Systematic Survey of the α–α Interaction

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A systematic survey of the subject of the α–α interaction is made. The early observations of α emission and resulting theoretical descriptions, especially the α-particle model of nuclei, are described, as are more recent α–α scattering experiments. The latter were designed to obtain information about the nature of the α–α interaction and about the energy levels of ^{8}Be by studying the resonance behavior of the phase shifts. The phenomenological approach, i.e., the attempt to construct the α–α potential which reproduces the experimental phase shifts, is described as are detailed theoretical constructions of the repulsive inner part and attactive outer part of the α–α interaction. Studies of how the α–α interaction provides better understanding of the structure of α-clustered nuclei and also of the fundamental forces between the constituents of these nuclei are discussed. Suggestions for the direction of further efforts are made.

CONTENTS

1. INTRODUCTION

Even early reviews of the α-particle model (αPM) of nuclei [(Rosenfeld (1948), Dennison (1954), Glassgold and Galonsky (1956)] recognized the importance of the α–α interaction. These reviews described the qualitative features of the αPM, but these features had to wait until the middle of the last decade to be verified by scattering experiments which had a direct bearing on the α–α interaction.

Recently, considerable interest has been shown in the cluster model of nuclei and hypernuclei. Such a model often involves two or more α's, and one needs to use an α–α potential that is in accord with fundamental studies and scattering results and, at the same time, can be handled with mathematical convenience in the problem. It thus seems very useful to make a systematic survey of the subject of the α–α interaction. To this end, we shall follow the historical development in this field. Some of the earlier scattering results and their interpretation are incorrect, and are only included for completeness. The entire subject of the α–α interaction occupies so important a role in the nuclear structural problems that we feel the gradual development of the subject is well worth following. Although the emphasis of this paper is on the two-body α–α interaction and not on the αPM, we begin with a brief review of the latter.

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With the discovery of radioactivity by Becquerel and the Curies in 1896–1898, the α particle was discovered. Subsequently, some of its properties were studied by Rutherford and his collaborators [Rutherford and Geiger (1908), Rutherford and Royds (1909)], who found that the α rays are positively charged (each α particle carrying two units of electronic charge) and are nothing but ^{4}He nuclei. Some other important properties of the α particle are

(1) It has an intrinsic spin and parity of 0^+ and thus obeys Bose–Einstein statistics; the wave function describing two α particles must remain symmetric with respect to exchange of particles.
(2) It has a radius of $\approx$1.44 fm.
(3) It is a tightly bound system with a binding energy of 28 MeV.

The αPM, which takes into account the above features, originated from the α emission of nuclei.

2. THEORY OF α EMISSION

The Coulomb force increases with size as the square of the charge number Z^2, but the nuclear binding increases approximately as the mass number A. Thus, the Coulomb repulsion dominates in heavy nuclei, causing α emission. Geiger and Nuttal (1911, 1912) established the rule that the α emitters with large disintegration energies have short half-lives, and conversely. Quantum mechanics found one of its first applications in the hands of Gamow (1928) and of Condon and Gurney (1928, 1929) in explaining this rule. This theory assumes that the α particle is preformed inside the parent nucleus and moves in a spherical well determined by the daughter nucleus. Assuming the spherical potential $V(r)$ between the α particle and the daughter nucleus, shown schematically in Fig. 1, the penetration probability P of an α particle of energy E meeting a barrier provided by the spherical region $a<r<b$ is given by

$$P=\exp\left\{-\frac{2}{\hbar}\int_a^b\{2M[V(r)-E]\}^{1/2}dr\right\}.$$

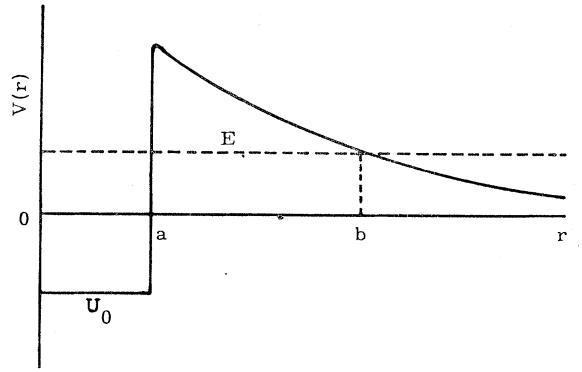

FIG. 1. $V(r)$ denotes the potential energy between a daughter nucleus (i.e., the parent nucleus minus one alpha particle) and an alpha-particle system and goes to zero when r, the distance between the two bodies, is very large. E denotes the disintegration energy.

Large E means a short life; the Geiger–Nuttal rule is explained. However, this success is rather misleading. Although certain nuclei emit α particle and this emission can be explained by quantum mechanics, there is not much other evidence to support the αPM. It was replaced by the subsequently developed n–p model of nuclei [Chadwick (1932)]. The study of nuclear reactions has shown that neutrons and protons are also emitted from some heavy nuclei in highly excited states. An improved understanding of the nuclear forces tells us that the α particle cannot remain in the nucleus for a long time and still keep its identity. In a modified version, the α particles exist as stable units only for a short time [Wefelmeiser (1937a, 1937b), von Weizsäcker (1938), Fano (1937), Wheeler (1937b)]; they then break up into their constituents, which again rearrange themselves into new α-particle structures. This version still owes its origin to the exceptional stability of the α particle. Wheeler (1937b) pointed out that for the αPM to hold, the frequency of vibration of the α particle should be greater than the frequency of nuclear exchange, but calculations of Grönblom and Marshak (1939) and of Margenau (1941) show that these two frequencies are comparable. In support of the αPM, 90% of the binding energy of the light nuclei comes from the constituents of α particles and 10% from the classical bond between α particles. Also, nuclei having nucleons besides α particles give a rather small binding energy and increase slowly until another α particle is formed. (Elements between ^{4}He and ^{8}Be show this trend.) However, the αPM cannot explain the binding energy of heavier nuclei having an odd nucleon which plays a more important role. The model was aptly defended by Herzenberg (1955), whose points will be discussed in dealing with the mechanism of the α–α interaction.

Like all other two-body interactions, the α–α interaction has been studied both from phenomenological and fundamental points of view. These two approaches meet, as usual, on the common ground of the "phase shift." Before we focus on this ground, we present some of the factors leading to the recent α–α scattering experiments.

3. α–α SCATTERING

A. Review of the Earlier α–α Scattering Results

The first α–α scattering was performed when Rutherford and Chadwick (1927) investigated the scattering of α particles from He nuclei. Up to 1939, the only sources available for α scattering were natural α emitters, e.g., radium, thorium, and polonium. The succeeding development of the cyclotron, Van der Graaff, and other high-energy accelerators has provided α particles with higher, more controlled bombarding energies. One can broadly divide the whole range of α–α scattering experiments into two periods: (1) 1927–1939, when the natural emitters were used as sources, and (2) 1940 onwards, when the accelerators were used for the source beam.

The α–α scattering experiments performed during the period 1927–1939 were all based on the "annular-ring method." A small chamber with an annular-ring scattering volume was used. Gas pressures were high compared with those now used. The energies of the α particles emitted from the radioactive nuclei at the scattering volume were determined by range measurements. The scattering of α particles from He nuclei was investigated as a function of energy by slowing down natural α particles with absorbers. The measurements were crude; the intensity of the beam was rather low and spreads in energy and angle were rather large.

Some of the basic ideas behind the α–α scattering experiments originated from Chadwick and Bieler (1921), who investigated the scattering of α particles from hydrogen nuclei. They attempted an explanation of this scattering in terms of the Coulomb field between the two particles. However, at close collision distances (<4 fm) the force between the two particles increased much more rapidly with decrease of distance than could be accounted for by the inverse square law of force. Chadwick and Bieler interpreted the divergence from the classical Coulomb scattering as follows: A point-charge structure was assumed for the H nucleus and, as a first approximation, an elastic oblate spheroid of semiaxes about 8 and 4 fm moving in the direction of the minor axes was assumed for the α particle. An H nucleus projected towards such an α particle would move under the ordinary Coulomb electrostatic forces until it reached a spheroidal surface of the above dimensions. Then it would experience a powerful field of force and recoil as from a hard elastic body. This model agreed only roughly with the experimental observations; a close comparison could not be made because of the

difficulty in calculating the collision relations for an oblate spheroid.

The collision of α particles with He nuclei could give further information of the force field in the immediate neighborhood of the He nucleus. Since both particles have the same structure, there would be no need to assume a structure for one to deduce that of the other.

In view of these considerations, Rutherford and Chadwick (1927) performed an experiment to scatter α particles off He nuclei. The results showed that the collision relations for these particles were similar to those holding for α–hydrogen scattering. They observed that at large distances of collision the force between the particles was given by Coulomb's law, but there was an indication of a departure from Coulomb's law at closer distances. This departure was attributed to the structure of the particles; Rutherford and Chadwick (1927) tentatively explained it by invoking some strong additional forces which increased much more rapidly with distance than an ordinary inverse square Coulomb field.

At the same time, some thought was given to the adequacy of the description of this phenomenon in terms of the old mechanics; the new mechanics had already been established. Oppenheimer (1927) and later Gordon (1928) showed that the scattering of particles by an inverse square field is the same in the new mechanics as in the classical theory. Mott (1930) showed that this is not necessarily true for two identical particles, since the wave functions used must be antisymmetrical or symmetrical in the coordinates of the two particles, and thus the scattering laws may very well be affected. According to Mott's theory, the formula giving the number of α particles scattered from He nuclei into a given solid angle $d\Omega(\theta, \phi)$ will be

$$dI = \frac{(2\epsilon^2)^4}{m^4v^4}\left[\csc^4\theta + \csc^4\left(\tfrac{1}{2}\pi-\theta\right) + 2\csc^2\theta\csc^2\left(\tfrac{1}{2}\pi-\theta\right)2\left(\eta\log\frac{1+\cos 2\theta}{1-\cos 2\theta}\right)\right] \times 2\sin\theta d\theta d\phi,$$

where $\eta=(2\epsilon)^2/\hbar v=(4/137)(e/v)$, v is the relative velocity of the two α's, θ is the angle through which the line joining the α's is deflected, and ϕ is the azimuthal angle. The ratio of quantum-theory scattering to classical scattering is thus seen to have a maximum value of 2 at 45° for all velocities of the α particles. Mott also predicted some maxima and minima in the scattering cross section. Unfortunately, in the experiments of Rutherford and Chadwick, these are masked by the structure effect. To show them, the α particles would have to be so slow that the structure effect would be negligible.

Chadwick (1930) was the first experimentalist to verify the validity of Mott's predictions by doing an α scattering experiment in He at very low energies ($\approx$1 MeV). Here the experimental results approach the value of twice the classical scattering as the energy of incident α particles decreases, i.e., the results approach more and more closely to quantum-theory scattering. Chadwick found, by comparing the amount of scattering observed in the collision experiments, that the forces between the particles vary little from Coulomb forces. Thus, the deviation from classical scattering could not be ascribed to a divergence from Coulomb forces; it was then attributed to a failure of the classical theory. This was indeed borne out by Blackett and Champion (1931), who performed an α scattering experiment in a Wilson cloud chamber and found the minimum in the scattering cross section at about 25°, in excellent agreement with Mott's theory. However, there were indications of deviations from Mott's theory, and experiments were carried out later by Wright (1932), Mohr and Pringle (1937), and Devons (1939) to study these deviations. An explanation was also sought in terms of the effects of nuclear forces, a more definite nomenclature for the so-called additional forces. By this time, the idea of the nucleus being composed of neutrons and protons was gaining ground, and the concept of the nuclear force (responsible for keeping the neutrons and protons together in the nucleus) was introduced. It was reasonable, therefore, to assume that this nuclear force would provide a nuclear potential between the α particles, and thus the α–α scattering should be governed not only by the Coulomb forces but also by the nuclear forces. One would then expect that the effect of the nuclear forces would be to cause a change in the phase of the initial wave describing the incident α particles. This phase change, usually called the phase shift and denoted by δ, was seen to contain all the essential information about the nuclear potential. Phase-shift measurements thus became the main concern of the later experiments.

B. More Recent α–α Scattering Results

The purposes of the α–α scattering experiments have been, first, to get information about the nature of the α–α interaction, and second, to investigate the energy levels of ^{8}Be by studying the resonance behavior of the phase shifts. ^{8}Be, which is unstable and dissolves into two α particles, provides a typical and interesting example of a system of two α particles. It is expected that the low-lying states of this sytem are determined fairly well by only the relative motion of the α particles because of their extreme tightness. Information on the levels of ^{8}Be can also be obtained from a number of nuclear reactions. A schematic diagram of the energy levels of ^{8}Be is shown in Fig. 2. The position of the ground state has been altered slightly by a recent reliable measurement (discussed later in this paper).

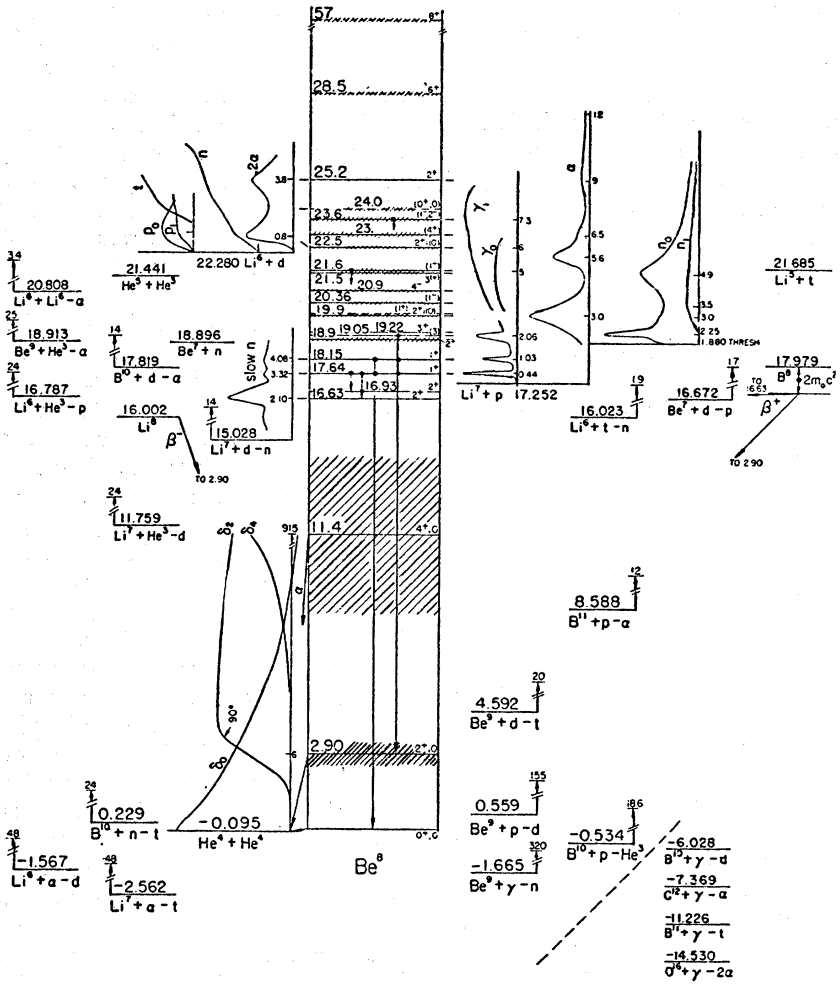

FIG. 2. Energy levels of ^{8}Be [reproduced from Lauritsen and Ajzenberg-Selove (1966) by courtesy of *Nuclear Physics*].

It is of interest to see if the resonances in α–α scattering are observed at the energies corresponding to the low-energy states of ^{8}Be. The usual procedure in this connection is to make a partial-wave analysis of the phase shifts δ_l for a given partial wave with angular momentum l. Because of the simplifying features of the α particles (namely the zero spin and the high internal binding energy), the phase shifts are reduced to a minimum, and one can analyze the scattering in terms of only real phase shifts up to a laboratory bombarding energy of 35 MeV. Appendix I gives the usual method of phase-shift analysis as employed by the experimentalists.

We shall present the α–α scattering results energywise, although this will disturb the chronological order a bit. Table I shows the list of experiments to date and the energy ranges covered.

The first postwar experiment was done by Cowie, Heydenburg, Temmer, and Little (1952) at the Department of Terrestrial Magnetism, Carnegie Institution of Washington (the data obtained at this institute on α–α scattering will be called DTM data), who measured the differential cross section for α scattering in helium in the energy range of 400 to 950 keV (lab) over the angular range of 10° to 45° (lab). They found that at an energy of 850 keV the deviations from Mott scattering amounted to as much as 8%.

A more complete account of α–α scattering in the low-energy region was given by Heydenberg and Temmer (1956), who covered the region from 150 keV to 3 MeV (lab) between laboratory angles of 10° and 80°. By doing these experiments they wanted, first, to verify the Mott formula in detail in the energy region where there was no nuclear effect; second, to explain and explore the region of the ground state of ^{8}Be; and finally, to study the 3-MeV level of ^{8}Be. Table II shows their S- and D-wave phase shifts. Below 400 keV, the nuclear interaction was not found to play an important role, and Mott's formula for Coulomb scattering was verified in detail. (The method of verification of Mott's formula consists in observing the departure of the ratio of the actual cross section to the theoretical Mott cross

TABLE I. A list of the postwar experiments on the scattering of alpha particles in helium.

Energy range in lab system (MeV)	Source of particles	Angles of observation of scattering in lab system	References
0.15–3.0	Van der Graaff (D.T.M., Washington)	10°–80°	Heydenberg and Temmer (1956)
3.0–6.0	Van der Graaff (Rice)	15°–45°	Russell, Phillips, and Reich (1956)
4.0–12.0	Tandem (O.N.R., Caltech)	15°–45°	Tombrello and Senhouse (1963)
5.0–9.0	Van der Graaff (Rice)	15°16.5′, 27°22′ and 35°3.5′	Jones, Phillips, and Miller (1960)
6.4–7.8	Cyclotron (Yale)	7°30′–45°	Berk, Steigert, and Salinger (1960)
	Cyclotron (Yale)	10°–45°	Dunning, Smith, and Steigert (1961)
10–20	Tandem (Heidelberg)	For each bombarding energy, data were taken at 11 pairs of angles (right–left)	Werner and Zimmerer (1964)
12.3–22.9	Cyclotron (Illinois)	10°30′–45°	Nilson, Jentschke, Briggs, Kerman, and Snyder (1958)
12.9–21.6	Cyclotron (Indiana)	15°–45°	Steigert and Sampson (1953)
20, 20.4	Cyclotron (St. Louis)	7°–60°	Mather (1951)
	Cyclotron (St. Louis)	30° and 45°	Braden, Carter, and Ford (1951)
23–38	Cyclotron (Birmingham)	15°–45°	Bredin, Burcham, Evans, Gibson, McKee, Prowse, Rotblat, and Snyder (1959)
23–51	Cyclotron (88″, Berkeley)	15.32°, 27.75°, 45°	Shield, Conzett, Darriulat, Pugh, and Slobodrian (1964)
		0–45°	
30	Cyclotron (M.I.T.)	45°–90° 270°–360°	Graves (1952)
33.5–35.5	Cyclotron (INSU, Tokyo)	15°–45°	Chiba, Conzett, Morinaga, Mutsuro, Shoda, and Kimura (1961)
38.5	Cyclotron (Birmingham)	15°–45°	Burcham, Gibson, Prowse, and Rotblat (1957)
37–47	Cyclotron (60″ and 80″, Berkeley)	7°30′–45°	Conzett, Igo, Shaw, and Slobodrian (1960)
			Conzett, Slobodrian, Yamabe, and Shield (1964)
			Conzett, Shield, Slobodrian, and Yamabe (1964)
51	Cyclotron (IKO, Amsterdam)	7°30′–50°	Van Niftrik, Brockman, and Van Oers (1964)
53–120	Cyclotron (88″, Berkeley)	5°–45°	Darriulat, Igo, Pugh, and Holmgren (1965)

section from unity as a function of energy.) Above 400 keV, nuclear interactions come into play; δ_0 decreases from approximately 180° at 400 keV to about 120° at 3 MeV. The D-wave phase shift does not appear below 2 MeV and reaches the value of 2.5° at 3 MeV. These results agree with those of the Rice Institute (presented below) at the 3-MeV point of overlap.

Hydenberg and Temmer also gave some estimates for the width Γ_0 of the ground state and its lifetime τ, namely $\Gamma_0 \leqslant 3.5$ eV and $\tau \approx 2\times10^{-16}$ sec; these were calculated by a method described in the Appendix of the Heydenberg and Temmer paper. Previous estimates were $\tau \leqslant 2\times10^{-14}$ sec and $\tau \leqslant 4\times10^{-15}$ sec [Crussard, (1950), Hodgson (1952), Treacy (1955)]. Thus, it was inferred that 2×10^{-16} sec $\leqslant \tau \leqslant 4\times10^{-15}$ sec. These estimates have been changed by the results of more reliable experiments.

At about the same time, Russell, Phillips, and Reich (1956) performed α–α scattering experiments in the Rice Institute for laboratory bombarding energies of 3 to 6 MeV at lab angles of 15°, 17.5°, 20°, 27°22′, and 35°3.5′. [An angle of 27°22′ corresponds to a zero of the second-order Legendre polynomial $P_2(\cos\theta)$.] Their phase-shifts plot is reproduced in Fig. 3. Their 27°22′ curve was found to be a monotonic function of bombarding energy; but the variations in the other excitation functions led Russell *et al.* to assign a J(spin) value of 2^+ for the 2.9-MeV state.

The single-level dispersion theory of Wigner and Eisenbud (1947) was applied to this state, and the level parameters of the state were obtained by fitting the D-wave resonance. To sketch the method of extracting the level parameters only very briefly, the nuclear phase shift δ_l is expressed in the dispersion theory as $\delta_l = \delta_{\lambda,R} - \Phi_l$ where, following the notations of Jones, Phillips, and Miller (1960),

$$\Phi_l = \tan^{-1}[F_l/G_l]_{\rho=kR} = \text{hard-sphere phase shift}$$

and

$$\delta_{\lambda,R} = \tan^{-1}[\tfrac{1}{2}\Gamma_{\lambda,l}/(E_{\lambda l}+\Delta_{\lambda l}-E_{\text{c.m.}})];$$

F_l, G_l are the regular and irregular Columb wave functions and $E_{\text{c.m.}}$ and k are the energy in the c.m. system and the wavenumber $(2\mu E/\hbar^2)^{1/2}$, respectively;

TABLE II. The experimental α–α scattering phase shifts in the energy range 0–23 MeV (lab) [Heydenberg and Temmer (1956), Tombrello and Senhouse (1963), Nilson, Jentschke, Briggs, Kerman, and Snyder (1958)]. All phase shifts are in degrees.

	Phase shifts			
Energy	δ_0	δ_2	δ_4	δ_6
0.400	0±0.5			
0.600	178±1			
0.850	175±1			
0.950	173±1			
1.00	171±1			
1.50	159±1			
2.00	148±1	0.0±0.1		
2.50	137.5±1	1.0±0.2		
3.00	128.4±1	2.5±0.3		
3.84	114.1±1	7.5±1	−0.1	
5.26	96.6±2	37.5±2	0.2	
6.47	79.5±2	80.8±2	−0.1	
6.96	75.9±3	92.7±3	−0.1	
7.47	71.4±4	102.1±4	0	
7.88	68.0±4	107.5±4	0	
8.87	59.4±4	113.8±3	0	
9.88	51.6±4	115.2±2	0±1	
10.88	45.6±4	116.3±2	0±1	
11.88	41.0±4	114.9±2	0±1	
12.3	29±4	103±8	3.0±1.5	
15.2	11±4	100±8	5.2±2	
17.8	7±2	104±4	24.1±2	0
20.4	−1.6±2	97.5±4	27.7±2	0.54
21.65	−8.8±2	94.7±2	41.8±2	0.13
21.8	−6.9±2	94.8±2	47.0±2	1.03
22.25	−10.2±2	93.3±2	48.1±2	0.09
22.81	−9.4±2	91.7±2	56.4±2	1.07
22.9	−10.7±2	94.0±2	59.2±2	1.09

$E_{\lambda l}$ is the constant expansion parameter. The c.m. reduced width γ_λ^2 and the c.m. laboratory width $\Gamma_{\lambda,l}$ are related by

$$\tfrac{1}{2}\Gamma_{\lambda l}=(\rho\gamma_\lambda^2/A_l^2)_{\rho=kR},$$

where

$$A_l^2=F_l^2+G_l^2$$

and

$$\Delta_{\lambda,l}=-\gamma_\lambda^2(g_l+l)_{\rho=kR};$$

and

$$g_l=\rho[F_l^{-1}(\partial F_l/\partial\rho)-A_l^{-2}(G_l/F_l)]_{\rho=kR};$$

R is the nuclear radius. The resonance energy $E_{\rm res}$ is defined as that at which $\delta_{\lambda,R}=\frac{1}{2}\pi$. Values of excitation energy, reduced width, etc., are then obtained (for a given hard-sphere radius) by fitting the phase shifts to the above dispersion-theory formulas.

The level parameters for the D state as obtained by Russell, Phillips, and Reich (1956) are shown in Table III, where the level parameters obtained by other authors for the various states are also shown. These widths seem consistent with a two-body model of ^{8}Be. Russell *et al.* applied the Landau K-function formalism to get the width of the ground state of ^{8}Be. The Landau K function for S-wave scattering of charged particles is defined by

$$K=[\pi\cot\delta_0/(e^{2\pi\eta}-1)]+h(\eta), \qquad \text{(E1)}$$

where δ_0 is the S-wave phase shift and η the energy parameter $4e^2/\hbar v$. The function $h(\eta)$ is given by

$$h(\eta)=\eta^2\sum_{n=1}^{\infty}[n(n^2+\eta^2)]^{-1}-\ln\eta-0.5772;$$

K may be expanded in the effective-range approximation as

$$K=(2\eta k)^{-1}(-a^{-1}+\tfrac{1}{2}r_0k^2-Pr_0^3k^4+Qr_0^5k^6\cdots)$$
$$=A+BE_L+CE_L^2+DE_L^3. \qquad \text{(E2)}$$

The parameters a, r_0, P, Q may be determined by fitting (E1) by (E2), using a standard least-squares program. The width in the c.m. system is calculated from the expression

$$\Gamma_{\rm c.m.}=\pi\Big/\left[(e^{2\pi\eta}-1)\left(\frac{dh}{dE_L}-\frac{dK}{dE_L}\right)\right]_{E_L=E_R(\rm lab)}. \qquad \text{(E3)}$$

Note that at resonance ($\delta=\frac{1}{2}\pi$), $K=h(\eta)$. Russell, Phillips, and Reich (1956) obtained a value of 4.5±3 eV for the width of the ground state.

Analysis of the experiment by Dunning, Smith, and Steigert (1961) at lab energies of 6.43, 6.84, and 7.78 MeV gave the values of the phase shifts as $\delta_0=82.5°$, $\delta_2=77.5°$; $\delta_0=80°$, $\delta_2=82°$; and $\delta_0=70°$, $\delta_2=104°$, respectively. The S-wave phase shifts are in excellent agreement with those to be discussed below of Jones, Phillips, and Miller (1960) and Berk, Steigert, and Salinger (1960), but their D-wave phase shifts are somewhat smaller. However, this is not very significant because the cross section is rather insensitive to the D-wave phase shift near this energy region.

Berk, Steigert, and Salinger (1960) attempted to fill in the gap between the 6-MeV Rice data and the Illinois data above 12 MeV by doing a scattering experiment at 7.56 MeV, and obtained $\delta_0=70°$ and

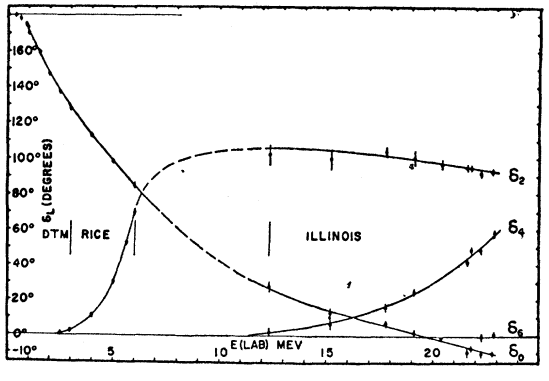

FIG. 3. α–α scattering phase shifts below 24 MeV showing the DTM [Heydenberg and Temmer (1956)], the Rice Institute [Russell, Phillips, and Reich (1956)], and the Illinois [Nilson, Jentschke, Briggs, Kerman, and Snyder (1958)] data. The smooth curves have been drawn arbitrarily through the data points. [Figure reproduced from Russell *et al.* (1956) by courtesy of *The Physical Review*.]

TABLE III. Level parameters as used by various authors for the D, G and other excited states of ^{8}Be. These parameters were obtained by fitting the α–α scattering resonances with the single-level dispersion relation. The term θ_l^2 is the ratio of the reduced width to the Wigner limit $3\hbar^2/2\mu R^2$, as given by the first sum rule of Wigner and Treichman (1952). For the comparison of the experimental phase shifts to those obtained with the use of parameters, see the references shown against each set of parameters. (The general feature of the dispersion-theory fits is that a larger radius gives a fit on the low-energy side of the resonance while, for the smaller radii, the fit on the high-energy side improves.

l	Nuclear radius R(fm)	Excitation energy[a] E_{exc} (MeV, c.m.)	Reduced width $\gamma_{\lambda,l}^2$ (MeV, c.m.)	θ_l^2	References
2	5.0	2.9	0.9	0.7	Russell, Phillips, and Reich (1956); Nilson, Jentschke, Briggs, Kerman, and Snyder (1958)
2	3.5	2.9	0.9	0.4	Nilson, Jentschke, Briggs, Kerman, and Snyder (1958)
2	3.5	3.1	3.5	1.32	Jones, Phillips, and Miller (1960)
2	3.5	3.18	3.36	1.27	Tombrello and Senhouse (1963)
4	4.44	11.7	2.0	1.26	Nilson, Jentschke, Briggs, Kerman, and Snyder (1958)
6	3.5	36.7	2.3	1.14	Darriulat, Igo, Pugh, and Holmgren (1965)
6	4.5	27.6	2.4	0.66	Darriulat *et al.* (1965)
6	4.5	29.4	3.3	0.48	Darriulat *et al.* (1965)
6	5.0	24.7	2.0	0.66	Darriulat *et al.* (1965)
8	4.5	56.9	6.5	0.24	Darriulat *et al.* (1965)

[a] The excitation energy of each level is with respect to the ground state energy of ^{8}Be. The resonance energy E_{res} is obtained by adding to the excitation energy E_{exc} the ground-state energy for which the value of 0.096 MeV (c.m.) was used in these calculations.

$\delta_2 = 100°$. These values are in good agreement with those found by Jones *et al.*

The next higher-energy experiments were performed in the range of 5 to 9 MeV by Jones, Phillips, and Miller (1960) at lab angles of 15°16.5′, 27°22′, and 35°3.5′. [The first and third angles correspond to zeroes of $P_4(\cos\theta)$, and the second to a zero of $P_2(\cos\theta)$.] The phase shifts as a function of energy are shown in Fig. 4. Their analysis shows evidence of the resonant state at $\approx$3 MeV, in keeping with the previous observations. The behavior of the S-wave phase shift was found to be the same as that observed in the early experiments. Jones *et al.* also examined the apparent inconsistency of the 12.3- and 15.2-MeV phase shifts of Nilson, Jentschke, Briggs, Kerman, and Snyder (1958). Assuming a different set of phase shifts at these energies, they still obtained nearly the same cross section as Nilson *et al.*, which shows that the cross section is not very sensitive to the choice of phase shifts in this region.

Looking at these level parameters (Table III), one notices that the width of the D state is large compared with the Wigner limit ($3\hbar^2/2\mu R^2$). This indicates that it is wholly of α–α parentage and thus supports the αPM of ^{8}Be.

Tombrello and Senhouse (1963) performed α–α scattering experiments in the energy range of 3.8 to

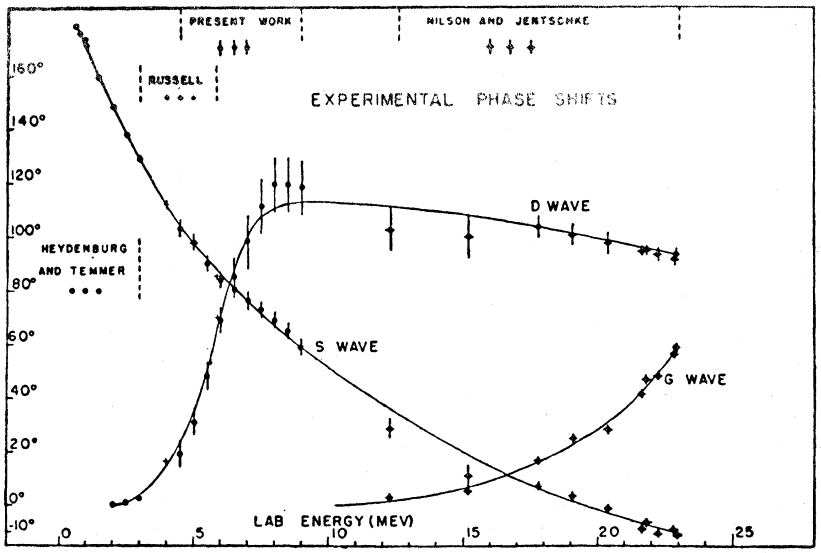

FIG. 4. α–α scattering phase shifts derived by Jones, Phillips, and Reich (1960) are shown with those derived by Heydenberg and Temmer (1956); Russell, Phillips, and Reich (1956); and Nilson, Jentschke, Briggs, Kerman, and Snyder (1958). The phase shifts are plotted as a function of laboratory energy. The smooth curves are drawn arbitrarily through the data points. In the figure, "present work" refers to the results of Jones, Phillips, and Miller (1960). [Figure reproduced from Jones, Phillips, and Miller (1960) by courtesy of *The Physical Review*.]

TABLE IV. A list of the main reaction channels that open at high bombarding energies in the initial α–α channel. The threshold energies of these channels are given in the lab coordinate system.

Reaction channel	Threshold energy (MeV)
^{7}Li+p	34.73
^{7}Be+n	39.68
^{4}He+n+^{3}He	41.21
^{4}He+n+^{3}He	43.1
^{3}He+^{5}He	43.6
t+^{5}Li	43.6
^{4}He+2d	44.77
^{6}Li+d	44.82
^{6}Li+n+p	49.28
He+p+d	54.2
He+2p	54.8
^{5}Li+n+d	56.2
^{6}Be+2n	60
^{3}He+d+t	76.47
4n+4p	113.61

12 MeV, covering the so-far untouched region of 9 to 12 MeV. The phase shifts they obtained are shown in Table II. At 3.8 and 6 MeV, their phase shifts agree with those of Russell *et al.* Below 9 MeV, δ_0 shows increasing disagreement with increasing energy, but the disagreement is within the uncertainties of the results. The phase shifts of Berk *et al.* at 7.56 MeV and of Dunning *et al.* at 7.78 MeV are in good agreement with those of Tombrello and Senhouse. The results of the latter are also consistent with those of Heydenberg and Temmer and of Nilson *et al.*, except for the two points of Nilson at 12.3 and 15.2 MeV.

The level parameters of Tombrello and Senhouse for the 2^+ state (see Table III) are in excellent agreement with those obtained by Jones *et al.* These level parameters were chosen primarily as a compromise between the disagreeing values at high and low energies. However, since the dispersion-theory fit to the data was not very impressive, Tombrello and Senhouse pointed out the doubtfulness of the significance of such a parametrization. The variety of parameters obtained by different groups for this state introduces such a doubt.

The energy range to be discussed next is 12.3 to 22.9 MeV. Absolute differential cross sections for α–α scattering have been measured at 10 energies between 12.3 and 22.9 MeV by Nilson, Jentschke, Briggs, Kerman, and Snyder (1958) at the University of Illinois. The angular range they covered was 11° to 50° (lab). They observed no broad S state in ^{8}Be for excitation energies between 0.5 and 11.45 MeV, in disagreement with the results of Steigert and Sampson (1953) who obtained a 7.5-MeV 0^+ state from their analysis. From their results there were, however, clear indications of the 2.9-MeV 2^+ state and a broad 4^+ state in the neighborhood of 11 MeV with a reduced width of about 2 MeV. The I-wave phase shift was first observed at 20 MeV and was found to be positive. An experimentally observed δ_6 indicated that an I state may exist at a high energy.

The experimental S-, D-, and G-wave phase shifts of Nilson *et al.* are given in Table II, the level parameters they used for the D-wave phase shifts being included in Table III. An interesting observation made by Nilson *et al.* for these parameters was that no one set of single-level parameters reproduced the experimental values over the entire range of 0–22 MeV [Nilson, Jentschke, Briggs, Kerman, and Snyder (1958)].

Two scattering experiments were performed at Washington University by Mather (1951) at 20 MeV and by Braden, Carter, and Ford (1951) at 20.4 MeV. Their experimental results show reasonably good agreement with those of Nilson *et al.* at 20.4 MeV. They did not give any phase-shift values due to insufficient data.

Steigert and Sampson (1953) performed α–α scattering experiments in the energy region almost overlapping that of Nilson *et al.* The region they covered was from 12.88 to 21.62 MeV. These results are found to agree qualitatively as well as quantitatively with those of Nilson *et al.* except for the indication of the 7.55-MeV 0^+ state in Steigert and Sampson's results. They accounted for the steep rise in their S-wave phase shifts by invoking a resonance level at 7.55±0.08 MeV with a width Γ of 1.2±0.4 MeV. There was a large error in the width measurement because there were few points available to give the resonant shape. However, this evidence for the 7.55-MeV level is at variance with many particle reactions and with the ^{8}Be energy-level schemes predicted either by an αPM or a central-force model. [The recent scattering experiments of Werner and Zimmerer (1964) in the energy range of 10–20 MeV also did not support the assumption of the 7.5-MeV level in ^{8}Be.] The D wave showed no evidence of any resonance behavior in the energy region covered. For the G wave, Steigert and Sampson obtained the value of the well-known level as 10.9±0.4 MeV and a width Γ of 1.2±0.4 MeV.

Bredin, Burcham, Evans, Gibson, McKee, Prowse, Rotblat, and Snyder (1959) performed α–α scattering in the range 23.1 to 38.4 MeV. The 23.1-MeV results agree with those of Nilson *et al.* at 22.8 MeV, but there is some disagreement between the results for 38.4 MeV and those of Farrel and Yavin (unpublished). There was also considerable difference between the results at about 30 MeV and those of Graves (1951, 1952). δ_6 was found to continue increasing negatively to the highest beam energy used (38.4 MeV), and there was some indication of a broad level in the S state at an excitation energy between 15 and 20 MeV. This could not be accounted for by the dispersion theory. The phase shift δ_6 continued smoothly between 23 and 38.4 MeV. They also found the 4^+ level at 11.4±0.3 MeV. The phase shifts δ_6 and δ_8 were not accurately determined and seemed to show no resonance behavior.

In connection with checking the apparatus and experimental arrangement for ^{3}He–α scattering, Chiba, Conzett, Morinaga, Mutsuro, Shoda, and Kimura (1961) investigated α–α scattering at 32.5 and 35.5

TABLE V. The real and imaginary parts of the phase shifts as deduced by Darriulat, Igo, Pugh, and Holmgren (1965) from their measurements on the elastic scattering of α particles by helium between 53 and 120 MeV (lab). The reaction cross section σ_R (in millibarns) and the value of $\epsilon(=\chi^2/N$, χ^2 being defined in Appendix I and N being the number of data points) are also shown. [Table reproduced from Darriulat, Igo, Pugh and Holmgren (1965) by courtesy of *The Physical Review*.]

Energy (MeV)	ϵ	σ_R(mb)	Re(δ_0)	Re(δ_2)	Re(δ_4)	Re(δ_6)	Re(δ_8)	Re(δ_{10})	Re(δ_{12})
53.40	2.2	649.9	−75.2±2.4	47.9±1.7	137.9±1.3	27.5±0.6	2.0±0.5		
58.49	1.4	687.9	−83.7±2.2	45.6±1.6	138.9±1.5	41.8±0.7	4.0±0.4	0.6±0.4	
63.91	3.4	800.2	−92.5±3.6	38.0±1.8	142.1±1.5	54.2±1.1	6.4±0.5	1.2±0.5	
69.91	1.0	859.3	−97.2±1.8	33.3±1.1	136.1±1.2	63.2±0.8	8.9±0.4	2.4±0.4	
77.55	2.6	848.1	−109.0±4.4	23.4±2.1	136.8±1.9	73.6±2.6	11.7±0.7	2.5±0.7	
77.55	2.7	879.0	−120.9±3.8	16.4±2.1	137.0±1.8	76.6±2.7	9.8±0.7	3.6±0.6	
99.60	1.1	791.5	−129.5±5.7	−2.0±1.7	128.0±1.7	86.7±2.0	21.5±0.7	4.3±0.5	
99.60	1.5	820.7	−140.7±6.4	−4.0±1.9	132.4±1.5	88.9±2.7	17.9±0.9	4.9±0.6	
99.60	1.9	833.3	−126.0±4.3	−6.4±1.9	133.6±1.9	90.0±2.7	16.9±1.2	6.8±0.6	
119.86	1.4	823.9	−161.5±6.3	−16.0±1.7	130.3±1.8	93.8±2.8	26.0±1.4	7.0±0.9	1.7±0.8

Energy (MeV)	ϵ	σ_R(mb)	Im(δ_0)	Im(δ_2)	Im(δ_4)	Im(δ_6)	Im(δ_8)	Im(δ_{10})	Im(δ_{12})
53.40	2.2	649.9	12.1±3.1	22.1±1.7	16.3±1.1	3.2±0.5	0±0.4		
58.49	1.4	687.9	10.7±2.3	19.2±1.3	16.4±0.9	6.9±0.6	0±0.4	0±0.4	
63.91	3.4	800.2	14.2±3.2	18.4±1.9	18.7±1.4	15.8±1.0	0±0.6	0±0.4	
69.91	1.0	859.3	9.6±2.0	17.9±1.0	20.3±1.0	20.3±0.8	1.9±0.4	0±0.2	
77.55	2.6	848.1	18.0±4.6	19.8±2.3	20.1±1.8	27.3±1.5	2.5±0.7	0±0.5	
77.55	2.7	879.0	12.2±3.7	18.2±2.1	12.9±1.3	32.8±2.0	4.7±0.8	0±0.5	
99.60	1.1	791.5	27.4±6.0	17.0±1.6	20.9±1.6	28.1±1.7	8.5±0.8	0±0.5	
99.60	1.5	820.7	26.2±6.9	15.7±1.8	13.2±1.8	27.0±1.8	11.0±1.2	1.0±0.6	
99.60	1.9	833.3	10.8±4.4	13.2±2.0	11.2±1.3	30.3±2.5	15.5±1.4	0.4±0.7	
119.86	1.4	823.9	15.7±3.9	15.6±2.5	13.8±1.6	26.6±1.4	18.3±1.2	3.7±0.6	0±0.5

MeV. They found that the angular distribution of α–α scattering changes appreciably with energy in this energy range. Their results were in agreement with those of Bredin *et al.* (1959) and of Burcham, Gibson, Prowse, and Rotblat (1957). Burcham *et al.* measured the angular distribution of α–α scattering at 38.5 MeV and obtained a c.m. cross section of 110 mb/sr at 90° falling to a sharp minimum of about 0.5 mb/sr at 64°. Their approximate analysis in terms of real phase shifts showed that δ_0 and δ_2 are large while δ_4, δ_6, δ_8, are small at this energy.

The next higher-energy scattering was performed by Conzett, Igo, Shaw, and Slobodrian (1960). Absolute differential cross sections had been obtained at 36.8, 38.8, 40.8, 41.9, 44.4, 46.1, 47.1, and 47.3 MeV. The single prominent minimum seen at 36.85 and 38.83 MeV gave way to two minima at the higher energies. This transformation from one to two minima with increasing energy is also present in the 12–23-MeV α–α data where resonance scattering from a virtual excited state (4^+) around 11 MeV in ^{8}Be is observed.

A phase-shift analysis by Berztiss (thesis) of the α–α elastic-scattering data of Bredin *et al.* (1959) and Conzett *et al.* (1960) between 23 and 47 MeV suggested rapid energy variation of the S, D, and G phase shifts near 40 MeV. Since the threshold for the reaction $\alpha+\alpha\rightarrow{}^7\text{Li}+p$ is at 34.73 MeV (lab) and since other reactions open near 40 MeV, there was some speculation that this energy variation might be due to threshold effects. This situation seemed to need analysis of experimental data more closely spaced in energy. In this connection, Conzett, Slobodrain, Yamabe, and Schield (1964) measured differential cross sections for α–α scattering from 8° to 50° (lab) at nine energies between 37 and 43 MeV with an energy resolution better than 200 keV and an angular resolution of 0.25°. From an analysis of these experimental data, Conzett, Schield, Slobodrian, and Yamabe (1964) obtained two sets of phase shifts consistent with the experimental data. The phase shifts oscillated rapidly with energy. The oscillations were thought to be related to the opening of inelastic channels and, most probably, to the neutron channel ($\alpha+\alpha\rightarrow{}^7\text{Be}+n$) which proceeds with a rapidly increasing cross section near threshold.

Excitation functions for α–α elastic scattering between laboratory bombarding energies of 23 and 51 MeV were measured by Shield, Conzett, Darriulat, Pugh, and Slobodrian (1964) at lab angles of 45°, 27.75°, and 15.32°. Sharp resonant oscillations in the excitation functions for 15.32° and 45° were found corresponding to the levels in ^{8}Be at 16.6 and 16.9 MeV for which spin 2 and even parity were assigned.

Darriulat, Igo, Pugh, and Holmgren (1965) performed α–α scattering experiments covering the range 53 to 120 MeV. Since a number of reaction channels open at $E(\text{lab})\geqslant 34.73$ MeV (as shown with the corresponding threshold energies in Table IV), the phase shifts are no longer real. Hence, Darriulat *et al.* analyzed the data in terms of complex phase shifts and obtained the phase shifts shown in Table V. The real parts vary smoothly with energy, and the $l=0$, 2 and 4 phase shifts follow the trends at lower energies. A broad $l=6$ resonance is found at $\approx$26 MeV (c.m.); again, the real parts show some fluctuations around

40 MeV, a plausible reason for the fluctuations being discussed in connection with the analysis of Conzett, Schield, Slobodrian, and Yamabe (1964). As can be seen from Table V, the imaginary parts of the phase shifts vary smoothly with angular momentum and energy. The absorption experienced by the $l=0, 2, 4$ partial waves at 53.4 MeV and at higher energies are about the same. The $l=6$ and the $l=8$ imaginary phase shifts, however, increase with energy in a manner almost similar to the corresponding real phase shifts. Darriulat, Igo, Pugh, and Holmgren (1965) also found that the effects of Im δ_l (absorption) on the elastic differential cross sections is small at small angles but increases rapidly at larger angles. Since at 53.4 MeV the effect of absorption was found to be already large, Darriulat *et al.* correctly pointed out that the phase shifts obtained at lower energies, but above the threshold energies, for the inelastic processes should be viewed with due caution since these were obtained from previous analyses using only real phase shifts. Thus, although Darriulat *et al.* found a broad $l=6$ resonance at about 26 MeV (c.m.), their conclusions for the $l=6$ partial wave are weakened because they were based on the use of the above-mentioned phase shifts. Darriulat *et al.* found evidence for an $l=8$ resonance on the basis of the complex phase-shift analysis of their data. They also found in the analysis of the differential cross sections that the real parts of the phase shifts are rather insensitive to the values used for the imaginary parts. If the same finding is also true for the low-energy analysis, then the conclusions for the $l=6$ wave may not be substantially changed. Darriulat *et al.* also tried dispersion-theory fits for the real parts of the phase shifts. Table III gives several sets of parameters which fit the phase shifts in various ways. No single value of the hard-sphere radius gave a reasonable fit for either the $l=6$ or the $l=8$ resonance. However, using a hard-sphere radius of 4.5 fm, all the five resonances so far observed were compared, and it was found that the energies and reduced widths of the resonances given by dispersion-theory fits are approximately proportional to $J(J+1)$.

The results discussed so far may be summarized as follows: Below 400 keV there does not seem to be any evidence of nuclear interaction. The S-wave phase shift starts from this point with a value of 180° and decreases monotonically with energy. It passes through zero at about 22 MeV and then becomes negative. The D wave comes into play at about 2.5 MeV and goes to a maximum of 120° at about 8 MeV and then starts decreasing. The G wave does not start before 4 MeV and then increases with energy. The I wave is first observed at 20 MeV and is positive. Reaction channels open at $E(\text{lab}) \gtrsim 35$ MeV when the phase shift is no longer real; the imaginary part accounts for the reaction, while the real part describes elastic scattering.

So far, three levels are definitely established; the 0^+ level at $\approx$92 keV, the 2^+ level at $\approx$3 MeV, and the 4^+ level at $\approx$11 MeV. Indications for higher levels have also been found and discussed. [Sources of information on the various levels of ^{8}Be are given by Lauritsen and Ajzenberg-Selove (1966).]

4. PHENOMENOLOGICAL ANALYSIS OF α–α SCATTERING

So far, we have discussed the α–α scattering results and the parameters that have been extracted from them. We now present the phenomenological approach, in which one attempts to construct an α–α potential which reproduces the experimental phase shifts. The method adopted in most of these approaches is outlined in Appendix II.

One of the earliest phenomenological α–α potentials was proposed by Haefner (1951) and was of the form

$$\begin{aligned} V(r) &= 4e^2/r && r>R \\ &= -D+q^2\hbar^2/2\mu r^2 && r<R. \end{aligned} \qquad \text{(P1)}$$

Haefner used this potential to determine the properties of ^{8}Be assuming that it is a 2α system. In this potential, r is the α–α separation, D is a constant (the well-depth parameter), μ is the reduced mass, and q^2 is a parameter. This potential is repulsive for small r (representing the effect of the Pauli principle operating between nucleons of the α clusters), attractive for intermediate r, and is Coulombic outside r_0.

Using the notation of Haefner, the radial functions for this potential for $l=0$, 2 and for $r<R$ are (for the choice of $q^2=30$)

$$\begin{aligned} R_0(r) &= A_0 j_5(k_0 r) && r<R, \\ R_2(r) &= A_2 J_6(k_0 r)/(k_0 r)^{1/2} && r<R, \end{aligned} \qquad \text{(P2)}$$

with $k_0^2=k^2+2\mu D/\hbar^2$, $k^2=2\mu E/\hbar^2$, and E, the relative energy of the system. The functions $R_L(r)$ for $r>R$ are

$$\begin{aligned} R_0(r) &= B_0[f_0(kr)\cos\delta_0+g_0(kr)\sin\delta_0] && r>R \\ R_2(r) &= B_2[f_2(kr)\cos\delta_2+g_2(kr)\sin\delta_2] && r>R, \end{aligned} \qquad \text{(P3)}$$

where $f_l(kr)=F_l(kr)/kr$, $g_l(kr)=G_l(kr)/kr$, and $F_l(kr)$ and $G_l(kr)$ are, as before, the regular and irregular Coulomb functions.

The matching of the functions (P2) and (P3) for each l at $r=R$ by requiring the continuity of the functions and their first derivatives, determines δ_l and A_l/B_l as functions of the energy. Haefner found the maximum of A_l/B_l to be a good criterion for a virtual level of angular momentum l. He also used the fact that when the maximum occurs at those energies for which the irregular wave function is much larger than the regular wave function, $\delta_l=90°$. (In the Breit–Wigner formulation, this means that there is no potential scattering and the phase shift is given by the resonant term alone, giving $\delta_l=90°$ at resonance. For values of $R=4.50$–4.75 fm, the maximum of the ratio occurred in the neighborhood of 3 MeV.)

Using the then-available data on the ground state of ^{8}Be, namely a virtual state (presumed to be a $^1S_0$ state) at an energy of $\approx$90 keV above that for an infinite separation of two α particles, Haefner observed that $g_0(kR) \gg f_0(kR)$ at this energy and obtained for the position of the ground state of zero angular momentum (given by $\delta_0 = 90°$) the relation

$$[k_0 R j_5'(k_0R)/j_5(k_0R) = kRg_0'(kR)/g_0(kR)]_{E=90\text{ keV}}. \tag{P4}$$

Equation (P4) gives D as a function of R. The maximum of A_2/B_2 as a function of R was also determined and for values of R from 4.0 to 5.0 fm this maximum ranged from $E \simeq 2.7$ to 3.8 MeV.

Later, Nilson, Jentschke, Briggs, Kerman, and Snyder (1958) used the Haefner potential in the analysis of their α–α scattering results and determined D from the (then) position of the ground state a $E = 96$ keV. It turned out that D was between 19 and 50 MeV depending on the choice of R. As we have already seen, for values of R between 4 and 5 fm, Haefner found the maximum in A_2/B_2 to be between $E = 2.7$ and 3.8 MeV, which included the D state at 2.9 MeV, known later. Nilson *et al.* extended Haefner's model to include $l = 4$ and found that the energy range in which A_4/B_4 attains its maximum value for values of R between 4.13 and 5.08 fm was from 9 to 12 MeV. This included the G state, which they determined from the dispersion-theory analysis. A quantitative test of the 2α model of Haefner seemed to be to see how well it can reproduce the experimental phase shifts, and in this connection Nilson *et al.* calculated, with Haefner's potential, the various phase shifts δ_0, δ_2, δ_4 and found that the best agreement with the low-energy experimental phase shifts requires a small value of R (3.49 fm); the D-wave phase shifts did not rise enough to reproduce the experimental $l = 2$ phase shifts beyond 12 MeV. A good fit to the G-wave experimental phase shifts was obtained with $R = 4.44$ fm. As pointed out by Nilson *et al.*, a crucial test of the potential would be a comparison between the experimental and theoretical δ_4 beyond 22.9 MeV, but no such data existed at that time. However, although Haefner's potential does not exactly reproduce S-, D-, and G-wave phase shifts, the similarities between the experimental and rotational model phase shifts seemed to indicate that these states could be described in terms of a two-body interaction.

About the same time as Nilson *et al.*, Humphrey (1957) was able to reproduce the α–α scattering phase shifts for the entire range of 0 to 22 MeV with a modified Haefner potential. His best fit required $R = 3.75$ fm and an l-dependent well depth D_l ($D_0 = 21$ MeV, $D_2 = 25$ MeV, and $D_4 = 32$ MeV).

Van der Spuy and Pienaar (1958) made a phenomenological analysis of α–α scattering up to a bombarding energy of about 6 MeV. They investigated whether a velocity-independent, two-body α–α interaction can represent the phase-shift data and also what features of the interaction are predicted phenomenologically. They considered the following α–α potential:

$$\begin{aligned} V_{\alpha\alpha} &= \infty, && r \leqslant r_1 \\ &= -V, && r_1 < r \leqslant r_N \\ &= 4e^2/r, && r > r_N \end{aligned} \tag{P5}$$

which is characterized by three parameters: the hard-core radius r_1, the nuclear-interaction range r_N, and the well depth V. Using the radial Schrödinger equation for two α particles,

$$\frac{d^2R_l(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V_{\alpha\alpha}(r) - \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}\right]R_l(r) = 0,$$

with the potential given by Eq. (P5). Van der Spuy and Pienaar deduced for the inside ($r < r_N$) solution the logarithmic derivative

$$f_l = \left[r\frac{dR_l(r)}{dr} \Big/ R_l(r)\right]_{r=r_N},$$

with

$$f_0 = Kr_N \cot K(r_n - r_1),$$

$$f_2 = -2 + \frac{x_N^2}{3} + \frac{x_N^2}{3[(3/x_N^2 - 1) - (3/x_N)\cot(x_N + \phi)]},$$

where

$$\cot(Kr_1 + \phi) = (Kr_1)^{-1} - \tfrac{1}{3}Kr_1, \qquad x_N = Kr_N$$

$$K^2 = k^2 + (2\mu V/\hbar^2), \qquad k^2 = 2\mu E/\hbar^2.$$

Thus, they selected a set of (r_N, r_1, V) values to get the best fit of the above inside log derivatives to the outside log derivatives calculated from the experimentally observed phase shifts at selected values of r_N. For the S wave they obtained the best compromise fit for $r_N = 4$ fm, $r_1 = 1.7$ fm, and $V = 7.2$ MeV (including the 94.5-keV resonance point). The fit, exluding the resonance point, required $r_N = 4$ fm, $r_1 = 1.8$ fm, and $V = 7.9$ MeV. For the D wave, the best compromise fit was for $r_N = 4$ fm, $r_1 = 1.8$ fm, $V = 10.5$ MeV. Thus, for both S and D waves the best fit required the same r_N and r_1, but different well depths, namely $V_S = 7.9$ MeV and $V_D = 10$ MeV. The difference between V_S and V_D increases for lower r_N because of the centrifugal potential.

Thus, the investigation of Van der Spuy and Pienaar indicated for the square-well analysis that even at very low energies ($E < 6$ MeV) one needs a velocity-dependent interaction with a core radius of about 1.8 fm. However, if the velocity dependence of the core can be schematized as an l dependence of the core, then the analysis would indicate that Potential (P5) with $r_N = 4$ fm, $r_{1S} = 2.05$ fm, $r_{1D} = 0.08$ fm, $V = 9.5$ MeV, would yield the best compromise fit. Thus, the results

of Van der Spuy and Pienaar suggested that it is not possible, even at very low energies, to obtain a velocity-independent α–α potential fitting both S and D waves.

H. Wittern (1959) derived a semiphenomenological potential from a consideration of the single nucleons given by the Pauli principle and from the character of the shell-model wave functions for the actual levels of the ^{8}Be compound nucleus and their excitation functions. Wittern's analysis points to the same conclusions about l dependence of the α–α potential as those of Van der Spuy and Pienaar.

Igo (1960) made an optical-model analysis of the elastic α–α scattering for bombarding energies ranging from 23.1 to 47.1 MeV. He used a complex potential

$$\frac{V+iW}{1+\exp\left[(r-r_0)/d\right]}, \tag{P6}$$

where the parameters V, W, r_0, and d were required, by agreement with the angular distributions taken at eight different energies, to be -112 and -1 MeV (for bombarding energies near 40 MeV), 1.8 and 0.6 ± 0.1 fm, respectively. The real parts of the phase shifts obtained for this potential were in good agreement with the preliminary values of Snyder [see Burcham *et al.* (1958)] in the energy range 23–42 MeV. The imaginary parts of the phase shifts were close to zero. The introduction of a nonzero W, necessary to reproduce the total reaction cross section, had a negligible effect on the real part of the phase shifts. Using the same potential form (P6), Van Niftrik, Brockman, and Van Oers (1964) obtained an optical-model fit to their angular distribution data of α–α elastic scattering at a primary energy of 51 MeV. The corresponding values of the optical-model parameters are $V=-122$ MeV, $W=-11$ MeV, $r_0=1.81$ fm, and $d=0.74$ fm. The calculated reaction cross section is 770 mb. The Coulomb potential was included by both Igo (1960) and Van Niftrik *et al.* with different assumptions about the charge distribution.

Endo, Shimodaya, and Hiura (1964) made an investigation of the extent to which α–α scattering can be explained in terms of an energy-independent potential and used the following l-dependent potential:

$$V_{\alpha\alpha}{}^{(l)}=V_1{}^{(l)}\exp\left(-\mu_1{}^{(l)}r^2\right)+V_0\exp\left(-\mu_0 r^2\right) \qquad r\gtrsim r_1{}^{(l)}, \tag{P7}$$

where $r_1{}^{(l)}$ is the hard-core radius. $V_{\alpha\alpha}{}^{(l)}$ at $r=r_1{}^{(l)}$ was taken to be zero. Potential (P7) is thus characterized for a given l by two l-dependent free parameters $V_1{}^{(l)}$, $\mu_1{}^{(l)}$ of the repulsive part; the parameters V_0 and μ_0 of the attractive part are fixed to -325 MeV and 0.29 fm^{-2} from the theoretical investigation of Shimodaya, Tamagaki, and Tanaka (1962) which we shall discuss in Sec. 5. A fairly good fit to the experimental phase shifts through the excitation of 40 MeV (lab) requires

$$V_1{}^{(0)}=371\text{ MeV}, \quad \mu_1{}^{(0)}=0.342\text{ fm}^{-2}, \quad r_1{}^{(0)}=1.6\text{ fm};$$
$$V_1{}^{(2)}=373\text{ MeV}, \quad \mu_1{}^{(2)}=0.360\text{ fm}^{-2}, \quad r_1{}^{(2)}=1.4\text{ fm};$$
$$V_1{}^{(4)}=357\text{ MeV}, \quad \mu_1{}^{(4)}=0.435\text{ fm}^{-2}, \quad r_1{}^{(4)}=0.8\text{ fm}.$$

The phase shifts of Endo *et al.* for $E(\text{lab})\approx 40$–50 MeV agree well for low l with those obtained by Igo (1960) who used a purely attractive real part in his optical potential. Endo *et al.* argued that the use of such a completely real part may not be realistic. However, Igo's analysis was perhaps justified, as discussed below.

Recently, Darriulat, Igo, Pugh, and Holmgren (1965), in an attempt to fit the real parts of the α–α scattering phase shifts for energies between 53 and 120 MeV, introduced a complex Saxon–Woods potential of the form

$$V_{\alpha\alpha}(r)=u_1\{1+\exp\left[(r-r_1)/a_1\right]\}^{-1}$$
$$-u_2\{1+\exp\left[(r-r_2)/a_2\right]\}^{-1}$$
$$-iW\{1+\exp\left[(r-r_3)/a_3\right]\}^{-1}+V_C(r),$$

where the first term represents the repulsive core and the second, the larger-ranged attractive potential; the third term accounts for inelastic processes and the fourth term is the Coulomb potential due to a uniformly charged sphere of radius R_C.

Like previous authors, Darriulat *et al.* also tried to fit the phase shifts for all partial waves using the same potential but failed. Then they took recourse to an l-dependent potential and used a different set of parameters for each partial wave. The values of the parameters of the real potentials that reproduced the real parts of the phase shifts are $U_1=150$ MeV, $a_1=0.1\pm0.005$ fm, $r_1=1.65\pm0.03$ fm, $U_2=9.2\pm0.5$ MeV, $a_2=0.4\pm0.1$ fm, $r_2=3.72\pm0.07$ fm for the S-wave; the corresponding values for the D and G waves being 150, 0.05 ± 0.03, 1.63 ± 0.03, 16.0 ± 0.2, 0.3 ± 0.05, 3.55 and 220, 0.05, 1.2, 71 ± 1, 0.46 ± 0.03, 2.48 ± 0.02, respectively. The values of the repulsive-core parameters for the G wave were found to be some sort of upper limit, since the G-wave phase shifts could be reproduced even without these. In view of the fact that the imaginary parts of the phase shifts were not experimentally well determined and also that the real parts of the phase shifts were not found to be sensitive to the parameters of the imaginary part of the α–α potential, the latter were rather arbitrarily fixed to $W=5$ MeV, $E>40$ MeV (and $W=0$, $E<40$ MeV), $r_3=r_2$, $a_3=a_2$. As pointed out by Dariullat *et al.*, these potentials suffer somewhat from the ambiguity which is characteristic of any Saxon–Woods parametrization. When the real parts of these potentials are plotted as functions of the α–α separation, the tails of the potentials for $l=0$, 2, 4 partial waves are almost the same (in agreement with theoretical expectations), but the $l=6$

and the $l=8$ potentials are different in this region. However, the latter finding should not be disturbing since the $l=6$ potential was mainly determined from phase shifts between 35 and 47 MeV and, for reasons mentioned in Sec. 3.B, should not be taken very seriously. Also, the $l=8$ potential was constructed from scanty experimental information and should also be treated with caution.

Summarizing, the α–α potentials that have been constructed from the experimental phase shifts show some common features: The α–α potential is l dependent but is independent of the incident energy. The ranges of the inner repulsive part and the outer attractive part are of the order of 2 and 5 fm, respectively. The repulsive part becomes weaker for higher l, while the attractive part becomes stronger. Thus the phenomenological analyses of α–α scattering establish beyond doubt that a static α–α potential common to all l does not exist. Nevertheless, the attractive part may be taken as common to all l. The l dependence enters through the repulsive part and may be regarded as a simple form of velocity dependence. [In fact, Ali and Bodmer (1966) have suggested a procedure for the construction of such potentials in a fairly unique way; we postpone its discussion until Sec. 6, when we deal with the prospect for further studies in the α–α interaction.] As remarked by Van der Spuy, admission of more general velocity-dependent effects seems to make the purely phenomenological analyses rather arbitrary unless one knows the actual type of velocity dependence from more fundamental considerations. Thus, studies of the fundamental model of the α–α interaction are extremely useful not only for understanding the essential features of the interaction but also for making the qualitative features of the phenomenological α–α potentials meaningful. Before these studies are dealt with in Sec. 6, the investigations concerning the effective-range theory of α–α scattering are discussed. These followed a recent measurement of the ground state of ^{8}Be and have some relevance to phenomenological studies of the α–α interaction.

Recently, Benn, Dally, Müller, Pixley, Staub, and Winkler (1966, 1967) have been able to make a rather direct and precise measurement of the width ($\Gamma_r=6.8\pm 1.7$ eV) and position (E_R(c.m.) $=92.12\pm0.05$ keV) of the ground state of ^{8}Be, whose lifetime τ has been determined to be $[0.97(+0.32, -0.19)]\times10^{-16}$ sec. Russell, Phillips, and Reich (1956) obtained $\Gamma=4.5\pm3$ eV with the effective-range theory. Later, Barker and Treacy (1962) repeated this calculation using the same expansion and found a value of 6.8 ± 0.6 eV, which is in better agreement with the recent experimental value. However, these results were obtained with the old value of the ground-state position of 94 keV, which was shown to be inaccurate by the experiments of Benn *et al.* and of Reichert, Staub, Stüssi, and Zamboni (1966). The correct resonance energy is important in obtaining the ground-state width because of the rather rapid change in the Coulomb penetration with energy. Further calculations using the new value of the resonance energy were first reported by Tombrello (1966) who found that the uncertainties in the parameters of the effective range expansion (which arise due to the uncertainties associated with the measured S-wave phase shifts) cause large uncertainties in the width. Tombrello concluded that, on the basis of the available data, the effective-range theory of S-wave α–α scattering does not enable the width of the ^{8}Be ground state to be determined accurately. Rasche (1967) has pointed out that Tombrello used only one-half of the experimental information (E_R) and tried to make predictions for the other half (Γ). Rasche used both E_R and Γ to determine the effective-range expansion parameters and observed that the inclusion of Γ in the analysis of low-energy phase shifts (δ_0) considerably reduces the uncertainties in the effective-range expansion coefficients A, B, C, D [see Eq. (E2)]. Rasche has apparently been able to put a better limit on Γ [$\Gamma_{\text{c.m.}}=6.4(+0.8, -0.5)$ eV] which is even narrower than the direct experimental limit quoted by Benn, *et al.*

Kermode (1967b, 1967c, 1968) has recently presented an analysis of S-wave phase shifts using a hard-core effective-range formula previously developed by Kermode, (1965). According to Kermode's analysis, one uses the experimental phase shifts δ_0 to calculate a function Y_0 defined by

$$Y_0=k[(G_0'+F_0'\cot\delta_0)/(G_0+F_0\cot\delta_0)]_{a_0},$$

where F_0 and G_0 are again the S-wave regular and irregular Coulomb wave functions, a_0 is the hard-core radius, and k is the relative momentum (c.m.). Y_0 is then fitted to the effective-range expansion

$$Y_0=A+Bk^2+Ck^4$$

and, finally, defining the width as the difference in the energies for which the phase shifts are 45° and 135°, one calculates it from

$$\Gamma=2k_r\left[G_0^2\left(\frac{dz_0}{dE}-\frac{dY_0}{dE}\right)\right]^{-1}_{E_R}=2\left(\frac{d\delta_0}{dE}\right)^{-1}_{E_R},$$

where $z_0=kG_0'/G_0$. Using a hard-core radius of 1.7 fm [which is the same as that used in the potential of Van der Spuy and Pienaar (1958)], Kermode obtained a value of 6.14 ± 0.04 eV, in reasonable agreement with the measurements of Benn *et al.*

Use of the phenomenological α–α potential has been made by Ali and Afzal (1967) to reproduce the new ground-state parameters of ^{8}Be. Their observation has been that an S-state α–α potential which reproduces phase shifts in good agreement with experiment may give a fair representation of the ground state of ^{8}Be.

More recently, Kermode (1967b, 1967c, 1968) has shown that the effective-range formula for charged

particles with no hard core, used by other authors, is not good for α–α scattering. Kermode finds that the hard-core, effective-range formula describes rather well the low-energy S-, D-, and G-wave α–α scattering. He also finds it possible to define a hard-core radius which takes approximately the same value (1.7 fm) for all three partial waves, in apparent inconsistency with the results of the phenomenological analyses. However, the latter finding is rather difficult to understand, for one hopes on fundamental grounds that if the α–α potential is at all velocity dependent, this dependence is more and more reflected in its inner part. If one can schematize the velocity dependence as an l dependence, then it seems more plausible that the α–α interaction should be characterized by an l-dependent inner core (and an outer attractive part common to all l) rather than an l-independent one.

5. FUNDAMENTAL STUDIES OF THE α–α INTERACTION

By fundamental studies of the α–α interaction we mean those studies which begin with an eight-nucleon (four protons and four neutrons) system and try to develop an interaction between two α clusters, starting from first principles and the basic two-nucleon forces. The earliest of such theoretical studies began with the celebrated "Resonating Group Formalism" of Wheeler (1937b). This formalism regards the neutrons and protons in the nucleus as being divided into various groups (e.g., α particles) which do not maintain their identity forever but undergo continual changes, redistributing themselves into new groups. One then seeks from group-theoretical considerations which of the groupings are important for a particular state of a given nucleus. The wave function of the composite nucleus is written as a totally antisymmetrized combination of the wave functions for the various possible groups in the nucleons. Following Wheeler's method (which, in fact, has paved the way for detailed analysis of configurations involving two groups) one obtains for the relative motion of two groups an integral equation in which appears an interaction generated from two-nucleon forces. This consists of two parts: a direct part which involves no particle exchange between the two groups and another part appearing in the form of a nonlocal kernel interaction containing terms corresponding to the exchange of one, two, or more nucleons between the groups. Wheeler's formalism has been followed up until quite recently by various authors in different contexts, and we shall later present direct applications of the method only as far as the α–α interaction is concerned.

Besides expanding the theory of resonating-group formalism, Wheeler also used a simple αPM (which assumes the α–α forces to be given beforehand) along molecular lines [Wheeler (1937b)] to describe low states of excitation between the α particles. The symmetric function of the centers of gravity of the α particles was written as the product of rotational and vibrational wave functions, and the results of the molecular theory were applied to determine the allowed rotational and vibrational quantum numbers for ^{8}Be and for ^{12}C and ^{16}O. Margenau (1941) pointed out that because of some essential differences between the nuclear and the molecular cases (e.g., the differences in the natures of the interactions between atoms and those between nuclei), the introduction of molecular viewpoints should be viewed with caution. Margenau calculated the interaction energy of two α particles as the difference between the total energy and the energy of two isolated α particles, each with a wave function built up about a fixed point. The calculations of Margenau were essentially the same as those of Heisenberg (1935) who allowed the mass centers of the interacting α particles to oscillate with an arbitrarily chosen amplitude about two fixed points. In Margenau's calculations the mass center was described by the same parameter that described the motion of a nucleon inside an α particle. Expressions for the interaction energy contained Heisenberg's mass-center oscillation parameter explicitly and Margenau's implicitly and were therefore difficult to interpret.

Wheeler (1941) was also the first to give a theoretical interpretation of the α–α scattering measurements up to 1939. Following Taylor (1931, 1932), he analyzed the results in terms of the phase shifts of the $l=0$, 2, 4 partial waves up to 7 MeV. His analysis demonstrated a rapid phase variation at about 3 MeV (c.m.) for $l=0$. Thus, Wheeler was forced to assign zero spin to the 2.9-MeV state in ^{8}Be, which is now definitely known to be a 2^+ state. This does not imply that Wheeler's method of analysis was strong, but that his calculations were based on inaccurate phase-shift values.

The intention of the early theoretical works was to see whether the basic two-body α–α interaction has a short-range repulsion and an attractive part over the α–α spacings of 3–4 fm (corresponding to the average spacing of two α's in a nucleus). The repulsion is needed to prevent α's from coming too close to each other and overlapping, making the αPM of nuclei meaningless. A rough criterion that overlap shall always be small was given by Herzenberg (1955, 1957): $R_0<\frac{1}{2}[s_\alpha-\langle(\delta s_\alpha^2)\rangle^{1/2}]$ where s_α is the average separation of two neighboring α particles in an α-particle nucleus and $\langle(\delta s_\alpha^2)\rangle^{1/2}$ is the root-mean-square deviation of s_α. R_0 is a certain distance from the center of the α particle up to the radius at which the density distribution in the α particle was assumed to be constant. From the relation of R_0 with r_α (the radius of the α particle) and from some rather rough estimates for s_α and $\langle\delta s_\alpha^2\rangle$ in ^{8}Be, Herzenberg deduced that $r_\alpha<1.57$ fm. This is indeed consistent with the measured radius (1.44 fm) of the α particle as was available later from the experiments of Hofstadter (1956).

The nonoverlapping of α particles in ^{8}Be should be reflected also in the repulsive nature of the α–α potential. The repulsive part of the theoretically determined α–α

interaction is not a static potential but has a velocity-dependent character; thus, one cannot represent the repulsion graphically. One may, however, plot the effective phenomenological α–α potentials and get some idea of the extent to which the α particles overlap in states of ^{8}Be. This point has been illustrated in Sec. 6.

The attractive part of the basic two-body α–α interaction is needed to sustain the nuclei against electrostatic repulsion. These theoretical works also sought to establish whether the α–α interaction is additive, i.e., whether the force between two α's is independent of the presence of neighboring α's. Heisenberg's idea was that the short-ranged repulsion (which corresponds to the exchange of nucleons between the α particles) could be accounted for by exchange forces, while the additive attraction would be due to van der Waals forces. But Margenau's calculations showed that the second-order forces between two α's, which correspond to the van der Waals force in molecular interactions, were of shorter range than the exchange forces and hence could not be additive. The total first-order interaction between α particles (i.e., sum of direct and exchange interactions) was found by Margenau to be repulsive. Thus, strong polarization forces (when two α's come very close to each other's field of forces they tend to distort each other so as to minimize the total energy, thus producing an attraction) had to be invoked, which went against the spirit of the αPM of nuclei.

Margenau also chose a simple model, in keeping with the α–α interaction composed of first- and second-order forces deduced by him, to obtain the S-wave phase shifts. He used an infinite repulsive potential for α–α separation from 0 to r_1, an attractive square well from r_1 to 4.5 fm, and Coulomb repulsion outside 4.5 fm. The phase shifts obtained at low energies for various values of r_1 agreed with the (then) phase shifts but did not show resonance effects, and Margenau concluded that the two-body interaction was incompatible with the scattering data.

Inglis (1941) showed that if the nucleon–nucleon interaction has a Yukawa rather than a Gaussian tail and if there is no appreciable nonexchange part in this tail, then the first-order forces would provide an attractive region. Later, Edwards (1952) pointed out that the attraction found by Inglis was not due to the shape of the potential but to the fact that the tail did not have an appreciable nonexchange part. Edwards calculated the binding energy of the ^{8}Be nucleus using the variational α–α wave function

$$\sum_P r_0^2 \exp\left(-Cr_0^2 - D\sum_{i,j=1}^{4} r_{ij}^2 - D\sum_{\alpha,\beta=5}^{8} r_{\alpha\beta}^2\right) \times \text{spin and charge functions},$$

where C and D are parameters and

$$r_0^2 = \left[\tfrac{1}{4}(r_1+r_2+r_3+r_4) - \tfrac{1}{4}(r_5+r_6+r_7+r_8)\right]^2,$$

the r's being the coordinates of the nucleons of the α clusters. Edwards' calculation showed that it was not possible to bind ^{8}Be unless the saturation conditions had been given up. However, as pointed out later by Herzenberg (1955, 1957), the significance of a calculation with a ground-state boundary condition is not clear since the ^{8}Be state is virtual. Thus, a more satisfactory treatment of the virtual levels of ^{8}Be should use the reduction of an eight-body problem to a two-particle equation of motion containing an effective interaction between two α clusters.

So much for the historical part of the theoretical trend in the development of the subject of α–α interaction. Let us now look at more current pictures of the subject. The recent theoretical analyses of the α–α interaction followed a great impulse given to this field by Herzenberg in a series of papers (1955, 1957) in which the αPM of nuclei was revived, and the several interesting features of the α–α interaction were explored. Herzenberg made an anatomical separation of the α–α interaction into four parts: (a) a direct interaction V_D due to the direct (nonexchange) tail of the nucleon–nucleon potential, (b) an exchange interaction V_X due to the exchange of nucleons between the α particles, (c) a polarization interaction V_{pol} (whose origin we have already explained), and (d) the Coulomb interaction E_C. The first three terms are analogous to the long-range Coulomb interaction of ions, electron exchange, and van der Waals forces, respectively, in atomic physics. Herzenberg made explicit calculations for these components with the use of a basic nucleon–nucleon force having the charge-independent form

$$V_{ij} = V_c(r_{ij})\,(a_0 + a_\sigma \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j + a_\tau \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j + a_{\sigma\tau}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j) + V_T(r_{ij})\,(a_{0T} + a_{\tau T}\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)\,(\boldsymbol{\sigma}_i\cdot\hat{\mathbf{r}}_{ij}\boldsymbol{\sigma}_j\cdot\hat{\mathbf{r}}_{ij} - \tfrac{1}{3}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)$$

(where r_{ij} is the distance between nucleons i and j, $\hat{\mathbf{r}}_{ij}$ the unit vector pointing from nucleon i to nucleon j, and the $\boldsymbol{\sigma}_i$ are $2/\hbar$ times the intrinsic spin operators) and a wave function in which each of the two interacting α particles is confined to the neighborhood of a fixed point. The important observations of Herzenberg were that V_D and V_{pol}, which provide the attractive part of the α–α interaction, are additive (in what follows, only the two-body interaction, in which additivity is of no concern, is dealt with) and that V_D and V_{pol} have a larger range than does V_X. V_X is velocity dependent and could account for the repulsion required by the αPM. The additivity of V_X depends on the existence of a direct tail in the N–N interaction. In the absence of this tail, the only source of attraction is V_{pol} which, however, is rather weak for central forces between nucleons. Herzenberg also found that the strength of the tail in the Serber forces provides the right order of magnitude of the α–α attraction. Thus, considerable emphasis is required on the direct part of the N–N interaction. In earlier calculations this was regarded as going against the usual saturation conditions. Herzenberg argued that "saturation con-

ditions" should not really stand in the way of the explanation of the α–α interaction by V_D because, first, nuclear forces should give saturation only within the known range of nuclei so that an attractive V_D may still be possible, and second, V_D is due to a fringing nuclear field extending outside the α particle, and thus, two α particles can attract each other without overlapping. Herzenberg remarked that the nuclear forces may be modified at short distances to give saturation and still allow a tail to give the α particle a fringing field. Considering the features of the component parts of the α–α interaction, Herzenberg suggested that the complete α–α interaction may be represented by a potential which has a static part corresponding to the direct part of the α–α interaction and an effective repulsive part which takes into account the repulsive nature of V_X. Indeed, in constructing α–α potentials from α–α scattering data, these features of Herzenberg's investigation, as well as those of the more recent theoretical works, have been taken care of at least qualitatively. Following Herzenberg, several authors have attempted to study the two-body α–α interaction in great detail using the resonating-group approach. The development of Van der Spuy (1959), outlined below, is typical.

Let us consider a system of eight nucleons (four protons and four neutrons) and imagine that they can be divided into various possible two-α groups. Starting from one pair of α particles, other groups of two α's may then be generated by allowing exchange of nucleons between the pair. These nucleon exchanges must, however, be governed by the Pauli principle. Thus, following the resonating-group formalism, one assumes that the wave function of the eight-nucleon system is given by

$$\psi=\mathcal{A}[\psi_a(12,34)\psi_b(56,78)R(r_{a,b})], \qquad \text{(F1)}$$

where $\mathcal{A}$ is the antisymmetrization operator given by

$$\mathcal{A}\equiv[1-H(25)-H(26)-H(15)-H(16)+H(15)H(26)][1-H(47)-H(48)-H(37)-H(38)+H(37)H(48)].$$

In (F1) the neutrons have been labeled as 1256 and the protons as 3478. $H(25)$ is the Heisenberg exchange operator, exchanging both space and spin coordinates of particles 2 and 5. The ψ_a's are the internal wave functions of the α particles and have the form $\psi(12,34)=\chi(12,34)\phi(12,34)$, where χ is the antisymmetric spin part and ϕ the symmetric space part of ψ. $R(r)$ describes the relative motion of two α particles. One assumes that the two α groups are unpolarized. This assumption is justified if one considers low α–α energies of scattering so that the particles do not interpenetrate. The basic idea is to start from (F1) and derive an equation of motion for $R(r)$ with the α–α interaction built up from fundamental two-nucleon forces. One achieves this in the following way.

Starting with the Hamiltonian

$$H=\frac{-\hbar^2}{2M}\sum_{i=1}^{8}\nabla_i^2+\sum_{(1\cdots 8)}V(i,j), \qquad \text{(F2)}$$

where $V(i,j)$, the two-nucleon force, is given by

$$V(i,j)=[w+mP_x(i,j)+bP_\sigma(i,j)+hH(i,j)]V(i,j),$$

and the eight-body Schrödinger equation is

$$H\psi=E\psi. \qquad \text{(F3)}$$

To obtain an equation for the relative wave function $R(r)$, the nucleon coordinates must be integrated out. Thus, (F3) is multiplied by $\psi_a^*(12,34)\psi_b(56,78)$ and integrated over the space and spin variables of all the nucleons, keeping $\mathbf{r}$ constant. Use is made of the identity

$$\int_{\text{space spin}}\chi(12,34)\phi^*(12,34)\times[T_\alpha-E_\alpha+\sum_{(1234)}V(i,j)]\chi(12,34)\phi(12,34)=0,$$

where T_α is the kinetic-energy operator in the c.m. system of the α particle; ϕ is the symmetrical, spacial α wave function, which for simplicity may be assumed to be

$$\phi(12,34)\sim\exp[-\alpha(r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2)].$$

The fact that the internal wave function does not depend on the inter-α separation implies the assumption that the α particles are not polarized during collision.

One obtains, using certain symmetries, the equation

$$\left[\frac{\hbar^2}{4M}\nabla^2+(E-2E_\alpha)-V_D(r)\right]R(\mathbf{r})=\int dr'K(\mathbf{r},\mathbf{r}')R(\mathbf{r}'). \qquad \text{(F4)}$$

$V_D(r)$ is the direct interaction (i.e., in the absence of nucleon exchange) originating from the identity element of the antisymmetrization operator $\mathcal{A}$ and is given by

$$V_D(r)=4(4w+2b-2h-m)V_0[16\alpha/(16\alpha+3\beta)]^{3/2}\times\exp[-16\alpha\beta r^2/(16\alpha+3\beta)]. \qquad \text{(F5)}$$

V_0 and β characterize the radial dependence of the nucleon–nucleon interaction v_{ij} which is taken as

$$v(ij)=-V_0\exp(-\beta r_{ij}^2). \qquad \text{(F6)}$$

This potential was chosen to obtain an analytical expression for the kernel since Gaussian forms integrate easily. The direct part $V_C(r)$ of the Coulomb inter-

action $[4e^2/r \text{ erf } (4r\frac{1}{3}^{1/2}\alpha)]$ is also added to $V_D(r)$ to give a total direct part $V_D'(r)=V_D(r)+V_c(r)$.

Equation (F4) shows that in order to know $R(\mathbf{r})$ at the point $\mathbf{r}$ in space, one needs to know $R(\mathbf{r}')$ at all other points $\mathbf{r}'$ in space. Thus, (F4) describes a nonlocal process; the kernel of the integral equation $K(\mathbf{r}, \mathbf{r}')$, involving both $\mathbf{r}$ and $\mathbf{r}'$, represents a nonlocal interaction and is symmetric. The exact analytical expression for this kernel is a rather complicated function and is given by Van der Spuy (1959). Mathematically speaking, the origin of the kernel interaction lies in the antisymmetrization of the wave function. But since antisymmetrization means physically that particles having the same spin and charge should not come too close to one another, it is expected that $K_l(\mathbf{r}, \mathbf{r}')$ should incorporate the character of a repulsion. We shall return shortly to this point. To get more insight into the kernel, one can separate out partial waves by making the following expansions in terms of Legendre polynomials:

$$R(\mathbf{r})=r^{-1}\sum f_l(r)P_l(\cos\theta) \qquad \text{(F7)}$$

and

$$K_l(\mathbf{r}, \mathbf{r}')=(4\pi rr')^{-1}\sum_{l=0}^{\infty}(2l+1)k_l(r, r')P_l(\cos\theta).$$

Thus,

$$k_l(r, r')=2\pi rr'\int_{-1}^{1}K(\mathbf{r}, \mathbf{r}')P_l(\cos\theta)\,d(\cos\theta), \qquad \text{(F8)}$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. With (F7) and (F8), Eq. (F4) reduces to

$$\left[\frac{\hbar^2}{4M}\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right)+E-V_D(r)\right]f_l(r)$$
$$=\int_0^{\infty}k_l(r, r')f_l(r')\,dr'. \qquad \text{(F9)}$$

The price for the antisymmetrization of the wave function in order not to violate the Pauli principle is to solve the integral equation (F9) with a complicated kernel function. Although this kernel can be computed numerically, it is rather difficult to interpret Eq. (F9) except in general terms. Van der Spuy used the Taylor series expansion

$$\int_0^{\infty}k_l(r, r')f_l(r')\,dr'$$
$$=\sum_n\left[\int_0^{\infty}k_l(r, r')\frac{(r'-r)^n}{n!}dr'\right]\frac{d^n}{dr^n}f_l(r)$$
$$=\sum_n V_{nl}(r)\,(d^n/dr^n)f_l(r), \qquad \text{(F10)}$$

where $V_{nl}(r)$ is the nth moment of the kernel $k_l(r, r')$. The $n=0$ term of (F10) gives $V_{0l}(r)f_l(r)$, an l-dependent direct part which can be added to the l-independent direct part $V_D(r)f_l(r)$. The next meaningful moment is for $n=2$ which gives $V_{2l}(r)[d^2f_l(r)/dr^2]$. Since the latter term involving the curvature (i.e., the second derivative of the relative wave function) can be added to the kinetic-energy term in the left-hand side of (F9), it leads to a correction in α mass in interaction. A similar situation is observed for nucleons in nuclear matter [Bethe (1956)]. The effective α mass increases if $V_{2l}(r)$ is positive. Investigation of the effect of this term on α–α scattering has not been made.

Van der Spuy made explicit evaluations of the direct and kernel interactions as a function of α–α separation and found that the direct interaction has a range of $\approx$5 fm, while the range of the kernel (exchange) interactions is about 4 fm. Van der Spuy (1956) attempted to calculate α–α scattering phase shifts, but used only one set of parameters, namely those that were found from the analysis of the n-α problem; $\alpha=0.08387$, $\beta=0.2853$, $V_0=-47.62$ MeV, $x=w+m=0.8$, $y=4w+2b-2h-m=1.620$. V_{00} is the predominant V_{n0} term; also $|V_T(r)=V_D(r)+V_c(r)+V_{0l}(r)|$ is small compared with $|V_D'(r)|$ or $V_{00}(r)$ up to 2–3 fm, indicating a low magnitude of the core potential. For the D wave, V_{02} was larger than the other terms in V_{n2} but was not predominant, and the D wave was found to have a velocity-dependent interaction. $V_T(r)$ was found to be negative for the D wave all the way from the origin. This is somewhat disturbing since in order to make the α–α interaction meaningful, considerable repulsion at short distances is necessary to prevent the two α's from dissolving into eight nucleons. One thus suspects the accuracy of $V_T(r)$ at short distances.

The scattering phase shifts were calculated by Butcher and McNamee (1959) within the framework of the resonating-group theory. But their calculations were with a nucleon–nucleon force which did not describe the S-wave scattering data. As pointed out by Schmid and Wildermuth (1961), their force mixture ($\frac{2}{3}$ Serber plus $\frac{1}{3}$ Rosenfeld) seemed to contradict the results of other analyses. Nevertheless, theirs were the first calculations which took a complete account of the exchange part of the Coulomb interaction.

The kernel interaction in (F9) was also derived by Schmid and Wildermuth by varying the relative wave function $R(r_a-r_b)$, according to the Ritz variational principle, in the expression

$$\delta\int(\psi^*H\psi-\lambda\psi^*\psi)\,d\tau=0.$$

Schmid and Wildermuth solved Eq. (F9) on an IBM 7090 computer. The nucleon–nucleon interaction they chose was of the same form as that used by Van der Spuy and may be written once more:

$$V_{ij}=-V_0\exp(-\beta r_{ij}^2)[w-\tfrac{1}{4}m(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(1+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)$$
$$+\tfrac{1}{2}b(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)-\tfrac{1}{2}h(1+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)]. \qquad \text{(F11)}$$

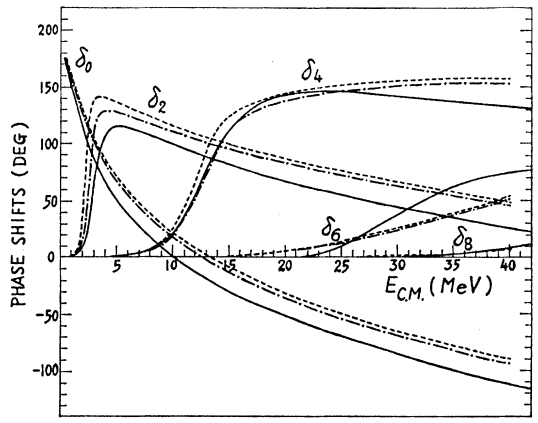

FIG. 5. The energy dependence of the phase shifts calculated by Okai and Park (1966) is shown by the dashed curves for a pure Serber force and by the dash and dot curves for the mixed force, respectively. The solid curves represent the real parts of the experimentol phase shifts. [Figure reproduced from Okai and Park (1966) by courtesy of *The Physical Review.*]

The numerical solution for $f(r)$ determined from Eq. (F9) for this potential was matched to Coulomb wave functions, the phase shifts being determined in the matching procedure. As we have seen earlier from the analysis of Van der Spuy (1959), the parameters to be adjusted in the calculations are the matter density of the α particle, which enters through the internal wave function of the α particle, and the parameters V_0, β, w, m, b, and h of the nucleon–nucleon interaction of the type (F11). Since by normalization $w+m+b+h=1$, the nucleon–nucleon parameters are reduced to five. Schmid and Wildermuth's calculations were with

$$V_0=72.98 \text{ MeV}, \qquad \beta=0.46 \text{ fm}^{-2};$$

$$w+m-b-h=0.63, \qquad w+m+b+h=1.$$

With this choice, the potential (F11) yields values of singlet and triplet effective ranges, singlet n–p scattering length, and deuteron binding energy in good agreement with experiment. As shown in the work of Pearlstein, Tang, and Wildermuth (1960), the exact form of the two-nucleon potential is not very important as long as it reproduces the two-body scattering data sufficiently well. Schmid and Wildermuth calculated the α–α scattering phase shift as a function of the force mixture consisting of a combination of pure Serber force ($w=m$, $b=h$) and a pure Rosenfeld force ($m=2b$, $h=2w$). For a pure Rosenfeld force, the direct nuclear part of the α–α interaction [see Eq. (F5)] vanishes. Thus, the phase shifts are expected to increase with a decrease in the Rosenfeld force.

The best agreement obtained with the experimental phase shifts was a 94% Serber force and a 6% Rosenfeld force. However, although their agreement for $l=2$ and 4 was good, for $l=0$ their curve was below the experimental points. Their resonance value was 0.4 MeV higher than the (then) experimental value of about 95 keV. They remarked that if the potential depth in (F11) were increased by a small amount, their curve would rise and fit the experimental points. However, the fact that among all authors employing the resonating-group formalism only Schmid and Wildermuth claimed success in reproducing experimental data needed investigation. It was discovered later by Okai and Park (1966) from a rather detailed cluster-model analysis of α–α scattering that the discrepancies of Schmid and Wildermuth's results from other calculations were in some cases due to an error in the numerical computation. Failing to reproduce the results of Schmid and Wildermuth, Okai and Park employed rather detailed computing techniques and calculated the phase shifts for s, d, g, i, k partial waves through the excitation energy of 40 MeV. Their main aim was to reproduce the energy dependence of the i and k waves on the basis of the resonating α-cluster model. To this end, they extended the previous cluster-model analysis to i and k waves and to higher bombarding energies. Although at these high energies (up to 40 MeV c.m.) channels other than the initial α–α channel were open, Okai's and Park's calculations were based on the one-channel approximation. They used the same nucleon-nucleon potential as used by Schmid and Wildermuth, but their calculations were more comprehensive and differed from those of the latter on a number of points. They calculated the Coulomb exchange terms explicitly and verified them to be the same as those obtained by Butcher and McNamee. They did some calculations including the Coulomb exchange terms in the kernel interaction; this was found to increase the phase shifts only slightly from those obtainable with the direct Coulomb interaction. The increase, it is understood, was due to the fact that the Coulomb exchange terms in the nonlocal kernel have an opposite (i.e., attractive) sign relative to the direct repulsive Coulomb part and, hence, increase the effective interaction and the phase shifts. Figure 5 gives the phase-shift-versus-energy plots for various l values, as obtained by Okai and Park. A comparison with the real parts of the experimental phase shifts is also shown. For S and D waves, the calculated phase shifts are systematically larger by about 20% than the experimental ones, the agreement for the G wave being slightly better below 25 MeV. There is a discrepancy for all partial waves except for $l=8$, for which the phase shifts are rather reasonable. For a pure Serber force, the phase shifts are consistently larger for all partial waves. As the Rosenfeld force is introduced and gradually increased, leading to an enhancement of the repulsion in the nonlocal kernel interaction, the phase shifts decrease. This has been confirmed by calculations with 6% and 10% Rosenfeld forces. However, Okai and Park observe that to get agreement with the experimental phase shifts, one

would need an inordinate amount of Rosenfeld component in the force mixture. Thus there was some indication that the only repulsion present in the short-range exchange interaction in the kernel arising from the antisymmetrization of the wave function is not sufficient to reproduce the experimental values; it thus seems necessary to invoke the explicit inclusion of a hard core in the nucleon–nucleon interaction. The preliminary calculations of Waghmare (1964) show that a soft-core N–N interaction leads to a soft-core repulsion in the α–α potential at least 1.6 times higher than the attractive depth at the origin. One thus hopes that a hard-core nucleon–nucleon interaction would strengthen the repulsion in the α–α interaction.

Okai and Park also investigated the energy independence of the α–α potentials. They plotted the scattering wave functions for different l values and for different incident energies as functions of α–α separation and found that the zero points $r_c^{(0)}$ and $r_c^{(2)}$ of the S- and D-wave functions, respectively, remain fairly unchanged over a wide range of incident energies. These zero points are within a distance of 3 fm, which is close to the contact distance (i.e., twice the root-mean-square radius of a free α particle) of the two α particles. [See Fig. 3 of Okai and Park (1966).] Comparing these zero points with the hard-core radii in the range of 1.4–2.1 fm and 1.2–1.8 fm for the S and D waves, respectively [as required by Endo, Shimodaya, and Hiura (1964)], the l-dependent, but energy-independent, core radii coincide with the zero points of the scattering function within the contact distance. However, since the wave functions within the contact distance do not vanish, the repulsive hard core, in the strict sense, does not exist; the core is seen to be rather soft. From the analysis of Okai and Park, it thus seems that the kernel interaction is not sufficiently strong to be replaced by a local hard-core potential.

At this point one may ask, even if the nonlocal kernel interaction may be replaced by a soft-core local α–α potential, to what extent would the local potential be equivalent to the nonlocal one. This question has not yet been fully answered. Some attention has already been given to it by Shimodaya, Tamagaki, and Tanaka (1962). Using essentially the same resonating-group formalism, these authors investigated an effective α–α potential from the standpoint of the cluster model of ^{8}Be by making use of reliable properties of the pion-theoretical nucleon–nucleon potential which consists of the one-pion exchange potential V_1 (OPEP), the two-pion exchange potential V_2 (TPEP), and the shorter-ranged potential V_3. It can be written as

$$V_{ij}=-(\tfrac{1}{3})(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)V_1(r_{ij}) + \sum_{S,\pi}P(S,\pi)[{}^{2S+1}V_2^{\pi}(r_{ij})+{}^{2S+1}V_3^{\pi}(r_{ij})],$$

where $P(S,\pi)$ is the projection operator for the state with spin S and parity π, and ${}^{2S+1}V^{\pi}(r_{ij})$ is the potential in this state. For explicit expressions for $V_n(r_{ij})$, see Shimodaya, Tamagaki, and Tanaka (1962). Their equation of relative α–α motion is written in the form

$$[E_r-T_r]\chi_l(r)=V_D(r)\chi_l(r)+\int_0^{\infty}w_l^{(\mathrm{pot})}(r,r')\chi_l(r')\,dr' + \int_0^{\infty}W_l^{(\mathrm{kin})}(r,r')\chi_l(r')\,dr', \quad \text{(F12)}$$

where $\chi_l(r)$ is r times the radial function $f_l(r)$ in Eq. (F9);

$$T_r=\frac{-\hbar^2}{4M}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{\partial}{\partial r}\right)+\frac{\hbar^2}{4M}\frac{l(l+1)}{r^2}.$$

In (F12), $V_D(r)$ is the direct potential as before, independent of $\chi_l(r)$, and $W_l^{(\mathrm{pot})}$ and $W_l^{(\mathrm{kin})}$ are the kernels of the exchange-potential and kinetic-energy terms, respectively. An "effective" local α–α potential V_l^{eff} may be defined as

$$[E_r-T_r]\chi_l(r)=V_l^{\mathrm{eff}}(r)\chi_l(r),$$

where

$$V_l^{\mathrm{eff}}(r)=V_D(r)+[\chi_l(r)]^{-1}\times\left\{\int_0^{\infty}[(W_l^{(\mathrm{pot})}(r,r')+W_l^{(\mathrm{kin})}(r,r')]\chi_l(r')\,dr'\right\}. \quad \text{(F13)}$$

Obviously, the "effective" potential is defined only where $\chi_l(r)$ is not equal to zero. The definition of the equivalent local potential (ELP) makes some sense if one argues that α–α scattering may be described phenomenologically by a local, energy-independent potential and that the kernel is a peaked function near $r=r'$; the nonlocality of the exchange terms is perhaps not too serious. Calculations of ELP require, however, the knowledge of $\chi_l(r)$. Shimodaya *et al.* pointed out that at least the qualitative features of $\chi_l(r)$ can be obtained by studying how $\chi_l(r)$ is damped by Pauli-principle requirements in the α-cluster structure of ^{8}Be. [The relation between the shell model and the cluster model of ^{8}Be, or of other α-clustered nuclei like ^{12}C and ^{16}O, was shown by Perring and Skyrme (1956). The α-particle wave functions for the ground states of ^{8}Be and of ^{12}C and ^{16}O, when antisymmetrized, become identical with the shell-model wave functions. The interested reader is referred in this connection to Biel (1957), Wildermuth and Kanellopoulos (1958a, 1958b), and Neudachin and Smirnov (1957) and to other relevant references cited in the book by Wildermuth and McClure (1966) on the "Cluster Model of Atomic Nuclei."]

Rewriting the antisymmetrized harmonic-oscillator shell-model wave function for the $(1S)^4(1P)^4$ con-

figuration into that of two ground α clusters, one obtains

$$\chi_l(r) = r^4 \exp(-\gamma r^2). \quad \text{(F14)}$$

The r^4 factor in (F14) corresponds at small distances to a repulsive potential like $20/r^2$; the latter together with the centrifugal potential give the effective repulsive potential

$$[20+l(l+1)]/r^2 = l_{\text{eff}}[(l_{\text{eff}}+1)/r^2], \quad \text{(F15)}$$

which determines the behavior of χ_l's at small distances. A form of $\chi_l(r)$ which is consistent with (F15) and with the approximate asymptotic form of $\chi_l(R)$ [i.e., $r\chi_l(r) \simeq 1$] for the ground state of ^{8}Be can be written as

$$\chi_l(r) = [1-\exp(-\nu r^2)]^p/r \approx r^{2p-1} \quad \text{for small } r$$
$$\approx r^{-1} \quad \text{for large } r. \quad \text{(F16)}$$

This form, chosen merely for convenience in calculation, was used by Shimodaya *et al.* with the values of the parameters $p=3$, $\nu=0.15$ fm^{-2} for $l=0$ and 2, and $p=4$, $\nu=0.15$ fm^{-2} for $l=4$. These parameters were chosen to correspond approximately to the solution in the presence of a phenomenological hard-core α–α potential of hard-core radius 2 fm, well depth -10 MeV and range 5 fm, as suggested by Van der Spuy and Pienaar (1958). The exchange potentials in (F13) calculated with this $\chi_l(r)$ show repulsion in the region where two α particles overlap, i.e., for $r \lesssim 2$ fm. Shimodaya *et al.* observed that the main contribution to the exchange potential comes from the TPEP one-pair exchange term and the kinetic-energy terms. The latter are of shorter range compared to $V_D(r)$. The total exchange potential behaves like a repulsive core with a radius of approximately 2 fm. This repulsion is due to the extreme tightness of the α particles, which cannot overlap unless quite high energies are spent to excite them. The total effective α–α potentials have stronger attraction and weaker repulsion inside for larger l, the l dependence resulting from the exchange potential. As pointed out by Shimodaya *et al.*, this means weak angular correlation between the exchange and nonexchange wave functions because the overlap of the two α particles is small. Thus, the effective potential $V_{\alpha\alpha}{}^l(r)$ and a relative wave function $\chi_l(r)$, which is consistent with the Pauli principle, are apparently consistent also with each other. [In a recent letter, Tamagaki and Tanaka (1965) discuss the connection between the nodal behavior of the α–α scattering function and the hard core of the α–α interaction.] Although the effective local α–α potential has nearly all the features demanded by phenomenological analyses of the interaction, these features are only qualitative, and it is not clear if the effective α–α potential so obtained will give a satisfactory fit to the experimental phase shifts. Moreover, the choice of parameters of $\chi_l(r)$, although appropriate for demonstrating the relative difference between the $l=0$ and $l=2$ potentials, is presumably not very realistic. Thus, within the framework of the approach of Shimodaya *et al.*, a somewhat more suitable procedure to derive the equivalent local α–α potential $V_{\alpha\alpha,l}{}^{eqv}$ seems to be to keep p and ν as free parameters along with the nucleon–nucleon force parameters and determine these by requiring $V_{\alpha\alpha,l}{}^{eqv}$ to reproduce the phase shifts. One could perhaps employ a self-consistent procedure; i.e., one would make a guess for $\chi_l(r)$ and calculate the exchange potentials, which in turn would be used to calculate a fresh $\chi_l(r)$, and so on, until self-consistency is attained for $\chi_l(r)$ and the exchange potentials. A comparison of $\chi_l(r)$ thus determined with the $\chi_l(r)$ of Eq. (F14) would perhaps give a better understanding of the interplay between the core of the α–α interaction and the damping of the relative α–α motion.

6. RECENT OUTLOOKS AND DISCUSSIONS

Knowing the general theoretical features of the α–α interaction, one may ask how best these features can be included in constructing phenomenological α–α potentials which are of interest in the α-cluster models of light nuclei and hypernuclei, etc. As is well known, one can construct a number of phase-invariant potentials; i.e., one may just assume an α–α potential shape (e.g., a square well or Gaussian shape) with a certain number of free parameters in it and see if this potential fits the phase shifts. The various potentials discussed in the section on phenomenological α–α potentials have in fact been derived this way. Recently, Ali and Bodmer (1966) adopted a somewhat different and more satisfactory approach. They constructed acceptable S-, D-, and G-wave potentials, adopting the philosophy that a common attractive part of these potentials can best be constructed from a consideration of the G-wave phase shifts which are sensitive mainly to the attractive part. Especially at low α–α bombarding energies, the large centrifugal barrier for the G wave masks the inner repulsive part. For the nuclear part of the phase shifts, the following superposition of repulsive and attractive Gaussian shapes was used:

$$V_{\alpha\alpha}{}^{(N)}(r) = V_R \exp(-\mu_R{}^2 r^2) - V_A \exp(-\mu_A{}^2 r^2),$$

where (V_R, μ_R) and (V_A, μ_A) are the depths and inverse ranges of the repulsive and attractive parts, respectively. (The advantage of using a Gaussian shape is that a comparison with the theoretical expectations for the direct part may be made; the latter becomes Gaussian in shape if, for convenience in calculation, a Gaussian shape for the N–N interaction is assumed.) To investigate the effect of the centrifugal barrier, only the attractive part was considered (i.e., $V_R=0$) for the G wave. For a given μ_A, V_A was varied to give agreement to the experimental phases. It was found possible to obtain two attractive parts [labeled

d_4 and e_4 by Ali and Bodmer (1966) with $V_A=$ 130 MeV, $\mu_A=0.475$ fm^{-1} and $V_A=150$ MeV, $\mu_A=$ 0.5 fm^{-1}] which alone reproduced the phase shifts reasonably well (see Fig. 6). Keeping these attractive parts the same for $l=0$ and $l=2$, the repulsive part was varied to obtain agreement with S- and D-wave phase shifts. The best S-wave potentials, denoted by d_0 and e_0 by Ali and Bodmer, were for $V_R=500$ MeV, $\mu_R=0.7$ fm^{-1} for d_4 and $V_R=1050$ MeV, $\mu_R=0.8$ fm^{-1} for e_4, respectively. The phase shifts reproduced by these potentials are shown in Fig. 7 and compared with the experimental ones. The parameters of the best D-wave potentials, denoted by d_2 and e_2 (see Fig. 8, where the D-wave phase shifts are shown), are $V_R=$ 320 MeV, $\mu_A=0.7$ fm^{-1} for d_4 and $V_R=640$ MeV, $\mu_R=0.8$ fm^{-1} for e_4, respectively. It was found that with

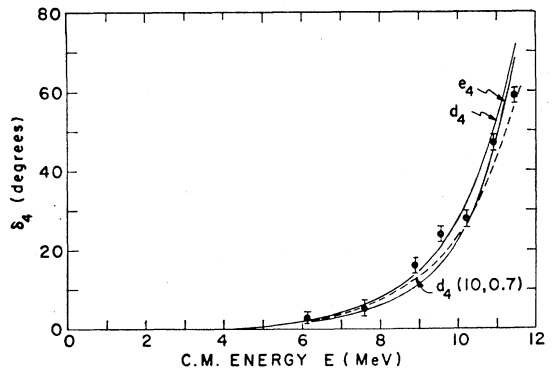

FIG. 6. The $l=4$ phase shifts δ_4 of Ali and Bodmer (1966) as a function of the c.m. energy E for the purely attractive potentials d_4 and e_4 as explained in Sec. 6 and for the potential d_4 (10, 0.7). The latter has the repulsive part $V_R=10$ MeV, $\mu_R=0.7$ fm^{-1} in addition to the attractive part d_4. The experimental values are shown with error bars. [Figure reproduced from Ali and Bodmer (1966) by courtesy of *Nuclear Physics.*]

attractive parts giving less satisfactory fits to δ_4 than d_4 and e_4, it was not possible to construct suitable potentials, especially for the D wave. Only very small repulsive parts are permissible for the G waves (permissible limits for the $l=4$ repulsive parts are $V_R\lesssim 10$ MeV for $\mu_R=0.7$ fm^{-1} for the attractive part d_4 and $V_R\lesssim 10$ MeV for $\mu_R=0.8$ fm^{-1} for e_4) to yield phase shifts consistent with the experimental ones, and d_4 and e_4, when plotted as functions of r, are quite similar. These facts strongly indicate that the approach of Ali and Bodmer uniquely determines the common attractive part of $l=0$, 2, and 4. One could always try to determine a common attractive part for $l=0$ and 2 by trial and error. However, since the results for δ_4, especially at low energies, are expected to be most sensitive to just the attractive tail of $V_{\alpha\alpha}(N)$, it seemed reasonable to try to determine this part by starting with δ_4.

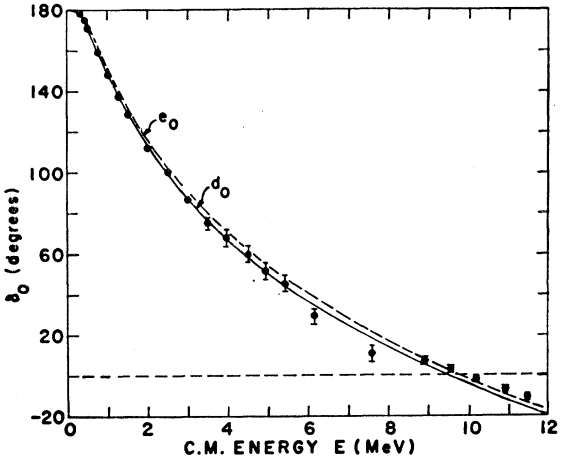

FIG. 7. The $l=0$ phase shifts δ_0 as obtained by Ali and Bodmer (1966) as a function of the c.m. energy E for the potentials d_0 and e_0 of Ali and Bodmer (1966) (explained in Sec. 6). The experimental values are shown with error bars except where the errors fall within the circles. [Figure reproduced from Ali and Bodmer (1966) by courtesy of *Nuclear Physics.*]

The results obtained by Ali and Bodmer, besides agreeing with the results of Darriulat *et al.* (1965), have all the features of the analysis of Shimodaya *et al.* (1962); the potential is local but l dependent and is weakened in going from $l=0$ to $l=4$, and an outer attractive tail, independent of l exists.

A plot of the effective local α–α potentials d_0, d_2, d_4 plus the corresponding centrifugal barriers for different l values is shown in Fig. 9. It is seen that with the decrease of the l value, the size of the repulsive core of the nuclear potential, as compared with the root-mean-square radius R_α ($=1.44$ fm) of the α particle,

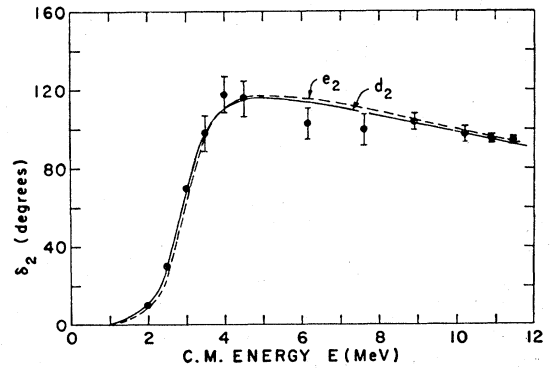

FIG. 8. The $l=2$ phase shifts δ_2 as obtained by Ali and Bodmer (1966) as a function of the c.m. energy E for the potentials d_2 and e_2 of Ali and Bodmer (explained also in Sec. 6). The experimental values are shown with error bars except where the errors fall within the circles. [Figure reproduced from Ali and Bodmer (1966) by courtesy of *Nuclear Physics.*]

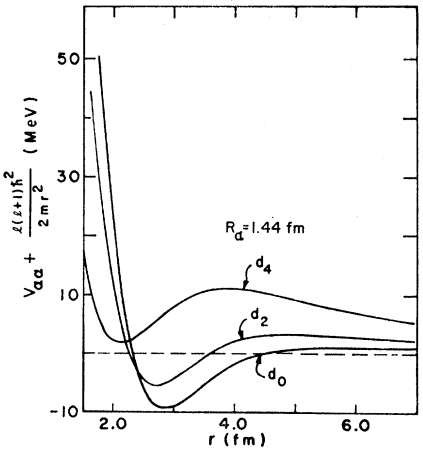

FIG. 9. The potentials d_0, d_2, and d_4 of Ali and Bodmer (1966) with inclusion of the Coulomb potential and the appropriate centrifugal barriers as functions of the α–α separation r. R_α denotes the root-mean-square (rms) radius of the α particle. [Figure reproduced from Ali and Bodmer (1966) by courtesy of *Nuclear Physics.*]

gets larger, ensuring less and less overlap of the particles. Evidence about the nonappreciability of overlap of two α particles, especially in the ground state of ^{8}Be, has been obtained from studies on $_\Lambda{}^9$Be. The investigations of Bodmer and Ali (1964, 1965), Tang and Herndon (1965), Ali, Murphy, and Bodmer (1965), and Herndon and Tang (1966) show that the rms separation of the α particles in the configuration $_\Lambda{}^9$Be (in which the Λ is expected to compress the ^{8}Be core assumed to be in its ground state) has a value of about 3.6 fm which is larger than twice the value of R_α, indicating that the α particles do not overlap appreciably.

Furthermore, it is remarkable that the best S-state α–α potential B of Bodmer and Ali (V_R=400 MeV, μ_R=0.635 fm^{-1}, V_A=160 MeV, μ_A=0.475 fm^{-1}), obtained by employing the Λ as a probe into the details of the α–α interaction in the cluster model of $_\Lambda{}^9$Be, is close to d_0, the best S-state α–α potential with the attractive part d_4, mentioned earlier. In fact, the correct Λ binding in $_\Lambda{}^9$Be was produced for those α–α potentials which not only yield the S-wave resonance correctly, but also reproduce the experimental phase shifts. Thus, from different points of view, it seems encouraging to employ accurate determinations of δ_4 (and of higher phase shifts, which would be even more appropriate) at low energies ($\lesssim$10 MeV) in the elucidation of the attractive tail of $V_{\alpha\alpha}(N)$.

Another interesting idea that arose out of the above considerations was to see whether the attractive tail of the α–α potential can be employed as a probe into just the central spin-independent, isospin-independent part of the nuclear force. Since for the α particle $T=0^+$, $J=0^+$, only the central direct component v (i.e., the term independent of spin and isospin) will contribute to the direct part of $V_{\alpha\alpha}(N)$, the other spin and isospin terms averaging to zero. The tensor force gives a small contribution because the α-particle wave function includes a small amount of D state in addition to the dominant S state. Thus, the tail of $V_{\alpha\alpha}(N)$ selects the attractive tail of v alone and is expected to be a selective probe into the direct component of the nuclear force.

Ali and Bodmer considered the relevance of these ideas to a one-boson exchange model for the nucleon–nucleon interaction [Bryan and Scott (1964)]. In this model, the bosons contributing to v are a σ meson ($T=0$, $J=0^+$) and mesons with $T=0$, $J=1^-$. Of the latter, the lightest is the ω meson (m_ω=5.6 m_π) which gives rise to a short-range repulsion in v, whereas the former ($m_\sigma\simeq 3m_\pi$) will give rise to the required strong and relatively long-range attractive part of v. The important point thus seems to be to extract from $V_{\alpha\alpha}{}^{(N)}$ information about m_σ as well as the σNN coupling constant.

Considering for the normalized spacial part of the α-wave function the form

$$(3/\pi a^2)^{9/4} \exp\left(-3\sum_{ij} r_{ij}^2/32a^2\right)$$

which gives a Gaussian distribution of the nucleons proportional to exp $(-r^2/a^2)$, which, in turn, gives an excellent fit to the electron scattering experiments [Hofstadter (1956), Burleson and Kendall (1960)], and assuming for simplicity the Gaussian potential V_0 exp $(-r^2/\beta^2)$ for the long-range attractive part of v, one gets

$$16V_0[1+(2a^2/\beta^2)] \exp[-r^2/(2a^2+\beta^2)]$$

for the direct part of $V_{\alpha\alpha}{}^{(N)}$.

Identifying this with the phenomenological shape used for the attractive tail of $V_{\alpha\alpha}{}^{(N)}$ which is proportional to exp $(-\mu_A{}^2r^2)$, one has $\mu_A{}^2=2a^2+\beta^2$. With $a=1.18\pm0.05$ fm and Ali and Bodmer's results for μ_A, namely 0.475 fm$^{-1}\lesssim\mu_A\lesssim$0.5 fm, one gets

$$0.97 \text{ fm} \lesssim \beta \lesssim 1.39 \text{ fm}.$$

One can relate β^{-1} to the range of a N–N Yukawa potential proportional to r^{-1} exp $(-\mu r)$ by obtaining that value of μ which gives the same intrinsic range b as the Gaussian interaction. Using the relations $\beta=b/(2.06)^{1/2}$ and $\mu^{-1}=b/2.12$, one obtains

$$0.66 \text{ fm} \lesssim \mu^{-1} \lesssim 0.94 \text{ fm}.$$

It is noteworthy that the values of μ^{-1} arrived at from the analysis of α–α scattering are significantly less than the one-pion range and are close to the value corresponding to a mass of $2m_\pi$ for the exchange particle for the N–N interaction. As pointed out by Ali and

Bodmer, in order to attach more significance to these results, precision measurements of high-l (i.e., $l=4$ or higher) phase shifts would be necessary.

Recently, interesting work has been reported by Benn and Scharf (1967) in which they attacked the inverse problem of constructing unique α–α phenomenological potentials from the scattering phase shifts. Using the Marchenko version of the fundamental Gel'fand–Levital solution of the inverse problem in scattering theory [Marchenko and Agranovich (1963), Gel'fand and Levitan (1951), Faddeev (1963)], they constructed S-, D-, and G-wave α–α potentials which for consistency gave exactly the phases from which they were constructed. The features of the α–α interaction found from earlier phenomenological studies were also present in their potentials, which claimed quantitative certainty because of the way they were determined. However, their G-wave potential, although giving correct phase shifts, shows oscillations [Benn and Scharf, (1967)] when plotted as a function of distance and does not seem to be physically reasonable. A tentative explanation given by Benn and Scharf for the oscillations was that in the energy region of 25–45 MeV, from which comes the main contribution to the G phase shifts, not many consistent phase shifts exist. Furthermore, the inelastic processes already open in this region, and hence the use of the two-body potential model becomes doubtful. Thus, for the determination of the G-wave α–α potential, one seems to need to restrict oneself to rather low laboratory bombarding energies ($\lesssim$24 MeV) as already emphasized by the analysis of Ali and Bodmer (1966).

More recently, Payne, using the one-boson exchange model of the N–N interaction which allows for the coupling of σ, ω, and ϕ mesons to the nucleon, calculated α–α scattering from threshold to 100 MeV laboratory energy using both the Schrödinger equation and the N/D equations. The phase shift in the N/D approach is given by

$$(Cq^{2l+1})^{-1}\exp(i\delta_l)\sin\delta_l=N_l(S)/D_l(S),$$

where

$$C=\prod_{j=1}^{l}\left(1+\frac{\eta^2}{j^2}\right)\frac{2\pi\eta}{e^{2\pi\eta}-1};$$

η is defined as before and q is the momentum. For details about N, D see Payne or Scotti and Wong (1965). Following Preist (1965), who also investigated the contribution of meson exchange to the α–α potential using only ω and σ mesons, Payne neglected the exchange forces in the calculation of the α–α potential. Payne observed that while the N/D equations do not give very accurate results, an extension of the earlier calculation of Preist with the Schrödinger-equation solution enabled him to obtain the best possible fit to the experimental data with $g_\sigma^2=2.66$, $g_\omega^2=3.07$, and $g_\varphi^2=2.95$. The values of the coupling constants which fit the N–N data are found to be $g_\sigma^2=3.05$, $g_\omega^2=2.77$, $g_\varphi^2=2.26$ [Scotti and Wong (1965)]. Since the one-boson exchange potential is a nonrelativistic reduction of the field-theoretical potential, and since there is some ambiguity in this reduction, the discrepancies between the above sets of parameters are not large. Payne, however, does not find a good fit to the D- and G-wave data, especially below 10 and 30 MeV, respectively. The calculated phase shifts are systematically larger than the experimental ones, indicating that some extra repulsion is needed for these waves. Whether this repulsion could be obtained by including exchange effects or, more likely, a hard core in the N–N interaction is not clear. A careful analysis in this direction would be illuminating.

Thus, studies of the α–α interaction may not only give a better understanding of the structures of ^{8}Be and other α-clustered nuclei, but may also provide information about the fundamental forces between the constituents of these nuclei. However, to fulfill the purposes of such studies, one must understand in greater detail the important features of the α–α interaction; i.e., the appearance of "hard" or "quasi-hard" core in the interaction and the question of replacement of the nonlocal α–α interaction by an equivalent local one. [The equivalent local potential as defined by Eq. (F13) is a rather trivially equivalent one; it has poles whenever $\chi_l(r)$ is zero.] Recently, Fiedeldey (1967) has given a general prescription for the construction of equivalent local potentials (ELP) which may be applied with profit to the α–α problem. Fiedeldey proposed that in order to define an ELP without poles one may relax the requirement that the equivalent local wave function equals the nonlocal wave function everywhere and assume that

$$u_l(r)_{r\to\infty}\to v_l(r),$$

where $v_l(r)$ is the solution of the Schrödinger equation with the ELP $U_l(r)$;

$$(d^2v_l/dr^2)+\{k^2-[l(l+1)/r^2]\}v_l=U_l(r)v_l. \quad \text{(R1)}$$

The relationship between $u_l(r)$ and $v_l(r)$ is given by

$$u_l(r)=g_l(r)v_l(r), \quad \text{(R2)}$$

with $g_l(r)_{r\to\infty}=1$.

A large number of ELP's satisfying Eq. (R2) may be constructed, but Fiedeldey has suggested a procedure for constructing one which is without poles and is smooth [$g_l(r)>0$]. According to this procedure, one obtains for a suitable ELP the expression

$$U_l(r)=-\frac{g_l''}{g_l}+g_l^{-1}\int_0^\infty K_l(r,r')g_l(r')\alpha_l(r,r')\,dr',$$

with

$$g_l'(r)=\tfrac{1}{2}\int_0^\infty K_l(r,r')\,g_l(r')\beta_l(r,r')\,dr'$$

and α_l, β_l defined by the equations

$$\alpha_l(r,r')=v_l'(r)\omega_l(r')-v_l(r')\omega_l'(r),$$

$$\beta_l(r,r')=\omega_l(r)v_l(r')-v_l(r)\omega_l(r'); \qquad \text{(R3)}$$

$v_l(r)$ and $\omega_l(r)$ are, respectively, the regular and irregular solutions of Eq. (R1) with the Wronskian

$$v_l'(r)\omega_l(r)-\omega_l'(r)v_l(r)=1.$$

Thus, one could use initial approximations for $U_l^{(0)}(r)$, $g_l^{(0)}(r)$, solve Eq. (R1) to get $v_l^{(0)}$ and $\omega_l^{(0)}$, calculate $\alpha_l^{(0)}(r,r')\beta_l^{(0)}(r,r')$ from Eqs. (R3) and use these α_l, β_l in calculating $u_l^{(1)}$. This $U_l^{(1)}$ could then be used as an input in (R1), and the whole procedure repeated. This iterative scheme could be carried on until self-consistency is attained for $U_l(r)$ and $g_l(r)$. The ELP thus obtained could be used to calculate the phase shifts δ_l, and by comparison of these phase shifts with those given by the nonlocal potential, the adequacy of the above procedure could be tested. One may wonder if one can do without an effective α–α potential, in view of the fact that the phase shifts can be obtained directly by solving an integrodifferential equation (F9) without using such an α–α potential. This may be true for the α–α system but for problems involving α clusters, other than α–α scattering, incorporation of the interaction terms in the form of an effective α–α potential is indispensable; the use of the theoretically obtained α–α interaction in these systems would be too complicated. Recently, Clark and Wang (1966) dealt with the theory of α matter, in which the importance of applying effective α–α potentials also becomes evident.

Some of the features of the effective local α–α potentials which seem to need further study are its energy dependence, the contribution of polarization forces to it, and the dependence of its repulsive part on the hard core of the N–N interaction. Regarding the first aspect, the phenomenological analyses and the fundamental studies of the α–α interaction present two rather different situations. The phenomenological analyses admit an l-dependent (indicating only *implicit* energy dependence through the connection of l with energy for a given impact parameter) but energy-independent potential, whereas the theoretically derived α–α potential is both l dependent and *explicitly* dependent on the relative α–α energy. It has been pointed out by Okai and Park (1966) and also by Abe *et al.* at the 1967 Tokyo Conference that the nodal behavior of the relative α–α wave function is *almost* energy independent. Thus, there are indications that the explicit energy dependence of the α–α interaction is perhaps not very serious, and one may just take l dependence as the simplest type of permissible velocity dependence. However, further work in this direction would be stimulating.

It is important to note that in order to get information about the direct part of the N–N interaction from the phenomenological tail of the α–α potential, it is necessary to assume that this tail can be identified with the tail of the direct part of $V_{\alpha\alpha}^{(N)}$ *arising from nuclear forces*. This assumption is strictly justified only if the latter tail is not complicated by polarization forces; i.e., if polarization forces are sufficiently small in the region of the tail. It may be argued on general grounds that because of the extreme rigidity of the α particles, the polarization forces are expected to be small. In fact, the calculations of Herzenberg and Roberts (1957) for the polarization forces ($V_{\rm pol}$) indicate that, at least at low energies, $V_{\rm pol}$ is small in the region of the tail of $V_{\alpha\alpha}^{(N)}$. Their calculations were based on central N–N interactions, whereas the most important long-range contribution to $V_{\rm pol}$ is expected to come from the strong and long-range tensor component of the one-pion exchange potential. Nevertheless, they did make some estimates for a Yukawa interaction with a range of μ_π^{-1}. For the nucleon–nucleus case, Drell (1955) estimated the contribution to $V_{\rm pol}$ from a one-pion exchange tensor forces and showed it to be small. However, although the polarization forces do not seem to obscure the tail of $V_{\alpha\alpha}^{(N)}$, detailed and more realistic calculations of the former would be useful in making the role of the direct part as a possible probe into the N–N interaction more significant.

An important study would be the relation between the hard core of the N–N interaction and the repulsion in the α–α interaction. There is repulsion in the latter at short distance which, in the resonating-group approach, is due to the nonlocal term $K_l(r,r')$ which originates from the Pauli principle. Since this repulsion is not strong enough to be replaced by a hard-core phenomenological α–α potential, and since the phase shifts calculated with the fundamental α–α interaction are above the experimental ones, a need to increase the effective repulsion in the α–α interaction is justified; it seems that the required extra repulsion could be obtained by explicitly invoking a hard core in the N–N interaction. Studies in this direction are expected to throw considerable light on the repulsive character of the α–α interaction. All of the fundamental studies made so far have used a local N–N interaction, whereas the recent developments in the nuclear-matter problem (such as described by Bethe at the Nuclear Structure Conference in Tokyo in September 1967) suggest the necessity that the N–N interaction be nonlocal. It would be of considerable interest to see the effect of the nonlocality of the N–N interaction on the effective

two-body α–α interaction. It is expected that the kernel $K_l(r, r')$ and, hence, the phase-shift calculations, would now be tediously complicated by the introduction of a hard core in the *N–N* interaction as well as by nonlocality of the latter; thus, proper numerical procedures may have to be sought.

The phase-shift calculations, within the framework of the resonating-group method, have been made for various partial waves at high energies (through the excitation energy of 40 MeV) and good comparisons have been observed with the real part of the *experimental phase shifts** (which are necessarily complex, because the lowest reaction threshold is at $E=34.73$ MeV for $^4\text{He}+^4\text{He}\rightarrow^7\text{Li}+p$). However, the interaction used was purely real; it was derived under assumptions (e.g., no-polarization approximation for the α particle) which are valid only at low energies when the scattering is entirely elastic. (The opening of inelastic and reaction channels not only introduces imaginary parts of the phase shifts but also affects the real parts of the phase shifts for the entrance channel.) Thus, such comparisons are not strictly justified, and at these high energies the phase shifts must be regarded as an indication only. A more complete theory would have to consider a multichannel formulation of the collision problem in the cluster-model approximation.

Abe, Endo, and Tamagaki (1967) have recently reported their study of the high-energy behavior of the repulsive core at 915 MeV. Using two-nucleon amplitudes in the impulse approximation, they constructed the α–α potential $V_{\alpha\alpha}(r)=-(120+i150)\exp(-0.387\,r^2)$ consistent with the elastic data at this energy. It is seen that the repulsive core, which manifests itself as a hard internal structure of the composite system at low energies, disappears and becomes absorptive at high energies.

The fundamental studies of the α–α interaction may serve as a useful basis for understanding interactions between other complex nuclei. Recently, considerable interest has developed in nucleus–nucleus scattering [Block and Malik (1967)] where one also speaks of an effective nucleus–nucleus interaction. In cases where these nuclei have a cluster structure (e.g., ^{12}C–^{12}C or ^{16}O–^{16}O scattering), the same resonating-group formalism which is applied to the α–α problem may be used, although the numerical complications in such cases would be considerably larger. However, the basis features of the interactions between these relatively complex nuclei are expected to be essentially the same as those observed in the case of a system of two spinless, tight α particles. Thus, a deeper understanding of the properties of the interaction between two α clusters is expected to shed considerable light on the structure of many of the light nuclei.

* The same observation has also been made by Ali and Bodmer with the use of purely real phenomenological α–α potentials.

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APPENDIX I: METHOD OF PHASE-SHIFT ANALYSIS

Starting with the two-body Schrödinger equation for the α–α system and following the usual partial-wave analysis, one obtains for the scattering amplitude the expression

$$f(\theta)=(2ik)^{-1}\sum_{l=0}^{\infty}(2l+1)[\exp(2i\delta_l)-1]P_l(\cos\theta).$$

Because of the symmetric nature of the α–α wave function, the components of the scattering amplitude with odd l make no contribution, while the components with even l are doubled. Thus,

$$f(\theta)=\frac{i}{ik}\sum_{l\ \text{even}}(2l+1)[\exp(2i\delta_l)-1]P_l(\cos\theta). \quad \text{(AI.1)}$$

When the Coulomb interaction is taken into consideration, the differential cross section for scattering is given by

$$d\sigma/d\Omega=|f(\theta)|^2, \quad \text{(AI.2)}$$

with

$$f(\theta)=f_{\text{CS}}(\theta)+f_{\text{NS}}(\theta)$$

and

$$f_{\text{CS}}(\theta)=f_c(\theta)+f_c(\pi-\theta),$$

where

$$f_c(\theta)=\frac{-\eta}{2k\sin^2\frac{1}{2}\theta}\exp[-i\eta\log(\sin^2\tfrac{1}{2}\theta)+2i\sigma_0]$$

and

$$f_{\text{NS}}(\theta)=(ik)^{-1}\sum_{l\ \text{even}}(2l+1)(\exp(2i\delta_l)-1]\times[\exp(2i\sigma_l)]P_l(\cos\theta); \quad \text{(AI.3)}$$

$f_{\text{NS}}(\theta)$ and $f_{\text{CS}}(\theta)$ denote the nuclear and the Coulomb part, respectively, of the scattering amplitude. The Coulomb phase shift of rank l is defined by

$$\sigma_l=\arg\Gamma(l+1+i\eta),$$

η being defined in the text.

The phase-shift analysis may be done in two ways. One is the graphical method which was first employed by Wheeler (1941) for analyzing the early scattering experiments of Mohr and Pringle (1937) and of Devons (1939).

In this method it is instructive to express the differential cross section $\sigma(\theta)$ in the form

$$\left[\frac{\sigma_{\text{c.m.}}(\theta)M^2v^4}{z^2e^4}\right]^{1/2}=\csc^2\theta\exp(-i\eta\log_e\sin^2\theta)$$
$$+\sec^2\theta\exp(-i\eta\log_e\cos^2\theta)+(2i/\eta)\sum_{l\text{ even}}(2l+1)$$
$$\times P_l(\cos\theta)\{\exp[2i(\sigma_l-\sigma_0)]\}[\exp(2i\delta_l-1)] \quad \text{(AI.4)}$$

in an obvious notation. A solution of Eq. (AI.4) for δ_l is sought by treating each term as a vector in the complex plane. This equation will usually be satisfied by more than one set of δ_l. When the number of partial waves to be included become large, the graphical method becomes rather laborious and one seeks a second method of doing the analysis, i.e., by an iterative procedure using a computer.

There are several slightly different versions of the second method. We sketch one which has been used by Darriulat, Igo, Pugh, and Holmgren (1965). An initial set $(\delta_l, 0)$ is chosen. For this set, the calculation of χ^2 (chi squared) defined by

$$\chi^2=\frac{[X_i(\text{expt})-X_i(\text{theor})]^2}{[\Delta X_i(\text{expt})]^2+\{\Delta\theta_i[dX_i(\text{expt})/d\theta_i]\}^2} \quad \text{(AI.5)}$$

is made. In (AI.5), $X_i=(d\sigma/d\Omega)(\theta_i, E)$, ΔX_i being the uncertainty in the measured value of X_i and $\Delta\theta_i$ the uncertainty in the angle at which X_i is measured. The theoretical cross section is calculated by using formulas (AI.2) and (AI.3). A new set of values for the δ_l is chosen, using the recursion formulas

$$\delta_{l,n+1}=\delta_{l,n}-H\left(\frac{\partial\chi^2}{\partial\delta_l}\right)_n\Bigg/\left[\sum_l\left(\frac{\partial\chi^2}{\partial\delta_l}\right)^2\right]^{1/2},$$

where H is a suitable steplength fixed beforehand. The procedure is continued in iteration until $\chi_{n+1}^2>\chi_n^2$, when the required solution for the δ_l is assumed to have been obtained.

APPENDIX II: PHASE-SHIFT CALCULATIONS

The radial part of the Schrodinger equation for two α particles may be written

$$\frac{d^2f_l(r)}{dr^2}+\left[k^2-U(r)-\frac{l(l+1)}{r^2}\right]f_l(r)=0, \quad \text{(AII.1)}$$

where $k^2=2\mu E/\hbar^2$, μ being the reduced mass of the two α particles, E their relative energy, and $U(r)=(2\mu/\hbar^2)V_{\alpha\alpha}(r)$. Beyond some large enough distance R, the nuclear part of the complete α–α potential $V_{\alpha\alpha}(r)$ becomes negligible and the Coulomb part dominates. The required solutions of (AII.1) are those which vanish at the origin and behave asymptotically as an incoming Coulomb distorted plane wave plus outgoing Coulomb plus nuclear distorted spherical wave. The solutions of (AII.1) in the region $r\geq R$ are then

$$[f_l(k,r)]_{r\geq R}=\exp[i(\delta_l+\sigma_l)]/k$$
$$\times[\cos\delta_l F_l(k,r)+\sin\delta_l G_l(k,r)], \quad \text{(AII.2)}$$

the notations being the same as used earlier.

The solutions for $r<R$ are generated as follows: As a starting value (near the origin or the hard core) $f_l(k,r)$ is taken to behave like r^{l+1}. Using this starting value, (AII.1) is integrated numerically, using standard procedures, out to $r=R$. The solutions generated in this way are multiples of the solutions we require, i.e., $\hat{f}_l=c_l f_l$, and hence the phase shift δ_l is independent of c_l. One readily obtains

$$\tan\delta_l=\left[\frac{kF_l'(k,r)-F(k,r)L(k,r)}{G_l(k,r)L(k,r)-kG_l'(k,r)}\right]_{r=R}, \quad \text{(AII.3)}$$

where the prime denotes differentiation with respect to $\rho(=kr)$, and $L=\hat{f}_l'/f_l$ is the log derivative of the function $\hat{f}_l$ at the joining radius $r=R$.

In deriving (AII.3), use is made of the property of continuity of the wave function and its log derivative at $r=R$. For given values of the parameters ρ(at $r=R$) and η, the Coulomb wave functions and their derivatives F_l, G_l, F_l' and G_l' may be calculated numerically by using, e.g., the series expansion method of Froberg (1955). The conditions for the series expansions to be valid are $\eta\rho<50$, $\rho\leq 10$. Tables of Abramowitz (1952) and of Tubis (1957) may be useful; see also Bloch (1951).

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