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Articles

Self-Adjoint Ladder Operators. III

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The method of self-adjoint ladder operators, developed in Parts I and II, is applied to the solution of the generalized angular-momentum problem. This reveals many interesting aspects of this approach to eigenvalue problems and, in particular, its relationship to addition of angular momentum. The complete set of irreducible unitary representations of the underlying algebra is obtained and also the corresponding Clebsch–Gordan (Wigner) coefficients for the addition of spin and angular momentum in a space of arbitrary dimension.

I. INTRODUCTION

In Parts I and II of this series,[1] hereafter denoted by I and II, we described the self-adjoint ladder operator approach to eigenvalue problems and its application to several systems of physical interest. Here it is used to study the generalized n-dimensional angular-momentum problem. This is essentially the problem of determining the irreducible unitary representations of the Lie algebra of $0(n)$ in the canonical group chain

$$0(n)\supset 0(n-1)\supset\cdots\supset 0(2)$$

and has applications to the study of the hadron mass spectrum,[2] in nuclear spectroscopy,[3] and in many-body theory.[4] It is moreover a system which is of particular interest in the development of the techniques which have so far been employed in the discussion of the orbital angular-momentum problem (I). The reason for this can be readily understood from the following brief resumé of the basic arguments involved.

Our primary aim is to derive irreducible representations for certain Lie algebras. The method for doing this which is both elegant and elementary is that which makes use of operators (known as ladder operators) which raise and lower the eigenvalues of a selected set of commuting elements of the algebra which we refer to as the observables. This is the basis of the approach used by Cartan[5] who was thus able to derive the irreducible representations for all the semisimple Lie algebras. However in Cartan's analysis a loss of generality arises because not all the possible sets of observables are considered; indeed, only those formed by taking linear combinations of the elements of the algebra are selected. As a consequence, not all the possible sets of irreducible representations are obtained, though those which are do have the property of being unitarily equivalent to any other. While this is sufficient for many purposes, it is sometimes desirable to derive, in explicit form, representations other than those obtained by the Cartan formalism. Such a consideration motivates the present analysis. The representation with which we are concerned is characterized by choosing the observables from the Casimir invariants of suitable subalgebras of the main algebra. As such it is not the most convenient to handle mathematically; but it is often the one of greatest physical importance.

Selecting a representation of the form described above presents us with the problem of determining a ladder operator for the Casimir invariant(s) of a given subalgebra. Now as these are at least quadratic in the elements of the subalgebra no simple ladder operator exists.[6] In the present method, this difficulty is overcome by factorizing the Casimir invariant and then determining ladder operators for the resulting factors. This factorization implements a decomposition of the eigenspace of the Casimir invariant into a pair of subspaces and the action of the ladder operator involves not only a change in the eigenvalue but also an interchange of subspaces, the direction of which determines the sign of the stepping process. As was explained in I, it is this additional property which results in a single self-adjoint ladder operator rather than, as in the Cartan formalism, a mutually adjoint pair.

The factorization described above is of physical importance because it is directly related to the solution by factorization of several important second-order differential equations occurring in mathematical physics.[7] This was illustrated in I and II by several examples. It was also pointed out that the ladder operator, being self-adjoint, could sometimes be interpreted as a constant of the motion. This was exemplified by a study

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[1] A. Joseph, Rev. Mod. Phys. **39**, 829 (1967); C. A. Coulson and A. Joseph, *ibid.* **39**, 838 (1967).
[2] P. Budini, Nuovo Cimento **44A**, 363 (1966).
[3] K. T. Hecht, Phys. Rev. **139**, B749 (1965) and references therein.
[4] P. Kramer and M. Moshinsky, Nucl. Phys. **82**, 241 (1966).
[5] S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic Press, Inc., New York, 1962), Chap. III, pp. 130–161. A simplified account can be found in a review by G. de Franceschi, and L. Maiani, Fortschr. Physik **13**, 318 (1965).
[6] A construction of ladder operators as polynomial functions of the generators of the algebra has been developed by J. C. Nagel and M. Moshinsky, J. Math. Phys. **6**, 682 (1965); see also Ref. 9.
[7] H. R. Coish, Can. J. Phys. **34**, 343 (1956); L. Infeld and T. E. Hull, Rev. Mod. Phys. **23**, 21 (1956).

of the Dirac–Kepler problem for which the present analysis enabled one to compare the degeneracy of this system with that of its nonrelativistic analog. It was also found that the subspaces described above came to have a physical interpretation as spin eigenfunctions.

The interest in the eigenvalue problem, with which we are presently concerned, arises out of the fact that there is not just one, but several Casimir invariants which must be factorized simultaneously. This leads to not just a pair of eigensubspaces, but 2^m of these, where m is the number of Casimir invariants in the subalgebra. Correspondingly the ladder operator decomposes into m self-adjoint parts each of which steps an eigenvalue and interchanges two of the subspaces.

Insofar as providing the irreducible representations of $0(n)$ in the form described above, the results obtained are not new. Gel'fand and Zetlin[8] first gave the matrix elements for the elements of the Lie algebra in this representation. Their results have been rederived by several other authors[9,10] Our work is perhaps most closely related to that given in a report by Louck,[10] referred to hereafter as L, and with which we make frequent comparison. The advantage of our present method of solution, though it lies partly in its greater simplicity, is essentially a conceptual one. It is also of interest because it represents a generalization of the factorization procedure used in solving many of the second-order differential equations occurring in physics. Moreover by relating it to the theory of addition of angular momentum we are able to derive that part of the Clebsch–Gordon (Wigner) coefficients corresponding to the addition of spin and angular momentum in a space of arbitrary dimension.

II. THE ALGEBRAIC IDENTITIES

The Angular-Momentum Algebra

We define as in I, Eqs. (2.1) and (2.2), a set of linear operators, which we refer to as the angular-momentum components, by the following algebraic identities:

$$\mathfrak{L}_{jk}=-\mathfrak{L}_{kj}, \tag{2.1a}$$

$$[\mathfrak{L}_{jk}, \mathfrak{L}_{jl}]=i\mathfrak{L}_{kl}, \tag{2.1b}$$

$$[\mathfrak{L}_{jk}, \mathfrak{L}_{lm}]=0 \quad \text{for} \quad jklm\neq, \tag{2.1c}$$

where the indices j, k, l, m take the values $1, 2\cdots n$. It should be noted that not all of the above relations are independent; indeed (2.1c) is implied by (2.1a, b). This is an elementary consequence of the Jacobi identity.

[8] I. M. Gel'fand and M. L. Zetlin, Dokl. Akad. Nauk. USSR **71**, 1017 (1950).

[9] S. C. Pang and K. T. Hecht, J. Math. Phys. **8**, 1233 (1967); M. K. F. Wong, *ibid.* **8**, 1899 (1967). The noncompact case has been studied by J. Niederle, J. Math. Phys. **8**, 1921 (1967).

[10] J. D. Louck, Los Alamos Scientific Laboratory Report, LA 2451 (1960).

The angular-momentum components are just the infinitesimal generators (i.e., they form the Lie algebra) of the group $0(n)$ of $n\times n$ real orthogonal matrices. This has two connected components,[11] and that which includes the identity is the subgroup $S0(n)$ [also denoted by $0^+(n)$ and $R(n)$] of matrices of determinant $+1$. Other groups may also have the same Lie algebra.

In the following we study representations of the angular momentum algebra described by linear transformations on Hilbert space, such that the individual elements $\mathfrak{L}_{jk}$ are self-adjoint. These may be regarded as unitary representations of $S0(n)$. However, because this has a doubly connected group manifold, not only single-valued, but also double-valued representations are obtained. The latter are nevertheless of physical importance because phase (of the wave function) is not itself an observable.[12]

The above equations describe a generalization of orbital angular momentum defined in I, Sec. 2. Thus in I, the operator which may be conveniently denoted by $\mathfrak{L}_{jklm}$, where

$$\begin{aligned}\mathfrak{L}_{jklm}&=\mathfrak{L}_{jk}\mathfrak{L}_{lm}+\mathfrak{L}_{lj}\mathfrak{L}_{km}+\mathfrak{L}_{kl}\mathfrak{L}_{jm} \quad &\text{for} \quad jklm\neq \\ &=0 &\text{otherwise,}\end{aligned} \tag{2.2}$$

was set identically zero. This resulted in a considerable simplification, and in particular was responsible for the existence of a simple ladder operator. In the present case we shall find that $\mathfrak{L}_{jklm}$, and operators like it, define additional Casimir invariants. The details of construction of these have been given in L, pp. 18–40. Before we can describe the modifications in the ladder operator necessitated by these changes, a brief review of this construction is required.

The Casimir Invariants

Let us first observe that $\mathfrak{L}_{jklm}$ is formed by taking a bilinear combination of the basic angular-momentum operators and cyclically permutating the indices j, k, l, keeping m fixed. Second, that $\mathfrak{L}_{jklm}$, like its precursor $\mathfrak{L}_{jk}$, is skew-symmetric (antisymmetric with respect to interchange of any pair of indices). These remarks suggest the following generalization. We define

$$\begin{aligned}\mathfrak{L}_{j_1j_2,\cdots,j_{2f}}&=\sum_P \mathfrak{L}_{j_1j_2,\cdots,j_{2f-2}}\mathfrak{L}_{j_{2f-1}j_{2f}} \quad &\text{all symbols different} \\ &=0 &\text{otherwise,}\end{aligned} \tag{2.3}$$

where the summation is over the $(2f-1)$ cyclic permutations of the indices $j_1j_2\cdots j_{2f-1}$, keeping j_{2f} fixed. This is just the recurrence relation given in L, Eq. (1.19). At present it may seem to arise in a rather *ad hoc*

[11] C. Chevalley, *The Theory of Lie Groups* (Princeton University Press, Princeton, N.J., 1946), Chap. II, Sec. V, p. 37.

[12] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publications, Inc., Princeton, N.J., 1931), Chap. III, Sec. 16, pp. 180–184 and references therein.

manner though we soon see that there is a sense in which it appears naturally. These operators, which are easily seen to be skew-symmetric, are referred to as the generalized skew-symmetric angular-momentum components (L, p. 20).

Let us recall that in I the Casimir invariant for the orbital angular-momentum algebra was represented by the total squared orbital angular momentum which took the form[13]

$$\mathfrak{L}_n^2 = \frac{1}{2}\sum_{j,k=1}^{n} \mathfrak{L}_{jk}^2 \equiv \sum_{j<k}^{n} \mathfrak{L}_{jk}^2. \tag{2.4}$$

The obvious generalization of (2.4) is afforded by a summation over the square skew-symmetric components (2.3). We thus define

$$\mathfrak{L}_n^{2f} = \frac{1}{(2f)!}\sum_{j_1,j_2,\cdots,j_{2f}=1}^{n} \mathfrak{L}_{j_1j_2,\cdots,j_{2f}}{}^2 \equiv \sum_{j_1<j_2\cdots<j_{2f}}^{n} \mathfrak{L}_{j_1j_2,\cdots,j_{2f}}{}^2, \tag{2.5}$$

operators which, as Louck as shown (L, pp. 27–40), form a commuting set of Casimir invariants, a result which may be succinctly expressed

$$[\mathfrak{L}_n^{2f}, \mathfrak{L}_{jk}] = 0, \tag{2.6}$$

for all j, $k=1, 2\cdots n$; all $f=1, 2\cdots [n/2]$, where $[n/2]$ denotes the largest integer less than $(n/2)$. When n is even we must choose $\mathfrak{L}_{j_1j_2,\cdots,j_n}$ rather than its square $\mathfrak{L}_n{}^n$ for this set of invariants to be complete.

Some further properties of these operators are discussed in L. In the present development we establish (2.6), though not directly, but instead as a consequence of other identities. Indeed much of the analysis which follows is not based on the Casimir invariants themselves; but on the Casimir factors, quantities that we define shortly. It sufficies for the present to remark that for the n-dimensional problem there are at most $[n/2]$ independent Casimir invariants. This is a consequence of the fact that the process of construction of the generalized skew-symmetric components $\mathfrak{L}_{j_1j_2,\cdots,j_{2f}}$ must stop when $2f \leqslant n$. It is responsible for the increase in the number of the Casmir invariants as n is raised by two units and this results in there being a slight difference in the nature of the eigenvalue problem for odd and even n.

The Spin Algebra

In order to construct the ladder operators for the generalized angular-momentum problem, we make use of the spin algebra introduced in I, Sec. 3. We give here, for completeness, a brief summary of their relevant properties.

We define [compare Eqs. (2.1a–c)] the linear operators σ_{jk}: j, $k=1, 2\cdots n$, by the algebraic identities

$$\sigma_{jk} = -\sigma_{kj}, \tag{2.7a}$$

$$\sigma_{jk}^2 = 1 \tag{2.7b}$$

$$\sigma_{jk}\sigma_{jl} = i\sigma_{kl}, \quad \text{for} \quad jkl\neq \tag{2.7c}$$

$$[\sigma_{jk}, \sigma_{lm}] = 0, \quad \text{for} \quad jklm\neq. \tag{2.7d}$$

These are in addition chosen to be self-adjoint. As in the case of (2.1) the last of these relations follows from the other three. It is easily seen that (2.1a, c) imply the anticommutation relation

$$(\sigma_{jk}\sigma_{jl} + \sigma_{jl}\sigma_{jk}) \equiv [\sigma_{jk}, \sigma_{jl}]_+ = 0 \quad \text{for} \quad jkl\neq, \tag{2.8}$$

which is instrumental in providing the ladder operation. We also have, as shown in I, that

$$\sigma_{jk}\sigma_{lm} = \sigma_{lj}\sigma_{km} = \sigma_{kl}\sigma_{jm} \quad \text{for} \quad jklm\neq, \tag{2.9}$$

a relationship which immediately suggests that we might construct generalized skew-symmetric components which are related to the basic spin operators σ_{jk} in a manner analogous to the relationship between the generalized skew-symmetric angular-momentum components and the basic angular-momentum operators. Thus we define

$$\begin{aligned}\sigma_{j_1j_2\cdots j_{2f}} &= \sigma_{j_1j_2}\cdot\sigma_{j_3j_4}\cdots\sigma_{j_{2f-1}j_{2f}} && \text{all symbols different}\\ &= 0 && \text{otherwise,}\end{aligned} \tag{2.10}$$

operators which are easily seen to be self-adjoint, unitary, and skew-symmetric. They also satisfy "contraction" relations similar to (2.7c). Finally the spin and angular-momentum algebras are chosen to be independent in the sense that any element of one commutes with any element of the other [I, Eq. (3.7)].

The Factorized Casimir Invariants (Casimir Factors)

The next step in the analysis of the angular-momentum problem is the factorization of the Casimir invariants $\mathfrak{L}_n^{2f}$ using the spin operators defined above. As explained in the Introduction, the object of the procedure to derive operators, to be known as Casimir factors, on which a ladder action can be defined. Now we showed in I, Eq. (4.4), that

$$\mathfrak{L}_n^2 = \Big(\sum_{j<k}^{n} \sigma_{jk}\mathfrak{L}_{jk}\Big)\Big(\sum_{l<m}^{n} \sigma_{lm}\mathfrak{L}_{lm} + n - 2\Big),$$

an expression which illustrates the factorization of the total squared orbital angular momentum. In the present

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[13] In this notation

$$\sum_{j<k}^{n}$$

represents the double summation

$$\sum_{k=2}^{n}\sum_{j=1}^{k-1}$$

Similarly,

$$\sum_{i_1<i_2\cdots<i_{2f}}^{n}$$

represents an analogous $2f$-fold summation.

context, this result no longer holds and is replaced by

$$\mathfrak{L}_n^2=(\sum_{j<k}^{n}\sigma_{jk}\mathfrak{L}_{jk})(\sum_{l<m}^{n}\sigma_{lm}\mathfrak{L}_{lm}+n-2)$$
$$-2\sum_{j<k<l<m}^{n}\sigma_{jklm}\mathfrak{L}_{jklm}. \quad (2.11)$$

The appearance of a new operator in the expression for $\mathfrak{L}_n^2$ can be accounted for by the fact that the $\mathfrak{L}_{jklm}$ are no longer identically zero. Its existence suggests possible candidates for the Casimir factors, as the summation involved is easily generalized by making use of the skew-symmetric components in both the angular-momentum and spin algebras. Thus we define

$$\mathfrak{L}_n^{(f)}=\sum_{j_1<j_2,\cdots,<j_{2f}}^{n}\sigma_{j_1j_2,\cdots,j_{2f}}\mathfrak{L}_{j_1j_2,\cdots,j_{2f}}, \quad (2.12)$$

as the Casimir factors. We shall soon see to what extent they fill this role.

Let us observe that, on account of Eq. (2.3) and (2.9), Eq. (2.12) may be rewritten in the form

$$\mathfrak{L}_n^{(f)}=(2^f f!)^{-1}\sum_{j_1,\cdots,j_{2f}=1}^{n}{}'\sigma_{j_1j_2}\sigma_{j_3j_4}\cdots\sigma_{j_{2f-1}j_{2f}}$$
$$\times\mathfrak{L}_{j_1j_2}\cdots\mathfrak{L}_{j_{2f-1}j_{2f}}, \quad (2.13)$$

where the prime denotes that, in the summation, no two indices may be set equal. The significance of this result is twofold. First it shows that, because we only meet the $\mathfrak{L}_{j_1j_2,\cdots,j_{2f}}$ in summations of the above kind, it is unnecessary to carry out computations which directly involve these operators and therefore we need consider only the basic angular momentum components $\mathfrak{L}_{jk}$. Second it shows that the $\mathfrak{L}_{j_1j_2,\cdots,j_{2f}}$ arise in a natural fashion, being defined by an identification of terms in Eqs. (2.12) and (2.13). These properties underlie the computational and conceptual advantages gained by use of the spin algebra.

The Casimir factors, like their precursors, form a set of independent operators which mutually commute. The latter result may be expressed

$$[\mathfrak{L}_n^{(f)},\mathfrak{L}_n^{(g)}]=0, \quad (2.14)$$

for all $f, g=1, 2\cdots[n/2]$. This does not follow trivially owing to the nature of the summations involved; but may be verified using the algebraic properties of the spin and angular-momentum operators [Eqs. (2.1a–c) and (2.7a, b, d)].

It is convenient to include in our list of Casimir factors the trivial invariant

$$\mathfrak{L}_n^{(0)}=1,$$

where 1 is the identity operator. It may also be considered to be one of the Casimir invariants and in this context is written without the brackets in the superscript.

A brief study of the angular-momentum and spin algebras shows that it is by no means easy to effect a factorization of the Casimir invariants in terms of the $\mathfrak{L}_n^{(g)}$. Nevertheless we are able to show, though by an indirect argument, that such a factorization is possible and furthermore to derive the coefficients in this decomposition.

The Casimir factors, unlike their precursors, do not commute when different values of n are involved. That is,

$$[\mathfrak{L}_n^{(f)},\mathfrak{L}_m^{(g)}]\neq0,$$

whenever $n\neq m$ and $(f\neq g)\neq0$. We find this result is essential to the existence of ladder operator. It is unfortunate that it may introduce manipulative difficulties.

The Ladder Operator

It was shown in I, Sec. 4 that the ladder operator for the orbital angular-momentum problem could be expressed as the difference of the Casimir factors in $(n+1)$ and n dimensions. This result, of intrinsic interest in itself, enabled us to determine the eigenfunctions of the Casimir factors needed in describing the action of the ladder operator. It may be directly generalized to the present situation in the following manner. We define the ladder operators $L_{n+1}^{(f)}$: $f=0, 1, 2\cdots n$ by

$$L_{n+1}^{(f)}=\mathfrak{L}_{n+1}^{(f)}-\mathfrak{L}_n^{(f)}. \quad (2.15)$$

This expression is equivalent to either of the following:

$$L_{n+1}^{(f)}=[2^{f-1}(f-1)!]^{-1}\sum_{j_1,\cdots,j_{2f-1}=1}^{n}{}'\sigma_{j_1j_2}\cdots\sigma_{j_{2f-1}n+1}$$
$$\times\mathfrak{L}_{j_1j_2}\cdots\mathfrak{L}_{j_{2f-1}n+1} \quad (2.16a)$$

$$L_{n+1}^{(f)}=\sum_{j_1<j_2\cdots<j_{2f-1}}^{n}\sigma_{j_1j_2\cdots j_{2f-1}n+1}\mathfrak{L}_{j_1j_2,\cdots,j_{2f-1}n+1}, \quad (2.16b)$$

as may be readily verified.

Like the Casimir factors these operators commute for fixed n, that is

$$[L_{n+1}^{(f)},L_{n+1}^{(g)}]=0, \quad (2.17)$$

for all $f, g=1, 2\cdots[n/2]$. This result, which like (2.14) is nontrivial, may be verified by use of the algebraic properties of the spin and angular momentum operators. When combined, Eqs. (2.14) and (2.17) imply [recalling the definition of the $L_{n+1}^{(f)}$] that

$$[\mathfrak{L}_{n+1}^{(f)},\mathfrak{L}_n^{(g)}]=[\mathfrak{L}_{n+1}^{(g)},\mathfrak{L}_n^{(f)}] \quad (2.18)$$

for all $f, g=0, 1\cdots[n/2]$. This relates the commutators between the Casimir factors for n and $(n+1)$.

Not all of the $L_{n+1}^{(f)}$ are important in describing the ladder operation. Indeed only the simplest of these finds extensive use, namely, $L_{n+1}^{(1)}$. It may be expressed as [compare I, Eqs. (4.1) and (4.5)]

$$L_{n+1}^{(1)}=\sum_{j=1}^{n}\sigma_{jn+1}\mathfrak{L}_{jn+1},$$

and satisfies

$$(L_{n+1}^{(1)})^2=\mathfrak{L}_{n+1}^2-\mathfrak{L}_n^2-\mathfrak{L}_n^{(1)}.$$

This last result implies that

$$[(L_{n+1}^{(1)})^2,\mathfrak{L}_n^{(f)}]=0, \tag{2.19}$$

for all $f=1, 2\cdots[n/2]$. Consequently $(L_{n+1}^{(1)})^2$ has no nonzero matrix elements between different eigenvalues of the Casimir invariants. $L_{n+1}^{(1)}$ itself is trivially self-adjoint and we show that it may be decomposed into $[n/2]$ self-adjoint parts, each of which steps an eigenvalue. Similar properties are shared by the remaining ladder operators $L_{n+1}^{(f)}: f=2, 3\cdots[n/2]$ though we are not to be concerned with this result in the present development.

In an earlier paragraph we remarked that the eigenvalue problem differs slightly for odd and even n. A further indication of this behavior is afforded by the identity

$$L_n^{(n/2)}=\mathfrak{L}_n^{(n/2)},$$

which holds when n is even. This result implies, as we shall see, that when n is even, the basic ladder operator $L_n^{(1)}$ decomposes to give, in addition to the $(n/2)$ terms which step the eigenvalues, a further term which commutes with all the $L_{n-1}^{(f)}$. This feature, which was not present in the orbital problem, is responsible for the appearance of a new Casimir invariant as n is increased by two units.

The ladder operators are related through the important (and easily verified) identity

$$[\mathfrak{L}_n^{(f)},L_{n+1}^{(1)}]_+=2L_{n+1}^{(f+1)}-(n+1-2f)L_{n+1}^{(f)}, \tag{2.20}$$

which holds for all integer n, $n>1$, and all $f=1, 2\cdots[n/2]$, if we set $L_n^{(n/2+1)}$ identically zero when n is even. It is mainly from this result that we are able to deduce the nature of the ladder operation. It should be noted that an anticommutation bracket appears in (2.20) rather than a commutator. This, as we pointed out in I, Sec. 1, is an essential characteristic and is responsible for the existence of a self-adjoint ladder operator.

The Dirac Operators

It was shown in I, Sec. 6, that the orbital angular-momentum problem admits certain operators which commute with the ladder operator L_{n+1}. These were referred to as the Dirac operators as they were found to be identical with the "total angular momentum operators" introduced by Dirac[14] as constants of the motion for spin-$\frac{1}{2}$ particles in a central (relativistic) field. In the present context the operators $J_{jk}: j, k=1, 2\cdots n$ defined by

$$J_{jk}=\mathfrak{L}_{jk}+\tfrac{1}{2}\sigma_{jk}, \tag{2.21}$$

which we continue to refer to as the Dirac operators, satisfy

$$[J_{jk},L_{n+1}^{(f)}]=0, \tag{2.22}$$

for all $j, k=1, 2\cdots n$; $f=1, 2\cdots[n/2]$. Thus, they possess the same invariance properties in the generalized angular-momentum problem as in the orbital one. This result finds application both to the understanding of the nature of the ladder oparation and to the determination of the eigenfunctions of the Casimir factors.

We now define

$$J_n^{(f)}=(2^f f!)^{-1}\sum_{j_1,\cdots,j_{2f}=1}^{n}{}' \sigma_{j_1j_2}\cdots\sigma_{j_{2f-1}j_{2f}}J_{j_1j_2}\cdots J_{j_{2f-1}j_{2f}} \tag{2.23a}$$

$$=(2^f f!)^{-1}\sum_{j_1,\cdots,j_{2f}=1}^{n}{}' (\sigma_{j_1j_2}\mathfrak{L}_{j_1j_2}+\tfrac{1}{2})\cdots \times(\sigma_{j_{2f-1}j_{2f}}\mathfrak{L}_{j_{2f-1}j_{2f}}+\tfrac{1}{2}), \tag{2.23b}$$

operators which can be seen to be the analogs of the Casimir factors. From the second of the above expressions it is readily shown that the $J_n^{(f)}$ are related to the $\mathfrak{L}_n^{(f)}$ through the identity

$$J_n^{(f)}=\sum_{g=0}^{f}(\tfrac{1}{2})^{2(f-g)}\frac{(n-2g)!}{(n-2f)!(f-g)!}\mathfrak{L}_n^{(g)}. \tag{2.24}$$

This proves useful in describing the action of the ladder. It also shows, recalling (2.14), that the $J_n^{(f)}$: $f=1, 2\cdots[n/2]$ commute for fixed n.

Next we define

$$J_n^{2f}=\sum_{j_1<j_2,\cdots<j_{2f}}^{n}J_{j_1j_2,\cdots,j_{2f}}^2, \tag{2.25}$$

where the $J_{j_1j_2\cdots j_{2f}}$ are formed by a recurrence relation identical to (2.3). The J_n^{2f} are analogous to the Casimir invariants. They may also be related to the Casimir factors by taking suitable bilinear combinations of the latter. We shall postpone a discussion of this to a later stage (see Sec. 5). For the present it sufficies to remark that in view of (2.22) these operators commute with the ladder operators, that is,

$$[J_n^{2f},L_{n+1}^{(g)}]=0, \tag{2.26}$$

for all $f, g=1, 2\cdots[n/2]$. For this reason they are conveniently termed the "Ladder invariants."

Finally, we remark that whereas the Casimir invariants of the subalgebra (defined by choosing $j, k\leqslant n$) commute with all the elements of the subalgebra, the Ladder invariants commute with ladder operators formed from the additional elements of the

[14] P. A. M. Dirac, Proc. Roy. Soc. (London) **A117**, 610 (1928).

algebra obtained when n is increased by one unit. The significance of this result will later become apparent.

III. THE $(2n+1)$-DIMENSIONAL PROBLEM

The bifurcation in the nature of the eigenvalue problem for odd and even values of n necessitates a separate computation for each. We start with the $(2n+1)$-dimensional problem, that is we assume that the eigenvalue problem has been solved in $2n$ dimensions and deduce, by construction of suitable ladder operators, its solution in $(2n+1)$ dimensions. As the number of Casimir invariants is the same in both $2n$ and $(2n+1)$ dimensions, this part of the eigenvalue problem is the simpler.

Let us start with some intuitive considerations which help to indicate the direction in which the argument must proceed. We remarked earlier that Eq. (2.20) is instrumental in the construction of the ladder operator. Replacing n by $2n$ and $L_{2n+1}^{(f)}$ by $(\mathfrak{L}_{2n+1}^{(f)}-\mathfrak{L}_{2n}^{(f)})$ it becomes

$$[\mathfrak{L}_{2n}^{(f)}, L_{2n+1}^{(1)}]_+ = 2\{(\mathfrak{L}_{2n+1}^{(f+1)}-\mathfrak{L}_{2n}^{(f+1)}) - (n-f+\tfrac{1}{2})(\mathfrak{L}_{2n+1}^{(f)}-\mathfrak{L}_{2n}^{(f)})\}, \quad (3.1)$$

as result which, recalling that

$$\mathfrak{L}_{2n}^{(0)}=\mathfrak{L}_{2n+1}^{(0)}=1, \qquad \mathfrak{L}_{2n}^{(n+1)}=\mathfrak{L}_{2n+1}^{(n+1)}=0 \quad (3.2)$$

can be seen to hold for all $f=0, 1\cdots n$.

A brief study of this set of equations reveals that an approximate summation will eliminate the terms appearing on the right-hand side, leaving an operator which anticommutes with $L_{2n+1}^{(1)}$. This may be expressed

$$\left[\left(\sum_{f=0}^{n}\frac{\Gamma(n-f+\frac{1}{2})}{\Gamma(n+\frac{1}{2})}\mathfrak{L}_{2n}^{(f)}\right), L_{2n+1}^{(1)}\right]_+ = 0, \quad (3.3)$$

in which Γ denotes the gamma function.

This shows that $L_{2n+1}^{(1)}$ changes the sign of the eigenvalue of the above linear combination of Casimir factors. In the orbital angular-momentum problem (I, Sec. 4) an analogous result [I, Eq. (4.3)], when combined with the expression for the total squared orbital angular momentum [I, Eq. (4.4)], enabled us to interpret L_{2n+1} as a ladder operator and thus to solve the eigenvalue problem. In the present system, there are n-independent Casimir factors and this provides n-independent eigenvalues which are to be stepped by $L_{2n+1}^{(1)}$. Equation (3.3) is insufficient, in itself, to imply such an action; yet there is reason to suppose that it can be inferred from a suitable number of anticommutation relations obtained by decomposition of (3.3). For this reason we express the bracketed sum appearing in (3.3) as a product of n (commuting) operators $\bar{k}_{2n}^{(f)}$; $f=1, 2\cdots n$

$$\sum_{f=0}^{n}\frac{\Gamma(n-f+\frac{1}{2})}{\Gamma(n+\frac{1}{2})}\mathfrak{L}_n^{(f)}=\prod_{f=1}^{n}\bar{k}_{2n}^{(f)}, \quad (3.4)$$

and substitute the resulting expression in (3.3)

$$[(\prod_{f=1}^{n}\bar{k}_{2n}^{(f)}), L_{2n+1}^{(1)}]_+ = 0. \quad (3.5)$$

This now suggests the following conjecture; namely, that complementary to the product decomposition (3.4), the ladder operator $L_{2n+1}^{(1)}$ reduces as a sum of terms each of which anticommutes with one of the $\bar{k}_{2n}^{(f)}$ and commutes with the others. That is

$$L_{2n+1}^{(1)} = \sum_{f=1}^{n} l_{2n+1}^{(f)} \quad (3.6)$$

such that

$$[\bar{k}_{2n}^{(f)}, l_{2n+1}^{(f)}]_+ = 0 \quad (3.7a)$$

for all $f=1, 2\cdots n$, and

$$[\bar{k}_{2n}^{(f)}, l_{2n+1}^{(g)}] = 0, \quad (3.7b)$$

for all $f(\neq g)=1, 2\cdots n$.

It is easily seen that such a result, though not implied by (3.3), does not contradict it. Moreover it is from this that we shall be able to deduce that the $l_{2n+1}^{(f)}$ behave as self-adjoint ladder operators for the eigenvalues of the $\bar{k}_{2n}^{(f)}$ and hence for the Casimir factors. Our first objective must therefore be to establish:

Theorem 1. From the spin and angular-momentum algebras defined in Sec. 2, it is possible to construct a set of commuting self-adjoint operators $\bar{k}_{2n}^{(f)}: f=1, 2\cdots n$, such that there exists a unique decomposition of $L_{2n+1}^{(1)}$ into self-adjoint components $l_{2n+1}^{(f)}: f=1, 2\cdots n$ satisfying (3.6) and (3.7).

Proof. The proof divides into two parts. In the first, the operators described above are constructed. In the second, they are shown to possess the required properties.

The Decomposition of the Casimir Factors

The above theorem does not give a definite prescription for the construction of the $\bar{k}_{2n}^{(f)}$. Let us however observe two features of Eq. (3.4), an identity which they must satisfy. The first is that the expression on the right-hand side of (3.4) represents one of the n-symmetric functions which can be formed from the $\bar{k}_{2n}^{(f)}$. If we could similarly relate the remaining $(n-1)$ symmetric functions to the Casimir factors then, to within a permutation, the $\bar{k}_{2n}^{(f)}$ would be uniquely determined. This last result is an elementary consequence of the fact that a polynomial of degree n over the complex field has precisely n roots. Applied here it shows that the eigenvalues of the $\bar{k}_{2n}^{(f)}$ (recall that these operators are to be self-adjoint and to mutually commute) are, to within a permutation, uniquely determined by the eigenvalues of the Casimir factors. This in turn determines the operators themselves uniquely.[15]

[15] Strictly speaking only uniqueness to within equivalence is implied. However, this distinction is unimportant for our present purpose.

The second point concerning (3.4) to which we must draw attention is that if the $\bar{k}_{2n}^{(f)}$ are to be treated on a equal footing then they must be chosen to be linear in the basic angular momentum components $\mathfrak{L}_{jk}$. Taken together these observations suggest that in order to define the $\bar{k}_{2n}^{(f)}$ we require an expression of the form

$$\bar{K}_{2n}^{(f)}=\sum_{g=0}^{f}\bar{a}_{2n}^{f,f-g}\mathfrak{L}_{2n}^{(g)}, \tag{3.8}$$

in which the $\bar{a}_{2n}^{f,f-g}$ are real or possibly complex numbers and the $\bar{K}_{2n}^{(f)}$ are the symmetric functions taken over the $\bar{k}_{2n}^{(f)}$. These may be conveniently defined by equating coefficients of powers of x in the expression

$$\prod_{f=1}^{n}(x+\bar{k}_{2n}^{(f)})=\sum_{f=0}^{n}x^{n-f}\bar{K}_{2n}^{(f)}. \tag{3.9}$$

Since the Casimir factors are self-adjoint and commute, the $\bar{k}_{2n}^{(f)}$ may also be chosen to have this property. Finally, in order to remove the permutation symmetry in the $\bar{k}_{2n}^{(f)}$, we introduce the ordering

$$(\bar{k}_{2n}^{(1)})^2>(\bar{k}_{2n}^{(2)})^2\cdots>(\bar{k}_{2n}^{(n)})^2>0. \tag{3.10}$$

This ensures a unique correspondence between these operators and the Casimir factors. It should be noted that (3.10) implicitly requires the eigenvalues of the $\bar{k}_{2n}^{(f)}$ to be distinct and nonzero. This assumption is permissible, according to the method of induction, as long as it can be shown to imply a corresponding result when n is increased by one and that it holds for $n=2$ (see Sec. 6, "The eigenvalues").

There now remains the problem of determining the coefficients $\bar{a}_{2n}^{f,g}$. A systematic computation of these would result from the requirement that (3.6) and (3.7) are satisfied. A strict derivation in this sense is not attempted. Instead we remark that since (3.1) determines, at least in part, the nature of the ladder operation, it is natural to choose the $\bar{a}_{2n}^{f,g}$ in such a manner that this equation adopts its simplest form. To this end we first relate the remaining Casimir factors $\mathfrak{L}_{2n+1}^{(f)}$ to symmetric functions in a manner analogous to (3.8). That is, we define[16]

$$K_{2n+1}^{(f)}=\sum_{g=0}^{f}a_{2n+1}^{f,f-g}\mathfrak{L}_{2n+1}^{(g)}, \tag{3.11}$$

where

$$\prod_{f=1}^{n}(x+k_{2n+1}^{(f)})=\sum_{f=0}^{n}x^{n-f}K_{2n+1}^{(f)}, \tag{3.12}$$

and

$$(k_{2n+1}^{(1)})^2\geqslant(k_{2n+1}^{(2)})^2\cdots\geqslant(k_{2n+1}^{(n)})^2\geqslant0. \tag{3.13}$$

Let us now choose $\bar{a}_{2n}^{f,g}$ and $a_{2n}^{f,g}$ so that (3.1) reduces to

$$[\bar{K}_{2n}^{(f)},L_{2n+1}^{(1)}]_+=2(K_{2n+1}^{(f+1)}-\bar{K}_{2n}^{(f+1)}), \tag{3.14}$$

for all $f=0,1\cdots n$, and where we have set [compare (3.2)]

$$K_{2n+1}^{(0)}=\bar{K}_{2n}^{(0)}=1,\qquad K_{2n+1}^{(n+1)}=\bar{K}_{2n}^{(n+1)}=0. \tag{3.15}$$

A first requirement of (3.14) is clearly

$$\bar{a}_{2n}^{f,g}=a_{2n+1}^{f,g}, \tag{3.16}$$

for all $f,g=0,1\cdots n$. To proceed further we first reduce the set of n equations represented by (3.14) to the single identity in x, namely

$$[\prod_{f=1}^{n}(x+\bar{k}_{2n}^{(f)}),L_{2n+1}^{(1)}]_+ =2x\{\prod_{f=1}^{n}(x+k_{2n+1}^{(f)})-\prod_{f=1}^{n}(x+\bar{k}_{2n}^{(f)})\}. \tag{3.17}$$

Then, in order to derive such an expression from (3.1), we replace (3.8) itself by an identity in x. This is achieved by introducing a set of linearly independent polynomials in x, denoted by $\bar{P}_f(x):f=0,1\cdots n$ and writing it in the form

$$\sum_{f=0}^{n}\bar{P}_{n-f}(x)\mathfrak{L}_{2n}^{(f)}=\prod_{f=1}^{n}(x+\bar{k}_{2n}^{(f)}). \tag{3.18}$$

Substitution of this expression in (3.17) and comparison with (3.1) enables one to derive, without much difficulty, the following recurrence relation for the polynomials

$$x\bar{P}_{n-f-1}(x)=\bar{P}_{n-f}(x)-(n-f-\tfrac{1}{2})\bar{P}_{n-f-1}(x).$$

We may assume, without loss of generality, that the polynomials are suitably "normalized." Thus setting

$$\bar{P}_0(x)=1, \tag{3.19}$$

we find that

$$\bar{P}_f(x)=\Gamma(x+f+\tfrac{1}{2})/\Gamma(x+\tfrac{1}{2}). \tag{3.20}$$

This result completely determines the coefficients $\bar{a}_{2n}^{f,g}$. Equating powers of x in Eqs. (3.8), (3.9), (3.18), and (3.20) it may be shown that

$$\bar{a}_{2n}^{f,g}=\sum_{j_1<j_2\cdots<j_g}^{n-(f-g)}\prod_{m=1}^{g}(j_m-\tfrac{1}{2}), \tag{3.21}$$

for all $f\geqslant g=1,2\cdots n$, and

$$\bar{a}_{2n}^{f,0}=1,$$

for all $f=1,2\cdots n$. It is unfortunate that these coefficients cannot be readily expressed in simple closed form. Similar difficulties are encountered in L (L, pp. 197–205). They are to some extent overcome by the above use of the polynomials $\bar{P}_f(x)$.

Equation (3.20) also determines the relationship between the $\bar{k}_{2n}^{(f)}$ and the Casimir factors. This takes

[16] Note the omission of the bar over $K_{2n+1}^{(f)}$ etc. The reason for this will shortly become clear [Sec. 4, Eq. (4.12)].

the form

$$\mathcal{L}_{2n}^{(f)}=\sum_{j_1<j_2\cdots<j_f}^{n}\prod_{m=1}^{f}(\mathcal{J}_{2n}^{(j_m)}+f-m), \qquad (3.22)$$

where the $\mathcal{J}_{2n}^{(f)}$ are operators defined by

$$\mathcal{J}_{2n}^{(f)}=(\bar{k}_{2n}^{(f)}-(n-f+\tfrac{1}{2})),$$

for all $f=1, 2\cdots n$, and similarly

$$\mathcal{L}_{2n+1}^{(f)}=\sum_{j_1<j_2\cdots<j_{2f}}^{n}\prod_{m=1}^{f}(\mathcal{J}_{2n+1}^{(j_m)}+f-m), \qquad (3.23)$$

where

$$\mathcal{J}_{2n+1}^{(f)}=(k_{2n+1}^{(f)}-(n-f+\tfrac{1}{2})),$$

for all $f=1, 2\cdots n$.

This result completes the description of the $\bar{k}_{2n}^{(f)}$.

The Decomposition of the Ladder Operator

Our next task is to establish the decomposition of $L_{2n+1}^{(1)}$ as implied by (3.6) and (3.7). We show that this may be effected by expressing its component parts, namely, the $l_{2n+1}^{(f)}$, in the form[17]

$$l_{2n+1}^{(f)}=\tfrac{1}{2}(\bar{k}_{2n}^{(f)})^{-1}[\bar{k}_{2n}^{(f)}, L_{2n+1}^{(1)}], \qquad (3.24)$$

where f takes the values $1, 2\cdots n$. Now if these elements are to satisfy (3.7) then each of them must commute with the $(\bar{k}_{2n}^{(f)})^2$, a property which must be shared by $L_{2n+1}^{(1)}$ itself. That is

$$[L_{2n+1}^{(1)}, (\bar{k}_{2n}^{(f)})^2]=0 \qquad (3.25)$$

for all $f=1, 2\cdots n$. To verify this identity we observe that

$$\tfrac{1}{2}\{[A, [B, L]_+]+[B, [A, L]_+]\}=[AB, L]$$

holds for arbitrary operators L, A, and B, given that A and B commute. The identification

$$A=\prod_{f=1}^{n}(x+\bar{k}_{2n}^{(f)}),$$

$$B=\prod_{f=1}^{n}(x-\bar{k}_{2n}^{(f)}),$$

$$L=L_{2n+1}^{(1)}$$

followed by substitution in (3.17), and use of the defining relations for the symmetric functions $\bar{K}_{2n}^{(r)}$, $K_{2n+1}^{(s)}$ gives

$$[\prod_{f=1}^{n}(x^2-(\bar{k}_{2n}^{(f)})^2), L_{2n+1}^{(1)}]$$

$$=2x\sum_{r,s=0}^{n}x^{2n-r-s}[\bar{K}_{2n}^{(r)}, K_{2n+1}^{(s)}]\{(-1)^r-(-1)^s\}. \qquad (3.26)$$

[17] The $(\bar{k}_{2n}^{(f)})^{-1}$ are well defined since we are able to show (by induction) that $(\bar{k}_{2n}^{(f)})^2>0$ for all $f=1, 2\cdots n$ [cf. Eq. (3.10) and Sec. 6, "The Eigenvalues"].

Now we have shown [Eq. (2.18)]

$$[\mathcal{L}_{2n}^{(f)}, \mathcal{L}_{2n+1}^{(g)}]=[\mathcal{L}_{2n}^{(g)}, \mathcal{L}_{2n+1}^{(f)}]$$

for all f and g. Multiplication of this expression on both sides by $\bar{a}_{2n}^{r,r-f}\cdot a_{2n+1}^{s,s-g}$ (recall 3.16) and summation over f and g gives

$$[\bar{K}_{2n}^{(r)}, K_{2n+1}^{(s)}]=[\bar{K}_{2n}^{(s)}, K_{2n+1}^{(r)}]. \qquad (3.27)$$

Substitution of this result in the right-hand side of (3.26) followed by interchange of the dummy indices r and s, shows that

$$[\prod_{f=1}^{n}(x^2-(\bar{k}_{2n}^{(f)})^2), L_{2n+1}^{(1)}]=0.$$

This identity when combined with the ordering of the $\bar{k}_{2n}^{(f)}$ implies (3.25) as required.

The result which we have just established enables us to deduce that a typical matrix element of $L_{2n+1}^{(1)}$, in a representation in which the $\bar{k}_{2n}^{(f)}$ are diagonal, is nonzero only if the eigenvalue of any given $\bar{k}_{2n}^{(f)}$ is unaltered or changes sign. This permits $L_{2n+1}^{(1)}$ to have at most 2^n nonzero matrix elements between one given function and any other. Equivalently (3.25) may be said to imply that $L_{2n+1}^{(1)}$ can be expressed as a sum of 2^n terms each of which commutes with a particular set of the $\bar{k}_{2n}^{(f)}$ and anticommutes with the rest. A typical term takes the form

$$(1/2^n)(\prod_{f=1}^{n}\bar{k}_{2n}^{(f)})^{-1}[\bar{k}_{2n}^{(1)}, [\bar{k}_{2n}^{(2)},\ldots[\bar{k}_{2n}^{(n)}, L_{2n+1}^{(1)}]\ldots]], \qquad (3.28)$$

in which the square brackets may represent anticommutation or commutation, making a total of 2^n such expressions. Now from the identities

$$[A, [A, L]_+]=[A, [A, L]]_+=[A^2, L],$$

which hold for arbitrary operators A and L, it is easily shown (recalling that the $\bar{k}_{2n}^{(f)}$ mutually commute) that a term described by (3.28) commutes with each $\bar{k}_{2n}^{(f)}$ appearing in an anticommutation bracket and anticommutes with each $\bar{k}_{2n}^{(f)}$ appearing in a commutation bracket. This illustrates the decomposition of the ladder operator $L_{2n+1}^{(1)}$, though we have yet to express it in the form described by (3.24). To do this we must first show that of the possible 2^n terms which can be constructed out of (3.28) only n are nonzero, a result which is a consequence of the identity

$$[[L_{2n+1}^{(1)}, \bar{k}_{2n}^{(f)}], \bar{k}_{2n}^{(g)}]=0, \qquad (3.29)$$

which holds for all $f\neq g=1, 2\cdots n$. This is established as follows:

Taking the commutator of (3.14) with $\bar{K}_{2n}^{(1)}$ gives

$$[\bar{K}_{2n}^{(1)}, [\bar{K}_{2n}^{(f)}, L_{2n+1}^{(1)}]_+]=2[\bar{K}_{2n}^{(1)}, K_{2n+1}^{(f+1)}], \qquad (3.30)$$

for all $f=0, 1\cdots n$.

Use of Eq. (3.27) with $r=(f+1)$ and $s=1$, followed by replacement of $K_{2n+1}^{(1)}$ by $L_{2n+1}^{(1)}$ [cf. (2.15), (3.8), and (3.11)] converts Eq. (3.30) to

$$[\bar{K}_{2n}^{(1)}, [\bar{K}_{2n}^{(f)}, L_{2n+1}^{(1)}]_+]=2[\bar{K}_{2n}^{(f+1)}, L_{2n+1}^{(1)}],$$

for all $f=0, 1\cdots n$. This set of $(n+1)$ equations may be replaced by a single identity in x, namely,

$$[\bar{K}_{2n}^{(1)}, [\prod_{f=1}^{n} (x+\bar{k}_{2n}^{(f)}), L_{2n+1}^{(1)}]_+] =2x[\prod_{f=1}^{n} (x+\bar{k}_{2n}^{(f)}), L_{2n+1}^{(1)}]. \quad (3.31)$$

We now substitute the eigenvalues $\bar{\lambda}_{2n}^{(f)}$ for the $\bar{k}_{2n}^{(f)}$ and matrix elements for $L_{2n+1}^{(1)}$. We may suppose that in a typical term m eigenvalues change sign and that the rest remain unaltered. The former may be denoted, without loss of generality, by $\bar{\lambda}_{2n}^{(f)}$: $f=1, 2\cdots m$. If the corresponding matrix element of $L_{2n+1}^{(1)}$ is *non*zero, it follows from (3.31) that

$$\sum_{g=1}^{m} \bar{\lambda}_{2n}^{(g)}\{\prod_{f=1}^{m} (x+\bar{\lambda}_{2n}^{(f)})+\prod_{f=1}^{m} (x-\bar{\lambda}_{2n}^{(f)})\} =x\{\prod_{f=1}^{m} (x+\bar{\lambda}_{2n}^{(f)})-\prod_{f=1}^{m} (x-\bar{\lambda}_{2n}^{(f)})\}. \quad (3.32)$$

Equation (3.32), being an identity in x, must hold in particular for $x=\bar{\lambda}_{2n}^{(h)}$. This implies

$$(\sum_{g(\neq h)=1}^{m} \bar{\lambda}_{2n}^{(g)})\cdot\prod_{f=1}^{m} (\bar{\lambda}_{2n}^{(h)}+\bar{\lambda}_{2n}^{(f)})=0,$$

an expression which further reduces, in view of the strict ordering (3.10) imposed on the $\bar{k}_{2n}^{(f)}$, to give

$$\sum_{g(\neq h)=1}^{m} \bar{\lambda}_{2n}^{(g)}=0.$$

Since this must hold for all h and the $\bar{\lambda}_{2n}^{(g)}$ are strictly positive (see Ref. 17), it follows that either $m=0$, or $m=1$. Consequently at most one eigenvalue can change sign in a typical nonzero matrix element of $L_{2n+1}^{(1)}$. This result implies (3.29). The case $m=0$, which corresponds to none of the eigenvalues changing sign, is excluded by (3.5). We may conclude that there are at most n nonzero terms described by (3.28) and further application of (3.29) enables one to show that these may be expressed in form given in (3.24). This establishes the decomposition of $L_{2n+1}^{(1)}$. By application of (3.25) and (3.29) one may verify that the components $l_{2n+1}^{(f)}$: $f=1, 2\cdots n$, satisfy (3.7), and since $L_{2n+1}^{(1)}$, $\bar{k}_{2n}^{(f)}$: $f=1, 2\cdots n$ are self-adjoint, they must also be self-adjoint. Finally, uniqueness follows as a direct consequence of the explicit expressions given for these operators. The theorem is thus proved.

Before continuing we wish to make a couple of remarks concerning the latter part of the above proof. First, the substitution for eigenvalues and matrix elements in (3.31), which resulted in a departure from the purely algebraic techniques previously used, should be viewed as a convenient rather than as an essential step. Thus when $n=3$, elementary, but tedious, manipulation of (3.31) reduces it to

$$[[[L_7^{(1)}, \bar{k}_6^{(1)}], \bar{k}_6^{(2)}], \bar{k}_6^{(3)}](\bar{k}_6^{(1)}+\bar{k}_6^{(2)}) \times(\bar{k}_6^{(2)}+\bar{k}_6^{(3)})(\bar{k}_6^{(3)}+\bar{k}_6^{(1)})=0,$$

a result which, combined with (3.5) and the strict ordering (3.10), implies (3.29). It would seem that a generalization of this procedure is possible, though not easy.

Second, we remark that Louck (L, pp. 132–137) makes use of an equation similar to (3.31) (with $x=-\frac{1}{2}$) to derive an equivalent result. We feel, however, that the analysis given fails to be entirely satisfactory in that it assumes an equation which is the equivalent of (3.32) (L, p.132, second equation on page) to be an identity in $\bar{\lambda}_{2n}^{(f)}$. This argument does not exclude the possibility of nonzero matrix elements of $L_{2n+1}^{(1)}$ involving more than one change of eigenvalue for specific values of $\bar{\lambda}_{2n}^{(f)}$. Moreover, since the equation referred to in L involves only two of the n Casimir invariants, it cannot, in itself, provide enough information for a selection rule in the n eigenvalues and hence Louck's analysis, though it obtains the correct result, is incomplete.

The Nonzero Matrix Elements of $L_{2n+1}^{(1)}$

In the following we derive an expression for the $l_{2n+1}^{(f)}$ from which the matrix elements of $L_{2n+1}^{(1)}$ can be readily obtained. It may be regarded as a direct generalization of I, Eq. (4.5).

Substitution for the $l_{2n+1}^{(f)}$ in (3.17) and use of (3.7) gives

$$\sum_{g=1}^{n} l_{2n+1}^{(g)} \prod_{f(\neq g)=1}^{n} (x+\bar{k}_{2n}^{(f)}) =\{\prod_{f=1}^{n} (x+k_{2n+1}^{(f)})-\prod_{f=1}^{n} (x+\bar{k}_{2n}^{(f)})\}. \quad (3.33)$$

This identity in x represents n equations which may be solved for the n unknowns $l_{2n+1}^{(f)}$: $f=1, 2\cdots n$. However, as the $k_{2n+1}^{(f)}$ do not commute with the $\bar{k}_{2n}^{(f)}$, the final expressions are rather clumsy and not very useful. We therefore derive instead relations for the $(l_{2n+1}^{(f)})^2$ as this difficulty is then avoided.

We return to a result established in Sec. 2. Here we showed, in Eq. (2.19), that the square of the ladder operator $L_{2n+1}^{(1)}$ commutes with all the Casimir factors. In the present context this implies

$$[(L_{2n+1}^{(1)})^2, \bar{k}_{2n}^{(f)}]=0, \quad (3.34)$$

for all $f=1, 2\cdots n$. Substitution in (3.24) and use of (3.29) then shows, after some manipulation, that

$$[l_{2n+1}^{(f)}, l_{2n+1}^{(g)}]_+=0, \quad (3.35)$$

for all $f\neq g=1, 2\cdots n$, and that

$$[(l_{2n+1}^{(f)})^2, \bar{k}_{2n}^{(g)}]=0, \qquad (3.36)$$

for all f, $g=1, 2\cdots n$. These equations show that if we square the summation appearing in (3.6), cross terms cancel, and $(L_{2n+1}^{(1)})^2$ may be expressed as the sum of squares $(l_{2n+1}^{(f)})^2$: $f=1, 2\cdots n$, none of which have any nonzero matrix elements between different eigenvalues of the Casimir factors. Intuitively this is precisely the result we should expect (3.34) to imply. We may use it [and (3.7)] to show that Eq. (3.33) may, after some reduction, be expressed in the form

$$\prod_{f=1}^{n} (x^2-(k_{2n+1}^{(f)})^2)-\prod_{f=1}^{n} (x^2-(\bar{k}_{2n}^{(f)})^2)$$

$$=-\sum_{g=1}^{n} (l_{2n+1}^{(g)})^2 \prod_{f(\neq g)=1}^{n} (x^2-(\bar{k}_{2n}^{(f)}))^2. \qquad (3.37)$$

It is easy to check that (3.36) implies that each of the terms appearing in (3.37) commutes. This result enables the "partial-fraction" decomposition [Eq. (B3)] of the left-hand side to be effected. Recalling (3.10) and equating powers in x, we obtain

$$(l_{2n+1}^{(f)})^2$$

$$=\{(k_{2n+1}^{(f)})^2-(\bar{k}_{2n}^{(f)})^2\} \prod_{g(\neq f)=1}^{n} \left\{\frac{(k_{2n+1}^{(g)})^2-(\bar{k}_{2n}^{(f)})^2}{(\bar{k}_{2n}^{(g)})^2-(\bar{k}_{2n}^{(f)})^2}\right\}, \qquad (3.38)$$

for all $f=1, 2\cdots n$. This exhibits the required expression for the $(l_{2n+1}^{(f)})^2$. We also obtain, from (3.33) and (3.37), the commutation relations

$$[(\bar{k}_{2n}^{(f)})^2, k_{2n+1}^{(g)}]=0 \qquad (3.39)$$

$$[\bar{k}_{2n}^{(f)}, (k_{2n+1}^{(g)})^2]=0, \qquad (3.40)$$

which hold for all f, $g=1, 2\cdots n$.

IV. THE $(2n+2)$-DIMENSIONAL PROBLEM

In the following we reapply the argument given in Sec. 3 to the $(2n+2)$-dimensional problem. The interest lies in the appearance of an extra Casimir invariant which introduces certain minor changes in the ladder operation. Naturally the same degree of detail will not be required and we do little more than simply state the main results.

As before we start from Eq. (2.20), but which we now write in the form

$$[\mathfrak{L}_{2n+1}^{(f)}, L_{2n+2}^{(1)}]_+=2\{(\mathfrak{L}_{2n+2}^{(f+1)}-\mathfrak{L}_{2n+1}^{(f+1)})$$
$$-(n-f+1)(\mathfrak{L}_{2n+2}^{(f)}-\mathfrak{L}_{2n+1}^{(f)})\}, \qquad (4.1)$$

where f takes the values $0, 1\cdots n$. At first sight this expression may seem identical with (3.1). In fact it differs slightly in that [contrast this with (3.2)]

$$\mathfrak{L}_{2n+2}^{(n+1)}\neq 0.$$

As a consequence it is no longer possible to find an operator which anticommutes with $L_{2n+2}^{(1)}$ by the method described previously, though in other respects the eigenvalue problem is unaltered. In fact we find that $L_{2n+2}^{(1)}$ decomposes to give, in addition to n self-adjoint ladder operators, a further term which commutes with all the Casimir factors. This is described in the following theorem.

Theorem 2. From the spin and angular-momentum algebras defined in Sec. 2, it is possible to construct a set of commuting self-adjoint operators $\bar{k}_{2n+1}^{(f)}$: $f=1, 2\cdots n$, such that there exists a unique decomposition of $L_{2n+2}^{(1)}$ into $(n+1)$ self-adjoint components $l_{2n+2}^{(f)}$: $f=1, 2\cdots(n+1)$, satisfying

$$L_{2n+2}^{(1)}=\sum_{f=1}^{n+1} l_{2n+2}^{(f)}, \qquad (4.2)$$

such that

$$[\bar{k}_{2n+1}^{(f)}, l_{2n+2}^{(f)}]_+=0, \qquad (4.3a)$$

for all $f=1, 2\cdots n$;

$$[\bar{k}_{2n+1}^{(f)}, l_{2n+2}^{(g)}]=0, \qquad (4.3b)$$

for all $f\neq g=1, 2\cdots n$; and

$$[\bar{k}_{2n+1}^{(f)}, l_{2n+2}^{(n+1)}]=0. \qquad (4.3c)$$

for all $f=1, 2\cdots n$.

Proof. As before the proof divides into two parts, starting with the construction of the $\bar{k}_{2n+1}^{(f)}$.

The Decomposition of the Casimir Factors

By analogy with (3.8) and (3.9) we define

$$\bar{K}_{2n+1}^{(f)}=\sum_{g=0}^{f} \bar{a}_{2n+1}^{f,f-g}\mathfrak{L}_{2n+1}^{(g)},$$

for all $f=0, 1, 2\cdots n$, with

$$\prod_{f=1}^{n} (x+\bar{k}_{2n+1}^{(f)})=\sum_{f=0}^{n} x^{n-f}\bar{K}_{2n+1}^{(f)}.$$

The coefficients $\bar{a}_{2n+1}^{f,g}$ turn out to be real numbers relating the symmetric functions $\bar{K}_{2n+1}^{(f)}$ to the Casimir factors $\mathfrak{L}_{2n+1}^{(f)}$. As before, we introduce the ordering

$$(\bar{k}_{2n+1}^{(1)})^2>(\bar{k}_{2n+1}^{(2)})^2\cdots>(\bar{k}_{2n+1}^{(n)})^2>0, \qquad (4.4)$$

which ensures a unique correspondence between the $\bar{k}_{2n+1}^{(f)}$ and the $\mathfrak{L}_{2n+1}^{(f)}$.

In a similar manner we define

$$K_{2n+2}^{(f)}=\sum_{g=0}^{f} a_{2n+2}^{f,f-g}\mathfrak{L}_{2n+2}^{(g)},$$

for all $f=0, 1\cdots(n+1)$, where

$$\prod_{f=1}^{n+1} (x+k_{2n+2}^{(f)})=\sum_{f=0}^{n+1} x^{n+1-f}K_{2n+2}^{(f)},$$

and

$$(k_{2n+2}^{(1)})^2 \geqslant (k_{2n+2}^{(2)})^2 \geqslant \cdots \geqslant (k_{2n+2}^{(n+1)})^2 \geqslant 0. \quad (4.5)$$

It should be noted that because of the existence of an extra Casimir factor, namely $\mathfrak{L}_{2n+2}^{(n+1)}$, an extra operator $k_{2n+2}^{(n+1)}$ and an extra symmetric function $K_{2n+2}^{(n+1)}$ appear in the decomposition.

We next choose the coefficients $\bar{a}_{2n+1}^{f,g}$, $a_{2n+2}^{f,g}$ such that (4.1) assumes the form [compare (3.14) and (3.15)]

$$[\bar{K}_{2n+1}^{(f)}, L_{2n+2}^{(1)}]_+ = 2(K_{2n+2}^{(f+1)} - \bar{K}_{2n+1}^{(f+1)}), \quad (4.6)$$

for all $f=0, 1, 2\cdots n$, and where

$$\bar{K}_{2n+1}^{(n+1)} = 0, \qquad K_{2n+2}^{(0)} = \bar{K}_{2n+1}^{(0)} = 1.$$

As before, this implies

$$\bar{a}_{2n+1}^{f,g} = a_{2n+2}^{f,g},$$

for all f, $g=1, 2\cdots n$. However we also require, essentially because $K_{2n+2}^{(n+1)}$ is nonzero, that[18]

$$a_{2n+2}^{n+1,0} = 1, \qquad a_{2n+2}^{n+1,f} = 0,$$

for all $f=1, 2\cdots n+1$.

Like (3.14), Eq. (4.6) may be reduced to a single identity in arbitrary variable, though the presence of the extra term, namely, $K_{2n+2}^{(n+1)}$, demands a slight modification. Thus we find that

$$[\prod_{f=1}^{n} (x+\bar{k}_{2n+1}^{(f)}), L_{2n+2}^{(1)}]_+$$
$$= 2\{\prod_{f=1}^{n+1} (x+k_{2n+2}^{(f)}) - x \prod_{f=1}^{n} (x+\bar{k}_{2n+1}^{(f)})\}, \quad (4.7)$$

a result which should be contrasted with (3.17). To derive such an expression from (4.1), we proceed as before letting $P_f(x): f=0, 1\cdots n$, denote a set of linearly independent polynomials in x, chosen such that

$$\sum_{f=0}^{n} P_{n-f}(x)\mathfrak{L}_{2n+1}^{(f)} = \prod_{f=1}^{n} (x+\bar{k}_{2n+1}^{(f)}).$$

Substitution of this expression in (4.7) followed by comparison with (4.1) gives the recurrence relation

$$xP_{n-f-1}(x) = P_{n-f}(x) - (n-f)P_{n-f-1}(x),$$

and the identity (cf. Ref. 18)

$$P_0(x) = 1, \quad (4.8)$$

from which it may easily be shown that

$$P_f(x) = \Gamma(x+f+1)/\Gamma(x+1), \quad (4.9)$$

for all $f=0, 1\cdots n$. This result completely determines the coefficients $\bar{a}_{2n+1}^{f,g}$, which take the form

$$\bar{a}_{2n+1}^{f,g} = \sum_{j_1<j_2\cdots<j_g}^{n-(f-g)} \prod_{m=1}^{g} j_m, \quad (4.10)$$

for all $f \geqslant g=1, 2\cdots n$, and

$$\bar{a}_{2n+1}^{f,0} = 1, \quad (4.11)$$

for all $f=1, 2\cdots n$.

Comparison of the polynomials $\bar{P}_f(x)$ and $P_f(x)$ shows that

$$\bar{k}_n^{(f)} = (k_n^{(f)} + \tfrac{1}{2}). \quad (4.12)$$

This result is found to be of considerable importance in the interpretation of $L_{2n+1}^{(1)}$ as a ladder operator. A corresponding identity

$$\bar{\lambda}_n^{(f)} = (\lambda_n^{(f)} + \tfrac{1}{2}), \quad (4.13)$$

which holds for all $f=1, 2\cdots[n/2]$, relates the eigenvalues $\bar{\lambda}_n^{(f)}$ of $\bar{k}_n^{(f)}$ and $\lambda_n^{(f)}$ of $k_n^{(f)}$.

The Decomposition of the Ladder Operator

By precisely the same argument as given in Sec. 3, we may show that

$$[L_{2n+2}^{(1)}, (\bar{k}_{2n+1}^{(f)})^2] = 0,$$

for all $f=1, 2\cdots n$, and that

$$[[L_{2n+2}^{(1)}, \bar{k}_{2n+1}^{(f)}], \bar{k}_{2n+1}^{(g)}] = 0,$$

for all $f \neq g=1, 2\cdots n$. It follows, as before, that we may express the $l_{2n+2}^{(f)}$ in the form

$$l_{2n+2}^{(f)} = \tfrac{1}{2}(\bar{k}_{2n+1}^{(f)})^{-1}[\bar{k}_{2n+1}^{(f)}, L_{2n+2}^{(1)}],$$

for all $f=1, 2\cdots n$, and

$$l_{2n+2}^{(n+2)} = \{\prod_{f=1}^{n} \tfrac{1}{2}(\bar{k}_{2n+1}^{(f)})^{-1}\}$$
$$\times[\bar{k}_{2n+1}^{(1)}, [\bar{k}_{2n+1}^{(2)}, \cdots [\bar{k}_{2n+1}^{(n)}, L_{2n+2}^{(1)}]_+\cdots]_+]_+.$$

This last term, which has no analog in the $(2n+1)$-dimensional problem, represents the commuting part of the ladder operator. The full details of the decomposition of $L_{2n+2}^{(1)}$, as expressed by (4.2) and (4.3a–c), may be readily verified from the above equations, as is also the fact that the $l_{2n+2}^{(f)}$ are self-adjoint. Uniqueness follows as before. The theorem is proved.

The Nonzero Matrix Elements of $L_{2n+2}^{(1)}$

We derive below an expression, which is the analog of (3.38), for the $l_{2n+2}^{(f)}$. It enables us to determine the nonzero matrix elements of $L_{2n+2}^{(1)}$.

[18] The choice $a_{2n+2}^{n+1,0}=1$ made here is arbitrary in that any real number will do. It is equivalent to the choice [Eq. (3.19)] made in normalizing the polynomials $\bar{P}_f(x)$. This is shown by (4.8).

Substitution of the $l_{2n+2}^{(f)}$ in (4.7) and use of (4.3) gives

$$x\sum_{g=1}^{n}\{l_{2n+2}^{(g)}\prod_{f(\neq g)=1}^{n}(x+\bar{k}_{2n+1}^{(f)})\}$$
$$+l_{2n+2}^{(n+1)}\prod_{f=1}^{n}(x+\bar{k}_{2n+1}^{(f)})$$
$$=\{\prod_{f=1}^{n+1}(x+k_{2n+2}^{(f)})-x\prod_{f=1}^{n}(x+\bar{k}_{2n+1}^{(f)})\}. \quad (4.14)$$

This identity in x can be expanded to give $(n+1)$ equations which determine the $(n+1)$ unknowns, $l_{2n+2}^{(f)}: f=1, 2\cdots(n+1)$. In particular identifying the coefficients of x^0 we obtain

$$l_{2n+2}^{(n+1)}=(\prod_{f=1}^{n+1}k_{2n+2}^{(f)})(\prod_{f=1}^{n}\bar{k}_{2n+1}^{(f)})^{-1}, \quad (4.15)$$

in which the order of the brackets may be interchanged even though the two sets of operators $\bar{k}_{2n+1}^{(f)}$ and $k_{2n+2}^{(f)}$ do not mutually commute. The remaining solutions are less interesting and for the same reasons as given in Sec. 3, we consider instead the corresponding expressions for the squares $(l_{2n+2}^{(f)})^2$: $f=1\cdots(n+1)$ which are simpler.

Let us first observe that (2.19) implies

$$[(L_{2n+2}^{(1)})^2, \bar{k}_{2n+1}^{(f)}]=0,$$

for all $f=1, 2\cdots n$. This may be used to show [cf. (3.35)] that

$$[l_{2n+2}^{(f)}, l_{2n+2}^{(g)}]_+=0,$$

for all $f\neq g=1, 2\cdots(n+1)$, and that

$$[(l_{2n+2}^{(f)})^2, k_{2n+1}^{(g)}]=0,$$

for all $f=1, 2\cdots(n+1)$: $g=1, 2\cdots n$. Combined with (4.14) this result implies that

$$\prod_{f=1}^{n+1}(x^2-(k_{2n+2}^{(f)})^2)-x^2\prod_{f=1}^{n}(x^2-(\bar{k}_{2n+1}^{(f)})^2)$$
$$=-x^2\{\sum_{g=1}^{n}(l_{2n+2}^{(g)})^2\cdot\prod_{f(\neq g)=1}^{n}(x^2-(\bar{k}_{2n+1}^{(f)})^2)\}$$
$$-(l_{2n+2}^{(n+1)})^2\prod_{f=1}^{n}(x^2-(\bar{k}_{2n+1}^{(f)})^2). \quad (4.16)$$

Equating coefficients of x^0 in (4.16) we obtain

$$(l_{2n+2}^{(n+1)})^2=\{\prod_{f=1}^{n}(\bar{k}_{2n+1}^{(f)})^{-2}\}\{\prod_{f=1}^{n+1}(k_{2n+2}^{(f)})^2\}. \quad (4.17)$$

This, when substituted back in (4.16), gives with a little manipulation

$$-\sum_{g=1}^{n}(l_{2n+2}^{(g)})^2\prod_{f(\neq g)=1}^{n}(x^2-(\bar{k}_{2n+1}^{(f)})^2)$$
$$=\prod_{f=1}^{n}(x^2-(k_{2n+2}^{(f)})^2)-\prod_{f=1}^{n}(x^2-(\bar{k}_{2n+2}^{(f)})^2)$$
$$-(-1)^n x^{2n-2}(K_{2n+2}^{(n+1)})^2\{\prod_{f=1}^{n}(x^{-2}-(k_{2n+2}^{(f)})^{-2})$$
$$-\prod_{f=1}^{n}(x^{-2}-(\bar{k}_{2n+1}^{(f)})^{-2})\}. \quad (4.18)$$

Each of the terms appearing in (4.18) commutes with any other and therefore conventional techniques of commutative algebra apply. In particular, decomposition of the right-hand side as a partial fraction [Eq. (B3)] gives, after some reduction

$$(l_{2n+2}^{(f)})^2$$
$$=\{(k_{2n+2}^{(f)})^2-(\bar{k}_{2n+1}^{(f)})^2\}\{1-(\bar{k}_{2n+1}^{(f)})^{-2}(k_{2n+2}^{(n+1)})^2\}$$
$$\times\prod_{g(\neq f)=1}^{n}\left\{\frac{(k_{2n+2}^{(g)})^2-(\bar{k}_{2n+1}^{(f)})^2}{(\bar{k}_{2n+1}^{(g)})^2-(\bar{k}_{2n+1}^{(f)})^2}\right\}. \quad (4.19)$$

for all $f=1, 2\cdots n$. This and (4.17) determine the $l_{2n+2}^{(f)}$. We may also show, as before, that

$$[k_{2n+2}^{(f)}, (\bar{k}_{2n+1}^{(g)})^2]=[(k_{2n+2}^{(f)})^2, \bar{k}_{2n+1}^{(g)}]=0, \quad (4.20)$$

for all $f=1, 2\cdots(n+1)$ and all $g=1, 2\cdots n$.

V. THE LADDER OPERATION

In this section we show how the anticommutation relations, Eqs. (3.7a) and (4.3a), may be used to demonstrate the ladder property which has been ascribed to $L_{n+1}^{(1)}$. This result enables us to assign eigenvalues to the $k_n^{(f)}$ and thus to derive the irreducible representations for the angular-momentum algebra.

The Casimir Invariants

We saw in I, Sec. 4 that a determination of the precise relationship between the Casimir invariant and its factor was an essential step in the interpretation of L_{n+1} as a ladder operator. This fact still holds true and consequently we must now try to derive the (now more complicated) expressions relating the Casimir invariants (2.5) to the Casimir factors (2.12). As before it is convenient to tackle the odd and even dimensional problems separately. We start with the former.

It is perhaps unfortunate that the direct approach proves unfruitful. Thus, an attempt to generalize (2.11), which is itself readily verified by explicit multiplication of the Casimir factors, rapidly produces indescribable confusion. With regard to an alternative solution, attention should be given to the important invariance property of the $(k_{2n+1}^{(f)})^2$ implied by (3.40). This shows, in view of (2.15), (3.8), (3.11), and (3.12), that these operators commute with the $L_{2n+1}^{(g)}$. As the latter are to have a ladder property, we might anticipate that the $(k_{2n+1}^{(f)})^2$, which taken together contain all the elements of $0(2n+1)$, will be diagonal in the representation we seek to construct. This makes them excellent candidates for the Casimir invariants, a qualification we now examine.

Though the $(k_{2n+1}^{(f)})^2$ are not explicitly defined in terms of the angular-momentum components, the symmetric functions over them are. These may be expressed (compare 3.12) in the following manner:

$$\prod_{f=1}^{n}((k_{2n+1}^{(f)})^2+x^2)=\sum_{f=0}^{n}x^{2n-2f}K_{2n+1}^{2f}. \qquad (5.1)$$

W now show the K_{2n+1}^{2f} to be linear combinations of Casimir invariants and derive the coefficients relating them.

From (3.16), (3.18), and (3.20) we obtain

$$\prod_{f=1}^{n}(k_{2n+1}^{(f)}+x)=\sum_{f=1}^{n}\frac{\Gamma(x+n-f+\frac{1}{2})}{\Gamma(x+\frac{1}{2})}\mathfrak{L}_{2n+1}^{(f)},$$

which, combined with the obvious identity

$$\prod_{f=1}^{n}((k_{2n+1}^{(f)})^2-x^2)=\prod_{f=1}^{n}(k_{2n+1}^{(f)}+x)\prod_{f=1}^{n}(k_{2n+1}^{(f)}-x),$$

implies that

$$\prod_{f=1}^{n}((k_{2n+1}^{(f)})^2-x^2)$$

$$=\sum_{f,g=0}^{n}\frac{\Gamma(x+n-f+\frac{1}{2})\Gamma(-x+n-g+\frac{1}{2})}{\Gamma(x+\frac{1}{2})\Gamma(-x+\frac{1}{2})}\mathfrak{L}_{2n+1}^{(f)}\mathfrak{L}_{2n+1}^{(g)}. \qquad (5.2)$$

In Appendix A, the double summation appearing in (5.2) is shown to commute with each of the spin operators σ_{jk}: j, $k=1, 2\cdots(2n+2)$. Now the only invariants which can be formed from this latter algebra are the identity and the generalized[19] skew-symmetric component $\sigma_{1,2\cdots(2n+2)}$. Since the index $(2n+2)$ is not present in the above summation, the only invariant which can appear in it is the identity. Consequently, in multiplying out the Casimir factors, all the terms involving the spin operators must cancel, leaving only those obtained by directly squaring expressions of the form

$$\sigma_{j_1j_2\cdots j_{2f}}\mathfrak{L}_{j_1j_2\cdots j_{2f}},$$

for all $f=1, 2\cdots n$.

This argument allows us to conclude that the following result holds:

$$\sum_{f,g=0}^{n}\frac{\Gamma(x+n-f+\frac{1}{2})\Gamma(-x+n-g+\frac{1}{2})}{\Gamma(x+\frac{1}{2})\Gamma(-x+\frac{1}{2})}\mathfrak{L}_{2n+1}^{(f)}\mathfrak{L}_{2n+1}^{(g)}$$

$$=\sum_{f=0}^{n}\frac{\Gamma(x+n-f+\frac{1}{2})\Gamma(-x+n-f+\frac{1}{2})}{\Gamma(x+\frac{1}{2})\Gamma(-x+\frac{1}{2})}\mathfrak{L}_{2n+1}^{2f}. \qquad (5.3)$$

This is the required relationship between the Casimir invariants and their factors. Moreover the absence of the spin operators from this expression quickly enables us to verify the invariance properties which have been attributed to the $\mathfrak{L}_{2n+1}^{2f}$. Thus recalling [Eq. (2.14)] that the Casimir factors commute, (5.3) is seen to imply

$$\left[\mathfrak{L}_{2n+1}^{(1)},\sum_{f=0}^{n}\frac{\Gamma(x+n-f+\frac{1}{2})\Gamma(-x+n-f+\frac{1}{2})}{\Gamma(x+\frac{1}{2})\Gamma(-x+\frac{1}{2})}\mathfrak{L}_{2n+1}^{2f}\right]=0$$

Equating coefficients of x^{2n-2f} and of the σ_{jk} (the latter can only appear in $\mathfrak{L}_{2n+1}^{(1)}$) we obtain

$$[\mathfrak{L}_{jk}, \mathfrak{L}_{2n+1}^{2f}]=0,$$

for all j, $k=1, 2\cdots(2n+1)$, for all $f=1, 2\cdots n$ and all positive integer values of n. This proves the invariance of $\mathfrak{L}_{2n+1}^{2f}$. A valuable feature of this derivation is that it clearly shows the natural manner in which the Casimir invariants acquire their particular form.

We can now relate the Casimir factors to the symmetric functions K_{2n+1}^{2f}. Indeed from Eqs. (5.1), (5.2), and (5.3), we immediately obtain

$$K_{2n+1}^{2f}=\sum_{g=0}^{f}B_{2n+1}^{f,f-g}\mathfrak{L}_{2n+1}^{2g},$$

for all $f=1, 2\cdots n$, where $B_{2n+1}^{f,f-g}$ is the coefficient of $(-x^2)^{n-f}$ in the expression

$$\Gamma(x+n-g+\tfrac{1}{2})\Gamma(-x+n-g+\tfrac{1}{2})/\Gamma(x+\tfrac{1}{2})\Gamma(-x+\tfrac{1}{2}).$$

From this it may be shown that

$$B_{2n+1}^{f,g}=\sum_{j_1<j_2\cdots<j_g}^{n-(f-g)}\prod_{m=1}^{g}(j_m-\tfrac{1}{2})^2, \qquad (5.4)$$

for all $f\geqslant g=1, 2\cdots n$, and

$$B_{2n+1}^{f,0}=1,$$

for all $f=1, 2\cdots n$. Like the $a_{2n+1}^{f,g}$, these coefficients do not appear to have a simple closed form.

A similar analysis may be applied to the $(2n+2)$-dimensional problem. Thus we define

$$\prod_{f=1}^{n+1}((k_{2n+2}^{(f)})^2+x^2)=\sum_{f=0}^{n+1}x^{2n-2f}K_{2n+2}^{2f},$$

and attempt to relate these symmetric functions to the Casimir invariants $\mathfrak{L}_{2n+2}^{2f}$. It is easily shown that

$$\prod_{f=1}^{n+1}((k_{2n+2}^{(f)})^2-x^2)$$

$$=\sum_{f,g=0}^{n+1}\frac{\Gamma(x+n-f+1)\Gamma(-x+n-g+1)}{\Gamma(x)\Gamma(-x)}\mathfrak{L}_{2n+2}^{(f)}\mathfrak{L}_{2n+2}^{(g)}, \qquad (5.5)$$

[19] H. Boerner, *Representations of Groups* (North-Holland Publ. Co., Amsterdam, 1963), Chap. VIII, pp. 265–287; R. Brauer and H. Weyl, Am. J. Math. **57**, 425 (1935).

which reduces, as in the previous discussion, to give

$$\prod_{f=1}^{n+1}((k_{2n+2}{}^{(f)})^2-x^2)$$

$$=\sum_{f=0}^{n+1}\frac{\Gamma(x+n-f+1)\Gamma(-x+n-f+1)}{\Gamma(x)\Gamma(-x)}\mathfrak{L}_{2n+2}{}^{2f}. \quad (5.6)$$

This result may be used to establish the invariance properties of the $\mathfrak{L}_{2n+1}{}^{2f}$. We may also derive from it an explicit relationship between these operators and the symmetric functions defined above. This is

$$K_{2n+2}{}^{2f}=\sum_{g=0}^{f}B_{2n+2}{}^{f,f-g}\mathfrak{L}_{2n+2}{}^{2g},$$

for all $f=1, 2\cdots(n+1)$; where

$$B_{2n+2}{}^{f,g}=\sum_{j_1<j_2\cdots<j_g}^{n-(f-g)}\prod_{m=1}^{g}(j_m)^2, \quad (5.7)$$

for all $f\geqslant g=1, 2\cdots n$;

$$B_{2n+2}{}^{f,0}=1, \qquad B_{2n+2}{}^{n+1,f}=0,$$

for all $f=1, 2\cdots(n+1)$.

The Dirac Operators

Before proceeding with an interpretation of the anticommutation relations, we digress slightly in order to discuss an invariance property [Eqs. (3.39) and (4.20)] of the $(\bar{k}_n{}^{(f)})^2$. This may be thought of as being complementary to the invariance of the $(k_{n+1}{}^{(f)})^2$, a result which proved to be invaluable in deriving the explicit form of the Casimir invariants. We start, as before, with the $(2n+1)$-dimensional problem.

Equation (3.39) may be shown to imply that the $(\bar{k}_{2n}{}^{(f)})^2$ commute with the ladder operators $L_{2n+1}{}^{(g)}$. However, unlike the $(k_{2n+1}{}^{(f)})^2$ which also share this property, the $(\bar{k}_{2n}{}^{(f)})^2$ are constructed from the angular-momentum components taken from the $0(2n)$ subalgebra of $0(2n+1)$. They cannot therefore be Casimir invariants. On the other hand, recalling the discussion of the Dirac operator given in Sec. 2, we see that they could be a suitable combination of ladder invariants. In verifying this conjecture, we must select the symmetric functions over the $(\bar{k}_{2n}{}^{(f)})^2$, as the latter are not related explicitly to the angular-momentum components. We therefore define in analogy with (5.1)

$$\prod_{f=1}^{n}((\bar{k}_{2n}{}^{(f)})^2+x^2)=\sum_{f=0}^{n}x^{2n-2f}\bar{K}_{2n}{}^{2f},$$

and seek to express the $\bar{K}_{2n}{}^{2f}$ in the form

$$\bar{K}_{2n}{}^{2f}=\sum_{g=0}^{f}\bar{B}_{2n}{}^{f,f-g}J_{2n}{}^{2f}, \quad (5.8)$$

where the coefficients $\bar{B}_{2n}{}^{f,g}$ are real, or possibly complex, numbers. In order to obtain such an expression, we must first attempt to write the $J_{2n}{}^{(f)}$ in terms of the $\bar{k}_{2n}{}^{(f)}$. The technique for doing this derives from the identity

$$\sum_{f=g}^{n}\frac{\Gamma(x+1)(2n-2g)!(\frac{1}{2})^{2(f-g)}}{\Gamma(x-n+f+1)(2n-2f)!(f-g)!}$$

$$=\Gamma(x+n-g+\tfrac{1}{2})/\Gamma(x+\tfrac{1}{2}), \quad (5.9)$$

the proof of which is given in Appendix B. Combined with (2.24) this gives

$$\sum_{f=0}^{n}\frac{\Gamma(x+1)}{\Gamma(x-n+f+1)}J_{2n}{}^{(f)}=\sum_{g=0}^{n}\frac{\Gamma(x+n-g+\frac{1}{2})}{\Gamma(x+\frac{1}{2})}\mathfrak{L}_{2n}{}^{(g)},$$

which, in view of (3.18) and (3.20), reduces to

$$\prod_{f=1}^{n}(\bar{k}_{2n}{}^{(f)}+x)=\sum_{f=0}^{n}\frac{\Gamma(x+1)}{\Gamma(x-n+f+1)}J_{2n}{}^{(f)},$$

as required. From this result we obtain after a little rearrangement

$$\prod_{f=1}^{n}((\bar{k}_{2n}{}^{(f)})^2-x^2)$$

$$=\sum_{f,g=0}^{n}(-1)^{f+g}\frac{\Gamma(x+n-g)\Gamma(-x+n-f)}{\Gamma(x)\Gamma(-x)}J_{2n}{}^{(f)}J_{2n}{}^{(g)}. \quad (5.10)$$

The double summation in the above expression should be compared to that appearing in (5.5). Apart from the trivial replacement of n by $(n+1)$, these differ only in that the term $(-1)^{f+g}$ is present in the former. Moreover a simplification of (5.10) parallel to that described by the successive equations (5.5) and (5.6) would show that the $(\bar{k}_{2n}{}^{(f)})^2$ are expressible, as we have already suggested, in terms of the ladder invariants. Add to these observations the easily proven fact that the J_{jk} satisfy the same algebraic relations as the $\mathfrak{L}_{jk}$, and we are immediately led to consider the identity

$$\sum_{f,g=0}^{n}(-1)^{f+g}\frac{\Gamma(x+n-g)\Gamma(-x+n-f)}{\Gamma(x)\Gamma(-x)}J_{2n}{}^{(f)}J_{2n}{}^{(g)}$$

$$=\sum_{f=0}^{n}\frac{\Gamma(x+n-f)\Gamma(-x+n-f)}{\Gamma(x)\Gamma(-x)}J_{2n}{}^{2f}, \quad (5.11)$$

which, apart from the factor of $(-1)^{(f+g)}$, is the exact analog of the expression relating (5.5) and (5.6). At first sight it may seem that an appeal to previous arguments would be sufficient to establish (5.11); but the presence of the extra factor frustrates such an attempt. Indeed the existence of this apparent discrepancy is extremely puzzling. Its explanation lies in the fact that whereas the spin and angular-momentum algebras are independent, the spin and Dirac algebras are not—elements of one being present in the other. This can implement certain changes in the algebraic identities when the J_{jk} replace the $\mathfrak{L}_{jk}$. In the latter part of Appendix A, we show that this is precisely what introduces the extra factor in (5.11), which, incidentally, we are thus able to verify.

Combining (5.10) and (5.11) we obtain

$$\prod_{f=1}^{n}\left((\bar{k}_{2n}^{(f)})^2-x^2\right)=\sum_{f=0}^{n}\frac{\Gamma(x+n-f)\Gamma(-x+n-f)}{\Gamma(x)\Gamma(-x)}J_{2n}^{2f}, \tag{5.12}$$

which confirms (5.8). Furthermore it is easily shown that

$$\bar{B}_{2n}^{f,g}=B_{2n}^{f,g},$$

for all f, $g=1, 2\cdots n$; where $B_{2n}^{f,g}$ is defined by (5.7).

Similar considerations apply to the $(2n+2)$-dimensional problem. The main result can be expressed as

$$\bar{K}_{2n+1}^{2f}=\sum_{g=0}^{f}B_{2n+1}^{f,f-g}J_{2n+1}^{2f},$$

for all $f=1, 2,\cdots, n$, where $B_{2n+1}^{f,g}$ is given by (5.4) and where the $\bar{K}_{2n+1}^{2f}$ denote the symmetric functions taken over the $\bar{k}_{2n+1}^{(f)}$.

The Ladder

We are now ready to show that $L_{n+1}^{(1)}$ behaves as a ladder operator towards the Casimir invariants of the $0(n)$ subalgebra of $0(n+1)$. In view of the results obtained in the first part of this section we may select the $(k_n^{(f)})^2$ to represent this set of Casimir invariants. A typical eigenvalue equation then takes the form

$$(k_n^{(f)})^2\psi(\lambda_n^{(f)})=(\lambda_n^{(f)})^2\psi(\lambda_n^{(f)}),$$

where the eigenvalues $\lambda_n^{(f)}$ are arbitrarily chosen to be nonnegative.

Following the analysis given in I, Sec. 4, we decompose the eigenspace of $(k_n^{(f)})^2$ into pair of "mutually adjoint" subspaces defined by

$$\begin{aligned}k_n^{(f)}\psi(\lambda_n^{(f)};a)&=\lambda_n^{(f)}\psi(\lambda_n^{(f)};a)\\ k_n^{(f)}\psi(\lambda_n^{(f)};b)&=-\lambda_n^{(f)}\psi(\lambda_n^{(f)};b),\end{aligned} \tag{5.13}$$

a factorization made possible by the use of the spin algebra.

We demand of (5.13) two prerequisites. The first is that this decomposition should not affect the requirement[20] that we seek a representation in which the Casimir invariants of $0(n+1)$ are diagonal. This condition is easily shown to be satisfied as follows. We recall that $(k_n^{(f)})^2$ are defined solely in terms of the angular-momentum components. Thus any eigenfunction of them, including those given in (5.13) above, is automatically an eigenfunction of the Casimir invariants. Consequently their diagonalization is unaffected by (5.13). This argument would fail if the $(k_n^{(f)})^2$ were to be replaced by the $(\bar{k}_n^{(f)})^2$ which involve both the spin and the angular-momentum algebras, Eq. (5.12). A similar breakdown was described in II for the Dirac-Kepler problem.

The second requirement we make of (5.13) is that, given that $\psi(\lambda_n^{(f)})$ exists, *both* $\psi(\lambda_n^{(f)};a)$ and $\psi(\lambda_n^{(f)};b)$ must also exist. In Sec. 6 this is shown to hold by explicit construction of the subspaces. It is a result needed if the ladder is to extend in *both* directions.

Now the action of $L_{n+1}^{(1)}$ on $k_n^{(f)}$ was shown to reduce to a set of anticommutation relations. A given equation of this type can be expressed in the form

$$[l_{n+1}^{(f)},\bar{k}_n^{(f)}]_+=0,$$

and combined with (4.12) and (5.13) implies

$$\begin{aligned}l_{n+1}^{(f)}\psi(\lambda_n^{(f)};a)&\sim\psi(\lambda_n^{(f)}+1;b)\\ l_{n+1}^{(f)}\psi(\lambda_n^{(f)}+1;b)&\sim\psi(\lambda_n^{(f)};a).\end{aligned} \tag{5.14}$$

This result demonstrates that $l_{n+1}^{(f)}$ both steps the eigenvalues of $(k_n^{(f)})^2$ and interchanges the subspaces described as eigenfunctions of the factorized operator $k_n^{(f)}$. This is essentially the conclusion reached in I, Sec. 4 for the orbital problem. In the present system (5.14) implies not just one, but a total of $[n/2]$ such ladders, though this generalization does not introduce any added difficulties. This result will now be used to determine the allowed eigenvalues of the Casimir invariants and thus to construct the irreducible representations.

VI. THE REPRESENTATION

A construction of the unitary irreducible representations belonging to the Lie algebra of $0(n)$ in the group chain $0(n)\supset 0(n-1)\supset\cdots\supset 0(2)$ is given below. The simplest part of this problem is the determination of the allowed values (eigenvalues) of the diagonal elements. These are, in the present case, the Casimir invariants of $0(n)$, and the Casimir invariants selected from the subalgebras of $0(n-1)$, $0(n-2)\cdots 0(2)$. Extensive details will not be necessary as the basic arguments are quite familiar.

The Eigenvalues

The fact that $l_{2n+1}^{(f)}$ is self-adjoint, itself a consequence of the unitarity of the representation, enables us to argue that the expression for the matrix elements of $(l_{2n+1}^{(f)})^2$, obtained from (3.38) by substitution of eigenvalues, must always be nonnegative. Combining this result with the action of $l_{2n+1}^{(f)}$ as a ladder operator for $(k_{2n}^{(f)})^2$ shows that the eigenvalues $(\lambda_{2n}^{(f)})^2$ of the latter are bounded above and below. In view of the ordering (3.10) and (3.13) imposed on the $\bar{k}_{2n}^{(f)}$ and the $k_{2n+1}^{(f)}$, it follows that these bounds take the form

$$(p_{2n}^{(f)}+\lambda_{2n+1}^{(f)})\geqslant\bar{\lambda}_{2n}^{(f)}\geqslant(\lambda_{2n+1}^{(f+1)}+q_{2n}^{(f)}+1), \tag{6.1}$$

for all $f=1, 2\cdots n$, and all integer n, $n>1$. The $p_{2n}^{(f)}$ and the $q_{2n}^{(f)}$ are nonnegative integers and the two sets

[20] This requirement is an immediate consequence of Schur's lemma and the fact that we are seeking *irreducible* representations. For an elementary discussion of this point, see S. S. Schweber, *Relativistic Quantum Field Theory* (Harper and Row, New York, 1964), Chap. I, p. 25.

of eigenvalues $\lambda_n^{(f)}$, $\bar{\lambda}_n^{(f)}$ are related through (4.13). Similarly in the $(2n+2)$-dimensional problem we obtain

$$(p_{2n+1}^{(f)}+\lambda_{2n+2}^{(f)})\geqslant\bar{\lambda}_{2n+1}^{(f)}\geqslant(\lambda_{2n+2}^{(f+1)}+q_{2n+1}^{(f)}+1), \tag{6.2}$$

for all $f=1, 2\cdots n$ and integer n, $n>1$. The above pattern of eigenvalues exemplifies the Weyl branching law developed originally for $U(n)$. A detailed study of this result and its application to calculation of matrix elements of tensor operators has been given by Biedenharn and Louck.[21]

From the analysis of the three-dimensional problem given in I, Sec. 4 it follows that $\lambda_2^{(1)}$ must take half-integer or integer values which may be arbitrarily chosen nonnegative. Applied to (6.1) and (6.2) this shows that all the $\lambda_n^{(f)}$ are half-integer or integer. They may also be chosen to be nonnegative since it is only their squares which determine the eigenvalues of the Casimir invariants.

It should be noted that (4.13), (6.1), and (6.2) imply that in any given set of eigenvalues $\lambda_n^{(f)}$: $f=1, 2\cdots[n/2]$ each of the members are distinct and nonzero. This enables us to justify the strict ordering described in (3.10) and (4.4). A similar strict ordering may replace that given in (3.13) and (4.5), except that in contrast to $\bar{k}_n^{[n/2]}$; $k_n^{[n/2]}$ may have a zero eigenvalue.

The bounds on the eigenvalues, as expressed by (6.1) and (6.2), can be given a simpler form in terms of the eigenvalues $j_n^{(f)}$ of the $\mathcal{J}_n^{(f)}$, operators defined by (3.22) and (3.23). This is

$$(p_n^{(f)}+j_{n+1}^{(f)})\geqslant j_n^{(f)}\geqslant(j_{n+1}^{(f+1)}+q_n^{(f)}),$$

where

$$j_n^{(f)}=\lambda_n^{(f)}-([n/2]-f)$$

for all $f=1, 2\cdots[n/2]$ and all integer n, $n>1$.

We may use these results and the expressions relating the symmetric functions over the $(k_n^{(f)})^2$ to the Casimir invariants, to derive eigenvalues for the latter. It is unfortunate that in general these do not have a simple form owing to the complexity of the coefficients involved, namely, the $B_n^{f,g}$. Exceptions are the quadratic invariant $\mathcal{L}_n^2$, and, when n is even, the invariant $\mathcal{L}_n^n$, the eigenvalues of which are given below

$$\mathcal{L}_n^2\psi(j_n^{(f)})=\{\sum_{f=1}^{[n/2]}j_{2n}^{(f)}(j_n^{(f)}+n-2f)\}\psi(j_n^{(f)})$$

$$\mathcal{L}_n^n\psi(j_n^{(f)})=\{\prod_{f=1}^{n/2}(j_n^{(f)}+n/2-f)^2\}\psi(j_n^{(f)}). \tag{6.3}$$

A comparison with the results given in L, pp. 100–109, can now be made. Complete agreement is reached except in that in L, the set of eigenvalues $\lambda_{2m}^{(m)}$: $m=1, 2\cdots n$, may assume not only positive, but also negative values. This discrepancy arises because the factorization procedure fails to distinguish between the negative range in the eigenvalues introduced by the spin algebra and that inherent in the angular-momentum algebra itself. In other words there may be Casimir invariants which can be factorized without recourse to the spin algebra. It is easy to see which these are, namely the set $\mathcal{L}_{2m}^{2m}$: $m=1, 2\cdots n$. They have as factors the skew-symmetric components $\mathcal{L}_{12,\cdots,2m}$: $m=1, 2\cdots n$, which are themselves invariants of the appropriate subalgebras. The set of eigenvalues $\lambda_{2m}^{(f)}:f\leqslant m=1, 2\cdots n$, is not therefore complete, though we can make it so by including in it a specification of the sign of the invariants $\mathcal{L}_{12,\cdots,2m}$. The simplest procedure for doing this is to allow for each m, one of the $\lambda_{2m}^{(f)}$: $f=1\cdots m$, to have both positive and negative values. In order to obtain agreement with L, we choose this to be $\lambda_{2m}^{(m)}$ and write

$$\mathcal{L}_{12,\cdots,2m}\psi(\lambda_{2m}^{(f)})=\{\prod_{f=1}^{m}\lambda_{2m}^{(f)}\}\psi(\lambda_{2m}^{(f)}). \tag{6.4}$$

Now this increase in the effective number of eigenvalues enlarges the representation and introduces a possible ambiguity in the nature of the stepping process. This can sometimes further result in the augmented representation not being irreducible.[22] though this does not happen in the present case. It is a difficulty which cannot be resolved by appealing to the methods of the factorization procedure. However, the following argument based on the identity

$$[\mathcal{L}_{2n}^2, [\mathcal{L}_{12,\cdots,2n}, \mathcal{L}_{2n\ 2n+1}]]=[\mathcal{L}_{2n\ 2n+1}, \mathcal{L}_{12,\cdots,2n}]_+ \tag{6.5}$$

developed by Louck (L, p. 120) provides a definitive solution.

The selection rule in question can be described by

$$(\lambda_{2n}^{(r)})'=\lambda_{2n}^{(r)}\pm\delta_{rn}:$$
$$\delta_{rn}=0, \quad r\neq n$$
$$=1, \quad r=n, \tag{6.6}$$

where the prime denotes the new eigenvalue. Because $\lambda_{2n}^{(n)}$ has been assigned a negative as well as a positive spectrum, a further possibility, namely,

$$(\lambda_{2n}^{(r)})'=\lambda_{2n}^{(r)} \quad \text{for} \quad r=1, 2\cdots(n-1)$$
$$=-(\lambda_{2n}^{(n)}\pm1) \quad \text{for} \quad r=n \tag{6.7}$$

is available. Now substitution for the appropriate

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[21] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publ. Inc., Princeton, N.J., 1931); Chap. V, Sec. 18, p. 390; L. C. Biedenharn and J. D. Louck, Commun. Math. Phys. **8**, 89 (1968), and Ref. therein.

[22] An example of this can be obtained from a study of the representations of the algebra generated by L_x, L_y, L_z, where

$$[L_x, L_y]_+=L_z \quad \text{cyclically},$$

using a factorization procedure involving the "spin" algebra defined by

$$[\sigma_x, \sigma_y]=0, \quad \sigma_x\sigma_y=\sigma_z \quad \text{cyclically}.$$

eigenvalues [cf. (6.3) and (6.4)] in (6.5) shows that in either case we must have

$$\{(\lambda_{2n}^{(n)})'+\lambda_{2n}^{(n)}\}\{[(\lambda_{2n}^{(n)})'-\lambda_{2n}^{(n)}]^2-1\}\prod_{r=1}^{n-1}\lambda_{2n}^{(r)}=0.$$

Recalling that $\lambda_{2n}^{(r)}>0$ for all $r=1, 2\cdots(n-1)$, this implies that either $\lambda_{2n}^{(n)}=0$, in which case (6.6) and (6.7) are indistinguishable or $[(\lambda_{2n}^{(n)})'-\lambda_{2n}^{(n)}]^2=1$ and (6.7) is excluded. A similar argument shows that matrix elements in which one of the other eigenvalues is stepped can be nonzero only if there is no change in the sign of $\lambda_{2n}^{(n)}$. Finally we remark that it should be clear that $|\lambda_{2n}^{(n)}|$ replaces $\lambda_{2n}^{(n)}$ as a nonnegative lower bound to $\bar{\lambda}_{2n-1}^{(n-1)}$ in (6.2). This completes the description of the eigenvalues.

The Eigenfunctions

Whereas the two sets of operators represented by the $(k_n^{(f)})^2$ and the $(k_{n+1}^{(f)})^2$ mutually commute and may thus be simultaneously diagonalized, this does not hold true for their factors. Nevertheless the eigenfunctions of the $k_{n+1}^{(f)}$ may be expressed as a linear combination of the eigenfunctions of the $k_n^{(f)}$ and in the following we obtain the coefficients in this decomposition. The result, which is of intrinsic interest in itself, is used to derive matrix elements of the angular-momentum components in this representation and moreover determines the Clebsch–Gordon coefficients for the addition of spin and angular momentum. It is a direct generalization of I, (4.15). We start with the $(2n+1)$-dimensional problem.

Let $\bar{\lambda}_{2n}$ denote the n-tuple $(\bar{\lambda}_{2n}^{(1)}, \bar{\lambda}_{2n}^{(2)},\cdots,\bar{\lambda}_{2n}^{(n)})$ and $\mathbf{i}$ the n-tuple $(i_1, i_2\cdots i_n)$. The elements of the former set will be recognized as the eigenvalues of the $\bar{k}_{2n}^{(f)}$ [cf. Eqs. (4.12) and (4.13)], whereas the latter are integers which may each take either of the values 0 or 1. Then we define the eigenfunctions $\psi(\bar{\lambda}_{2n};\mathbf{i})$ of the $\bar{k}_{2n}^{(f)}$ by the following set of equations

$$\bar{k}_{2n}^{(f)}\psi(\bar{\lambda}_{2n};\mathbf{i})=(-1)^{i_f}\bar{\lambda}_{2n}^{(f)}\psi(\bar{\lambda}_{2n};\mathbf{i}), \qquad (6.8)$$

where $f=1, 2\cdots n$. A comparison of this with (5.13) shows that when $i_f=0$, the corresponding eigenfunction lies in the "a" subspace (with respect to $k_{2n}^{(f)}$) and has eigenvalue $\lambda_{2n}^{(f)}$, whereas when $i_f=1$, it lies in the "b" subspace with eigenvalue $-(\lambda_{2n}^{(f)}+1)$. These functions are further taken to be eigenfunctions of all the operators diagonal in the present representation, though to avoid cumbersome notation this is not written explicitly. For a given set of eigenvalues, there are clearly 2^n such functions, and they represent the decomposition of the eigenspace of the Casimir invariants $(k_{2n}^{(f)})^2$ into its 2^n components.

Similarly we define the eigenfunctions $\Psi(\lambda_{2n+1};\mathbf{j})$ of the $k_{2n+1}^{(f)}$ by the equations

$$k_{2n+1}^{(f)}\Psi(\lambda_{2n+1};\mathbf{j})=(-1)^{j_f}\lambda_{2n+1}^{(f)}\Psi(\lambda_{2n+1};\mathbf{j}) \qquad (6.9)$$

which hold for $f=1, 2\cdots n$. It should be noted that these equations differ slightly from (6.8) in the omission of the bar over the eigenvalues and the operators. The need for this distinction soon becomes clear. Like the $\psi(\bar{\lambda}_{2n};\mathbf{i})$ these functions are also taken to be eigenfunctions of all the operators diagonal in the representation, and for a given set of eigenvalues form the 2^n components of the eigenspace of the $(k_{2n+1}^{(f)})^2$.

The $k_{2n+1}^{(f)}$ can be constructed from a combination of the $k_{2n}^{(f)}$ and the ladder operators $l_{2n+1}^{(f)}$, a result which is essentially a consequence of (2.15). Moreover the $l_{2n+1}^{(f)}$ interconvert amongst themselves the functions described by (6.8). Thus the $\Psi(\lambda_{2n+1};\mathbf{j})$ can be expressed in terms of the $\psi(\bar{\lambda}_{2n};\mathbf{i})$. (This argument is a direct generalization of that given in I, Sec. 4 which provides a simple example of the construction.) Explicitly we have the linear relation

$$\Psi(\lambda_{2n+1};\mathbf{j})=\sum_{i_1,i_2,\cdots,i_n=0}^{1}B(\mathbf{j},\mathbf{i})\psi(\bar{\lambda}_{2n};\mathbf{i}), \qquad (6.10)$$

where the $B(\mathbf{j},\mathbf{i})$ form the elements of a $2^n\times2^n$ matrix. These coefficients depend on the eigenvalues $\bar{\lambda}_{2n}^{(f)}$, $\lambda_{2n+1}^{(f)}$; but explicit reference to this is only made where necessary. They are evaluated as follows.

We apply both sides of the operator equation (3.33) to the function $\Psi(\lambda_{2n+1};\mathbf{j})$ and select out the component of $\psi(\bar{\lambda}_{2n};\mathbf{i})$. The resulting identity in x can be decomposed by a partial-fraction procedure (as used in Sec. 3) to give the n equations

$$\langle\psi(\bar{\lambda}_{2n};\mathbf{i}), l_{2n+1}^{(g)}\psi(\bar{\lambda}_{2n};\mathbf{i}+\mathbf{1}_g)\rangle\cdot B(\mathbf{j},\mathbf{i}+\mathbf{1}_g)=B(\mathbf{j},\mathbf{i})\{\prod_{f=1}^{n}((-1)^{j_f}\lambda_{2n+1}^{(f)}-(-1)^{i_g}\bar{\lambda}_{2n}^{(g)})/\prod_{f(\neq g)=1}^{n}((-1)^{i_f}\bar{\lambda}_{2n}^{(f)}-(-1)^{i_g}\bar{\lambda}_{2n}^{(g)})\}, \qquad (6.11)$$

which hold for $g=1, 2\cdots n$. In these $\mathbf{1}_g$ denotes the unit n-tuple with the gth component one. It is used to indicate that i_g has been increased by 1 (mod 2). Dirac notation has been used for the matrix elements of the $l_{2n+1}^{(f)}$: the inner product $\langle\cdot,\cdot\rangle$ being conjugate linear in the first term and linear in the second. The moduli of these matrix elements can be obtained from (3.38), though their phase is as yet indeterminate. This is because it depends on the precise form of the $\psi(\bar{\lambda}_{2n},\mathbf{i})$ though it cannot depend on the $\Psi(\lambda_{2n+1},\mathbf{j})$ and *a fortiori* must be independent of the i_f: $f=1, 2\cdots n$. On the other hand, the relative phase of the $B(\mathbf{j},\mathbf{i})$ for fixed $\mathbf{j}$ can always be chosen to be independent of the i_f. Recalling the ordering of the eigenvalues implied by (6.1), which determines the signs of the various terms, appearing in (6.11), we thus obtain

$$\frac{B(\mathbf{j},\mathbf{i}+\mathbf{1}_g)}{B(\mathbf{j},\mathbf{i})}=\Big(\prod_{f=1}^{g}(-1)^{i_g}\Big)\Bigg(\prod_{f=1}^{n}\left\{\frac{(-1)^{j_f}\lambda_{2n+1}^{(f)}+(-1)^{i_g+1}\bar{\lambda}_{2n}^{(g)}}{(-1)^{j_f}\lambda_{2n+1}^{(f)}+(-1)^{i_g}\bar{\lambda}_{2n}^{(g)}}\right\}\Bigg/\prod_{f(\neq g)=1}^{n}\left\{\frac{(-1)^{i_f}\bar{\lambda}_{2n}^{(f)}+(-1)^{i_g+1}\bar{\lambda}_{2n}^{(g)}}{(-1)^{i_f}\bar{\lambda}_{2n}^{(f)}+(-1)^{i_g}\bar{\lambda}_{2n}^{(g)}}\right\}\Bigg)^{1/2},$$

which holds for all $g=1, 2\cdots n$. It is not difficult to see that this is satisfied if the $B(\mathbf{j},\mathbf{i})$ take the form

$$B(\mathbf{j},\mathbf{i})=\Big(\prod_{g=f}^{n}\prod_{f=1}^{n}(-1)^{j_f i_g}\Big)\Big(\prod_{g=1}^{n}\prod_{f=1}^{n}\{(-1)^{j_f}\lambda_{2n+1}^{(f)}+(-1)^{i_g}\bar{\lambda}_{2n}^{(g)}\}\Big/\prod_{g=f}^{n}\prod_{f=1}^{n}\{(-1)^{j_f}\lambda_{2n+1}^{(f)}+(-1)^{j_g}\lambda_{2n+1}^{(g)}\}$$

$$\times\prod_{g=f+1}^{n}\prod_{f=1}^{n}\{(-1)^{i_f}\bar{\lambda}_{2n}^{(f)}+(-1)^{i_g}\bar{\lambda}_{2n}^{(g)}\}\Big)^{1/2}.\quad(6.12)$$

This expression has two arbitrary features. The first is that we have chosen to normalize the $B(\mathbf{j},\mathbf{i})$ in the sense that

$$\sum_{i_1,i_2,\cdots,i_n=0}^{1}|B(\mathbf{j},\mathbf{i})|^2=1,\quad(6.13)$$

holds for all values of j_f. [The proof of (6.13) is given in Appendix C]. The second is contained in the choice of phase of $B(\mathbf{j},\mathbf{i})$, which is determined only to within multiplication of each column and each row by a complex number of modulus one. With respect to the above choice of phase it can be shown, by substitution back in (6.11), that the phases of the matrix elements of the $l_{2n+1}^{(g)}$, are given by

$$\langle\psi(\bar{\lambda}_{2n};\mathbf{i}),\,l_{2n+1}^{(g)}\psi(\bar{\lambda}_{2n};\mathbf{i}+\mathbf{1}_g)\rangle=\{\prod_{f=1}^{g-1}(-1)^{i_f}\}\,|\langle\cdot,\cdot\rangle|.\quad(6.14)$$

The transformation described by (6.10) maps orthogonal vectors into orthogonal vectors. This fact combined with (6.13) implies that $B(\mathbf{j},\mathbf{i})$ is a unitary matrix and thus its inverse can be easily obtained. In handling this matrix some care has to be taken in multiplying out the terms under the square root sign. Difficulties can be avoided by always choosing the individual terms to be positive.

We now derive the corresponding result for the $(2n+2)$-dimensional problem. The argument used is similar; but has the additional complication introduced by the increase in number of Casimir invariants.

We define, by analogy with (6.8) and (6.9), the functions $\psi(\bar{\lambda}_{2n+1};\mathbf{i})$ and $\Psi(\lambda_{2n+2};\mathbf{j})$. Thus

$$\bar{k}_{2n+1}^{(f)}\psi(\bar{\lambda}_{2n+1};\mathbf{i})=(-1)^{i_f}\bar{\lambda}_{2n+1}^{(f)}\psi(\bar{\lambda}_{2n+1};\mathbf{i}),\quad(6.15)$$

for all $f=1, 2\cdots n$, and

$$k_{2n+2}^{(f)}\Psi(\lambda_{2n+2};\mathbf{j})=(-1)^{j_f}\lambda_{2n+2}^{(f)}\Psi(\lambda_{2n+2};\mathbf{j}),\quad(6.16)$$

for all $f=1, 2\cdots(n+1)$. It should be noted that $\mathbf{i}$ denotes the n-tuple $(i_1, i_2\cdots i_n)$, where $\mathbf{j}$ denotes the $(n+1)$-tuple $(j_1, j_2\cdots j_{n+1})$. A similar remark applies to the eigenvalues $\bar{\lambda}_{2n+1}$ and λ_{2n+2}. Apart from a notational change, which is convenient for our present purposes, these functions are exactly the same as those defined by Eqs. (6.8) and (6.9). This correspondence can be made explicit by the identity

$$\psi(\bar{\lambda}_n;\mathbf{i})\equiv\Psi(\lambda_n+\mathbf{i};\mathbf{i}),\quad(6.17)$$

which holds for all integer n, $n>1$.

We now seek to express the $\Psi(\lambda_{2n+2};\mathbf{j})$ as a linear combination of the $\psi(\bar{\lambda}_{2n+1};\,\mathbf{i})$. Unfortunately as it stands this is impossible, since (6.16) defines 2^{n+1} orthogonal functions whereas (6.15) defines only 2^n. Not unnaturally this difficulty can be resolved by appealing to the spin algebra which, we recall, has in $(2n+2)$-dimensional space the single nontrivial invariant $\sigma_{1,2\cdots(2n+2)}$. This has eigenvalues ±1, and we denote by α_{n+1} and β_{n+1} the corresponding eigenfunctions. These may be represented as spinors with two independent components (see Sec. 7). We choose their relative phases such that $\sigma_{2n+2,2n+3}$, which is a unitary operator anticommuting with $\sigma_{1,2\cdots(2n+2)}$, satisfies

$$\sigma_{2n+2,2n+3}\,|\,\alpha_{n+1}\rangle=|\,\beta_{n+1}\rangle.\quad(6.18)$$

We may use these functions to effect the required increase in the dimension of the space of the $\psi(\bar{\lambda}_{2n+1};\mathbf{i})$. The new functions take the form

$$\psi(\bar{\lambda}_{2n+1};\mathbf{i},0)=\alpha_{n+1}\psi(\bar{\lambda}_{2n+1};\mathbf{i}),$$
$$\psi(\bar{\lambda}_{2n+1};\mathbf{i},1)=\beta_{n+1}\psi(\bar{\lambda}_{2n+1};\mathbf{i}),\quad(6.19)$$

and for a fixed set of eigenvalues form a space of dimension 2^{n+1} as required. This above notation can be put in a more succinct and logical form by extending the n-tuple $(i_1, i_2\cdots i_n)$ to include the term i_{n+1}, choosing $i_{n+1}=0$ to denote multiplication by α_{n+1} and $i_{n+1}=1$ to denote multiplication by β_{n+1}. These augmented functions may be used to construct the $\Psi(\lambda_{2n+2},\mathbf{j})$, as shown in the following.

By analogy with (6.10) we write

$$\Psi(\lambda_{2n+2};\mathbf{j})=\sum_{i_1,i_2,\cdots,i_{n+1}=0}^{1}C(\mathbf{j},\mathbf{i})\psi(\bar{\lambda}_{2n+1};\mathbf{i}),\quad(6.20)$$

where the coefficients $C(\mathbf{j},\mathbf{i})$ form a $2^{n+1}\times2^{n+1}$ matrix, whose explicit form is determined below.

From (3.22) we have that

$$\sigma_{1,2,\cdots,(2n+2)}\mathcal{L}_{1,2,\cdots,(2n+2)}=\mathcal{L}_{2n+2}^{(n+1)}$$
$$=\prod_{f=1}^{n}k_{2n+2}^{(f)}$$

an identity which implies

$$\mathfrak{L}_{2n+2}{}^{(n+1)}\Psi(\boldsymbol{\lambda}_{2n+2};\,\mathbf{j})=\{\prod_{f=1}^{n+1}(-1)^{j_f}\lambda_{2n+2}{}^{(f)}\}\Psi(\boldsymbol{\lambda}_{2n+2};\,\mathbf{j}).$$

On the other hand, from (6.4) we obtain

$$\mathfrak{L}_{1,2,\cdots,(2n+2)}\Psi(\boldsymbol{\lambda}_{2n+2};\,\mathbf{j})=(\prod_{f=1}^{n+1}\lambda_{2n+2}{}^{(f)})\Psi(\boldsymbol{\lambda}_{2n+2};\,\mathbf{j}).$$

A comparison of these expressions shows that the eigenvalue assigned to $\sigma_{1,2\cdots(2n+2)}$ is just the product

$$\prod_{f=1}^{n+1}(-1)^{j_f}.$$

Applying the invariance property of this operator to (6.20) it follows that

$$C(\mathbf{j},\mathbf{i})=0, \tag{6.21}$$

When

$$(\sum_{f=1}^{n+1}j_f)+i_{n+1}=1\,(\mathrm{mod}\ 2).$$

The remaining (nonzero) coefficients may be obtained by the method used to derive the $B(\mathbf{j},\mathbf{i})$. In this, Eqs. (4.14), (4.15), and (4.19) replace (3.33) and (3.38) in the determination of the ratio $C(\mathbf{j},\,\mathbf{i}+\mathbf{1}_g)/C(\mathbf{j},\mathbf{i})$. The final result takes the form

$$C(\mathbf{j},\mathbf{i})=(\prod_{g=f}^{n}\prod_{f=1}^{n}(-1)^{j_fi_g})\{\prod_{f=1}^{n+1}\prod_{g=1}^{n}((-1)^{j_f}\lambda_{2n+2}{}^{(f)}+(-1)^{i_g}\bar{\lambda}_{2n+1}{}^{(g)})/\prod_{g=f+1}^{n+1}\prod_{f=1}^{n+1}((-1)^{j_f}\lambda_{2n+2}{}^{(f)}+(-1)^{j_g}\lambda_{2n+2}{}^{(g)})$$
$$\times\prod_{g=f}^{n}\prod_{f=1}^{n}((-1)^{i_f}\bar{\lambda}_{2n+1}{}^{(f)}+(-1)^{i_g}\bar{\lambda}_{2n+1}{}^{(g)})\}^{1/2}, \tag{6.22}$$

when

$$(\sum_{f=1}^{n+1}j_f)+i_{n+1}=0\ (\mathrm{mod}\ 2).$$

Like the $B(\mathbf{j},\mathbf{i})$, these coefficients have been normalized and thus form the elements of an unitary matrix; but one which is now of dimension 2^{n+1}. They differ from the $B(\mathbf{j},\,\mathbf{i})$ in the nature of the summations involved. Like the $B(\mathbf{j},\,\mathbf{i})$, their phases are determined only to within multiplication of each column and of each row by an arbitrary complex number of modulus one. With respect to this choice it may be shown that the phases of the matrix elements of the $l_{2n+2}{}^{(f)}$ are given by

$$\langle\psi(\bar{\lambda}_{2n+1};\,\mathbf{i}),l_{2n+2}{}^{(g)}\psi(\bar{\lambda}_{2n+1};\,\mathbf{i}+\mathbf{1}_g)\rangle$$
$$=(\prod_{f=1}^{g-1}(-1)^{i_f})\,|\,\langle\cdot,\cdot\rangle\,|\,. \tag{6.23}$$

This completes the description of the eigenfunctions.

The Matrix Elements

We derive in the following the matrix elements for the basic angular momentum operators $\mathfrak{L}_{jk}$ in this representation. In doing so we may restrict our attention to the $\mathfrak{L}_{mm+1}$: $m=1,2\cdots n$, as the matrix elements of the remaining components may be determined from the commutation relation

$$\mathfrak{L}_{jk}=(i)^{k-j-1}[\mathfrak{L}_{j\,j+1},[\mathfrak{L}_{j+1\,j+2},\cdots[\mathfrak{L}_{k-2\,k-1},\mathfrak{L}_{k-1\,k}]\cdots]]$$

which holds for all $j<k-1$.

The method we use is based on the identity

$$L_{n+1}{}^{(1)}=-[L_n{}^{(1)},M_n]_++M_n, \tag{6.24}$$

where

$$M_n=\sigma_{n\,n+1}\mathfrak{L}_{n\,n+1}, \tag{6.25}$$

which may be easily verified. This expression is effectively much simpler than appears at first sight, because we may evaluate its matrix elements with respect to the eigenfunctions of the Casimir factors in such a fashion that the anticommutator makes no contribution. This is made possible by the fact that the operators $L_n{}^{(1)}$ and M_n act independently. Thus the former, as we have seen, steps the Casimir invariants of $0(n-1)$. Whereas the latter, because it commutes with all the elements of this subalgebra, steps only the Casimir invariants of $0(n)$. We start with the $(2n+1)$-dimensional problem.

We consider first the matrix elements of M_{2n} corresponding to the single change of eigenvalue $\lambda_{2n}{}^{(g)}\rightarrow\lambda_{2n}{}^{(g)}+1$. In this we make use of (6.24) (with n replaced by $2n$) and, so as to reduce the contribution from the anticommutator, we restrict to eigenfunctions of the $\bar{k}_{2n-1}{}^{(f)}$ in a fixed subspace. Recalling that $L_{2n}{}^{(1)}$ can be expressed as a sum of ladder operators $l_{2n}{}^{(f)}$: $f=1,2\cdots n$, which interchange these subspaces (Theorem II), we obtain

$$\langle\psi(\boldsymbol{\lambda}_{2n}+\mathbf{1}_g,\,\bar{\boldsymbol{\lambda}}_{2n-1};\,\mathbf{i}+\mathbf{1}_n),\,L_{2n+1}{}^{(1)}\psi(\boldsymbol{\lambda}_{2n},\,\bar{\boldsymbol{\lambda}}_{2n-1};\,\mathbf{i})\rangle$$
$$=\langle\psi(\boldsymbol{\lambda}_{2n}+\mathbf{1}_g,\,\bar{\boldsymbol{\lambda}}_{2n-1};\,\mathbf{i}+\mathbf{1}_n),\,\{-[l_{2n}{}^{(n)},M_{2n}]_++M_{2n}\}$$
$$\times\psi(\boldsymbol{\lambda}_{2n},\,\bar{\boldsymbol{\lambda}}_{2n-1};\,\mathbf{i})\rangle, \tag{6.26}$$

where $l_{2n}{}^{(n)}$ is the commuting part of $L_{2n}{}^{(1)}$.

$$\psi(\boldsymbol{\lambda}_{2n}+\mathbf{1}_g,\,\bar{\boldsymbol{\lambda}}_{2n-1};\,\mathbf{i}+\mathbf{1}_n)\quad\text{and}\quad\psi(\boldsymbol{\lambda}_{2n},\,\bar{\boldsymbol{\lambda}}_{2n-1};\,\mathbf{i})$$

are eigenfunctions of the $\bar{k}_{2n-1}{}^{(f)}$ as defined by (6.15), in which we have made explicit reference to the eigenvalues $(\lambda_{2n}{}^{(f)})^2$ of the $(k_{2n}{}^{(f)})^2$. The index i_n which is included in these functions determines which of the spinors α_n, β_n appears in them. For nonzero matrix elements of M_{2n} and $L_{2n+1}{}^{(1)}$ this index must be increased by 1 (mod 2) because both these operators contain the spin component $\sigma_{2n,2n+1}$ which interchanges α_n and

β_n. Now if the phase of this transformation is determined by (6.18) then the matrix elements of M_{2n} will be independent of the value of i_n chosen. On the other hand reference to Eq. (4.15) and the fact that the eigenvalues $(-1)^{i_n}$ of $\sigma_{1,2\cdots 2n}$ must equal the product

$$\prod_{f=1}^{n}(-1)^{j_f}$$

shows that the (diagonal) matrix elements of $l_{2n}^{(n)}$ change sign with this change in i_n. Thus by summation over i_n in (6.26) we eliminate the contribution from the anticommutator $[l_{2n}^{(n)}, M_{2n}]_+$.

The left-hand side of (6.26) may be evaluated by expanding the eigenfunctions of the $\bar{k}_{2n-1}^{(f)}$ in terms of the eigenfunctions of the $k_{2n}^{(f)}$ whose matrix elements with $L_{2n+1}^{(1)}$ are known. Thus by making use of the unitarity of $C(\mathbf{j}, \mathbf{i})$ and the fact that its entries are real, we may write (6.26) in the form

$$2\langle\psi, M_{2n}\psi\rangle=\sum_{i_n=0}^{1}\sum_{j_1',\cdots,j_n'=0}^{1}\sum_{j_1,\cdots,j_n=0}^{1}C(\boldsymbol{\lambda}_{2n}+\mathbf{1}_g; \mathbf{j}', \mathbf{i}+\mathbf{1}_n)C(\boldsymbol{\lambda}_{2n}; \mathbf{j}, \mathbf{i})\langle\Psi(\boldsymbol{\lambda}_{2n}+\mathbf{1}_g; \mathbf{j}'), L_{2n+1}^{(1)}\Psi(\boldsymbol{\lambda}_{2n}; \mathbf{j})\rangle, \quad (6.27)$$

and substitute for the matrix elements of $L_{2n+1}^{(1)}$ which, recalling (3.38), (4.12), and (6.16), are easily seen to be given by

$$\begin{aligned}\langle\Psi(\boldsymbol{\lambda}_{2n}+\mathbf{1}_g; \mathbf{j}'), L_{2n+1}^{(1)}\Psi(\boldsymbol{\lambda}_{2n}; \mathbf{j})\rangle &= \Big(\prod_{f=1}^{g-1}(-1)^{j_f}\Big)\Big(\prod_{f=1}^{n}\{(\lambda_{2n+1}^{(f)})^2-(\bar{\lambda}_{2n}^{(g)})^2\}\Big/\prod_{f(\neq g)=1}^{n}\{(-1)^{j_f}\lambda_{2n}^{(f)}+\lambda_{2n}^{(g)}+1\}\\ &\quad\times\{(-1)^{j_f}\lambda_{2n}^{(f)}-\lambda_{2n}^{(g)}\}\Big)^{1/2} \quad \text{for } \mathbf{j}'=\mathbf{j}+\mathbf{1}_g \text{ and } j_g=0 \quad (6.28)\\ &=0 \qquad \text{otherwise.}\end{aligned}$$

[It should be noted that the above expression for the matrix elements of $L_{2n+1}^{(1)}$ differs slightly in appearance from those obtained when the alternative form of the eigenfunctions $\psi(\bar{\lambda}_{2n}, \mathbf{i})$ of the $\bar{k}_{2n}^{(f)}$ defined by (6.8) is used, cf., Eq. (6.17).]

Orthogonality of the $\Psi(\boldsymbol{\lambda}_{2n}; \mathbf{j})$ reduces the $2n$-fold summation over $\mathbf{j}'$ and $\mathbf{j}$ appearing in (6.27) to an $(n-1)$-fold summation over $\mathbf{j}$ excluding j_f. It should also be noted that the phase factor appearing in (6.28) is eliminated by equivalent terms appearing under the square root sign. Furthermore it is not difficult to see that all the terms involving $\bar{\lambda}_{2n}^{(g)}$ may be taken outside this summation, which by inclusion of the summation over i_n reduces to unity. This latter simplification is a direct consequence of the fact that the $C(\mathbf{j}, \mathbf{i})$ are normalized. We thus obtain the following expression for the matrix elements of M_{2n}:

$$\begin{aligned}&\langle\psi(\boldsymbol{\lambda}_{2n}+\mathbf{1}_g, \bar{\boldsymbol{\lambda}}_{2n-1}; \mathbf{i}+\mathbf{1}_n), M_{2n}\psi(\boldsymbol{\lambda}_{2n}, \bar{\boldsymbol{\lambda}}_{2n-1}; \mathbf{i})\rangle\\ &=\tfrac{1}{2}\Big(\prod_{f=1}^{n}\{(\lambda_{2n+1}^{(f)})^2-(\bar{\lambda}_{2n}^{(g)})^2\}\prod_{f=1}^{n-1}\{(\lambda_{2n-1}^{(f)}+i_f)^2-(\bar{\lambda}_{2n}^{(f)})^2\}\Big/\prod_{f(\neq g)=1}^{n}\{(\lambda_{2n}^{(f)})^2-(\lambda_{2n}^{(g)})^2\}\{(\lambda_{2n}^{(f)})^2-(\lambda_{2n}^{(g)}+1)^2\}\Big)^{1/2},\end{aligned} \quad (6.29)$$

which holds for all $g=1, 2\cdots n$, and all possible values of $\mathbf{i}$.

The appearance of the i_f in the right-hand side of (6.29) should occasion no surprise since $(\lambda_{2n-1}^{(f)}+i_f)^2$ is just the eigenvalue of $(k_{2n-1}^{(f)})^2$ associated with $\psi(\bar{\boldsymbol{\lambda}}_{2n-1}, \mathbf{i})$. Indeed it serves to show that the matrix elements of M_{2n} are independent of the choice of subspace of eigenfunctions of the $(k_{2n-1}^{(f)})^2$. This fact combined with (6.18) and (6.25) enables us to extract from (6.29) the matrix elements of $\mathfrak{L}_{2n\ 2n+1}$. These take the form

$$\begin{aligned}&\langle(\boldsymbol{\lambda}_{2n+1}, \boldsymbol{\lambda}_{2n}+\mathbf{1}_g, \boldsymbol{\lambda}_{2n-1}), \mathfrak{L}_{2n\ 2n+1}(\boldsymbol{\lambda}_{2n+1}, \boldsymbol{\lambda}_{2n}, \boldsymbol{\lambda}_{2n-1})\rangle\\ &=\tfrac{1}{2}\Big(\prod_{f=1}^{n}\{(\lambda_{2n+1}^{(f)})^2-(\lambda_{2n}^{(g)}+\tfrac{1}{2})^2\}\prod_{f=1}^{n-1}\{(\lambda_{2n-1}^{(f)})^2-(\lambda_{2n}^{(g)}+\tfrac{1}{2})^2\}\Big/\prod_{f(\neq g)=1}^{n}\{(\lambda_{2n}^{(f)})^2-(\lambda_{2n}^{(g)})^2\}\{(\lambda_{2n}^{(f)})^2-(\lambda_{2n}^{(g)}+1)^2\}\Big)^{1/2}\\ &=\langle(\boldsymbol{\lambda}_{2n+1}, \boldsymbol{\lambda}_{2n}, \boldsymbol{\lambda}_{2n-1}), \mathfrak{L}_{2n\ 2n+1}(\boldsymbol{\lambda}_{2n+1}, \boldsymbol{\lambda}_{2n}+\mathbf{1}_g, \boldsymbol{\lambda}_{2n-1})\rangle,\end{aligned} \quad (6.30)$$

where the second of these identities follows from the self-adjointness of $\mathfrak{L}_{2n\ 2n+1}$. The expected phase factors $\exp(i\phi_g)$: $g=1, 2\cdots n$ do not appear because of the choice of phase in (6.12), though they may be included if required.

From (6.24) and the known selection rules for $L_{2n+1}^{(1)}$, it is easy to see that there are no other nonzero matrix elements of $\mathfrak{L}_{2n\ 2n+1}$. Finally we remark that by suitable identification of terms, the above result combined with ordering of eigenvalues described by (6.1), can be shown to be in exact agreement with those given by Louck (L, pp. 117–118).

We now derive the corresponding results for the $(2n+2)$-dimensional problem. The argument is similar and for this reason extensive details are unnecessary.

From (6.24) (with n replaced by $2n+1$) we obtain by analogy with (6.26)

$$\langle\psi(\boldsymbol{\lambda}_{2n+1}+\mathbf{1}_g, \bar{\boldsymbol{\lambda}}_{2n}; \mathbf{i}), L_{2n+2}^{(1)}\psi(\boldsymbol{\lambda}_{2n+1}, \bar{\boldsymbol{\lambda}}_{2n}; \mathbf{i})\rangle=\langle\psi(\boldsymbol{\lambda}_{2n+1}+\mathbf{1}_g, \bar{\boldsymbol{\lambda}}_{2n}; \mathbf{i}), M_{2n+1}\psi(\boldsymbol{\lambda}_{2n+1}, \bar{\boldsymbol{\lambda}}_{2n}; \mathbf{i})\rangle, \quad (6.31)$$

where $\psi(\boldsymbol{\lambda}_{2n+1}+\mathbf{1}_g, \bar{\boldsymbol{\lambda}}_{2n-1}; \mathbf{i})$ and $\psi(\boldsymbol{\lambda}_{2n+1}, \bar{\boldsymbol{\lambda}}_{2n}; \mathbf{i})$ are eigenfunctions of the $\bar{k}_{2n}^{(f)}$ as defined by (6.8) in which, as before, we have made explicit reference to the eigenvalues $(\lambda_{2n+1}^{(f)})^2$ of the $(k_{2n+1}^{(f)})^2$. The above expression is less complicated than (6.26) owing to the fact that $L_{2n+1}^{(1)}$ does not contain a term which commutes with all the Casimir factors.

To evaluate the left-hand side of (6.31), we proceed as before expanding the eigenfunctions of the $\bar{k}_{2n}^{(f)}$ in terms of the eigenfunctions of the $k_{2n+1}^{(f)}$ whose matrix elements with $L_{2n+2}^{(1)}$ are known. Thus from (4.12), (4.19), (6.9), and (6.12) we obtain

$$\langle\psi(\boldsymbol{\lambda}_{2n+1}+\mathbf{1}_g,\bar{\lambda}_{2n};\mathbf{i}),M_{2n+1}\psi(\boldsymbol{\lambda}_{2n+1},\bar{\lambda}_{2n};\mathbf{i})\rangle$$

$$=\tfrac{1}{2}\Big(\prod_{f=1}^{n+1}\{(\lambda_{2n+2}^{(f)})^2-(\bar{\lambda}_{2n+1}^{(g)})^2\}\prod_{f=1}^{n}\{(\lambda_{2n}^{(f)}+i_f)^2-(\bar{\lambda}_{2n+1}^{(g)})^2\}/(\bar{\lambda}_{2n+1}^{(g)})^2\lambda_{2n+1}^{(g)}(\lambda_{2n+1}^{(g)}+1)$$

$$\times\prod_{f(\neq g)=1}^{n}\{(\lambda_{2n+1}^{(f)})^2-(\lambda_{2n+1}^{(g)}+1)^2\}\{(\lambda_{2n+1}^{(f)})^2-(\lambda_{2n+1}^{(g)})^2\}\Big)^{1/2},\quad(6.32)$$

which holds for all $g=1,2\cdots n$ and all possible values of $\mathbf{i}$. The nonzero off-diagonal matrix elements of $\mathfrak{L}_{2n+1\,2n+2}$ are thus given by

$$\langle(\boldsymbol{\lambda}_{2n+2},\boldsymbol{\lambda}_{2n+1}+\mathbf{1}_g,\boldsymbol{\lambda}_{2n}),\mathfrak{L}_{2n+1\,2n+2}(\boldsymbol{\lambda}_{2n+2},\boldsymbol{\lambda}_{2n+1},\boldsymbol{\lambda}_{2n})\rangle$$

$$=\Big(\prod_{f=1}^{n+1}\{(\lambda_{2n+2}^{(f)})^2-(\lambda_{2n+1}^{(g)}+\tfrac{1}{2})^2\}\prod_{f=1}^{n}\{(\lambda_{2n}^{(f)})^2-(\lambda_{2n+1}^{(g)}+\tfrac{1}{2})^2\}/(2\lambda_{2n+1}^{(g)}+1)^2\lambda_{2n+1}^{(g)}(\lambda_{2n+1}^{(g)}+1)$$

$$\times\prod_{f(\neq g)=1}^{n}\{(\lambda_{2n+1}^{(f)})^2-(\lambda_{2n+1}^{(g)}+1)^2\}\{(\lambda_{2n+1}^{(f)})^2-(\lambda_{2n+1}^{(g)})^2\}\Big)^{1/2}\quad(6.33)$$

$$=\langle(\boldsymbol{\lambda}_{2n+2},\boldsymbol{\lambda}_{2n+1},\boldsymbol{\lambda}_{2n}),\mathfrak{L}_{2n+1\,2n+2}(\boldsymbol{\lambda}_{2n+2},\boldsymbol{\lambda}_{2n+1}+\mathbf{1}_g,\boldsymbol{\lambda}_{2n})\rangle.$$

In addition to the above, $\mathfrak{L}_{2n+1\,2n+2}$ has also nonzero diagonal matrix elements which arise because $L_{2n+2}^{(1)}$ includes a term, namely $l_{2n+2}^{(n+1)}$, which commutes with all the Casimir factors. To derive these we merely repeat the above argument omitting $\mathbf{1}_g$, in the eigenfunctions appearing on the left-hand side of the Dirac bracket. This gives

$$\langle\psi(\boldsymbol{\lambda}_{2n+1},\bar{\lambda}_{2n};\mathbf{i}),M_{2n+1}\psi(\boldsymbol{\lambda}_{2n+1},\bar{\lambda}_{2n};\mathbf{i})\rangle=\sum_{j_1',\cdots,j_n'=0}^{1}{}'\sum_{j_1,\cdots,j_n=0}^{1}{}'B(\mathbf{j}',\mathbf{i})B(\mathbf{j},\mathbf{i})\langle\Psi(\boldsymbol{\lambda}_{2n+1};\mathbf{j}'),L_{2n+2}^{(1)}\Psi(\boldsymbol{\lambda}_{2n+1};\mathbf{j})\rangle.\quad(6.34)$$

On using the following identity derived by substitution of eigenvalues in (4.15)

$$\langle\Psi(\boldsymbol{\lambda}_{2n+1};\mathbf{j}'),L_{2n+2}^{(1)}\Psi(\boldsymbol{\lambda}_{2n+1};\mathbf{j})\rangle=\Big(\prod_{f=1}^{n+1}\lambda_{2n+2}^{(f)}\Big)\Big(\prod_{f=1}^{n}(-1)^{j_f}\lambda_{2n+1}^{(f)}+\tfrac{1}{2}\Big)^{-1}\quad\text{for}\quad\mathbf{j}'=\mathbf{j},\quad(6.35)$$

$$=0\qquad\text{otherwise}$$

and summing after the appropriate indices in (6.34) (see Appendix C), we obtain

$$\langle\psi(\bar{\lambda}_{2n};\mathbf{i}),M_{2n+1}\psi(\bar{\lambda}_{2n};\mathbf{i})\rangle=\{\prod_{f=1}^{n+1}\lambda_{2n+2}^{(f)}\}\prod_{f=1}^{n}\{((-1)^{i_f}\bar{\lambda}_{2n}^{(f)}-\tfrac{1}{2})/((\lambda_{2n+1}^{(f)})^2-\tfrac{1}{4})\},\quad(6.36)$$

whence the diagonal matrix elements of $\mathfrak{L}_{2n+1\,2n+2}$ take the form

$$\langle(\boldsymbol{\lambda}_{2n+2},\boldsymbol{\lambda}_{2n+1},\boldsymbol{\lambda}_{2n}),\mathfrak{L}_{2n+1\,2n+2}(\boldsymbol{\lambda}_{2n+2},\boldsymbol{\lambda}_{2n+1},\boldsymbol{\lambda}_{2n})\rangle=\{\prod_{f=1}^{n+1}\lambda_{2n+2}^{(f)}\}\prod_{f=1}^{n}\{\lambda_{2n}^{(f)}/((\lambda_{2n+1}^{(f)})^2-\tfrac{1}{4})\}.\quad(6.37)$$

This expression contains an indeterminacy if

$$\lambda_{2n+2}^{(n+1)}=\lambda_{2n+1}^{(n)}=\lambda_{2n}^{(n)}=0.$$

However in this case it is not difficult to see from (6.35) that the corresponding matrix element is identically zero. These results combined with ordering of the eigenvalues implied by (6.2) can be shown to be in precise agreement with those of Louck (L, pp. 116–117).

Finally we remark that for each distinct set of eigenvalues of the Casimir invariants, these matrix elements describe an irreducible representation which is finite dimensional. The irreducibility follows from the fact that the eigenvalues of the operators diagonal in the representation separate the states and the fact that we have exhibited a complete set of ladder (shift) operators which may be used to convert any one state into any other. It is finite dimensional because of the bounds on the eigenvalues. Moreover as these bounds were not arbitrary but a specific requirement derived from unitarity, it follows that all the irreducible representations are finite dimensional and that we have obtained all of them. This completes the description of the representation.

VII. THE REPRESENTATIONS OF THE SPIN ALGEBRA

In the following we show that our results may be used to construct all the irreducible representations of the spin algebra. Though these have been derived before,[19] the present analysis is nevertheless of interest as it helps to reveal the underlying structure of the factorization method, which we are thus able to relate to the theory of addition of angular momentum. These observations enable us to suggest possible means of generalizing this approach to the study of arbitrary Lie algebras and to derive the Clebsch–Gordon coefficients in the addition of spin and angular momentum in a space of arbitrary dimension.

A comparison of Eqs. (2.1a–c) with (2.7 a–d) leads to the conjecture that the spin algebra may be a special case of the angular momentum algebra.[23] To show this explicitly we merely have to verify that (2.1 a–c) and

$$\mathcal{L}_{jk}^2=\tfrac{1}{4}1, \tag{7.1}$$

where $j,\ k=1,\ 2\cdots n$, with the identification

$$\sigma_{jk}=2\mathcal{L}_{jk} \tag{7.2}$$

for all $j,\ k=1,\ 2\cdots n$, imply (2.7a–d) and vice-versa.

Combining (2.1b) and (7.1) it follows that

$$(\mathcal{L}_{jk}\mathcal{L}_{jl}\mathcal{L}_{jk}-\tfrac{1}{4}\mathcal{L}_{jl})=i\mathcal{L}_{jk}\mathcal{L}_{kl}$$

$$(\tfrac{1}{4}\mathcal{L}_{jl}-\mathcal{L}_{jk}\mathcal{L}_{jl}\mathcal{L}_{jk})=i\mathcal{L}_{kl}\mathcal{L}_{jk},$$

for all $j,\ k,\ l\neq$. Addition shows that for this range of indices the angular-momentum components anticommute. Substituting this back into (2.1b) gives

$$\mathcal{L}_{jk}\mathcal{L}_{jl}=\tfrac{1}{2}i\mathcal{L}_{kl}$$

as required. The rest is trivial.

This result has the following interpretation: the correct algebra to use in the factorization of the Casimir invariants of the angular-momentum algebra (2.1a–c) can be derived from the latter using (7.1) and (7.2). It would be of some interest to know whether this procedure could be profitably applied to other Lie algebras.

A further consequence of the above result is that it enables us to determine all the irreducible unitary representations of the spin algebra from our previous results concerning the angular momentum algebra. This is shown in the following.

The first step is to derive the form taken by the Casimir invariants when (7.1) holds. From (2.3), (2.5), and (7.1) we find this to be given by

$$\mathcal{L}_n^{2f}=[(2f)!]^{-1}\sum_{j_1,\cdots,j_{2f}=1}^{n}{}'(2^{2f})^{-1}\left(\frac{\Gamma(f+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right)^2 2^{2f}1,$$

$$=\frac{n!}{(2f)!(n-2f)!}\left(\frac{\Gamma(f+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right)^2 1,$$

for all $f=1,\ 2\cdots[n/2]$. Substituting this result in (5.6) and using the identity

$$\sum_{f=0}^{n}\left\{\frac{(2n)!}{(2f)!(2n-2f)!}\left(\frac{\Gamma(f+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right)^2\right.$$

$$\left.\times\frac{\Gamma(x+n-f)\Gamma(-x+n-f)}{\Gamma(x)\Gamma(-x)}\right\}$$

$$=\frac{\Gamma(x+n+\frac{1}{2})\Gamma(-x+n+\frac{1}{2})}{\Gamma(x+\frac{1}{2})\Gamma(-x+\frac{1}{2})}, \tag{7.3}$$

which is verified in Appendix B, we obtain

$$(k_{2n}^{(f)})^2=(n-f+\tfrac{1}{2})^2 1, \tag{7.4}$$

for all $f=1,\ 2\cdots n$. Similarly we find that

$$(k_{2n-1}^{(f)})^2=(n-f)^2 1, \tag{7.5}$$

for all $f=1,\ 2\cdots(n-1)$.

From the results of Sec. 6, we know that the eigenvalues of these operators are all positive with the exception of $\lambda_{2n}^{(n)}$ which may assume both positive and negative values corresponding to the sign of the invariant $\mathcal{L}_{1,2,\ldots,2n}$. Thus we obtain

$$\lambda_{2n-1}^{(f)}=(n-f),\qquad \lambda_{2n}^{(f)}=(n-f+\tfrac{1}{2}),$$

for all $f=1,\ 2\cdots(n-1)$ and

$$\lambda_{2n}^{(n)}=\pm\tfrac{1}{2}.$$

Substitution of these into (6.30), (6.33), and (6.37) gives the following expressions for the nonzero matrix elements of $\mathcal{L}_{2n-1\ 2n}$ and $\mathcal{L}_{2n\ 2n+1}$:

$$\langle(\lambda_{2n},\lambda_{2n-1},\lambda_{2n-2}),\mathcal{L}_{2n-1\ 2n}(\lambda_{2n},\lambda_{2n-1},\lambda_{2n-2})\rangle$$

$$=2\lambda_{2n}^{(n)}\lambda_{2n-2}^{(n-1)}, \tag{7.6}$$

$$\langle(\lambda_{2n+1},\lambda_{2n}\pm 1_n,\lambda_{2n-1})\mathcal{L}_{2n\ 2n+1}(\lambda_{2n+1},\lambda_{2n},\lambda_{2n-1})\rangle$$

$$=\tfrac{1}{2}(\tfrac{1}{2}\mp\lambda_{2n}^{(n)}). \tag{7.7}$$

We may use (7.1), (7.6), and (7.7) to write down matrix expressions for the spin algebra. These take the form

$$\sigma_{12}=\sigma_z\otimes 1_n,$$

$$\sigma_{2m\ 2m+1}=1_{m-1}\otimes\sigma_x\otimes 1_{n-m+1}, \tag{7.8}$$

$$\sigma_{2m+1\ 2m+2}=1_{m-1}\otimes\sigma_z\otimes\sigma_z\otimes 1_{n-m},$$

where $m=1,\ 2\cdots n$, $\otimes$ denotes the Kronecker product and 1_m the unit matrix of dimension 2^m. σ_x, σ_y, and σ_z are twice the Pauli spin matrices being defined as

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\qquad \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{7.9}$$

When the dimension of the space is odd, (7.8) describes the one and only irreducible representation (to within equivalence) of the spin algebra. On the other hand, for a space of dimension $(2n+2)$ there are two irreducible representations and these are indexed by the eigenvalues of the single nontrivial invariant $\sigma_{1,2,\cdots(2n+2)}$. From (7.8) this is seen to take the form

$$\sigma_{1,2,\cdots,(2n+2)}=1_n\otimes\sigma_z, \tag{7.10}$$

an expression which enables us to decompose the matrix representation (7.8) into its irreducible parts.

Inspection of (7.8) shows that the dimension of the spin matrices is doubled as n is increased by two units. This result blends in nicely with the corresponding increase in the number of Casimir invariants of the angular-momentum algebra. In particular it is just what we need to describe the decomposition of the eigenspace of these invariants by the Casimir factors. The form that this takes can be deduced from (6.19) in which the eigenfunctions α_{n+1}, β_{n+1} of the invariant $\sigma_{1,2,\cdots,(2n+1)}$ were introduced explicitly. Now from (7.10) it follows that these are spinors with two independent components. Consequently the eigenfunctions

[23] The connection between the Lie algebras is described in H. Boerner, *Representations of Groups* (North-Holland Publ. Co., Amsterdam, 1963), Chap. II, Sec. 5, pp. 36–42, Chap. VIII, Theorem 2.1, p. 269.

of the Casimir factors in $(2n+2)$-dimensional space are spinors having 2^{n+1} independent components which are in fact eigenfunctions of the Casimir invariants. They may be constructed explicitly by use of the matrix transformations described by $B(\mathbf{j}, \mathbf{i})$ and $C(\mathbf{j}, \mathbf{i})$ as defined in Sec. 6.

The above result is of some interest from the point of view of addition of angular momentum, that is the problem of determining the Clebsch–Gordon coefficients in the restriction of $0(n)\times 0(n)$ to $0(n)$. Let us recall the form taken by the Ladder invariants $(\bar{k}_{2n}^{(f)})^2$: $f=1, 2\cdots n$. From (2.21), (2.25), and (5.12) it is not difficult to see that these are just the Casimir invariants corresponding to the addition of spin and angular momentum. On account of (4.12) the eigenfunctions of these operators are precisely the eigenfunctions of the Casimir factors and their eigenvalues determine the different possible final quantum numbers. Now we have described above an explicit construction of these functions from the eigenfunctions of the Casimir invariants of the spin and angular-momentum algebras (and subalgebras). The coefficients in this expansion, obtained as above from the set of $B(\mathbf{j}, \mathbf{i})$ and $C(\mathbf{j}, \mathbf{i})$ matrices, are just Clebsch–Gordon coefficients for the addition of spin and angular momentum, for which explicit formulae are given in Appendix D. Though this is by no means the whole of the Clebsch–Gordon series for the addition of angular momentum, a generalization of the above procedure may help to solve this difficult problem.[24] Moreover it may prove useful in revealing some of the symmetries contained in this transformation, which even in three-space are not yet fully understood.[25] From the point of view of the factorization of the Casimir invariants of an arbitrary Lie algebra these results are particularly encouraging. For since the Clebsch–Gordon series can be defined for any Lie algebra there is no reason to suppose that there is something special about the angular-momentum algebra which permits the use of factorization techniques.

Finally we remark that from the spin-algebra representation we may derive the familiar mutually adjoint pairs of ladder operators which are customarily used in studying the angular-momentum problem. This was discussed in I, (1.13), for the three-dimensional space. As a further example we compute the form taken by $L_4^{(1)}$.

[24] For $n=3$, this problem was first solved by E. Wigner, *Gruppentheorie* (Frederic Vieweg and Sohn, Braunschweig, Germany, 1931), Chap. XVII, pp. 198–208, and later by G. Racah, Phys. Rev. **62**, 438 (1942) who put the results in a simpler and more symmetric form. A lucid account of Wigner's approach is given in L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon Press Ltd., London, 1958), Chap. XII, Sec. 97, pp. 353–360. Racah's method and related results can be found in A. Messiah, *Quantum Mechanics* (North-Holland Publ. Co., Amsterdam, 1962), Vol. II, Chap. XIII and Appendix C. For $n=4$, a complete solution is available on account of the isomorphism between the Lie algebras of $0(4)$ and $0(3)\times 0(3)$, L. C. Biedenharn, J. Math. Phys. **2**, 433 (1961) and references therein. For $n=5$, a partial solution has been given by K. T. Hecht, Nucl. Phys. **63**, 177 (1965).

[25] T. Regge, Nuovo Cimento **10**, 544 (1958); **11**, 116 (1959); V. Bargmann, Rev. Mod. Phys. **34**, 829 (1962).

From (2.7c), (7.9), and (7.10) it is easily shown that

$$\sigma_{14}=\sigma_x\otimes\sigma_z,$$
$$\sigma_{24}=\sigma_y\otimes\sigma_z,$$
$$\sigma_{34}=\sigma_z\otimes\sigma_z.$$

Substitution of this result in (2.16a) then gives

$$L_4^{(1)}=\begin{pmatrix}\mathfrak{L}_{34} & (\mathfrak{L}_{14}-i\mathfrak{L}_{24})\\ (\mathfrak{L}_{14}+i\mathfrak{L}_{24}) & -\mathfrak{L}_{34}\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The operators $(\mathfrak{L}_{14}\pm i\mathfrak{L}_{24})$ which respectively step up and down the eigenvalues of $\mathfrak{L}_{12}$ are immediately recognized. Similar results hold in higher-dimensional space. Pursuing this line of reasoning one might hope to reveal the connection between the ladder operators used here and those developed by Nagel, Moshinsky, and others.[6,9,21]

VIII. SUMMARY

We have derived all the irreducible unitary representations of the Lie algebra of $0(n)$ in the canonical group chain $0(n)\supset 0(n-1)\supset\cdots\supset 0(2)$ and the corresponding Clebsch–Gordon coefficients for the addition of spin and angular momentum.

In the Introduction the motivation for this study is outlined. In Sec. 2 the basic operators used throughout this paper are defined. These are: the angular-momentum components which form the Lie algebra of $0(n)$, the spin components, which are used to factorize the Casimir invariants, and the Dirac algebra which derives from the addition of spin and angular momentum. The ladder operators, Casimir factors, and ladder invariants are then constructed and their basic properties examined.

In Secs. 3 and 4 the $(2n+1)$- and $(2n+2)$-dimensional problems are separately studied, though attention is paid to the close relationship which exists between the two. It is shown in both problems that the Casimir factors can be expressed as symmetric functions over a commuting set $k_n^{(f)}: f=1, 2\cdots[n/2]$ of operators which have the important property of being linear in the basic angular momentum components. A corresponding decomposition of the simplest of the ladder operators is given, and the commutation relations between the two sets determined. From these results the selection rules for the matrix elements are derived. When the angular-momentum components are represented by differential operators then this procedure corresponds to the factorization of the resulting differential equations as is, for example, described in I, Sec. 5. In this respect it is unfortunate that we are unable to given explicit expressions for the $k_n^{(f)}$, though these operators are determined uniquely by the relations they satisfy.[26]

In Sec. 5, explicit relationships between the Casimir factors, the Casimir invariants and the ladder invariants are derived. These results are used to show that the

[26] When $n=4$, one may realize these operators explicitly by making use of the isomorphism $0(4)\cong 0(3)\times 0(3)$. This provides a pleasing description of the factorization procedure, and the ladder operators it involves.

anticommutation relations which exist between the components of the ladder operator $L_{2n+1}^{(1)}$ and the $\bar{k}_n^{(f)}$ imply that they may raise or lower the eigenvalues belonging to suitable eigenfunctions of the Casimir invariants.

In Sec. 6, the details of the irreducible representation are worked out. The argument divides into three main parts. First the eigenvalues of all operators diagonal in the representation are computed. In this a slight ambiguity, which is characteristic of the method, is noted. It concerns the factorization of the Casimir invariants $\mathfrak{L}_{2m}^{2m}$: $m=1, 2, \cdots, n$, of the $0(2m)$ subalgebras, which can be carried out without reference to the spin algebra. It results in the corresponding eigenvalues admitting a negative as well as a positive range and this augments the representation which has then to be shown to be still irreducible. It is of some interest that the factors $\mathfrak{L}_{1,2,\cdots,2m}$: $m=1, 2\cdots n$, which are also invariants of the 0(2m) subalgebras, have esesntially the same form as the nontrivial invariants $\sigma_{1,2,\cdots,2m}$: $m=1, 2\cdots n$ of the spin subalgebras. This is no mere coincidence. Indeed it reflects the fact that representations of the angular-momentum algebra are not constructed directly; but are derived by a procedure which involves addition of spin and angular momentum. Consequently the representations of the angular-momentum algebra are mixed in with those of the spin algebra. The difficulties that this introduces are resolved by the use of additional commutation relations which are independent of the spin algebra.

The second stage in the construction of the representation involves a determination of the relationship between the eigenfunctions of the Casimir factors in spaces of different dimension. This is described explicitly and in the third part these results are used to derive the matrix elements for all the basic angular-momentum components.

In Sec. 7, we show that the spin algebra is a special case of the angular-momentum algebra and use this result to derive all the irreducible unitary representations (to within equivalence) which may be afforded to it. The relationship between the present method of construction of ladder operators and the theory of addition of angular momentum is pointed out. This observation enables us to derive explicitly the Clebsch–Gordon coefficients for the addition of spin and angular momentum in a space of arbitrary dimension. It also suggests some further generalizations of the present theory, though these are not pursued in detail. Finally the ladder operators discussed here are related to those customarily used.

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APPENDIX A

In the following we show that the double summation $I_{2n+1}(x)$ defined by

$$I_{2n+1}(x) = \sum_{f,g=0}^{n} \frac{\Gamma(x+n-f+\frac{1}{2})\Gamma(-x+n-g+\frac{1}{2})}{\Gamma(x+\frac{1}{2})\Gamma(-x+\frac{1}{2})} \mathfrak{L}_{2n+1}^{(f)}\mathfrak{L}_{2n+1}^{(g)}, \tag{A1}$$

commutes with all the spin operators σ_{jk}: $j, k=1, 2\cdots(2n+2)$. That is

$$[I_{2n+1}(x), \sigma_{jk}]=0, \tag{A2}$$

and establish a similar result for the Dirac algebra.

From the defining relations for the $\mathfrak{L}_{2n+1}^{(f)}$ and the $L_{2n+1}^{(f)}$ it is easy to show that

$$[\sigma_{2n+1\ 2n+2}, \mathfrak{L}_{2n+1}^{(f)}]=2\sigma_{2n+1\ 2n+2}L_{2n+1}^{(f)},$$

for all $f=1, 2\cdots n$. Hence, recalling (2.14) and (2.15), it follows that

$$[\sigma_{2n+1\ 2n+2}, \mathfrak{L}_{2n+1}^{(f)}\mathfrak{L}_{2n+1}^{(g)}] = \sigma_{2n+1\ 2n+2}([\mathfrak{L}_{2n}^{(f)}, L_{2n+1}^{(g)}]_+ + [\mathfrak{L}_{2n}^{(g)}, L_{2n+1}^{(f)}]_+), \tag{A3}$$

for all $f, g=1, 2\cdots n$. Now from (2.20), with $n=2n$, we obtain, after some reduction

$$\sum_{f=0}^{n} \frac{\Gamma(x+n-f+\frac{1}{2})}{\Gamma(x+\frac{1}{2})}[\mathfrak{L}_{2n}^{(g)}, [\mathfrak{L}_{2n}^{(f)}, L_{2n+1}^{(1)}]_+]_+ = 2x\sum_{f=1}^{n} \frac{\Gamma(x+n-f+\frac{1}{2})}{\Gamma(x+\frac{1}{2})}[\mathfrak{L}_{2n}^{(g)}, L_{2n+1}^{(f)}]_+, \tag{A4}$$

for all $g=1, 2\cdots n$. A similar expression

$$\sum_{g=0}^{n} \frac{\Gamma(-x+n-g+\frac{1}{2})}{\Gamma(-x+\frac{1}{2})}[\mathfrak{L}_{2n}^{(f)}, [\mathfrak{L}_{2n}^{(g)}, L_{2n+1}^{(1)}]_+]_+ = -2x\sum_{g=1}^{n} \frac{\Gamma(-x+n-g+\frac{1}{2})}{\Gamma(-x+\frac{1}{2})}[\mathfrak{L}_{2n}^{(f)}, L_{2n+1}^{(g)}]_+, \tag{A5}$$

for all $f=1, 2\cdots n$, derives from (A4) by replacement of x by $-x$, and interchange of f and g.

The identity

$$[a, [b, c]_+]_+ = [b, [a, c]_+]_+,$$

which holds for arbitrary operators a, b, c given that a and b commute, implies, recalling (2.14), that

$$[\mathfrak{L}_{2n}^{(f)}, [\mathfrak{L}_{2n}^{(g)}, L_{2n+1}^{(1)}]_+]_+ = [\mathfrak{L}_{2n}^{(g)}, [\mathfrak{L}_{2n}^{(f)}, L_{2n+1}^{(1)}]_+]_+.$$

This result, suitably combined with (A4) and (A5), shows that those terms involving this double anticommutator can be eliminated, leaving

$$2x\sum_{f,g=0}^{n} \frac{\Gamma(x+n-f+\frac{1}{2})\Gamma(-x+n-g+\frac{1}{2})}{\Gamma(x+\frac{1}{2})\Gamma(-x+\frac{1}{2})} \times\{[\mathfrak{L}_{2n}^{(g)}, L_{2n+1}^{(f)}]_+ + [\mathfrak{L}_{2n}^{(f)}, L_{2n+1}^{(g)}]_+\}=0. \tag{A6}$$

From Eqs. (A1), (A3), and (A6) we thus obtain

$$[\sigma_{2n+1\ 2n+2}, I_{2n+1}(x)]=0. \qquad (A7)$$

Now since $I_{2n+1}(x)$ is symmetric in all the indices $1, 2\cdots(2n+1)$ used to label the spin and angular-momentum algebras, (A7) may be expressed in the more general form

$$[\sigma_{j\ 2n+2}, I_{2n+1}(x)]=0,$$

for all $j=1, 2\cdots 2n+1$, which, in view of (2.7c), establishes (A2).

The above result was used in Sec. 5 to verify (5.3). We now wish to establish the related identity, namely (5.11). Despite the close similarity between these expressions, the interdependence of the spin and Dirac algebras would seem to imply that the argument used in Sec. 5 is no longer appropriate. However we show that it is applicable if suitable modifications are made. We describe these below.

In attempting to establish (5.3) by direct computation, we make use of a contraction relation having the form

$$\sum_{j,k,l=1}^{n}{}' \sigma_{jk}\mathfrak{L}_{jk}\sigma_{jl}\mathfrak{L}_{jl}=-\tfrac{1}{2}(n-2)\sum_{k,l=1}^{n}{}' \sigma_{kl}\mathfrak{L}_{kl},$$

a result which is a direct consequence of (2.1b) and (2.7c).

Similarly in establishing (5.11), an analogous contraction relation is used. However, this now takes the form

$$\sum_{j,k,l=1}^{n}{}' \sigma_{jk}J_{jk}\sigma_{jl}J_{jl}=\tfrac{1}{2}(n-2)\sum_{k,l=1}^{n}{}' \sigma_{kl}J_{kl},$$

in which a plus sign has replaced a minus sign on the right-hand side. Since this is the only part of the computation for which the different algebraic properties of the $\mathfrak{L}_{jk}$ and the J_{jk} lead to different results, it is easy to see that (5.11) can be reproduced if the J_{jk} are replaced by the elements of a spin-independent algebra satisfying (2.1), but with (2.1b) replaced by

$$[\mathfrak{L}_{jk}', \mathfrak{L}_{jl}']=-i\mathfrak{L}_{kl}', \qquad (A8)$$

for all $j, k, l=1, 2\cdots n$.

Let us now recall that in establishing (A2) we made use of the algebraic properties of the angular-momentum components only insofar as (2.14) and (2.20) were employed. The alteration in this algebra implied by (A8) affects only the latter and results in a plus sign replacing the minus appearing on the right-hand side. Repetition of the argument, given in the first part of this Appendix, with this change shows that (A2) holds if the term $(-1)^{f+g}$ is included in the double summation. Using the independence of the spin and this modified angular momentum algebra, as in Sec. 5, it follows that this result implies (5.11) as required.

APPENDIX B

In the following we establish two identities involving summations over gamma functions, namely (5.9) and (7.3). We start with the former which is the simpler.

Applying Leibnitz's rule to n-fold differentiation of the product $z^{\alpha}z^{\beta}$ and setting $z=1$, we obtain

$$\sum_{f=0}^{n}\frac{n!}{f!(n-f)!}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+1-f)\Gamma(\beta+1-n+f)}=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+1-n)}, \qquad (B1)$$

where Γ denotes the gamma function. It should be noted that this holds even for negative integer values of $(\alpha+\beta)$ resulting in the gamma functions having negative integer exponents and consequent singularities.[27]

Replacing n by $(n-g)$ and the summation index f by $(n-f)$ in (B1) gives

$$\sum_{f=g}^{n}\frac{(n-g)!}{(n-f)!(f-g)!}\cdot\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+1-n+f)\Gamma(\beta+1+g-f)}=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+1-n+g)}. \qquad (B2)$$

Setting $\alpha=x$, $\beta=(n-g-1/2)$ in (B2) and rearranging we obtain (B1).

To verify (7.3) a little extra manipulation is required. We start from the partial-fraction[28] expansion

$$\prod_{j=1}^{n}(y-a_j)-\prod_{j=1}^{n}(y-b_j)=\sum_{j=1}^{n}(b_j-a_j)\prod_{k(\neq j)=1}^{n}\left(\frac{b_j-a_k}{b_j-b_k}\right)(y-b_k) \qquad (B3)$$

which holds identically in the continuous variable y. It will be noted that the right-hand side is a sum of polynomials of degree $(n-1)$ in y. We wish to rearrange this to give a sum of polynomials of decreasing degree. To do this we substitute the easily verified identity

$$\prod_{k(\neq j)=1}^{n}\left\{\frac{y-b_k}{b_j-b_k}\right\}=\sum_{l=j-1}^{n-1}\{\prod_{k=1}^{l}(y-b_k)/\prod_{k(\neq j)=1}^{l+1}(b_j-b_k)\}$$

into (B3). After a minor rearrangement of the summations involved this gives the required expression, namely,

$$\{\prod_{j=1}^{n}(y-a_j)-\prod_{j=1}^{n}(y-b_j)\}=\{\sum_{r=1}^{n-1}S_r\prod_{j=1}^{r}(y-b_j)\}+S_0, \qquad (B4)$$

where S_r takes the form

$$S_r=\sum_{j=1}^{r+1}[\prod_{k=1}^{n}(b_j-a_k)/\prod_{k(\neq j)=1}^{r+1}(b_j-b_k)], \qquad (B5)$$

for all $r=0, 1\cdots(n-1)$.

We now evaluate S_r for the special case when

$$a_j=(j-\tfrac{1}{2})^2, \qquad b_j=(j-1)^2,$$

[27] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, England, 1965), 4th ed. reprinted, Chap. XII, p. 236.

[28] W. L. Ferrar, *Higher Algebra* (Clarendon Press, Oxford, England, 1958), Chap. XXII, p. 214.

holds for all $j=1, 2\cdots n$. It is not difficult to see that these substitutions will give rise to gamma functions appearing in (B5). Indeed a fairly elementary manipulation of this expression reduces it to

$$S_r=\sum_{j=-r}^{r}\frac{\Gamma(j+n+\frac{1}{2})}{\Gamma(j-n+\frac{1}{2})}\frac{(-1)^{r-j}}{(r+j)!(r-j)!},\qquad \text{(B6)}$$

in which to obtain the summation over the range $-r$ to r we have made use of the symmetry in j. Equation (B6) may be rewritten in a form which admits application of (B1). This is

$$S_r=\frac{(-1)^{n-r}}{(2r)!}\left(\frac{\Gamma(\frac{1}{2})}{\Gamma(r-n+\frac{1}{2})}\right)^2\sum_{j=-r}^{r}\frac{(2r)!}{(r+j)!(r-j)!}\times\frac{\{\Gamma(r-n+\frac{1}{2})\}^2}{\Gamma(j-n+\frac{1}{2})\Gamma(-j-n+\frac{1}{2})}.$$

Now, choosing $\alpha=\beta=(r-n-\frac{1}{2})$ and rearranging the gamma functions on the right-hand side, we obtain

$$S_r=\left(\frac{\Gamma(n-r+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right)^2(-1)^{n-r}\cdot\frac{(2n)!}{(2n-2r)!(2r)!}.\qquad \text{(B7)}$$

Substitution of (B7) back into (B4) and setting $y=x^2$, reduces the latter to

$$\prod_{j=1}^{n}(x^2-(j-\tfrac{1}{2})^2)-\prod_{j=1}^{n}(x^2-(j-1)^2)=\sum_{r=1}^{n-1}\left\{\left(\frac{\Gamma(n-r+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right)^2\frac{(2n)!(-1)^{n-r}}{(2r)!(2n-2r)!}\times\prod_{j=1}^{r}(x^2-(j-1)^2)\right\}+\left(\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right)^2.$$

Replacement of the summation index r by $(n-f)$ and identification of the products over j as gamma functions gives (7.3) as required.

It is worth mentioning that the resulting summation over f involves at least five gamma functions. The above procedure has enabled us to carry out such a summation using only a summation over four gamma functions, namely (B1). It is as far as we know a new identity for gamma functions.

APPENDIX C

We establish two identities satisfied by the $B(\mathbf{j},\mathbf{i})$. The first is (6.13) which expresses the normalization of these coefficients. We prove it by induction.

Starting[29] with $n=1$, (6.13) reduces to

$$\sum_{i_1=0}^{1}\{(-1)^{j_1}\lambda_{2n+1}^{(1)}+(-1)^{i_1}\bar{\lambda}_{2n}^{(1)}\}((-1)^{j_1}2\lambda_{2n+1}^{(1)})^{-1}=0,$$

which holds trivially. Now consider the expression

$$\prod_{f=1}^{n-1}\{(\bar{\lambda}_{2n}^{(f)})^2-(\bar{\lambda}_{2n}^{(n)})^2\}[\sum_{i_1,\cdots,i_n=0}^{1}\{|B(\mathbf{j},\mathbf{i})|^2\}-1]=0.\qquad \text{(C1)}$$

This may be shown to be a polynomial of degree $(2n-1)$ in $\bar{\lambda}_{2n}^{(n)}$ as follows. Observe that the terms in $\bar{\lambda}_{2n}^{(n)}$ which appear in the denominator of $|B(\mathbf{j},\mathbf{i})|^2$ cancel, irrespective of the value of $\mathbf{i}$, with the product in curly brackets. This reduces the latter to a polynomial of degree $(n-1)$ in $\bar{\lambda}_{2n}^{(n)}$. Since the numerator is itself a polynomial of degree n, the over-all expression is a polynomial of degree $(2n-1)$.

We now show that the square-bracketed term appearing in (C1) has at least $2n$ zeros in $\bar{\lambda}_{2n}^{(n)}$. To do this we set $\bar{\lambda}_{2n}^{(n)}=\lambda_{2n+1}^{(n)}$ and observe that half the terms in this summation (namely, those corresponding to $i_n=j_n$) become identically zero. Furthermore the remainder reduces by cancellation in the numerator and the denominator to give an expression corresponding to the replacement of n by $(n-1)$. Thus if (6.13) holds for $(n-1)$, $\lambda_{2n+1}^{(n)}$ is a zero of (C1). Using the symmetry of this expression it is easy to see that this implies that the set $\pm\lambda_{2n+1}^{(f)}: f=1, 2\cdots n$ are the required $2n$ zeros in $\bar{\lambda}_{2n}^{(n)}$. Finally since a polynomial of degree $(2n-1)$ has at most $(2n-1)$ zeros, it follows that the square-bracketed term in (C1) is identically zero. This demonstrates the validity of (6.13).

A similar argument can be used to verify the identity

$$\sum_{j_1,\cdots,j_n=0}^{1}[|B(\mathbf{j},\mathbf{i})|^2/\prod_{f=1}^{n}\{(-1)^{j_f}\lambda_{2n+1}^{(f)}+\tfrac{1}{2}\}]-\prod_{f=1}^{n}\left\{\frac{((-1)^{i_f}\bar{\lambda}_{2n}^{(f)}-\frac{1}{2})}{((\lambda_{2n+1}^{(f)})^2-\frac{1}{4})}\right\}=0,\qquad \text{(C2)}$$

which is used to establish (6.36). Thus when $n=1$, we obtain

$$\sum_{j_1=0}^{1}\left\{\frac{((-1)^{j_1}\lambda_{2n+1}^{(1)}+(-1)^{i_1}\bar{\lambda}_{2n}^{(1)})}{2(-1)^{j_1}\lambda_{2n+1}^{(1)}((-1)^{j_1}\lambda_{2n+1}^{(1)}+\frac{1}{2})}\right\}-\left\{\frac{((-1)^{i_1}\bar{\lambda}_{2n}^{(1)}-\frac{1}{2})}{((\lambda_{2n+1}^{(1)})^2-\frac{1}{4}))}\right\}=0,$$

which holds trivially. Then multiply (C2) throughout by

$$\prod_{f=1}^{n-1}\{(-1)^{i_f}\bar{\lambda}_{2n}^{(f)}+(-1)^{i_n}\bar{\lambda}_{2n}^{(n)}\}.$$

Remembering that the summation is now over $\mathbf{j}$, we see that this reduces (C2) to a polynomial of degree $(2n-1)$ in $\bar{\lambda}_{2n}^{(n)}$. As before the fact that (C2) holds for n replaced by $(n-1)$ implies that this polynomial has the $2n$ zeros $\pm\lambda_{2n+1}^{(f)}: f=1, 2\cdots n$. Thus (C2) holds for all n.

The use of an induction procedure betrays a lack of understanding of the nature of these coefficients, which a more direct proof might reveal.

APPENDIX D

We write out below the Clebsch–Gordon coefficients for the addition of spin and angular momentum, as computed by the method described in Sec. 7. We use the following notation for the eigenvalues:

$$\boldsymbol{\lambda}_{2m}=(\lambda_{2m}^{(1)}, \lambda_{2m}^{(2)},\cdots,\lambda_{2m}^{(m)});$$
$$\boldsymbol{\lambda}_{2m+1}=(\lambda_{2m+1}^{(1)}, \lambda_{2m+1}^{(2)},\cdots,\lambda_{2m+1}^{(m)}),$$

[29] These choices of n do *not* apply to the subscripts in $\lambda_{2n+1}^{(f)}$ and $\bar{\lambda}_{2n}^{(f)}$.

for $m=1, 2\cdots n$. A similar meaning is given to $\bar{\lambda}_{2m}$, $\bar{\lambda}_{2m+1}$, [recall (4.13)]. In addition we define

$$\mathbf{i}_{2m}=(i_{2m}^{(1)}, i_{2m}^{(2)},\cdots, i_{2m}^{(m)});$$
$$\mathbf{i}_{2m+1}=(i_{2m+1}^{(1)}, i_{2m+1}^{(2)},\cdots, i_{2m+1}^{(m)}),$$

where $m=1, 2\cdots n$ and where the individual coefficients $i_r^{(k)}$ each take the values 0 and 1. Then combining Eqs. (6.10), (6.12), (6.17), (6.20), (6.21), and (6.22) we readily obtain the following expression for the Clebsch–Gordon coefficients $G(\cdots)$ for $0(2n)\times 0(2n)$:

$$G(\boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3,\cdots, \boldsymbol{\lambda}_n; \mathbf{i}_2, \mathbf{i}_3,\cdots, \mathbf{i}_{2n}; i_1^{(2)}, i_3^{(4)},\cdots, i_{2n-1}^{(2n)})=\{\prod_{m=1}^{n} C_m(\boldsymbol{\lambda}_{2m}, (\bar{\boldsymbol{\lambda}}_{2m-1}-\mathbf{i}_{2m-1}); \mathbf{i}_{2m}, \mathbf{i}_{2m-1}, i_{2m-1}^{(2m)})\}$$
$$\times\{\prod_{m=2}^{n} B_{m-1}(\boldsymbol{\lambda}_{2m-1}, (\bar{\boldsymbol{\lambda}}_{2m-2}-\mathbf{i}_{2m-2}); \mathbf{i}_{2m-1}, \mathbf{i}_{2m-2})\}, \quad \text{(D1)}$$

where

$$C_m(\boldsymbol{\lambda}_{2m}, \bar{\boldsymbol{\lambda}}_{2m-1}; \mathbf{i}_{2m}, \mathbf{i}_{2m-1}, i_{2m-1}^{(2m)})=\{\prod_{g=f}^{m-1}\prod_{f=1}^{m-1}(-1)^{i_{2m}^{(f)}\cdot i_{2m-1}^{(g)}}\}$$
$$\times(\prod_{f=1}^{m}\prod_{g=1}^{m-1}\{(-1)^{i_{2m}^{(f)}}\lambda_{2m}^{(f)}+(-1)^{i_{2m-1}^{(g)}}\bar{\lambda}_{2m-1}^{(g)}\}\Big/\prod_{g=f+1}^{m}\prod_{f=1}^{m}((-1)^{i_{2m}^{(f)}}\lambda_{2m}^{(f)}+(-1)^{i_{2m}^{(g)}}\lambda_{2m}^{(g)})$$
$$\times\prod_{g=f}^{m-1}\prod_{f=1}^{m-1}((-1)^{i_{2m-1}^{(f)}}\bar{\lambda}_{2m-1}^{(f)}+(-1)^{i_{2m-1}^{(g)}}\bar{\lambda}_{2m-1}^{(g)}))^{1/2},$$

when

$$i_{2m-1}^{(2m)}+\sum_{f=1}^{m} i_{2m}^{(f)}=0 \pmod 2,$$

where

$$C_m(\cdots)=0,$$

when

$$i_{2m-1}^{(2m)}+\sum_{f=1}^{m} i_{2m}^{(f)}=1 \pmod 2,$$

and where

$$B_m(\boldsymbol{\lambda}_{2m+1}, \bar{\boldsymbol{\lambda}}_{2m}; \mathbf{i}_{2m+1}, \mathbf{i}_{2m})=\{\prod_{g=f}^{m}\prod_{f=1}^{m}(-1)^{i_{2m+1}^{(f)}i_{2m}^{(g)}}\}$$
$$\times(\prod_{g=1}^{m}\prod_{f=1}^{m}\{(-1)^{i_{2m+1}^{(f)}}\lambda_{2m+1}^{(f)}+(-1)^{i_{2m}^{(g)}}\bar{\lambda}_{2m}^{(g)}\}\Big/\prod_{g=f}^{m}\prod_{f=1}^{m}\{(-1)^{i_{2m+1}^{(f)}}\lambda_{2m+1}^{(f)}+(-1)^{i_{2m+1}^{(g)}}\lambda_{2m+1}^{(g)}\}$$
$$\times\prod_{g=f+1}^{m}\prod_{f=1}^{m}\{(-1)^{i_{2m}^{(f)}}\bar{\lambda}_{2m}^{(f)}+(-1)^{i_{2m}^{(g)}}\bar{\lambda}_{2m}^{(g)}\})^{1/2}.$$

In (D1) the set $\boldsymbol{\lambda}_m$: $m=2, 3\cdots 2n$ determine the eigenvalues of the angular momentum operators as described in Sec. 6. The numbers $i_{2m-1}^{(2m)}$: $m=1, 2\cdots n$, which each take the values 0 and 1, determine the eigenvalues of the spin components $\sigma_{1,2,\ldots,2m}$: $m=1, 2\cdots 2n$, by the rule

$$\sigma_{1,2,\cdots,2m}|\cdots\rangle=(-1)^{i_{2m-1}^{(2m)}}|\cdots\rangle.$$

The eigenvalues $(\lambda_m^{(f)})'$ of the operators describing the addition of spin and angular momentum are determined by the set of integers $\mathbf{i}_m$: $m=2, 3\cdots 2n$. This takes the form

$$(\lambda_{2m}^{(f)})'=(\bar{\lambda}_{2m}^{(f)}-i_{2m}^{(f)});$$
$$(\lambda_{2m+1}^{(f)})'=(\bar{\lambda}_{2m+1}^{(f)}-i_{2m+1}^{(f)}), \quad \text{(D2)}$$

for $f=1\cdots m$; $m=1, 2\cdots n$. Like their precursors, the $(\lambda_m^{(f)})'$ must satisfy the ordering described by Eqs. (6.1) and (6.2). All other Clebsch–Gordon coefficients for spin coupling are identically zero. In determining the form taken by a final wave function with a given set of eigenvalues, one must sum over all possible starting wave functions which can give rise to that function as determined by (D2).

Finally we remark that the Clebsch–Gordon coefficients for $0(2n-1)\times(2n-1)$ can be determined from (D1) by omitting the factor $C_n(\cdots)$ in the product appearing on the right-hand side.