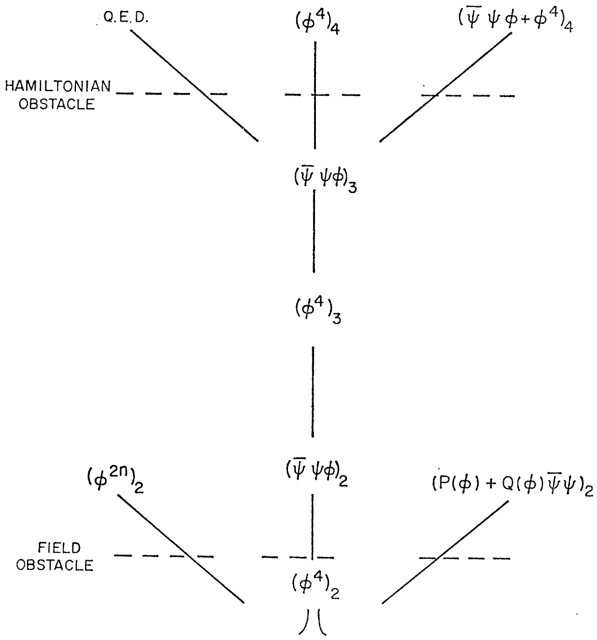


FIG. 2. The field-theory tree.

least divergent theory with a simple interaction. As we move up the field-theory tree, we meet obstacles at two levels. First we want to construct local field theories for models with controllable Hamiltonians. Above the roots of the $(\phi^4)_2$ theory lie other two-dimensional models,[13,14] i.e., $(\phi^{2n})_2$, $(\bar\psi\psi\phi)_2$, and $(P(\phi)+Q(\phi)\bar\psi\psi)_2$. For these theories, renormalized Hamiltonians H_n exist with a spatial cutoff.[15,16] However, no local field exists at present, just as none existed for $(\phi^4)_2$ only a year ago. More singular than these is $(\phi^4)_3$, and this Hamiltonian[17] is not known to be positive. One can also renormalize $(\phi^3)_4$.[18] The Yukawa theory in three dimensions is superrenormalizable but has not yet been fully treated.

The second real obstacle separates the superrenormalizable from the simply renormalizable theories—those that have renormalizable, but not superrenormalizable, Hamiltonians. No Hamiltonian operators are yet known for these.

The study of simple models provides a laboratory for quantum field theory. In this laboratory we can do experiments to test what a theory actually does. For instance, we can check the predictions of perturbation theory to see how badly the renormalization constants diverge. Up to now, all the predictions of perturbation theory hold. Once we have exact solutions for the Green's functions or the S-matrix elements, we can determine whether the Feynman perturbation series is asymptotic.

We can check for broken symmetries. The vacuum vector Ω is not known to be unique, and its uniqueness or nonuniqueness may help to determine whether broken-symmetry solutions exist.

We can test different axiom schemes against results obtained by constructing a model.[19,20] We could make sure, for example, that time translation is given by a unitary group of operators.

These constructive methods will lead, we hope, to new techniques for calculations. Meanwhile, we can practice climbing the field-theory tree.

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