REVIEWS OF MODERN PHYSICS VOLUME 41, NUMBER 3 JULY 1969

Effective Lagrangians and Field Algebras with Chiral Symmetry*

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This paper reviews recent developments of effective Lagrangians and field algebras as means of treating chiral symmetry and partially conserved axial current (PCAC) for the study of elementary particle physics. The techniques employed are developed in considerable detail. As examples, we concentrate primarily on spin 0 and 1, linear and nonlinear realizations of $SU(2)\times SU(2)$ and $SU(3)\times SU(3)$ and some of the significant predictions of the theory are derived. The paper contains an extensive discussion of an effective Lagrangian with nonets of real scalar, pseudoscalar, vector, and axial-vector mesons that illustrate the problems of broken $SU(3)\times SU(3)$.

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I. INTRODUCTION

The algebraic formulation of chiral symmetry by Gell-Mann (1962; 1964) has opened up a new, fertile field of study in elementary particle physics.‡ The postulated local commutation relations for the vector- and axial-current densities,

$$[j_0^{\alpha}(x),j_0^{\beta}(y)]_{x_0=y_0}=[j_{50}^{\alpha}(x),j_{50}^{\beta}(y)]_{x_0=y_0}$$
$$=if_{\alpha\beta\gamma}\,j_0^{\gamma}(x)\,\delta(\mathbf{x}-\mathbf{y}),$$
$$[j_0^{\alpha}(x),j_{50}^{\beta}(y)]_{x_0=y_0}=if_{\alpha\beta\gamma}\,j_{50}^{\gamma}(x)\,\delta(\mathbf{x}-\mathbf{y}), \qquad (1.1)$$

bring the problem of determining the current matrix elements into the realm of quantum field theory, with the accompanying difficulties that arise from multi-particle states. In most calculations, therefore, little use has been made of the locality of the current commutators. The content of (1.1) has only been used in integrated form in "Ward–Takahashi identities"* and a number of approximations have been made.

(i) It has generally been assumed that the divergence of the axial current, when used as the interpolating field for the pseudoscalar mesons, yields the smoothest possible off-shell continuation for matrix elements involving such mesons. In a Lagrangian model, the relation

$$\partial^{\mu}\mathbf{j}_{5\mu}(x)=(\text{const})\,\boldsymbol{\phi}(x), \qquad (1.2)$$

where $\boldsymbol{\phi}(x)$ is the pion field, certainly satisfies this condition in lowest-order perturbation theory. We shall refer to (1.2) as the partially conserved axial-current condition, or PCAC.†

(ii) The idea of the vector dominance of current matrix elements‡ has been implemented by the replacement of the currents by the interpolating fields of vector mesons (Kroll, Lee, and Zumino, 1967), treated as stable particles.

(iii) The most important assumption has been the replacement of the sum over intermediate states of certain quantum numbers by a sum over a few low-spin, single-particle states. With this assumption only contact terms (which may be polynomials in some of the scalar variables) and single-particle pole terms appear in the calculations.

* Supported in part by the U.S. Atomic Energy Commission under contract AT-(11-1)-1764.
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‡ A survey of this field may be found in the two excellent monographs of Adler and Dashen (1968) and Renner (1968).

* See the papers of Schnitzer and Weinberg (1967), Geffen (1967), Brown and West (1967), Das, Mathur, and Okubo (1967), Arnowitt, Friedman and Nath (1967), Gerstein and Schnitzer (1968), and Arnowitt, Friedman, Nath, and Suitor (1968). A recent discussion of chiral symmetry may be found in the articles of Dashen (1969) and Dashen and Weinstein (1969).
† Gell-mann and Levy (1960), Nambu (1960), and Chou Kuang-Chao (1961); Bernstein, Fubini, Gell-Mann, and Thirring (1960).
‡ Sakurai (1960); Gell-Mann and Zachariasen (1961).

A number of people* observed that these techniques could be more completely represented by appropriately constructed effective Lagrangians,† with which lowest-order perturbation calculations, not involving any closed loops, were to be carried out. The effective Lagrangian method suffers from the defect that if one wishes to include more particles, one has to start all over again. A corresponding advantage is that one is sure of consistency of the calculations, whatever the reaction involving the particles included at the start. Thus in a system involving π's, ρ's, and A's one must simultaneously deal with $\pi\pi$, $\pi\rho$, and πA scattering to ensure consistency in a current-algebra calculation: this is automatically satisfied by a Lagrangian. Effective Lagrangians are also useful models which can be used to test the particular assumptions made in any more conventional current-algebra calculation. In addition, there is still a great deal of arbitrariness in the construction of effective Lagrangians, especially for the case of $SU(3)\times SU(3)$. This is due partly to the intrinsic abitrariness of the approach and partly to the experimental uncertainty about the existence and properties of axial-vector and scalar mesons. The study of these Lagrangians, therefore, can give us a feeling for the limitations of the current-algebra approach and an understanding of the enormous variety of the assumptions made and predictions resulting that have recently been appearing in the literature.

Finally, a word about our references. This is a review of the use of effective Lagrangians and field algebras for exploring some of the consequences of current algebra and PCAC. Consequently, we have made no attempt to give a complete bibliography for all the work done on current algebras using non-Lagrangian methods. The references we do cite are only some of the many excellent current-algebra and Ward–Takahashi identity treatments of the problems we discuss. They are mentioned so that the reader can make the connection between these methods and the use of effective Lagrangians. For a more complete bibliography in current algebra the reader is referred to the monographs by Adler and Dashen (1968) and by Renner (1968) that were mentioned at the beginning of the Introduction, to the rapporteur talk by Weinberg (Vienna, 1968), and to the other references appended in a postbibliographic note.

The plan of the paper is the following: In Sec. II we discuss the Lagrangian formalism, i.e., how currents are defined so that their integrated densities obey the required algebraic properties. In this section we also discuss the tree-graph approximation and show that the symmetry of the Lagrangian is maintained by it. In Sec. III we discuss various ways of implementing chiral $SU(2)\times SU(2)$ for pions. We discuss in some detail a model containing an isoscalar 0^+ meson, and show that in the limit that the mass of this particle goes to infinity, we obtain a nonlinear realization of the symmetry. The uniqueness of the nonlinear realization is established, and the connection with the Goldstone bosons is briefly discussed. In Sec. IV we discuss the Yang–Mills Lagrangian and show that its use allows us to construct a theory in which the currents are proportional to the spin-1 meson fields. The commutation relations of the currents are discussed and the Schwinger terms derived.

In Sec. V the Yang–Mills theory, as modified above, is generalized to satisfy chiral symmetry by the inclusion of axial-vector mesons. Covariant derivatives for the scalar and pseudoscalar fields are constructed. These ideas are used in Sec. VI to construct a Lagrangian which is then studied in detail. We describe how a mixing between the pseudoscalar field and the axial-vector field can be diagonalized, and we derive expressions for the $\rho\pi\pi$ and $A\rho\pi$ couplings in agreement with current algebra results. We also show that a nonlinear realization for the axial field is possible, but not likely. In Sec. VII the Yang–Mills formalism is generalized to $SU(3)\times SU(3)$. A convenient 3×3 matrix notation for octets is introduced. The various ways in which octet $SU(3)$ symmetry breaking can be introduced into the theory are discussed. Section VIII generalizes the developments of Sec. III to unitary symmetry. The different invariants are discussed, as is octet symmetry breaking. The model is further considered in Sec. IX, where nonlinear realizations are treated. In Sec. X we discuss in some detail a Lagrangian containing nonets of scalar and pseudoscalar mesons interacting with nonets of vector and axial mesons. Some numerical results are presented.

In Sec. XI some miscellaneous topics, which could not be discussed in depth, are briefly touched upon. They are (i) the question of coupling to photons, (ii) chiral symmetry for baryons, and (iii) a new way of obtaining further relations between coupling constants, due to Weinberg. In the five appendices we discuss (i) the transformation properties of fields under $SU(2)\times SU(2)$, (ii) the transformation properties of fields under $SU(3)\times SU(3)$, (iii) our conventions for calculating decay widths, (iv) the Gell-Mann matrices, and (v) some mathematical material appropriate to the nonlinear realizations of transformation groups. The bibliography includes literature available to us through May 1969.

II. THE LAGRANGIAN FORMALISM

The equations of motion for a system described by a Lagrangian density function of certain fields $\phi_A(x)$

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* Weinberg (1967); Schwinger (1967); Wess and Zumino (1967); Bardeen and Lee (1968); Lee and Nieh (1968); Cronin (1967); Arnowitt, Friedman, and Nath (1967). Minamikawa and Miyamoto (1967); Shiozaki (1968); Yamaguchi (1968); and Sabo (1968).

† Some early papers are Schwinger (1957); Polkinghorne (1958); Gursey (1960; 1961); and Kramer, Rollnik, and Stech (1959).

and their derivatives

$$\mathcal{L}=\mathcal{L}(\phi_A(x),\,\partial_\mu\phi_A(x)) \tag{2.1}$$

are the Euler–Lagrange equations, which take the form

$$\partial_\mu[\partial\mathcal{L}/\partial(\partial_\mu\phi_A(x))]=\partial\mathcal{L}/\partial\phi_A(x) \tag{2.2}$$

when the Lagrangian density is of second order in the derivatives. The labels A here refer to internal symmetry states such as the i-spin state, as well as to the space–time components if the field is a tensor or a spinor.

The canonical momentum is defined by

$$\pi^A(x)=\partial\mathcal{L}/\partial_0(\phi_A(x)). \tag{2.3}$$

We shall later treat the fields as operators, and assume for them the canonical commutation relations

$$[\pi^A(x),\,\phi_B(y)]_{x_0=y_0}=-i\delta_B{}^A\delta(\mathbf{x}-\mathbf{y}). \tag{2.4}$$

The distinction between covariant and contravariant indices is irrelevant for the internal-symmetry indices but is necessary for the spatial indices as we are using the $(1,\,-1,\,-1,\,-1)$ metric.

The quantity of central interest to us is the current associated with some internal symmetry. We arrive at its definition by considering the fields to be altered by an infinitesimal space–time-dependent gauge transformation of the form

$$\phi_A(x)\rightarrow\phi_A(x)+\delta\phi_A(x)\equiv\phi_A(x)+iC_{ABC}\alpha_B(x)\phi_C(x), \tag{2.5}$$

so that

$$\partial_\mu\phi_A(x)\rightarrow\partial_\mu\phi_A(x)+iC_{ABC}\partial_\mu(\alpha_B(x)\phi_C(x)). \tag{2.6}$$

The constant C_{ABC} depends on the internal variables (such as i spin) alone, as does the function $\alpha_B(x)$. Thus this equation holds for every space–time component of the field. The change induced in the Lagrangian density by this transformation is

$$\begin{aligned}\delta\mathcal{L}&=\frac{\partial\mathcal{L}}{\partial\phi_A(x)}\,\delta\phi_A(x)+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A(x))}\,\partial_\mu\delta\phi_A(x)\\&=\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A(x))}\,\delta\phi_A(x)\right)\\&=\partial_\mu\left(i\alpha_B(x)\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A(x))}\,C_{ABC}\phi_C(x)\right).\end{aligned} \tag{2.7}$$

Here the equation of motion (2.2) has been used in the second step. For a constant gauge transformation (α_B independent of x) the *invariance* of the Lagrangian density implies the existence of a conserved current defined by

$$j_\mu{}^B(x)=-i[\partial\mathcal{L}/\partial(\partial^\mu\phi_A(x))]C_{ABC}\phi_C(x). \tag{2.8}$$

If the internal indices were separated from the space–time indices, it would be manifest that in (2.8) the latter are summed over. The proportionality constant $-i$ is determined by consideration of the unitary transformation that implements (2.5). Consider the generalized charge Q_B defined by

$$Q_B\equiv\int d^3x j_0{}^B(x), \tag{2.9}$$

in which the index B refers to the internal variable alone. Using (2.3) we may write this in the form

$$Q_B(x_0)=-iC_{ABC}\int_{x_0'=x_0}d^3x'\pi^A(x')\phi_C(x'). \tag{2.10}$$

Hence

$$\begin{aligned}[][Q_B(x_0),\,\phi_D(x)]&=-iC_{ABC}\int d^3x'[\pi^A(x'),\phi_D(x)]\phi_C(x')\\&=-C_{DBC}\phi_C(x).\end{aligned} \tag{2.11}$$

From this it follows that the unitary transformation

$$U=\exp[-i\alpha_B(x)Q_B] \tag{2.12}$$

with infinitesimal $\alpha_B(x)$ transforms $\phi_A(x)$ as follows:

$$\begin{aligned}U\phi_A(x)U^{-1}\phi_A(x)-i\alpha_B(x)[Q_B,\,\phi_A(x)]\\=\phi_A(x)+iC_{ABC}\alpha_B(x)\phi_C(x).\end{aligned} \tag{2.13}$$

This is just the transformation law postulated in (2.5).

With this definition of the current we may rewrite (2.7) in the form

$$\delta\mathcal{L}=-\alpha_B(x)\partial^\mu j_\mu{}^B(x)-\partial^\mu\alpha_B(x)j_\mu(x) \tag{2.14}$$

from which we obtain the important equations (Gell-Mann and Levy, 1960)

$$\begin{aligned}j_\mu{}^B(x)&=-\partial(\delta\mathcal{L})/\partial(\partial^\mu\alpha_B(x)),\\\partial^\mu j_\mu{}^B(x)&=-\partial(\delta\mathcal{L})/\partial\alpha_B(x).\end{aligned} \tag{2.15}$$

The coefficients C_{ABC} can be determined from the commutation relations for the Q_B required by the group property of the unitary transformations U. If the Lie algebra is characterized by

$$[Q_A,\,Q_B]=if_{ABC}Q_C, \tag{2.16}$$

then using (2.10) we have

$$\begin{aligned}[][Q_A,\,Q_B]&=-C_{MAN}C_{PBQ}\int d^3x\int d^3y\\&\quad\times[\pi^M(x)\phi_N(x),\,\pi^P(y)\phi_Q(y)]\\&=i\int(d^3x[C_{MAN}C_{PBM}\pi^P(x)\phi_N(x)\\&\quad-C_{MAN}C_{NBQ}\pi^M(x)\phi_Q(x)]\\&\equiv if_{ABC}\left[-iC_{MCQ}\int d^3x\pi^M(x)\phi_Q(x)\right]\end{aligned} \tag{2.17}$$

provided that

$$C_{MAN}C_{NBQ}-C_{MBN}C_{NAQ}=if_{ABC}C_{MCQ}.$$

We may write this in the form

$$(C_A)_{MN}(C_B)_{NQ}-(C_B)_{MN}(C_A)_{NQ}=if_{ABC}(C_C)_{MQ}. \quad (2.18)$$

Thus the C_{MAN} must be chosen to be the MN matrix element of a matrix representation C_A of the generalized charges Q_A. For example, if the Q_A obey the commutation relations of $SU(2)$, then

$$C_{MAN}=(T^A)_{MN}, \quad (2.19)$$

where the T^A are i-spin matrices. The multiplicity of possible labels is three here ($A=1, 2, 3$) and in general matches the number of generators of the transformations U (as the Q_A are sometimes called), as can be seen from (2.12) and (2.13). The indices M and N, on the other hand, have the multiplicity of the fields under consideration.

The scattering matrix for a physical system described by the Lagrangian density (2.1) and the canonical commutation relations (2.4) can *formally* be obtained by writing $\mathcal{L}$ in the form

$$\mathcal{L}(\phi_A, \partial_\mu\phi_A)=\mathcal{L}_0+\mathcal{L}_{\text{int}}, \quad (2.20)$$

where $\mathcal{L}_0$ is the Lagrangian density describing the system without interaction, and then "computing" the matrix elements of the operator

$$S=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int dx_1\cdots dx_n T(\mathcal{L}_{\text{int}}(x_1)\cdots\mathcal{L}_{\text{int}}(x_n)). \quad (2.21)$$

The terms in this series are given by the collection of all Feynman graphs computed according to rules appropriate to the spins of the particles involved and the vertices that appear in $\mathcal{L}_{\text{int}}(x)$. There are two important properties of this formal* series solution that should be noted. The first one has to do with the behavior of the scattering matrix under a point transformation of the fields

$$\phi_A(x)=\chi_A(x)g(\chi(x)); \qquad g(0)=1. \quad (2.22)$$

As was shown by Chisholm (1961) [see also Kamefuchi, O'Raifeartaigh, and Salam (1961)], the scattering matrix elements, in contrast to the Green's functions, are unchanged by such a local transformation that does not involve the derivatives of the fields. This result may be viewed as a concrete illustration of the general result that the scattering matrix, connecting the "in" and "out" fields, does not specify a unique local interpolating field (Haag, 1958; Nishijima, 1958; Zimmermann, 1958; and Borchers, 1960); if $\phi_A(x)$ is a suitable local interpolating field, so is a local function of it.

* The *formal* character of the expansion should be stressed as one frequently deals with unrenormalizable theories for which no real meaning can be ascribed to the series.

The second one has to do with the properties of the "tree-graph" approximation (no closed-loop graphs) to the scattering matrix. As was noted by a number of authors (Nambu, 1968; Lee and Nieh, 1968; Boulware and Brown, 1968; and Coleman, Wess, and Zumino, 1969), the "tree-graph" approximation may be considered as a first term in a systematic expansion of the scattering matrix. We outline the argument following the discussion of Coleman *et al.* Given a Lagrangian $\mathcal{L}(\phi)$ consider

$$\mathcal{L}(\phi, \lambda)\equiv(1/\lambda^2)\mathcal{L}(\lambda\phi). \quad (2.23)$$

A given connected Feynman diagram can be seen to have a definite power of the parameter λ associated with it. Let E be the number of its external lines, I the number of internal lines, L the number of loops (the number of internal integrations), V the number of vertices, and N_i ($i=1, 2,\cdots, V$) the number of lines attached to the ith vertex. Since a vertex with N_i lines comes from a term involving N_i field operators multiplied together, and there is a factor λ^{-2} in front of the Lagrangian, each vertex carries the power N_i-2 of the parameter. Thus the diagram carries a power given by

$$P=\sum_{i=1}^{V}(N_i-2). \quad (2.24)$$

Since a line is either an internal line or an external one,

$$\sum_{i=1}^{V}N_i=E+2I. \quad (2.25)$$

Thus

$$P=E+2I-2V. \quad (2.26)$$

The number of loops is given by

$$L=I-V+1, \quad (2.27)$$

so that

$$P=E+2L-2. \quad (2.28)$$

Thus for a given process, characterized by a fixed E, terms with different numbers of loops carry different powers of λ. The tree graphs, for which $L=0$, are thus the lowest order terms in a systematic expansion of powers of λ. If follows from this that invariance properties of the Lagrangian are maintained by the tree-graph contributions alone, as is the invariance of the tree-graph approximation to the scattering matrix under point transformations. For the latter it is only necessary to introduce λ into the transformation via

$$\phi_A=\chi_A g(\lambda\chi). \quad (2.29)$$

Then

$$\mathcal{L}(\chi g(\lambda\chi), \lambda)=(1/\lambda^2)\mathcal{L}(\lambda\chi g(\lambda\chi)). \quad (2.30)$$

Thus the power counting is the same, and a tree graph is again characterized by

$$P=E-2. \quad (2.31)$$

From the equality of the scattering matrices for $\mathfrak{L}(\phi$ and $\mathfrak{L}'(\phi)\equiv\mathfrak{L}(\phi g(\phi))$ follows the equality of the terms proportional to λ^{E-2}.*

A final comment which leads us naturally into the next section is that if all of the fields ϕ_A are independent, the transformation law (2.5) is a linear one, whereas if some of the fields are functions of the others, we speak of a nonlinear realization of the symmetry.

III. CHIRAL SYMMETRY FOR PIONS

The symmetry which will be considered here is one first proposed by Gell-Mann (1962), namely $SU(2)\times SU(2)$. The transformations are generated by a set of six generalized charges, Q^A and $Q_5{}^A$ $(A=1, 2, 3)$, which obey the commutation relations

$$[Q^A, Q^B]=ie_{ABC}Q^C,$$
$$[Q^A, Q_5{}^B]=ie_{ABC}Q_5{}^C,$$
$$[Q_5{}^A, Q_5{}^B]=ie_{ABC}Q^C. \qquad (3.1)$$

The nomenclature is related to the commutation relations

$$[Q_\pm{}^A, Q_\pm{}^B]=ie_{ABC}Q_\pm{}^C,$$
$$[Q_+{}^A, Q_-{}^B]=0, \qquad (3.2)$$

obeyed by the operators

$$Q_\pm{}^A=\tfrac{1}{2}(Q^A\pm Q_5{}^A). \qquad (3.3)$$

The construction of Lagrangians which are invariant under unitary transformations generated by these operators is made nontrivial by the requirement that parity be conserved, and

$$PQ_+{}^AP^{-1}=Q_-{}^A. \qquad (3.4)$$

Let us construct some simple invariant Lagrangians containing the pion field of i spin 1. The statement that the i spin of the pion is 1 is equivalent to writing

$$\delta\phi_A\equiv-i[\alpha_BQ^B, \phi_A]=e_{BAC}\alpha_B\phi_C$$
$$=-(\boldsymbol{\alpha}\times\boldsymbol{\phi})_A. \qquad (3.5)$$

This is just the infinitesimal version of

$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q})\phi_A\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q})=\phi_B[\exp(-i\boldsymbol{\alpha}\cdot\mathbf{T})]_{BA},$$

where the $\mathbf{T}$ are the 3×3-dimensional i-spin matrices with $(T^k)_{BA}=-ie_{kBA}$. In order to test the invariance of a Lagrangian, i.e., to see whether

$$[Q_5{}^A, \mathfrak{L}]=0, \qquad (3.6)$$

we need to know $[Q_5{}^A, \phi_B]$. This must be a scalar. A linear realization involving the pion field alone would require that

$$[Q_5{}^A, \phi_B]=0, \qquad (3.7)$$

which is inconsistent with (3.5) and the Jacobi identity

$$[Q_5{}^A, [Q_5{}^B, \phi_C]]-[Q_5{}^B, [Q_5{}^A, \phi_C]]=[[Q_5{}^A, Q_5{}^B], \phi_C]. \qquad (3.8)$$

Thus, to construct a linear realization of the symmetry known as *chiral symmetry*, we must introduce other fields.

The simplest possibility is to introduce a scalar field σ_A carrying i spin 1. We then have

$$\delta\boldsymbol{\sigma}=-(\boldsymbol{\alpha}\times\boldsymbol{\sigma}), \qquad (3.9)$$

and we propose that

$$\delta'\phi_A\equiv-i[\beta_BQ_5{}^B, \phi_A]=-(\boldsymbol{\beta}\times\boldsymbol{\sigma})_A,$$
$$\delta'\sigma_A=-(\boldsymbol{\beta}\times\boldsymbol{\phi})_A. \qquad (3.10)$$

It follows from these relations that

$$\delta'(\boldsymbol{\phi}^2+\boldsymbol{\sigma}^2)=-2\boldsymbol{\phi}\cdot\boldsymbol{\beta}\times\boldsymbol{\sigma}-2\boldsymbol{\sigma}\cdot\boldsymbol{\beta}\times\boldsymbol{\phi}=0. \quad (3.11)$$

Thus $\boldsymbol{\phi}^2+\boldsymbol{\sigma}^2$ is a *chiral invariant* and so, for constant gauge functions, is

$$\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi})^2+\tfrac{1}{2}(\partial_\mu\boldsymbol{\sigma})^2. \qquad (3.12)$$

Thus a satisfactory Lagrangian would be

$$\mathfrak{L}=\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi})^2+\tfrac{1}{2}(\partial_\mu\boldsymbol{\sigma})^2-\tfrac{1}{2}\mu^2(\boldsymbol{\phi}^2+\boldsymbol{\sigma}^2)$$
$$+\tfrac{1}{4}\lambda(\boldsymbol{\phi}^2+\boldsymbol{\sigma}^2)^2+\cdots \qquad (3.13)$$

to which we could also add any *even* function of the pseudoscalar invariant $\boldsymbol{\phi}\cdot\boldsymbol{\sigma}$.

For space–time-dependent gauge functions we have

$$\delta\mathfrak{L}=-\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\alpha}\times\boldsymbol{\phi}-\partial_\mu\boldsymbol{\sigma}\cdot\partial^\mu\boldsymbol{\alpha}\times\boldsymbol{\sigma} \qquad (3.14)$$

and

$$\delta'\mathfrak{L}=-\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\beta}\times\boldsymbol{\sigma}-\partial_\mu\boldsymbol{\sigma}\cdot\partial^\mu\boldsymbol{\beta}\times\boldsymbol{\phi}, \qquad (3.15)$$

from which we can determine the currents

$$\mathbf{j}_\mu=\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi}+\boldsymbol{\sigma}\times\partial_\mu\boldsymbol{\sigma},$$
$$\mathbf{j}_{5\mu}=\boldsymbol{\sigma}\times\partial_\mu\boldsymbol{\phi}+\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\sigma}. \qquad (3.16)$$

If we require these currents to have the same CP (or GP) transformation properties, then the $\boldsymbol{\sigma}$ must be an even G-parity field. Such a $\boldsymbol{\sigma}$ (with its "abnormal" charge conjunction properties) could not couple to two pions (statistics) or to three pions (G parity). Also since the G parity of the $N\bar{N}$ system with S, orbital angular momentum L, and i spin T is $(-1)^{S+L+T}$, there can be no coupling of the $\boldsymbol{\sigma}$ to a $T=1\ {}^3P_0$ state. Such a particle would be, to say the least, difficult to produce. If the $\boldsymbol{\sigma}$ had odd G parity, its simplest decay would be into five pions ($\boldsymbol{\sigma}\not\to3\pi$ because of spin and parity), but it could couple to nucleons. The axial current in this

* The parameter $\hbar$ (if one expresses the Lagrangian in now unfamiliar units!) quite naturally plays the role of λ in the expansion, so that the tree-graph approximation bears a formal resemblance to the WKB approximation, as noted by Nambu.

case would be a second class current (Weinberg, 1958) and with CP transformation properties opposite to the usual axial current could lead to CP violation (Maiani, 1968). These speculations are not really within the scope of this review, and we continue by considering an alternate possibility, that of including an i-spin-0 scalar field $\sigma(x)$. We require that

$$\delta'\boldsymbol{\phi}=\boldsymbol{\beta}\sigma,$$
$$\delta'\sigma=-\boldsymbol{\beta}\cdot\boldsymbol{\phi}. \tag{3.17}$$

[This implies that we are dealing with a $(\frac{1}{2},\frac{1}{2})$ representation of $SU(2)\times SU(2)$. See Appendix A for a discussion of this.] Again $\boldsymbol{\phi}^2+\sigma^2$ can be seen to be invariant, and a chiral-invariant Lagrangian is

$$\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi})^2+\tfrac{1}{2}(\partial_\mu\sigma)^2-\tfrac{1}{2}\mu^2(\boldsymbol{\phi}^2+\sigma^2)+\tfrac{1}{4}\lambda(\boldsymbol{\phi}^2+\sigma^2)^2+\cdots. \tag{3.18}$$

The σ can couple to any even number of pions* as well as to baryons and is a perfectly acceptable particle. It can also couple to the vacuum, i.e., it can have a nonvanishing vacuum expectation value without violating Lorentz invariance, parity or G parity. Such a nonvanishing vacuum expectation value can, of course, only arise when chiral symmetry is broken, since of the fields $(\sigma,\boldsymbol{\phi})$ that transform among themselves, one is singled out. We shall see that if the symmetry is broken by a term of the form

$$\mathcal{L}_{SB}=f_\pi m_\pi^2\sigma, \tag{3.19}$$

then necessarily $\langle\sigma\rangle_0\neq 0$. The above choice of symmetry breaking is particularly interesting to us because from

$$\delta'\mathcal{L}_{SB}=-f_\pi m_\pi^2\boldsymbol{\beta}\cdot\boldsymbol{\phi} \tag{3.20}$$

it follows that

$$\partial^\mu\mathbf{j}_{5\mu}=f_\pi m_\pi^2\boldsymbol{\phi}. \tag{3.21}$$

This, however, is just the PCAC condition referred to in the Introduction. With this choice of symmetry breaking, the total Lagrangian (3.18)+(3.19) can be treated in a conventional way, except for the fact that the term linear in σ leads to "tadpole graphs" (Fig. 1).† To eliminate these graphs we write

$$\sigma(x)=\langle\sigma(x)\rangle_0+\sigma'(x)\equiv\sigma_0+\sigma'(x) \tag{3.22}$$

and determine σ_0 from the condition that terms linear in σ' disappear from the Lagrangian. With our Lagrangian this leads to the condition

$$-\mu^2\sigma_0+\lambda\sigma_0^3+f_\pi m_\pi^2=0. \tag{3.23}$$

The coefficients of $-\frac{1}{2}\boldsymbol{\phi}^2$ and $-\frac{1}{2}\sigma'^2$ may be identified as the squares of the masses so that

$$m_\pi^2=\mu^2-\lambda\sigma_0^2,$$
$$m_\sigma^2=\mu^2-3\lambda\sigma_0^2. \tag{3.24}$$

Note that λ may be expressed in terms of f_π and the masses, $\lambda=(m_\pi^2-m_\sigma^2)/2f_\pi^2$. Also, $m_\sigma^2-3m_\pi^2=-2\mu^2$, which implies $\mu^2<0$ if the σ is massive. Combining (3.24) with (3.23) we find that

$$\sigma_0=f_\pi. \tag{3.25}$$

Thus symmetry breaking introduces a $\sigma\pi$ mass splitting absent in the symmetric Lagrangian. One may use this Lagrangian to calculate $\pi\pi$ scattering in the tree-graph approximation. The graphs that contribute are shown in Fig. 2, and the amplitude is given by*

$$T_{\gamma\delta,\alpha\beta}=-\frac{1}{(2\pi)^6}\left[\delta_{\alpha\beta}\delta_{\gamma\delta}\left(\frac{4\lambda^2f_\pi^2}{m_\sigma^2-s}+2\lambda\right)+\delta_{\alpha\gamma}\delta_{\beta\delta}\times\left(\frac{4\lambda^2f_\pi^2}{m_\sigma^2-t}+2\lambda\right)+\delta_{\alpha\delta}\delta_{\beta\gamma}\left(\frac{4\lambda^2f_\pi^2}{m_\sigma^2-u}+2\lambda\right)\right], \tag{3.26}$$

so that the predicted scattering lengths are†

$$a_0=\frac{1}{16\pi}\frac{1}{m_\pi}\left(10\lambda+\frac{12\lambda^2f_\pi^2}{m_\sigma^2-4m_\pi^2}+\frac{8\lambda^2f_\pi^2}{m_\sigma^2}\right),$$
$$a_2=\frac{1}{16\pi}\frac{1}{m_\pi}\left(4\lambda+\frac{8\lambda^2f_\pi^2}{m_\sigma^2}\right). \tag{3.27}$$

Let us now turn to the possibility of a *nonlinear realization* in which we do not admit the existence of fields other than that of the pion (Bardeen and Lee, 1968; Schwinger, 1968; Weinberg, 1968). The most general form for $\delta'\boldsymbol{\phi}$ is

$$\delta'\boldsymbol{\phi}=\boldsymbol{\beta}f_1(\boldsymbol{\phi}^2)+\boldsymbol{\phi}(\boldsymbol{\beta}\cdot\boldsymbol{\phi})f_2(\boldsymbol{\phi}^2) \tag{3.28}$$

or, equivalently,

$$[Q_5^k,\phi_l]=i\delta_{kl}f_1(\boldsymbol{\phi}^2)+i\phi_k\phi_lf_2(\boldsymbol{\phi}^2). \tag{3.29}$$

There is a relation between the functions f_1 and f_2 that follows from the Jacobi identity (3.8), which takes the form

$$[\delta_{(\beta)}',\delta_{(\alpha)}']\boldsymbol{\phi}=-(\boldsymbol{\beta}\times\boldsymbol{\alpha})\times\boldsymbol{\phi}. \tag{3.30}$$

After some simple algebraic manipulations one finds that the relation

$$1+2f_1(x)[df_1(x)/dx]+2xf_2(x)[df_1(x)/dx]-f_1(x)f_2(x)=0, \tag{3.31}$$

* Only if additional symmetry-breaking terms are added.

† Tadpole graphs are discussed by Coleman and Glashow (1964).

* It is easily checked that when one of the pion four-momenta vanishes, so that $s=t=u=m_\pi^2$, the amplitude goes to zero as required by the Adler (1965) condition.

† We use the normalization of states $\langle p'\mid p\rangle=2p_0\delta(\mathbf{p}-\mathbf{p}')$ as in Gasiorowicz (1966). The scattering amplitude $f(W,\theta)$ is related to T by $f=-(8\pi^5)\,T/W$, where W is the center-of-mass total energy. We also use $\delta_{\alpha\beta}\delta_{\gamma\delta}=3P_0$; $\delta_{\alpha\gamma}\delta_{\beta\delta}=P_2+P_1+P_0$; $\delta_{\alpha\delta}\delta_{\beta\gamma}=P_2-P_1+P_0$, where the P_i are i-spin projection operators.

where $x=\boldsymbol{\phi}^2$, must be satisfied. Thus, given one of the functions we can always find the other. In the special case that

$$f_2(x)=0 \tag{3.32}$$

the differential equation has the simple solution

$$f_1(x)=[f_1^2(0)-x]^{1/2}. \tag{3.33}$$

This is, in fact, the most general case because if $f_2(x)\neq 0$, then it is always possible to find a new field $\boldsymbol{\pi}$ related to $\boldsymbol{\phi}$ by

$$\boldsymbol{\phi}=\boldsymbol{\pi}g(\boldsymbol{\pi}^2) \tag{3.34}$$

in terms of which the Jacobi identity is satisfied with the form

$$\delta'\boldsymbol{\pi}=\boldsymbol{\beta}f_3(\boldsymbol{\pi}^2). \tag{3.35}$$

Since

$$\begin{aligned}\delta'\boldsymbol{\phi}&=\boldsymbol{\beta}f_1(x)+\boldsymbol{\pi}(\boldsymbol{\beta}\cdot\boldsymbol{\pi})g^2(y)f_2(x)\\&=\delta'\boldsymbol{\pi}g(y)+2\boldsymbol{\pi}g'(y)(\boldsymbol{\pi}\cdot\delta'\boldsymbol{\pi}),\end{aligned}$$

where the prime denotes differentiation with respect to $y=\boldsymbol{\pi}^2$, it follows that

$$(f_3y)g(y)=f_1(x);\qquad 2f_3(y)g'(y)=g^2(y)f_2(x). \tag{3.36}$$

From (3.34) we have

$$x=yg^2(y),$$

so that

$$\frac{dg}{dy}=\frac{g^2(y)(dg/dx)}{1-2yg(dg/dx)}.$$

From (3.36) and (3.31) we have

$$\frac{f_2(x)}{f_1(x)}=2\frac{dg}{dy}\frac{1}{g^3(y)}=\frac{1+df_1^2(x)/dx}{f_1^2(x)-x[df_1^2(x)/dx]}.$$

Hence

$$g(y)=c[x+f_1^2(x)]^{1/2};$$

i.e.,

$$\boldsymbol{\pi}^2=f_1^2(0)(\boldsymbol{\phi}^2/\{\boldsymbol{\phi}^2+[f_1(\boldsymbol{\phi}^2)]^2\}), \tag{3.37}$$

so that in a power-series expansion $\boldsymbol{\pi}^2=\boldsymbol{\phi}^2+\cdots$.

The case $f_2=0$ is equivalent to the σ model considered earlier, with the proviso that $\sigma(x)$ is not an independent field, but satisfies

$$\sigma^2(x)+\boldsymbol{\phi}^2(x)=f_\pi^2. \tag{3.38}$$

We have set $f_1(0)=f_\pi$. This guarantees the correct form for the pion mass term in (3.44) below. This is the nonlinear model of Gell-Mann and Levy (1960). The most general Lagrangian containing no more than two derivatives is

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi})^2+\tfrac{1}{2}(\partial_\mu\sigma)^2, \tag{3.39}$$

since $\sigma^2+\boldsymbol{\phi}^2$ is not merely a chiral-invariant operator but is actually a numerical constant. With this Lagrangian the currents turn out to be

$$\begin{aligned}\mathbf{j}_\mu&=\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi},\\ \mathbf{j}_{5\mu}&=\boldsymbol{\phi}\partial_\mu\sigma-\sigma\partial_\mu\boldsymbol{\phi}.\end{aligned} \tag{3.40}$$

It is interesting to notice that

$$\begin{aligned}\mathbf{j}_\mu\cdot\mathbf{j}^\mu+\mathbf{j}_{5\mu}\cdot\mathbf{j}_5^\mu&=\boldsymbol{\phi}^2(\partial_\mu\boldsymbol{\phi})^2-(\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi})^2+\boldsymbol{\phi}^2(\partial_\mu\sigma)^2\\&\quad+\sigma^2(\partial_\mu\boldsymbol{\phi})^2-2\sigma\partial_\mu\sigma(\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi})\\&=(\boldsymbol{\phi}^2+\sigma^2)((\partial_\mu\boldsymbol{\phi})^2+(\partial_\mu\sigma)^2)\\&=2f_\pi^2\mathcal{L}.\end{aligned} \tag{3.41}$$

More generally, a simple calculation of the energy momentum tensor yields*

$$\begin{aligned}T_{\mu\nu}&=[\partial\mathcal{L}/\partial(\partial^\mu\boldsymbol{\phi})]\cdot\partial_\nu\boldsymbol{\phi}-g_{\mu\nu}\mathcal{L}\\&=(1/f_\pi^2)[\mathbf{j}_\mu\cdot\mathbf{j}_\nu+\mathbf{j}_{5\mu}\cdot\mathbf{j}_{5\nu}-\tfrac{1}{2}g_{\mu\nu}(\mathbf{j}_\alpha\cdot\mathbf{j}^\alpha+\mathbf{j}_{5\alpha}\cdot\mathbf{j}_5^\alpha)].\end{aligned} \tag{3.42}$$

Let us now continue by introducing symmetry breaking in the form

$$\mathcal{L}_{SB}=m_\pi^2f_\pi\sigma. \tag{3.43}$$

This implies the PCAC condition (3.21). In terms of the pion field the total Lagrangian reads

$$\begin{aligned}\mathcal{L}&=\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi})^2+\tfrac{1}{2}[(\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi})^2/(f_\pi^2-\boldsymbol{\phi}^2)]+f_\pi m_\pi^2(f_\pi^2-\boldsymbol{\phi}^2)^{1/2}\\&=\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi})^2-\tfrac{1}{2}m_\pi^2\boldsymbol{\phi}^2+(1/2f_\pi^2)(\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi})^2\\&\quad-(m_\pi^2/8f_\pi^2)(\boldsymbol{\phi}^2)^2+\cdots.\end{aligned} \tag{3.44}$$

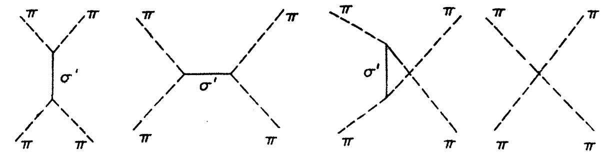

FIG. 2. Tree graphs for $\pi\pi$ scattering including a real σ meson.

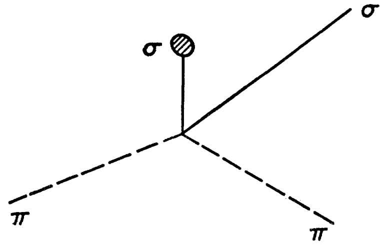

FIG. 1. A typical tadpole graph.

* This form appears in the current model of Sugawara (1968). Because the Schwinger terms are not c numbers, however, the model discussed above has a different structure. See also Sommerfield (1968).

The terms quartic in the meson field may be used to calculate the pion–pion scattering amplitude (Lee and Nieh, 1968)

$$T(p_3,\gamma,p_3,\delta;p_1,\alpha,p_2,\beta)=[1/(2\pi)^6](1/f_\pi^2)$$
$$\times[\delta_{\alpha\beta}\delta_{\gamma\delta}(m_\pi^2-s)+\delta_{\alpha\gamma}\delta_{\beta\delta}(m_\pi^2-t)+\delta_{\alpha\delta}\delta_{\beta\gamma}(m_\pi^2-u)]. \tag{3.45}$$

From this one finds the scattering lengths

$$a_0=(7/16\pi)(m_\pi/f_\pi^2),$$
$$a_2=-(1/8\pi)(m_\pi/f_\pi^2) \tag{3.46}$$

in agreement with those obtained by Weinberg (1966) using current algebra, PCAC, and smoothness conditions which in effect correspond to those implied by restricting the number of derivatives that appear in the Lagrangian.

It has been observed by Weinberg (1967) [see also Bardeen and Lee (1969)] that the scattering matrix of the nonlinear realization may be obtained from that of the linear σ model if the mass of the σ is allowed to become very large. The limit $m_\sigma^2\to\infty$ is obtained if $\mu^2\to-\infty$ and $\lambda\to-\infty$ such that m_π^2 remains finite (3.24). In that limit

$$\lambda/m_\sigma^2\to-\tfrac{1}{2}f_\pi^2,$$
$$\mu^2/m_\sigma^2\to-\tfrac{1}{2}, \tag{3.47}$$

and

$$T_{\gamma\delta,\alpha\beta}\to\frac{1}{(2\pi)^6}\left\{\delta_{\alpha\beta}\delta_{\gamma\delta}\left[\frac{(m_\sigma^2/f_\pi)^2}{m_\sigma^2}\left(1-\frac{s}{m_\sigma^2}\right)+\frac{m_\pi^2-m_\sigma^2}{f_\pi^2}\right]+\cdots\right\}$$
$$=\frac{1}{(2\pi)^6}\left[\delta_{\alpha\beta}\delta_{\gamma\delta}\left(\frac{m_\pi^2-s}{f_\pi^2}\right)+\cdots\right]. \tag{3.48}$$

This is just the result obtained in Eq. (3.45) above. One may in fact show that the Lagrangian (3.18) goes over into (3.39) in that limit, although the proof, strictly speaking, only holds in the tree-graph approximation. If in the equation of motion for the σ field

$$\Box\sigma=-\mu^2\sigma+\lambda\sigma(\sigma^2+\boldsymbol{\phi}^2) \tag{3.49}$$

one takes the limit $\lambda\approx\mu^2/f_\pi^2\to\infty$, then (3.38) necessarily results, provided the momenta are small compared with the σ mass. If there are closed loops, one integrates over the momenta and the limits become very delicate.

The three models discussed in this section show different ways in which chiral symmetry can be satisfied. The symmetry requirement that for the Hamiltonian H

$$[Q_5^i,H]=0 \tag{3.50}$$

implies that for a discrete state, such as the one-pion state $|\,\mathbf{p},a\rangle$,

$$HQ_5^i\,|\,\mathbf{p},a\rangle=Q_5^iH\,|\,\mathbf{p},a\rangle=\omega_pQ_5^i\,|\,\mathbf{p},a\rangle; \tag{3.51}$$

i.e., the state $Q_5^i\,|\,\mathbf{p},a\rangle$ must be a discrete positive parity state, unless $Q_5^i\,|\,\mathbf{p},a\rangle=0$. If $\phi_{\mathbf{p},a}^+$ denotes the creation operator for the state $|\,\mathbf{p},a\rangle$ so that

$$|\,\mathbf{p},a\rangle=\phi_{\mathbf{p},a}^+\,|\,0\rangle,$$

then

$$Q_5^i\,|\,\mathbf{p},a\rangle=[Q_5^i,\phi_{\mathbf{p},a}^+]\,|\,0\rangle,$$

provided the vacuum is a unique invariant state for which

$$Q_5^i\,|\,0\rangle=0. \tag{3.52}$$

In the first model

$$[Q_5^i,\phi_{\mathbf{p},a}^+]=ie_{iak}\sigma_{\mathbf{p},k}^+. \tag{3.53}$$

In this case we have genuine parity doublets; the $\boldsymbol{\sigma}$ is degenerate in mass with the pion. The commutation relation (3.53) yields i-spin 1 for the state $Q_5^i\,|\,\mathbf{p},a\rangle$. If there were an i-spin-2 component, then a subsequent application of Q_5^j would yield an i-spin-2 pion, degenerate with the ordinary pion, and this is not acceptable.

In the second model

$$[Q_5^i,\phi_{\mathbf{p},a}^+]=i\delta_{ai}\sigma_{\mathbf{p}}^+.$$

Here the neutral σ meson, degenerate in mass with the pion triplet, in effect acts as a parity partner for all of them.

In the nonlinear model $\sigma(x)$ is just a shorthand notation for

$$\sigma(x)=f_\pi-(1/2f_\pi)\boldsymbol{\phi}^2(x)-(1/8f_\pi^3)(\boldsymbol{\phi}^2(x))^2+\cdots. \tag{3.54}$$

Thus*

$$Q_5^i\,|\,\mathbf{p},a\rangle=i\delta_{ai}\sigma_{\mathbf{p}}^+\,|\,0\rangle \tag{3.55}$$

yields on the right-hand side a state containing one, two,$\cdots$ pairs of pions, whose total energy and momentum are $(\omega_p,\mathbf{p})$. This is only possible if the pions are massless. Indeed, the chiral symmetric Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi})^2+\tfrac{1}{2}(\partial_\mu\sigma)^2$$

does not contain a pion mass term. The pion mass arises as a result of symmetry breaking. Once there is symmetry breaking, then in the last two models

$$\langle\sigma(x)\rangle_0\equiv\sigma_0\neq0. \tag{3.56}$$

This implies that (3.52) can no longer be true since

$$\langle0\,|\,[Q_5^i,\phi_{\mathbf{p},a}^+]\,|\,0\rangle=i\delta_{ai}\sigma_0. \tag{3.57}$$

* Here $\sigma_{\mathbf{p}}^+$ stands for the p-Fourier component of the negative-frequency part of the operator $\sigma(x)$.

There is nothing pathological about this, since $\partial_\mu \mathbf{j}_5^\mu \neq 0$ implies that Q_5^i is no longer a constant of the motion. The definition of $\sigma(x)$ in (3.54) for the nonlinear model implies that in this case, even without an explicit symmetry-breaking term in the Lagrangian,

$$Q_5^i\,|\,0\rangle \neq 0. \qquad (3.58)$$

This comes about because $\sigma_0 = f_\pi \neq 0$ in (3.57). We speak of this as "spontaneous symmetry breaking" and it has been shown (Goldstone, 1961; Goldstone, Salam, and Weinberg, 1962; Bludman and Klein, 1963) that in many situations such a spontaneous symmetry breakdown is accompanied by the appearance of massless particles, the "Goldstone bosons." [This point of view, which in the symmetry limit identifies pions as massless Goldstone bosons is originally due to Nambu and collaborators (Nambu and Jona-Lasinio, 1961; Nambu and Lurie, 1962) and has recently been stressed by Dashen (1969). For a recent review and bibliography, see Kibble (1967).] In the real-σ model, too, spontaneous symmetry breaking can occur. Without the symmetry-breaking "driving term," i.e., with $f_\pi = 0$, the condition that eliminates tadpoles, Eq. (3.23) reads

$$\sigma_0(\lambda\sigma_0^2 - \mu^2) = 0. \qquad (3.59)$$

The nontrivial solution gives $\sigma_0 \neq 0$ and, by (3.24), yields massless pions.*

IV. VECTOR MESONS AND FIELD ALGEBRA

The inclusion of vector mesons in a Lagrangian enormously increases the number of possible couplings and hence the number of arbitrary constants in the theory. A very powerful restriction is provided by the requirement that the current be proportional to the vector-meson field (Kroll, Lee, and Zumino, 1967). This is the most direct way of implementing the experimentally well-supported idea of vector-meson dominance of the current matrix elements† (Sakurai, 1960; Gell-Mann and Zachariasen, 1961). To see how this restriction works, we note that ordinarily an i-spin-1 vector meson would transform according to

$$\delta \boldsymbol{\varrho}_\mu(x) = -\boldsymbol{\alpha} \times \boldsymbol{\varrho}_\mu(x). \qquad (4.1)$$

Consequently

$$\delta(\partial_\mu \boldsymbol{\varrho}_\nu - \partial_\nu \boldsymbol{\varrho}_\mu) = -\boldsymbol{\alpha} \times (\partial_\mu \boldsymbol{\varrho}_\nu - \partial_\nu \boldsymbol{\varrho}_\mu) - (\partial_\mu \boldsymbol{\alpha} \times \boldsymbol{\varrho}_\nu - \partial_\nu \boldsymbol{\alpha} \times \boldsymbol{\varrho}_\mu), \qquad (4.2)$$

so that for the conventional kinetic-energy term of the Lagrangian, $-\frac{1}{4}(\partial_\mu \boldsymbol{\varrho}_\nu - \partial_\nu \boldsymbol{\varrho}_\mu)^2$, we have

$$\delta \mathcal{L}_{\text{kin}} = \partial_\mu \boldsymbol{\alpha} \times \boldsymbol{\varrho}_\nu \cdot (\partial^\mu \boldsymbol{\varrho}^\nu - \partial^\nu \boldsymbol{\varrho}^\mu). \qquad (4.3)$$

This leads to

$$\mathbf{j}_\mu = -\boldsymbol{\varrho}^\nu \times (\partial_\mu \boldsymbol{\varrho}_\nu - \partial_\nu \boldsymbol{\varrho}_\mu) \qquad (4.4)$$

which is not what we want.

To get the field–current proportionality (hereafter called *field algebra*) we recall the work of Yang and Mills (1954) [see also Utiyama (1956); Bludman (1955)], who explored the question of constructing theories in which the notion of space–time-dependent gauges is extended to i spin. [See also Glashow and Gell-Mann (1961) for a generalization to other non-abelian gauge theories.] They showed that if the transformation law for the vector field is written as*

$$\delta \boldsymbol{\varrho}_\mu(x) = -\boldsymbol{\alpha} \times \boldsymbol{\varrho}_\mu(x) + (1/\gamma_0)\partial_\mu \boldsymbol{\alpha}, \qquad (4.5)$$

then the field $\mathbf{f}_{\mu\nu}(x)$, defined by

$$\mathbf{f}_{\mu\nu}(x) = \partial_\mu \boldsymbol{\varrho}_\nu(x) - \partial_\nu \boldsymbol{\varrho}_\mu(x) + \gamma_0 \boldsymbol{\varrho}_\mu(x) \times \boldsymbol{\varrho}_\nu(x), \qquad (4.6)$$

can easily be seen to transform under (4.5) according to

$$\delta \mathbf{f}_{\mu\nu}(x) = -\boldsymbol{\alpha} \times \mathbf{f}_{\mu\nu}(x). \qquad (4.7)$$

Consequently with

$$\mathcal{L}_0 = -\tfrac{1}{4}\mathbf{f}_{\mu\nu} \cdot \mathbf{f}^{\mu\nu}, \qquad (4.8)$$

we have

$$\delta \mathcal{L}_0 = \tfrac{1}{2}\mathbf{f}_{\mu\nu} \cdot \boldsymbol{\alpha} \times \mathbf{f}^{\mu\nu} = 0. \qquad (4.9)$$

If to this totally invariant term we add

$$\mathcal{L}_1 = \tfrac{1}{2}m_0^2 \boldsymbol{\varrho}_\mu \cdot \boldsymbol{\varrho}^\mu, \qquad (4.10)$$

we see that

$$\delta \mathcal{L}_1 = m_0^2 \boldsymbol{\varrho}_\mu \cdot [-\boldsymbol{\alpha} \times \boldsymbol{\varrho}^\mu + (1/\gamma_0)\partial^\mu \boldsymbol{\alpha}] = (m_0^2/\gamma_0)\, \boldsymbol{\varrho}_\mu \cdot \partial^\mu \boldsymbol{\alpha} \qquad (4.11)$$

from which it follows that (Lee and Zumino, 1967)

$$\mathbf{j}_\mu = -(m_0^2/\gamma_0)\, \boldsymbol{\varrho}_\mu. \qquad (4.12)$$

We now study the *canonical commutation relations* and their implication for the commutation relations among the currents. We start with

$$\mathcal{L} = -\tfrac{1}{4}\mathbf{f}_{\mu\nu} \cdot \mathbf{f}^{\mu\nu} + \tfrac{1}{2}m_0^2 \boldsymbol{\varrho}_\mu \cdot \boldsymbol{\varrho}^\mu + \mathcal{L}_2(\boldsymbol{\varrho}). \qquad (4.13)$$

The momentum conjugate to $\boldsymbol{\varrho}_\nu$ is given by

$$\boldsymbol{\pi}^\nu(x) = \partial\mathcal{L}/\partial(\partial_0 \boldsymbol{\varrho}_\nu(x)) = -\mathbf{f}^{0\nu}(x), \qquad (4.14)$$

so that

$$\boldsymbol{\pi}^0(x) = 0,$$
$$\boldsymbol{\pi}^i(x) = -\mathbf{f}^{0i}(x) = \mathbf{f}^{i0}(x). \qquad (4.15)$$

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* In a recent paper, B. Lee (1969) has discussed the renormalization of the linear σ model and shown that this feature persists in higher orders.

† For a recent review of the experimental situation, see Joos (1967). The paper of Kroll, Lee, and Zumino (1967) contains a detailed bibliography on vector meson dominance.

* The Jacobi identity for i-spin rotations acting on a field can be written as $[\delta_\alpha, \delta_\beta]\chi = -\delta_{\alpha\times\beta}\chi$ and for a vector field, because of the distributive properties of the derivative, (4.5) as well as (4.1) satisfy the condition.

The canonical commutation relations are

$$[\pi_a{}^i(x),\, \rho_{bj}(y)]_{x_0=y_0} = -i\delta_{ab}\delta_j{}^i\delta(\mathbf{x}-\mathbf{y}). \quad (4.16)$$

In order to calculate the equal time commutators among the currents we must express ρ_{a0} in terms of the $\pi_a{}^i$. The equation of motion is

$$-\partial_\mu \mathbf{f}^{\mu\nu} = m_0{}^2 \boldsymbol{\varrho}^\nu - \gamma_0 \mathbf{f}^{\mu\nu} \times \boldsymbol{\varrho}_\mu + \partial\mathcal{L}_2/\partial\boldsymbol{\varrho}_\nu. \quad (4.17)$$

If we set $\nu=0$ we get

$$-\partial_i \mathbf{f}^{i0} = m_0{}^2\boldsymbol{\varrho}^0 - \gamma_0 \mathbf{f}^{i0} \times \boldsymbol{\varrho}_i + (\partial\mathcal{L}_2/\partial\boldsymbol{\varrho}_0);$$

i.e.,

$$\boldsymbol{\varrho}^0 = -\frac{1}{m_0{}^2}\partial_i \boldsymbol{\pi}^i - \frac{\gamma_0}{m_0{}^2}\boldsymbol{\varrho}_i \times \boldsymbol{\pi}^i - \frac{1}{m_0{}^2}\frac{\partial\mathcal{L}}{\partial\boldsymbol{\varrho}_0}. \quad (4.18)$$

In the last term we shall only consider those forms of $\mathcal{L}_2$ in which the only dependence on $\boldsymbol{\varrho}_\mu$ is through terms like

$$\tfrac{1}{2}(\partial_\mu \boldsymbol{\chi} + \gamma_0 \boldsymbol{\varrho}_\mu \times \boldsymbol{\chi})^2.$$

Thus,

$$\partial\mathcal{L}_2/\partial\boldsymbol{\varrho}_0 = \gamma_0\boldsymbol{\chi} \times (\partial_0\boldsymbol{\chi} + \gamma_0\boldsymbol{\varrho}_0 \times \boldsymbol{\chi}) = \gamma_0\boldsymbol{\chi} \times [\partial\mathcal{L}_2/\partial(\partial_0\boldsymbol{\chi})];$$

i.e., the last term in (4.18) only involves the fields $\boldsymbol{\chi}$ and their conjugate momenta, which commute with ρ_i and π^i. We may thus ignore the last term in calculating the commutation relation,

$$\begin{aligned}[\rho_a{}^0(x),\, \rho_b{}^j(y)]_{x_0=y_0} &= [-(1/m_0{}^2)\partial^i\pi_{ia}(x) \\ &\quad -(\gamma_0/m_0{}^2)\, e_{amn}\rho_m{}^i(x)\pi_{in}(x),\, \rho_b{}^j(y)] \\ &= -(1/m_0{}^2)\partial^i(-i\delta_i{}^j\delta_{ab}\delta(\mathbf{x}-\mathbf{y})) \\ &\quad -(\gamma_0/m_0{}^2)\, e_{amn}\rho_m{}^i(x)(-i\delta_i{}^j\delta_{nb}\delta(\mathbf{x}-\mathbf{y})) \\ &= (i/m_0{}^2)\,\delta_{ab}\partial^j\delta(\mathbf{x}-\mathbf{y}) \\ &\quad +(i\gamma_0/m_0{}^2)\, e_{amb}\rho_m{}^j(x)\,\delta(\mathbf{x}-\mathbf{y}). \quad (4.19)\end{aligned}$$

Multiplication by $(m_0{}^2/\gamma_0)^2$ yields [Lee, Weinberg, and Zumino (1967)]

$$\begin{aligned}[j_a{}^0(x),\, j_b{}^k(y)]_{x_0=y_0} &= ie_{abc}\, j_c{}^k(x)\,\delta(\mathbf{x}-\mathbf{y}) \\ &\quad +(im_0{}^2/\gamma_0{}^2)\,\delta_{ab}\partial^k\delta(\mathbf{x}-\mathbf{y}). \quad (4.20)\end{aligned}$$

The second term is a c-number "Schwinger term" (Schwinger, 1959; Goto and Imamura, 1955). The existence of such terms is required on very general grounds, but their form is usually not known or not very well defined. It is one of the attractions of field algebra that the Schwinger terms are well defined. Schwinger terms have been calculated for a renormalizable theory by Johnson and Low (1966), Hamprecht (1967), Polkinghorne (1967), and given a general discussion by Gross and Jackiw (1967).

It can be easily shown that

$$[j_a{}^0(x),\, j_b{}^0(y)]_{x_0=y_0} = ie_{abc}\delta(\mathbf{x}-\mathbf{y})\, j_c{}^0(x). \quad (4.21)$$

It is of course evident that

$$[j_a{}^k(x),\, j_b{}^l(y)]_{x_0=y_0} = 0. \quad (4.22)$$

If we wish to couple the vector mesons to other fields (e.g., the pion field) and at the same time preserve field algebra, we cannot include terms like $\frac{1}{2}(\partial_\mu\boldsymbol{\phi})^2$ in the Lagrangian, since

$$\delta(\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi})^2) = \partial^\mu\boldsymbol{\phi}\cdot(-\partial_\mu\boldsymbol{\alpha}\times\boldsymbol{\phi}) \quad (4.23)$$

leads to additional current terms. If, however, we consider

$$D_\mu\boldsymbol{\phi} = \partial_\mu\boldsymbol{\phi} + \gamma_0\boldsymbol{\varrho}_\mu \times \boldsymbol{\phi}, \quad (4.24)$$

we see that

$$\begin{aligned}\delta(D_\mu\boldsymbol{\phi}) &= -\partial_\mu\boldsymbol{\alpha}\times\boldsymbol{\phi} - \boldsymbol{\alpha}\times\partial_\mu\boldsymbol{\phi} - \gamma_0(\boldsymbol{\varrho}_\mu \times (\boldsymbol{\alpha}\times\boldsymbol{\phi})) \\ &\quad + \gamma_0\left(\frac{1}{\gamma_0}\partial_\mu\boldsymbol{\alpha} - \boldsymbol{\alpha}\times\boldsymbol{\varrho}_\mu\right)\times\boldsymbol{\phi} \\ &= -\boldsymbol{\alpha}\times D_\mu\boldsymbol{\phi}, \quad (4.25)\end{aligned}$$

so that the coupling via the "covariant derivative," as in the case of $\mathcal{L}_2$ discussed above, preserves the field–current proportionality. A useful identity is

$$\begin{aligned}[D_\mu, D_\nu]\boldsymbol{\chi} &= D_\mu(\partial_\nu\boldsymbol{\chi} + \gamma_0\boldsymbol{\varrho}_\nu\times\boldsymbol{\chi}) - (\mu\leftrightarrow\nu) \\ &= \partial_\mu(\partial_\nu\boldsymbol{\chi} + \gamma_0\boldsymbol{\varrho}_\nu\times\boldsymbol{\chi}) \\ &\quad + \gamma_0\boldsymbol{\varrho}_\mu\times(\partial_\nu\boldsymbol{\chi} + \gamma_0\boldsymbol{\varrho}_\nu\times\boldsymbol{\chi}) - (\mu\leftrightarrow\nu) \\ &= \gamma_0\mathbf{f}_{\mu\nu}\times\boldsymbol{\chi}, \quad (4.26)\end{aligned}$$

valid for all isovector fields $\boldsymbol{\chi}$.

V. VECTOR MESONS AND CHIRAL SYMMETRY

If chiral symmetry is to be implemented, the triplet of i-spin-1 mesons can be supplemented by a triplet of axial mesons, for which we will also require field–current proportionality.* We define the fields

$$\mathbf{V}_\mu{}^{(\pm)} = \boldsymbol{\varrho}_\mu \pm \mathbf{a}_\mu, \quad (5.1)$$

where $\mathbf{a}_\mu$ are axial-meson fields. With the help of

$$\mathcal{L}_0{}^{(\pm)} = -\tfrac{1}{4}(\mathbf{f}_{\mu\nu}{}^{(\pm)})^2 + \tfrac{1}{2}m_0{}^2(\mathbf{V}_\mu{}^{(\pm)})^2, \quad (5.2)$$

we construct

$$\mathcal{L}_0 = \tfrac{1}{2}(\mathcal{L}_0{}^{(+)} + \mathcal{L}_0{}^{(-)}), \quad (5.3)$$

a combination which is invariant under reflections. This Lagrangian, when expressed in terms of $\boldsymbol{\varrho}_\mu$ and $\mathbf{a}_\mu$, yields

$$\begin{aligned}\mathcal{L}_0 &= -\tfrac{1}{4}(\partial_\mu\boldsymbol{\varrho}_\nu - \partial_\nu\boldsymbol{\varrho}_\mu + \gamma_0\boldsymbol{\varrho}_\mu\times\boldsymbol{\varrho}_\nu + \gamma_0\mathbf{a}_\mu\times\mathbf{a}_\nu)^2 \\ &\quad -\tfrac{1}{4}(\partial_\mu\mathbf{a}_\nu - \partial_\nu\mathbf{a}_\mu + \gamma_0\boldsymbol{\varrho}_\mu\times\mathbf{a}_\nu - \gamma_0\boldsymbol{\varrho}_\nu\times\mathbf{a}_\mu)^2 \\ &\quad + \tfrac{1}{2}m_0{}^2(\boldsymbol{\varrho}_\mu{}^2 + \mathbf{a}_\mu{}^2). \quad (5.4)\end{aligned}$$

We introduce the notation

$$\begin{aligned}\mathbf{F}_{\mu\nu} &\equiv \partial_\mu\boldsymbol{\varrho}_\nu - \partial_\nu\boldsymbol{\varrho}_\mu + \gamma_0\boldsymbol{\varrho}_\mu\times\boldsymbol{\varrho}_\nu + \gamma_0\mathbf{a}_\mu\times\mathbf{a}_\nu, \\ \mathbf{G}_{\mu\nu} &\equiv \partial_\mu\mathbf{a}_\nu + \gamma_0\boldsymbol{\varrho}_\mu\times\mathbf{a}_\nu - \partial_\nu\mathbf{a}_\mu - \gamma_0\boldsymbol{\varrho}_\nu\times\mathbf{a}_\mu, \quad (5.5)\end{aligned}$$

so that

$$\mathcal{L}_0 = -\tfrac{1}{4}(\mathbf{F}_{\mu\nu})^2 - \tfrac{1}{4}(\mathbf{G}_{\mu\nu})^2 + \tfrac{1}{2}m_0{}^2(\boldsymbol{\varrho}_\mu{}^2 + \mathbf{a}_\mu{}^2). \quad (5.6)$$

---

* This corresponds to assigning the spin-1 mesons to the reducible representation (1, 0)⊕(0, 1) of SU(2)×SU(2).

The gauge transformations now are

$$\delta\boldsymbol{\rho}_\mu = -\boldsymbol{\alpha}\times\boldsymbol{\rho}_\mu + (1/\gamma_0)\partial_\mu\boldsymbol{\alpha},$$
$$\delta\mathbf{a}_\mu = -\boldsymbol{\alpha}\times\mathbf{a}_\mu,$$
$$\delta'\boldsymbol{\rho}_\mu = -\boldsymbol{\beta}\times\mathbf{a}_\mu,$$
$$\delta'\mathbf{a}_\mu = -\boldsymbol{\beta}\times\boldsymbol{\rho}_\mu + (1/\gamma_0)\partial_\mu\boldsymbol{\beta}, \quad (5.7)$$

and it is easy to show that

$$\delta'\mathbf{F}_{\mu\nu} = -\boldsymbol{\beta}\times\mathbf{G}_{\mu\nu},$$
$$\delta'\mathbf{G}_{\mu\nu} = -\boldsymbol{\beta}\times\mathbf{F}_{\mu\nu}. \quad (5.8)$$

Thus the first two terms in $\mathcal{L}_0$ are invariant, while the mass term yields

$$\delta\mathcal{L}_0 = (m_0^2/\gamma_0)\,\boldsymbol{\rho}_\mu\cdot\partial^\mu\boldsymbol{\alpha},$$
$$\delta'\mathcal{L}_0 = (m_0^2/\gamma_0)\,\mathbf{a}_\mu\cdot\partial^\mu\boldsymbol{\beta}, \quad (5.9)$$

showing that field algebra is satisfied by both the vector and the axial currents. Both currents are evidently conserved and the proportionality constant relating current to field is the same in both cases. [This is the field-algebraic statement of "Weinberg's Second Sum Rule," (Weinberg, 1967).] The current commutation relations can be obtained from the canonical commutation relations. For the vector mesons

$$\pi_V{}^k = \partial\mathcal{L}/\partial(\partial_0\rho_k) = -\mathbf{F}^{0k} = \mathbf{F}^{k0}, \quad (5.10)$$

and for the axial mesons

$$\pi_A{}^k = \partial\mathcal{L}/\partial(\partial_0\mathbf{a}_k) = -\mathbf{G}^{0k} = \mathbf{G}^{k0}. \quad (5.11)$$

The equations of motion for the axial field are

$$-\partial_\mu\mathbf{G}^{\mu\nu} = \gamma_0\mathbf{F}^{\mu\nu}\times\mathbf{a}_\mu + \gamma_0\mathbf{G}^{\mu\nu}\times\boldsymbol{\rho}_\mu + m_0^2\mathbf{a}^\nu, \quad (5.12)$$

from which we obtain

$$\mathbf{a}^0 = -\frac{1}{m_0^2}\partial_k\boldsymbol{\pi}_A{}^k - \frac{\gamma_0}{m_0^2}\boldsymbol{\pi}_V{}^k\times\mathbf{a}_k - \frac{\gamma_0}{m_0^2}\boldsymbol{\pi}_A{}^k\times\boldsymbol{\rho}_k. \quad (5.13)$$

The commutation relations can easily be calculated with the help of this expression. We limit ourselves to the remark that the coefficient of $\partial_k\boldsymbol{\pi}_A{}^k$ is the same as in Eq. (4.18) so that the Schwinger terms for the axial-current commutation relations are the same as for the vector-current commutators (Weinberg, 1967). This does not hold for all models. For the σ model, for example, the coefficients of $-i\partial_k\delta(\mathbf{x}-\mathbf{y})$ in $[j_a{}^0(x), j_b{}^k(y)]$ and $[j_{5a}{}^0(x), j_{5b}{}^k(y)]$ are $\phi_a\phi_b-\delta_{ab}\boldsymbol{\phi}^2$ anp $\phi_a\phi_b-\delta_{ab}\sigma^2$, respectively.

Let us now turn to the coupling of the vector and axial mesons to pions (and the σ as an independent or dependent field). With

$$\delta\sigma = 0;\qquad \delta\boldsymbol{\phi} = -\boldsymbol{\alpha}\times\boldsymbol{\phi} \quad (5.14)$$

and

$$\delta'\sigma = -\boldsymbol{\beta}\cdot\boldsymbol{\phi};\qquad \delta'\boldsymbol{\phi} = \boldsymbol{\beta}\sigma, \quad (5.15)$$

we get

$$\delta(\partial_\mu\sigma) = 0;\qquad \delta(D_\mu\boldsymbol{\phi}) = -\boldsymbol{\alpha}\times D_\mu\boldsymbol{\phi}. \quad (5.16)$$

The complication comes in the axial transformation. We see, for example, that

$$\delta'(\partial_\mu\sigma) = -\partial_\mu\boldsymbol{\beta}\cdot\boldsymbol{\phi} - \boldsymbol{\beta}\cdot\partial_\mu\boldsymbol{\phi}. \quad (5.17)$$

To compensate for the $\partial_\mu\boldsymbol{\beta}$ term we need a scalar term involving $\mathbf{a}_\mu$ linearly. The possibility of $\mathbf{a}_\mu\cdot\boldsymbol{\phi}$ suggests itself. We find that

$$\delta'(\mathbf{a}_\mu\cdot\boldsymbol{\phi}) = (1/\gamma_0)\partial_\mu\boldsymbol{\beta}\cdot\boldsymbol{\phi} - \boldsymbol{\beta}\times\boldsymbol{\rho}_\mu\cdot\boldsymbol{\phi} + \mathbf{a}_\mu\cdot\boldsymbol{\beta}\sigma. \quad (5.18)$$

Hence

$$\delta'(\partial_\mu\sigma + \gamma_0\mathbf{a}_\mu\cdot\boldsymbol{\phi}) = -\boldsymbol{\beta}\cdot(D_\mu\boldsymbol{\phi} - \gamma_0\sigma\mathbf{a}_\mu). \quad (5.19)$$

Similarly, a short calculation shows that

$$\delta'(D_\mu\boldsymbol{\phi} - \gamma_0\sigma\mathbf{a}_\mu) = \boldsymbol{\beta}(\partial_\mu\sigma + \gamma_0\mathbf{a}_\mu\cdot\boldsymbol{\phi}). \quad (5.20)$$

Thus, with (Lee and Nieh, 1968)

$$\Delta_\mu\sigma \equiv \partial_\mu\sigma + \gamma_0\mathbf{a}_\mu\cdot\boldsymbol{\phi},$$
$$\Delta_\mu\boldsymbol{\phi} \equiv \partial_\mu\boldsymbol{\phi} + \gamma_0\boldsymbol{\rho}_\mu\times\boldsymbol{\phi} - \gamma_0\sigma\mathbf{a}_\mu, \quad (5.21)$$

we find that

$$\delta'\Delta_\mu\sigma = -\boldsymbol{\beta}\cdot\Delta_\mu\boldsymbol{\phi},$$
$$\delta'\Delta_\mu\boldsymbol{\phi} = \boldsymbol{\beta}\Delta_\mu\sigma, \quad (5.22)$$

and thus

$$\mathcal{L}_1 = \tfrac{1}{2}(\Delta_\mu\sigma)^2 + \tfrac{1}{2}(\Delta_\mu\boldsymbol{\phi})^2 - \tfrac{1}{2}\mu^2(\boldsymbol{\phi}^2+\sigma^2) + \cdots \quad (5.23)$$

is chiral invariant.

The combination of $\mathcal{L}_0$ from (5.6) and the above term give the simplest chiral-invariant Lagrangian. We can, however, construct other invariant terms which involve more derivatives. For example, using (5.8) and (5.22) we can check that

$$\delta'(\mathbf{F}_{\mu\nu}\cdot\Delta^\mu\boldsymbol{\phi}\times\Delta^\nu\boldsymbol{\phi}) = 2(\boldsymbol{\beta}\cdot\Delta^\nu\boldsymbol{\phi})(\mathbf{G}_{\mu\nu}\cdot\Delta^\mu\boldsymbol{\phi}) - 2\boldsymbol{\beta}\cdot\mathbf{F}_{\mu\nu}\times\Delta^\nu\boldsymbol{\phi}\Delta^\mu\sigma, \quad (5.24)$$

where use has been made of the antisymmetry of $\mathbf{F}_{\mu\nu}$ and $\mathbf{G}_{\mu\nu}$. Similarly,

$$\delta'(\mathbf{G}_{\mu\nu}\cdot\Delta^\mu\boldsymbol{\phi}\Delta^\nu\sigma) = -\beta\cdot\mathbf{F}_{\mu\nu}\times\Delta^\mu\boldsymbol{\phi}\Delta^\nu\sigma - (\mathbf{G}_{\mu\nu}\cdot\Delta^\mu\boldsymbol{\phi})(\beta\cdot\Delta^\nu\boldsymbol{\phi}). \quad (5.25)$$

Hence

$$\delta'(\tfrac{1}{2}\mathbf{F}_{\mu\nu}\cdot\Delta^\mu\boldsymbol{\phi}\times\Delta^\nu\boldsymbol{\phi} + \mathbf{G}_{\mu\nu}\cdot\Delta^\mu\boldsymbol{\phi}\Delta^\nu\sigma) = 0. \quad (5.26)$$

We shall actually carry this term along because it satisfies all the requirements we have imposed. [This is the counterpart of the κ term in Wess and Zumino (1967) in Sec. V, for example.]

In order to break chiral symmetry in a way which leads to the PCAC condition we again add

$$\mathcal{L}_{SB} = m_\pi^2 f\sigma. \quad (5.27)$$

The constant $\tilde{f}$ will be determined later. It is clear that

$$\partial_\mu \mathbf{j}_5{}^\mu = m_\pi{}^2 \tilde{f} \boldsymbol{\phi} \tag{5.28}$$

but, as we shall see in the next section, the pion field will have to be renormalized. It is there that we shall work out the consequences of the form of

$$\begin{aligned}\mathfrak{L} = &-\tfrac{1}{4}(\partial_\mu \boldsymbol{\rho}_\nu - \partial_\nu \boldsymbol{\rho}_\mu + \gamma_0 \boldsymbol{\rho}_\mu \times \boldsymbol{\rho}_\nu + \gamma_0 \mathbf{a}_\mu \times \mathbf{a}_\nu)^2 \\ &-\tfrac{1}{4}(\partial_\mu \mathbf{a}_\nu - \partial_\nu \mathbf{a}_\mu + \gamma_0 \boldsymbol{\rho}_\mu \times \mathbf{a}_\nu - \gamma_0 \boldsymbol{\rho}_\nu \times \mathbf{a}_\mu)^2 \\ &+\tfrac{1}{2} m_0{}^2 (\boldsymbol{\rho}_\mu{}^2 + \mathbf{a}_\mu{}^2) + \tfrac{1}{2}(\partial_\mu \sigma + \gamma_0 \mathbf{a}_\mu \cdot \boldsymbol{\phi})^2 \\ &+\tfrac{1}{2}(\partial_\mu \boldsymbol{\phi} + \gamma_0 \boldsymbol{\rho}_\mu \times \boldsymbol{\phi} - \gamma_0 \sigma \mathbf{a}_\mu)^2 \\ &+\tfrac{1}{2}\kappa(\partial_\mu \boldsymbol{\rho}_\nu - \partial_\nu \boldsymbol{\rho}_\mu + \gamma_0 \boldsymbol{\rho}_\mu \times \boldsymbol{\rho}_\nu + \gamma_0 \mathbf{a}_\mu \times \mathbf{a}_\nu) \cdot \Delta^\mu \boldsymbol{\phi} \times \Delta^\nu \boldsymbol{\phi} \\ &+\kappa(\partial_\mu \mathbf{a}_\nu - \partial_\nu \mathbf{a}_\mu + \gamma_0 \boldsymbol{\rho}_\mu \times \mathbf{a}_\nu - \gamma_0 \boldsymbol{\rho}_\nu \times \mathbf{a}_\mu) \cdot \Delta^\mu \boldsymbol{\phi} \Delta^\nu \sigma \\ &+ m_\pi{}^2 \tilde{f} \sigma, \end{aligned} \tag{5.29}$$

in which we shall also take

$$\sigma \equiv \sigma_0 + \sigma' = \sigma_0 - (1/2\sigma_0) \boldsymbol{\phi}^2 + \cdots \tag{5.30}$$

for convenience. Including real scalar mesons will change very little.

VI. COUPLING CONSTANTS IN $SU(2)\times SU(2)$

The predictions of the Lagrangian (5.29) are easily obtained provided that note be taken of the following feature:

When $\sigma = \sigma_0 + \sigma'$ is inserted into $\Delta_\mu \boldsymbol{\phi}$, then

$$\Delta_\mu \boldsymbol{\phi} = D_\mu \boldsymbol{\phi} - \gamma_0 \sigma_0 \mathbf{a}_\mu - \gamma_0 \sigma' \mathbf{a}_\mu. \tag{6.1}$$

The square of this contains a cross term of the form $\mathbf{a}_\mu \cdot D^\mu \boldsymbol{\phi}$, and this leads to mixing between the axial meson and the pion as soon as the symmetry is broken. To get rid of this cross term we introduce a new axial field $\mathbf{A}_\mu(x)$ by means of

$$\mathbf{a}_\mu = \mathbf{A}_\mu + \xi D_\mu \boldsymbol{\phi}. \tag{6.2}$$

The Lagrangian in (5.29) now takes the form

$$\begin{aligned}\mathfrak{L} = &-\tfrac{1}{4}(\mathbf{f}_{\mu\nu} + \gamma_0 \mathbf{A}_\mu \times \mathbf{A}_\nu + \gamma_0 \xi \mathbf{A}_\mu \times D_\nu \boldsymbol{\phi} + \gamma_0 \xi D_\mu \boldsymbol{\phi} \times \mathbf{A}_\nu \\ &+\gamma_0 \xi^2 D_\mu \boldsymbol{\phi} \times D_\nu \boldsymbol{\phi})^2 - \tfrac{1}{4}(D_\mu \mathbf{A}_\nu - D_\nu \mathbf{A}_\mu + \xi \gamma_0 \mathbf{f}_{\mu\nu} \times \boldsymbol{\phi})^2 \\ &+\tfrac{1}{2} m_0{}^2 \boldsymbol{\rho}_\mu{}^2 + \tfrac{1}{2} m_0{}^2 (\mathbf{A}_\mu + \xi D_\mu \boldsymbol{\phi})^2 \\ &+\tfrac{1}{2}(\partial_\mu \sigma' + \gamma_0 \boldsymbol{\phi} \cdot \mathbf{A}_\mu + \gamma_0 \xi \boldsymbol{\phi} \cdot D_\mu \boldsymbol{\phi})^2 \\ &+\tfrac{1}{2}[(1-\gamma_0 \xi \sigma_0) D_\mu \boldsymbol{\phi} - \gamma_0 \xi \sigma' D_\mu \boldsymbol{\phi} - \gamma_0 \sigma_0 \mathbf{A}_\mu - \gamma_0 \sigma' \mathbf{A}_\mu]^2 \\ &+\tfrac{1}{2}\kappa(\mathbf{f}_{\mu\nu} + \gamma_0 \mathbf{A}_\mu \times \mathbf{A}_\nu + \gamma_0 \xi \mathbf{A}_\mu \times D_\nu \boldsymbol{\phi} \\ &+\gamma_0 \xi D_\mu \boldsymbol{\phi} \times \mathbf{A}_\nu + \gamma_0 \xi^2 D_\mu \boldsymbol{\phi} \times D_\nu \boldsymbol{\phi}) \\ &\cdot[(1-\gamma_0 \xi \sigma_0) D_\mu \boldsymbol{\phi} - \gamma_0 \sigma_0 \mathbf{A}^\mu - \gamma_0 \xi \sigma' D^\mu \boldsymbol{\phi} - \gamma_0 \sigma' \mathbf{A}^\mu] \\ &\times[(1-\gamma_0 \xi \sigma_0) D^\nu \boldsymbol{\phi} - \gamma_0 \sigma_0 \mathbf{A}^\nu - \gamma_0 \xi \sigma' D^\nu \boldsymbol{\phi} - \gamma_0 \sigma' \mathbf{A}^\nu] \\ &+\kappa(D_\mu \mathbf{A}_\nu - D_\nu \mathbf{A}_\mu + \gamma_0 \xi \mathbf{f}_{\mu\nu} \times \boldsymbol{\phi}) \\ &\cdot[(1-\gamma_0 \xi \sigma_0) D^\mu \boldsymbol{\phi} - \gamma_0 \sigma_0 \mathbf{A}^\mu - \gamma_0 \xi \sigma' D^\mu \boldsymbol{\phi} - \gamma_0 \sigma' \mathbf{A}^\mu] \\ &\times(\partial^\nu \sigma' + \gamma_0 \boldsymbol{\phi} \cdot \mathbf{A}^\nu + \gamma_0 \xi \boldsymbol{\phi} \cdot D^\nu \boldsymbol{\phi}) + \tilde{f} m_\pi{}^2 \sigma'. \end{aligned} \tag{6.3}$$

The cross term $\mathbf{A}_\mu \cdot D^\mu \boldsymbol{\phi}$ can now be eliminated by a proper choice of ξ. It appears in the axial-vector mass term and in $(\Delta_\mu \boldsymbol{\phi})^2$. The condition is that

$$m_0{}^2 \xi - \gamma_0 \sigma_0 (1 - \gamma_0 \xi \sigma_0) = 0;$$

i.e.,

$$\xi = \gamma_0 \sigma_0 / [m_0{}^2 + (\gamma_0 \sigma_0)^2]. \tag{6.4}$$

The coefficient of $\frac{1}{2}(D_\mu \boldsymbol{\phi})^2$ is now

$$(1 - \gamma_0 \xi \sigma_0)^2 + m_0{}^2 \xi^2 = m_0{}^2 / [m_0{}^2 + (\gamma_0 \sigma_0)^2]. \tag{6.5}$$

If we now define the *renormalized* meson field $\boldsymbol{\phi}_r$ by

$$\boldsymbol{\phi}_r = Z^{-1/2} \boldsymbol{\phi}, \tag{6.6}$$

then the coefficient of $\frac{1}{2}(D_\mu \boldsymbol{\phi}_r)^2$ will be unity, provided that

$$Z = [m_0{}^2 + (\gamma_0 \sigma_0)^2] / m_0{}^2. \tag{6.7}$$

This renormalization of the field brings the kinetic-energy term into standard form, which is necessary to allow us to interpret the Fourier components of $\boldsymbol{\phi}_r(x)$ as creation or annihilation operators for properly normalized one-meson states.

The PCAC condition now reads

$$\partial^\mu \mathbf{j}_{5\mu} = m_\pi{}^2 \tilde{f} Z^{1/2} \boldsymbol{\phi}_r \equiv m_\pi{}^2 f_\pi \boldsymbol{\phi}_r, \tag{6.8}$$

so that

$$\tilde{f} = Z^{-1/2} f_\pi. \tag{6.9}$$

The pion mass term appears only in σ'. The coefficient of $-\frac{1}{2}\boldsymbol{\phi}_r{}^2$ is to be $m_\pi{}^2$. Thus

$$m_\pi{}^2 = m_\pi{}^2 f_\pi Z^{1/2} / \sigma_0;$$

i.e.,

$$\sigma_0 = f_\pi Z^{1/2}. \tag{6.10}$$

Next we consider the coefficient of $\frac{1}{2}\mathbf{A}_\mu{}^2$ which we call $m_A{}^2$. We see that

$$m_A{}^2 = m_0{}^2 + (\gamma_0 \sigma_0)^2. \tag{6.11}$$

The coefficient of $\frac{1}{2}\boldsymbol{\rho}_\mu{}^2$ is the square of the ρ mass, so that $m_0{}^2 = m_\rho{}^2$ and hence

$$Z = (m_A / m_\rho)^2. \tag{6.12}$$

Note that (6.7) and (6.10) combine to give

$$\gamma_0{}^2 f_\pi{}^2 / m_\rho{}^2 = 1 - m_\rho{}^2 / m_A{}^2. \tag{6.13}$$

If we set $m_A = \sqrt{2} m_\rho$ in Eq. (6.13), we obtain

$$2\gamma_0{}^2 f_\pi{}^2 / m_\rho{}^2 = 1, \tag{6.14}$$

which is known as the KSFR relation (Kawarabayashi and Suzuki, 1966; Riazzuddin and Fayazuddin, 1966). This relation does not follow from our effective Lagrangian which incorporates ρ dominance, PCAC, and the current commutation relations. It does seem to hold, *approximately*, using experimentally determined values for f_π and $m_\rho{}^2$, the relation predicts $\gamma_0{}^2/4\pi \simeq 2.6$, a reasonable, if slightly large, value. Several authors have obtained the KSFR relation by using an additional assumption. Sakurai (1966) assumes that the pion–nucleon scattering length (a_1-a_3) is given by ρ exchange. Wess and Zumino (1967) make the equivalent assumption for their effective Lagrangian involving nucleons. The same result can be obtained by explaining the $\pi\pi$ scattering lengths by ρ exchange (Ademollo, 1966). In all cases the coupling constant γ_0 appears rather than $\gamma_{\rho\pi\pi}$ on the mass shell. Brown and Goble (1968) on the other hand obtain the KSFR relation by $\gamma_{\rho\pi\pi}$ replacing γ_0. They use the current algebra determination of the P-wave scattering length and use an effective range approximation to extrapolate the scattering amplitude to the ρ resonance. Since $\gamma_{\rho\pi\pi}\neq\gamma_0$ in effective-Lagrangian models, Brown and Goble's assumption about the extrapolation of the P-wave scattering amplitude would have to be modified when $\gamma_{\rho\pi\pi}\neq\gamma_0$, because $\gamma_{\rho\pi\pi}$ has a dependence on the mass of the ρ [see Eq. (6.17) below].

We now proceed to calculate the $\rho\pi\pi$ and $A\rho\pi$ couplings. The relevant terms in (6.3) are

$$-\tfrac{1}{4}(\partial_\mu \boldsymbol{\varrho}_\nu-\partial_\nu\boldsymbol{\varrho}_\mu+2\gamma_0\xi Z^{1/2}\mathbf{A}_\mu\times D_\nu\boldsymbol{\phi}_r+\gamma_0\xi^2 Z D_\mu\boldsymbol{\phi}_r\times D_\nu\boldsymbol{\phi}_r)^2$$
$$-\tfrac{1}{4}(\partial_\mu\mathbf{A}_\nu-\partial_\nu\mathbf{A}_\mu+2\gamma_0\xi Z^{1/2}\partial_\mu\boldsymbol{\varrho}_\nu\times\boldsymbol{\phi}_r)^2$$
$$+\tfrac{1}{2}(\partial_\mu\boldsymbol{\phi}_r+\gamma_0\boldsymbol{\varrho}_\mu\times\boldsymbol{\phi}_r)^2$$
$$+\tfrac{1}{2}\kappa(\partial_\mu\boldsymbol{\varrho}_\nu-\partial_\nu\boldsymbol{\varrho}_\mu)\cdot[Z(1-\gamma_0\xi\sigma_0)^2D^\mu\boldsymbol{\phi}_r\times D^\nu\boldsymbol{\phi}_r$$
$$-Z^{1/2}\gamma_0\sigma_0(1-\gamma_0\xi\sigma_0)(D^\mu\boldsymbol{\phi}_r\times\mathbf{A}^\nu+\mathbf{A}^\mu\times D^\nu\boldsymbol{\phi}_r)].$$

Upon integrating by parts we find that the term linear in $\boldsymbol{\varrho}_\mu$ and bilinear in $\boldsymbol{\phi}_r$ is

$$\gamma_0\boldsymbol{\varrho}_\mu\cdot\boldsymbol{\phi}_r\times\partial^\mu\boldsymbol{\phi}_r+[\tfrac{1}{2}\gamma_0\xi^2Z-\tfrac{1}{2}\kappa Z(1-\gamma_0\xi\sigma_0)^2]$$
$$\times\Box\boldsymbol{\varrho}_\mu\cdot\boldsymbol{\phi}_r\times\partial^\mu\boldsymbol{\phi}_r.$$

In the tree-graph approximation for the process $\rho\rightarrow 2\pi$ we can replace $-\Box\boldsymbol{\varrho}_\mu$ by $p^2\boldsymbol{\varrho}_\mu$, where p is the four momentum of the ρ. We get*†

$$\gamma_{\rho\pi\pi}=\gamma_0\left[1-\tfrac{1}{2}p^2\left(\frac{m_A^2-m_\rho^2}{m_A^2m_\rho^2}-\frac{\kappa m_\rho^2}{\gamma_0}\frac{1}{m_A^2}\right)\right]. \quad (6.15)$$

On the mass shell

$$\gamma_{\rho\pi\pi}=\gamma_0\left(1-\frac{m_A^2-m_\rho^2}{2m_A^2}+\frac{1}{2}\frac{\kappa m_\rho^2}{\gamma_0}\frac{m_\rho^2}{m_A^2}\right), \quad (6.16)$$

whereas for $p^2=0$ we get

$$\gamma_{\rho\pi\pi}=\gamma_0, \quad (6.17)$$

a result that is required in order to obtain the correct normalization of the matrix element of the i-spin current between two pion states with zero momentum transfer.

The terms linear in $\mathbf{A}_\mu$, $\boldsymbol{\varrho}_\mu$, and $\boldsymbol{\phi}_r$ can similarly be obtained. We leave out some of the algebra and just write the resulting coupling:

$$-\gamma_0\xi Z^{1/2}\Box\boldsymbol{\varrho}_\mu\cdot\mathbf{A}^\mu\times\boldsymbol{\phi}_r,$$
$$-\kappa Z^{1/2}\gamma_0\sigma_0(1-\gamma_0\sigma_0\xi)[\Box\boldsymbol{\varrho}_\mu\cdot\mathbf{A}^\mu\times\boldsymbol{\phi}_r$$
$$+(\partial_\mu\boldsymbol{\varrho}_\nu-\partial_\nu\boldsymbol{\varrho}_\mu)\cdot\partial^\mu\mathbf{A}_\nu\times\boldsymbol{\phi}_r]. \quad (6.18)$$

On the mass shell the effective-Lagrangian term is

$$\mathcal{L}_{A\rho\pi}=\gamma_0 m_\rho\left(1-\frac{m_\rho^2}{m_A^2}\right)^{1/2}\left(1+\frac{\kappa m_\rho^2}{\gamma_0}\right)\boldsymbol{\varrho}_\mu\cdot\mathbf{A}^\mu\times\boldsymbol{\phi}_r$$
$$-\frac{\gamma_0}{m_\rho}\left(1-\frac{m_\rho^2}{m_A^2}\right)^{1/2}\cdot\frac{\kappa m_\rho^2}{\gamma_0}(\partial_\mu\boldsymbol{\varrho}_\nu-\partial_\nu\boldsymbol{\varrho}_\mu)\cdot\partial^\mu\mathbf{A}^\nu\times\boldsymbol{\phi}_r. \quad (6.19)$$

The first term is an S-wave coupling of the form

$$\gamma_0 m_\rho\left(1-\frac{m_\rho^2}{m_A^2}\right)^{1/2}\left(1+\frac{\kappa m_\rho^2}{\gamma_0}\right)\epsilon_\rho\cdot\epsilon_A, \quad (6.20)$$

and the second term includes both S- and D-wave couplings of the form

$$-\frac{\gamma_0}{m_\rho}\left(1-\frac{m_\rho^2}{m_A^2}\right)^{1/2}\frac{\kappa m_\rho^2}{\gamma_0}(p\cdot Q\epsilon_A\cdot\epsilon_\rho-p\cdot\epsilon_A Q\cdot\epsilon_\rho), \quad (6.21)$$

where Q is the four momentum of the A. The calculation of the decay rates is discussed in Appendix C.

In conclusion we may ask whether chiral invariance, except for the PCAC symmetry-breaking term, can be maintained without the existence of an axial field. We proceed as in our discussion of the σ field and try

$$\mathbf{a}_\mu=f_1\boldsymbol{\varrho}_\mu\times\boldsymbol{\phi}+f_2\partial_\mu\boldsymbol{\phi}, \quad (6.22)$$

where f_1 and f_2 are assumed to be functions of $\boldsymbol{\phi}^2$ alone. Familiar manipulations show that with

$$f_1=1/\sigma, \qquad f_2=1/\gamma_0\sigma, \quad (6.23)$$

the Jacobi identity is satisfied, so that

$$\mathbf{a}_\mu=(1/\gamma_0\sigma)(\partial_\mu\boldsymbol{\phi}+\gamma_0\boldsymbol{\varrho}_\mu\times\boldsymbol{\phi})=(1/\gamma_0\sigma)D_\mu\boldsymbol{\phi}. \quad (6.24)$$

It is also possible to see this by noting that

$$\delta'[(1/\gamma_0\sigma)D_\mu\boldsymbol{\phi}]=(1/\gamma_0)\partial_\mu\boldsymbol{\beta}-\boldsymbol{\beta}\times\boldsymbol{\varrho}_\mu, \quad (6.25)$$

and this is just the transformation law for the axial field. With (6.24) and using $\sigma\partial_\mu\sigma+\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi}=0$, we see that in this case

$$\Delta_\mu\boldsymbol{\phi}=0; \qquad \Delta_\mu\sigma=0. \quad (6.26)$$

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* The implications of this form for the shape of electromagnetically produced ρ's have been analyzed by Schwinger (1968) and by Geffen and Walsh (1968). Since these analyses have been made, based on the Novosibirsk $e^++e^-\rightarrow\pi^++\pi^-$ colliding-beam data, ORSAY has reported their measurements of this reaction at the Vienna high-energy conference. They obtained cross sections which yield somewhat different values for γ_ρ and $\gamma_{\rho\pi\pi}$ Than were obtained by Schwinger or Geffen and Walsh. Consequently, the experimental situation is as yet unresolved.

† This is in agreement with the effective Lagrangian results of Wess and Zumino (1967) and the current-algebra calculations of Schnitzer and Weinberg (1967), Brown and West (1968), Arnowitt, Friedman, and Nath (1967), and others.

Thus the original Lagrangian (6.3) becomes

$$\mathcal{L}=-\frac{1}{4}\left(\mathbf{f}_{\mu\nu}+\frac{1}{\gamma_0\sigma}D_\mu\boldsymbol{\phi}\times D_\nu\boldsymbol{\phi}\right)^2$$

$$-\frac{1}{4}\left(\frac{1}{\sigma}\mathbf{f}_{\mu\nu}\times\boldsymbol{\phi}+\partial_\mu\frac{1}{\gamma_0\sigma}D_\nu\boldsymbol{\phi}-\partial_\nu\frac{1}{\gamma_0\sigma}D_\mu\boldsymbol{\phi}\right)^2$$

$$+\frac{1}{2}m_\rho^2\left[\boldsymbol{\varrho}_\mu^2+\frac{1}{\gamma_0^2\sigma^2}(D_\mu\boldsymbol{\phi})^2\right]+m_\pi^2\tilde{f}\sigma. \quad (6.27)$$

Note that there is no κ term. The following results emerge after some simple calculations:

(i) The coefficient of $\frac{1}{2}(\partial_\mu\boldsymbol{\phi}_r)^2$ can be made unity if the renormalized meson field is defined by

$$\boldsymbol{\phi}_r=|m_\rho/\gamma_0\sigma_0|\,\boldsymbol{\phi}. \quad (6.28)$$

(ii) The PCAC condition allows us to make the identification

$$\tilde{f}=f_\pi(\gamma_0\sigma_0/m_\rho). \quad (6.29)$$

(iii) The requirement that the coefficient of $-\frac{1}{2}\boldsymbol{\phi}_r^2$ be m_π^2 implies that

$$f_\pi\sigma_0=(m_\rho/\gamma_0)^2. \quad (6.30)$$

The only quantity of interest, the $\rho\pi\pi$ coupling constant, is obtained from

$$\gamma_0[\boldsymbol{\rho}_\mu\cdot\boldsymbol{\phi}_r\times\partial^\mu\boldsymbol{\phi}_r+(1/2m_\rho^2)\Box\boldsymbol{\varrho}_\mu\cdot\boldsymbol{\phi}_r\times\partial^\mu\boldsymbol{\phi}_r], \quad (6.31)$$

which on the mass shell leads to the relation $\gamma_{\rho\pi\pi}/\gamma_0=\frac{1}{2}$, which is in disagreement with experiment. We are thus led to the conclusion that within the framework of the effective Lagrangian approach, a *real* axial meson with $T=1$ is required.

VII. TRANSFORMATION PROPERTIES UNDER $SU(3)\times SU(3)$

A straightforward generalization (Glashow and Gell-Mann, 1961) of (5.5) suggests the following field quantities:

$$F_{\mu\nu}{}^i=\partial_\mu V_\nu{}^i-\partial_\nu V_\mu{}^i+\gamma_0 f_{ijk}(V_\mu{}^j V_\nu{}^k+\mathcal{A}_\mu{}^j\mathcal{A}_\nu{}^k) \quad (7.1)$$

and

$$G_{\mu\nu}{}^i=\partial_\mu\mathcal{A}_\nu{}^i-\partial_\nu\mathcal{A}_\mu{}^i+\gamma_0 f_{ijk}(V_\mu{}^j\mathcal{A}_\nu{}^k-V_\nu{}^j\mathcal{A}_\mu{}^k), \quad (7.2)$$

where the f_{ijk} are the structure constants of $SU(3)$ as defined by Gell-Mann (1962). These are listed in Appendix D together with some other relations of interest in $SU(3)$. With the variations

$$\delta V_\mu{}^i=(1/\gamma_0)\partial_\mu\alpha^i-f_{ijk}\alpha^j V_\mu{}^k,$$

$$\delta\mathcal{A}_\mu{}^i=-f_{ijk}\alpha^j\mathcal{A}_\mu{}^k, \quad (7.3)$$

and the use of the identity*

$$f_{kmn}f_{ijk}-f_{kmj}f_{ink}=f_{imk}f_{kjn}, \quad (7.4)$$

we can show that

$$\delta F_{\mu\nu}{}^i=-f_{ijk}\alpha^j F_{\mu\nu}{}^k,$$

$$\delta G_{\mu\nu}{}^i=-f_{ijk}\alpha^j G_{\mu\nu}{}^k. \quad (7.5)$$

Since $f_{0ij}=0$ it follows that the *ninth* vector and axial mesons are completely decoupled and only undergo the trivial gauge transformation. For the axial transformations we take

$$\delta' V_\mu{}^i=-f_{ijk}\beta^j\mathcal{A}^k \quad (7.6)$$

and

$$\delta'\mathcal{A}_\mu{}^i=(1/\gamma_0)\partial_\mu\beta^i-f_{ijk}\beta^j V_\mu{}^k. \quad (7.7)$$

This leads to

$$\delta' F_{\mu\nu}{}^i=-f_{ijk}\beta^j G_{\mu\nu}{}^k,$$

$$\delta' G_{\mu\nu}{}^i=-f_{ijk}\beta^j F_{\mu\nu}{}^k. \quad (7.8)$$

It follows that $F_{\mu\nu}{}^iF^{\mu\nu i}$ and $G_{\mu\nu}{}^iG^{\mu\nu i}$ are separately invariant under the vector transformations, while it is only the combinations

$$-\tfrac{1}{4}(F_{\mu\nu}{}^iF^{\mu\nu i}+G_{\mu\nu}{}^iG^{\mu\nu i}) \quad (7.9)$$

and

$$\tfrac{1}{2}m_0^2(V_\mu{}^iV^{\mu i}+\mathcal{A}_\mu{}^i\mathcal{A}^{\mu i}) \quad (7.10)$$

that are invariant under chiral transformations, with the latter only invariant under constant gauge transformations. The sum of the two terms yields a chiral-invariant Lagrangian *with the currents satisfying the field algebra.*

Before continuing with the very convenient 3×3 matrix formalism and the transformation of the spin-0 mesons, let us briefly discuss symmetry breaking for $SU(3)$ alone. We shall introduce *octet* breaking in the form

$$\mathcal{L}=-\tfrac{1}{4}f_{\mu\nu}{}^i(\delta_{ij}+\sqrt{3}\xi d_{8ij})f^{\mu\nu j}$$

$$+\tfrac{1}{2}m_0^2V_\mu{}^i(\delta_{ij}+\sqrt{3}\xi' d_{8ij})V^{\mu j}, \quad (7.11)$$

with ξ and ξ' giving the magnitude of the symmetry breaking in the two terms. Terms with f_{8ij} are excluded by symmetry. For the d_{ijk}, see Appendix D. For present purposes we shall write this Lagrangian by introducing an 8×8 (or 9×9) matrix D_8 defined by

$$(D_8)_{ij}=d_{8ij}. \quad (7.12)$$

Then

$$\mathcal{L}=-\tfrac{1}{4}f_{\mu\nu}{}^T(1+\sqrt{3}\xi D_8)f^{\mu\nu}+\tfrac{1}{2}m_0^2V_\mu{}^T(1+\sqrt{3}\xi' D_8)V^\mu \quad (7.13)$$

---

* This identity and others we shall use later follow from one of the following:

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0,$$
$$\{A,\{B,C\}\}-\{B,\{C,A\}\}+[C,[A,B]]=0,$$
$$\{C,[A,B]\}-\{A,[B,C]\}+[B,\{C,A\}]=0,$$

by replacing A, B, C by λ_i, λ_j, λ_k, using the commutation or anticommutation relations, and multiplying by λ_m and taking traces.

with $f_{\mu\nu}$ and V_μ forming an eight- (or nine-) component column vector and $f_{\mu\nu}^T$, V_μ^T forming an eight- (or nine-) component row vector. The matrix $\sqrt{3}D_8$ is given by

$$\sqrt{3}D_8=\begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & -1 \end{pmatrix} \tag{7.14}$$

for an octet theory (the boxes refer to rows and columns labeled 1, 2, 3; 4, 5, 6, 7; 8; respectively), or

$$\sqrt{3}D_8=\begin{pmatrix} 1 & & & \\ & -\frac{1}{2} & & \\ & & -1 & \sqrt{2} \\ & & \sqrt{2} & 0 \end{pmatrix} \tag{7.15}$$

for a nonet theory. (By a nonet theory we mean one in which the ninth vector meson is degenerate in mass with the octet in the absence of symmetry breaking.) As noted above, the ninth vector meson does not transform under $SU(3)$ and it is completely decoupled in the symmetric Lagrangian. For a nonet theory the (8, 0) submatrix can be diagonalized:

$$U^{-1}\begin{pmatrix} -1 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}U=\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.16}$$

Thus with the appropriate mixing* of the (8, 0) states the Lagrangian (7.13) now reads

$$\mathcal{L}=-\tfrac{1}{4}(1+\xi)\sum_{i=0}^{3}(f_{\mu\nu}^i)^2-\tfrac{1}{4}(1-\tfrac{1}{2}\xi)\sum_{i=4}^{7}(f_{\mu\nu}^i)^2 -\tfrac{1}{4}(1-2\xi)(f_{\mu\nu}^8)^2+\tfrac{1}{2}m_0^2(1+\xi')\sum_{i=0}^{3}(V_\mu^i)^2 +\tfrac{1}{2}m_0^2(1-\tfrac{1}{2}\xi')\sum_{i=4}^{7}(V_\mu^i)^2+\tfrac{1}{2}m_0^2(1-2\xi')(V_\mu^8)^2. \tag{7.17}$$

To be able to use the conventional commutation relations we must renormalize the above fields so that the kinetic energy terms have unit coefficients again. We thus write

$$\hat{V}_\mu^i=\begin{pmatrix} (1+\xi)^{1/2} \\ (1-\frac{1}{2}\xi)^{1/2} \\ (1-2\xi)^{1/2} \end{pmatrix}V_\mu^i \qquad \begin{matrix} i=0, 1, 2, 3 \\ i=4, 5, 6, 7. \\ i=8 \end{matrix} \tag{7.18}$$

The current now has the form

$$j_\mu^i=-\frac{m_0^2}{\gamma_0}\begin{pmatrix} 1+\xi' \\ 1-\frac{1}{2}\xi' \\ 1-2\xi' \end{pmatrix}V_\mu^i =-\frac{m_0^2}{\gamma_0}\begin{pmatrix} (1+\xi')(1+\xi)^{-1/2} \\ (1-\frac{1}{2}\xi')(1-\frac{1}{2}\xi)^{-1/2} \\ (1-2\xi')(1-2\xi)^{-1/2} \end{pmatrix}\hat{V}_\mu^i \tag{7.19}$$

and

$$\hat{\pi}_k^i=\hat{f}_{k0}^i. \tag{7.20}$$

This implies a change in the nonlinear term in $f_{\mu\nu}^i$. If $V_\mu^i=S^i\hat{V}_\mu^i$, then

$$f_{\mu\nu}^i=S_i(\partial_\mu\hat{V}_\nu^i-\partial_\nu\hat{V}_\mu^i+(S_jS_k/S_i)f_{ijk}V_\mu^jV_\nu^k)\equiv S_i f_{\mu\nu}^i.$$

Since we now have

$$\mathcal{L}=-\tfrac{1}{4}\hat{f}_{\mu\nu}^i\hat{f}^{\mu\nu i}+\tfrac{1}{2}m_0^2\frac{1+\xi'}{1+\xi}\sum_{i=0}^{3}(\hat{V}_\mu^i)^2+\cdots, \tag{7.21}$$

it follows that

$$-\partial^\mu\hat{f}_{\mu\nu}^i=m_0^2\begin{pmatrix} (1+\xi')(1+\xi)^{-1} \\ (1-\frac{1}{2}\xi')(1-\frac{1}{2}\xi)^{-1} \\ (1-2\xi')(1-2\xi)^{-1} \end{pmatrix}\hat{V}_\nu^i+\cdots \tag{7.22}$$

and therefore

$$\hat{V}_0^i=-\frac{1}{m_0^2}\begin{bmatrix} (1+\xi)(1+\xi')^{-1} \\ (1-\frac{1}{2}\xi)(1-\frac{1}{2}\xi')^{-1} \\ (1-2\xi)(1-2\xi')^{-1} \end{bmatrix}\nabla\cdot\hat{\pi}^i+\cdots. \tag{7.23}$$

Thus the commutator

$$[j_0^k, j_m^l]=\left(\frac{m_0^2}{\gamma_0}\right)^2\begin{bmatrix} (1+\xi')^2(1+\xi)^{-1} \\ (1-\frac{1}{2}\xi')^2(1-\frac{1}{2}\xi)^{-1} \\ (1-2\xi')^2(1-2\xi)^{-1} \end{bmatrix}[\hat{V}_0^k, \hat{V}_m^l] \tag{7.24}$$

has Schwinger terms of the form

$$\frac{im_0^2}{\gamma_0^2}\begin{bmatrix} 1+\xi' \\ 1-\frac{1}{2}\xi' \\ 1-2\xi' \end{bmatrix}\partial_m\delta(\mathbf{x}-\mathbf{y}). \tag{7.25}$$

Thus if there is symmetry breaking in the mass terms, the Schwinger terms are *not* invariant under $SU(3)$, and one cannot obtain a generalization of the Weinberg sum rules. It is an attractive hypothesis to assume that $\xi'=0$ (Oakes and Sakurai, 1967; Kimmel, 1968). One

* Mixing was first noted by Glashow (1963) and Sakurai (1963). The possibility of various kinds of mixing was first discussed in detail by Coleman and Schnitzer (1964). The most recent discussion is that of Kroll, Lee, and Zumino (1967) and Kimmel (1968).

consequence of this assumption is, as can be read off from (7.23), that (Coleman and Schnitzer, 1964)

$$m_\omega^2 = m_\rho^2 = m_0^2/(1+\xi);$$
$$m_{K^*}^2 = m_0^2/(1-\tfrac{1}{2}\xi);$$
$$m_\phi^2 = m_0^2/(1-2\xi), \qquad (7.26)$$

the Gell-Mann–Okubo mass formula for m^{-2} with the nonet mixing angle (Okubo, 1963) which is satisfied to within 5%. The $\omega-\rho$ mass degeneracy is independent of the choice of ξ' or ξ. It follows from our decision to take the symmetry breaking proportional to d_{8ij}.

If the ninth vector meson is split off by the addition of a term like

$$-\tfrac{1}{4}\alpha f_{\mu\nu}{}^0 f^{\mu\nu 0} + \tfrac{1}{2}m_0^2\beta V_\mu{}^0 V^{\mu 0}, \qquad (7.27)$$

then the (8, 0) submatrix in (7.11) takes the form

$$-\tfrac{1}{4}f_{\mu\nu}{}^T\begin{pmatrix}1-\xi & \sqrt{2}\xi\\ \sqrt{2}\xi & 1+\alpha\end{pmatrix}f^{\mu\nu} + \tfrac{1}{2}m_0^2 V_\mu{}^T\begin{pmatrix}1-\xi' & \sqrt{2}\xi'\\ \sqrt{2}\xi' & 1+\beta\end{pmatrix}V^\mu. \qquad (7.28)$$

The two matrices cannot be diagonalized simultaneously. The matrices that do diagonalize them are of the form

$$U = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \qquad (7.29)$$

with $\tan 2\theta = 2\sqrt{2}\xi/(\alpha+\xi)$ for the first matrix and $\tan 2\theta = 2\sqrt{2}\xi'/(\beta+\xi')$ for the second. The procedure for dealing with this situation is the following. First we transform the fields according to

$$V_\mu' = U^{-1}V_\mu \qquad (7.30)$$

with U chosen such that

$$U^T\begin{pmatrix}1-\xi & \sqrt{2}\xi\\ \sqrt{2}\xi & 1+\alpha\end{pmatrix}U = \begin{pmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{pmatrix}; \qquad (7.31)$$

$\lambda_\pm = 1+\tfrac{1}{2}(\alpha-\xi) \pm \tfrac{1}{2}[(\alpha+\xi)^2 + 8\xi^2]^{1/2}$ are the eigenvalues of the matrix appearing in the kinetic-energy term in (7.28). Next, the fields are renormalized by choosing

$$\hat{V}_\mu = \begin{pmatrix}\lambda_+^{1/2} & 0\\ 0 & \lambda_-^{1/2}\end{pmatrix}V_\mu' \equiv \mathbf{Z}^{1/2}V_\mu' \qquad (7.32)$$

so that the kinetic-energy terms have unit coefficient. The mass matrix appearing with the renormalized fields now becomes

$$m_0^2\mathbf{Z}^{-1/2}U^T\begin{pmatrix}1-\xi' & \sqrt{2}\xi'\\ \sqrt{2}\xi' & 1+\beta\end{pmatrix}U\mathbf{Z}^{-1/2}. \qquad (7.33)$$

The eigenvalues of this matrix yield the masses of the "mixed" vector mesons.

It turns out to be very convenient to work with a 3×3 matrix formulation of the equations. We define the matrix V_μ by

$$V_\mu = \frac{1}{\sqrt{2}}\sum_{i=0}^{8}\lambda_i V_\mu{}^i. \qquad (7.34)$$

The matrices λ_i and some of their properties are listed in Appendix D. Since

$$f_{ijk}\lambda_k = -(i/2)[\lambda_i, \lambda_j],$$
$$d_{ijk}\lambda_k = (1/2)\{\lambda_i, \lambda_j\}, \qquad (7.35)$$

we can write the field strengths in the form

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - (i\gamma_0/\sqrt{2})[V_\mu, V_\nu] - (i\gamma_0/\sqrt{2})[\mathcal{A}_\mu, \mathcal{A}_\nu] \qquad (7.36)$$

and

$$G_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - (i\gamma_0/\sqrt{2})[V_\mu, \mathcal{A}_\nu] + (i\gamma_0/\sqrt{2})[V_\nu, \mathcal{A}_\mu]. \qquad (7.37)$$

The transformation laws read

$$\delta V_\mu = (1/\gamma_0)\partial_\mu\alpha + (i/\sqrt{2})[\alpha, V_\mu],$$
$$\delta\mathcal{A}_\mu = (i/\sqrt{2})[\alpha, \mathcal{A}_\mu],$$
$$\delta' V_\mu = (i/\sqrt{2})[\beta, \mathcal{A}_\mu],$$
$$\delta'\mathcal{A}_\mu = (1/\gamma_0)\partial_\mu\beta + (i/\sqrt{2})[\beta, V_\mu], \qquad (7.38)$$

etc. The chiral-invariant Lagrangian (7.9), (7.10) is seen to be

$$-\tfrac{1}{4}\,\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu} + G_{\mu\nu}G^{\mu\nu}) + \tfrac{1}{2}m_0^2\,\mathrm{Tr}\,(V_\mu V^\mu + \mathcal{A}_\mu\mathcal{A}^\mu) \qquad (7.39)$$

when use is made of

$$\mathrm{Tr}\,\lambda_i\lambda_j = 2\delta_{ij}. \qquad (7.40)$$

To construct symmetry-breaking terms like (7.11) we observe that

$$\mathrm{Tr}\,(F_{\mu\nu}\lambda_8 F^{\mu\nu}) = \tfrac{1}{2}F_{\mu\nu}{}^i F^{\mu\nu j}\,\mathrm{Tr}\,(\lambda_i\lambda_8\lambda_j)$$
$$= d_{8ij}F_{\mu\nu}{}^i F^{\mu\nu j}. \qquad (7.41)$$

We shall construct symmetry-breaking terms somewhat differently in Sec. IX. We note that a term like (7.27) can also be written in terms of the $F_{\mu\nu}$. It is given by

$$-(1/12)\alpha(\mathrm{Tr}\,F_{\mu\nu})^2 + (1/6)\beta m_0^2(\mathrm{Tr}\,V_\mu)^2. \qquad (7.42)$$

Let us now turn to the spin-0 fields. We deal with a set of nine scalar fields σ_i and nine pseudoscalar fields ϕ_i. [We are assuming that the mesons belong to the representation $(\mathbf{3}, \bar{\mathbf{3}}) + (\bar{\mathbf{3}}, \mathbf{3})$ of $SU(3)\times SU(3)$. This is discussed in Appendix B.] For these fields we take the transformation laws to be

$$\delta\phi_i = -f_{ijk}\alpha_j\phi_k,$$
$$\delta\sigma_i = -f_{ijk}\alpha_j\sigma_k, \qquad (7.43)$$

and

$$\delta'\phi_i = d_{ijk}\beta_j\sigma_k,$$
$$\delta'\sigma_i = -d_{ijk}\beta_j\phi_k. \qquad (7.44)$$

In 3×3 matrix notation these have the form

$$\delta\phi = (i/\sqrt{2})[\alpha, \phi],$$
$$\delta\sigma = (i/\sqrt{2})[\alpha, \sigma], \qquad (7.45)$$

and

$$\delta'\phi = (1/\sqrt{2})\{\beta, \sigma\},$$
$$\delta'\sigma = -(1/\sqrt{2})\{\beta, \phi\}. \qquad (7.46)$$

As a generalization of $\Delta_\mu\phi$ and $\Delta_\mu\sigma$ in (5.21), we write

$$\Delta_\mu\phi = \partial_\mu\phi - (i\gamma_0/\sqrt{2})[V_\mu, \phi] - (\gamma_0/\sqrt{2})\{\mathcal{A}_\mu, \sigma\},$$
$$\Delta_\mu\sigma = \partial_\mu\sigma - (i\gamma_0/\sqrt{2})[V_\mu, \sigma] + (\gamma_0/\sqrt{2})\{\mathcal{A}_\mu, \phi\}. \qquad (7.47)$$

Thus, for example,

$$\delta'(\Delta_\mu\phi) = (1/\sqrt{2})\{\partial_\mu\beta, \sigma\} + (1/\sqrt{2})\{\beta, \partial_\mu\sigma\} + \tfrac{1}{2}\gamma_0[[\beta, \mathcal{A}_\mu], \phi] - \tfrac{1}{2}(i\gamma_0)[V_\mu, \{\beta, \sigma\}] + \tfrac{1}{2}\gamma_0\{\mathcal{A}_\mu, \{\beta, \phi\}\} - (\gamma_0/\sqrt{2})\{\sigma, (1/\gamma_0)\partial_\mu\beta + (i/\sqrt{2})[\beta, V_\mu]\}.$$

Simple manipulations of identities lead to

$$\delta'(\Delta_\mu\phi) = (1/\sqrt{2})\{\beta, \partial_\mu\sigma\} - \tfrac{1}{2}(i\gamma_0)\{\beta, [V_\mu, \sigma]\} + \tfrac{1}{2}\gamma_0\{\beta, \{\phi, \mathcal{A}_\mu\}\}$$
$$= (1/\sqrt{2})\{\beta, \Delta_\mu\sigma\}. \qquad (7.48)$$

Similarly,

$$\delta'(\Delta_\mu\sigma) = -(1/\sqrt{2})\{\beta, \Delta_\mu\phi\}. \qquad (7.49)$$

Consequently,

$$\delta' \operatorname{Tr}(\Delta_\mu\phi\Delta^\mu\phi + \Delta_\mu\sigma\Delta^\mu\sigma) = 0. \qquad (7.50)$$

Thus

$$\tfrac{1}{2}\operatorname{Tr}[(\Delta_\mu\sigma)^2 + (\Delta_\mu\phi)^2] \qquad (7.51)$$

may serve as a chiral-invariant kinetic-energy term for the spinless mesons.

To construct other invariants it is useful to consider the quantities

$$B = \sigma + i\phi,$$
$$B^\dagger = \sigma - i\phi. \qquad (7.52)$$

It follows from (7.46) that

$$\delta' B = (i/\sqrt{2})\{\beta, B\} \qquad (7.53)$$

and

$$\delta' B^\dagger = -(i/\sqrt{2})\{\beta, B^\dagger\}. \qquad (7.54)$$

Hence

$$\delta'(BB^\dagger) = (i/\sqrt{2})\{\beta, B\}B^\dagger - (i/\sqrt{2})B\{\beta, B^\dagger\}$$
$$= (i/\sqrt{2})[\beta, BB^\dagger], \qquad (7.55)$$

and similarly

$$\delta'(B^\dagger B) = -(i/\sqrt{2})[\beta, B^\dagger B]. \qquad (7.56)$$

We can thus immediately see that

$$\delta' \operatorname{Tr} BB^\dagger = 0. \qquad (7.57)$$

Thus

$$\operatorname{Tr} BB^\dagger = \operatorname{Tr}(\sigma^2 + \phi^2 + i[\phi, \sigma]) = \operatorname{Tr}(\sigma^2 + \phi^2) \qquad (7.58)$$

is a chiral invariant. Similarly, since

$$\delta'(BB^\dagger)^2 = (i/\sqrt{2})BB^\dagger[\beta, BB^\dagger] + (i/\sqrt{2})[\beta, BB^\dagger]BB^\dagger$$
$$= (i/\sqrt{2})[\beta, (BB^\dagger)^2],$$

we see that

$$\operatorname{Tr}(BB^\dagger)^2 = \operatorname{Tr}(\sigma^2 + \phi^2 - i[\sigma, \phi])^2 \qquad (7.59)$$

is a chiral invariant, and so is the even parity part of it, which has the form

$$\operatorname{Tr}(\sigma^4 + \phi^4 + 4\sigma^2\phi^2 - 2\sigma\phi\sigma\phi). \qquad (7.60)$$

Terms like $\operatorname{Tr}(BB^\dagger)^n$ for $n \geq 3$ can be expressed in terms of the lower invariants.* Finally, we note that for transformations that do not involve the ninth meson

$$\mathcal{L}_D = \det(\sigma + i\phi) + \det(\sigma - i\phi) \qquad (7.61)$$

is also a chiral invariant (Levy, 1967).†

Symmetry-breaking terms which lead to PCAC are of the form

$$\mathcal{L}_{SB} = f_0\sigma_0 + f_8\sigma_8 = (1/\sqrt{2})f_0 \operatorname{Tr}(\lambda_0\sigma) + (1/\sqrt{2})f_8 \operatorname{Tr}(\lambda_8\sigma) = \operatorname{Tr}(f\sigma). \qquad (7.62)$$

Thus

$$\delta'\mathcal{L}_{SB} = -\operatorname{Tr}(f\{(1/\sqrt{2})\beta, \phi\}) = -(1/\sqrt{2}) \operatorname{Tr}(\beta\{f, \phi\}) \qquad (7.63)$$

and

$$\delta\mathcal{L}_{SB} = (i/\sqrt{2}) \operatorname{Tr}(\alpha[\phi, f]), \qquad (7.64)$$

so that‡

$$\partial^\mu j_\mu{}^k = f_8 f_{8kn}\sigma^n \qquad (7.65)$$

and

$$\partial^\mu j_{5\mu}{}^k = (f_0 d_{0kn} + f_8 d_{8kn})\phi^n. \qquad (7.66)$$

---

* To establish this one may use Burgoyne's identity (Coleman, 1965) which states that for traceless matrices

$$\operatorname{Tr}(ABCD + ABDC + ACBD + ACDB + ADBC + ADCB) = \operatorname{Tr}(AD)\operatorname{Tr}(BC) + \operatorname{Tr}(AB)\operatorname{Tr}(CD) + \operatorname{Tr}(AC)\operatorname{Tr}(BD).$$

† See Appendix B.

‡ The association of a nonconserved vector current with a scalar field was first noted by Bernstein and Weinberg (1960).

We shall see that there are indications that both f_0 and f_8 do not vanish, i.e., there is chiral symmetry breaking as well as octet $SU(3)$ breaking. All the other f's must, of course, vanish if i spin and hypercharge are to remain good symmetries. We postpone mention of nonlinear realizations to the next section.

VIII. SPIN-ZERO MESONS IN $SU(3)\times SU(3)$

We begin by considering a chiral-invariant Lagrangian which consists of the kinetic-energy terms

$$\mathcal{L}_{KE}=\tfrac{1}{2}\operatorname{Tr}(\partial_\mu\sigma)^2+\tfrac{1}{2}\operatorname{Tr}(\partial_\mu\phi)^2, \tag{8.1}$$

a mass term

$$\mathcal{L}_M=-\tfrac{1}{2}\mu^2\operatorname{Tr}(\phi^2+\sigma^2), \tag{8.2}$$

and a coupling term of the form

$$\mathcal{L}_C=\mathcal{L}_C(X,Y,Z), \tag{8.3}$$

where

$$\begin{aligned}X&=\operatorname{Tr}(\phi^2+\sigma^2),\\ Y&=\operatorname{Tr}(\sigma^4+\phi^4+4\sigma^2\phi^2-2\sigma\phi\sigma\phi),\\ Z&=\det(\sigma+i\phi)+\det(\sigma-i\phi).\end{aligned} \tag{8.4}$$

We should note the following points:

(a) The presence of Z implies that the 0 component of the axial current is not conserved and that PCAC is not satisfied for that current. We shall see in a moment that this violation is required if the only symmetry-breaking terms are of the form given in (7.62).

(b) This is still not the most general Lagrangian involving two derivatives. It is clear from the transformation properties of the fields that the following combinations are also invariant:

$$\begin{aligned}\tfrac{1}{2}\operatorname{Tr}&(\partial_\mu BB^\dagger\partial^\mu BB^\dagger+\partial_\mu B^\dagger B\partial^\mu B^\dagger B)\\ &=\operatorname{Tr}(\partial_\mu\sigma\sigma\partial^\mu\sigma\sigma+\partial_\mu\phi\phi\partial^\mu\phi\phi+2\partial_\mu\sigma\sigma\partial^\mu\phi\phi\\ &\quad+2\partial_\mu\sigma\phi\partial^\mu\phi\sigma-\sigma\partial_\mu\phi\sigma\partial^\mu\phi-\partial_\mu\sigma\phi\partial^\mu\sigma\phi)\end{aligned} \tag{8.5}$$

and

$$\begin{aligned}\tfrac{1}{2}\operatorname{Tr}&(\partial_\mu B\partial^\mu B^\dagger BB^\dagger+\partial_\mu B^\dagger\partial^\mu BB^\dagger B)\\ &=\operatorname{Tr}\{(\sigma^2+\phi^2)(\partial_\mu\sigma\partial^\mu\sigma+\partial_\mu\phi\partial^\mu\phi)\\ &\quad+[\partial_\mu\sigma,\partial^\mu\phi][\phi,\sigma]\}.\end{aligned} \tag{8.6}$$

If the symmetry breaking enters through

$$\mathcal{L}_{SB}=\operatorname{Tr}(f,\sigma), \tag{8.7}$$

then we have

$$\begin{aligned}\partial^\mu j_{5\mu}{}^{\pi}&=[(\tfrac{2}{3})^{1/2}f_0+(\tfrac{1}{3})^{1/2}f_8]\phi_\pi\equiv m_\pi^2 f_\pi\phi_\pi,\\ \partial^\mu j_{5\mu}{}^{K}&=[(\tfrac{2}{3})^{1/2}f_0-\tfrac{1}{2}(\tfrac{1}{3})^{1/2}f_8]\phi_K\equiv m_K^2 f_K\phi_K.\end{aligned} \tag{8.8}$$

The experimental facts that $f_\pi\approx f_K$ and $m_\pi^2\ll m_K^2$ thus imply that

$$f_8\approx-\sqrt{2}f_0. \tag{8.9}$$

Thus $SU(3)$ symmetry breaking (caused by f_8) is of the same order of magnitude as chiral symmetry breaking (caused by f_8 and f_0). As was pointed out by Gell-Mann, Oakes, and Renner (1968), this suggests that the chain of symmetry violations is more likely to be

$$SU(3)\times SU(3)\to SU(2)\times SU(2)\to SU(2),$$

rather than

$$SU(3)\times SU(3)\to SU(3)\to SU(2).$$

Let us continue by eliminating the tadpole graphs by writing

$$\sigma=\Sigma+\sigma' \tag{8.9}$$

with

$$\Sigma\equiv\langle\sigma\rangle_0=\begin{pmatrix}a&0&0\\0&a&0\\0&0&b\end{pmatrix} \tag{8.10}$$

and

$$\begin{aligned}\Sigma^2&=(a+b)\Sigma-abI,\\ \Sigma^3&=(a^2+ab+b^2)\Sigma-ab(a+b)I.\end{aligned} \tag{8.11}$$

We find that

$$X=\operatorname{Tr}\phi^2+\operatorname{Tr}\sigma'^2+2\operatorname{Tr}\sigma'\Sigma+(2a^2+b^2) \tag{8.12}$$

and

$$\begin{aligned}Y&=2a^4+b^4+4(a^2+ab+b^2)\operatorname{Tr}\Sigma\sigma'-4ab(a+b)\operatorname{Tr}\sigma'\\ &\quad+4(a+b)\operatorname{Tr}\Sigma\sigma'^2-4ab\operatorname{Tr}\sigma'^2+4(a+b)\operatorname{Tr}\Sigma\phi^2\\ &\quad-4ab\operatorname{Tr}\phi^2+2\operatorname{Tr}\sigma'\Sigma\sigma'\Sigma-2\operatorname{Tr}\phi\Sigma\phi\Sigma\\ &\quad+4\operatorname{Tr}(\Sigma\{\phi^2,\sigma'\})-4\operatorname{Tr}\Sigma\phi\sigma'\phi+4\operatorname{Tr}\Sigma\sigma'^3\\ &\quad+\operatorname{Tr}\sigma'^4+\operatorname{Tr}\phi^4+4\operatorname{Tr}\sigma'^2\phi^2-2\operatorname{Tr}\sigma'\phi\sigma'\phi.\end{aligned} \tag{8.13}$$

Also, using (B.19) we find that, aside from a numerical factor,

$$\begin{aligned}Z&=6a^2b+6a(a+b)\operatorname{Tr}\sigma'-6a\operatorname{Tr}\Sigma\sigma'\\ &\quad+3(2a+b)(\operatorname{Tr}\sigma')^2-6\operatorname{Tr}\sigma'\operatorname{Tr}\Sigma\sigma'-3(2a+b)\operatorname{Tr}\sigma'^2\\ &\quad+6\operatorname{Tr}\Sigma\sigma'^2-3(2a+b)(\operatorname{Tr}\phi)^2+6\operatorname{Tr}\phi\operatorname{Tr}\Sigma\phi\\ &\quad+3(2a+b)\operatorname{Tr}\phi^2-6\operatorname{Tr}\Sigma\phi^2+(\operatorname{Tr}\sigma')^3-3\operatorname{Tr}\sigma'\operatorname{Tr}\sigma'^2\\ &\quad-3\operatorname{Tr}\sigma'(\operatorname{Tr}\phi)^2-6\operatorname{Tr}\sigma'\phi^2+3\operatorname{Tr}\sigma'\operatorname{Tr}\phi^2\\ &\quad+6\operatorname{Tr}\phi\operatorname{Tr}\sigma'\phi+2\operatorname{Tr}\sigma'^3.\end{aligned} \tag{8.14}$$

With the notation

$$\begin{aligned}&\partial\mathcal{L}_C(2a^2+b^2,2a^4+b^4,6a^2b)/\partial X\equiv L_x,\\ &\partial\mathcal{L}_C(2a^2+b^2,2a^4+b^4,6a^2b)/\partial Y\equiv L_y,\\ &\partial\mathcal{L}_C(2a^2+b^2,2a^4+b^4,6a^2b)/\partial Z\equiv L_z,\\ &\partial^2\mathcal{L}_C(2a^2+b^2,2a^4+b^4,6a^2b)/\partial X\partial Y\equiv L_{xy},\ \text{etc.},\end{aligned} \tag{8.15}$$

we see that the condition that there be no terms linear in σ' reduces to

$$\operatorname{Tr} f\sigma'-\mu^2 \operatorname{Tr} \Sigma\sigma'+2L_x \operatorname{Tr} \sigma'\Sigma$$
$$+L_y[4(a^2+ab+b^2) \operatorname{Tr} \Sigma\sigma'-4ab(a+b) \operatorname{Tr} \sigma']$$
$$+L_z[6a(a+b) \operatorname{Tr} \sigma'-6a \operatorname{Tr} \Sigma\sigma']=0. \quad (8.16)$$

If we write

$$f=u\Sigma+vI, \quad (8.17)$$

we obtain the following equations for u and v:

$$u-\mu^2+2L_x+4(a^2+ab+b^2)L_y-6aL_z=0,$$
$$v-4ab(a+b)L_y+6a(a+b)L_z=0. \quad (8.18)$$

Note that

$$(\tfrac{2}{3})^{1/2}f_0+(\tfrac{1}{3})^{1/2}f_8=m_\pi^2 f_\pi=\sqrt{2}(au+v)$$
$$=\sqrt{2}a(\mu^2-2L_x-4a^2L_y-6bL_z),$$
$$(\tfrac{2}{3})^{1/2}f_0-\tfrac{1}{2}(\tfrac{1}{3})^{1/2}f_8=m_K^2 f_K=\sqrt{2}[\tfrac{1}{2}(a+b)u+v]$$
$$=\sqrt{2}\tfrac{1}{2}(a+b)[\mu^2-2L_x$$
$$-4(a^2-ab+b^2)L_y-6aL_z]. \quad (8.19)$$

We can also exhibit the mass terms. The terms quadratic in the pseudoscalar fields are given by

$$-\tfrac{1}{2}\mu^2 \operatorname{Tr} \phi^2+L_x \operatorname{Tr} \phi^2$$
$$+L_y[4(a+b) \operatorname{Tr} \Sigma\phi^2-4ab \operatorname{Tr} \phi^2-2 \operatorname{Tr} \Sigma\phi\Sigma\phi]$$
$$+L_z[6 \operatorname{Tr} \phi \operatorname{Tr} \Sigma\phi-6 \operatorname{Tr} \Sigma\phi^2+3(2a+b) \operatorname{Tr} \phi^2$$
$$-3(2a+b)(\operatorname{Tr} \phi)^2]. \quad (8.20)$$

The terms quadratic in the scalar fields are somewhat more complicated:

$$-\tfrac{1}{2}\mu^2 \operatorname{Tr} \sigma'^2+L_x \operatorname{Tr} \sigma'^2+L_y[4(a+b) \operatorname{Tr} \Sigma\sigma'^2$$
$$-4ab \operatorname{Tr} \sigma'^2+2 \operatorname{Tr} \Sigma\sigma'\Sigma\sigma']$$
$$+L_z[6 \operatorname{Tr} \Sigma\sigma'^2-6 \operatorname{Tr} \sigma' \operatorname{Tr} \Sigma\sigma'+3(2a+b) \operatorname{Tr} \sigma')^2$$
$$-3(2a+b) \operatorname{Tr} \sigma'^2]+2L_{xx}(\operatorname{Tr} \Sigma\sigma')^2$$
$$+8L_{yy}[(a^2+ab+b^2) \operatorname{Tr} \Sigma\sigma'-ab(a+b) \operatorname{Tr} \sigma']^2$$
$$+18L_{zz}[a(a+b) \operatorname{Tr} \sigma'-a \operatorname{Tr} \Sigma\sigma']^2$$
$$+8L_{xy} \operatorname{Tr} \Sigma\sigma'[(a^2+ab+b^2) \operatorname{Tr} \Sigma\sigma'$$
$$-ab(a+b) \operatorname{Tr} \sigma']+12L_{xz} \operatorname{Tr} \Sigma\sigma'$$
$$\times[a(a+b) \operatorname{Tr} \sigma'-a \operatorname{Tr} \Sigma\sigma']$$
$$+24L_{yz}[(a^2+ab+b^2) \operatorname{Tr} \Sigma\sigma'-ab(a+b) \operatorname{Tr} \sigma']$$
$$\times[a(a+b) \operatorname{Tr} \sigma'-a \operatorname{Tr} \Sigma\sigma']. \quad (8.21)$$

We now observe the following: If $\mathfrak{L}_C$ is independent of Z then the terms quadratic in the meson field appear in the form

$$\operatorname{Tr} \phi^2=\phi_{\pi^0}^2+2\phi_{\pi^+}\phi_{\pi^-}+2\phi_{K^+}\phi_{K^-}+2\phi_{K^0}\phi_{\bar{K}^0}+\phi_8^2+\phi_0^2,$$
$$\operatorname{Tr} \Sigma\phi^2=a(\phi_{\pi^0}^2+2\phi_{\pi^+}\phi_{\pi^-})$$
$$+\tfrac{1}{2}(a+b)(2\phi_{K^+}\phi_{K^-}+2\phi_{K^0}\phi_{\bar{K}^0})$$
$$+2a(3^{-1/2}\phi_0+6^{-1/2}\phi_8)^2+b[3^{-1/2}\phi_0-2(6^{-1/2})\phi_0]^2,$$
$$\operatorname{Tr} \Sigma\phi\Sigma\phi=a^2(\phi_{\pi^0}^2+2\phi_{\pi^+}\phi_{\pi^-})+ab(2\phi_{K^+}\phi_{K^-}+2\phi_{K^0}\phi_{\bar{K}^0})$$
$$+2a^2(3^{-1/2}\phi_0+6^{-1/2}\phi_8)^2+b^2[3^{-1/2}\phi_0-2(6^{-1/2})\phi_8]^2. \quad (8.22)$$

The mass terms may be diagonalized by the introduction of new fields,

$$\phi_\eta=(\tfrac{2}{3})^{1/2}\phi_0+(\tfrac{1}{3})^{1/2}\phi_8,$$
$$\phi_X=(\tfrac{1}{3})^{1/2}\phi_0-(\tfrac{2}{3})^{1/2}\phi_8. \quad (8.23)$$

We now observe that the combination

$$\phi_{\pi^0}^2+2\phi_{\pi^+}\phi_{\pi^-}+\phi_\eta^2 \quad (8.24)$$

appears in all terms, i.e., in the absence of a Z dependence the pions, and one of the isoscalar mesons are degenerate. This is far from true experimentally, and we thus see that we must, in effect, abandon PCAC for the ninth axial current.

If we use (8.22) to work out the pion and K masses from (8.20), we find that

$$m_\pi^2=\mu^2-2L_x-4a^2L_y-6bL_z,$$
$$m_K^2=\mu^2-2L_x-4(a^2-ab+b^2)L_y-6aL_z. \quad (8.25)$$

Comparison with (8.19) shows that

$$f_\pi=\sqrt{2}a,$$
$$f_K=(a+b)/\sqrt{2}. \quad (8.26)$$

Thus the ratio

$$f_K/f_\pi=(a+b)/2a \quad (8.27)$$

differs from unity if $b\neq a$. This condition is essential, however, to ensure that $m_K^2\neq m_\pi^2$.

We shall not bother to study this model further. Given the masses of the pseudoscalar mesons, π, K, η, η', and the masses of the scalar mesons $\sigma_\pi, \sigma_K, \sigma_\eta, \sigma_{\eta'}$ (if we were sure about them) and f_π, f_K we could only determine 10 of the 12 quantities $\mu^2, a, b, L_x, \cdots, L_{zz}$ that have appeared so far. Fixing some of them* and making models is not really the subject of this paper. Instead, we shall next turn to the subject of nonlinear realizations of $SU(3)\times SU(3)$.

* If we demand renormalizability we can set $L_{xx}=L_{xy}=\cdots=L_{zz}=0$. This was done by Levy (1967) and Gasiorowicz and Geffen (1968). Such a criterion has certainly not played any role in the development of effective Lagrangians.

IX. NONLINEAR REALIZATIONS OF $SU(3)\times SU(3)$

In our discussion of chiral symmetry for pions (Sec. III) we saw that we could arrive at a nonlinear realization of the symmetry in two ways that turned out to be equivalent:

(1) We found that the chiral invariant

$$\sigma^2(x)+\phi^2(x)$$

could be used to define $\sigma(x)$ as a nonlinear function of the pion field.

(2) We obtained the same result by letting $m_\sigma\rightarrow\infty$.

A more systematic approach to the problem has been developed by Coleman, Wess, and Zumino (1969) and Isham(1969),* and we shall briefly refer to it in Appendix E. At this stage, however, we shall try to follow the above procedures in discussing the case of $SU(3)\times SU(3)$.

Let us recall the results from Sec. VII that with

$$B=\sigma+i\phi,$$
$$B^\dagger=\sigma-i\phi, \tag{9.1}$$

we have

$$\delta' BB^\dagger=(i/\sqrt{2})[\beta, BB^\dagger] \tag{9.2}$$

and

$$\delta' B^\dagger B=-(i/\sqrt{2})[\beta, B^\dagger B]. \tag{9.3}$$

If we now *assume* that

$$\sigma=\sum_{n=0}^{\infty} C_n\phi^n, \tag{9.4}$$

i.e., that

$$[\sigma, \phi]=0, \tag{9.5}$$

then

$$B^\dagger B=BB^\dagger, \tag{9.6}$$

and it follows that

$$\delta' BB^\dagger=0. \tag{9.7}$$

This implies that we may write

$$BB^\dagger=\sigma^2+\phi^2=F^2 I, \tag{9.8}$$

the analog of (3.38). With this nonlinear relation we find that X and Y defined in (8.4) are both numerical constants, so that we have

$$\mathcal{L}=\tfrac{1}{2}\operatorname{Tr}(\partial_\mu\sigma)^2+\tfrac{1}{2}\operatorname{Tr}(\partial_\mu\phi)^2+\mathcal{L}_G(Z)+\operatorname{Tr}(f\sigma) \tag{9.9}$$

in which we make the replacement

$$\sigma=FI-(1/2F)\phi^2-(1/8F^3)\phi^4+\cdots. \tag{9.10}$$

We see from this expression that

$$\sigma_k=\sqrt{3}F\delta_{0k}-(1/2\sqrt{2}F)d_{klm}\phi_l\phi_m-\cdots. \tag{9.11}$$

In particular

$$\langle\sigma_0\rangle_0=\sqrt{3}F, \tag{9.12}$$

but

$$\langle\sigma_8\rangle_0=0. \tag{9.13}$$

The last result implies that in the nonlinear realization *in which no scalar particles exist*, $b=a$; i.e.,

$$f_K=f_\pi. \tag{9.14}$$

In this model

$$\Sigma=\begin{pmatrix} a & & \\ & a & \\ & & a\end{pmatrix}=F\cdot I \tag{9.15}$$

and

$$Z=\text{const}+6a^2\operatorname{Tr}\sigma+3a^2\operatorname{Tr}\phi^2-3a^2(\operatorname{Tr}\phi)^2+\cdots$$
$$=\text{const}-3a^2(\sqrt{3}X_0)^2+\cdots. \tag{9.16}$$

Thus the mass terms come from the term quadratic in the field in Z and from $\operatorname{Tr}(f\sigma)$, when (9.10) is inserted. There are three unknown coefficients in the theory: f_0, f_8, and the coefficient of the Z term ($\sqrt{2}F=f_K=f_\pi$ and is fixed). When these are evaluated in terms of the masses of π, K, and η^0, it turns out that the mass of the X^0 is predicted to lie at 1640 MeV, which is quite unsatisfactory.

It turns out that another nonlinear realization* of interest emerges if we look at what happens when we let some of the parameters go to infinity. If we look at Eqs. (8.20) and (8.21) and introduce the notation

$$\mu^2-2L_x\equiv\mu_1^2,$$
$$4a^2L_y\equiv\xi\mu_1^2,$$
$$6aL_z\equiv\eta\mu_1^2,$$
$$b/a\equiv r, \tag{9.17}$$

we find the following expressions for the masses:

$$m_\pi^2=\mu_1^2(1-\xi-r\eta),$$
$$m_K^2=\mu_1^2[1-(1-r+r^2)\xi-\eta],$$
$$m_{\sigma\pi}^2=\mu_1^2(1-3\xi+r\eta),$$
$$m_{\sigma K}^2=\mu_1^2[1-(1+r+r^2)\xi+\eta], \tag{9.18}$$

while the masses of the two i-spin-0 pseudoscalar

* The application to $SU(3)\times SU(3)$ is discussed by Callan, Coleman, Wess, and Zumino (1969) and by Bardeen and Lee (1969), as well as by Isham (1969), Dietz and Honerkamp (1968), Macfarlane and Weisz (1968), and Macfarlane, Sudbery, and Weisz (1969).

* There are some "uninteresting" realizations. This topic is thoroughly discussed in Bardeen and Lee (1969).

particles can be obtained by diagonalizing

$$\tfrac{1}{2}\phi_8^2\mu_1^2[1-\tfrac{1}{3}(1+2r^2)\xi-\tfrac{1}{3}(4-r)\eta] +\tfrac{1}{2}\phi_0^2\mu_1^2[1-\tfrac{1}{3}(2+r^2)\xi+\tfrac{1}{3}(4+2r)\eta] +\tfrac{1}{3}\sqrt{2}\phi_8\phi_0\mu_1^2[(r^2-1)\xi+(r-1)\eta]. \quad (9.19)$$

Let us now ask what happens when $\mu_1^2\to\infty$. If we want to keep the pion and K masses finite, we must have

$$1-\xi-r\eta=0,$$
$$1-(1-r+r^2)\xi-\eta=0. \quad (9.20)$$

If $r\neq 1$ this implies that

$$\xi=1/(1+r^2); \qquad \eta=r/(1+r^2). \quad (9.21)$$

With these values we find that:

(a) The mass m_{σ_K} of the scalar K remains finite.

(b) The mass of one of the pseudoscalar i-spin-0 particles remains finite while the other goes to infinity.

(c) The mass of the scalar i-spin-1 particle goes to infinity. The masses of the i-spin-0 scalar particles involves undetermined parameters $(L_{xx}, L_{xy},\cdots)$ and can be made finite or infinite. If the latter choice is made, we have a theory in which there exist an octet of pseudoscalar mesons and a set of scalar K mesons. The absence of a ninth pseudoscalar meson need not disturb us: we are, after all, working with $SU(3)$ and not $U(3)$, and there is no need to connect the ninth meson with the octet. The scalar K mesons play the role of "Goldstone bosons" corresponding to the nonconserved strangeness changing vector current (Glashow and Weinberg, 1968). With this choice of parameters we find

$$f=am_{\sigma_K}^2\begin{pmatrix}1&&\\&1&\\&&r\end{pmatrix}+a(m_\pi^2-m_{\sigma_K}^2)\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \quad (9.22)$$

and, as before,

$$f_\pi=\sqrt{2}a,$$
$$f_K=\sqrt{2}a[\tfrac{1}{2}(1+r)]=\tfrac{1}{2}(1+r)f_\pi. \quad (9.23)$$

It follows from (9.22) that

$$f_8=(1/\sqrt{3})(1-r)m_{\sigma_K}^2f_\pi. \quad (9.24)$$

Hence the coefficient of the scalar field on the right-hand side of (7.65) is

$$(1/\sqrt{3})(1-r)m_{\sigma_K}^2f_{845}f_\pi=\tfrac{1}{2}(1-r)m_{\sigma_K}^2f_\pi;$$

i.e., the scalar K meson (κ) decay constant is given by

$$f_\kappa=\tfrac{1}{2}(1-r)f_\pi. \quad (9.25)$$

Hence we get, combining (9.23) and (9.25),

$$f_K+f_\kappa=f_\pi. \quad (9.26)$$

It is interesting to compare (9.26) with a result obtained first by Glashow and Weinberg (Glashow and Weinberg, 1968)* by satisfying two- and three-point-function Ward–Takahashi identities with meson poles, including the kappa meson. They found that if the $SU(3)\times SU(3)$ symmetry is broken in the manner assumed for the Lagrangian discussed here, then the second-order $SU(3)$ breaking correction to the K_{e3} form factor is given by

$$f_+(0)=(f_\pi^2+f_K^2-f_\kappa^2)/2f_Kf_\pi. \quad (9.27)$$

For the particular class of effective Lagrangians considered here so far, with derivative couplings of the form of Eqs. (8.5) and (8.6) omitted, the strangeness changing vector current defined does not renormalize f_+ so long as we restrict ourselves, in the spirit of effective Lagrangians, to tree graphs. Consequently, we would obtain $f_+(0)=1$. Since these effective Lagrangians are particular models satisfying the assumptions made by Glashow and Weinberg, (9.27) should also be valid. It is easy to see that (9.26) is a solution to $f_+(0)=(f_\pi^2+f_K^2-f_\kappa^2)/2f_Kf_\pi=1$. More generally, however, as will be seen in the next section, the addition of the terms (8.5) and (8.6) and the inclusion of spin-1 fields introduce the derivative couplings which renormalize the fields and modify the currents. As a result, $f_+(0)\neq 1$ and Eq. (9.26) is modified. On the other hand, unless terms with many derivatives are added, our effective Lagrangians should satisfy Eq. (9.27).

It is worth making a point here about the experimental determination of the value of f_K or, more conveniently, the ratio f_K/f_π. One could determine f_K directly from $K_{\mu 2}$ decay provided the axial-vector Cabibbo angle were known precisely enough. Unfortunately, this angle is not known at present. Until forced to do otherwise, we can define the axial-vector Cabibbo angle to be equal to the vector angle. Nevertheless, this angle cannot be precisely determined until we learn the renormalization effects for the various semileptonic decay form factors. This is a problem for the theorist. The best we can do at present is to determine the Cabibbo angle, using K_{e3} decay, in terms of the form factor $f_+(0)$.† This yields the relation based on

* A preliminary, but more extensive discussion of this work may be found in Glashow (1968). See also Glashow, Schnitzer, and Weinberg (1967), and Nieh (1967).

† The cosine of the Cabibbo angle could be determined from accurate measurements of nonstrangeness changing semileptonic decays. The cosine is so close to 1, however, that the poorly known radiative corrections produce large uncertainties in the angle. Nevertheless, present analyses of the beta decay of ^{14}O has led to the value $\sin\theta=0.1\pm0.01$, with a more recent measurement yielding $\sin\theta=0.19\pm0.01$. A value of $\sin\theta=0.2$ combined with the most recent average of K_{e3}^+ decay rates, fixing $f_+(0)\sin\theta=0.221\pm0.003$, predicts $f_+(0)=1.11\pm0.06$. This value for $f_+(0)$ is inconsistent with the models discussed here which favor $f_+(0)\lesssim 1$. Despite the uncertainties in this method of evaluating θ, unless the K_{e3} decay rate is in error, we would be hard pressed to reconcile the ^{14}O data with a value of $f_+(0)$ less than 1 by more than a few percent.

$\pi_{\mu 2}$, $K_{\mu 2}$, and K_{e3} decay rates,

$$f_K/f_\pi f_+(0)=1.25[1-0.024/f_+^2(0)].$$

A deviation of $f_+(0)$ from 1 by 20% only yields a correction of 1% to the right-hand side. $f_+(0)\sim 0.95$ yields*

$$f_K/f_\pi=1.22f_+(0). \qquad (9.28)$$

Any consistent effective Lagrangian approach to $SU(3)\times SU(3)$ symmetry breaking should predict values for f_K/f_π for $f_+(0)$ that reasonably satisfy Eq. (9.28).

In order to calculate matrix elements we may proceed in one of several ways. One is to calculate the matrix element with all the scalar mesons present and then take the appropriate limits

$$\mu_1^2\to\infty, \qquad \xi\to 1/(1+r^2), \qquad \eta\to r\xi \qquad (9.29)$$

in such a way that the masses that remain finite approach their experimental value. This approach was illustrated for the $\pi\pi$ scattering length in Sec. III [Eq. (3.48)]. Another approach is to find an expression for the fields that represent particles, as in Eq. (9.8), for example. Such a relation can be obtained by taking the limit (9.29) in the equation of motion, as was illustrated in Eq. (3.49). When this is done for our Lagrangian (8.1)–(8.4), a horrible implicit equation for σ emerges. It turns out that the relation implied by that equation can be greatly simplified if a transformation to a new set of fields is made. The transformation looks as follows.

If we write the Lagrangian (8.1)–(8.4) in terms of the fields B and $B^\dagger$, it takes the form

$$\mathcal{L}=\tfrac{1}{2}\,\mathrm{Tr}\,(\partial_\mu B\partial^\mu B^\dagger)-(\mu^2/2)\,\mathrm{Tr}\,BB^\dagger$$
$$+\mathcal{L}_C(X,Y,Z)+\tfrac{1}{2}\,\mathrm{Tr}\,[f(B+B^\dagger)] \qquad (9.30)$$

with

$$X=\mathrm{Tr}\,(BB^\dagger),$$
$$Y=\mathrm{Tr}\,(BB^\dagger BB^\dagger),$$
$$Z=\det B+\det B^\dagger. \qquad (9.31)$$

Now the first nonlinear realization which we considered satisfied

$$BB^\dagger=B^\dagger B=F^2I \qquad (9.32)$$

and contained only pseudoscalar fields. A parametrization that satisfies (9.32) [such as suggested by Chang and Gursey (1967) and Brown (1967), for example] is

$$B=Fe^{2iP}, \qquad (9.33)$$

where

$$P=(1/\sqrt{2})\sum_i \lambda_i P_i \qquad (9.34)$$

and the P_i are the nine pseudoscalar fields that enter the theory. (We have the freedom to keep the ninth pseudoscalar meson or not. Since X and Y are numerical constants, P_0 only enters in Z and we can make the mass of P_0 whatever we like. This is similar to the decoupling of the ninth vector meson noted in Sec. VII.) For the second nonlinear realization Bardeen and Lee (1969) have shown that the form

$$B=e^{iP}e^{iS}\Sigma e^{-iS}e^{iP} \qquad (9.35)$$

should be taken. Here, because of the form of Σ, only the strange scalar fields S_i $(i=4, 5, 6, 7)$ remain, and all eight pseudoscalar fields P_i remain. Note that since

$$BB^\dagger=e^{iP}e^{iS}\Sigma^2e^{-iS}e^{-iP}, \qquad (9.36)$$

it follows that $\mathrm{Tr}\,BB^\dagger$ and $\mathrm{Tr}\,BB^\dagger BB^\dagger$ are numerical constants. Also, if we ignore the ninth pseudoscalar meson P_0, $\det B$ is a numerical constant, so that the Lagrangian becomes, in the absence of terms like Eqs. (8.5) and (8.6),

$$\mathcal{L}=\tfrac{1}{2}\,\mathrm{Tr}\,\partial_\mu B\partial^\mu B^\dagger+\tfrac{1}{2}\,\mathrm{Tr}\,f(B+B^\dagger). \qquad (9.37)$$

X. THE "SUPER LAGRANGIAN"

So far we have discussed effective Lagrangians with $SU(3)\times SU(3)$ symmetry that have been constructed with either spin-1 or spin-0 fields. In this section we shall examine the effects of coupling the spin-0 and -1 fields together into a "Super Lagrangian" satisfying PCAC and the algebra of fields.

We will generalize the $SU(2)\times SU(2)$ effective Lagrangian of Sec. V to $SU(3)\times SU(3)$, with the additional difference that the scalar fields will describe *real* 0^+ mesons. There is growing experimental evidence for 0^+ resonances, a $T=0$ σ meson with mass around 700 MeV, with the rest of the nonet clustered around 1 GeV. Their existence is still in doubt, but so is that of some of the axial mesons. Why not, therefore, treat the 0^+ and the 1^+ mesons on the same footing? Thus in generalizing Eq. (5.29) to $SU(3)\times SU(3)$ we no longer take $\sigma^2+\phi^2$ as a c number; we will therefore add invariant meson–meson terms of the type discussed in Sec. VII to $\mathcal{L}_0$, and, for added flexibility, include the invariant 0^+–0^- interaction terms given by Eqs. (8.5) and (8.6) (with ∂_μ replaced by the chiral-covariant derivative Δ_μ). Next we generalize Eq. (5.26) to $SU(3)\times SU(3)$. To do this, we make use of the simple chiral transformation properties of $F_{\mu\nu}\pm G_{\mu\nu}$, $\Delta^\mu B\Delta^\nu B^+$, and $\Delta^\mu B^+\Delta^\nu B$:

$$\delta'(F_{\mu\nu}\pm G_{\mu\nu})=\pm(i/\sqrt{2})[\beta, F_{\mu\nu}\pm G_{\mu\nu}],$$
$$\delta'\Delta^\mu B\Delta^\nu B^\dagger=(i/\sqrt{2})[\beta, \Delta^\mu B\Delta^\nu B^\dagger],$$
$$\delta'\Delta^\mu B^\dagger\Delta^\nu B=-(i/\sqrt{2})[\beta, \Delta^\mu B^\dagger, \Delta^\nu B]. \qquad (10.1)$$

Clearly

$$i\,\mathrm{Tr}\,[(F_{\mu\nu}+G_{\mu\nu})\Delta^\mu B\Delta^\nu B^\dagger+(F_{\mu\nu}-G_{\mu\nu})\Delta^\mu B^\dagger\Delta^\nu B]$$

is parity and $SU(3)\times SU(3)$ invariant and Hermitian. Since $B=\sigma+i\phi$, this is easily seen to reduce to (5.26) for the case of $SU(2)\times SU(2)$.

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* The value $f_K/f_\pi f_+(0)=1.28$ that has been widely quoted in the literature was based on an earlier value for the K_{e3}^+ decay rate. More recent measurements have increased this rate by 7%.

The chiral-invariant Lagrangian that we have constructed so far appears as

$$\mathcal{L}_0=-\tfrac{1}{4}\,\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}+G_{\mu\nu}G^{\mu\nu})+\tfrac{1}{2}m_0^2\,\mathrm{Tr}\,(V_\mu V^\mu+\mathcal{A}_\mu\mathcal{A}^\mu)$$
$$+\tfrac{1}{2}\,\mathrm{Tr}\,(\Delta_\mu\sigma\Delta^\mu\sigma+\Delta_\mu\phi\Delta^\mu\phi)-\tfrac{1}{2}\mu^2\,\mathrm{Tr}\,(\sigma^2+\phi^2)$$
$$+\tfrac{1}{4}\lambda[\mathrm{Tr}\,(\phi^2+\sigma^2)]^2+\tfrac{1}{8}\lambda'\,\mathrm{Tr}\,[(BB^\dagger)^2+(B^\dagger B)^2]$$
$$+\tfrac{1}{4}\eta_1\,\mathrm{Tr}\,(\Delta_\mu B\Delta^\mu B^\dagger BB^\dagger+\Delta_\mu B^\dagger\Delta^\mu BB^\dagger B)$$
$$+\tfrac{1}{4}\eta_2\,\mathrm{Tr}\,(\Delta_\mu BB^\dagger\Delta^\mu BB^\dagger+\Delta_\mu B^\dagger B\Delta^\mu B^\dagger B)$$
$$+(i\gamma_0\delta/2m_0^2)\,\mathrm{Tr}\,[(F_{\mu\nu}+G_{\mu\nu})\Delta^\mu B\Delta^\nu B^\dagger$$
$$+(F_{\mu\nu}-G_{\mu\nu})\Delta^\mu B^\dagger\Delta^\nu B]. \quad (10.2)$$

We have by no means exhausted the possibilities for adding chiral-invariant terms to $\mathcal{L}$. We could, for example, add higher powers of the coefficients of μ^2 and λ'.* We *must*, however, add one additional term, as we will see in a moment. Since the observations on symmetry breaking made in Sec. VIII still apply, we must add to $\mathcal{L}_0$ the symmetry-breaking terms

$$\mathcal{L}_1=\mathrm{Tr}\,(f\sigma)+\tfrac{1}{3}\mu g(\det B+\det B^\dagger) \quad (10.3)$$

given in Eqs. (7.61) and (7.62). $\mathcal{L}_1$ leads to PCAC for eight of the nine axial currents and introduces nonvanishing vacuum expectation values for σ_0 and σ_8 that lead to symmetry-breaking effects in $\mathcal{L}_0$. It is not hard to see, however, that (10.1)+(10.2) maintains the mass degeneracy of the nonstrange vector mesons. The K^* are shifted by contributions from $\tfrac{1}{2}\,\mathrm{Tr}\,(\Delta_\mu\sigma\Delta^\mu\sigma)$ analogous to the $A_1-\rho$ mass splitting seen in $SU(2)\times SU(2)$; this mass shift is interesting and we will come back to it later. The axial mesons are all shifted by the mass term arising from $\tfrac{1}{2}\,\mathrm{Tr}\,(\Delta_\mu\phi\Delta^\mu\phi)$. Since the ninth-vector and axial-vector fields do not transform under $SU(3)\times SU(3)$, as pointed out earlier, we could add terms like those in Eq. (7.42) to split off the ninth-vector and axial-vector mesons from the nonet, and these would be enough to permit a fitting of the spin-1 meson masses since, in fact, $m_\rho\simeq m_\omega$. Such a procedure would, however, identify the ϕ meson as a pure unitary singlet, with the decay $\phi\to K\bar{K}$ occurring purely as a result of a $SU(3)$ breaking interaction not contained in our Lagrangian. This is very unattractive, since the coupling is consistent with the conventional $SU(3)$ mixing approach.

The conventional $SU(3)$ symmetry-breaking term $\mathrm{Tr}\,(V_\mu\lambda_8V^\mu+\mathcal{A}_\mu\lambda_8\mathcal{A}^\mu)$, which is the simple chiral extension of the mass term in Eq. (7.13), will produce ϕ–ω mixing. It has the disadvantage of destroying the equality of the Schwinger terms in $SU(3)\times SU(3)$ and, more important, is chiral invariant only under $SU(2)\times SU(2)$, so that PCAC is only satisfied by the pion. A kinetic-energy mixing term $\mathrm{Tr}\,(F_{\mu\nu}\lambda_8F^{\mu\nu}+G_{\mu\nu}\lambda_8G^{\mu\nu})$ also violates PCAC for the K and η although it maintains the equality of the Schwinger terms. While we cannot rule out the possibility that either or both of these symmetry-breaking terms will be necessary, it is interesting that ϕ–ω mixing can be obtained without them by adding the chiral invariant

$$\mathrm{Tr}\,[(F_{\mu\nu}+G_{\mu\nu})(F^{\mu\nu}+G^{\mu\nu})BB^\dagger$$
$$+(F_{\mu\nu}-G_{\mu\nu})(F^{\mu\nu}-G^{\mu\nu})B^\dagger B] \quad (10.4)$$

to $\mathcal{L}_0$. When σ_0, σ_8 acquire nonzero vacuum expectation values through the addition of $\mathcal{L}_1$, $BB^\dagger$ and $B^\dagger B$ acquire constant terms which produce, in (10.4), vector and axial-vector *kinetic-energy* mixing terms. A chiral-invariant term like $\mathrm{Tr}\,[(V_\mu+\mathcal{A}_\mu)(V^\mu+\mathcal{A}^\mu)BB^\dagger+(V_\mu-\mathcal{A}_\mu)(V^\mu-\mathcal{A}^\mu)B^\dagger B]$ which yields mass-mixing terms after symmetry breaking, contributes to the currents, breaking the current–field proportionality. We see, therefore, that in the effective-Lagrangian approach, the combined conditions of field–current proportionality and PCAC for the pseudoscalar meson octet requires that ϕ–ω mixing arises in the kinetic-energy term only (Mitter and Swank, 1968; Gasiorowicz and Geffen, 1968, unpublished). The field–current proportionality, it will be recalled, has the attractive feature of generating c-number Schwinger terms. The additional term (10.4), which preserves field algebra and PCAC, yields mixing and hence $SU(3)$ breaking in the vector-meson nonet, while maintaining the equality of all the Schwinger terms in $SU(3)\times SU(3)$. This is a very interesting result and lends support to the conjecture made by many authors that this equality was indeed correct.

We can now complete our "Super Lagrangian" by adding to $\mathcal{L}_0$ [Eq. (10.2)] and $\mathcal{L}_1$ [Eq. (10.3)] a final contribution $\mathcal{L}_2$, consisting of (10.4) and an additional kinetic-energy term which breaks $V_\pi-V_X(\rho-\omega)$ and $A_\pi-A_X$ degeneracies:

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2 \quad (10.5)$$

with

$$\mathcal{L}_2=\tfrac{1}{8}\xi\,\mathrm{Tr}\,[(F_{\mu\nu}+G_{\mu\nu})^2BB^\dagger+(F_{\mu\nu}-G_{\mu\nu})^2B^\dagger B]$$
$$-\tfrac{1}{12}\xi_V(\mathrm{Tr}\,F_{\mu\nu})^2-\tfrac{1}{12}\xi_A(\mathrm{Tr}\,G_{\mu\nu})^2. \quad (10.6)$$

A similar Lagrangian has been discussed by Mitter and Swank (1968) who also obtained some of the results given below. We repeat that the Lagrangian is by no means unique, as there are many additional terms that could be added to (10.5), terms that preserve PCAC and the field algebra; in particular, the meson–meson part of $\mathcal{L}_0$ contains only a very special form of $\mathcal{L}_C$ [as defined in Eq. (8.3)]. Our aim in this section, however, is not to write the most general Lagrangian, but rather to provide the techniques and to give some insight into the problems of $SU(3)\times SU(3)$ in a field algebra.

In what follows we shall outline the steps taken to obtain the final expressions for the meson masses and renormalization constants. We shall then discuss the couplings of the vector mesons to the vector currents (related to the ρ, ω, $\phi-\gamma$ couplings), and then show

* If one wishes to go to the limiting case of the scalar masses (other than the σ_π) going to infinity, such higher powers are necessary.

how the many parameters in the Lagrangian are surprisingly and severely restricted by the requirement that the particle masses and the value of $f_K/f_\pi f_+(0)$ be fitted to their experimental values.

As shown earlier, the tadpoles can be eliminated by introducing new scalar fields σ', with zero vacuum expectation values:

$$\sigma=\Sigma+\sigma'; \qquad \Sigma=\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}. \tag{10.7}$$

The relation between f, written in the form $u\Sigma+vI$, and Σ is unaffected by the presence of vector mesons and is therefore the same as obtained for the spin 0 Lagrangian of Sec. VIII. With our choice of the meson–meson interaction it reads

$$u=\mu^2-\lambda(2a^2+b^2)-\lambda'(a^2+ab+b^2)+2\mu ga,$$
$$v=a(a+b)(\lambda'b-2\mu g). \tag{10.8}$$

In order to obtain expressions for the masses of the various particles we must examine the quadratic terms and the kinetic-energy terms for the various fields.

(1) The quadratic terms in the scalar fields are again the same as in Sec. VIII. They read*

$$\begin{aligned}
&-\tfrac{1}{2}\sigma_\pi^2(\mu_1^2-3a^2\lambda'+2\mu gb)\\
&\quad-\tfrac{1}{2}\sigma_K^2[\mu_1^2-\lambda'(a^2+ab+b^2)+2\mu ga]\\
&\quad-\tfrac{1}{2}\sigma_8^2[\mu_1^2-\tfrac{4}{3}\lambda(a-b)^2-\lambda'(a^2+2b^2)+\tfrac{2}{3}\mu g(4a-b)]\\
&\quad-\tfrac{1}{2}\sigma_0^2[\mu_1^2-\tfrac{2}{3}\lambda(2a+b)^2-\lambda'(2a^2+b^2)-\tfrac{4}{3}\mu g(2a+b)]\\
&\quad+\tfrac{2}{3}\sqrt{2}\sigma_8\sigma_0(a-b)[\lambda(2a+b)+\tfrac{3}{2}\lambda'(a+b)-\mu g],
\end{aligned} \tag{10.9}$$

where we have written

$$\mu_1^2=\mu^2-\lambda(2a^2+b^2). \tag{10.10}$$

Similarly, the terms quadratic in the pseudoscalar fields are unchanged and have the form

$$\begin{aligned}
&-\tfrac{1}{2}\phi_\pi^2(\mu_1^2-\lambda'a^2-2\mu gb)\\
&\quad-\tfrac{1}{2}\phi_K^2[\mu_1^2-\lambda'(a^2-ab+b^2)-2\mu ga]\\
&\quad-\tfrac{1}{2}\phi_8^2[\mu_1^2-\tfrac{1}{3}\lambda'(a^2+2b^2)-\tfrac{2}{3}\mu g(4a-b)]\\
&\quad-\tfrac{1}{2}\phi_0^2[\mu_1^2-\tfrac{1}{3}\lambda'(2a^2+b^2)+\tfrac{4}{3}\mu g(2a+b)]\\
&\quad+\tfrac{2}{3}\sqrt{2}\phi_8\phi_0(a-b)[\tfrac{1}{2}\lambda'(a+b)+\mu g].
\end{aligned} \tag{10.11}$$

Before we can find expressions for the spin-0 meson masses, we must first isolate their kinetic-energy terms in $\mathcal{L}$ and renormalize the fields. One contribution to the kinetic energy comes from the coupling to the vector and axial mesons: the case of such a coupling of π to A_1 in $SU(2)\times SU(2)$ was treated in Sec. VI. The second contribution comes from the terms proportional to η_1 and η_2 in $\mathcal{L}_0$ when the substitution (10.7) is made.

(2) Consider first the generalization of Sec. VI, the elimination of the direct pseudoscalar–axial-vector-meson couplings for the nonets. We do this by introducing a new axial-vector field A_μ, defined by

$$A_\mu=\mathcal{A}_\mu-\mathfrak{D}_\mu\phi. \tag{10.12}$$

The matrix $\mathfrak{D}_\mu\phi$ (not the transpose) can be defined as follows: if $D_\mu\phi$ is symbolically denoted by

$$D_\mu\phi=\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \tag{10.13}$$

then

$$\mathfrak{D}_\mu\phi=\begin{pmatrix} \alpha\cdot & \alpha\cdot & \beta\cdot \\ \alpha\cdot & \alpha\cdot & \beta\cdot \\ \beta\cdot & \beta\cdot & \gamma\cdot \end{pmatrix}; \tag{10.14}$$

i.e., the (11), (12), (21), (22) components of the matrix are multiplied by α; the (13), (23), (31), (32) components, by β; and the (33) component, by γ. The matrix in (10.14) is the most general form permitted that preserves i-spin and hypercharge invariance. The chiral-covariant derivative now becomes

$$\begin{aligned}
\Delta_\mu\phi&=D_\mu\phi-(1/\sqrt{2})\gamma_0\{\mathcal{A}_\mu,\sigma\}\\
&=D_\mu\phi-(1/\sqrt{2})\gamma_0\{A_\mu+\mathfrak{D}_\mu\phi,\Sigma+\sigma'\}\\
&=D_\mu\phi-(1/\sqrt{2})\gamma_0\{\mathfrak{D}_\mu\phi,\Sigma\}\\
&\qquad-(1/\sqrt{2})\gamma_0\{A_\mu,\Sigma\}+\cdots.
\end{aligned} \tag{10.15}$$

The first two terms on the right-hand side, which will contribute to the pseudoscalar kinetic energy, can be written as

$$D_\mu\phi-(1/\sqrt{2})\gamma_0\{\mathfrak{D}_\mu\phi,\Sigma\}=\begin{pmatrix} [1-(1/\sqrt{2})\gamma_0(2a\alpha)]\cdot & [1-(1/\sqrt{2})\gamma_0(2a\alpha)]\cdot & [1-(1/\sqrt{2})\gamma_0(a+b)\beta]\cdot \\ [1-(1/\sqrt{2})\gamma_0(2a\alpha)]\cdot & [1-(1/\sqrt{2})\gamma_0(2a\alpha)]\cdot & [1-(1/\sqrt{2})\gamma_0(a+b)\beta]\cdot \\ [1-(1/\sqrt{2})\gamma_0(a+b)\beta]\cdot & [1-(1/\sqrt{2})\gamma_0(a+b)\beta]\cdot & [1-(1/\sqrt{2})\gamma_0(2b\gamma)]\cdot \end{pmatrix}. \tag{10.16}$$

* We use the notation $\sigma_\pi^2=\sigma_{\pi^0}^2+2\sigma_{\pi^+}\sigma_{\pi^-}$ and $\sigma_K^2=2\sigma_{K^+}\sigma_{K^-}+2\sigma_{K_0}\sigma_{\bar{K}_0}$ and similarly for the pseudoscalar, vector, and axial mesons.

The terms in $\mathcal{L}$ that will contribute to the pseudoscalar-meson kinetic energy after the substitutions (10.7), (10.12), and (10.15) are

$$\tfrac{1}{2}\,\mathrm{Tr}\,\Delta_\mu\phi\Delta^\mu\phi+\tfrac{1}{2}m_0^2\,\mathrm{Tr}\,\mathcal{A}_\mu\mathcal{A}^\mu+\tfrac{1}{2}\eta_1\,\mathrm{Tr}\,(\Delta_\mu\phi\Delta^\mu\phi\sigma^2)-\tfrac{1}{2}\eta_2\,\mathrm{Tr}\,(\Delta_\mu\phi\sigma\Delta^\mu\phi\sigma). \quad (10.17)$$

The constants α, β, and γ defined in Eq. (10.14) are determined by the condition that the new fields A_μ have no direct coupling to $\partial_\mu\phi$, i.e., that coefficients of terms like $A_\mu{}^i\partial^\mu\phi^j$ vanish. The proper choice is

$$m_0\alpha=\frac{f[1+(\eta_1-\eta_2)a^2]}{1+f^2[1+(\eta_1-\eta_2)a^2]}, \quad (10.18)$$

$$m_0\beta=\frac{\tfrac{1}{2}(r+1)f[1+\tfrac{1}{2}\eta_1(a^2+b^2)-\eta_2ab]}{1+\tfrac{1}{4}(r+1)^2f^2[1+\tfrac{1}{2}\eta_1(a^2+b^2)-\eta_2ab]}, \quad (10.19)$$

$$m_0\gamma=\frac{rf[1+(\eta_1-\eta_2)b^2]}{1+r^2f^2[1+(\eta_1-\eta_2)b^2]}, \quad (10.20)$$

with $f\equiv\sqrt{2}\gamma_0a/m_0$, $r\equiv b/a$.

(3) Unlike the case of $SU(2)\times SU(2)$, where we deal with only a $T=0$ σ field, the extension to $SU(3)\times SU(3)$ with nine scalar fields results in a direct coupling term between the K^* field, $V_\mu{}^K$, and the scalar K field (the kappa meson) σ_K. This arises from $i[V_\mu,\sigma]$ in $\Delta_\mu\sigma$ [Eq. (7.47)] when σ_8 acquires a nonvanishing vacuum expectation value. We eliminate this coupling by defining new vector fields U_μ by

$$V_\mu=U_\mu+i\kappa[\Delta_\mu\sigma',\Sigma], \quad (10.21)$$

where

$$\Delta_\mu\sigma'=\Delta_\mu\sigma+(i\gamma_0/\sqrt{2})[V_\mu,\Sigma]. \quad (10.22)$$

Since Σ contains only the diagonal matrices λ_0 and λ_8, and $[\lambda_8,\lambda_j]=0$ unless $j=4, 5, 6, 7$, only the strangeness ±1 parts of U_μ differ from V_μ. In using (10.22) to eliminate $\Delta_\mu\sigma$ in $\mathcal{L}$, (10.21) must be used to introduce U_μ into (10.22). Thus

$$\Delta_\mu\sigma=\Delta_\mu\sigma'+(\kappa\gamma_0/\sqrt{2})[[\Delta_\mu\sigma',\Sigma],\Sigma]-(i\gamma_0/\sqrt{2})[U_\mu,\Sigma]. \quad (10.23)$$

If $\Delta_\mu\sigma'$ is denoted by the matrix

$$\Delta_\mu\sigma'=\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}, \quad (10.24)$$

then

$$[\Delta_\mu\sigma',\Sigma]=\begin{pmatrix}0&0&(b-a)\cdot\\0&0&(b-a)\cdot\\(a-b)\cdot&(a-b)\cdot&0\end{pmatrix}. \quad (10.25)$$

Elimination of the $\Delta_\mu\sigma'U^\mu$ cross terms from the kinetic-energy terms

$$\tfrac{1}{2}\,\mathrm{Tr}\,\Delta_\mu\sigma\Delta^\mu\sigma+\tfrac{1}{2}\eta_1\,\mathrm{Tr}\,(\Sigma^2\Delta_\mu\sigma\Delta^\mu\sigma)+\tfrac{1}{2}\eta_2\,\mathrm{Tr}\,(\Delta_\mu\sigma\Sigma\Delta^\mu\sigma\Sigma) \quad (10.26)$$

and the vector-meson mass term $\tfrac{1}{2}m_0^2\,\mathrm{Tr}\,V_\mu V^\mu$ yields the expression

$$m_0^2\kappa=\frac{-(1/\sqrt{2})\gamma_0[1+\tfrac{1}{2}\eta_1(a^2+b^2)+\eta_2ab]}{1+\tfrac{1}{4}f^2(r-1)^2[1+\tfrac{1}{2}\eta_1(a^2+b^2)+\eta_2ab]}. \quad (10.27)$$

(4) Given α, β, γ, and κ, it is straightforward though tedious to rewrite $\mathcal{L}$ in terms of the new fields σ', A_μ, and U_μ. In isolating the pseudoscalar and scalar kinetic-energy terms there is mixing between the 0 and 8 components which can be removed by introducing the new fields ϕ_η, ϕ_X, σ_η', σ_X', related to the old fields by the nonet transformation discussed in Sec. VII, i.e.,

$$\phi_\eta=(\tfrac{2}{3})^{1/2}\phi_0+(\tfrac{1}{3})^{1/2}\phi_8,$$
$$\phi_X=(\tfrac{1}{3})^{1/2}\phi_0-(\tfrac{2}{3})^{1/2}\phi_8, \quad (10.28)$$

and similarly for the scalar mesons. The kinetic-energy terms now turn out to be

$$\tfrac{1}{2}[(D_\mu\phi_\pi)^2+(D_\mu\phi_\eta)^2]\frac{1+(\eta_1-\eta_2)a^2}{1+f^2[1+(\eta_1-\eta_2)a^2]}+\tfrac{1}{2}(D_\mu\phi_K)^2\frac{1+\tfrac{1}{2}\eta_1(a^2+b^2)-\eta_2ab}{1+\tfrac{1}{4}(r+1)^2f^2[1+\tfrac{1}{2}\eta_1(a^2+b^2)-\eta_2ab]}+\tfrac{1}{2}(D_\mu\phi_X)^2\frac{1+(\eta_1-\eta_2)b^2}{1+r^2f^2[1+(\eta_1-\eta_2)b^2]} \quad (10.29)$$

for the pseudoscalar mesons, and

$$\tfrac{1}{2}[(\Delta_\mu\sigma_\pi')^2+(\Delta_\mu\sigma_\eta')^2][1+(\eta_1+\eta_2)a^2]+\tfrac{1}{2}(\Delta_\mu\sigma_K')^2\frac{1+\tfrac{1}{2}\eta_1(a^2+b^2)+\eta_2ab}{1+\tfrac{1}{4}(r-1)^2f^2[1+\tfrac{1}{2}\eta_1(a^2+b^2)+\eta_2ab]}+\tfrac{1}{2}(\Delta_\mu\sigma_X')^2[1+(\eta_1+\eta_2)b^2] \quad (10.30)$$

for the scalar mesons.

To make the coefficients of the kinetic-energy terms unity we introduce renormalized fields

$$\hat{\phi}_\pi=Z_{\phi\pi}{}^{-1/2}\phi_\pi,\qquad \hat{\sigma}_\pi=Z_{\sigma\pi}{}^{-1/2}\sigma_\pi,\qquad \hat{\phi}_K=Z_{\phi K}{}^{-1/2}\phi_K,\cdots. \quad (10.31)$$

The $Z_{\phi\pi}{}^{-1}$, $Z_{\sigma\pi}{}^{-1}$, etc., are given by the coefficients of $\tfrac{1}{2}(\Delta_\mu\phi_\pi)^2$, $\tfrac{1}{2}(\Delta_\mu\sigma_\pi')^2$, etc., in Eqs. (10.29) and (10.30). With the notation

$$\Gamma_\pi\equiv1+f^2[1+(\eta_1-\eta_2)a^2],$$
$$\Gamma_K\equiv1+\tfrac{1}{4}(r+1)^2f^2[1+\tfrac{1}{2}\eta_1(a^2+b^2)-\eta_2ab],$$
$$\Gamma_X\equiv1+r^2f^2[1+(\eta_1-\eta_2)b^2],$$
$$\Gamma_\kappa\equiv1+\tfrac{1}{4}(r-1)^2f^2[1+\tfrac{1}{2}\eta_1(a^2+b^2)+\eta_2ab], \quad (10.32a)$$

these may be rewritten as

$$Z_{\phi\pi}^{-1}=Z_{\phi\eta}^{-1}=\Gamma_\pi^{-1}[1+(\eta_1-\eta_2)a^2],$$
$$Z_{\phi K}^{-1}=\Gamma_K^{-1}[1+\tfrac{1}{2}\eta_1(a^2+b^2)-\eta_2 ab],$$
$$Z_{\phi X}^{-1}=\Gamma_X^{-1}[1+(\eta_1-\eta_2)b^2],$$
$$Z_{\sigma\pi}^{-1}=Z_{\sigma\eta}^{-1}=1+(\eta_1+\eta_2)a^2,$$
$$Z_{\sigma X}^{-1}=1+(\eta_1+\eta_2)b^2,$$
$$Z_{\sigma K}^{-1}=\Gamma_\kappa^{-1}[1+\tfrac{1}{2}\eta_1(a^2+b^2)+\eta_2 ab]. \quad (10.32\text{b})$$

When the renormalized fields are introduced into (10.9) and (10.11), we find that the masses are

$$m_\pi^2=Z_{\phi\pi}(\mu_1^2-\lambda' a^2-2\mu g b),$$
$$m_K^2=Z_{\phi K}[\mu_1^2-\lambda'(a^2-ab+b^2)-2\mu g a],$$
$$m_{\sigma\pi}^2=Z_{\sigma\pi}(\mu_1^2-3\lambda' a^2+2\mu g b),$$
$$m_{\sigma K}^2=Z_{\sigma K}[\mu_1^2-\lambda'(a^2+ab+b^2)+2\mu g a], \quad (10.33)$$

while the masses of the isoscalar spin-0 mesons are obtained by diagonalizing

$$\tfrac{1}{2}\hat{\phi}_X^2(\mu_1^2-\lambda' b^2)Z_{\phi X}+\tfrac{1}{2}\hat{\phi}_\eta^2(\mu_1^2-\lambda' a^2+2\mu g b)Z_{\phi\eta}+2\sqrt{2}\mu g a(Z_{\phi\eta}Z_{\phi X})^{1/2}\hat{\phi}_X\hat{\phi}_\eta \quad (10.34)$$

and

$$\tfrac{1}{2}\hat{\sigma}_X^2(\mu_1^2-2\lambda b^2-3\lambda' b^2)Z_{\sigma X}+\tfrac{1}{2}\hat{\sigma}_\eta^2(\mu_1^2-4\lambda a^2-3\lambda' a^2-2\mu g b)Z_{\sigma\eta}-2\sqrt{2}a(\lambda b+\mu g)(Z_{\sigma X}Z_{\sigma\eta})^{1/2}\hat{\sigma}_X\hat{\sigma}_\eta, \quad (10.35)$$

respectively.

(5) The vector and axial-vector fields are treated similarly. Just as in the case of π–A_1 mixing treated in our earlier discussion of $SU(2)\times SU(2)$, the spin-0 kinetic-energy terms also contribute to the masses of the axial-vector mesons and to the mass of the K^* vector meson. In addition we must take into account the contribution to the spin-1 kinetic-energy terms that comes from $\mathfrak{L}_2$ [Eq. (10.6)], which implies that here too the kinetic-energy terms have to be renormalized.* After some computation one finds that the vector-meson kinetic-energy terms are

$$-\tfrac{1}{4}(1-\xi a^2)(F_\pi^{\mu\nu})^2-\tfrac{1}{4}[1-\tfrac{1}{2}\xi(a^2+b^2)](F_K^{\mu\nu})^2-\tfrac{1}{4}[1-\tfrac{1}{2}\xi(a^2+b^2)+\tfrac{1}{6}(b^2-a^2)(\mu_V-\lambda_V)](F_\eta^{\mu\nu})^2-\tfrac{1}{4}[1-\tfrac{1}{2}\xi(a^2+b^2)+\tfrac{1}{6}(b^2-a^2)(\mu_V+\lambda_V)](F_X^{\mu\nu})^2 \quad (10.36)$$

with

$$\mu_V=3\xi_V/(b^2-a^2), \quad \lambda_V=[8\xi^2+(\xi+\mu_V)^2]^{1/2}. \quad (10.37)$$

The fields U_X, U_η that enter into $F_{\mu\nu}{}^i$ are related to U_0, U_8 by the mixing angle found in Sec. VII [see Eqs. (7.29), (7.30), (7.31)], with μ_V replacing α:

$$U_8=\cos\theta U_\eta+\sin\theta U_X,$$
$$U_0=-\sin\theta U_\eta+\cos\theta U_X, \quad (10.38)$$
$$\tan 2\theta=2\sqrt{2}\xi/(\xi+\mu_V). \quad (10.39)$$

Actually, with the exception of U_K, the U and V fields are identical. The axial-vector kinetic-energy terms and mixing are obtained by replacing ξ_V by ξ_A, and hence μ_V by μ_A in the above expressions. Analogously to Eq. (10.31) the renormalized spin-1 fields are given by

$$\hat{V}_\pi=Z_{V\pi}^{-1/2}U_\pi,$$
$$\hat{A}_\pi=Z_{A\pi}^{-1/2}A_\pi,$$
$$\vdots \quad (10.40)$$

and

$$Z_{V\pi}^{-1}=Z_{A\pi}^{-1}=1-t,$$
$$Z_{VK}^{-1}=Z_{AK}^{-1}=1-\tfrac{1}{2}(1+r^2)t,$$
$$Z_{V\eta}^{-1}=Z_{VK}^{-1}-\tfrac{1}{6}(r^2-1)([8t^2+(t+s_V)^2]^{1/2}-s_V),$$
$$Z_{VX}^{-1}=Z_{VK}^{-1}+\tfrac{1}{6}(r^2-1)([8t^2+(t+s_V)^2]^{1/2}+s_V), \quad (10.41)$$

where $t=\xi a^2$, $s_V=\mu_V a^2$, and $s_A=\mu_A a^2$. For $Z_{A\eta}^{-1}$ and Z_{AX}^{-1}, replace s_V by s_A in $Z_{V\eta}^{-1}$ and Z_{VX}^{-1}. With the exception of the K^* meson, the vector-meson masses are only changed by the renormalization constants

$$m_V^2=Z_V m_0^2 \qquad [V=\pi,\eta,X(\equiv\rho,\phi,\omega)],$$
$$m_{K^*}^2=Z_{VK}\Gamma_\kappa m_0^2; \quad (10.42)$$

the axial-vector mesons have masses given by

$$m_{A\pi}^2=Z_{V\pi}\Gamma_\pi m_0^2,$$
$$m_{AK}^2=Z_{VK}\Gamma_K m_0^2, \quad (10.43)$$

while $m_{A\eta}^2$ and m_{AX}^2 are obtained by solving

$$m_{AX}^2+m_{A\eta}^2=\tfrac{1}{2}m_0^2(\Gamma_\pi+\Gamma_X)(Z_{AX}+Z_{A\eta})+m_1^2(Z_{AX}-Z_{A\eta}),$$
$$m_{AX}^2-m_{A\eta}^2=\{\tfrac{1}{4}(\Gamma_\pi+\Gamma_X)^2(Z_{AX}-Z_{A\eta})^2m_0^4+[32Z_{A\eta}Z_{AX}\mu_A^2/(9\xi+\mu_A)^2]m_1^4\}^{1/2},$$
$$m_1^2/m_0^2=\tfrac{1}{6}(r^2-1)f^2(9\xi+\mu_A)[8\xi^2+(\xi+\mu_A)^2]^{-1/2}\times[1+(\eta_1-\eta_2)(b^2+a^2)]. \quad (10.44)$$

(6) Before discussing these various mass formulas, it will be valuable to use the results of the last two paragraphs to discuss the vacuum-to-one-particle matrix elements of the currents, i.e., f_π, f_K, f_κ, F_ρ, etc. The quantities f_π and f_K can easily be obtained by combining Eqs. (8.17), (8.19), (10.8), (10.31), and (10.33). In fact they are given by just Eq. (9.23) with a correction

* For simplicity we have omitted terms like $-\xi_V'/12(\text{Tr } V_\mu)^2$ and $-\xi_A'/12(\text{Tr } A_\mu)^2$ from $\mathfrak{L}_2$.

made for the renormalization of the fields:

$$f_\pi=\sqrt{2}aZ_{\phi\pi}^{-1/2},$$
$$f_K=(1/\sqrt{2})(a+b)Z_{\phi K}^{-1/2}. \tag{10.45}$$

We omit the expressions for f_η and f_X here since they are not presently observable. The scalar K decay constant, f_κ is similarly found to be [see Eq. (9.25)]

$$f_\kappa=(1/\sqrt{2})(a-b)Z_{\sigma K}^{-1/2}. \tag{10.46}$$

A glance at Eq. (9.27) shows that $f_+(0)$, the zero-momentum-transfer K_{e3} form factor, is no longer unity because of the renormalizations of the fields. It should be noted that (9.27), obtained by Glashow and Weinberg, must still be valid for the super Lagrangian. The relation $f_+(0)=(f_\pi^2+f_K^2-f_\kappa^2)/2f_Kf_\pi$ follows if one assumes PCAC (or the use of the divergence of the axial currents to extrapolate to zero π and K masses), $SU(3)\times SU(3)$, current commutation relations, and, most important, that $f_+(0)$, when extrapolated off the π and K mass shells, does not depend on the extrapolated values of these masses. In an effective-Lagrangian calculation of $f_\pm(0)$ in the tree-graph approximation, only the graph involving the K^*, but not the kappa, σ_K, contributes to f_+. It is easy to see from the Lagrangian that the $K^*K\pi$ interaction must have the structure

$$g_1K_\mu^*(\phi_K\overleftrightarrow{\partial^\mu}\phi_\pi)+g_2(\partial_\mu K_\nu^*-\partial_\nu K_\mu^*)\partial^\mu\phi_K\partial^\nu\phi_\pi,$$

analogous to the structure of the $\rho\pi\pi$ interaction in Sec. VI. Direct calculations show that such terms do not yield any K or π mass dependence in f_+, though they do give a mass dependence (proportional to $p_K^2-p_\pi^2$) for f_-. In this connection it is important to note that the PCAC conditions $\partial^\mu j_{5\mu}^\pi=m_\pi^2f_\pi\hat{\phi}_\pi$, etc., which follow from the effective Lagrangian do not ensure the absence of a dependence on the masses of the pseudoscalar mesons for every extrapolation.

Consequently, the experimental decay rates for $\pi_{\mu2}$, $K_{\mu2}$, and K_{e3} [Eq. (9.28)] imply that*

$$\tfrac{1}{2}[1+(f_\pi^2/f_K^2)-(f_\kappa^2/f_K^2)]=0.82\ (\pm5\%) \tag{10.47}$$

if the axial and vector Cabibbo angles are taken as equal. This numerical condition, we shall see, severely restricts the values of the parameters in our Lagrangian.

Consider next the matrix elements of the currents connecting the vacuum to one-particle spin-1 states. The currents are given by the field algebra

$$j_\mu{}^i=-(m_0^2/\gamma_0)V_\mu{}^i,$$
$$j_{5\mu}{}^i=-(m_0^2/\gamma_0)\mathfrak{A}_\mu{}^i, \tag{10.48}$$

where $V_\mu{}^i$, $\mathfrak{A}_\mu{}^i$ are the original unrenormalized and unmixed fields. We examine the case of vector currents; similar results follow for the axial currents.* In terms of the renormalized, physical vector fields, Eq. (10.48) becomes

$$j_\mu{}^i=-(m_0^2/\gamma_0)Z_{V\pi}{}^{1/2}\hat{V}_\mu{}^i,\qquad i=1,2,3(\rho),$$
$$j_\mu{}^i=-(m_0^2/\gamma_0)Z_{VK}{}^{1/2}V_\mu{}^i+(m_0^2/\gamma_0)(2/\sqrt{3})\kappa$$
$$\times(a-b)f_{ij8}\Delta_\mu\sigma_j{}',\qquad i=4,\cdots7(K^*,\kappa),$$
$$j_\mu{}^8=-(m_0^2/\gamma_0)[Z_{V\eta}{}^{1/2}\cos\theta\hat{V}_\mu{}^\eta+Z_{VX}{}^{1/2}\sin\theta\hat{V}_\mu{}^X]$$
$$(\omega,\phi),$$
$$j_\mu{}^0=-(m_0^2/\gamma_0)[-Z_{V\eta}{}^{1/2}\sin\theta\hat{V}_\mu{}^\eta+Z_{VX}{}^{1/2}\cos\theta\hat{V}_\mu{}^X] \tag{10.49}$$

with $\tan 2\theta=2\sqrt{2}t/(t+s_V)$. With the constants F_ρ, F_{K^*}, etc. (sometimes called G_ρ, $G_{K^*},\cdots$, g_ρ, $g_{K^*},\cdots$, m_ρ^2/γ_ρ, $\cdots$, etc.), defined in the usual way,

$$\langle 0\,|\,j_\mu{}^{(i)}\,|\,\rho^i,\epsilon\rangle=F_\rho\epsilon_\mu,\ \langle 0\,|\,j_\mu{}^{(8)}\,|\,\phi,\epsilon\rangle=F_\phi\epsilon_\mu,\ \text{etc.} \tag{10.50}$$

Eq. (10.49) yields

$$F_\rho=-Z_{V\pi}{}^{1/2}(m_0^2/\gamma_0),$$
$$F_{K^*}=-Z_{VK}{}^{1/2}(m_0^2/\gamma_0),$$
$$F_\omega=-Z_{VX}{}^{1/2}\sin\theta(m_0^2/\gamma_0),$$
$$F_\phi=-Z_{V\eta}{}^{1/2}\cos\theta(m_0^2/\gamma_0). \tag{10.51}$$

We assume that the hadronic electromagnetic current is given by $j_\mu{}^{(3)}+(1/\sqrt{3})j_\mu{}^{(8)}$ and therefore omit consideration of matrix elements of $j_\mu{}^{(0)}$. With this assumption F_ρ^2, F_ω^2, F_ϕ^2 can be obtained from the leptonic decay rates of the vector mesons, e.g., $\rho\to e^+e^-$. With the help of the mass relations (10.42) we can calculate ratios like $(Z_{V\eta}/Z_{V\pi})^{1/2}$ and finally obtain

$$F_\rho=-m_\rho m_0/\gamma_0,$$
$$F_{K^*}=-m_{K^*}m_0/\Gamma_\kappa\gamma_0,$$
$$F_\omega=-\sin\theta(m_\omega m_0/\gamma_0),$$
$$F_\phi=-\cos\theta(m_\phi m_0/\gamma_0). \tag{10.52}$$

The coupling constant γ_ρ is defined by† $F_\rho=-m_\rho^2/\gamma_\rho$, so that $m_0/\gamma_0=m_\rho/\gamma_\rho$. Equation (10.52) is not surprising if we note that the c-number Schwinger terms in our Lagrangian are just F_ρ^2/m_ρ^2 and $(F_\phi^2/m_\phi^2)+(F_\omega^2/m_\omega^2)$ for the $i=1,2,3$ and $i=8$ vector currents, respectively. From (10.52) it follows that

$$\frac{F_\phi}{F_\rho}=\left(\frac{m_\phi}{m_\rho}\right)\cos\theta,\qquad \frac{F_\omega}{F_\rho}=\left(\frac{m_\omega}{m_\rho}\right)\sin\theta, \tag{10.53}$$

* The experimental errors in the decay rates are small (<2%) but the radiative corrections are not known and we estimate that they could make the right-hand side of (10.47) uncertain by as much as 5%.

* Equation (10.48) extends to the unitary singlet currents in the absence of terms like $(\mathrm{Tr}\,V_\mu)^2$ in the Lagrangian.

† Some authors use a γ_ρ which is one-half the value defined here.

so that θ is often referred to as the generalized mixing angle; it differs from the standard angle because of renormalization effects and is fixed by the two mass ratios m_ρ^2/m_ϕ^2 and m_ρ^2/m_ω^2 which determine t and s_V (or ξ and μ_V) by (10.41) and (10.42). With new variables defined by

$$x=\tfrac{1}{2}(r^2-1)[t/(1-t)],$$
$$y=\tfrac{1}{2}(r^2-1)[s_V/(1-t)],$$

we obtain

$$\tfrac{1}{2}[(m_\rho^2/m_\omega^2)+(m_\rho^2/m_\phi^2)]=1-x+\tfrac{1}{3}y,$$
$$[(m_\rho^2/m_\omega^2)-(m_\rho^2/m_\phi^2)]=\tfrac{1}{3}[8x^2+(y+x)^2]^{1/2}. \quad (10.54)$$

The solution of this equation is straightforward, though sensitive to $m_\rho^2-m_\omega^2$. With the presently accepted values $m_\rho=765$ MeV, $m_\omega=783$ MeV, and $m_\phi=1019$ MeV, it turns out that $\theta=40.3°$. This is to be compared with the standard nonet angle $\theta=35.3°$ for $m_\rho=m_\omega$ (i.e., $s_V=0$). A generalized mixing angle of 40° agrees quite well with the present experimental value obtained from ρ, ω, and ϕ leptonic decay rates measured at Orsay and Desy, $\theta_{\text{exptl}}\cong 40°$ (Vienna, 1968). The "super-Lagrangian" predictions for F_ρ, F_ω, and F_ϕ come close to those of Das *et al.* (Das, Mathur, and Okubo, 1967), who assumed the equality of all the vector Schwinger terms, assumed a Gell-Mann–Okubo type of relation for the F_V^2, and neglected all 0^+ contributions to the V_μ^K Schwinger terms (effectively taking $f_\kappa^2=0$).

The importance of the 0^+ contributions can more accurately be gauged if we note that the assumptions that go into the construction of the "super Lagrangian" correspond very closely to those made by Oakes and Sakurai (Oakes and Sakurai, 1967), who take a kinetic-energy mixing-mass formula in addition to the Weinberg sum rule and treat $f_\kappa^2=0$. The equation that they obtain for the generalized mixing angle is

$$\frac{\cos^2\theta}{m_\phi^2}+\frac{\sin^2\theta}{m_\omega^2}=\frac{4}{3m_{K^*}^2}-\frac{1}{3m_\rho^2}, \quad (10.55)$$

which yields $\theta=28°$. The "super Lagrangian" mixing angle satisfies a similar equation [these results only follow if there is no $(\text{Tr }V_\mu)^2$ mass term in the Lagrangian]:

$$\frac{\cos^2\theta}{m_\phi^2}+\frac{\sin^2\theta}{m_\omega^2}=\frac{4\Gamma_\kappa}{3m_{K^*}^2}-\frac{1}{3m_\rho^2}, \quad (10.56)$$

obtained by combining (10.39), (10.41), and (10.42). The two equations coincide for $\Gamma_\kappa=1$. That this indeed corresponds to $f_\kappa^2=0$ can be seen as follows: Writing

$$\Gamma_\kappa=1+\epsilon \qquad (\epsilon\simeq 0.08) \quad (10.57)$$

(the numerical value follows from the value of θ and the masses), we find that f_κ^2 is proportional to ϵ by equating the $V_\mu^{(K)}$ and $V_\mu^{(\pi)}$ Schwinger terms,

$$(F_{K^*}^2/m_{K^*}^2)+f_\kappa^2=F_\rho^2/m_\rho^2, \quad (10.58)$$

and inserting the expressions for F_{K^*} and F_ρ from (10.52) which yields

$$f_\kappa^2=(F_\rho^2/m_\rho^2)[\epsilon/(1+\epsilon)]. \quad (10.59)$$

If we write $F_\rho=m_\rho^2/\gamma_\rho$, we may cast the above equation into the form

$$f_\kappa^2/f_\pi^2=(m_\rho^2/2f_\pi^2\gamma_\rho^2)[2\epsilon/(1+\epsilon)]. \quad (10.60)$$

The first factor on the right-hand side is unity when the KSFR relation holds, or when the A_1 mass is used [Eq. (10.43) or Eq. (6.13) with γ_0 replaced by γ_ρ], it is given by

$$m_\rho^2/2f_\pi^2\gamma_\rho^2=m_{A\pi}^2/2(m_{A\pi}^2-m_\rho^2). \quad (10.61)$$

The present experimental value for γ_ρ ($\gamma_\rho^2/4\pi\approx 2$) corresponds to a value of $m_{A\pi}\simeq 980$ MeV rather than $m_{A\pi}\simeq\sqrt{2}m_\rho=1080$ MeV, but the experimental situation for both $m_{A\pi}$ and γ_ρ is still uncertain.

The value of f_κ^2/f_π^2 is thus of the order of 0.16. This is much smaller than the estimate of Glashow and Weinberg (Glashow and Weinberg, 1968), who obtain $f_\kappa^2/f_\pi^2\simeq 0.34$ by setting $m_{A\pi}=\sqrt{2}m_\rho$ and $m_{AK}=\sqrt{2}m_{K^*}$. As pointed out earlier, all the equations used by Glashow and Weinberg hold for the "super Lagrangian" so that our relations for the much better known vector-meson masses tend to rule out the axial-vector mass choices made by them, though the addition of a $(\text{Tr }V_\mu)^2$ term would give us more freedom to increase ϵ. As we shall see, and as also noted by Glashow and Weinberg, an increase in f_κ^2 leads to a lower value of the kappa mass. In view of the lack of evidence for a lower-mass kappa and a suggestion that the $K\pi$ 0^+, $T=\tfrac{1}{2}$ phase shift rises through 90° in the 1–1.2-GeV region, we prefer to use the low value of ϵ to fit the parameters in the "super Lagrangian."

(7) Let us now consider the problem of fitting the spin-0 masses using Eqs. (10.32) and (10.33) and the results of paragraph (6). The equations are really simpler than they look. Using our result that $m_0/\gamma_0=m_\rho/\gamma_\rho$ we see that (10.45) implies $f^2=\gamma_\rho^2 f_\pi^2 Z_{\phi\pi}/m_\rho^2$, so that if we define

$$\Lambda_\pi=1+(\eta_1-\eta_2)a^2, \quad (10.62)$$

then Γ_π and $Z_{\phi\pi}$ become

$$\Gamma_\pi=1+f^2\Lambda_\pi; \qquad Z_{\phi\pi}=(1+f^2\Lambda_\pi)/\Lambda_\pi.$$

We may thus write

$$f^2=\frac{f_\pi^2\gamma_\rho^2/m_\rho^2}{1-f_\pi^2\gamma_\rho^2/m_\rho^2}\frac{1}{\Lambda_\pi}. \quad (10.63)$$

We saw in the last paragraph that $f_\pi^2\gamma_\rho^2/m_\rho^2$ may be related to experiment or to $m_{A\pi}^2/m_\rho^2$; in view of the experimental uncertainties it is probably safe to say that $f_\pi^2\gamma_\rho^2/m_\rho^2$ is in the neighborhood of $\tfrac{1}{2}$ to within 25%.

We define the parameter

$$A=\frac{f_\pi^2\gamma_\rho^2/m_\rho^2}{1-f_\pi^2\gamma_\rho^2/m_\rho^2}=\frac{m_{A\pi}^2}{m_\rho^2}-1 \qquad (10.64)$$

which can vary between 1 and $\frac{2}{3}$. Another parameter that we know something about from the work in the last paragraph is $\Gamma_\kappa=1+\epsilon$; the vector-meson mass equations yield $\epsilon\simeq 0.08$; we may want to vary it a little but still keep it small. As a third input we shall take

$$f_\pi f_+(0)/f_K\equiv 1-\omega=0.82\pm0.04. \qquad (10.65)$$

Folding in the arbitrarily chosen 5% error discussed in the last paragraph, we see that ω can vary between 0.14 and 0.22. With the help of Eq. (9.27) for $f_+(0)$ and (10.60) for f_κ^2/f_π^2, we may use (10.64) to express f_K^2/f_π^2 in terms of ϵ, A, and ω. Indeed ϵ, A, and ω determine all parameters of Eq. (10.32), except for $r=b/a$. Although the three input parameters vary only over relatively narrow ranges, it turns out that the solutions are particularly sensitive to their values (especially ϵ and ω). The requirement that the renormalization constants Z_ϕ and Z_σ remain positive (and hence the renormalized fields Hermitian) severely restricts the allowable range of the parameter r. This can be seen as follows: Let us rewrite our equations in simplified form by introducing the quantities

$$R=\tfrac{1}{4}(r+1)^2[1+\tfrac{1}{2}(r^2-1)(\eta_1-\eta_2)a^2/\Lambda_\pi],$$
$$Q=\tfrac{1}{4}(r+1)^2[\tfrac{1}{2}(r-1)^2\eta_2a^2/\Lambda_\pi],$$
$$\chi=(r-1)/(r+1), \qquad (10.66)$$

which satisfy the following relations

$$\chi^2R+Q=\frac{\epsilon}{A};\qquad R+Q=\frac{f_K^2/f_\pi^2}{1-A(f_K^2/f_\pi^2-1)};$$
$$\frac{f_K^2}{f_\pi^2}=\frac{1-\epsilon/A}{(1-2\omega)(1+\epsilon)}. \qquad (10.67)$$

We can thus solve of R and Q in terms of ϵ, A, and ω. The definition (10.62) of Λ_π implies that

$$\frac{4R}{(r+1)^2}=\frac{1}{2}\left[r^2+1-\frac{r^2-1}{\Lambda_\pi}\right]=\frac{R+Q-\epsilon/A}{r}\equiv\frac{N_0}{r}. \qquad (10.68)$$

The requirement that $\Lambda_\pi>0$ [see (10.63)] leads to the inequality

$$r(r^2+1)>2N_0. \qquad (10.69)$$

As an example of the scale involved, we see that with ϵ small ($\epsilon\lesssim 0.10$), f_K^2/f_π^2 tends to be close to 1.5, so that with A close to unity, N_0 varies between 2 and 4. For the lower value, (10.69) then implies that $r>1.4$. We cannot choose r very much larger because then some other renormalization constant may become negative. The first one to do so is $Z_{\sigma X}$. After some tedious algebra we find the following inequality for r:

$$(4r+2-2r^2)N_0-r(r+1)^2+[16r^2/(r-1)^2](\epsilon/A)>0. \qquad (10.70)$$

To illustrate the implications of this let us set $r=2$. Then (10.70) and (10.69) combined imply that $\epsilon/A>\frac{1}{8}$. We cannot vary N_0 as much as we like since ω is so restricted and N_0 decreases as ϵ increases. Hence with $\epsilon\sim 0.08$ and $A\sim 1$ we may expect r to lie in the range $1.4<r<1.8$.*

The restrictions on the parameters arise from the conflict between the fact that the vector-meson masses come close to fitting a nonet theory and the very large deviation of $f_\pi f_+(0)/f_K$ from unity (if a single Cabibbo angle is assumed). The effective Lagrangian does not naturally lead to a nonet theory unless symmetry-breaking effects are small, while the value of $f_\pi f_+(0)/f_K$ indicates large symmetry breaking. For values of $f_\pi f_+(0)/f_K$ closer to unity, the ranges of parameters become much more acceptable.

Let us now consider the kappa-meson mass since this plays a key role in $SU(3)$ symmetry breaking. We can simplify our algebra by taking $m_\pi^2=0$ in (10.33). We then find

$$\frac{m_{\sigma K}^2}{m_K^2}=\frac{A}{1+A}\frac{f_K^2}{f_\pi^2}\frac{r-1}{r+1}\frac{1+\epsilon}{\epsilon}. \qquad (10.71)$$

With $A/(1+A)\lesssim\frac{1}{2}$, $f_K^2/f_\pi^2\lesssim 1.6$, and $r\lesssim 1.8$ we have

$$m_{\sigma K}^2/m_K^2\lesssim 0.23(1+\epsilon)/\epsilon.$$

For $\epsilon=0.08$ given by the vector-meson masses, $m_{\sigma K}<$ 865 MeV. The much larger value for ϵ (and smaller value for f_K^2/f_π^2) required by Glashow and Weinberg yields a lower value for this bound. If we want $m_{\sigma K}\gtrsim$ 1 GeV, we must take $\epsilon<0.08$, i.e., relax somewhat the precise fit of the vector-meson masses. This is reasonable: for example, a value of $\epsilon=0.05$ would be consistent with Eq. (10.65) with nonet symmetry ($\theta=35°$ is within the experimental errors) and $m_\phi=1019$ MeV, $m_{K^*}=$ 891 MeV, and $m_\rho=m_\omega=770$ MeV. If one further chooses $A=1$, i.e., $m_{A\pi}=1080$ MeV and $\omega=0.2$, $r=1.7$, then one finds $m_{\sigma K}^2=4.1m_K^2$. The K and π masses can be fitted, but the $T=0$ mesons are slightly off: $m_\eta=565$ MeV, $m_X=948$ MeV. A computer search of solutions was made with $A=1$, fitting the pseudoscalar and vector masses (but taking $s_V=0$ so that $m_\omega=m_\rho$) and the pseudoscalar meson masses (allowing a 1% variation in the η and X^0 masses). The value taken for ϵ is flexible, however, since it is so sensitive to the value taken for the ρ mass. $f_K/f_\pi f_+(0)$ was varied between 1.15 and 1.25

* It is interesting to note that $|\langle 0|\sigma_8|0\rangle/\langle 0|\sigma_0|0\rangle|=\sqrt{2}|(1-r)/(2+r)|\sim\frac{1}{4}$, i.e., the octet tadpole is quite a bit smaller than the chiral symmetry violating, SU(3) preserving tadpole (Gell-Mann, Oakes, and Renner, 1968).

and we required that $m_{\sigma\pi} \geq 900$ MeV. The only acceptable solutions with $m_{\sigma K} \geq 1$ Gev were found with $f_K/f_\pi f_+(0) \leq 1.21$. In all cases, $1-f_+(0) \sim 0.02$ showing that the second-order $SU(3)$ symmetry-breaking deviation of $f_+(0)$ from 1 (Ademollo and Gatto, 1964) is very small. The fact that $f_+(0) < 1$ is in agreement with the arguments of Bjorken and Quinn (1968). None of these solutions can be regarded as entirely satisfactory since the K_A mass is consistently found close to a value of 1.65 GeV, quite far from the presently favored range of 1.2 to 1.3 GeV. Taking a smaller value for the A_1 mass, however, would greatly improve the value for m_{AK}. It was not possible to find any solutions when the X is identified as the $E(1420)$. Given the sensitivity of the equations to the input data, the results of a computer search are not very meaningful, at least until better values of the scalar and axial meson masses are obtained; perhaps fits to decay widths will be helpful.

It may be that the trouble lies with the conventional approach, which ignores representation mixing [for a discussion of this see Gilman and Harari (1968)]: in the symmetry limit, the spin 0 mesons fit into a $(\mathbf{3}, \bar{\mathbf{3}}) \oplus (\bar{\mathbf{3}}, \mathbf{3})$ representation (nonet), while the spin 1 mesons fit into the $(\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8})$ representation, which is not a nonet. The observed facts, $m_\omega \simeq m_\rho$, $m_\pi \ll m_\eta$, m_X are quite different, a situation that seems to be difficult for the "super Lagrangian" to adjust to. There certainly seem to be limitations to the accuracy to which these methods may be used in the study of symmetry breaking in $SU(3) \times SU(3)$.

XI. MISCELLANEOUS TOPICS

A. Coupling to Photons

There is increasing evidence for the utility of the notion of the "vector dominance" of the electromagnetic current, i.e., of the connection between the processes

$$\gamma + A \to B$$

and

$$\begin{pmatrix} \rho \\ \omega \\ \phi \end{pmatrix}_{\text{transverse}} + A \to B.$$

This connection is implicit in what we have done, as long as we (i) accept the identification of the hadronic electric current in terms of $SU(3)$ generating currents (Coleman and Glashow, 1961; Cabibbo and Gatto, 1961)

$$j_\mu^{(\text{el})} = e[j_\mu^{(3)} + (1/\sqrt{3}) j_\mu^{(8)}] \tag{11.1}$$

with a possible additional unitary singlet term (Nauenberg, 1964) and (ii) implement vector dominance of the $SU(3)$ currents, as was done in Secs. IV and X, for example. In processes involving more than one photon, gauge invariance of the photon couplings is not trivial (e.g., the need for "sea-gull graphs" in Compton scattering), and it is therefore useful to exhibit explicitly how electromagnetic couplings enter effective Lagrangians.

Recall that the transformation law for the vector mesons, obtained in Sec. VII, is

$$\delta V_\mu = (1/\gamma_0) \partial_\mu \alpha + (i/\sqrt{2}) [\alpha, V_\mu]. \tag{11.2}$$

When the "direction" of α is in the (3, 8) plane, then the transformation law for the vector mesons V^3 and V^8 simplifies to

$$\delta V_\mu^k = (1/\gamma_0) \partial_\mu \alpha^k; \qquad k = 3, 8. \tag{11.3}$$

The electromagnetic potentials $e\Phi_\mu$ must appear in the Lagrangian in such a way that under the transformation

$$e\Phi_\mu \to e\Phi_\mu + \partial_\mu \chi, \tag{11.4}$$

the Lagrangian remains invariant. A trivial way to satisfy this is by using the field strengths $\Phi_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu$ as in the free-photon Lagrangian

$$\mathcal{L}_\gamma = -\tfrac{1}{4} \Phi_{\mu\nu} \Phi^{\mu\nu}. \tag{11.5}$$

Another "minimal" coupling is obtained by noting that if

$$\alpha^k = C^k \chi, \tag{11.6}$$

then the combinations

$$V_\mu^k - (eC^k/\gamma_0) \Phi_\mu; \qquad k = 3, 8 \tag{11.7}$$

are invariant under a combination of (11.4) and (11.2) with the special choice of vector gauge function given by (11.6). Now the transformation (11.2) leaves all but the vector-meson mass terms invariant. Hence the gauge-invariant coupling of the electromagnetic field may be achieved by replacing $\tfrac{1}{2} m_0^2 (V_\mu^{(k)})^2$ by $\tfrac{1}{2} m_0^2 (V_\mu^{(k)} - e/\gamma_0 C^{(k)} \Phi_\mu)^2$ for $k = 3$, 8 and adding $-\tfrac{1}{4}(\partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu)^2$ to the Lagrangian. With the usual definition of the electric current by

$$j_\mu^{(\text{el})} = \partial \mathcal{L} / \partial \Phi^\mu, \tag{11.8}$$

we have

$$j_\mu^{(\text{el})} = -\frac{eC^{(3)}}{\gamma_0} \frac{\partial \mathcal{L}}{\partial V^{\mu(3)}} - \frac{eC^{(8)}}{\gamma_0} \frac{\partial \mathcal{L}}{\partial V^{\mu(8)}}$$
$$= -\frac{eC^{(3)} m_0^2}{\gamma_0} V_\mu^{(3)} - \frac{eC^{(8)} m_0^2}{\gamma_0} V_\mu^{(8)} + \cdots. \tag{11.9}$$

For photon–vector-meson couplings (corresponding to calculating $\langle V | j_\mu^{(\text{el})} | 0 \rangle$ in the "tree-graph" approximation), we see that since

$$j_\mu^{(k)} = -(m_0^2/\gamma_0) V_\mu^{(k)} \tag{11.10}$$

the condition (11.1) requires that we choose

$$C^{(3)} = 1; \qquad C^{(8)} = 1/\sqrt{3}. \tag{11.11}$$

The direct coupling from the mass term leads, in general, to a photon–vector-meson "mixing." This is diagonalized by the introduction of new vector-meson fields corresponding to the unitary spin components $k=3$ and 8. We forego a detailed discussion and instead refer the reader to the papers of Kroll, Lee, and Zumino (1967), Lee and Zumino (1967), Schwinger (1964; 1968), Lee and Nieh (1968), Wick and Zumino (1967), and Gerstein, Lee, Nieh, and Schnitzer (1967) for further study.

B. Baryons

In order to do justice to the subject matter in a paper of reasonable length, we have concentrated on treating systems of spin 0 and 1 in great detail. Since many applications of effective Lagrangians involve spin-$\frac{1}{2}$ particles, and since there is a very large literature dealing with this subject, we feel that completeness demands a brief description of the complications caused by spin-$\frac{1}{2}$ particles.

A spin-$\frac{1}{2}$ octet transforms as follows under $SU(3)$:

$$\delta\psi_A = -i\psi_B(\alpha\cdot F)_{BA} = -f_{ACB}\alpha_C\psi_B. \quad (11.12)$$

We take the transformation law for the chiral transformation to be

$$\delta'\psi_A = -f_{ACB}\alpha_C\gamma_5\psi_B. \quad (11.13)$$

We may combine the two transformation laws and write them in 3×3 matrix form

$$\delta\psi_\pm = (i/\sqrt{2})[\alpha, \psi_\pm] \quad (11.14)$$

and

$$\delta'\psi_\pm = \pm(i/\sqrt{2})[\alpha, \psi_\pm], \quad (11.15)$$

where

$$\psi_\pm = (1\pm\gamma_5)\psi. \quad (11.16)$$

Note that

$$\bar{\psi}_\pm = \bar{\psi}(1\mp\gamma_5) \quad (11.17)$$

and it transforms like $\psi_\pm$, i.e.,

$$\delta\bar{\psi}_\pm = (i/\sqrt{2})[\alpha, \bar{\psi}_\pm] \quad (11.18)$$

and

$$\delta'\bar{\psi}_\pm = \pm(i/\sqrt{2})[\alpha, \bar{\psi}_\pm]. \quad (11.19)$$

Thus it is easy to see that terms like Tr $(\bar{\psi}_\pm\psi_\pm)$ are chiral invariant. Since $\gamma_5^2=1$ it follows that

$$\bar{\psi}_\pm\psi_\pm = 0. \quad (11.20)$$

Terms like

$$\bar{\psi}_\pm\gamma^\mu\partial_\mu\psi_\pm,\ \bar{\psi}_\pm\gamma_\mu\psi_\pm, \quad (11.21)$$

do not vanish. Thus, as is well known, mass terms, which, unlike kinetic-energy terms, appear in the combination $\bar{\psi}_+\psi_- + \bar{\psi}_-\psi_+$, break chiral symmetry. This does not cause any serious difficulties because we may use the spin 0 fields B and $B^\dagger$ [Eq. (7.52)] to construct spin $\frac{1}{2}$ objects for which mass terms emerge when chiral symmetry is broken through:

$$\sigma \to \sigma' + \Sigma. \quad (11.22)$$

We now observe that

$$\delta'\psi_+ B = (i/\sqrt{2})[\alpha, \psi_+]B + (i/\sqrt{2})\psi_+\{\alpha, B\}$$
$$= (i/\sqrt{2})\{\alpha, \psi_+ B\} \quad (11.23)$$

and

$$\delta' B\psi_- = (i/\sqrt{2})\{\alpha, B\psi_-\}; \quad (11.24)$$

i.e., these combinations transform like B. Similarly,

$$\delta' B^\dagger\psi_+ = -(i/\sqrt{2})\{\alpha, B^\dagger\psi_+\} \quad (11.25)$$

and

$$\delta'\psi_- B^\dagger = -(i/\sqrt{2})\{\alpha, \psi_- B^\dagger\}, \quad (11.26)$$

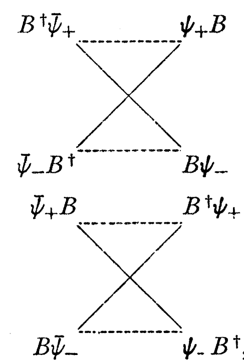
so that these combinations transform like $B^\dagger$. The adjoints of these four combinations are

$$B^\dagger\bar{\psi}_+,\ \bar{\psi}_- B^\dagger \sim B^\dagger \quad (11.27)$$

and

$$\bar{\psi}_+ B,\ B\bar{\psi}_- \sim B. \quad (11.28)$$

Possible invariants are schematically shown below



(11.29)

where the solid lines indicate "mass"-type invariants such as

$$\mathrm{Tr}\, B^\dagger\bar{\psi}_+ B\psi_-, \qquad \mathrm{Tr}\, \bar{\psi}_- B^\dagger\psi_+ B,$$

and the dotted lines indicate "kinetic-energy"-type invariants such as

$$\mathrm{Tr}\, B^\dagger\bar{\psi}_+\gamma^\mu\partial_\mu\psi_+ B, \cdots$$

or terms which are invariants when coupled to vector or axial mesons such as

$$\mathrm{Tr}\,[B^\dagger\bar{\psi}_+\gamma^\mu\psi_+ B(V_\mu - \mathcal{A}_\mu)] \quad (11.30)$$

and

$$\mathrm{Tr}\,[B^\dagger\bar{\psi}_+\sigma_{\mu\nu}B\psi_-(F^{\mu\nu} - G^{\mu\nu})].$$

Note that, in addition to $\bar{\psi}_\pm\psi_\pm = 0$, $\bar{\psi}_\pm\sigma_{\mu\nu}\psi_\pm = 0$ as well. The number of invariant couplings is restricted by parity:

$$P\psi_\pm P^{-1} = \psi_\mp,$$
$$PBP^{-1} = B^\dagger. \quad (11.31)$$

We may also use nonlinear realizations for the spin-0 fields.

When parity is taken into account, there is only one possible "mass"-type term,

$$\tfrac{1}{4}\,\mathrm{Tr}\,(\bar{\psi}_+B\psi_-B^\dagger+\bar{\psi}_-B^\dagger\psi_+B)$$
$$=\mathrm{Tr}\,(\bar{\psi}\sigma\psi\sigma+\bar{\psi}\phi\psi\phi)$$
$$+i\,\mathrm{Tr}\,(\bar{\psi}\gamma_5\sigma\psi\phi-\bar{\psi}\gamma_5\phi\psi\sigma). \quad (11.32)$$

This term does give a contribution to the mass when we let $\sigma\rightarrow\sigma'+\Sigma$. If we wish to preserve field algebra, we must use covariant derivatives for the "kinetic-energy"-type terms. We note that (7.47) implies that

$$\Delta_\mu(\sigma\pm i\phi)=\partial_\mu(\sigma\pm i\phi)-(i\gamma_0/\sqrt{2})[V_\mu,\sigma\pm i\phi]$$
$$\mp(i\gamma_0/\sqrt{2})\{\mathcal{A}_\mu,\sigma\pm i\phi\} \quad (11.33)$$

transforms like B and $B^\dagger$, respectively. Hence

$$\Delta_\mu(\psi_+B)=\partial_\mu(\psi_+B)-(i\gamma_0/\sqrt{2})[V_\mu,\psi_+B]$$
$$-(i\gamma_0/\sqrt{2})\{\mathcal{A}_\mu,\psi_+B\} \quad (11.34)$$

will, for example, transform like B.

To illustrate the kind of couplings that emerge, let us consider the following, quite general Lagrangian involving baryons and spin zero mesons:

$$\mathcal{L}=\tfrac{1}{4}i\,\mathrm{Tr}\,(\bar{\psi}_+\gamma_\mu\partial^\mu\psi_++\bar{\psi}_-\gamma_\mu\partial^\mu\psi_-)$$
$$+\tfrac{1}{4}a\,\mathrm{Tr}\,(\bar{\psi}_+B\psi_-B^\dagger+\bar{\psi}_-B^\dagger\psi_+B)$$
$$+\tfrac{1}{4}ib_1\,\mathrm{Tr}\,(\bar{\psi}_+\gamma_\mu\psi_+\partial^\mu BB^\dagger+\cdots)$$
$$+\tfrac{1}{4}ib_2\,\mathrm{Tr}\,(\bar{\psi}_+\gamma_\mu\psi_+B\partial^\mu B^\dagger+\cdots)$$
$$+\tfrac{1}{4}ib_3\,\mathrm{Tr}\,(\bar{\psi}_+\partial^\mu BB^\dagger\gamma_\mu\psi_++\cdots)$$
$$+\tfrac{1}{4}ib_4\,\mathrm{Tr}\,(\bar{\psi}_+B\partial^\mu B^\dagger\gamma_\mu\psi_++\cdots)$$
$$+\mathcal{L}_{\mathrm{mesonic}}. \quad (11.35)$$

The coefficient of the leading term can be chosen to be unity by appropriately renormalizing the baryon field. For the same reason we do not gain any generality (for what we want to discuss) by including terms like $\mathrm{Tr}\,(\bar{\psi}_+\gamma_\mu\partial^\mu\psi_+BB^\dagger+\cdots)$. Such terms are necessary, however, if we are to obtain the observed values of the baryon octet masses. When this Lagrangian is written in terms of the σ, ϕ, and ψ fields, and the shift $\sigma\rightarrow\sigma'+\Sigma$ is made, the terms that give the mass and contribute to meson–baryon scattering are

$$i\,\mathrm{Tr}\,(\bar{\psi}\gamma_\mu\partial^\mu\psi)+a\,\mathrm{Tr}\,(\bar{\psi}\Sigma\psi\Sigma)+a\,\mathrm{Tr}\,(\bar{\psi}\phi\psi\phi)$$
$$+a\,\mathrm{Tr}\,(\bar{\psi}\Sigma\psi\sigma')+a\,\mathrm{Tr}\,(\bar{\psi}\sigma'\psi\Sigma)+ia\,\mathrm{Tr}\,(\bar{\psi}\gamma_5\Sigma\psi\phi)$$
$$-ia\,\mathrm{Tr}\,(\bar{\psi}\gamma_5\phi\psi\Sigma)+ib_1\,\mathrm{Tr}\,(\bar{\psi}\gamma_\mu\psi\partial^\mu\phi\phi)$$
$$-b_1\,\mathrm{Tr}\,(\bar{\psi}\gamma_\mu\gamma_5\psi\partial^\mu\phi\Sigma)+ib_2\,\mathrm{Tr}\,(\bar{\psi}\gamma_\mu\psi\phi\partial^\mu\phi)$$
$$+b_2\,\mathrm{Tr}\,(\bar{\psi}\gamma_\mu\gamma_5\psi\Sigma\partial^\mu\phi)+ib_3\,\mathrm{Tr}\,(\bar{\psi}\gamma_\mu\partial^\mu\phi\phi\psi)$$
$$-b_3\,\mathrm{Tr}\,(\bar{\psi}\gamma_\mu\gamma_5\partial^\mu\phi\psi\Sigma)+ib_4\,\mathrm{Tr}\,(\bar{\psi}\gamma_\mu\phi\partial^\mu\phi\psi)$$
$$+b_4\,\mathrm{Tr}\,(\bar{\psi}\gamma_\mu\gamma_5\Sigma\partial^\mu\phi\psi). \quad (11.36)$$

Let us, for the sake of simplicity, restrict ourselves to $SU(2)\times SU(2)$, which we do by writing

$$\psi=\begin{pmatrix}0 & p\\ 0 & n\end{pmatrix};$$

$$B=\begin{pmatrix}(\sigma+\pi^0)/\sqrt{2} & \pi^+\\ \pi^- & (\sigma-\pi^0)/\sqrt{2}\end{pmatrix}=(1/\sqrt{2})\,(\sigma+\boldsymbol{\tau}\cdot\boldsymbol{\phi}). \quad (11.37)$$

We take $\sigma=(f_\pi^2-\boldsymbol{\phi}^2)^{1/2}$; i.e., $\Sigma=(f_\pi/\sqrt{2})I$ and $\sigma'=-\boldsymbol{\phi}^2/2\sqrt{2}f_\pi+\cdots$. The relevant terms in (11.36) now are

$$i\bar{\psi}\gamma_\mu\partial^\mu\psi+\tfrac{1}{2}af_\pi^2\bar{\psi}\psi-\tfrac{1}{2}a\bar{\psi}\psi\boldsymbol{\phi}^2-\tfrac{1}{2}iaf_\pi\bar{\psi}\gamma_5\boldsymbol{\tau}\psi\cdot\boldsymbol{\phi}$$
$$+\tfrac{1}{2}(b_3-b_4)\bar{\psi}\gamma_\mu\boldsymbol{\tau}\psi\cdot\boldsymbol{\phi}\times\partial^\mu\boldsymbol{\phi}$$
$$+\tfrac{1}{2}(b_4-b_3)f_\pi\bar{\psi}\gamma_\mu\gamma_5\boldsymbol{\tau}\psi\cdot\partial^\mu\boldsymbol{\phi}. \quad (11.38)$$

We note the following:

(a) The nucleon mass is

$$m=-\tfrac{1}{2}af_\pi^2. \quad (11.39)$$

(b) Making use of the fact that

$$\delta(\partial^\mu\psi)=\tfrac{1}{2}i\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\alpha}\psi+\cdots,$$
$$\delta'(\partial^\mu\psi)=\tfrac{1}{2}i\boldsymbol{\tau}\cdot\partial^\mu\boldsymbol{\beta}\gamma_5\psi+\cdots,$$
$$\delta'(\partial^\mu\boldsymbol{\phi})=f_\pi\partial^\mu\boldsymbol{\beta}+\cdots, \quad (11.40)$$

we see that the baryonic part of the vector current is

$$\tfrac{1}{2}\bar{\psi}\gamma_\mu\boldsymbol{\tau}\psi \quad (11.41)$$

and the baryonic part of the axial current is

$$\tfrac{1}{2}\bar{\psi}\gamma_\mu\gamma_5\boldsymbol{\tau}\psi[1-(b_4-b_3)f_\pi^2], \quad (11.42)$$

so that

$$G_A/G_V=1-(b_4-b_3)f_\pi^2. \quad (11.43)$$

(c) The coefficient of $i\bar{\psi}\gamma_5\boldsymbol{\tau}\psi\cdot\boldsymbol{\phi}$ obtained from the fourth term of (11.38) and the last one (upon integrating by parts and omitting irrelevant terms) is

$$g=-\tfrac{1}{2}af_\pi-(b_4-b_3)mf_\pi$$
$$=(m/f_\pi)[1-(b_4-b_3)f_\pi^2]$$
$$=(m/f_\pi)(G_A/G_V). \quad (11.44)$$

This is just the relation of Goldberger and Treiman (1958).

We may eliminate the pseudoscalar coupling term $\bar{\psi}\gamma_5\boldsymbol{\tau}\psi\cdot\boldsymbol{\phi}$ by introducing a new nucleon field N defined by

$$\psi=\exp\,(i\lambda\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi})N \quad (11.45)$$

with $\lambda=1/2f_\pi$. The transformed Lagrangian now has the form

$$i\bar{N}\gamma_\mu\partial^\mu N - m\bar{N}N - (g/2m)\bar{N}\gamma_\mu\gamma_5\boldsymbol{\tau}N\cdot\partial^\mu\boldsymbol{\phi}$$
$$-(1/2f_\pi^2)\bar{N}\gamma_\mu\boldsymbol{\tau}N\cdot\boldsymbol{\phi}\times\partial^\mu\boldsymbol{\phi}+\cdots. \quad (11.46)$$

We have omitted a pair term $\bar{N}N\boldsymbol{\phi}^2$ since we only want to look at the amplitude proportional to $\frac{1}{2}[\tau_\beta, \tau_\alpha]$. In the soft-pion limit the gradient term does not contribute and the remaining term yields the threshold result

$$\tfrac{1}{3}(a_1-a_3)=(\mu^2/8\pi f_\pi^2)(1+\mu/m)^{-1}. \quad (11.47)$$

This is the relation of Adler (1965) and Weisberger (1965).

There is a large literature dealing with the implications of chiral symmetry for the interactions of baryons. We quote the papers that have come to our attention: in addition to the pioneering papers of Schwinger (1957), Kramer, Rollnik, and Stech (1959), Gell-Mann and Levy (1960), and Gursey (1960), we refer the reader to the papers of Weinberg (1967), Schwinger (1967), Wess and Zumino (1967), Bardeen and Lee (1968), Cronin (1967), Chang and Gursey (1967), Brown (1967), Mani, Tomozawa, and Yao (1967), Lee (1968), Kramer (1968), and Schechter, Ueda, and Venturi (1969).

C. Representation Mixing

The techniques developed above lead, in the calculation of three-point functions (vertices), to the same results as have been obtained by the standard current algebra treatments with their pole dominance and smoothness assumptions (Schnitzer and Weinberg, 1967; Arnowitt, Friedman, and Nath, 1967; Brown and West, 1968; Gerstein, Schnitzer, and Weinberg, 1968; Gerstein and Schnitzer, 1968). We have also seen in several examples (the $\pi\pi$ scattering lengths in Sec. III and the threshold theorem form of the Adler–Weisberger relation in Sec. XI.B) that threshold results for the four-point functions are similarly reproduced. There exists, however, a class of sum rules which are derived on the basis of assumptions which have no place in the Lagrangian formalism: these are integral sum rules whose convergence is justified by an appeal to the Regge model for high-energy behavior. An example of such a sum rule is the Adler–Weisberger relation in the original form

$$\left(\frac{G_A}{G_V}\right)^{-2}=1+\frac{2m^2}{\pi g^2}\int_0^\infty\frac{d\nu}{\nu}\left[\sigma_{\text{tot}}^{\pi^-}(\nu)-\sigma_{\text{tot}}^{\pi^+}(\nu)\right], \quad (11.48)$$

which, after a resonance-saturation assumption (which does have a Lagrangian counterpart in the tree-graph approximation), yields further relations between coupling constants. Such a sum rule does not have anything to do with chiral symmetry. The above equation (11.48) follows from Eq. (11.47) in the previous subsection, and from the assumption that the pion–nucleon scattering amplitude corresponding to a $T=1$ exchange in the t channel needs no subtractions in a dispersion-relation representation. Nevertheless, like PCAC and vector dominance, it might be interesting to "build in" the extra assumption and extract additional information from it. This has recently been suggested by Weinberg (1969) and by Wess and Zumino (unpublished). The procedure suggested is the following:

(a) Scattering amplitudes are constructed in the usual tree-graph approximation. These have a polynomial dependence on the energy that comes from the derivatives in the vertices, from the energy dependence of the coupling constants, and from the propagators.

(b) The energy-independent part of the amplitude is set equal to zero for those amplitudes which should go to zero at high energies according to the Regge model. (The higher powers of the energy are not discussed in this heuristic prescription.)

The results obtained by Weinberg in considering the forward scattering of *massless* pions ($\pi+A\to\pi'+B$) are the following:

(1) A matrix element of the axial current, $[X_\alpha(\lambda)]_{BA}$, which depends on the helicity λ and the i-spin labels of the initial and final target states A and B, is defined.* The requirement that the $T=1$ exchange amplitude vanishes at high energies (i.e., has a vanishing energy-independent part) leads to

$$[X_\alpha(\lambda), X_\beta(\lambda)]=ie_{\alpha\beta\gamma}T_\gamma. \quad (11.49)$$

This implies that the one-particle states of any given helicity must form representations (not necessarily irreducible) of $SU(2)\times SU(2)$.

(2) If a mass matrix is defined

$$(M^2)_{BA}=m_A^2\delta_{BA}, \quad (11.50)$$

then the requirement that the $T=2$ exchange amplitude has the required behavior implies that

$$[X_\alpha(\lambda), [X_\beta(\lambda), M^2]]\propto\delta_{\alpha\beta}. \quad (11.51)$$

It is then shown that this relation implies that the mass matrix may be written in the form

$$M^2=m_0^2(\lambda)+m_4^2(\lambda), \quad (11.52)$$

where $m_0^2(\lambda)$ is a chiral invariant [i.e., it commutes with $X_\alpha(\lambda)$] and $m_4^2(\lambda)$ is the fourth component of a chiral four vector, i.e., it transforms like the σ in the $(\frac{1}{2},\frac{1}{2})$ representation $(\boldsymbol{\phi},\sigma)$. If the $T=0$ amplitude also

* Weinberg defines $(X_\alpha(\lambda))_{BA}$ by $\langle B, p', \lambda' \mid j_{50}^\alpha+j_{53}^\alpha \mid A, p, \lambda\rangle \propto E\delta_{\lambda\lambda'}(X_\alpha(\lambda))_{BA}$, where A and B are one-particle states with the given helicities and momenta in the z direction, α is the isovector index, and $E=p+(p^2+m_A^2)^{1/2}=p'+(p'^2+m_B^2)^{1/2}$. This definition has the advantage that the definition of $X_\alpha(\lambda)$ is invariant under boosts in the z direction, since for a four vector

$$\exp(i\zeta K_3)(V_0\pm V_3)\exp(-i\zeta K_3)=e^{\pm\zeta}(V_0\pm V_3) \qquad (\sinh\zeta=p/m).$$

went to zero at high energies, $m_4^2(\lambda)$ would vanish and then M^2 would be a chiral invariant, i.e., all the masses would have to be equal. In the real world this is not the case, and therefore, particle states of definite mass must belong to *reducible* representations of $SU(2)\times SU(2)$. If it is further assumed that differences of the forward-scattering amplitude for different helicities vanish at high energies, then $m_0^2(\lambda)$ and $m_4^2(\lambda)$ are independent of helicity.

(3) Weinberg also shows that for $\lambda=0$ states of self-charge conjugate isomultiplets (i.e., nonstrange bosons), all the vectors in the reducible representations of $SU(2)\times SU(2)$ must have the same value of the quantum number $GP(-1)^J$ (where P is the parity). Thus a representation of $SU(2)\times SU(2)$ may contain the particles π, ρ, σ, A_1, f,··· or η, ω, ϕ, X, B,···. If we want to build up a reducible representation that contains the pion and a certain number of its chiral "partners," none of which have $T=2$, we must construct it out of irreducible representations which contain only $T=0$ and $T=1$, i.e., out of $(0,0)$, $(\frac{1}{2},\frac{1}{2})(1,0)\oplus(0,1)$, and $(1,0)\ominus(0,1)$. The equivalence of $SU(2)\times SU(2)$ and $SO(4)$ allows us to use the nomenclature appropriate to the rotation group, and the irreducible representations are labeled scalar, vector, and tensor (magnetic and electric). If we want to have a mass term (quadratic in the fields) that transforms as a vector V, the fields must contain V and S or T at least once since in the square, V must appear in the reduction. The simplest reducible representation $S\oplus V$ turns out to be uninteresting (it contains two scalar isosinglets and the pion). The next nontrivial representation is $V\oplus T$, and it contains a σ, the pion, ρ, and A_1. Straightforward algebraic manipulations of the $SO(4)$ algebra, for which we have no space, leads to the conclusion that the π and the A_1 are both mixtures of V and the "electric" part of T, the σ belongs to V, and ρ, to the "magnetic" part of T. The decay rates for $\rho\rightarrow 2\pi$, $A_1\rightarrow\rho\pi$, etc., depend on the mixing angle between V_k and T_{k4}. There is a V mass and a T mass, and the mass matrix depends on these and the mixing angle. When the mixing angle is chosen to be 45°, which is in agreement with experiment for the ρ width, and the pion mass is set equal to zero, one finds that $m_\rho=m_\sigma$ and $m_{A_1}{}^2=m_\rho{}^2+m_\sigma{}^2$, in agreement with what seems to be observed.

For a simple illustration we may consider pion-nucleon scattering. With the choice of

$$-\lambda=\frac{(G_A/G_V-1)}{2f_\pi}$$

in (11.45), the couplings that contribute to the i-spin antisymmetric amplitude ($\sim\frac{1}{2}[\tau_\beta,\tau_\alpha]$) are

$$g\bar{\psi}\gamma_5\boldsymbol{\tau}\psi\cdot\boldsymbol{\phi}$$

and

$$\left[\frac{(G_A/G_V-1)}{2f_\pi^2}\right]\bar{\psi}\gamma_\mu\boldsymbol{\tau}\psi\cdot\boldsymbol{\phi}\times\partial^\mu\boldsymbol{\phi}.$$

In the limit $s\rightarrow\infty$ the forward-scattering amplitude only has a contribution coming from the second term, since the γ_5 terms are proportional to $(s-m^2)^{-1}$ and $(u-m^2)^{-1}$. Setting this equal to zero leads to

$$G_A/G_V=1, \qquad (11.53)$$

an algebraic relation not contained in the theory without the high-energy assumption. If the Lagrangian is enlarged to include the Δ decuplet, the expression for G_A/G_V is altered in a way which depends on the width of the Δ.

In view of the heuristic nature of the proposal for the new "rules" which are to accompany the tree-graph approximation, it is a little early to assess their soundness. They do provide a way of bringing in representation mixing and of reproducing the very interesting results of Gilman and Harari (1968) based on current-algebra and superconvergence relations, and they may suggest ways of avoiding the difficulties that we found in our discussion of the Super Lagrangian in Sec. X.

The two approaches to calculating vertex functions, direct tree-graph calculation or configuration mixing, differ dramatically in their treatments of $A\rho\pi$ and $\omega\rho\pi$ couplings. The two methods predict radically different $\rho\pi$ angular distributions for $A\rightarrow\rho+\pi$: Tree graphs yield a predominantly S-wave $\rho\pi$ state, and configuration mixing, a D-wave state. It has so far not been possible to reconcile the two predictions. The problem with $\omega\rho\pi$ coupling is that while such an interaction plays an important role in the superconvergence relations, there is no way of introducing a $\omega\rho\pi$ vertex into an effective Lagrangian without violating either PCAC or the field algebra. It is evident from (10.1)–(10.6), for example, that our Lagrangian does not contain a term that describes the $\omega\rho\pi$ vertex. We do not expect to see such a term directly, since the ρ and ω belong to $(\mathbf{8},\mathbf{1})+(\mathbf{1},\mathbf{8})$ [with a $(\mathbf{1},\mathbf{1})$ admixture] and the pion belongs to $(\mathbf{3},\bar{\mathbf{3}})+(\bar{\mathbf{3}},\mathbf{3})$. This is also the case for the A meson, but there the symmetry breaking induced an $A\rho\pi$ coupling. We cannot construct a coupling which, when $\sigma\rightarrow\sigma'+\Sigma$, leads to a term of the type

$$\mathrm{Tr}\,(F_{\mu\nu}\tilde{F}^{\mu\nu}\phi). \qquad (11.54)$$

The reason for this is that the dual field $\tilde{F}^{\mu\nu}\equiv\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ has the same parity properties as the field $G^{\mu\nu}$, but has opposite charge conjugation transformation properties. For example, a coupling like

$$\mathrm{Tr}\,(F_{\mu\nu}{}^{(+)}\tilde{F}^{(+)\mu\nu}BB^\dagger-F_{\mu\nu}{}^{(-)}\tilde{F}^{(-)\mu\nu}B^\dagger B)$$

$$F_{\mu\nu}{}^{(\pm)}=F_{\mu\nu}\pm G_{\mu\nu}, \qquad (11.55)$$

which is chiral invariant, has its ω and ρ terms appear in the combination

$$-ai\,\mathrm{Tr}\,(F_{\mu\nu}\tilde{F}^{\mu\nu}[\sigma,\phi]). \qquad (11.56)$$

Thus when $\sigma\rightarrow\sigma'+\Sigma$, only the K mesons will have a ωKK^* interaction. We can, of course, write down a coupling like

$$\mathrm{Tr}\,(F_{\mu\nu}{}^{(+)}\tilde{F}^{(+)\mu\nu}B - F_{\mu\nu}{}^{(-)}\tilde{F}^{(-)\mu\nu}B^{\dagger}) \tag{11.57}$$

which leads to the desired vertex, but such a term does not satisfy the PCAC condition (Arnowitt, Friedman, and Nath, 1969; see also Brown and West, 1968, for a discussion using dispersion-relation techniques). This difficulty of reconciling PCAC with the PVV vertex was first noted for the process $\pi^0 \rightarrow 2\gamma$ (Sutherland, 1967) where it was shown that if $\partial^\mu \mathbf{A}_\mu$ were used to extrapolate the $\pi^0 \rightarrow 2\gamma$ amplitude off the pion mass shell, then the amplitude vanishes as the pion mass goes to zero. For an effective Lagrangian where $\partial^\mu \mathbf{A}_\mu \propto \boldsymbol{\phi}$ no such off mass-shell dependence can appear so that the $\pi^0 \rightarrow 2\gamma$ vertex must vanish in this case.

A term of the form (11.57) which violates PCAC does give rise to a nonvanishing $\pi^0 \rightarrow 2\gamma$ vertex. If $\partial^\mu \mathbf{A}_\mu$ were still used as the pion interpolating field, then again $\pi^0 \rightarrow 2\gamma$ would vanish for zero pion mass.

It is also possible to generate an $\omega\rho\pi$ coupling from a chiral-invariant term (maintaining PCAC) but which violates the field algebra:

$$\mathrm{Tr}\,(V_\mu{}^{(+)}\tilde{F}^{(+)\mu\nu}\Delta_\nu BB^{\dagger} - V_\mu{}^{(-)}\tilde{F}^{(-)\mu\nu}\Delta_\nu B^{\dagger}B), \tag{11.58}$$

The nonzero vacuum expectation values of σ_0 and σ_8 contributing to B and $B^{\dagger}$, yield the $\omega\rho\pi$ coupling. Despite the presence of a term like Eq. (11.58), however, the $\pi^0 \rightarrow 2\gamma$ vertex still vanishes, as it must in any effective-Lagrangian model preserving PCAC. The conventional contribution to $\pi^0 \rightarrow 2\gamma$ arising from $\omega\rho\pi$ (Gell-Mann, Sharp, and Wagner, 1962) is canceled by what are equivalent to subtraction terms arising from the breaking by (11.58) of the field-current identity. This alternative, therefore, is unsatisfactory if we wish to link π^0 decay to the $\omega\rho\pi$ interaction.

One interpretation of this difficulty is that PVV couplings do not fit into the framework of the $SU(3)\times SU(3)$ field algebra because it is a coupling which must be generated through the intermediary of baryon (or quark) loops. Such triangle graphs are excluded from the effective-Lagrangian treatment. It has been recently observed by Adler (1969) that in electrodynamics (i.e., a quarklike model) a careful examination of the divergence of the axial current appearing in such triangle graphs shows that the formal current-algebra manipulations are not correct and that an effective PVV coupling does appear.

An alternative approach, yielding the same result, is to deal more carefully with the operator definition of the axial current as a product of field operators. When this is done in a gauge-invariant way, an effective $\pi\gamma\gamma$ coupling appears (Schwinger, 1951; Hagen, 1969). Again, this is a subtlety not included in the effective-Lagrangian, tree-graph approximation.

XII. CONCLUSIONS

We have described in considerable detail the method of effective Lagrangians to provide the reader with a useful tool for studying the consequences of broken chiral symmetries and of field algebras. We have discussed some of the significant results obtained by many authors using these techniques or the equivalent approach of applying pole dominance and smoothness assumptions to two-, three-, and four-point functions. While effective Lagrangians easily reproduce the soft-pion results, i.e., π–N scattering lengths, the real test of its applicability awaits the experimental verification of its "hard-pion" predictions.

For $SU(2)\times SU(2)$, the relation between $\gamma_{\rho\pi\pi}$ and γ_ρ (or γ_0 in Sec. VI) given by Eq. (6.16), when compared with present experimental determinations of $\gamma_{\rho\pi\pi}$ and γ_ρ, clearly requires the existence of the $A_1(A_\pi)$ axial-vector meson.* Given γ_ρ and $\gamma_{\rho\pi\pi}$ both the mass of the A_1 [Eq. (6.13) or (10.61)] and its decay properties [see Eq. (6.19) and Appendix C] are determined. Experimental verifications of these predictions are crucial tests of the model, especially in light of the quite different predictions of Gilman and Harari discussed in Sec. XI.

The question of the existence of scalar particles becomes important when generalizing effective Lagrangians to treat $SU(3)\times SU(3)$. One attractive possibility, discussed in Sec. IX, is that only some of the scalar fields, such as the σ_K, represent real scalar particles. We treat in more detail in Sec. X the example of an $SU(3)\times SU(3)$ field algebra where all the fields are represented by particles. Two particular problems are discussed; V–γ couplings and Weinberg's sum rule in $SU(3)\times SU(3)$ (equality of Schwinger terms), and the problem of fitting the known meson masses given the "observed" value of $f_\pi f_+(0)/f_K$ so different from unity. In the former we have seen how the effective-Lagrangian model demonstrates the connection between Weinberg's sum rule, the field–current identity, and PCAC. It is interesting that our super Lagrangian, with Schwinger terms equal, generates kinetic-energy mixing for the F_V and a generalized mixing angle θ compatible with present experiments. An important feature of the model is the explicit correction to the results of Oakes and Sakurai coming from the kappa meson. Our results show how θ can be sensitive to second-order $SU(3)$ breaking terms in an $SU(3)\times SU(3)$ symmetry-breaking model.

The problem of fitting the meson masses appearing in the super Lagrangian to their observed values proved to be surprisingly nontrivial. Indeed, no precise fit to the spin-0^- and -1^- nonets could be made without having presently unacceptably low values for some of the scalar meson masses or taking $f_\pi f_+(0)/f_K$ closer to one than given by experiment. Three possibilities require further exploration: (i) A more complicated choice for $\mathcal{L}_C$ in Eq. (8.6) is required. (ii) The masses given by the effective Lagrangian cannot be too

* As in the limit $m_\sigma \rightarrow \infty$ discussed in Sec. IX, the limit $m_A \rightarrow \infty$ is equivalent to a nonlinear realization for $\mathfrak{A}_\mu$ [Eq. (6.24)]. Taking $m_A \rightarrow \infty$ in (6.16) yields $\gamma_{\rho\pi\pi} = \frac{1}{2}\gamma_0$.

precisely fit. (iii) If radiative corrections could be taken into account or if θ_A differs slightly from θ_V, the value for $f_\pi f_+(0)/f_K$ that should be used is in fact closer to 1. In connection with (ii), it has long been a puzzle to some theorists why large $SU(3)$ breaking mass shifts don't arise due to the large variations in the widths of decaying multiplets because the decay products (π, K, $\bar{K}$, η, for example) have such large mass splittings. Since this effect comes about through closed-loop graphs, perhaps an effective Lagrangian cannot include such corrections. Whatever the explanation is, it is significant that there should exist such severe restrictions on the choice of parameters in the model.

Finally, even if experiments do confirm the predictions of some appropriately constructed Lagrangian, we would still be dealing with a model with a limited range of applicability. We have seen that even in the context of spin-0 and -1 particles, the $\omega\rho\pi$ (and consequently $\omega\pi\gamma$) coupling is not included in the model. More fundamentally, such problems as treating unitarity and the high-energy behavior of amplitudes and the associated problem of dealing with the many higher-spin multiplets on Regge trajectories do not at present seem relevant to an effective-Lagrangian approach. Nevertheless within the relevant range of experience, effective Lagrangians are a useful tool and can give valuable insight into the nature of broken chiral symmetry.

ACKNOWLEDGMENTS

We have benefited much from discussions with Dr. W. A. Bardeen and Professor H. Joos, Professor B. W. Lee, and Professor H. Suura. We gratefully acknowledge the hospitality of the High Energy Theory Groups at the Argonne National Laboratory, the Los Alamos Scientific Laboratory, the Lawrence Radiation Laboratory, and the University of California at Los Angeles, where we worked on this project. One of us (S. G.) would like to thank the Theory Group at the Deutsches Elektronen Synchrotron (DESY) for their hospitality and stimulation during 1968–1969.

APPENDIX A: TRANSFORMATIONS OF FIELDS UNDER $SU(2)\times SU(2)$

We deal with transformations generated by the "charges"

$$\mathbf{Q}_\pm=\tfrac{1}{2}(\mathbf{Q}\pm\mathbf{Q}_5) \tag{A1}$$

which satisfy the commutation relations

$$[Q_\pm^a, Q_\pm^b]=ie_{abc}Q_\pm^c,$$
$$[Q_+^a, Q_-^b]=0, \tag{A2}$$

and are related by the parity operator

$$P\mathbf{Q}_+P^{-1}=\mathbf{Q}_-. \tag{A3}$$

The transformation properties of an irreducible tensor of i-spin t are given by

$$U_+\phi_AU_+^{-1}=\phi_B[\exp(-i\boldsymbol{\alpha}\cdot\mathbf{T})]_{BA}, \tag{A4}$$

where

$$U_+=\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q}_+) \tag{A5}$$

and the $\mathbf{T}$ form a $(2t+1)\times(2t+1)$ matrix representation of the charges $\mathbf{Q}_+$, say. If we want to specify the transformation properties of a field under $SU(2)\times SU(2)$, we must also write the response of the field to

$$U_-=\exp(-i\boldsymbol{\beta}\cdot\mathbf{Q}_-). \tag{A6}$$

A field that transforms as

$$U_+\phi_{AB'}U_+^{-1}=\phi_{CB'}[\exp(-i\boldsymbol{\alpha}\cdot\mathbf{T})]_{CA} \tag{A7}$$

and

$$U_-\phi_{AB'}U_-^{-1}=\phi_{AD'}[\exp(-i\boldsymbol{\beta}\cdot\mathbf{T}')]_{D'B'}, \tag{A8}$$

where the $\mathbf{T}'$ form a $(2t'+1)\times(2t'+1)$ matrix representation of the charges, is said to transform as a (t, t') representation of $SU(2)\times SU(2)$. The transformation law (A8) is equivalent to

$$U_-\phi_{AB'}U_-^{-1}=[\exp(i\boldsymbol{\beta}\cdot\mathbf{T}')]_{B'D'}\phi_{AD'} \tag{A9}$$

because there exists a matrix C which has the property that

$$C\mathbf{T}^*C^{-1}=-\mathbf{T}. \tag{A10}$$

In the representation in which T_2 is imaginary,

$$C_{AB}=[\exp(-i\pi T_2')]_{AB}=(-1)^{t'-B}\delta_{A,-B}. \tag{A11}$$

Consider, as a first example, a field transforming as $(1, 0)$. This implies that

$$U_+\phi_AU_+^{-1}=\phi_B[\exp(-i\boldsymbol{\alpha}\cdot\mathbf{T})]_{BA},$$
$$U_-\phi_AU_-^{-1}=\phi_A. \tag{A12}$$

For the field $\psi_{A'}$ defined by

$$\psi_{A'}\equiv P\phi_AP^{-1}, \tag{A13}$$

we have

$$U_-\psi_{A'}U_-^{-1}=\psi_{B'}[\exp(-i\boldsymbol{\alpha}\cdot\mathbf{T})]_{B'A'},$$
$$U_+\psi_{A'}U_+^{-1}=\psi_{A'}; \tag{A14}$$

i.e., we see that $\psi_{A'}$ transforms as $(0, 1)$. We can combine these to write

$$\begin{aligned}\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q}_+)\exp(-i\boldsymbol{\beta}\cdot\mathbf{Q}_-)\phi_A\exp(i\boldsymbol{\beta}\cdot\mathbf{Q}_-)\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q}_+)\\ =\exp[-\tfrac{1}{2}i(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\mathbf{Q}-\tfrac{1}{2}i(\boldsymbol{\alpha}-\boldsymbol{\beta})\cdot\mathbf{Q}_5]\phi_A\\ \times\exp[\tfrac{1}{2}i(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\mathbf{Q}+\tfrac{1}{2}i(\boldsymbol{\alpha}-\boldsymbol{\beta})\cdot\mathbf{Q}_5]\\ =\phi_B[\exp(-i\boldsymbol{\alpha}\cdot\mathbf{T})]_{BA},\end{aligned} \tag{A15}$$

and

$$\exp\left[-\tfrac{1}{2}i(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\mathbf{Q}-\tfrac{1}{2}i(\boldsymbol{\alpha}-\boldsymbol{\beta})\cdot\mathbf{Q}_5\right]\psi_{A'}\times\exp\left[\tfrac{1}{2}i(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\mathbf{Q}+\tfrac{1}{2}i(\boldsymbol{\alpha}-\boldsymbol{\beta})\cdot\mathbf{Q}_5\right]=\psi_{B'}[\exp(-i\boldsymbol{\beta}\cdot\mathbf{T})]_{B'A'}. \quad (A16)$$

If we now set $\boldsymbol{\alpha}=\boldsymbol{\beta}$ and take $\boldsymbol{\alpha}$ infinitesimal, we get

$$-i[\boldsymbol{\alpha}\cdot\mathbf{Q},\phi_A]=-i\phi_B(\boldsymbol{\alpha}\cdot\mathbf{T})_{BA},$$
$$-i[\boldsymbol{\alpha}\cdot\mathbf{Q},\psi_{A'}]=-i\psi_{B'}(\boldsymbol{\alpha}\cdot\mathbf{T})_{B'A'}. \quad (A17)$$

If we set $\boldsymbol{\beta}=-\boldsymbol{\alpha}$, we get

$$-i[\boldsymbol{\alpha}\cdot\mathbf{Q}_5,\phi_A]=-i\phi_B(\boldsymbol{\alpha}\cdot\mathbf{T})_{BA},$$
$$-i[\boldsymbol{\alpha}\cdot\mathbf{Q}_5,\psi_{A'}]=i\psi_{B'}(\boldsymbol{\alpha}\cdot\mathbf{T})_{B'A'}. \quad (A18)$$

Using

$$(I^k)_{BA}=-ie_{kBA} \quad (A19)$$

for the 3×3-dimensional matrix representation of the charges, we see that (A17) and (A18) may be written in the form

$$\delta\boldsymbol{\phi}_A=-(\boldsymbol{\alpha}\times\boldsymbol{\phi})_A,$$
$$\delta\boldsymbol{\psi}_{A'}=-(\boldsymbol{\alpha}\times\boldsymbol{\psi})_{A'},$$
$$\delta'\boldsymbol{\phi}_A=-(\boldsymbol{\alpha}\times\boldsymbol{\phi})_A,$$
$$\delta'\boldsymbol{\psi}_{A'}=(\boldsymbol{\alpha}\times\boldsymbol{\psi})_{A'}. \quad (A20)$$

Hence the even- and odd-parity combinations $\phi_A\pm\psi_{A'}$, denoted by Σ_A and π_A, respectively, satisfy

$$\delta\boldsymbol{\Sigma}=-\boldsymbol{\alpha}\times\boldsymbol{\Sigma},$$
$$\delta\boldsymbol{\pi}=-\boldsymbol{\alpha}\times\boldsymbol{\pi},$$
$$\delta'\boldsymbol{\Sigma}=-\boldsymbol{\alpha}\times\boldsymbol{\pi},$$
$$\delta'\boldsymbol{\pi}=-\boldsymbol{\alpha}\times\boldsymbol{\Sigma}. \quad (A21)$$

These are just the transformation laws (3.9) and (3.10).

As a second example we will consider a field that transforms as $(\frac{1}{2},\frac{1}{2})$. Making use of the equivalence expressed in Eq. (A9), we write

$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q}_+)M_{AB'}\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q}_+)=[\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2)]_{AC}M_{CB'} \quad (A22)$$

and

$$\exp(-i\boldsymbol{\beta}\cdot\mathbf{Q}_-)M_{AB'}\exp(i\boldsymbol{\beta}\cdot\mathbf{Q}_-)=M_{AC'}[\exp(-i\boldsymbol{\beta}\cdot\boldsymbol{\tau}/2)]_{C'B'}. \quad (A23)$$

It follows from this that

$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q})M\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q})=\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2)M\exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2),$$
$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q}_5)M\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q}_5)=\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2)M\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2). \quad (A24)$$

If we now write the 2×2 matrix M in the form

$$M=\Sigma 1+i\boldsymbol{\tau}\cdot\boldsymbol{\pi}, \quad (A25)$$

then the above relations lead to

$$\delta(\Sigma 1+i\boldsymbol{\tau}\cdot\boldsymbol{\pi})=-i\boldsymbol{\tau}\cdot\boldsymbol{\alpha}\times\boldsymbol{\pi},$$
$$\delta'(\Sigma 1+i\boldsymbol{\tau}\cdot\boldsymbol{\pi})=i\Sigma\boldsymbol{\alpha}\cdot\boldsymbol{\tau}-\boldsymbol{\alpha}\cdot\boldsymbol{\pi}1. \quad (A26)$$

These are just the relations that tell us that Σ has i spin 0, $\boldsymbol{\pi}$ has i spin 1, and (3.17) is satisfied.

Under the parity transformation,

$$PMP^{-1}=M^\dagger \quad (A27)$$

and

$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q}_5)M^\dagger\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q}_5)=\exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2)M^\dagger\exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2). \quad (A28)$$

Hence

$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q}_5)MM^\dagger\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q}_5)=\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2)MM^\dagger\exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2). \quad (A29)$$

From this it follows that Tr MM^+ is chiral invariant. Actually, since

$$MM^\dagger=M^\dagger M, \quad (A30)$$

it follows that

$$MM^\dagger=M^\dagger M=(\Sigma^2+\boldsymbol{\pi}^2)1 \quad (A31)$$

is chiral invariant. For a nonlinear realization we may set

$$MM^\dagger=(\Sigma^2+\boldsymbol{\pi}^2)1=f_\pi^2(\text{const})1. \quad (A32)$$

A parametrization equivalent to that of (A25), namely

$$M=f_\pi\exp(i\boldsymbol{\tau}\cdot\mathbf{P}), \quad (A33)$$

where $\mathbf{P}$ is a new (equally good) pseudoscalar field, has frequently been used (Chang and Gursey, 1967; Brown, 1967).

APPENDIX B: TRANSFORMATIONS OF FIELDS UNDER $SU(3)\times SU(3)$

The development here exactly parallels that of Appendix A. The equivalence expressed in (A9) no longer holds, so that the transformation laws have to be specified a little more carefully. If we label the $SU(3)$ representation by the multiplicity (e.g., **1**, **3**, $\bar{\mathbf{3}}$, **8**,$\cdots$) (this will cause no confusion in the cases of interest here) then the transformation law for a field transforming under $SU(3)\times SU(3)$ according to $(\mathbf{m},\bar{\mathbf{n}})$ is

$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q}_+)\exp(-i\boldsymbol{\beta}\cdot\mathbf{Q}_-)M_{AB'}\times\exp(i\boldsymbol{\beta}\cdot\mathbf{Q}_-)\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q}_+)=[\exp(i\boldsymbol{\alpha}\cdot\mathbf{F})]_{AC}M_{CD'}[\exp(-i\boldsymbol{\beta}\cdot\mathbf{F}')]_{D'B'}, \quad (B1)$$

where the $\mathbf{F}$ and $\mathbf{F}'$ are matrix representations of the $SU(3)$ generators in the $\mathbf{m}$ and $\mathbf{n}$ representations, respectively. If the field transforms under $(\mathbf{m}, \mathbf{n})$, then $\mathbf{F}'$ must be taken to be matrices in the $\bar{\mathbf{n}}$ representation.

The matrices satisfy the commutation relations

$$[F^a, F^b]=if_{abc}F^c \qquad (a, b, c=1, 2, \cdots, 8). \quad \text{(B2)}$$

The fact that the fields $F_{\mu\nu}{}^i$ and $G_{\mu\nu}{}^i$ [see (7.5) and (7.8)] transform under $SU(3)\times SU(3)$ as $(\mathbf{8}, \mathbf{1})+(\mathbf{1}, \mathbf{8})$ is established straightforwardly if use is made of the fact that in the 8 representation (the adjoint representation) the matrices $\mathbf{F}$ can be written in terms of the structure constants

$$(F^k)_{AB}=-if_{kAB}. \quad \text{(B3)}$$

Let us now consider the representation $(\mathbf{3}, \bar{\mathbf{3}})+(\bar{\mathbf{3}}, \mathbf{3})$. For a field that transforms as $(\mathbf{3}, \bar{\mathbf{3}})$ we have

$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q}_+)\exp(-i\boldsymbol{\beta}\cdot\mathbf{Q}_-)M_{AB'}$$
$$\times\exp(i\boldsymbol{\beta}\cdot\mathbf{Q}_-)\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q}_+)$$
$$=[\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}/2)]_{AC}M_{CD'}[\exp(-i\boldsymbol{\beta}\cdot\boldsymbol{\lambda}/2)]_{D'B'} \quad \text{(B4)}$$

since the $\frac{1}{2}\lambda_i$'s form a 3×3-dimensional representation of the generators. The matrices $-\frac{1}{2}\lambda_i^*$ satisfy the same commutation relations but are not equivalent to the set $\frac{1}{2}\lambda_i$. Thus they yield a representation $\bar{\mathbf{3}}$ of the generators. Hence, for a field transforming as $(\bar{\mathbf{3}}, \mathbf{3})$ we have

$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q}_+)\exp(-i\boldsymbol{\beta}\cdot\mathbf{Q}_-)N_{A'B}$$
$$\times\exp(i\boldsymbol{\beta}\cdot\mathbf{Q}_-)\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q}_+)$$
$$=[\exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}^*/2)]_{A'C'}N_{C'D}[\exp(i\boldsymbol{\beta}\cdot\boldsymbol{\lambda}^*/2)]_{DB}$$
$$=[\exp(i\boldsymbol{\beta}\cdot\boldsymbol{\lambda}/2)]_{BD}N_{C'D}[\exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}/2)]_{C'A'}. \quad \text{(B5)}$$

The last line follows from the Hermiticity of the λ matrices. Thus the matrix N^T has the same transformation properties as M except that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are interchanged. This however implies that

$$N^T=PMP^{-1}. \quad \text{(B6)}$$

We can deduce from (B4) and (B5) that

$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q})M\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q})$$
$$=\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}/2)M\exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}/2),$$
$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q}_5)M\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q}_5)$$
$$=\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}/2)M\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}/2), \quad \text{(B7)}$$

which allows us to make the identification

$$M=B \quad \text{(B8)}$$

with B defined as in (7.52). Similarly,

$$N^T=B^\dagger. \quad \text{(B9)}$$

We easily show that

$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q}_5)BB^\dagger\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q}_5)$$
$$=\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}/2)BB^\dagger\exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}/2),$$
$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q}_5)B^\dagger B\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q}_5)$$
$$=\exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}/2)B^\dagger B\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}/2). \quad \text{(B10)}$$

Thus if

$$B^\dagger B=BB^\dagger, \quad \text{(B11)}$$

which implies that

$$\sigma=\Sigma C_n\phi^n, \quad \text{(B12)}$$

we may set

$$B^\dagger B=BB^\dagger=\sigma^2+\phi^2=(\text{const})1. \quad \text{(B13)}$$

Under these circumstances, corresponding to the existence of a pseudoscalar nonet only, we may also use the parametrization

$$B=e^{iP} \quad \text{(B14)}$$

with P an equivalent nonet pseudoscalar field.

We also note that (B7) implies that

$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q})\det M\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q})=\det M. \quad \text{(B15)}$$

However,

$$\exp(-i\boldsymbol{\alpha}\cdot\mathbf{Q}_5)\det M\exp(i\boldsymbol{\alpha}\cdot\mathbf{Q}_5)$$
$$=\det[\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\lambda})M]$$
$$=\exp[i(6^{1/2})\alpha_0]\det M. \quad \text{(B16)}$$

Hence

$$2I=\det B+\det B^\dagger \quad \text{(B17)}$$

is chiral invariant, provided $\alpha_0=0$. Making use of the fact that

$$6\det U=e_{abc}e_{klm}U_{ak}U_{bl}U_{cm}$$
$$=(\text{Tr }U)^3-3\text{ Tr }U\text{ Tr }U^2+2\text{ Tr }U^3, \quad \text{(B18)}$$

we can write

$$6I=(\text{Tr }\sigma)^3-3\text{ Tr }\sigma\text{ Tr }\sigma^2-3\text{ Tr }\sigma(\text{Tr }\phi)^2+3\text{ Tr }\sigma\text{ Tr }\phi^2$$
$$+6\text{ Tr }\phi\text{ Tr }\sigma\phi+2\text{ Tr }\sigma^3-6\text{ Tr }\sigma\phi^2. \quad \text{(B19)}$$

APPENDIX C: DECAY RATES

Consider the decay

$$A(Q)\rightarrow B(p)+C(q)$$

with the quantities in the parentheses denoting the four momenta of the particles. From the Feynman rules we calculate

$$\mathcal{R}=i(2\pi)^4\delta(Q-p-q)\langle p, q\mid\mathcal{L}_{ABC}\mid Q\rangle, \quad \text{(C1)}$$

and we write

$$\mathfrak{M} = -(2\pi)^{9/2}\langle p, q \mid \mathcal{L}_{ABC} \mid Q\rangle. \quad \text{(C2)}$$

With the normalization of states $\langle p' \mid p\rangle = 2p_0\delta(\mathbf{p}'-\mathbf{p})$ we get

$$d\Gamma = (2\pi)^4\delta(Q-p-q)\,\frac{(2\pi)^3}{2Q_0}\,\frac{1}{(2\pi)^9}$$

$$\times\left(\frac{1}{2J_A+1}\sum_{\text{spins}} |\mathfrak{M}|^2\right)\frac{d^3q}{2q_0}\frac{d^3p}{2p_0}. \quad \text{(C3)}$$

Hence

$$\Gamma = \frac{1}{8\pi}\frac{1}{2J_A+1}\frac{\sum_{\text{spins}}|\mathfrak{M}|^2}{m_A^2}p_{\text{c.m.}}. \quad \text{(C4)}$$

Here $p_{\text{c.m.}}$ is the center-of-mass momentum of the decay products and is given by

$$2m_A p_{\text{c.m.}} = [\lambda(m_A^2, m_B^2, m_C^2)]^{1/2}, \quad \text{(C5)}$$

where

$$\lambda(x, y, z) = x^2+y^2+z^2-2xy-2yz-2zx.$$

We now consider two examples:

(I) For the decay

$$\rho\to\pi+\pi$$

the matrix element is given by

$$\mathfrak{M} = \gamma_{\rho\pi\pi}\epsilon^\mu(\rho)(p_\mu - q_\mu). \quad \text{(C6)}$$

Hence

$$\sum_{\text{spins}}|\mathfrak{M}|^2 = \gamma_{\rho\pi\pi}^2(p-q)_\mu(p-q)_\nu[(Q^\mu Q^\nu/m_\rho^2)-g^{\mu\nu}]$$

$$= -\gamma_{\rho\pi\pi}^2(p-q)^2.$$

Noting that here

$$p_{\text{c.m.}} = \tfrac{1}{2}(m_\rho^2-4m_\pi^2)^{1/2} = [-\tfrac{1}{4}(p-q)^2]^{1/2},$$

we get

$$\Gamma = \frac{2}{3}\frac{\gamma_{\rho\pi\pi}^2}{4\pi}\frac{p_{\text{c.m.}}^3}{m_\rho^2} \simeq 52\frac{\gamma_{\rho\pi\pi}^2}{4\pi}\,\text{MeV}. \quad \text{(C7)}$$

With $m_\rho = 765$ MeV and $m_A \simeq \sqrt{2}m_\rho$, we get

$$\Gamma = (\gamma_0^2/4\pi)52(\tfrac{3}{4}+\tfrac{1}{4}\delta)^2\,\text{MeV}, \quad \text{(C8)}$$

where we have denoted $\kappa m_\rho^2/\gamma_0$ by δ.

(II) For either of the decays

$$A^0\to\rho^\pm+\pi^\mp$$

we write the matrix element in the form

$$\mathfrak{M} = \epsilon_\mu{}^A\epsilon_\nu{}^\rho[Ag^{\mu\nu}-B(p^\mu Q^\nu/m_A^2)]. \quad \text{(C9)}$$

Then

$$\sum_{\text{spins}}|\mathfrak{M}|^2 = \left(Ag^{\mu\nu}-B\frac{p^\mu Q^\nu}{m_A^2}\right)\left(Ag^{\alpha\beta}-B\frac{p^\alpha Q^\beta}{m_A^2}\right)$$

$$\times\left(\frac{Q_\mu Q_\alpha}{m_A^2}-g_{\mu\alpha}\right)\left(\frac{p_\nu p_\beta}{m_\rho^2}-g_{\nu\beta}\right)$$

$$= 2A^2\left(1+\frac{m_A^2}{2m_\rho^2}r^2\right)+2ABr\left(1-\frac{m_A^2}{m_\rho^2}r^2\right)$$

$$+2B^2\left(\frac{m_A^2}{2m_\rho^2}r^4-r^2+\frac{m_\rho^2}{2m_A^2}\right), \quad \text{(C10)}$$

where

$$r = (p\cdot Q)/2m_A^2 = (m_A^2+m_\rho^2-m_\pi^2)/m_A^2. \quad \text{(C11)}$$

The results of Sec. VI are that

$$A = \gamma_0 m_\rho(1-m_\rho^2/m_A^2)^{1/2}\{1+[1-(m_A^2/m_\rho^2)r]\delta\};$$

$$B = -\gamma_0 m_\rho(1-m_\rho^2/m_A^2)^{1/2}(m_A^2/m_\rho^2)\delta. \quad \text{(C12)}$$

Hence

$$\Gamma_{A\rho\pi} \simeq 64(\gamma_0^2/4\pi)(1-0.88\delta+0.19\delta^2), \quad \text{(C13)}$$

so that

$$\frac{\Gamma_{A\rho\pi}}{\Gamma_{\rho\pi\pi}} \simeq 2.2\,\frac{1-0.88\delta+0.19\delta^2}{1+0.66\delta+0.11\delta^2}. \quad \text{(C14)}$$

The value of δ is not known very accurately. The determination from the ρ width is uncertain to the extent that this width is uncertain. Its determination from the A width is complicated by the fact that there is a contribution from the $A\to3\pi$ [see Gerstein and Schnitzer (1968) for a discussion of this] background, and, if there is a σ, from the decay

$$A\to\sigma+\pi.$$

For the latter, if the matrix element is given by

$$\mathfrak{M} = \gamma_{A\sigma\pi}\epsilon_A{}^\mu\cdot q_\mu, \quad \text{(C15)}$$

then

$$\sum_{\text{spins}}|\mathfrak{M}|^2 = \gamma_{A\sigma\pi}^2[(m_A^2-m_\sigma^2+m_\pi^2)^2-4m_A^2m_\pi^2]/4m_A^2. \quad \text{(C16)}$$

For the Lagrangian in (6.3) one finds that

$$\gamma_{A\sigma\pi} = 2\gamma_0(m_\rho/m_A)[1-\tfrac{1}{2}\delta(m_A^2/m_\rho^2)], \quad \text{(C17)}$$

so that if we take $m_\sigma = m_\rho$ for simplicity, we get

$$\Gamma_{A\sigma\pi} \simeq 4(\gamma_0^2/4\pi)(1-\delta)^2\,\text{MeV}. \quad \text{(C18)}$$

One could also determine δ from the helicity of the ρ in the decay of the A_1, since the matrix elements are

$$\mathfrak{M}_{11} = A,$$

$$\mathfrak{M}_{00} = [(E_\rho/m_\rho)A-(p_\rho^2/m_Am_\rho)B]. \quad \text{(C19)}$$

Of some relevance may be the fact that apparently the A's are not being photoproduced.

APPENDIX D: THE GELL-MANN MATRICES*

For completeness we list the canonical Gell-Mann matrices:

$$\lambda_0=(\tfrac{2}{3})^{1/2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix};\quad \lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix};$$

$$\lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix};\quad \lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix};$$

$$\lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix};\quad \lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix};$$

$$\lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix};\quad \lambda_7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix};$$

$$\lambda_8=(\tfrac{1}{3})^{1/2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}. \qquad \text{(D1)}$$

They satisfy the relations

$$\mathrm{Tr}\,\lambda_i\lambda_j=2\delta_{ij};$$
$$[\lambda_i,\lambda_j]=2if_{ijk}\lambda_k;$$
$$\{\lambda_i,\lambda_j\}=2d_{ijk}\lambda_k. \qquad \text{(D2)}$$

The f_{ijk} are antisymmetric under the interchange of any two subscripts. Their values are listed below:

ijk	f_{ijk}
123	1
147	1/2
156	−1/2
246	1/2
257	1/2
345	1/2
367	−1/2
458	√3/2
678	√3/2

All others vanish.

The coefficients d_{ijk} are symmetric under the interchange of any two subscripts. The values of the d_{ijk} are listed below:

ijk	d_{ijk}
118	1/√3
146	1/2
157	1/2
228	1/√3
247	−1/2
256	1/2
338	1/√3
344	1/2
355	1/2
366	−1/2
377	−1/2
448	−1/2√3
558	−1/2√3
668	−1/2√3
778	−1/2√3
888	−1/√3

and

$$d_{0ij}=(\tfrac{2}{3})^{1/2}\delta_{ij}.$$

APPENDIX E: NONLINEAR REALIZATIONS

In a recent paper, Coleman, Wess, and Zumino (1969) have given a complete discussion of nonlinear realizations for compact, semisimple Lie groups. We shall not attempt to do anything more than outline the general argument. The point of departure is the equivalence of all fields obtained by transformations of the type shown in Eq. (2.22). This strongly suggests a geometrical approach* to the problem. Quite generally then, the fields (ϕ,ψ) will be taken as coordinates on a manifold, and under the group G a transformation law for these coordinates is given:

$$g(\phi,\psi)=(\phi',\psi'). \qquad \text{(E1)}$$

For this transformation law the group properties must hold, e.g.,

$$\begin{aligned}g_1(g(\phi,\psi))&=g_1(\phi',\psi')\\&=(\phi'',\psi'')\\&=(g_1g)(\phi,\psi).\end{aligned} \qquad \text{(E2)}$$

The transformation of fields in (2.22) suggests that the origin is unchanged under the change of coordinates, i.e., it is distinguished for physical reasons. Those elements g of G for which

$$g(0,0)=(0,0) \qquad \text{(E3)}$$

form a subgroup H, the stability group of the origin.

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* For further properties of the λ matrices see Macfarlane, Sudbery, and Weisz (1968).

* This geometrical view is also espoused by Finkelstein (1968), Isham (1969), Meetz (1968), Volkov (1968), and Hiida, Ohnuki, and Yamaguchi (1968).

If wo first consider the special case that there are no fields ψ, we may, with the help of H (which is assumed given), give a simple realization of G on the manifold. In this case there is only one orbit* (i.e., every point is connected to the origin by some $g\in G$), and it is possible to represent the manifold by the cosets of the stability group. Recall that the set of cosets, G/H, is given by xH, where x runs through the group. Now for $y\in G$, yxH is again a coset, i.e., a group element takes us from one coset to another. We may thus take G/H as our manifold and as coordinates take those parameters that parametrize the cosets.

For the group of interest to us, every element of G may be written in the product form

$$g=\exp(-i\boldsymbol{\xi}\cdot\bar{\mathbf{Q}})\exp(-i\mathbf{u}\cdot\mathbf{Q}), \tag{E4}$$

where the $\mathbf{Q}$ are the generators of the subgroup H, and the $\bar{\mathbf{Q}}$ are the generators orthogonal to the $\mathbf{Q}$. Hence ξ's parametrize the cosets and can now be taken as coordinates for our manifold. Their transformation law is given as follows: for an arbitrary group element g_0, $g_0\exp(-i\boldsymbol{\xi}\cdot\bar{\mathbf{Q}})$ is again a group element. We may therefore write

$$g_0\exp(-i\boldsymbol{\xi}\cdot\bar{\mathbf{Q}})=\exp(-i\boldsymbol{\xi}'\cdot\bar{\mathbf{Q}})\exp(-i\mathbf{u}'\cdot\mathbf{Q}), \tag{E5}$$

where

$$\boldsymbol{\xi}'=\boldsymbol{\xi}'(\boldsymbol{\xi},g_0),$$
$$\mathbf{u}'=\mathbf{u}'(\boldsymbol{\xi},g_0). \tag{E6}$$

Thus the transformation

$$g_0:\quad \boldsymbol{\xi}\rightarrow\boldsymbol{\xi}'(\boldsymbol{\xi},g_0) \tag{E7}$$

is the desired nonlinear realization. The authors of the paper cited above show that when there are other fields ψ present, it is always possible to bring their transformation law into the form

$$g_0:\quad \psi\rightarrow D[\exp(-i\mathbf{u}'\cdot\mathbf{Q})]\psi. \tag{E8}$$

$D(h)$ is a linear unitary representation of the subgroup H. The transformation law of the ψ as well as that of the $\boldsymbol{\xi}$ has the property that it is linear, i.e., independent of $\boldsymbol{\xi}$ (the meson field) when g_0 is restricted to the subgroup H. In that case

$$h\exp(-i\boldsymbol{\xi}\cdot\mathbf{Q})=[h\exp(-i\boldsymbol{\xi}\cdot\bar{\mathbf{Q}})h^{-1}]h$$
$$=[\exp(-i\boldsymbol{\xi}\cdot h\bar{\mathbf{Q}}h^{-1})]h$$
$$=[\exp(-i\boldsymbol{\xi}'\cdot\bar{\mathbf{Q}})]h. \tag{E9}$$

Hence $\exp(-i\mathbf{u}'\cdot\mathbf{Q})=h$ and (E8) reads

$$h:\quad \psi\rightarrow D(h)\psi. \tag{E10}$$

* In fact, our separation of fields into ϕ and ψ is made on the basis that if $\psi=0$, there is only one orbit.

From (E.9) we see that

$$\boldsymbol{\xi}'=D^{(b)}(h)\boldsymbol{\xi}, \tag{E11}$$

where $D^{(b)}(h)$ is the adjoint representation restricted to H and $\mathbf{Q}$.

The important result is that any nonlinear realization can, by a coordinate transformation, be cast into the form given by (E7) and (E8), and that the realization depends only on the subgroup H and its representation $D(h)$. A special application of this standard procedure may be found in the paper by Bardeen and Lee (1969), who use it to derive Eq. (9.34).

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Postbibliographic Note

Chiral symmetry continues to be an extremely active field of theoretical research. As noted by Renner (1968), over 500 papers on this subject had been published by mid-1967. It is not feasible to try to list all the papers that have appeared since that time. We here list a number of references, which, when combined with Adler and Dashen (1968), and Renner (1968), will enable the reader to construct a fairly full bibliography on chiral symmetry.

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