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Present Status of the Nuclear Three-Body Problem*

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The emphasis in this review is on the extent to which theoretical studies have accounted for the empirical data on the three-nucleon systems. The ground, excited, and continuum states of the nuclear three-body systems are discussed, and a summary is given of the interpretation of the electromagnetic- and weak-interaction properties.

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I. INTRODUCTION

The basic motivation for the theoretical investigation of the three-nucleon problem is to use the three-body data to add to our knowledge of the interaction between nucleons. The first and *only* success in this direction occurred in 1935: at a time when the two-nucleon data was completely consistent with a zero-range force, Thomas (1935) showed that such an interaction would give an infinite triton binding energy. Since then, there has been steady progress in our understanding of the three-body equations and systems, matched by an equally steady but faster growth in the study of the two-nucleon system and in the complexity of the assumed two-nucleon interaction. Only in the last year has there been any sign that realistic three-nucleon calculations are becoming feasible. Such calculations can in principle clarify two fundamental issues: first, the accuracy of the existing potential-model predictions for the off-the-energy-shell elements of the two-nucleon scattering matrix; and second, the existence and strength of any explicit three-nucleon forces.

By far the most important three-nucleon observable is the triton binding energy. The early calculations of this quantity used variational methods* and the equivalent two-body method† to estimate the triton binding energy for a variety of simple interactions. It was soon established that the triton is much more sensitive than the deuteron to the details of the nuclear force. Rarita and Present (1937) showed that simple central nuclear potentials cannot simultaneously account for the binding energies of the deuteron, triton, and α particle. This result was subsequently attributed to the presence of noncentral nuclear forces. The first quantitative calculation of the triton binding energy with tensor forces was by Gerjuoy and Schwinger (1942). Later calculations established that a central plus tensor force was sufficiently flexible to fit both the *existing* two-nucleon data and the triton binding energy (Clapp, 1949; Hu and Hsu, 1951; Pease and Feshbach, 1952). By 1950 the two-nucleon data indicated the presence of short-range repulsion, and the calculations of Feshbach and Rubinow (1955) and Ohmura (formerly Kikuta) *et al.* (Kikuta, Morita, and Yamada, 1957; 1956; Ohmura, 1959) demonstrated that the triton binding energy was sensitive to this property of the two-nucleon interaction. The task of solving the three-nucleon problem with potentials that contain both noncentral and short-range repulsive terms, and that fit the observed two-body data, has only recently been carried through using-

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* Variational calculations of the triton binding energy with simple central potentials have been far too numerous to quote here. The earliest such calculations are those of Thomas (1935) and Rarita and Present (1937). Calculations including hard cores are those of Kikuta, Morita, and Yamada (1957); Ohmura (1959); Ohmura and Ohmura (1962); Tang, Schmid, and Herndon (1965); van Wageningen and Kok (1967); Blatt and Derrick (1958).

† The equivalent two-body method has been revived recently despite its major disadvantage—that it is a finite approximation yielding no information on its own accuracy. For recent calculations using the method, see van Wageningen and Kok (1967); Fiedeldey, Erens, van Wageningen, Homan, and Kok (1968); Bodmer and Ali (1964); Kok (unpublished, 1968).

variational techniques to solve the Schrödinger equation (Delves, Blatt, Pask, and Davies, 1969). The difficulties encountered are inherent in the form of the interaction rather than in the method of solution.

An alternative formulation of the problem is given by the Faddeev equations, and a number of recent calculations have used direct finite-difference methods for the solution of these equations. For local potentials, this involves the solution of a set of coupled integral equations in at least two continuous variables; so far, only the simplest case of three spinless particles has been solved (Osborn, 1967; Humberston, Hall, and Osborn, 1968). However, the problem is simplified to the solution of coupled integral equations in just *one* continuous variable if separable nonlocal interactions are assumed; i.e., the partial-wave projection of the potential is expressed as

$$V_l(r, r') = \sum_n \lambda_n g_n(r) g_n(r').$$

This approach, which was pioneered by Mitra (1962), Kharchenko (1962), and Sitenko and Kharchenko (1963), has the chief advantage of permitting a simplification of both the bound-state and scattering problems, two problems which hitherto had only been investigated separately. Furthermore, it provides a simple framework in which to investigate the sensitivity of three-nucleon properties to the variation of particular two-nucleon parameters. However, it is evident that this approach is unsuitable for testing most models for the nucleon–nucleon interaction. There is no particular physical interest in making a separable expansion for the interaction, and it is probably preferable to try and tackle the three-body problem with local (or at least asymptotically local) potentials; one would then include the most reliable feature of the nucleon–nucleon interaction: the one-pion-exchange component. The basic idea of the separable potential approach, the introduction of a particular interaction chosen to simplify the three-body equation, was predated (by some years) by the work of Skorniakov and Ter-Martirosian (1956).

In spite of our limited knowledge of the three-nucleon, bound-state wave functions, the study of magnetic moments (Sachs and Schwinger, 1946; Villars, 1947) and charge and magnetic form factors (Collard, Hofstader, Hughes, Johansson, Yearian, Day, and Wagner, 1964; Schiff, 1964; Gibson and Schiff, 1965; Gibson, 1965) has led to useful results. There is clear evidence for appreciable modifications of the magnetic properties (though none for the charge properties) of nucleons within the three-particle nuclei. Such modifications, often referred to as interaction or exchange effects, are due primarily to electromagnetic interactions with virtual mesons. In addition, it seems likely that mesonic effects are significant in the β decay of ^{3}He (Blin-Stoyle and Papageorgiou, 1964; Blin-Stoyle, 1964). However, a full analysis of mesonic effects requires not only a reliable theory of strong interactions but also a three-nucleon wave function which bears at least some correspondence to the interactions from which the mesonic currents are derived.

The uncertainties in the structure of the three-nucleon continuum wave functions allow only preliminary and incomplete interpretations of the neutron–deuteron scattering and radiative-capture reactions and the electromagnetic disintegrations of the three-particle nuclei. However, Faddeev's theory of three-particle scattering (Faddeev, 1963; 1961) and the recognition that the gross properties of the positive-energy, three-nucleon wave functions could be obtained using simple two-nucleon interaction mechanisms of a separable nonlocal type (Yamaguchi and Yamaguchi, 1954),* has provided a simple basis for future work in this field (Amado, 1963; Lovelace, 1964; Aaron, Amado, and Yam, 1966; 1965; 1964; Phillips, 1968a; 1968b; 1966; Barbour and Phillips, 1968).

The purpose of this article is to present a (necessarily rather condensed) picture of the physical significance of existing three-nucleon calculations with local or separable interactions, and of the electromagnetic and weak interactions of the three-nucleon systems. Though the complexities of the three-body problem and of the two-nucleon interaction lead to difficulties in interpreting the details of the quantitative results, we shall see that the *gross* features of both the three-nucleon-scattering and bound-state data can indeed be understood in terms of the main features of the nonrelativistic interaction between two nucleons. Furthermore, it is possible to gain some qualitative ideas as to the dependence of three-nucleon observables, particularly the triton binding energy, on various aspects of the two-nucleon interaction. Unfortunately, most of the existing three-nucleon data, with the exception of the triton energy and the doublet S-wave neutron–deuteron scattering length, are rather insensitive to the details of the nuclear force. Three-body calculations also provide insight into the possible structure of the three-nucleon wave functions. It is this information which provides the basis for the interpretation of the electromagnetic and weak-interaction properties of the three-particle nuclei.

We do not attempt in this article to cover the mathematical techniques behind the three-body calculations; a number of detailed reviews of these and related subjects have already appeared in the literature (Noyes, 1968; Faddeev, 1965; Watson and Nuttall, 1967; Duck, 1968). We refer the reader to Faddeev (1965) and Watson and Nuttall (1967) for the subtleties of the mathematical aspects of three-body scattering theory;

* The Yamaguchi potential is the most commonly used nonlocal, separable potential. It can be adjusted to fit the $^1S_0$ and $^3S_1$ effective-range parameters, and if a tensor component is included, the deuteron quadrupole moment and D-state probability. This procedure leads to an approximate fit to the medium-energy nucleon–nucleon data, but does not give the change of sign of the S-wave phase shifts at high energies.

to Duck (1968) for the details of the separable-interaction approach; and to Noyes (1968) for a discussion of the significance of properties of the two-nucleon interaction to the nuclear three-body problem.

II. GROUND STATES OF THREE NUCLEONS

A. General Features

Because of the existence of noncentral nuclear forces, the only absolute space–time quantum numbers of the three-particle nuclei are the total angular momentum J, its projection J_z, and the parity π. Conservation of charge implies that T_z is also conserved; and for a completely charge-independent Hamiltonian, the isotopic spin T^2 is also a good quantum number. However, in the real world the Coulomb interaction in ^3He, and the breakdown of charge independence implied by the difference in the proton and neutron masses and magnetic moments, as well as any explicit charge dependence of the nuclear potential, all imply that T^2 is only an approximate quantum number. Thus the complete wave function of ^3He or ^3H may be written as a linear combination of terms with various values for the orbital angular momentum L, the total spin S, and the isotopic spin T. However, the dominant component of the three-nucleon wave function, *the principal S state*, can be derived from the assumption of spin- and isospin-independent, central nucleon–nucleon forces. This state is completely symmetric under interchange of the space coordinates of the particles and has the quantum numbers

$$(J, L, S, \pi, T) = (1/2, 0, 1/2, +, 1/2).$$

This result is consistent with experiment: First, the isotopic spin is $T=1/2$, since ^3He and ^3H are particle-stable nuclei of roughly equal binding energies, whereas the trineutron and ^3Li are almost certainly not stable at all (see Sec. III). Second, the total angular momentum is indeed $J=1/2$ by direct measurement of the optical hyperfine structure of tritium (Li, 1951); with $J=1/2$ the electric quadrupole moment is necessarily zero. Third, the experimental results on the magnetic moments, electromagnetic form factors, the photodisintegration of ^3H and ^3He, and the inverse processes (see Sec. V) strongly indicate that the dominant component of the ground state is spatially symmetric with $L=0$.

Although the quantum numbers L, S, and T are not conserved with actual nuclear forces, they can still be used to simplify and systematize the construction of the most general three-nucleon wave function. Two distinct ways of doing this have appeared in the literature. The first, due originally to Gerjuoy and Schwinger (1942), is based on the observation that all three-nucleon wave functions with $J=1/2$ can be constructed by operating on the principal S-state function by means of scalar functions (i.e., functions which commute with $\mathbf{J}$) of the spin and space vectors associated with the system. Sachs (1953) extended the Gerjuoy–Schwinger classification by including the isospin formalism. This classification omitted three (rather unimportant) states, which were included later in the similar construction of Cohen and Willis (1962). The second classification, due primarily to Derrick and Blatt (1958), factors out the rotational symmetry of the wave function by introducing a set of Euler angles to describe the relevant rotations. Similar classifications have been given by Shimamoto (1959) and by Feshbach (1942). The arbitrary (J, π, T) three-nucleon state has been considered by Kalotas and Delves (1964). We give here a brief resumé of the salient properties of the Derrick and Blatt classification, which was historically the first to be brought to systematic completion and which has been used in much of the later work.

The wave function depends on the position vectors $\mathbf{r}_i$, the spin coordinates ζ_i, and the isospin coordinates η_i of the particle; $i=1, 2, 3$. Removal of the dependence on the center-of-mass vector leaves six coordinates to describe the spatial motion of the particles. The three interparticle distances r_{12}, r_{23}, and r_{31} may be taken as three of these six coordinates. The remaining three coordinates may be chosen as the Euler angles α, β, and γ which define the orientation of the triangle formed by vectors $\mathbf{r}_i$. Our aim is to separate the dependence of the wave function on the interparticle distances r_{ij} from its much simpler dependence on the spin, isospin, and Euler-angle variables by making an expansion in a convenient complete set of spin–isospin–Euler-angle functions Y_i:

$$\Psi = \sum_i f_i(r_{12}, r_{23}, r_{31}) Y_i(\alpha\beta\gamma, \zeta_1\zeta_2\zeta_3, \eta_1\eta_2\eta_3). \quad (2.1)$$

We start by recalling the permutation symmetries available to an arbitrary three-particle wave function.

Each partition of the number 3, namely $3=3$, $3=2+1$, and $3=1+1+1$, is associated with a different irreducible representation of the permutation group on three objects. The partition $3=3$ gives rise to the completely symmetric representation, i.e., all permutations are "represented" by $+1$. The partition $3=1+1+1$ gives rise to the completely antisymmetric representation. To shorten our notation, we denote the symmetric representation by the letter s; the antisymmetric representation by the letter a. The third irreducible representation, corresponding to the partition on $3=2+1$, is less generally familiar; it will be denoted by the letter m for "mixed." Given any function $F(1, 2, 3)$ of three things, we can decompose it into three mutually orthogonal functions:

$$F = F_s + F_m + F_a.$$

The function F_s is completely symmetric,

$$F_s(1, 2, 3) = \tfrac{1}{6}[F(1, 2, 3) + F(2, 1, 3) + F(1, 3, 2) + F(3, 2, 1) + F(2, 3, 1) + F(3, 1, 2)], \quad (2.2)$$

F_a is completely antisymmetric,

$$F_a(1,2,3)=\tfrac{1}{6}[F(1,2,3)-F(2,1,3)-F(1,3,2)-F(3,2,1)+F(2,3,1)+F(3,1,2)], \quad (2.3)$$

and F_m is given by

$$F_m(1,2,3)=\tfrac{1}{6}[4F(1,2,3)-2F(2,3,1)-2F(3,1,2)]. \quad (2.4)$$

The arbitrary function $F(1,2,3)$ may be written as the sum of two functions $G(12,3)$ and $H(12,3)$ which are, respectively, even and odd under the interchange of 1 and 2. The equation (2.4) can then be used to define the functions G_m and H_m. Each of these functions can be further decomposed into a part which is symmetric and a part which is antisymmetric under the interchange of 2 and 3. We have, for example,

$$G_{m,1}=\tfrac{1}{6}[G(12,3)+G(31,2)-2G(23,1)], \quad (2.4a)$$

$$G_{m,2}=\tfrac{1}{2}[G(12,3)-G(31,2)], \quad (2.4b)$$

where $G_{m,1}$ and $G_{m,2}$ or $H_{m,1}$ and $H_{m,2}$ are the basis functions for the mixed representation of the permutation group, the permutation (i,j) being represented by a 2-by-2 matrix.

For the internal function f_i [Eq. (2.1)] all permutation symmetry types are possible. Since the spin states are constructed from just two linearly independent functions (the two components of the nucleon spinors), only states of symmetric and mixed permutation symmetry are possible, and these states correspond, respectively, to total spin of 3/2 and 1/2. Similarly, the isospin 3/2 and 1/2 states have symmetric and mixed permutation symmetry. The Euler-angle functions, which determine the orientation in space of the triangle formed by the three particles, are the conventional rotation matrices which represent the rotation necessary to bring a system of axes fixed to the triangle (the body axes) into coincidence with a space-fixed system of axes. The analytic properties of these functions are well known [see, for example, Kalotas and Delves (1964) and the references cited there]. For a given orbital angular momentum L there are $(2L+1)^2$ independent Euler-angle functions:

$$D_{\mu M}{}^{L}(\alpha,\beta,\gamma); \qquad \mu, M=-L, -L+1, \cdots +L.$$

The suffix M denotes the projection of L on the space-fixed axes, and the suffix μ the projection on the body-fixed axes. The symmetry properties of the $D_{\mu M}{}^{L}$ depend on the choice of body axes. If the body axes of Kalotas and Delves are adopted, symmetrized Euler-angle functions can be defined which are either completely symmetric or antisymmetric; the mixed representation does not occur. These functions have the form

$$Y_M{}^L(p_E,\mu)=A_{\mu,p_E}D_{\mu M}{}^L+B_{\mu,p_E}D_{-\mu M}{}^L,$$

where $p_E=a$ or s and $\mu=0, 1, \cdots, L$. The coefficients A and B are given in Kalotas and Delves (1964). The parity of the functions is $(-1)^{\mu}$. For each value of μ, except $\mu=0$, there is one symmetric and one antisymmetric function, while for $\mu=0$ there is only one function, which is symmetric or antisymmetric accordingly as L is even or odd. Hence, for example, we obtain the following even-parity functions:

$L=0$: $Y_0{}^0(s,0)$;

$L=1$: $Y_M{}^1(a,0)$; $\quad M=-1, 0, +1$;

$L=2$: $Y_M{}^2(s,0)$, $\quad Y_M{}^2(s,2)$, $\quad Y_M{}^2(a,2)$; $\quad M=-2$ to $+2$.

Thus, apart from the trivial degeneracy in M, there are five independent Euler-angle functions for the $J^P=1/2^+$ three-nucleon system.

We now have to construct the spin–isospin–Euler-angle functions Y_i of Eq. (2.1). These may be built up to have definite permutation symmetry and definite total angular momentum. Functions of symmetric, anti-symmetric, and mixed permutation symmetry are possible; we write schematically as Y_s, Y_a, and $Y_{m,1}$ and $Y_{m,2}$. An over-all antisymmetric wave function is written by multiplying these angular functions by an internal function of appropriate adjoint symmetry. We obtain three different types of products, each of which is separately antisymmetric; they are

$$f_sY_a, \; f_aY_s,$$

and

$$-f_{m,2}Y_{m,1}+f_{m,1}Y_{m,2}.$$

We note that the third combination is antisymmetric, but that the separate terms $-f_{m,2}Y_{m,1}$ and $f_{m,1}Y_{m,2}$ by themselves are not.

For the three-nucleon $J=1/2$, $\pi=+$, and $T=1/2$ system the result of the construction is summarized in Table I. There are 10 states: three S states, four P states and three D states. In this table, an entry m refers to a pair of functions of mixed symmetry; thus the expansion (2.1) contains 16 terms and the three-body Schrödinger equation has been reduced from a partial differential equation in six independent variables to a set of 16 coupled partial differential equations with only three independent variables.

B. Qualitative Discussion of the Triton Bound-State

We do not expect all of the 10 states to be equally important in the triton. First let us consider the S states: The principal S state has a completely symmetric internal function. It follows immediately that the expectation values of the odd-parity nuclear interactions are indentically zero. Further, the spatial distribution of each nucleon is independent of whether it corresponds to a neutron or a proton. Thus, the effective interaction in the principal S state is the isospin-independent part of the even-parity nuclear forces. By the Pauli principle, such an interaction is also spin independent, and hence the effective force in the principal S states of the three-particle nuclei is

$$V_{\text{eff}} \approx \tfrac{1}{2}({}^3V_c^+ + {}^1V_c^+), \tag{2.5}$$

where ${}^3V_c^+$ is the central force in the triplet-even state of the two-body system and ${}^1V_c^+$ is the central force in the singlet-even state. There is also a very small contribution from the spin-independent part of the quadratic $\mathbf{L}\cdot\mathbf{S}$ force. The principal S state is expected to remain the most important even in the presence of strong tensor forces; in practice it contributes more than 90% of the normalization of the wave function.

The second S state has a fully antisymmetric internal function, which is therefore zero whenever the triangle formed by the three particles is isosceles. The more zeros a function has, the larger its derivatives and hence the larger its kinetic energy; accordingly, the coupling of this state into the ground-state wave function is expected to be completely unimportant. This is well borne out by recent variational estimates (Delves, Blatt, Pask, and Davies, 1969), which give a probability of around 0.003% for this state.

The third S state has an internal function of mixed symmetry and is important enough to have attained a notation all to itself: it is usually referred to as the S' state. It is mixed in to first order by the difference between the singlet and triplet potentials, and since this difference is small, we do not expect more than a few percent S' state. The exact S' probability $P(S')$ turns out to affect the theoretical predictions for a number of observable parameters; these include the charge form factors of ^{3}H and ^{3}He (Sec. V.B), the radiative–deuteron capture (Sec. V.D) and the β decay of the triton (Sec. VI.A). Estimates for $P(S')$ range from 0.1% to 4% and depend quite strongly on the form of the potential used (Ohmura, 1969).

The P states are all expected to be very small. They are coupled in to first order by the $\mathbf{L}\cdot\mathbf{S}$ potential and to second order by the tensor potential. Estimates show that the $\mathbf{L}\cdot\mathbf{S}$ force is rather unimportant in the triton (Derrick, 1960), and this result is borne out in practice. To a good first approximation, P states and $\mathbf{L}\cdot\mathbf{S}$ forces may be neglected.

The D states on the other hand cannot be neglected. They are coupled to the principal S state by the noncentral part of the two-nucleon potential, i.e., chiefly by the tensor potential; and this potential is very strong in any current fit to the two-body data. The triton D-state probability may be as high as 9%; calculations using separable and local potentials both lead to D-state probabilities of about 6% and 9%, respectively, for potentials yielding deuteron D states of 4% and 7%, respectively (see Table III). It was noted long ago that the tensor force reduces the ratio of the theoretical binding energy of light nuclei to the binding energy of the deuteron (Inglis, 1939). There are two effects responsible for this result: First, in the triton a nucleon pair has approximately a probability of one-half of being in a singlet spin state in which the tensor force does not act at all. Second, because of the large spatial extent of the deuteron, the centrifugal barrier, which acts against the admixture of D states, is less effective in the deuteron than it is in the triton. Any actual nuclear potential to be used in triton calculations must be adjusted so that it at least fits the deuteron binding energy. Thus a potential with a large tensor force (e.g., the Hamada–Johnston potential) will give less binding for the triton than a potential with smaller tensor component.

There is also the possibility of $T=3/2$ states generated by Coulomb forces, charge-dependent nuclear forces, and the neutron–proton mass difference. Of these states the most important is a mixed-symmetry S state in ^{3}He. Recent detailed calculations (Ohmura, 1967, 1969; Bell, unpublished) indicate that a $T=3/2$ admixture of 0.01% to 0.001% is likely.

We now sum up: Because of the dominance of the principal S state, the main effective central force is the average of the singlet-even and triplet-even central forces (2.5). Although there is a contribution from the quadratic $\mathbf{L}\cdot\mathbf{S}$ force, the operator (2.5) gives the only significant expectation value in the principal S state of the triton. The other forces act only indirectly, by admixing the D, S', and P states. In determining the binding energy, the triplet-even tensor force which admixes the D states is the most important. Next in importance is the difference between the triplet-even and singlet-even forces; this difference, a Bartlett force,

TABLE I. Classification of triton wave functions.

Spectroscopic classification	L	S	Permutation symmetry: Internal	Euler angles	Spin–isospin	$\|\mu\|$
${}^2S_{1/2}$	0	1/2	s	s	a	0
	0	1/2	a	s	s	0
	0	1/2	m	s	m	0
${}^2P_{1/2}$	1	1/2	s	a	s	0
	1	1/2	a	a	a	0
${}^4P_{1/2}$	1	1/2	m	a	m	0
	1	3/2	m	a	m	0
${}^4D_{1/2}$	2	3/2	m	s	m	0
	2	3/2	m	s	m	2
	2	3/2	m	a	m	2

TABLE II. Expectation values of the various components of the Hamada–Johnston Hamiltonian in ^{3}H (Humberston, Hawkins, Hennell, and Wallace, 1968); units are megaelectron volts. The calculation has not converged.

Triton energy	−2.79
Kinetic energy	+75.71
Central	−40.36
Tensor	−35.79
Spin–orbit	−0.99
Quadratic spin–orbit	−1.53

admixes the S' state. Of least importance are the $\mathbf{L}\cdot\mathbf{S}$ force, the quadratic $\mathbf{L}\cdot\mathbf{S}$ force, and odd-parity forces. Tables II and III give weight to these statements. Table II lists the expectation values of the various components of the Hamada–Johnston potential (Hamada and Johnston, 1962) obtained in variational calculations (Delves and Blatt, 1967; Humberston, Hawkins, Hennell, and Wallace, 1968; Davies, 1967a; 1967b). (The Hamada–Johnston potential, which includes one-pion exchange and a repulsive hard core, has central, tensor, spin–orbit, and quadratic spin–orbit components and provides a semiquantitative fit to the two-nucleon data.) Table III gives the theoretical probability densities of the various triton admixtures for a number of two-nucleon interactions. The variational results (Delves, Blatt, Pask, and Davies, 1969) are for the Hamada–Johnston potential. The separable-potential results (Phillips, unpublished; see Phillips, 1968; Borysowicz and Dabrowski, 1967)* correspond to approximate fits to the two-nucleon data, but indicate the sort of variations that are to be expected as the parameters of the two-nucleon interaction are varied.

C. Calculations of the Triton Binding Energy

A most important physical problem is to determine the dependence of the theoretical triton binding energy on the various properties of the two-nucleon interaction. Among the important properties of this interaction which are not uniquely determined by two-nucleon experiments are: The short- and medium-range radial dependence of the interaction and in particular the short-range repulsion; the relative strengths of the tensor and central components; and the off-the-energy-shell behavior of the scattering amplitudes, which in part is related to the locality or nonlocality of phenomenological potentials. These properties and their effect on the triton binding energy are usually considered independently and within the context of simple models for the nucleon–nucleon interaction. The results of such an approach should not be taken too literally.

* A later calculation using the same potential (Jaffe and Reiner, 1968) gave binding energies that differed from those of Borysowicz and Dabrowski (1967) by 0.4 MeV.

The effect of short-range repulsion in central potential models on the theoretical binding energy is illustrated in Fig. 1. The variational calculations of Ohmura (1959) and Tang, Schmid, and Herndon (1965) assumed local exponential potentials with a hard core; the low-energy parameters were taken to be $a_s=-23.69$ fm, $r_s=2.7$ fm, $a_t=5.28$ fm, and $r_t=1.70$ fm. The Ohmura results are upper bounds and those of Tang *et al.* (1965), were estimated using both upper and lower bounds. The Borysowicz–Dabrowski (1967) calculation used a nonlocal separable potential with a hard-shell repulsion. Because of the nonlocality of this potential, the probability density inside the core radius is finite but small. The low-energy parameters of the potential differ from the exponential potential only in the value of the singlet-effective range which was taken to be $r_s=2.5$ fm. The results corresponding to the local and nonlocal potentials are very similar and indicate a fairly rapid decrease in the magnitude of the binding energy with increasing core radius (van Wageningen and Kok, 1967). There are two opposing effects at work which lead to this net result: The introduction of the hard core eliminates part of the configuration space of the three-particle system, causing an increase in the kinetic energy; it increases by a factor of 2 as the core radius increased from 0 to 0.6 fm (Kikuta, Morita, and Yamada, 1957; 1956; Ohmura, 1959). However, there is a compensating effect in the potential energy: In order to retain the fit to the two-body data, the attrac-

TABLE III. Theoretical predictions for the probability densities of $-S'$, D, and P states of the triton. The variational results are taken from Delves, Blatt, Pask, and Davies (1969) and the separable-model results from Phillips (unpublished) and Borysowicz and Dabrowski (1967).

Two-nucleon potential		$P(S')$%	$P(D)$%	$P(P)$%
Hamada–Johnston		2	9	0.03
Noncentral separable; no repulsion				
Deuteron D state %	Singlet effective range (fm)			
0.0	2.85	1.60	0.0	0.0
0.0	2.70	1.40	0.0	0.0
4.0	2.70	1.32	5.77	0.02
5.5	2.70	1.29	8.05	
7.0	2.70	1.27	10.0	
Central separable; with short-range repulsion				
Core radius (fm)	Singlet-effective range (fm)			
0	2.5	1.1	0.0	0.0
0.4	2.5	2.0	0.0	0.0

tion just outside the repulsive core must be increased (i.e., a "sticky core" is introduced). If a tensor force is included, the central force approaches the "sticky core" limit more rapidly (Biedenharn, Blatt, and Kalos, 1958). Therefore, a more realistic calculation including such effects may indicate that Fig. 1 overestimates the effect of the hard core on the triton binding energy. Indeed, an extreme (and of course unrealistic) calculation including only the *central part* of a (central and tensor) potential leads to an *increasing* triton binding energy with increasing hard-core radius (Blatt and Derrick, 1958).

One can simulate short-range repulsion in the nucleon–nucleon interaction in various ways; e.g., local hard-core and soft-core potentials, strongly velocity-dependent potentials, and nonlocal separable potentials with either smooth or sharp repulsion. (A nonlocal potential may, of course, be represented as a velocity-dependent potential.) Tabakin (1965) has investigated the sensitivity of the theoretical triton binding energy to the type of short-range repulsion in a separable potential model. Using two spin-independent potential models fitted to the same on-shell scattering data, he found that the use of a hard-shell repulsion results in 5% less binding than a smooth repulsion. However, a calculation by Folk and Bonnem (1965) found that a velocity-dependent or a hard-core representation of the repulsion gave essentially the same binding energy if the potentials have the same S-wave phase shifts. Similarly, the equivalence of local hard-core and soft-core potentials has been demonstrated in a simple model (Afnan and Tang, 1968).

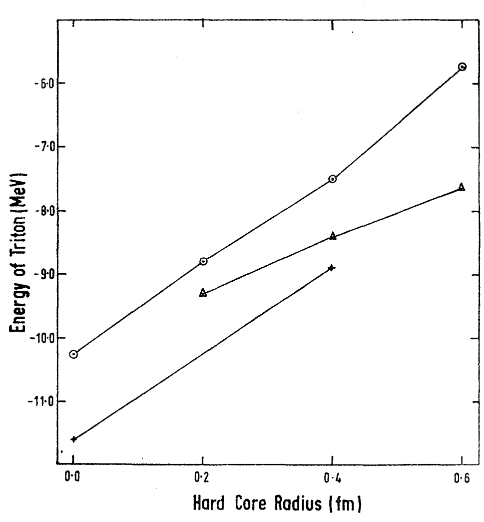

Fig. 1. The effect of short-range repulsion on the energy of the triton. The abscissa represents the hard-core radius. The circles and triangles refer to two sets of variational calculations with local exponential potentials: the circles, by Kikuta, Morita, and Yamada (1957) and Ohmura (1959); the triangles, by Tang, Schmid, and Herndon (1965). The crosses correspond to a direct calculation using nonlocal separable potentials by Borysowicz and Dabrowski (1967).

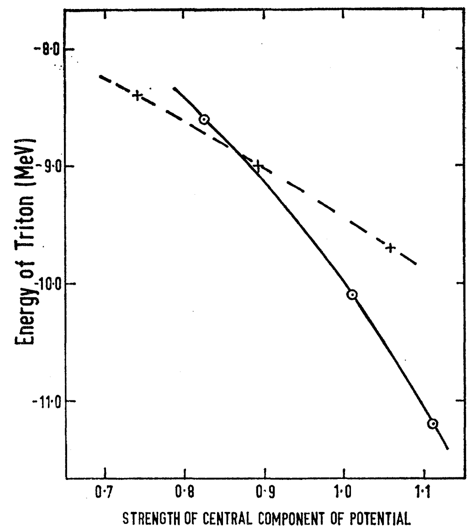

Fig. 2. The effect of the strength of the tensor force on the energy of the triton. The abscissa represents the central potential strength $S(^3V_c{}^+)$. An increase in $S(^3V_c{}^+)$ corresponds to a decrease in the tensor-potential strength. The circles refer to a variational calculation (Blatt, Derrick, and Lyness, 1963) with local Pease–Feshbach potentials and the crosses to a direct calculation (Phillips, 1968c) with nonlocal separable potentials.

An early investigation of the sensitivity of the triton binding energy to the properties of the nucleon–nucleon tensor force is due to Pease and Feshbach (1952). This variational calculation assumed a Yukawa-type central-plus-tensor potential with no hard core. The range of the tensor component was varied, keeping all the low-energy scattering parameters roughly constant with the exception of the triplet-effective range. The calculation was repeated with an improved wave function by Blatt, Derrick, and Lyness (1963). The results show that, as expected, the binding energy decreases quite rapidly as the fraction of the triplet interaction contributed by the tensor force is increased. The inclusion of tensor forces in a separable model calculation is straightforward and leads to qualitatively similar results. Purely attractive S-wave Yamaguchi interactions (Yamaguchi and Yamaguchi, 1954) overbind the triton by about 3 MeV; the actual value of the binding energy depends quite strongly on the singlet-effective range. But the inclusion of a tensor force, for which the deuteron D state is 4%, leads to a reduction of about 2 MeV in the binding energy (Sitenko and Kharchenko, 1965; Bhakar and Mitra, 1965). These calculations have been extended so as to include the predictions corresponding to a deuteron D-state probability of 5.5% to 7%, all other low-energy parameters being kept constant (Phillips, 1968).

The main effect of the tensor force in the local and nonlocal calculations is shown vividly in Fig. 2, which plots the energy of the triton against the nondimensional strength parameter of the central component of the potential. For both the Pease–Feshbach potential

(Pease and Feshbach, 1952) and the separable Yamaguchi potential (Yamaguchi and Yamaguchi, 1954), the binding energy is a strongly decreasing function of the tensor strength.

The importance of the off-the-energy-shell behavior of the nucleon–nucleon scattering amplitudes is illustrated by recent calculations of Kok, Erens, and van Wageningen (1968). It is possible to construct a local potential, a generalized Bargmann potential (Newton, 1960), to fit the phase shifts of a nonlocal separable potential at *all* energies. Kok *et al.* found that a separable Yamaguchi potential binds a system of three identical, spinless particles with an energy of -12.4755 MeV and the equivalent generalized Bargmann potential binds with an energy of -10.90 MeV; this difference in the off-the-energy shell behavior leads to a difference of $\sim 14\%$ in the three-particle binding energy. This result suggests that the unknown behavior of the two-body, t-matrix off shell leads to an uncertainty of ~ 1.2 MeV in the triton binding energy.

It is evident from this discussion that the triton binding energy is likely to be sensitive to various two-nucleon properties which cannot easily be determined. A detailed description of the nucleon–nucleon short-range repulsion is beyond the scope of any nonrelativistic theory: Deuteron photodisintegration is sensitive to the ratio of the tensor-to-central components of the nuclear force, and nucleon–nucleon bremsstrahlung may give information on the off-shell behavior of the amplitudes; but the interpretations of both these phenomena are clouded by the problem of mesonic exchange corrections to the electromagnetic Hamiltonian. Accordingly, the testing of phenomenological potentials in the three-nucleon system is likely to play an important role in the understanding of the fundamentals of the nucleon–nucleon interaction. However, the interpretation of such calculations requires reliable estimates of the effects of three-body forces and relativistic corrections.*

With a more ambitious attempt to calculate the triton binding energy for an interaction yielding a detailed fit to the two-body-data, the computational difficulties rise sharply. Separable potential calculations have been carried out by Schrenk and Mitra (Schrenk and Mitra, 1967; Mitra, Schrenk, and Bhasin, 1966) and by Dabrowski and Dworzecki (1968; to be published) using (central and tensor) attractive interactions in the triplet state and a central potential, including also a repulsive term in the singlet state. Their results are roughly comparable: both calculations *overbind* the triton by 0.3–1.5 MeV. Since the interactions used do not provide a detailed fit to the two-body data, it is as yet too early to assess the relevance of this overbinding when compared with the underbinding (by a similar amount) for a local phenomenological potential (see below).

Investigations of the nuclear three-body problem using "realistic" local phenomonological potentials involve the use of variational techniques. A sequence of such calculations have been carried out* for a number of local potentials fitted to the two-nucleon data and containing hard cores and noncentral components.

The earlier calculations in this sequence served to show two things. First, as expected, the calculated binding energy decreased sharply with increasing tensor potential strength. Second, the results show in distressing clarity the "sticky core" effect referred to earlier. As the tensor strength is increased, the two-body data appear to require (at least in the potential models considered) that the central potential become rapidly very deep over a narrow region outside the hard-core radius. The Hamada–Johnston central potential (Hamada and Johnston, 1962), for instance, reaches a peak attractive depth of well over 1000 MeV in both the singlet and triplet states, over a narrow range of less than 0.5 fm outside the hard core. This ill-mannered behavior makes it very difficult to construct adequate variational wave functions for these potentials, and this is reflected in the clear lack of convergence of the early calculations.† However, later work appears to have overcome the difficulty (Delves, Blatt, Pask, and Davies, 1969), partly by introducing terms in the trial function which involve explicitly the deuteron wave function (and hence reproduce the very strong two-body correlations implied by the sticky core nature of the potential), and partly by the brute-force method of including many terms in the trial function. The upper bounds $E_u(N)$, found by Delves *et al.* for the Hamada–Johnston potential with trial functions containing up to $N=40$ terms, are shown in Fig. 3. From this figure one would estimate that $E(N)$ has converged to the exact value $E_u(\infty)$ for the potential used to within at most a few tenths of a megaelectron volt. This is consistent with the lower bounds $E_L(N)$ computed from the same wave functions, which are shown in Fig. 4. These are still very far from the upper bounds, values of $E_L(40)\sim -60$ MeV being the best available. However, if we define a parameter η,

$$\eta(N)=[E_L(N)-E]/[E-E_u(N)], \qquad (2.6)$$

then it is well understood that, for large N, values of $\eta \gg 1$ are to be expected (Delves, Blatt, Pask and Davies, 1969); the upper bound is much closer to the

* Recent papers on three-body forces include: Loiseau and Nogami (1967); Pask (1967); Brown, Green, and Gerace (1968); McKellar and Rajaraman (1968). Relativistic effects have been considered by: Primakoff (1947); Gupta, Bhakar, and Mitra (1965).

* Delves and Blatt (1967); Davies (1967a; 1967b) Blatt, Derrick, and Lyness (1963); Derrick and Blatt (1960); Blatt and Delves (1960); Delves, Blatt, Pask, and Davies (1969).

† We do not wish to blame the variational approach for this. The form of the potential leads to similar difficulties for a direct finite-difference solution of the equations; such a solution appears still quite impracticable today.

exact eigenvalue than the lower. For interactions with hard cores, we expect values of several hundreds for η. If we are willing to guess η, we can of course solve Eq. (2.6) for the eigenvalue E; more realistically, a lower bound on η leads to a lower bound on E. Such "lower bounds" are plotted in Fig. 3 for the assumptions $\eta=50$, $\eta=100$ and show that the direct lower bounds of Fig. 4 are quite consistent with the apparent convergence of the upper bounds. The final values quoted by Delves *et al.*, including an estimate of the numerical accuracy of the calculation, are

$$E_{^3H}(\text{H.J.}) = -6.7\pm1.0 \text{ MeV},$$

$$P(S')=2\%, \qquad P(P)\simeq0.03\%, \qquad P(D)=9\%.$$

The S'- and D-state probabilities as a function of N are shown in Fig. 5.

A value of around 7 MeV for the binding energy leaves a discrepancy of 1.5 MeV with experiment, to be attributed to a three-body potential, relativistic effects, or a "defect" in the Hamada–Johnston potential. Existing estimates of the contribution from three-body potentials and relativistic effects are in the range 1.5 to 2 MeV,* but these estimates are not sufficiently complete to be definitive. A single calculation yields no information on the sensitivity of the triton binding energy to the detailed structure of the potential, and it is important that this sensitivity should be explored. This is especially so since there exist potential fits (Bressel, Kerman, and Rouben, 1968; Reid, 1968) to the two-body data, whose radial dependence is very much less singular than that of the Hamada–Johnston potential.

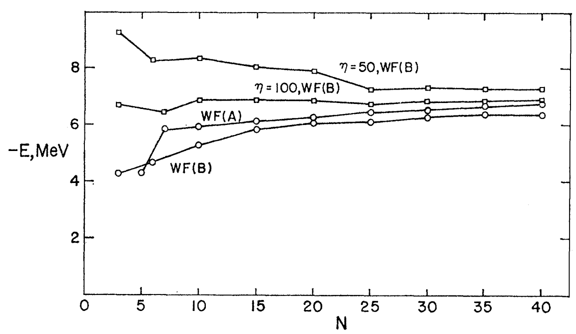

Fig. 3. Upper (○) and modified lower (□) bounds on the triton binding energy $E_u(N)$ for two distinct wave functions A and B. The lower bounds are defined by Eq. (2.6) of the text. Figures 3–5 are taken from Delves, Blatt, Pask, and Davies (1969) and relate to calculations with the Hamada–Johnston potential (Hamada and Johnston, 1962).

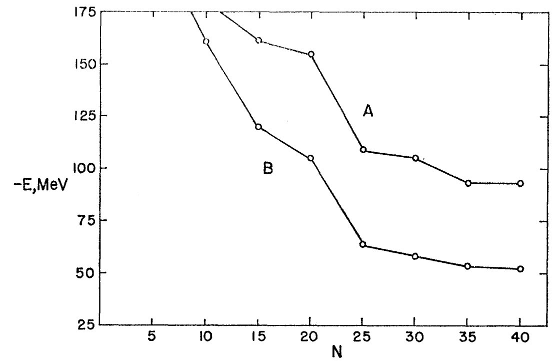

Fig. 4. Unmodified Temple lower bounds for the wave functions A and B of Fig. 3.

III. POSSIBLE EXCITED STATES OF THREE NUCLEONS

A number of searches have been made for possible excited bound and unbound resonant states of three nucleons. The most interesting possibility is a bound state of three neutrons (^3n). Initial evidence (Adjačič, Cerineo, Lakovic, Paić, Šlaus, and Tomaš, 1965) in the ^{3}H(n, p) $3n$ reaction for a bound trineutron has not been supported by later experimental work on the same reaction (Thornton, Blair, Jones, and Willard, 1966; Fuschini, Maroni, Uguzzori, Verodini, and Vitale, 1967). No evidence was found in the ^{7}Li$+n$ reaction (Fukikawa and Morinaga, 1968) [^{7}Li$(n, {}^3n)$ ^{5}Li or ^{7}Li(n, p) ^{7}He$\rightarrow\alpha+{}^3n$] or in the ^{4}He$+\pi^-$ reaction (Kaufman, Perez-Mandez, and Sperinde, 1968); in this last reaction an upper limit of $0.074+0.015$ μb/sr was established for the cross section for a trineutron with energy -3 to $+3$ MeV. Further, there is no evidence in the ^{3}He$(p, n)3p$ reaction for excited $3p$ states (Anderson, Wong, McClure, and Pohl, 1965; Cookson, 1966), nor for a nnp state in ^{3}He(n, p) nnp (Antolkovic, Cerineo, Paić, Tomaš, Ajdjačič, Lalovic, van Oers, and Šlaus, 1966). With regard to possible excited ppn states, Kim, Bunch, Devins, and Foster (1966) reported three narrow peaks in the proton spectrum in ^{3}He(p, p') ppn

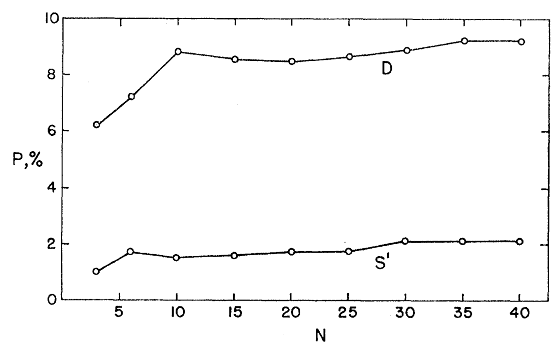

Fig. 5. The triton S'- and D-state probabilities for wave function B.

* Recent papers on three-body forces include: Loiseau and Nogami (1967); Pask (1967); Brown, Green, and Gerace (1968); McKellar and Rajaman (1968). Relativistic effects have been considered by: Primakoff (1947); Gupta, Bhakar, and Mitra (1965).

reactions. But Cerny, Detraz, Pugh, and Šlaus (unpublished) found no evidence for such peaks in the same reaction at a slightly lower energy, and inelastic electron scattering on ^{3}He gave no evidence for excited states for excitation energies up to 17 MeV (Frosch, Crannell, McCarthy, Rand, Safrata, Suelzle, and Yearian, 1967).

Theoretical information on three-nucleon excited states is very limited. Calculations with purely attractive, spin-independent central forces give rise to a $T=1/2$, $J=1/2$ S-wave bound excited state which is present whenever the two-body subsystem is bound; this is the case both for local (Osborn, 1967; Humberston, Hall, and Osborn, 1968) and separable interactions (Aaron, Amado, and Yam, 1964a). The wave function of this excited state corresponds to a nucleon in an extended orbit about a compact two-nucleon subsystem (Osborn, 1967; Humberston, Hall, and Osborn, 1968), and we shall see in Sec. IV.A that the state is an essentially two-body one, owing its existence to the single-particle exchange diagram between the odd particle and the bound two-body subsystem. The inclusion of spin-dependent and tensor forces weakens the attraction in the three-nucleon system, and for this case neither local nor separable potential calculations show any evidence for a $T=1/2$ bound excited state.

The most favorable system in which to look for a $T=3/2$ state is the three-neutron system. It is easy to show that, provided the spin–isospin independence of nuclear forces [i.e., $SU(4)$ symmetry] is a fair approximation, any bound n^3 is primarily supported by the triplet-odd, two-nucleon interaction. With spin–isospin independent nuclear forces, the $T=3/2$, $S=1/2$ nnn state and the $T=1/2$, $S=3/2$ nnp state are degenerate. Thus an $S=1/2$ trineutron should be reflected in the $S=3/2$ nnp system. The phase-shift analysis (van Oers and Brockman, 1967) and separable-model calculations (Aaron, Amado, and Yam, 1965; 1964b; Phillips, 1966) are in complete agreement in their predictions for the $S=3/2$ neutron–deutron scattering amplitude; the S-wave and D-wave phase shifts are negative, and the P-wave phase shift, though positive, is less than 34° in the energy range 0 to 14 MeV. In order to admit the possibility of a strong attraction in the $S=3/2$ nnp state and hence the possibility of bound ^3n with $S=1/2$, it would be necessary to disregard the phase-shift analysis and include strong odd-parity nuclear forces in the separable-model calculations. Alternatively, there may be a $T=3/2$, $S=3/2$ trineutron bound state; such a state is completely dependent on the triplet-odd nuclear force. Thus it seems likely that the existence of a trineutron depends on the strength of the triplet-odd and in particular the ^3P nucleon–nucleon interaction. As emphasized by Mitra and Bhasin (1966) (see also Mitra, 1966), this has the effect of disentangling the question of the existence of a ^3n from that of a ^4n. Similarly, the evidence (Okamoto and Davies, 1967) against the existence of ^3n provided by the systematics of the neutron separation energies in light nuclei with $N=3$ is not conclusive. It should also be noted that noncentral forces play an important part in the triplet-odd, two-nucleon state. Accordingly, it may be a poor approximation to assume that the orbital angular momentum and total spin of a trineutron system are constants of motion.

There have been a number of calculations which looked for a three-neutron bound state. A calculation by Okamoto and Davies (1967) found that the Pease–Feshbach potential (Pease and Feshbach, 1952) was too weak (by a factor of 2) to bind the $P_{1/2}$ three-neutron state. (S and L were assumed to be constants of motion.) The Pease–Feshbach potential is independent of the parity of the two-nucleon state, and the ^3P scattering for this potential is primarily determined by the fit of the potential parameters to the ^3S data. As a result, the attraction in the ^3P two-nucleon state is greater than that given by more modern potentials. The $^2P_{1/2}$ and $^4P_{1/2}$ three-neutron states were considered by Barbi (1967) using central, local, exponential potentials with and without hard cores; this calculation has been repeated by Bell and Delves (to be published) using much more refined trial functions. In both calculations ^3n was found to be unbound for acceptable values of the potential parameters.

These local-potential results disagree with a calculation using P-state, central, *separable* potentials by Mitra and Bhasin (1966), who concluded that, for a reasonable choice of the potential strength, ^3n is bound. However, this calculation appears to contain a numerical error; an application of the lower-bound technique by Hall and Post (1967) shows very simply that, for the potential strengths quoted, the trineutron cannot be bound. We also note that the features of the two-nucleon S state (the deuteron bound state and the singlet antibound state), which lead us to expect a separable model to give a reasonable representation of the S-wave interaction (Amado, 1963; Lovelace, 1964), are completely missing in the P states.

In conclusion, the bulk of the experimental and theoretical evidence is against the existence of a bound ^3n. In addition, it seems possible that there are no excited states at all of ^{3}H and ^{3}He.

IV. THE THREE-NUCLEON CONTINUUM STATES

A. Zero-Energy Neutron–Deuteron Scattering

The theoretical problems associated with the calculation of the neutron–deuteron scattering amplitudes are far less severe for energies below the threshold for three-particle breakup. In fact for these energies calculations using simple two-nucleon interactions may be sufficiently precise that meaningful comparisons may be made between the scattering and the bound-state results.

Of particular importance is the zero-energy scattering which is parametrized by two numbers, the doublet and quartet scattering lengths, ${}^2a$ and ${}^4a$, respectively. The experimental situation was for many years both static and ambiguous; the data was consistent with either of two sets of scattering lengths (Hurst and Alcock, 1951):

$$\text{Set A:}\quad {}^2a=0.7\pm0.3\text{ fm},\quad {}^4a=6.4\pm0.1\text{ fm};$$

$$\text{Set B:}\quad {}^2a=8.3\pm0.1\text{ fm},\quad {}^4a=2.6\pm0.2\text{ fm}.\quad (4.1)$$

Much of the early theoretical work (that is, up to and including 1964–1965) was devoted to removing this ambiguity in the experimental data. This work came down heavily in favor of set A, and a recent experiment (Alkimenkov, Luschikov, Nikolenko, Taran, and Shapiro, 1967) involving the scattering of polarized neutrons from polarized deuterons confirmed that ${}^4a$ is larger than ${}^2a$. However, an analysis (van Oers and Seagrave, 1967; Seagrave and van Oers, 1967) of the latest experimental results (Donaldson, Bartolini, and Otuski, 1966; Bartolini, Donaldson, and Groves, 1968; Gissler, 1963) revise set A significantly to read*

$$\text{Set A}':\quad {}^2a=0.15\pm0.05\text{ fm},\quad {}^4a=6.13\pm0.04\text{ fm}.$$

The difference between the doublet scattering lengths of sets A and A′ is quite large; for a given potential shape a change of ${}^2a$ from 0.7 to 0.1 fm is accompanied typically by an increase of the order of 0.5 to 1 MeV in the triton binding energy. Independent confirmation of set A′ would therefore be very desirable. The set A′ is determined from the recent measurements of the coherent neutron scattering amplitude (Bartolini, Donaldson, and Groves, 1969),

$$f_{\text{coh}}=(\sigma_{\text{coh}}/4\pi)^{1/2}={}^4a+{}^2a/2=6.21\pm0.04\text{ fm},$$

and the spin-incoherent cross section (Gissler, 1963)

$$(3/\sqrt{2})f_{\text{inc}}=\sqrt{2}(\sigma_{\text{inc}}/4\pi)^{1/2}={}^4a-{}^2a=5.99\pm0.06\text{ fm}.$$

These equations are presented graphically in Fig. 6. It should be noted that the new solution, set A′, corresponds to a total cross section for free neutrons of $\sigma_{\text{free}}=3.154\pm0.04$ b. The set-A results, which are indicated by the shaded rectangle, are based on the measurements of the ratio of ${}^2a/{}^4a=0.12\pm0.04$ (Hurst and Alcock, 1951) (obtained by scattering from ortho- and paradeuterium) and the value of $\sigma_t=3.44\pm0.06$ b for the total cross section for epithermal neutrons (Fermi and Marshall, 1949). As emphasized by Seagrave and van Oers (1967), the epithermal value for the cross section is in disagreement with the value $\sigma_{\text{free}}=3.2\pm0.1$ b obtained from extrapolating the elastic cross section to zero energy and the value given above of $\sigma_{\text{free}}=3.154\pm0.04$ b; it is likely that corrections due

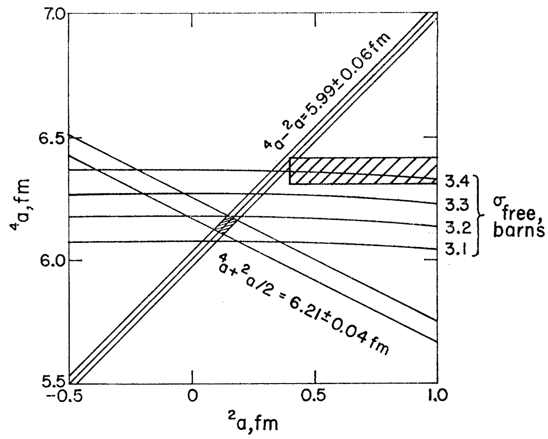

FIG. 6. Plot of ${}^4a$ vs ${}^2a$ showing the intersections of the two linear relations derived from the coherent and incoherent n–d date, together with portions of the ellipses corresponding to the indicated values for σ_{free}. The set-A values for ${}^2a$ and ${}^4a$ are shown as a shaded rectangle. This is taken in part from van Oers and Seagrave (1967).

to residual binding effects could account for this discrepancy (Løvseth, 1962). However, in order to reconcile the Hurst–Alcock experiment (Hurst and Alcock, 1967) with the new data, it would be necessary to double the quoted error in the value for ${}^2a/{}^4a$.

For *central* potentials, variational calculations* all give results for the quartet scattering length in good agreement with set A (or set A′). But for the doublet state widely varying results for ${}^2a$ were obtained. This is not at all surprising; the symmetry of the quartet state ensures that the particles stay well apart, and hence the quartet scattering length is relatively insensitive to the details of the force law and to details of the trial function used. This is not so for the doublet state, and early calculations used inadequate trial functions for this state. Of the more recent variational calculations, that of Pett (1967) gave a preliminary value for ${}^2a$; its good agreement with set A′ is certainly accidental, since a better trial function should lower the scattering length ${}^2a$ to near that of Humberston (1964), who obtained a value of −2 fm for a central Yukawa potential. That this value is algebraically too small for either set A or A′ is not surprising, since the actual potential used certainly strongly overbinds the triton.

The separable-model results† for *central* potentials give quartet scattering lengths in good agreement with set A (or set A′). The absence of short-range repulsion and tensor forces in these calculations result in too much attraction in the ${}^2S_{1/2}$ state. Consequently, the values

* The figures quoted in the text are those which follow from the later results of Bartolini *et al.* (1968) for the n–D coherent-scattering length. Note that the abstract of this reference misquoted ${}^2a=0.13\pm0.05$ fm.

* Sartori and Rubinow (1958); Burke and Haas (1959); Efimov (1959); Humberston (1964); Pett (1967).

† Aaron, Amado, and Yam (1965, 1964b); Phillips (1966a); Mitra and Bhasin (1963); Sitenko and Kharchenko (1963).

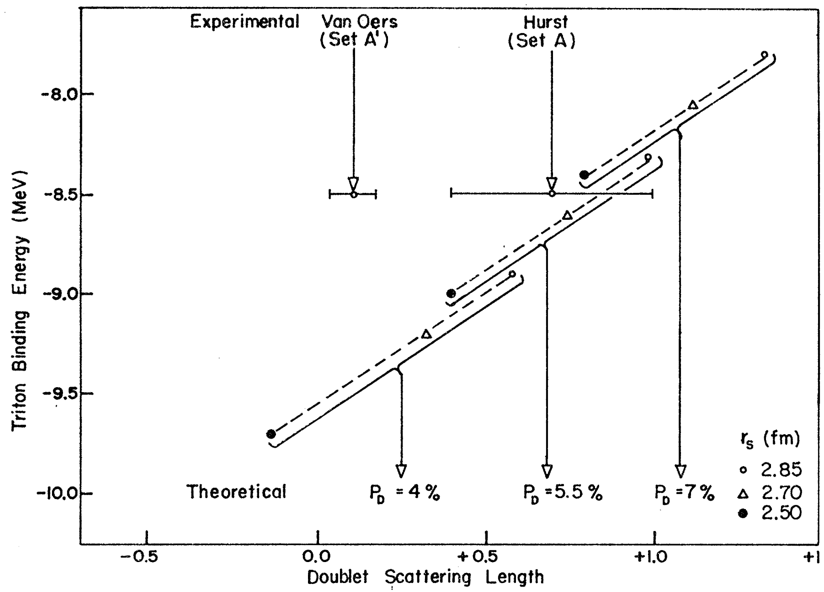

FIG. 7. Separable-potential predictions for ^2a and the triton energy for various sets of assumed deuteron D-state probabilities P_D and singlet two-nucleon effective range r_s; taken from Phillips (1968c).

for the doublet scattering length are algebraically too small; they range from -1 to -2 fm.

The triton binding energy is known to be sensitive to the potential shape; for example, Yukawa and separable potentials fitted to the same low-energy data give binding energies of 12.76 and 11.24 MeV, respectively (Noyes and Osborn, 1968). Thus, it is instructive to compare Humberston's (1964) Yukawa potential result for ^2a with those obtained by solving the neutron–deuteron scattering problem exactly, using central separable potentials. A separable-model calculation by Sitenko, Kharchenko, and Petrov (1966) illustrates the way that ^2a varies as the singlet scattering length and effective range are varied. Humberston's result corresponds to a rather weak singlet interaction with $a_s = -7.42$ fm and $r_s = 3.25$ fm and unfortunately lies outside the region considered by Sitenko. Nevertheless, it is evident that a separable-model calculation using the same low-energy parameters as Humberston leads to a considerably larger (algebraically) scattering length $^2a \sim -0.3$ fm. For a separable potential, a change in ^2a from -0.3 to -2.0 fm is accompanied by an increase of the order of 2 MeV in the triton binding energy. Thus the doublet scattering length seems to be at least as sensitive (and maybe more sensitive) as the binding energy to the shape of the potential.

The separable-model calculations have been extended so as to include a tensor component (Sitenko and Kharchenko, 1965; Phillips, 1968; Mitra, Schrenk, and Bhasin, 1966). In the calculation of Phillips (1968) the parameters of the potential were adjusted to fit the deuteron binding energy and quadrupole moment, the triplet scattering length, and the singlet scattering length; the deuteron D-state probability P_D and the rather poorly determined singlet-effective range r_s were allowed to vary. The predictions of this calculation for both the doublet scattering length and triton binding energy are shown in Fig. 7. We see that it is possible to choose P_D and r_s to fit both the experimental binding energy and the doublet scattering length of set A, but not of set A′. Taken at its face value, this calculation makes the set A′ very inconvenient, and hence it is important to try to estimate the magnitude of the effects not included in the calculation. The most important of these effects is the repulsive core. It is quite conceivable that the core is more effective in reducing the binding energy than in increasing the scattering length, thus making it possible to fit the set A′, provided low values for P_D and r_s are taken. Accordingly, it is interesting to look at the recent separable calculation of Schrenk and Mitra (1967) which does include a repulsive soft core (but only in the singlet state). This calculation also fails to fit simultaneously the binding energy and the doublet scattering length of set A′ and again favors the set A.

Variational calculations have also been carried out on ^2a with local potentials which fit the two-body data (Davies, 1967a; 1967b; Delves and Blatt, 1967; Delves, Lyness, and Blatt, 1964; Delves, Blatt, Pask, and Davies, 1969). These calculations yield an upper bound on ^2a, and the most accurate results for the Hamada–Johnston (H.J.) potential (Hamada and Johnston, 1962) to date are shown in Fig. 8, taken from Delves, Blatt, Pask, and Davies (1969). The numerical accuracy for ^2a is not very high, and the final value quoted is

$$^2a = 1.2 \pm 1.0 \text{ fm} \quad \text{(H.J. potential)}.$$

This value is associated with a triton binding for the same potential of around 7 MeV. If we estimate the change in ^2a associated with an increase in the binding

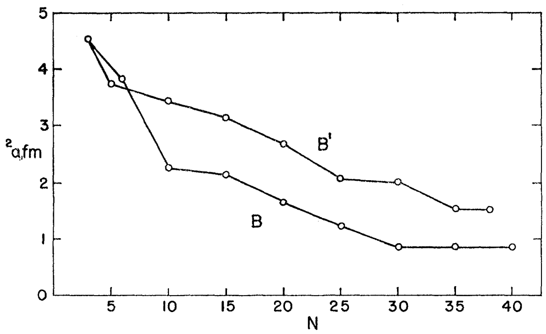

FIG. 8. Doublet scattering length ^2a for two distinct trial functions B and B'. N is the number of terms in the trial function.

energy to the observed 8.4 MeV, we find an extrapolated value

$$^2a = 0.8 \pm 1.0 \text{ fm} \quad (E_T = -8.4 \text{ MeV}).$$

That is, neither separable nor local potentials fitted to the triton binding energy appear to like set A′, although it is still within the error bounds quoted above. However, we see below that the relation between the doublet scattering length and triton binding energy is nontrivial and likely to be highly potential dependent; it is therefore perhaps too early to read any significance into this difficulty.

B. Structure of the Low-Energy N–D Scattering Amplitudes

At nonzero energies the even-parity $J=1/2$ state contains two coupled channels; the S-state doublet channel is coupled to the D-state quartet scattering state. Hence the scattering matrix is 2×2 and is characterized by three real parameters: two eigenphase shifts and a mixing parameter. One of these eigenphase shifts, the doublet phase shift δ_α, contains the doublet scattering length. There is similarly a quartet alpha phase shift belonging to the $J=3/2$ even-parity state which contains the quartet scattering length; and these scattering lengths are defined by the well-known

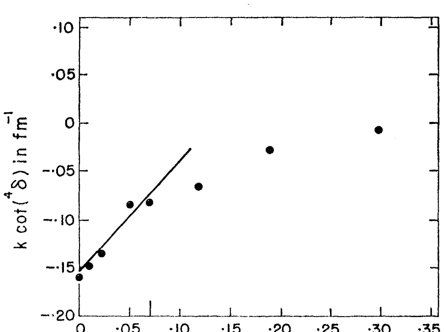

FIG. 9. The quartet phase shift k cot ($^4\delta$); The unbroken line is the calculated N/D result. The experimental points are from van Oers and Seagrave (1967).

effective-range expansion

$$k \cot \delta = -1/a + \tfrac{1}{2} r_0 k^2 + \cdots. \tag{4.2}$$

Van Oers and Seagrave (1967) have carried out an approximate phase-shift analysis of low-energy n–d scattering; their results for the quartet and doublet S-wave phase shifts are shown in Figs. 9 and 10. The quartet phase shifts are fitted well for small k^2 by an expansion of the form (4.2) although the effective range r_0 is not well determined. (The phase shifts at very low energies are not accurately enough known, while at higher energies the assumptions which have gone into this preliminary phase shift analysis are questionable.) A straight-line fit to the data shown corresponds to the parameters

$$^4r_0 \sim 2.2 \text{ fm}, \qquad ^4a = 6.6 \text{ fm}.$$

The doublet k cot ($^2\delta$), on the other hand, shows no

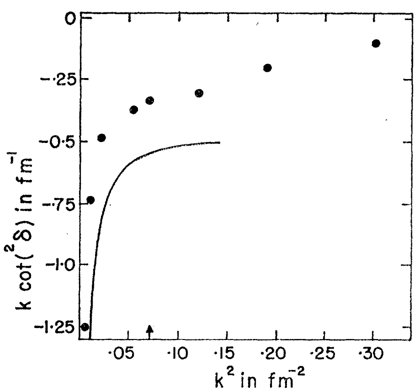

FIG. 10. The doublet phase shift k cot ($^2\delta$). The unbroken line is the calculated N/D result fitted to a scattering length of $^2a = 0.11$ fm. The experimental points are from the phase-shift analysis of van Oers and Seagrave (1967).

recognizable linear portion. On the contrary, the experimental points strongly suggest that a continuation of k cot ($^2\delta$) below threshold will contain a pole for a small imaginary value of k. This suggestion agrees with the calculated doublet phase shifts for a separable model (Phillips, unpublished; see Phillips and Barton, 1969; Barton and Phillips, to be published). The existence of such a singularity has been suggested previously (Delves, 1960a)* from an analysis of the n–d wave function in a resonating-group calculation, and on the grounds that, in the absence of a singularity, the measured small value of ^2a is inconsistent with either the triton binding energy or the observed electric-dipole, n–d photodisintegration cross section.

We should expect a pole in $k \cot \delta$ whenever the attraction between the two scattering systems is

* The original suggestion for a pole in k cot δ was made by J. A. Gammel and G. L. Baker in a private communication quoted by Delves (1960a).

sufficiently large. If we consider the t matrix

$$t(k)=k^{-1}\sin\delta e^{i\delta},$$

we see that a *pole* in $k\cot\delta$ corresponds to a zero in t. Moreover, t is real below threshold (at least over a sufficiently small range determined by the range of the potentials—see below) and has a pole at each bound state of the system, with positive residue. It therefore necessarily has a zero (a pole in $k\cot\delta$) between each bound state. We saw earlier that the three-body system fails to support a bound-excited state only because the singlet and triplet interactions differ. Since the singlet interaction is only slightly less strong than the triplet, we expect the pole in $k\cot\delta$ to remain, although the excited state has moved up above the scattering threshold (Delves, 1968). This argument gives a qualitative explanation of the existence of a pole in $k\cot\delta$; however, the structure of the scattering amplitude below threshold is more complex than such an outline might suggest. In a potential scattering model with short-range forces, the scattering length passes through zero and becomes negative as the excited state becomes unbound. The observed doublet n–d scattering length is *positive*, and hence the potential-model picture is incomplete. The reason has been pointed out by Phillips and Barton (1969), Barton and Phillips (to be published), and independently by Reiner (1969): For a nonnegligible energy range around threshold, both the doublet and quartet scattering amplitudes are dominated by an exceedingly long-range interaction generated by the exchange of a proton between two alternative deuteron states.

The on-the-energy-shell partial-wave amplitudes for n–d scattering have right-hand- and left-hand-cut singularities. The two-body and three-body unitarity relations imply that there are two right-hand cuts with branch points at $k^2=0$ and $k^2=4/3\alpha_d^2$, where α_d^2 is the deuteron binding energy (i.e., $\alpha_d=0.2316$ fm^{-1}), and $\mathbf{k}$ is the relative n–d momentum. The position of the left-hand-cut singularities are determined by the range of the forces. In n–d scattering the longest-range force is due to proton exchange, and the associated cut is close to the scattering region. To see this we observe that the position of the cut is determined by the condition for an intermediate nnp state with the same energy as the initial and final n–d states. If $\mathbf{p}_n$ and $\mathbf{p}_n'$ are, respectively, the momenta of bound and incoming neutrons in the deuteron rest frame, this condition gives $p_n=i\alpha_d$ and $p_n'^2+2\mathbf{p}_n'\cdot\mathbf{p}_n-3p_n^2=0$. That is, $p_n'^2$ must lie between p_n^2 and $9p_n^2$; and since $\mathbf{k}=2/3\mathbf{p}_n'$, the partial-wave amplitude is expected to have a cut with branch points at $k^2=-4\alpha_d^2/9$ and $k^2=-4\alpha_d^2$.

It is useful to introduce the notation $z=3k^2/4\alpha_d^2$. Then the two-body and three-body unitarity branch points lie at $z=0$ and $z=1$, respectively; the branch points of the proton-exchange cut are at $z=-\frac{1}{3}$ and $z=-3$; the triton pole lies at $z=-2.9$; and all other left-hand singularities are much further from the origin than $-\frac{1}{3}$ Barton and Phillips, to be published). Thus, it is expected that to a reasonable accuracy, some of the features of the on-shell, S-wave n–d amplitudes near the elastic threshold can be understood in a two-body N/D approach dominated by the proton-exchange interaction.

It is straightforward to calculate the contribution for the proton-exchange diagram. (Note that the above condition, $p_n=i\alpha_d$, implies that the strength of this interaction is determined by the deuteron wave function outside the range of the nuclear forces, i.e., by the two-nucleon $^3S_1$ effective range parameters.) Barton and Phillips give an approximate solution for the N/D equations with a proton-exchange interaction. For three spinless particles the solution explicitly displays a bound excited state which should come into existence as soon as the "deuteron" becomes bound. This result provides a simple physical interpretation of the numerical results of Osborn (1967) and Aaron, Amado, and Yam (1964a) for spinless particles interacting via local and nonlocal potentials.

For particles with spin, the sign of the proton-exchange force depends on the total spin of the system. If the appropriate spin-coupling constants are inserted, the S-wave exchange force is repulsive in the quartet spin state and attractive in the doublet. The long-range repulsion in the quartet state implies that at low energies the particles cannot penetrate to the region where the other forces of shorter range can act. Thus the low-energy quartet amplitude, and in particular ^4a, are almost completely specified by the parameters of the proton-exchange diagram, i.e., by the two-nucleon $^3S_1$ effective-range parameters. The two-body model, without adjustable parameters, gives $^4a\cong 6.3$ fm and the phase shifts shown in Fig. 9 (Barton and Phillips, to be published). This calculation for the quartet state bears a strong resemblance to that of Skorniakov and Ter-Matirosiam (1956). These authors set up a very similar integral equation for the t-matrix under the assumption of zero-range forces, and showed that a quartet scattering length $^4a=5.1$ fm followed by using only the deuteron binding energy as input.

In the doublet state the proton-exchange force is attractive, and shorter-range forces are important. The latter are represented by Barton and Phillips by an additional constant background interaction to give a closed expression for $k\cot\delta$ containing one adjustable parameter which is fitted to the measured scattering length. This expression, which is plotted in Fig. 10, indicates that the pole in $k\cot\delta$ just below threshold is an automatic result of fitting the observed doublet scattering length and of the attractive proton-exchange force.

These simple calculations have quite far-reaching implications when we consider the information available about the two-nucleon interaction from the data. In particular, we see that once the deuteron binding energy is known, the spin 3/2 channel near threshold (including ^4a) offers *no new information whatsoever* about nuclear forces. By contrast, in the spin 1/2 channel, the value of ^2a is determined by real three-body effects and could well be sensitive to details of the two-body force which are not easily detected in the two-body system. The same is true of the triton binding energy. But even in this channel, once a numerically small value of ^2a has been achieved, the subsequent energy variation of $k \cot(^2\delta)$ follows automatically and therefore yields *no new qualitative* information. To gather any new information on the nuclear forces which is not included in a calculation of the triton binding energy and doublet scattering length, we must either move to higher energies (around or above the breakup threshold) or seek a precision fit that is better than the 10%–20% accuracy implicit in the model described.

C. Behavior near the Breakup Threshold

Above the threshold for deuteron breakup, the physical scattering matrix contains also the breakup channel. The opening of this channel also induces a singularity in the other matrix elements and, in particular, in the eigenphaseshift δ_α. The singularities induced are analogous to the familiar "Wigner cusps" induced in elastic scattering by the opening of an excited elastic-scattering channel, but they are not nearly so prominent. The inelastic scattering cross section has the following behavior near the threshold at $k^2=0$:

$$\sigma(\text{breakup}) \propto k^4. \qquad (4.3)$$

There is a corresponding k^4 singularity in $\cot \delta_\alpha$ at $k=0$ (Delves, 1961). For the $J^P=1/2^+$ or $3/2^+$ state and in the crude approximation in which coupled D states are ignored, the S-state scattering matrix contains the elastic channel and an infinite number of breakup channels; but near threshold only the lowest of the breakup channels, the phase-space channel, survives (Delves, 1960b). If we formally drop all except the elastic-scattering and the phase-space channel, the resulting 2×2 matrix is characterized by eigenphase shifts δ_α(elastic) and δ_β(breakup) and a mixing parameter ϵ:

$$S=\begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}\begin{pmatrix} \exp(2i\delta_\alpha) & 0 \\ 0 & \exp(2i\delta_\beta) \end{pmatrix} \times\begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix}. \qquad (4.4)$$

Above threshold, the δ_α and ϵ have well-behaved (real) expansions (Delves, 1961):

$$\cot\delta_\alpha=\delta_{\alpha0}+\delta_{\alpha2}k^2+\delta_{\alpha4}k^4+\cdots,$$
$$\tan\epsilon=k^2(\epsilon_0+\epsilon_2k^2+\cdots). \qquad (4.5)$$

Now *below* threshold it is the (1×1) scattering matrix $\exp(i2\delta_\alpha)$ which is unitary, and $\cot\delta_\alpha$ has the expansion

$$\cot\delta_\alpha-\delta_{\alpha0}+\delta_{\alpha2}k^2+[\delta_{\alpha4}-\epsilon_0^2(1-i)]k^4+\cdots \qquad (4.6)$$

That is, there is a finite singularity in $d^2\cot\delta_\alpha/d(k^2)^2$ at $k=0$, and this carries over to a finite singularity in the second derivative of the elastic-scattering cross section at the breakup threshold:

$$(d^2\sigma_{\text{el}}/dE^2)_{E=0+}-(d^2\sigma_{\text{el}}/dE^2)_{E=0-}=\text{finite}. \qquad (4.7)$$

This weak predicted singularity is in contradiction to the numerical results of Aaron, Amado, and Yam (1964a). The calculations of these authors show a discontinuity in $d\sigma/dE$ at the breakup threshold for a model of three spinless particles. This discontinuity occurs in a situation when the eigenphaseshift is passing through π in the region of the inelastic threshold, in which case the power expansions (4.5) and (4.6) are no longer meaningful.

D. Elastic Scattering and Breakup

We turn now to the subject of the interpretation of the mass of experimental data on the elastic scattering of neutrons and of protons by deuterons and of the corresponding breakup reactions. Unfortunately, nearly all of the theoretical work on these processes has been exploratory, containing so many simplifying assumptions that it is not possible to interpret the results other than in the most qualitative manner.

There exists a lengthy sequence of papers, chiefly by Massey and his school, examining the elastic scattering problem using the resonating-group approximation [see, for instance, De Borde and Massey (1966)* and references cited there]. The resonating-group method yields an approximation of unknown quality, and perhaps the major achievement of this series of papers is that they show that a reasonable fit is possible to the low-energy, elastic-scattering cross sections and angular distributions, using a suitable "equivalent central potential." This equivalent potential has exchange properties similar to that needed to fit the two-nucleon data, but a rather larger range. In view of the crudity of the calculations, we might feel justified in concluding that the main features of the low-energy, elastic-scattering process are not strongly dependent on the details of the two-nucleon interaction (such as noncentral forces and repulsive cores) and the three-body aspects of the problem. This conclusion agrees well of course with that given by the N/D calculation discussed earlier; we remark that, at least at low energies, the observed scattering is dominated by the quartet state.

The resonating-group and also variational calculations with simple trial functions involve gross approx-

* This article gives a review of the resonating-group calculations carried out on n–d scattering.

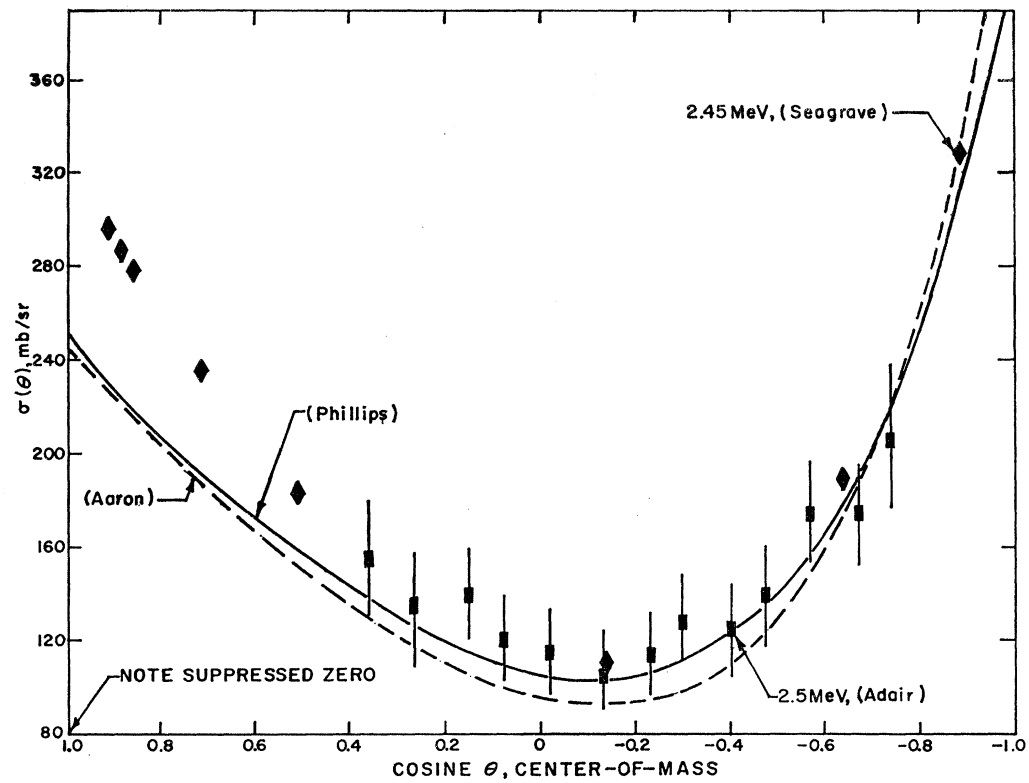

FIG. 11. Neutron–deuteron elastic-scattering angular distributions at neutron laboratory energy of 2.45 MeV; comparison of theory (Aaron, Amado, and Yam, 1966; Phillips, 1966a) with experiment (Adair, Kazaki, and Walt, 1953; Seagrave and Henkel, 1955).

imations to the three-particle aspects of the problem. These approximations are made in the cause of computational convenience, and for energies below the inelastic threshold it has been clear how to improve the calculations, given enough computing power. However, above the breakup threshold, there has been a more serious block. Until the work of Faddeev (1963, 1961), it was not in fact known how to set up a mathematically sound theory of three-particle scattering above the breakup threshold, and it is only since this work that any significant progress has been made. The direction in which computational progress has been made since is well known. The Faddeev equations are a set of coupled three-dimensional integral equations and are not tractable on current computers.* A particular form for the two-body interaction, the *separable* approximation, reduces the Faddeev equation to a small set of one-dimensional integral equations. This is a drastic approximation to the *interaction*, but it permits the exact treatment of the three-particle aspects of the problem. We might hope to obtain significant results in this way, provided that the separable interaction can give at least a first approximation to reality. Indeed, Amado (1963) and Lovelace (1964) have shown that, at least in the two-nucleon S states, it can. The basic assumption is that the singularities associated with the deuteron bound state and the singlet antibound state dominate the off-the-energy, two-nucleon amplitude that occurs in the Faddeev equations. The separable approximation yields simple analytic expressions for the two-nucleon amplitudes which have the correct analytic structure in the neighborhood of the two-nucleon bound and antibound states; and, in addition, the two-particle unitarity relation is satisfied, leading to equations which obey the three-particle unitarity relations. There has been only one definitive test of the accuracy of approximations of this kind (Kok, Erens, and van Wageningen, 1968): The *local* Hulthén potential and the separable *nonlocal* Yamaguchi potential may be adjusted to give exactly the same energy and wave function for the two-body bound state, and as a result the corresponding two-body scatter ng amplitudes have identical structure in the vicinity of the two-body, bound-state pole. Kok *et al.* found that a Yamaguchi potential binds a system of three identical, spinless particles with an energy of −12.4755 MeV; the equivalent Hulthén potential binds with an energy of −14.59 MeV. That is, the properties of the scattering amplitudes in the neighborhood of the two-body, bound-

* Such statements are always dangerously temporary. At least one direct attack on these three-dimensional integral equations with local potentials is now in progress.

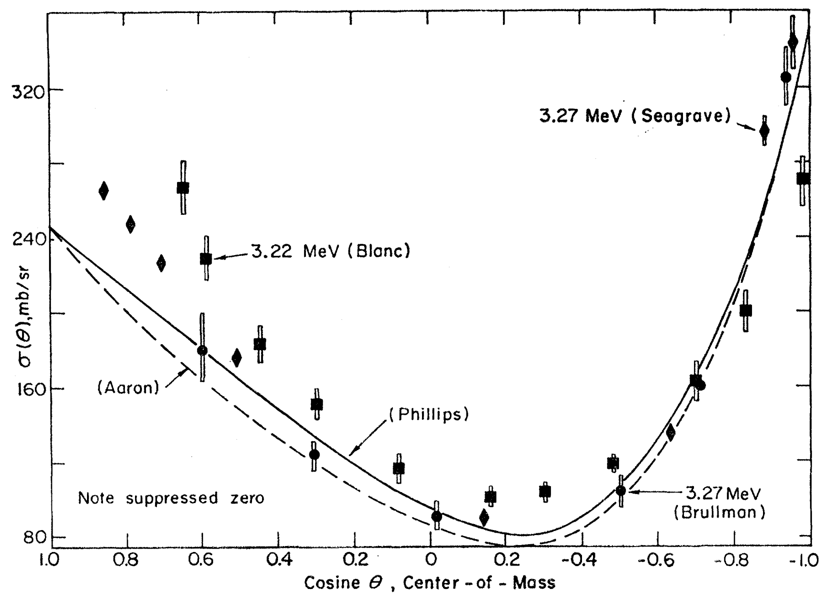

FIG. 12. Neutron–deuteron elastic-scattering angular distributions at neutron laboratory energy of 3.27 MeV; comparison of theory (Aaron, Amado, and Yam, 1966; Phillips, 1966a) with experiment (Seagrave and Cranberg, 1957; Blanc, Cambon, and Verdonne, 1966; Brüllmann, Gerber, Meier, and Scherrer, 1959).

state pole are sufficient to determine the three-particle binding energy to an accuracy of only ~85%. Inspection of the structure of the kernel of the Faddeev equations suggests that the accuracy of the separable approximation is better if the low-energy, three-particle, continuum states are considered. However, reliable estimates of this accuracy have been prevented by the difficulties encountered when using local potentials in continuum-state calculations.

The existing calculations of the neutron–deuteron elastic and inelastic cross sections by Aaron, Amado, and Yam (1966; 1965; 1964b) and Phillips (1966) have made the simplest possible assumption about the separable interactions, including only central $^1S_0$ and $^3S_1$ components. Such interactions give rise to too much attraction in the doublet neutron–deuteron state. In order to bring the theoretical doublet scattering length into agreement with experiment, it is necessary to introduce at least one phenomenological parameter. Aaron *et al.* take this parameter to be the deuteron wave-function renormalization parameter Z, i.e., the deuteron is not entirely a composite structure resulting from a $^3S_1$ central interaction. Phillips introduces a fictitious three-body potential. Both of these approaches have the merit of simplicity and the important advantage that the three-particle unitarity relations are obeyed. Both also allow the triton binding energy and the doublet scattering length of set A to be fitted simultaneously.

The predictions for the angular distributions in elastic neuteron–deuteron scattering at energies 2.45, 3.27, and 14 MeV are compared with the experimental results in Figs. 11 to 13. Apart from some differences at forward angles between the results of Blanc, Cambon, and Verdenne (1966) with those of Seagrave and Cranberg (1957) and Brüllmann, Gerber, Meier, and Scherrer (1959), the experimental data are in reasonable agreement. The discrepancy between the two calculations at 14 MeV is primarily due to the fact that an insufficient number of partial waves were included in the calculation of Phillips. The phase-shift-analysis results (van Oers and Brockman, 1967) are also included in Fig. 13. The separable-model calculations are equivalent to solving coupled Lippmann–Schwinger equations with effective nucleon–deuteron and nucleon–singlet potentials. These potentials are nonlocal, energy dependent, and correspond to the exchange of a nucleon from one three-body configuration to another. The basic exchange nature of the interaction produces the strong backward peaking in Figs. 11 to 13. The complete solution of the three-body equations includes the unitarity corrections, which in turn result in a large imaginary part for the forward amplitude and the forward peaking; the enhancement of the forward peak at 14 MeV due to the presence of the inelastic channel is quite marked. The fact that the agreement is less good in the forward direction is to be expected; here high impact parameter collisions involving the interaction in states other than the two-nucleon S states are certainly important. Nevertheless, these results suggest that the major features of the elastic scattering can be reproduced by extremely simple nuclear forces. In addition, the total cross sections, which are related to the elastic amplitudes by the optical theorem, are in agreement with the latest experimental values (Glasgow and Foster, 1967). We again caution that it is important not to be too dazzled by the agreement with experiment. At low and moderate energies, the scattering is dominated by the quartet state. In this state the nucleons are kept far apart so that the fine

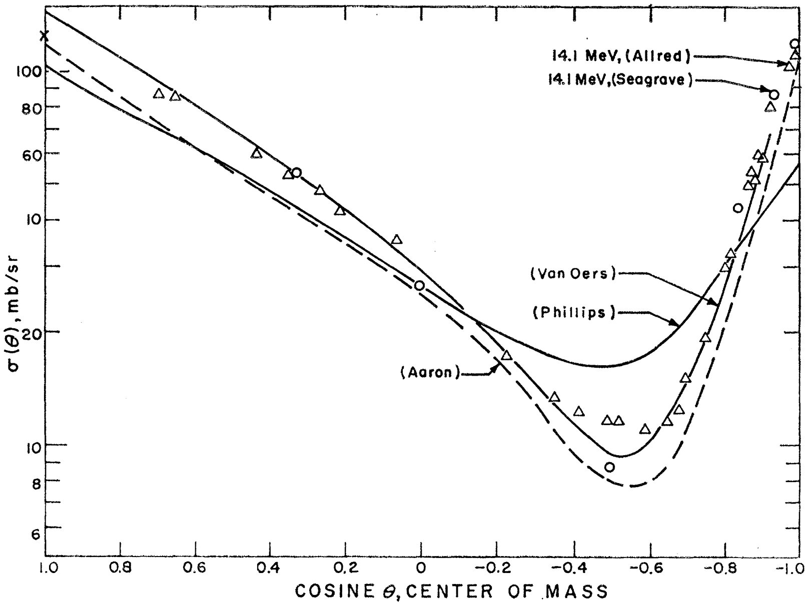

Fig. 13. Neutron–deuteron elastic-scattering angular distributions at neutron laboratory energy of 14 MeV; comparison of theoretical phase-shift analysis (Aaron, Amado, and Yam, 1966; Phillips, 1966) with experiment (Seagrave and Cranberg, 1957; Allred, Armstrong, and Rosen, 1953).

details of the interaction are unimportant and the distortion of the deuteron is small. Thus the elastic and total neutron–deuteron cross sections are not a particularly useful test of a dynamical three-body calculation.

Calculations have also been carried out on the breakup reaction $n+d \rightarrow n+n+p$ (Aaron, Amado, and Yam, 1966; Phillips, 1966). The basic mechanism of the separable-model calculations of the breakup reaction corresponds to an isobar model; breakup occurs as the result of the production, propagation, and subsequent decay of virtual deuterons and singlet antibound states. In this case the numerical problems associated with the complex singularities of the three-body equations require special treatment. One possible technique, first used by Hetherington and Schick (1965), is to solve the integral equations for complex momentum and then continue the solutions to the physical region. The calculation of Aaron and Amado adopted this technique and must be considered more reliable than that of Phillips.

The various sets of experimental data* on the breakup reaction are, where comparable, in agreement with each other. In Figs. 14 and 15 the results of Aaron and Amado for the energy spectrum of the breakup reaction are compared with some of the experimental data (Cerineo *et al.*, 1964; Ilakovac *et al.*, 1963). The general features of the spectrum are given quite well. The agreement is not as good as that obtained for the elastic-scattering cross sections, and this is probably due to the fact that the breakup reaction is more sensitive to the doublet state and hence to the details of the nuclear interaction. The fact that the theoretical cross section is too low at small angles and too high at large ones is presumably due to the neglect of the high-momentum components of the interaction.

Thus, the separable-model calculations can reproduce the gross features of the low-energy, three-nucleon system; in so doing, they indicate that sophisticated three-body dynamics are essential and that the details of the nuclear interaction are relatively unimportant.

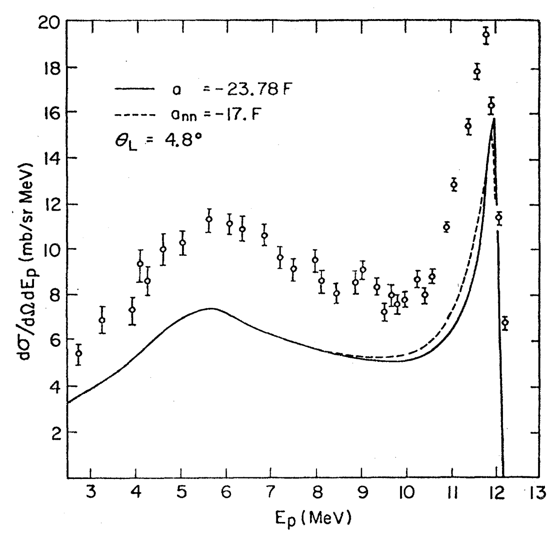

Fig. 14. The proton energy spectrum for neutron–deuteron breakup for laboratory proton angle 4.80; comparison of theory (Aaron, Amado, and Yam, 1966) and experiment (Cerineo, Dakovac, Šlaus, Tomaš, and Valkovic, 1964). The solid and dotted lines indicate the effect of varying the neutron–neutron scattering lengths from −17 to −24 fm.

* Cerineo, Ilakovac, Šlaus, Tomaš, and Valkovic (1964); Ilakovac, Kuo, Petravic, Šlaus, and Tomaš (1963); Voitovetskii, Korsunskii, and Pazhin (1965); Bar-Avraham, Fox, Porath, Adam, and Frieder (1967); Debertin, Hofmann, and Rossle (to be published).

We emphasize how inconvenient this is for a program aiming to add to our knowledge of the two-nucleon interaction from three-body calculations. The only features of the three-body system which have been unambiguously shown to be sensitive to the details of the interaction are the binding energy E_T and the doublet scattering length ${}^2a$ (even the triton D-state probability appears to be essentially numerically determined by the deuteron D-state probability). There is even no convincing evidence that E_T and ${}^2a$ are in any effective sense independent parameters, although current difficulties in fitting set A′ for ${}^2a$ with local potentials indicate that they may be.

This conclusion does *not* mean that there is *no* information to be gained from three-body calculations: it does mean that, as for the two-nucleon analysis, progress will not come easily, but will depend on detailed and accurate fitting to the experimental data.

E. The Neutron–Neutron Scattering Length and the $D(n, p)$ 2n Reaction

One two-body potential which is hardly known at all from direct experiments is that between two neutrons. There have therefore been a number of attempts to extract information on the 2n potential from three-body systems.

A strong final-state interaction between neutrons with small relative momentum is primarily responsible for the peak in the spectrum of the $D(n, p)2n$ reaction near the maximum proton energy. Figure 14 shows the effect of varying the ${}^1S_0$ neutron–neutron scattering length (a_{nn}) in the separable-interaction calculation: it appears that it would be difficult to distinguish the two cases, $a_{nn}=-23.78$ and -17.00 fm, experimentally. (The fact that the results of Fig. 14 correspond to varying a_{nn} in the final propagator only is unlikely to affect this conclusion). Nevertheless, this reaction has been considered as a possible source of information on neutron–neutron scattering. The usual assumption that the shape of the final-state interaction peak is insensitive to the neutron–proton interaction requires justification. If the production mechanism for the $p+(2n)$ system (i.e., the mechanism that produces a proton and a noninteracting neutron–neutron system) were short range, then it would be possible to use the Watson–Migdal final-state interaction theory (Watson, 1952; Migdal, 1955). The cross section for small neutron–neutron momentum k would then have the form

$$\frac{d^2}{d\Omega_p dE_p}=\frac{Ck}{|\, ik+\frac{1}{2}r_{nn}k^2-1/a_{nn}\,|^2}.$$

The factor k in the numerator is a phase-space term and C is a slowly varying function of k. However, a long-range production mechanism for the $p+(2n)$ system would imply that it is more difficult to isolate the two neutrons and, in so doing, avoid the complexities of the three-body aspects of the problem. The difference between simple examples of short-range and long-range reaction mechanisms has been discussed by Phillips (1964).

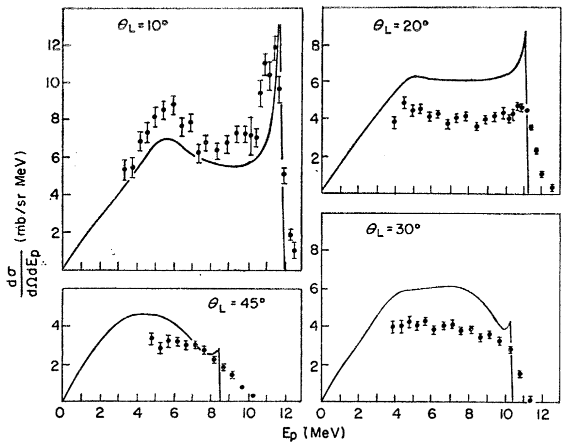

Fig. 15. The proton energy spectrum for neutron–deuteron breakup at various values for the laboratory proton angle θ_L; comparison of theory (Aaron, Amado, and Yam, 1966) with experiment (Ilakovac, Kuo, Petravic, Šlaus, and Tomaš, 1963).

Several values for a_{nn} have been derived from the $D(n, p)$ 2n reaction (Ilakovac *et al.*, 1963; Voitovetskii *et al.*, 1965; Bar-Avraham, Fox, Porath, Adam, and Freider, 1967; Slobodrian, Conzett, and Resmini, 1968). As pointed out by Slobodrian *et al.*, the experimental data analyzed in these papers are self-consistent within the experimental errors, statistical uncertainties, and the difference in resolution. But in spite of this, the results for the neutron–neutron scattering length show a considerable spread: $a_{nn}=-21.7\pm1$ fm (Ilakovac *et al.*, 1963), $-23.6(+2.0, -1.6)$ fm (Voitovetskii *et al.*, 1965), -14.0 ± 3 fm (Bar-Avraham *et al.*, 1967), and $-16.7(+2.6, -3.0)$ fm (Slobodrian *et al.*, 1968). The differences in the values obtained for a_{nn} must be due to theoretical inaccuracies in the analysis. In fact, the calculations of Ilakovac *et al.*, Voitovetskii *et al.*, and Bar-Avraham *et al.* correspond to various production mechanisms for the $p+(2n)$ system, all of which have long range and are likely to strongly influence the shape of the energy spectrum. Cerineo, Ilakovac, Šlaus, and Tomaš (1963) use the Born approximation and include terms that correspond to the pickup of a neutron by the incident neutron and the knockout of a proton by the incident neutron. Both of these reaction mechanisms have a range determined approximately by the size of the deuteron. The calculation of Bar-Avraham *et al.* (1967) uses an impulse approximation which is roughly equivalent to the proton knockout term of Cerineo *et al.* Voitovetskii *et al.* (1965) used a technique based on Feynman diagrams, including terms corresponding to an interaction range given by the deuteron size.

The most reasonable determination of a_{nn} from the

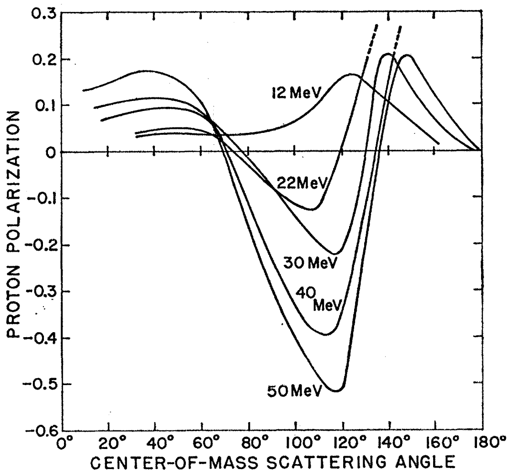

FIG. 16. General behavior of existing data on proton polarization in proton–deuteron elastic scattering up to 50 MeV; taken from Noyes (1966).

nucleon–deuteron reaction is due to Slobodrian *et al.* (1968). This involves the comparative analysis of the $D(n, p)$ $2n$ and $D(p, n)$ $2p$ reactions and rests on the assumption that the mechanism producing the nucleon and the singlet dinucleon system is the same in both reactions. However, very little is known about the effect of Coulomb forces in a three-nucleon scattering state and this assumption may not be valid.

The separable-interaction calculations, which are exact for the interaction used, indicate that it is a formidable task to estimate a_{nn} from the neutron–deuteron reaction. In the three-nucleon final state the interactions between the various pairs of particles are coherent. The information about the interaction of a given pair is in principle distributed over the entire amplitude. In the Faddeev theory the amplitude for the production of the $p+(2n)$ system is simply the sum of all ways in which the three nucleons can scatter without ending with a neutron–neutron interaction. In general, there will be long-range terms in the $p+(2n)$ production mechanism corresponding to outgoing neutron–proton scattered waves in the $n+(np)$ channel. [This is most easily seen by considering the Faddeev equations in configuration space (Noyes, 1968).] The importance of such terms will vary from model to model. But they are expected to be significant in the separable interaction calculations since the basic process is a nucleon exchange from one nucleon–dinucleon system to another. This is borne out by the results of Aaron, Amado, and Yam (1966). They found that the $nd\rightarrow p+(2n)$ amplitude varies rapidly in the region near the high-energy end of the proton spectrum. This amplitude with a final-state neutron–neutron interaction gave a broad unpronounced peak. But the inclusion of the terms in which a neutron and a proton interacted last led to an enhancement of the peak. This result and the general question of how three-particle amplitudes depend on the interaction of given pairs of particles has been discussed by Amado (1967).

F. Polarization in Nucleon–Deuteron Scattering

We conclude this discussion of the continuum states with a brief look at the polarization data. Early experimental work on this subject was contradictory (White, Chisholm, and Brown, 1958; Bucher, Bererley, Cobb, and Hereford, 1959; Shafroth, Chalmers, Strait, and Segel, 1960). However, the experimental situation has now settled down: there are quite good measurements of the neutron polarization in neutron–deuteron scattering,* of the protons in proton–deuteron scattering,† and of the tensor polarization of the deuteron in proton–deuteron scattering.‡ As emphasized by Noyes (1966), this data must certainly contain much valuable information on the two- and three-particle interactions; we reproduce Fig. 16 to emphasize the amount of excellent information going begging. However, apart from an oversimplified model calculation at 3 MeV by Delves (1962; 1959) and a simple diffraction model calculation at 40 MeV by Hufner and de-Shalit (1965), attempts at theoretical interpretation are completely lacking. Hopefully, this lack will be remedied in the next few years.

V. ELECTROMAGNETIC PROPERTIES OF THREE-NUCLEON SYSTEMS

A. Magnetic Moments of ^{3}H and ^{3}He

The magnetic moments of the three-particle nuclei demonstrate the importance of the distortion of the electromagnetic nucleon structure in the presence of nuclear interactions (i.e., the interaction effect). The interaction effect also plays an important role in the analysis of the magnetic form factors (Sec. V.B) and in the photocapture and disintegration reactions (Secs. V.D and V.E). In the absence of interaction currents, the magnetic moment operator is

$$\mathbf{M}=\sum_{i=1}^{3}\{\tfrac{1}{2}[1-\tau_z(i)]\mu_n\boldsymbol{\sigma}(i)+\tfrac{1}{2}[1+\tau_z(i)][\mu_p\boldsymbol{\sigma}(i)+\mathbf{L}(i)]\}.$$

For the principal S state there is no orbital contribution, and since the spatial symmetry implies that the two

* Brüllmann, Gerber, Meier, and Scherrer (1959); Cranberg (1959); Walter and Kelsey (1963); Malonify, Simmons, Perkins, and Walter (1966).

† Conzett, Igo, and Knox (1964); Conzett, Goldberg, Shield, Slobodrian, and Yanabe (1964); Skakun, Strashinskii, and Kyucharev (1964); Chalmers, Cox, Seth, and Strait (1965); Clegg (1966); Gruebler, Haeberli, and Extermann (1966); McKee, Clark, Slobodrian, and Tirrol (1968, 1966).

‡ Young, Ivanovitch, and Olsen (1965); Young and Ivanovitch (1966); Extermann (1966).

like nucleons are in a singlet spin state, the spin contribution comes entirely from the spin magnetic moment of the unlike nucleon (i.e., the proton for ^{3}H and the neutron for ^{3}He). Furthermore, under fairly reasonable assumptions regarding the structure of ground-state wave functions, the inclusion of other symmetry admixtures reduces the ^{3}H magnetic moment and increases the ^{3}He magnetic moment (Sachs and Schwinger, 1946; Sachs, 1953). This disagrees with the experimental results:

$$\mu(^3\text{H}) = \mu_p + 0.186 \text{ nuclear magnetons}$$

and

$$\mu(^3\text{He}) = \mu_n - 0.215 \text{ nuclear magnetons.}$$

It is useful to express these discrepancies in terms of the isovector and isoscalar magnetic moments of the ^{3}He–^{3}H doublet. Assuming that the probability densities of the principal S state, S' state, and D states are $P(S)$, $P(S')$, and $P(D)$, one obtains the following for the isoscalar and isovector moments:

$$\mu_s = \tfrac{1}{2}[\mu(^3\text{He}) + \mu(^3\text{H})]$$
$$= \tfrac{1}{2}(\mu_p + \mu_n)[P(S) + P(S') - P(D)] + \tfrac{1}{2}P(D),$$
$$\mu_v = \tfrac{1}{2}[\mu(^3\text{He}) - \mu(^3\text{H})]$$
$$= \tfrac{1}{2}(\mu_p - \mu_n)[P(S) - \tfrac{1}{3}P(S') + \tfrac{1}{3}P(D)] - \tfrac{1}{6}P(D).$$

For $P(D) = 6\%$ and $P(S') = 1.2\%$ (see Table III), $\mu_s = 0.417$, which is in reasonable agreement with the experimental value of 0.425, but the isovector moment $\mu_v = -2.20$ differs considerably from the experimental value of -2.55.

If there are charge-exchange, momentum-dependent, or nonlocal terms in the nucleon–nucleon interaction, the position of the nucleon can no longer be considered as a point of constant charge, and additional electromagnetic interaction currents are necessary in order to allow for charge conservation. This in turn leads to corrections to the magnetic moments of nuclei.

By far the most important correction (the exchange magnetic moment) arises from the interaction of the electromagnetic field with a charged pion exchanged between nucleons. Isovector and isoscalar photons have G parity of $+1$ and -1, respectively, and a system of n pions has G parity of $(-1)^n$. Thus, the coupling of the photon to the one-pion exchange component of the nucleon–nucleon interaction results in a correction to the isovector magnetic moment. Villars (1947), using lowest-order perturbation theory, showed that the magnetic-moment contribution due to one-pion exchange accounts for the sign and order of magnitude of the discrepancy in the isovector magnetic moment. Corrections to the isoscalar magnetic moment depend on the shorter-range two-pion and ρ-meson exchange components and are much more difficult to estimate, but they are expected to be an order of magnitude smaller than the corrections to the isovector moment. In the absence of a complete meson-theoretic treatment, phenomenological representations of interaction currents and relativistic corrections have also been considered (Sachs, 1953). It should be emphasized that a complete understanding of these phenomena requires not only a reliable theory of strong interactions, but also a three-nucleon wave function which bears at least some relation to the interactions from which the mesonic currents are derived. Existing treatments of the magnetic moments should be considered as order of magnitude estimates. Furthermore, it seems unlikely that the understanding of these phenomena will improve significantly in the near future.

B. The Electromagnetic Form Factors of ^{3}H and ^{3}He

To first order in the electromagnetic interaction, the elastic scattering of an electron from a particle with angular momentum 1/2 is given by the Rosenbluth formula (Rosenbluth, 1950). This formula expresses the results of electron scattering in terms of two functions of the momentum transfer, the charge and magnetic form factors of the target particle. These functions may be identified with the Fourier transform of the spatial distributions of the electric charge and magnetic moment of the target.

A number of very precise electron-scattering experiments from ^{3}H and ^{3}He have been performed (Collard, Hofstadter, Hughes, Johansson, Yearian, Day, and Wagner, 1964) and analyzed in this way (Schiff, 1964). The interpretation of the magnetic-form-factor results is uncertain because of the existence of appreciable magnetic-interaction currents; for example, the interaction-current contribution to the isovector magnetic moment is of the order of 0.3 nm. On the other hand, the electric-multipole interaction with photons at low energies depends largely on the charge distribution, i.e., directly on the nuclear wave function but not directly on the details of the nuclear interaction. Thus it is possible that the effects of charge interaction currents may be small (Siegert's theorem) (Siegert, 1937). Accordingly, we shall deal primarily with the charge-form-factor data. These data provide the most reliable and exacting information available on the properties of the three-nucleon bound-state wave functions.

If charge-exchange effects are absent and if the charge properties of free and bound nucleons are the same, the charge form factors of ^{3}He and ^{3}H are, in the impulse approximation, given by

$$ZF_{\text{ch}}(q) = \iint \exp(i\mathbf{q}\cdot\mathbf{x})\Psi^*\rho_{\text{ch}}(\mathbf{x}, \mathbf{r}_i)\Psi d^3\mathbf{x}\, d^3\mathbf{r}_i. \quad (5.1)$$

Here ρ_{ch} is the charge-density operator which depends on the spatial distribution functions for the charge densities of a free proton f_{ch}^p and a free neutron f_{ch}^n.

We have

$$\rho_{ch}(\mathbf{x}, \mathbf{r}_i) = \sum_{i=1}^{3} \{\tfrac{1}{2}[1+\tau_z(i)]f_{ch}^p(\mathbf{x}-\mathbf{r}_i) + \tfrac{1}{2}[1-\tau_z(i)]f_{ch}^n(\mathbf{x}-\mathbf{r}_i)\}. \quad (5.2)$$

The integration over $\mathbf{x}$ in (5.1) may be performed by changing variables from $\mathbf{x}$ to $\mathbf{x}-\mathbf{r}_i$. This causes the nucleon-charge form factors F_{ch}^p and F_{ch}^n to appear as multiplying factors. We introduce the vectors

$$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j,$$

$$\boldsymbol{\varrho}_k = \mathbf{r}_k - \tfrac{1}{2}(\mathbf{r}_i + \mathbf{r}_j).$$

Then for either ^{3}He or ^{3}H

$$ZF_{ch}(q) = \frac{3}{2} \iint d^3\boldsymbol{\varrho}_k\, d^3\mathbf{r}_{ij} \exp(i\tfrac{2}{3}\mathbf{q}\cdot\boldsymbol{\varrho}_k) \times \{[F_{ch}^p(q)+F_{ch}^n(q)]\Psi^*\Psi + [F_{ch}^p(q)-F_{ch}^n(q)]\Psi^*\tau_z(k)\Psi\}. \quad (5.3)$$

As in the case of the static magnetic moments, any treatment of the magnetic form factors must include interaction effects. In the impulse approximation for either ^{3}He or ^{3}H we have

$$\mu F_{mag}(q) = \frac{3}{2} \iint d^3\boldsymbol{\varrho}_k\, d^3\mathbf{r}_{ij} \exp(i\tfrac{2}{3}\mathbf{q}\cdot\boldsymbol{\varrho}_k) \times \{[F_{mag}^p(q)+F_{mag}^n(q)]\Psi^*\sigma(k)\Psi + [F_{mag}^p(q)-F_{mag}^n(q)]\Psi^*\tau_z(k)\sigma(k)\Psi + F_x(q)\}. \quad (5.4)$$

In the impulse approximation the nucleons are treated as free particles, and hence there is no orbital-angular-momentum term in (5.4). In the $q^2=0$ limit this leads to an uncertainty of the order of 2% in values for the static magnetic moments of ^{3}He and ^{3}H. The function $F_x(q)$ is the form factor arising from magnetic-interaction currents and is usually decomposed into an isoscalar and a dominant isovector term:

$$F_x^{^3\mathrm{He}}(q) = F_{xs} + F_{xv},$$

$$F_x^{^3\mathrm{H}}(q) = F_{xs} - F_{xv}.$$

In calculating the charge form factors of ^{3}H and ^{3}He, it is usual to assume that the wave functions of both nuclei are substantially the same; the charge-dependent effects, of which the most important is the Coulomb potential, are relatively weak and, to first order, do not affect the wave function. Yet, experimentally, the ^{3}H and ^{3}He form factors are not equal to each other. For small q^2 we have

$$F_{ch}(q) = 1 - \tfrac{1}{6} r_{ch}^2 q^2 + \cdots,$$

where r_{ch} is the root-mean-square radius of the charge distribution in the nucleus. The experiments give (Collard *et al.*, 1964)

$$r_{ch}(^3\mathrm{He}) = 1.87 \pm 0.05 \text{ fm},$$

$$r_{ch}(^3\mathrm{H}) = 1.70 \pm 0.05 \text{ fm}.$$

At higher momentum transfer the ^{3}He charge form factor continues to fall more rapidly with q than the ^{3}H form factor. These results are not necessarily inconsistent with the assumption of equality of the two wave functions, since the charge form factor measures the charge distribution. In ^{3}H and ^{3}He we have two like particles and one unlike; in ^{3}H, the unlike particle is charged, while in ^{3}He the like particles are charged. We would therefore expect the form factors to differ if the like particles have a different distribution from the unlike particle. In particular, if the like particles are spread over a bigger region of space than the unlike particle, then the observed difference in the form factors can be accounted for. Schiff (1964) has pointed out that this is to be expected, since the interaction between the like particles, which are necessarily in a singlet state, is weaker than that between the unlike particles, which are either in a singlet or triplet state. Thus, in this model the S' state, which is generated by the difference in the triplet and singlet two-nucleon interactions, is expected to provide a major contribution to the difference between the ^{3}H and ^{3}He charge form factors.

In terms of the classification of the three-nucleon wave function given in Sec. II.A, the most important admixtures are expected to be the S' and the D states. In addition, the $T=3/2$ mixed-symmetry S-state component, which is induced in the ^{3}He wave function by the Coulomb potential, may conceivably be important. It is straightforward to analyze the form factors in terms of the various admixtures; one can then hope to estimate the probabilities of the various states. The first such estimate (Schiff, 1964) assumed only the S' state contributed to the difference in the charge form factors and showed that, with the wave function used, a probability $P(S')$ of 4% was needed to explain the data. Dalitz and Thacker (1965) have indicated that the use of a wave function whose asymptotic form is correctly related to the binding energies of the nucleons of the system reduces the necessary S'-state probability. In addition, Levinger and Srivastava (1965) have emphasized the importance of the value of the neutron-charge form factor in the analysis. Subsequent calculations (Gibson and Schiff, 1965; Gibson, 1965) have included the effects of the D states and the $T=3/2$ state. These results indicate first that the D states contribute significantly to the form factors for small values of q^2, but cannot account for the experimental difference in the charge form factors; and second, that it is possible to reach agreement with the experiment with quite reasonable values for $P(S')$, $P(D)$, and $F^n(q)$ without invoking charge-interaction effects. This does not mean that interaction effects are not important, and in particular the possibility of the modification of the isovector form factor at high q^2 cannot be discounted (Sarker, 1965).

We give here a simplified analysis, neglecting the

D-state part of the bound-state wave functions. The three-nucleon wave functions are taken as

$$\Psi^{^3\text{He}}=uY_0+(v_2Y_1-v_1Y_2)+(v_2'Y_1'-v_1'Y_2') \quad (5.5)$$

and

$$\Psi^{^3\text{H}}=uY_0+(v_2Y_1-v_1Y_2), \quad (5.6)$$

where Y_0, (Y_1, Y_2), and (Y_1', Y_2') are the S, S', and $T=3/2$ spin–Euler-angle functions. For the two nuclei we assume identical S and S' states and include a $T=3/2$ admixture in the ^{3}He wave function only.

The function u is spatially symmetric. The S' functions v_1 and v_2 may be constructed from the function $g(ij,k)$, which is symmetric under interchange of particles i and j, to have the symmetry properties (2.4a) and (2.4b). We have

$$v_1=6^{-1/2}[g(12,3)+g(13,2)-2g(23,1)], \quad (5.7)$$

$$v_2=2^{-1/2}[g(12,3)-g(13,2)]. \quad (5.8)$$

The $T=3/2$ functions v_1' and v_2' may be constructed in an identical way from the function $g'(i,j,k)$.

Substitution of (5.5) and (5.6) into the equation for the charge form factor (5.3) and neglect of the (S', S') and $(T=3/2, T=3/2)$ terms gives

$$F_{\text{ch}}{}^{^3\text{He}}(q)=[F_{\text{ch}}{}^p(q)+\tfrac{1}{2}F_{\text{ch}}{}^n(q)]F_1(q) -\tfrac{1}{3}[F_{\text{ch}}{}^p(q)-F_{\text{ch}}{}^n(q)][F_2(q)+F_2'(q)], \quad (5.9)$$

$$F_{\text{ch}}{}^{^3\text{H}}(q)=[2F_{\text{ch}}{}^n(q)+F_{\text{ch}}{}^p(q)]F_1(q) +\tfrac{2}{3}[F_{\text{ch}}{}^p(q)-F_{\text{ch}}{}^n(q)]F_2(q). \quad (5.10)$$

The body form factors F_1, F_2, and F_2' are

$$F_1(q)=\iint d^3\mathbf{\varrho}\, d^3\mathbf{r} \exp(i\tfrac{2}{3}\mathbf{q}\cdot\mathbf{\varrho})u^2, \quad (5.11)$$

$$F_2(q)=-3\iint d^3\mathbf{\varrho}\, d^3\mathbf{r} \exp(i\tfrac{2}{3}\mathbf{q}\cdot\mathbf{\varrho})uv_1, \quad (5.12)$$

$$F_2'(q)=-3\iint d^3\mathbf{\varrho}\, d^3\mathbf{r} \exp(i\tfrac{2}{3}\mathbf{q}\cdot\mathbf{\varrho})uv_1', \quad (5.13)$$

where $\mathbf{\varrho}=\mathbf{\varrho}_1$ and $\mathbf{r}=\mathbf{r}_{23}$. It is possible to relate these form factors to the body form factors that specify the distribution of the like and unlike nucleons of the bound state (Schiff, 1964).

We look first at the difference between $F_{\text{ch}}{}^{^3\text{He}}$ and $F_{\text{ch}}{}^{^3\text{H}}$:

$$F_{\text{ch}}{}^{^3\text{He}}(q)-F_{\text{ch}}{}^{^3\text{H}}(q)=-F_{\text{ch}}{}^p(q)[F_2(q)+\tfrac{1}{3}F_2'(q)] -F_{\text{ch}}{}^n(q)[\tfrac{3}{2}F_1(q)-F_2(q)-\tfrac{1}{3}F_2'(q)]. \quad (5.14)$$

Now $F_{\text{ch}}{}^p$ and F_1 are large. Hence this difference is particularly sensitive to F_2 (the S' contribution), F_2' (the $T=3/2$ contribution), and the neutron-charge form factor. For zero neutron form factor and for simple analytic forms for the wave functions, the experimental values for (5.14) are approximately reproduced if $P(S')\simeq4\%$ and $P(T=3/2)=0$ (Schiff, 1964). If the shape of the S' and $T=3/2$ wave functions are the same, this fit to the charge data is unchanged, provided $P(S')^{1/2}+\frac{1}{3}P(T=3/2)^{1/2}$ is kept constant. (This fit to the form-factor data gives $\simeq0.75$ fm^2 for the difference in the squares of the charge radii; the experimental result is 0.607 ± 0.18 fm^2.)

We have not included the *D*-state contribution to the form factors. The analysis of Gibson (1965) includes a 6% *D* state. This state at low q^2 accounts for 20% of the difference in the charge form factors. A tolerable fit to the data is obtained with zero neutron form factors if $P(S')=2.5\%$ and $P(T=3/2)=0$ or $P(S')=2.0\%$ and $P(T=3/2)=0.25\%$. Recent calculations of the effect of the Coulomb force in a model consisting of three scalar nucleons indicate that a $T=3/2$ admixture of 0.01% to 0.001% is more likely than 0.25% (Ohmura, 1969, 1967; Bell and Delves, to be published). An admixture as high as 0.25% would therefore represent a significant breakdown of charge independence in the two-nucleon potential. Furthermore, a $P(S')$ of 2.5% is rather large in view of the evidence from variational calculations (Davies, 1967a; 1967b; Delves, Blatt, Pask, and Davies, 1969) and model calculations using separable two-nucleon interactions (Bhakar and Mitra, 1965; see also Table III). These results indicate that an S' probability of around 1.5%–2% would be more acceptable. The discrepancy, however, is not very large; moreover, the form-factor data may be made compatible with $P(S')\simeq1.5\%$ if the neutron form factor is not taken to be zero (Levinger and Srivastava, 1965; Gupta, Bhakar, and Mitra, 1967). Thus, if we take

$$F_{\text{ch}}{}^n(q)\simeq0.02q^2,$$

then the neutron form factor alone accounts for 30% of the experimental difference in the squares of the charge radii.

An alternative possibility is to introduce different wave functions for ^{3}He and ^{3}H, in addition to the difference implied by a $T=3/2$ state in ^{3}He. The most obvious difference is in the asymptotic behavior of the wave functions. For example, the asymptotic behavior of the wave function, obtained when the bound-state problem is solved using separable potentials, may be adjusted to correspond to the actual binding energies of ^{3}He or of ^{3}H. Keeping the principal S state only, the difference in the square of the charge radii that results from the different asymptotic form of ^{3}He and ^{3}H is 0.18 fm^2 (Phillips, unpublished); that is, 30% of the experimental value for this difference. A similar result occurs for the Dalitz–Thacker wave function (Dalitz and Thacker, 1965). However, wave functions of this kind almost certainly overemphasize the importance of the asymptotic behavior. This is borne out by recent calculations by Ohmura (1967). Assuming the existence of central spin-independent nuclear forces only, Ohmura

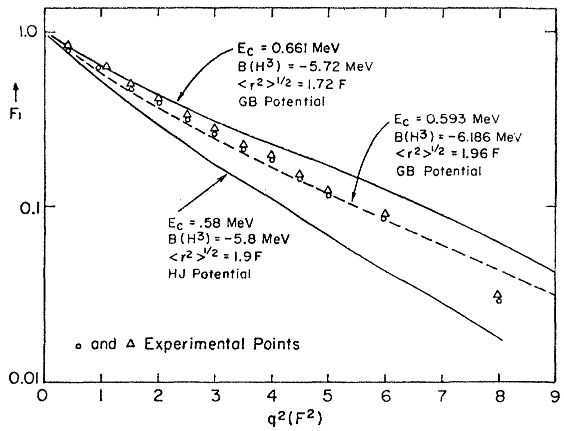

FIG. 17. The body form factor $F_1(q^2)$ of ^{3}Ge–^{3}H doublet predicted by a number of variational wave functions (Delves and Blatt, 1967; Davies, 1967a; 1967b). The experimental points taken from Levinger and Srivastiva (1965) were obtained by analyzing the data of Collard, Hoftadter, Hughes, Johansson, Yearian, Day, and Wagner (1964) with different values for the neutron-charge form factor.

estimates that the difference in the square of the charge radii resulting from the Coulomb potential is $\simeq 0.1$ fm^2. This calculation also predicts that the Coulomb expansion of the principal S state is three to five times more important in determining the radii than the distortion of the bound state. Thus, as far as the Coulomb potential is concerned, we expect only minor differences between the wave functions of ^{3}He and ^{3}H, and these differences are probably better represented by a modification of the principal S state than by admixing a $T=3/2$ state into the ^{3}He wave functions.

There remains the question of the absolute magnitude of the charge form factors. It turns out that this is easier to fit than the ^{3}H$-^3$He difference. This is because we have an initial condition $F(0)=1$, and an initial slope determined uniquely by the charge radius. Hence any wave function with the correct radius will fit at least the initial portion of the form-factor curve. Moreover, the curve, on a logarithmic scale, is essentially straight, and it is not until we reach $q^2 \approx 7$–8 fm^{-2} that any structure begins to appear. At these momentum transfers, the form factor appears to drop below the straight line predicted for a Gaussian charge distribution; this effect is what should be expected from the presence of a repulsive core in the two-body interaction. Figure 17 shows the experimental F_1 form factor, together with the predictions (Delves and Blatt, 1967; Davies, 1967a; 1967b) of a number of wave functions; these straddle the experimental curve, even though none represents a particularly good wave function. We also see that it is not enough to fit the triton binding energy; two of the curves presented correspond to wave functions yielding essentially the same energy, but quite different form factors.

To conclude, the charge-form-factor results correspond to three-nucleon, bound-state wave functions which are in basic agreement with those expected from three-body calculations. At the present level of sophistication there is no need to introduce interaction currents or relativistic corrections to the form factors. This is not the case for the magnetic form factors; here phenomenological interaction effects are necessary to fit both the isovector and isoscalar magnetic form factors (Gibson, 1965).

C. The Coulomb Energy of ^{3}He

The difference between the binding energy of ^{3}H and ^{3}He is experimentally found to be

$$\Delta = 0.764 \text{ MeV}.$$

If the nucleon–nucleon strong Hamiltonian is charge symmetric, this difference must be wholly ascribed to electromagnetic effects, of which the most important is the Coulomb repulsion between the protons in ^{3}He. It has been assumed in the past that charge symmetry of the nuclear Hamiltonian (as opposed to charge independence) is established quite strongly from the observed energy differences between mirror nuclei (Wilkinson and Hay, 1966; Okamoto, 1966) and from the observed approximate equality of the *nn* and *pp* scattering lengths (Heller, Signell, and Yoder, 1964; Haddock, Salter, Zeller, Czier, and Nygram, 1965). However, there are always difficulties inherent in reliably interpreting such quantitative results in complex nuclei; moreover, equality of the *nn* and *pp* scattering lengths is not alone sufficient to establish charge symmetry of the nuclear interaction.

The three-body nuclei should provide a very clear-cut test of charge symmetry, since they are simple enough for a fairly complete analysis of the problem. Direct variational calculations of the ^{3}H$-^3$He energy difference (Delves, Blatt, Pask, and Davies, 1969; Delves and Blatt, 1967), assuming charge symmetry and protons of finite size, lead to an estimate

$$E(\text{Coulomb}) \sim 0.6 \text{ MeV}.$$

These estimates are likely, however, to be on the low side since the wave functions used underbind the triton; hence they tend to be too spread out. This is confirmed by a comparison of the charge form factor and, in particular, the rms radii of the wave functions. A number of authors have therefore calculated the Coulomb energy with wave functions which are consistent with the observed electron-scattering form factors. The most recent result, by Okamoto and Lucas (1967), for the electromagnetic-binding-energy difference is 0.63 ± 0.03 MeV. This value corresponds to using a Gaussian wave function with two-nucleon soft cores and protons of finite size. It also includes corrections due to the neutron–proton mass difference, the magnetic interaction energy, corrections to vacuum polarization effects, and to the electric polarizability of the nucleons, which together result in a reduction of about 0.1 MeV in the energy difference. The final result of these calculations is a discrepancy of about 0.13 MeV in the binding-energy difference of ^{3}H and ^{3}He. If we accept this, we conclude that the *n–n* interaction is about 1% stronger than the *p–p* interaction (Okamoto, 1966). Alternatively, charge-asymmetric three-body forces may exist in appreciable strength.

An argument against this conclusion, i.e., that there is an apparent breakdown of charge symmetry in the three-body system, comes from the separable-model energy calculations of Mitra and his coworkers (Gupta and Mitra, 1967). They find a Coulomb energy that is *too high*, and they claim that the inclusion of the repulsive core effects, which they have omitted so far, should bring the Coulomb energy into agreement with experiment. However, it has been recently suggested (Okamoto and Lucas, 1968) that this high value for the Coulomb energy may be due to the use of a $^1S_0$ nucleon–nucleon interaction which is too attractive.

The crucial point in this problem is the extent to which the charge form factor of ^{3}He determines the Coulomb energy. We give here a qualitative discussion of this point. The Coulomb energy is

$$\Delta E_c=\langle\Psi^{^3\mathrm{He}}\mid\sum_{i\neq j}C_{ij}\mid\Psi^{^3\mathrm{He}}\rangle-\langle\Psi^{^3\mathrm{H}}\mid\sum_{i\neq j}C_{ij}\mid\Psi^{^3\mathrm{H}}\rangle, \tag{5.15}$$

where, allowing for extended charge distributions for the nucleons, we have

$$\langle\mathbf{r}_{ij},\boldsymbol{\rho}_k\mid C_{ij}\mid\mathbf{r}_{ij}',\boldsymbol{\rho}_k'\rangle=\frac{e^2}{2\pi^2}\int\frac{d^3\mathbf{p}}{p^2}\exp\left(i\mathbf{r}_{ij}\cdot\mathbf{p}\right)F_i(p)F_j(p)\times\delta_3(\mathbf{r}_{ij}-\mathbf{r}_{ij}')\delta_3(\rho_k-\rho_k'), \tag{5.16}$$

where

$$F_i(p)=\tfrac{1}{2}[1+\tau_z(i)]F^p(p)+\tfrac{1}{2}[1-\tau_z(i)]F^n(p).$$

Reversing the order of integration in (5.15) gives

$$\langle\Psi\mid\sum_{i\neq j}C_{ij}\mid\Psi\rangle=\frac{3}{4}\frac{e^2}{2\pi^2}\int\frac{d^3\mathbf{p}}{p^2}\iint d^3\mathbf{r}_{ij}\,d^3\boldsymbol{\rho}_k\times(\exp\left(i\mathbf{r}_{ij}\cdot\mathbf{p}\right)\Psi^*\{(F^p+F^n)^2+(F^{p^2}-F^{n^2})\times[\tau_z(i)+\tau_z(j)]+(F^p-F^n)^2\tau_z(i)\tau_z(j)\}\Psi). \tag{5.17}$$

Comparing Eq. (5.17) with Eq. (5.3), we note that the Coulomb energy involves the Fourier transform with regard to the variable **r**, but that the charge form factor depends on the transform with regard to the variable **ρ**. Thus the relation between the Coulomb energy and the form factor is, in principle, dependent on the details of the wave function. An analogous situation occurs for heavier mirror nuclei. But here the antisymmetrization of the wave function is more effective than in the three-nucleon system in lessening the importance of the details of the wave function.

If the wave function is such that the dependence on **ρ** and **r** are similar, one can relate the Coulomb energy to the form factor. For example, if in (5.5) and (5.6)

$$u=u(\sum_{i\neq j}r_{ij}^2)=u(2\rho_1^2+\tfrac{3}{2}r_{23}^2), \tag{5.18}$$

$$v_1=(r_{12}^2+r_{31}^2-2r_{23}^2)f(\sum_{i\neq j}r_{ij}^2), \tag{5.19}$$

and

$$v_1'=(r_{12}^2+r_{31}^2-2r_{23}^2)f'(\sum_{i\neq j}r_{ij}^2), \tag{5.20}$$

then we obtain a result similar to that of de La Rappelle (1964):

$$\Delta E_c=\frac{2e^2}{\pi}\int dp\{[F^{p^2}(p)-F^{n^2}(p)][F_1(\sqrt{3}p)-\tfrac{2}{3}F_2(\sqrt{3}p)]+[F^{p^2}(p)+2F^p(p)F^n(p)][-\tfrac{2}{3}F_2'(\sqrt{3}p)]\}, \tag{5.21}$$

where F_1, F_2, and F_2' are the body form factors occurring in Eqs. (5.9) to (5.13).

Inspection of (5.21) gives a number of results. First, the range of momentum transfer for which the form factors are known is in principle sufficient to determine the Coulomb energy to an accuracy of a few percent. Second, the S' and $T=3/2$ states and a nonzero neutron form factor that are necessary to explain the experimental difference between the charge form factors of ^{3}He and ^{3}H all result in a decrease in the Coulomb energy. Third, if F^n, F_2, and F_2' are neglected, it seems that the effect of decreasing F^p (i.e., increasing the size of the proton) is to decrease ΔE_c. In fact, ΔE_c increases, since, in order to retain the fit to $F^{^3\mathrm{He}}$, it is necessary to increase significantly the magnitude of F_1.

We emphasize that (5.21) is only valid if the wave functions have the form (5.18)–(5.20). The calculations of Okamoto and Lucas (1967) correspond to introducing a soft two-nucleon core in (5.18), i.e., a core in the **r** variable but not directly in the **ρ** variable. For this case the Coulomb energy is less than the right-hand side of (5.21). However, the functional dependence in (5.18) may not be a good starting point for the introduction of a two-nucleon soft core. In particular, virtual scattering within the bound state of a nucleon from a two-nucleon pair may result in a wave function which is more spread out with respect to the **ρ** variable and hence a Coulomb energy greater than the right-hand side of (5.21).

D. Radiative Capture of Thermal Neutrons by Deuterons

We look now at those processes connecting the bound- and scattering-state wave functions of three nucleons. In this section radiative capture of thermal neutrons by deuterons is considered and, in Secs. V.E and V.F, the photodisintegration and electrodisintegration of the three-particle nuclei are discussed. We emphasize that all these processes are extremely difficult to interpret, mainly as a result of the complexities of three-nucleon continuum states. In fact, the primary theoretical interest of these reactions is in the analysis of the scattering corrections, and very little can be deduced about the properties of the three-nucleon, bound-state wave functions.

It is instructive to compare the radiative neutron–deuteron capture process with neutron–proton capture:

$$n+p\rightarrow d+\gamma,$$

$$n+d\rightarrow {}^3\mathrm{H}+\gamma.$$

At low energies only $M1$ transitions are important, and the cross sections exhibit the characteristic $1/v$ behavior. For 22-km/sec neutrons the experimental cross sections (Cox, Wynchank, and Collie, 1965; Jurney and Motz, unpublished) are

$$\sigma_{np}=334.2\pm0.5 \text{ mb},$$

$$\sigma_{nd}=0.60\pm0.05 \text{ mb}.$$

The reason for this large difference in the cross sections of two apparently similar processes was first given by Schiff (1937) and is easily understood by sketching an outline of the calculations.

If the internal magnetic structure of the nucleons and any magnetic interaction effects are neglected, the relevant matrix element in n–p capture is given by

$$M_d=\langle\phi_d\,{}^3\chi\mid\mu_p\sigma_z(1)+\mu_n\sigma_z(2)\mid{}^1\phi^1\chi+{}^3\phi^3\chi\rangle,$$

where ${}^3\phi$, ${}^1\phi$ are the triplet and singlet parts of the initial-state wave function. Because of the orthogonality of (a), the triplet- and singlet-spin wave functions and (b) of the deuteron- and triplet-scattering-space wave functions (they are eigenfunctions of the same Hamiltonian at different energies), the only nonvanishing term is

$$M_d=\tfrac{1}{2}(\mu_p-\mu_n)\langle\phi_d\mid{}^1\phi\rangle.$$

This overlap would also be zero if the Hamiltonian were spin independent, since we should then have ${}^1\phi={}^3\phi$. However, the Hamiltonian is *not* spin independent; in fact, at low energies ${}^1\phi$ is very different from ${}^3\phi$, due to the singlet resonance, and the cross section that we find is large. The *experimental* cross section, however, is even larger, by some 10%; this discrepancy is attributed to the "interaction effect" (the distortion of the nucleon structure inside the nucleus) (Noyes, 1965, 1967).

For n–d capture the magnetic-dipole amplitude involves the overlap of the triton and the n–d wave functions

$$M_{^3\mathrm{H}}=\langle\Psi_{^3\mathrm{H}}\mid\mu_n[\sigma_z(1)+\sigma_z(2)]+\mu_p\sigma_z(3)\mid{}^2\Psi_{nd}+{}^4\Psi_{nd}\rangle.$$

If the Hamiltonian is spin and charge independent, the triton wave function is then the completely space-symmetric principal S state, in which the spins of the two neutrons are antiparallel. Since the principal S state is an eigenstate of the magnetic-dipole-transition operator, the matrix element vanishes. When we include spin-dependent effects, the triton wave function contains an S'-state component of mixed symmetry, and there is a finite probability that the two neutrons are in a triplet spin state. We then obtain nonzero transition rates from both the n–d quartet and doublet states to the S' triton state. Both of these transitions depend on the S' probability, and hence the cross section for n–d capture must be small. This small magnetic-dipole cross section is the basis of the suggestion that the n–d capture reaction may be a possible source of information on the strength of the parity nonconserving components of the nuclear interactions (Blin-Stoyle and Feshbach, 1961).

Of course, it is quite simple to make a plane-wave calculation of the magnetic-dipole cross section; if we do this, we find

$$P(S')\simeq0.7\pm0.5\%.$$

This is in reasonably good agreement with the available estimates of $P(S')$. However, this agreement is quite fortuitous. First, in the analysis of n–p capture and of magnetic moments and magnetic form factors of the three-nucleon bound states, there is strong evidence for large interaction or magnetic-exchange effects. Simple phenomenological treatments indicate that the neutron–deuteron capture amplitudes due to the interaction effect interfere destructively with the spin–magnetic-moment amplitude (Austern, 1951; Radha and Meister, 1964; 1965); this is true in the plane-wave and the zero-distortion approximations for the neutron–deuteron scattering state.

Second, neutron–deuteron scattering corrections are expected to be important. A two-body model for the neutron–deuteron system shows that the plane-wave approximation is quite inadequate (Barucchi, Bosco, and Nata, 1965; Erdas, Milani, Pompei, and Seatzu, 1966), and a complete separable-model calculation shows that there are indeed very large scattering corrections (Phillips, 1968a; 1968b). There is a considerable reduction in the quartet amplitude. The results for the doublet amplitude show that the distortion of the deuteron, which is included automatically by the exact treatment of the three-body aspects of the neutron–deuteron system, plays a very significant role. In fact, the distortion is sufficient to *reverse* the zero-distortion result (i.e., that the interaction effect and nucleon spin magnetic amplitudes interfere destructively).

Perhaps the most significant result is that, without the interaction effect, the experimental capture cross section cannot be reconciled with reasonable values for $P(S')$ once the three-particle aspects are adequately treated. This is in complete contrast to the encouraging results of a Born-approximation calculation. Since the interaction effects can be treated only phenomenologically, it becomes quite impossible to deduce $P(S')$ from the data being fitted.

E. Photodisintegration of Three-Particle Nuclei

We now consider the competing two-body and three-body breakup reactions:

$$\gamma + {}^3\text{He}({}^3\text{H}) \rightarrow d + p(n),$$

$$\gamma + {}^3\text{He}({}^3\text{H}) \rightarrow n + p + p(n).$$

[A comprehensive account of these reactions up to 1964 is given by Fetisov, Gorbunov, and Varflomameev (1965).] Unlike the situation in radiative neutron–deuteron capture, the nature of the electromagnetic-transition operator in the photodisintegration processes is well understood; the experimental angular distributions agree moderately well with the assumption that $E1$ transitions form the major contributions. Thus, given adequate initial- and final-state wave functions, it should be possible to reproduce the main features of the low-energy experimental data.

Dzhibuti, Mamasakhlisov, and Macharadze (1964)* have speculated as to whether the electric-dipole-transition operator should be represented as $\mathbf{E}\cdot\mathbf{r}$ or $\mathbf{A}\cdot\mathbf{p}$. However, insofar as the long-wavelength approximation for the photon is valid, $\mathbf{E}\cdot\mathbf{r}$ is expected to be the better representation of the $E1$ operator (Siegert, 1937).

There have been a number of calculations of the electric-dipole cross sections using simple models for the initial and final states. The early calculations (see Fetisov, Gorbunov, and Varflomameev, 1965) gave a reasonable fit to the *two-body* breakup cross sections, being, in general, some 20% to 30% below experiment. In most cases the final-state interactions were neglected, the justification being that the third nucleon is in a P wave relative to the deuteron and that the elastic ${}^2P_{1/2}$ nucleon–deuteron phase shift is small. The validity of this argument in a situation in which there are both distortion of the deuteron and inelastic effects is doubtful. Calculations which neglect the deuteron distortion suggest that final-state interactions are important (Eichmann, 1963; Fetisov, 1967). A calculation using separable potentials, and hence including both distortion and inelasticity, shows that final-state interactions increase the total cross section by about 25% (Barbour and Phillips, 1968).

The calculations on the three-body breakup reaction have had a much more colorful history. The crudest theory, which assumes plane-wave final states and $E1$ transitions, gives a theoretical cross section approximately three times the experimental cross section. The final-state interactions have been included approximately by a number of authors (Eichmann, 1963; Fetisov, 1967; Delves, 1962). Fetisov (1967) includes the final-state proton–proton interaction and finds that the neutron spectrum is predicted qualitatively, but the total cross section remains much too large. The electric-dipole transition operator has the effect of emphasizing the asymptotic form of the three-nucleon wave functions. Fetisov (1966) and Knight, O'Connell, and Prats (1967a; 1967b)* found that the use of a bound-state wave function with an asymptotic form determined by the binding energies of the nucleons of the system reduces the cross section by a large factor. Fetisov found that for the Dalitz–Thacker wave function (Dalitz and Thacker, 1965), theory and experiment are in rough (30%) agreement for both the two- and three-body breakup. However, an examination of the predictions of the Bremsstrahlung-weighted, electric-dipole sum rule shows that these calculations are inadequate (Barton, 1967; Gerasimov, 1967).

It is well known that under sufficiently restrictive conditions various sum rules for the $E1$ and $M1$ absorption cross sections may be derived. These sum rules are important because they do not depend on the final-state wave functions. The Bremsstrahlung-weighted sum rule has the additional advantage in that it is independent, within the limits of the validity of Siegert's theorem, of the details of the nuclear forces. We have

$$J_T = \int_0^\infty \sigma_{E1}(E_\gamma)\,\frac{dE_\gamma}{E_\gamma} = \tfrac{4}{3}\pi^2\alpha(\tfrac{2}{3}R_{pn})^2, \qquad (5.22)$$

where α is the fine-structure constant and R_{pn}, the rms distance between the center of mass of the neutron and proton distributions within the bound state. For a spatially symmetric wave function and for point protons, $(2/3)R_{pn}$ is the rms radius of the charge distribution. We have used the notation J_T to show that it refers to the total cross section. We can clearly split the sum rule into two-body and three-body breakup contributions:

$$J_T = J_{ND} + J_{3N}. \qquad (5.23)$$

If we put in the experimental results (for ^{3}He breakup) up to 170 MeV, we find (Fetisov, Gorbunov, and Varflomameev, 1965)

$$\int \sigma_{ND}\,\frac{dE_\gamma}{E_\gamma} = 1.34 \pm 0.05 \text{ mb},$$

$$\int \sigma_{3N}\,\frac{dE_\gamma}{E_\gamma} = 1.42 \pm 0.07 \text{ mb}.$$

Thus

$$\int \sigma_T\,\frac{dE_\gamma}{E_\gamma} = 2.76 \pm 0.18 \text{ mb}. \qquad (5.24)$$

These cross sections include also $E2$ and higher-order transitions. If we extract the estimated $E2$ contribution, we find (Fetisov, Gorbunov, and Varflomameev, 1965)

$$J_T = 2.53 \pm 0.19 \text{ mb} \simeq 2J_{ND} \simeq 2J_{3N}. \qquad (5.25)$$

* See also Carron (1968).

* See also O'Connell and Prats (1968).

This result and (5.22) for a spatially symmetric ^{3}He yield a charge radius of

$$R_c = 1.62 \pm 0.06 \text{ fm}.$$

If we include the finite-proton size, this becomes

$$R_c(\text{finite protons}) = 1.81 + 0.06 \text{ fm},$$

and is in good agreement with electron-scattering data (Collard, Hofstadter, Hughes, Johansson, Yearian, Day, and Wagner, 1964).

A rather surprising result is obtained if the sum rule (5.22) is considered for a model three-nucleon system in which the two-nucleon interaction responsible for the deuteron has been weakened such that the deuteron is unbound. In this model the two-body channel is absent and the calculation of the three-body breakup cross section is affected by changes in the three-nucleon bound and scattering states. The sum rule prediction is independent of the final state:

$$J_T' = J_{3N}' = \tfrac{4}{3}\pi^2\alpha(\tfrac{2}{3}R_{pn}')^2, \qquad (5.26)$$

where R_{pn}' is the new separation between the neutron and proton distributions. If we assume that the triplet two-nucleon force in this model has, say, the same effective strength as the real two-nucleon singlet force, then the difference between the real and the model three-nucleon bound-state wave functions is small and $R_{pn} \simeq R_{pn}'$. Combining (5.26) and (5.22) with the experimental result (5.25), we obtain

$$J_{3N}' \simeq 2J_{3N}. \qquad (5.27)$$

That is, the cross section for three-body breakup is extremely sensitive to whether the final-state interactions correspond to a bound or unbound deuteron. In other words, the existence or nonexistence of the two-body channel is strongly reflected in the cross section for the three-body channel. Thus, any calculation of the three-body photodisintegration must include a final-state wave function in which there is information concerning the existence of the deuteron and the two-body channel. This is the conclusion reached by Barton (1967) and Gersimov (1967) from an alternative split of the sum rule (5.22) into isospin $T=1/2$ and $T=3/2$ components.

The dynamical reason for the sensitivity of the final-state interactions to the potential responsible for the deuteron is that the amplitude A_{3N} for three-body breakup has a pole. If $\mathbf{p}$ is the relative momentum of a triplet two-nucleon subsystem and $\mathbf{q}$ is the momentum of a third nucleon with respect to this subsystem, then

$$\lim_{p^2 \to -\epsilon_d} (p^2+\epsilon_d) A_{3N}(\mathbf{p}, \mathbf{q}) = A_{ND}(\mathbf{q}) g_d(\mathbf{p}), \qquad (5.28)$$

where $A_{ND}(\mathbf{q})$ is the amplitude for two-body breakup, and $g_d(p)/(p^2+\epsilon_d)$ and ϵ_d are the wave function and binding energy of the deuteron.

A calculation by Barbour and Phillips (1968), using an exact separable model for the final-state interactions and hence incorporating (5.28), gives results for the three-body breakup which change markedly (as in 5.27) if the final-state interactions do not include information about the existence of the deuteron. Furthermore, for a particular bound-state wave function whose asymptotic behavior is determined by the binding energy of the nucleons, this calculation is in good agreement with both the two- and three-body breakup sections.

The angular distributions in the photodisintegration reaction are sensitive to the details of the three-nucleon wave functions and to the $E1$, $M1$, and $E2$ multipole transitions. A recent calculation of the two-body cross section (Gibson, 1967) included unretarded $E1$, $M1$, and $E2$ transitions and assumed a three-nucleon bound-state wave which contained S, S', and D admixtures. The leading retardation correction to the $E1$ transition is expected to be of the same order of magnitude as the $E2$ transition. Retardation corrections have been studied by Bailey, Griffiths, and Donnelly (1967) and Carron (1968). The absence in all these calculations of a reasonable treatment of the final-state interactions prevents a clearcut interpretation of the results.

F. Electrodisintegration of Three-Particle Nuclei

The inelastic-scattering experiments of Johansson (1964) measured the cross section of coincidences between scattered electrons and the ejected protons for the reactions

$$e + {}^3\text{He} \to d + p + e, \qquad (\text{A})$$

$$e + {}^3\text{He} \to n + p + p + e, \qquad (\text{B})$$

and for the reaction

$$e + {}^3\text{He} \to n + n + p + e. \qquad (\text{C})$$

These reactions were analyzed in the impulse approximation, keeping only those terms corresponding to the scattering from the ejected proton (Griffy and Oakes, 1965; 1964). The interactions between the ejected proton and the remaining two nucleons were neglected. The excitation energy is quite large and the neglect of final-state interactions in (A) may be a fair approximation. But in (B) and (C), where three nucleons share the energy, it is unlikely that interactions between them can be ignored. However, Griffy and Oakes primarily use their theory to relate Reactions (C) and (B) and thereby isolate the "experimental" data for the simpler Reaction (A). The variation of the cross section of (A) with the proton angle was studied for fixed electron angle and energy.

If the ejected proton is initially at rest, then for given final electron energy and angle there is a unique proton angle. However, the proton has an initial momentum distribution that is characteristic of the structure of the bound-state wave function. Thus the observed variation with proton angle reflects the momentum distribution of the proton in the bound state. The experimental data are in fact sensitive to the asymptotic region of the bound state and the theoretical analysis distinguishes markedly among various radial forms of the three-body wave functions (Griffy and Oaks, 1964).

The electron–proton coincidence cross sections can also be analyzed by means of dispersion-relation techniques (Griffy and Oakes, 1965). These techniques have been used only in their simplest form; proton-pole terms, which are determined by the asymptotic normalization of the bound-state wave function and the proton form factors, are assumed to dominate. This model reproduces the shape of the coincidence cross sections for Reactions (A) and (C) and fits the shape of the integrated cross sections (Gibson and West, 1967). However, if the pole approximation is normalized to the coincidence measurements, then the normalization of the integrated cross section exceeds that of the experimental points by 30% to 70%. A more detailed calculation of the electromagnetic vertex and the use of the impulse approximation fails to remove this discrepancy; qualitative agreement is obtained, however, if the ^{3}He coincidence data is reanalyzed without reference to the ^{3}H data (Gibson and West, 1967). We conclude that more detailed experiments and theoretical work are needed in order to exploit the potential of inelastic electron scattering as a source of information on the three-body bound and continuum states.

VI. WEAK INTERACTIONS OF THREE-NUCLEON SYSTEMS

A. Beta Decay of the Triton

The major physical interest in the beta decay of the triton is centered on the interpretation of the role of nuclear exchange effects. If exchange effects are not included, then the comparison of this decay with that of the neutron indicates that the impulse approximation for the ^{3}H decay rate needs to be enhanced; the actual magnitude of the enhancement depends quite sensitively on the probability densities for the S' and D states of the triton (Blin-Stoyle, 1964; Blin-Stoyle and Papageorgiou, 1965a; 1965b; Blin-Stoyle and Nair, 1966; Blin-Stoyle and Myo Tint, 1967; Freeman, Montague, Murray, White, and Burcham, 1964).

The ft value for an allowed decay is given by the following expression:

$$(ft)^{-1}=(m^5c^4/2\pi^3\hbar^7\ln 2)[G_V^2\mid M_V\mid^2+G_A^2\mid M_A\mid^2]. \tag{6.1}$$

In this equation G_V and G_A are the polar vector and axial vector coupling constants, while M_V and M_A are the matrix elements of the corresponding weak Hamiltonians. An analysis of 0^+ to 0^+ decays (Freeman *et al.*, 1964; Lee and Wu, 1965; Blin-Stoyle and Nair, 1966) yields a value for G_V:

$$G_V=1.4052\pm0.0049\times10^{-10}\text{ erg}\cdot\text{fm}^3.$$

The neutron decay yields a value for G_A. For this decay, the latest experimental ft value is (Christensen, Nielson, Baknee, Brown, and Rusted, 1967)

$$(ft)_n=1133\pm20\text{ sec},$$

in contrast to the previous long-standing value (Sosnovskii, Spivak, Prokofiev, Katikov, and Dobrynim, 1959)

$$(ft)_n=1228\pm35\text{ sec}.$$

We look ahead at the following discussions to note that the reasonable agreement with experiment found depends crucially on the acceptance of this new value for $(ft)_n$. It would be very desirable to have independent confirmation of this value.

Now for the neutron we have

$$M_V^2=1\quad\text{and}\quad M_A^2=3.$$

Hence

$$G_A^2/G_V^2=1.365\pm0.050.$$

The measured ft value for ^{3}H is (Porter, 1959)

$$(ft)_{^3\text{H}}=1137\pm20\text{ sec},$$

while, if we ignore the differences between the ^{3}H and ^{3}He wave functions, we again have immediately

$$M_V^2=1.$$

This result is independent of the structure of the ^{3}H nucleus. However, the axial-vector matrix element does depend on this structure; dropping terms depending on $T=3/2$ and P states, we have (Blatt, 1952)

$$M_A^2=3[P(S)-\tfrac{1}{3}P(S')+\tfrac{1}{3}P(D)].$$

The terms $P(S)$, $P(S')$, $P(D)$ are the probabilities of the principal S, the S', and the D states.

In principle, a comparison between the neutron and triton decay rate thus yields quite sharp information on the probabilities $P(S')$ and $P(D)$. In practice, however, there are a number of corrections which must be included first. The most straightforward are the radiative corrections to the measured (ft) values which change the raw figures quoted above (Berman and Sirlin, 1962)* to

$$(ft)_n=1120\pm21\text{ sec},$$

$$(ft)_{^3\text{H}}=1173\pm20\text{ sec}.$$

These lead to an "experimental" value of $\mid M_A\mid^2$:

$$\mid M_A\mid^2\text{ exptl}=2.83(+0.13,-0.12).$$

The "theoretical" value depends both on the probabilities $P(S')$ and $P(D)$ and on relativistic and meson exchange contributions. Relativistic effects alone lead to a small (2%) reduction in M_A (Blin-Stoyle and

* See also Blin-Stoyle and Nair (1966). The figures quoted are due to R. J. Blin-Stoyle (private communication).

TABLE IV. Values of the axial-vector matrix element $|M_A|^2$ for triton beta decay corresponding to various assumed S'- and D-state probabilities. The last column shows the required percentage enhancement of $|M_A|^2$ from relativistic and mesonic corrections which would yield agreement with the value $|M_A|^2_{\text{exptl}}=2.83(+0.13, -0.12)$.

$P(D)$	$P(S')$	M_A^2	Required exchange correction (%)
4	0	2.84	$+1.9\pm2.3$
6	0	2.76	$+3.4\pm2.3$
6	1.2	2.70	$+5.3\pm2.3$
9	2	2.47	$+9.2\pm2.4$

Papageorgiou, 1965b); however, the exchange effects are more difficult to estimate. We may make a comparison between theory and experiment in terms of the mesonic enhancement of $|M_A|^2$ required to yield agreement with the experimental value. The required enhancement is shown in Table IV for various values of $P(S')$ and $P(D)$. The figures there assume that there are no exchange corrections to M_V, which is true if the polar vector current is conserved. The percentage values (6, 1.2) and (9, 2) for the probabilities $[P(D), P(S')]$ bracket the range of values predicted by various separable and local potential models. The spread reflects closely the deuteron D-state probability in these models, the lower values corresponding to $P(D)=4\%$ and the higher to around 7%; and we see that a reliable assessment of the exchange contributions would yield valuable information on these models.

An estimate of the exchange contributions to M_A has been made by Blin-Stoyle and Tint (1967), who show that the assumption that the divergence of the axial-vector current is proportional to the pion field (i.e., partially conserved axial-vector current or PCAC theory) relates these effects to the observed cross section and polarization for the reaction

$$p+p\rightarrow d+\pi$$

for threshold pions. The net result of their comparison is a reduction rather than an enhancement of M_A; but this numerical result depends on the values assumed for both the production cross section and the deuteron polarization produced, and a recent remeasurement of the cross section (Rose, 1967) changes the previously accepted values (Woodruff, 1960) by a factor of 2. It would be desirable to have an independent remeasurement of this cross section and also of the (still more difficult) deuteron polarization. At least until these measurements have been repeated, we conclude that there is no current serious disagreement between theory and experiment. This conclusion is backed by earlier phenomenological treatments of the exchange contributions (Blin-Stoyle and Papageorgiou, 1965a; Blin-Stoyle and Nair, 1966), which show that effects in the range 6%–10% may well be expected.

B. Muon Capture in ^{3}He

The reaction

$$\mu+{}^3\text{He}\rightarrow{}^3\text{H}+\nu \tag{6.2}$$

differs from the inverse beta-decay reaction almost solely in the large energy release which occurs. It therefore yields, in principle, the same sort of information as does the beta-decay process, and has been used by a number of authors to discuss the μ–nucleon interaction. The capture reaction can go to either (a) the ground state of ^{3}H or (b) to a two-body scattering state or (c) to a three-body scattering state; the experimental capture rates are currently measured as (Zaimidoroga, Kulyukin, Pontecorvo, Sulyaer, Falomkin, Filippov, Trupk-Sitmikov, and Scheibokov, 1963; Auerbach, Esterling, Hill, Jenkins, Land, and Lipman, 1965; Clay, Keuffel, Wagner, and Edelstein, 1966)

$$\Gamma(\text{breakup})=665(+170, -430)\ \text{sec}^{-1},$$

$$\Gamma_{^3\text{H}}=1505\pm46\ \text{sec}^{-1}. \tag{6.3}$$

Of these two rates, the capture rate to the triton ground state is more accurately measured and is easier to interpret, although some *very* preliminary calculations have been carried out for both the two-body channel (b) (Pascual and Pascual, 1967) and three-body channel (c) (Yano, 1964; Wong and Pascual, 1967).

A review of early work on channel (a) has been given by Primakoff (1959). If the basic μ–N interaction is the same as the e–N interaction, then the capture process should be determined by the same matrix elements (6.1) as the triton beta decay; moreover, the coupling constants G_V and G_A should also be the same. However, the μ-capture process has three features which are not significant in beta decay and which stem from the large momentum transfers involved. First, relativistic corrections are significant, contributing perhaps some 3%–4% to the capture rate (Goulard and Primakoff, 1964) in a model-dependent direction. Second, the finite nuclear-size corrections also contribute a few percent (Fujii and Primakoff, 1959). Third, and the point of main interest, there is expected to be an appreciable induced pseudoscalar contribution to the capture rate characterized by a coupling constant G_p in an effective pseudoscalar interaction term. The Goldberger and Treiman prediction for the induced pseudoscalar coupling constant (Goldberger and Treiman, 1958) can also be obtained from the PCAC hypothesis (see, e.g., Kim and Primakoff, 1965):

$$G_p/G_A=7. \tag{6.4}$$

The measurements of μ capture in hydrogen are complicated by molecular effects and yield

$$7<G_p/G_A<15. \tag{6.5}$$

There have been a number of attempts to extract a value for this ratio from the ^{3}He capture rate (e.g., Fujii and Primakoff, 1959; Kim and Primakoff, 1965).

A recent and extensive calculation is that of Peachey (1968). He evaluates the matrix elements involved allowing for the relativistic and finite-size corrections and for exchange effects and using an explicit form for the ^{3}H wave function. The form used contains S'- and D-state components in addition to the principal S state, but with relatively simple radial dependence; the calculations were carried out with both Gauss and a modified Irving radial dependence for all states. The results were quite insensitive to the S' probability at around 1%, to the D-state probability between 5% and 9%, and to the size parameters used for these states. This is what we might expect since they form only a small correction to the whole capture rate. The results were sensitive to the size parameter of the principal S state; this was fixed by fitting the Coulomb energy of ^{3}He. A rather different size parameter results if the charge radius is fitted. Making allowance for this uncertainty and for the many others in the calculation, Peachey's final value is

$$G_p/G_A = 9 \pm 4, \tag{6.6}$$

which is consistent with the PCAC and with the hydrogen-capture estimate. A very similar calculation by Peterson (1968) leads to the rather pessimistic conclusion that the data determine G_p/G_A only within the limits $6 \leq G_p/G_A \leq 33$. However, this assumption depends on the very large exchange contributions allowed by Peterson from a fit to the ^{3}H beta decay, but using the "old" ft value for the neutron. If these exchange contributions are reduced, the allowed values of G_p/G_A in Peterson's calculations split into two bands, in agreement with the result of Peachey. The lower of these two bands yields again essentially Peachey's result.

VII. SUMMARY AND CONCLUSIONS

The primary aim of this article has been to explore the relation between the experimental data on the nuclear three-body system and the underlying interaction between nucleons. The difficulties of the three-body problem as such and the complexity of the two-nucleon interaction have meant that in most calculations drastic simplifying assumptions are made. Nevertheless, it is possible to account for many of the gross features of the three-nucleon-scattering and bound-state data, and to investigate the importance of the various components of the two-nucleon interaction. The three-body calculations also provide insight into the structure of those features of the three-nucleon wave functions which are essential for the interpretation of the electromagnetic- and weak-interaction properties of the three-particle nuclei.

In conclusion we would like to emphasize the following:

(1) The study of the three-nucleon ground state with simplified interactions illustrates that the triton energy is sensitive to several, but as yet poorly understood, features of the two-nucleon interaction; in particular, the relative strengths of the central and tensor components, the short-range repulsion, and the off-the-energy-shell properties.

(2) The variational calculations on the nuclear three-body system with the Hamada–Johnston potential show that the state of the art in computing is now such that we may make reliable three-body predictions for two-body potentials which have been fitted in detail to the two-body data. For the H.J. potential, the theoretical energy of the triton is -6.7 ± 1.0 MeV. Bearing in mind the uncertainties due to relativistic corrections, three-body forces and the quoted estimate of the numerical error, this value is sufficiently close to the experimental value of -8.49 MeV such that the acceptability of the Hamada–Johnston potential, from the point of view of the three-nucleon system, is not in question.

(3) The ultimate usefulness of three-nucleon calculations in discriminating between various fits to the two-nucleon data depends upon the understanding of the extent to which the system can be treated as a nonrelativistic three-body problem; reliable estimates of the corrections due to three-body forces and relativity do not exist at present.

(4) There is no clearcut evidence to support the existence of three-nucleon excited resonant states, or particle stable states other than ^{3}H and ^{3}He.

(5) The latest experimental values for the doublet and quartet neutron–deuteron scattering lengths, $^2a = 0.15 \pm 0.05$ fm and $^4a = 6.13 \pm 0.04$ fm, differ considerably from the older values $^2a = 0.7 \pm 0.3$ fm and $^4a = 6.4 \pm 0.1$ fm. For simple separable potentials, and possibly for the Hamada–Johnston potential, it is possible to reconcile the older value for the doublet scattering length, but not the newer value, with the experimental triton binding energy.

(6) The S-wave scattering of neutrons by deuteron below the inelastic threshold yields limited information: the absolute magnitude of the quartet amplitude and the energy dependence of both the quartet and the doublet amplitudes are approximately determined by the triplet-even, two-nucleon, effective-range parameters.

(7) Separable, central interactions can account for the main features of the elastic, and to a lesser extent the inelastic, cross sections for low- and moderate-energy neutron–deuteron scattering. It is important to realize that those features of the elastic-scattering data that depend on the quartet spin state are not a particularly useful test of a dynamical three-body calculation, since in the quartet state the details of interaction and distortion of the deuteron are unimportant.

(8) The existing estimates of the neutron–neutron scattering length from the $d(n, p)2n$ reaction are model dependent, and it seems unlikely that this reaction can give useful information of this kind.

(9) The nucleon–deuteron polarization data probably contains much valuable information on the two- and three-particle interactions but attempts at theoretical interpretation are almost completely lacking.

(10) The experimental charge form factors of ^{3}H and ^{3}He can be understood in terms of the charge properties of free nucleons and three-body wave functions whose properties are similar to those obtained in nonrelativistic three-nucleon calculations.

(11) The magnetic moments and form factors, and the predominately magnetic-dipole radiative capture of thermal neutrons by deuterons all imply that the electromagnetic properties of the nucleons are modified in the presence of nuclear interactions, i.e., magnetic-interaction effects.

(12) The analysis of the ^{3}He$-^3$H Coulomb energy difference suggests a possible deviation from charge symmetry of nuclear forces or possibly charge-interaction effects; until one has accurate ^{3}He and ^{3}H wave functions, it is not possible to discuss this question quantitatively.

(13) Final-state interactions are important in the analysis of the low-energy photodisintegration of ^{3}He. This and the limited accuracy of the experimental data prevent detailed comparison of theory with experiment.

(14) The analysis of the beta decay of the triton indicates that at present there is no serious disagreement between theory and experiment. The reconciliation of the experimental *ft* values for the neutron and the triton requires that relativistic and mesonic corrections give rise to a 4% to 9% enhancement of the square of the axial-vector matrix element that enters in the beta decay of the triton.

(15) Muon capture in ^{3}He permits an estimate of the induced pseudoscalar coupling constant which agrees with theoretical predictions and the value obtained for muon capture in hydrogen.

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