REVIEWS OF MODERN PHYSICS VOLUME 40, NUMBER 3 JULY 1968

# Stability of the Interface between Two Fluids in Relative Motion

RICHARD A. GERWIN
*Boeing Scientific Research Laboratories, Seattle, Washington*

Some basic properties of the Kelvin–Helmholtz instability are reviewed, and some papers dealing with various refinements of this phenomenon are discussed.

## I. INTRODUCTION

The problem reviewed here is the stability of a plane interface between two fluids. If they are in relative motion parallel to their interface, the latter may become unstable under certain conditions. This type of instability is commonly known as the Kelvin–Helmholtz instability.

There is a large body of recent literature on this subject and on the related topics of the stability of jets and of shear flows, both in the "ordinary" MHD and plasma situations.[2–33,36–38] The problem has been considered in connection with the stability of the earth's magnetosphere in the solar wind,[23b] the stability of streams of charged particles emanating from the sun,[32] the excitation of water waves by the wind,[4,7] the stability of jets[3,10,14] and of plasma jets,[29] as well as the stability of other kinds of laboratory plasmas.[26–28,36–38] The phenomenon is also thought to be a factor influencing the melting surface of an object falling into the atmosphere.[15]

## II. SURVEY OF PREVIOUS WORK

### A. Incompressible Fluids

The Kelvin–Helmholtz instability between incompressible fluids has been studied by Lamb,[1] where earlier references may be found. The situation under consideration is shown in Fig. 1. The equilibrium interface is the $(y,z)$ plane. Fluid 2, filling the region $x>0$, is flowing in the $z$ direction with velocity $u$. Without loss of generality, fluid 1 is assumed at rest. In the presence of a small-amplitude perturbation, the equilibrium surface suffers a normal displacement $\xi(y,z,t)$. For simplicity, it is here assumed that the two liquids have the same uniform density $\rho$ and that gravity, surface tension, and viscosity may be neglected.

The linearized equations governing the perturbation pressure $p_2$ and perturbation velocity $\mathbf{v}_2$ in fluid 2 are

$$\nabla\cdot\mathbf{v}_2=0, \tag{1}$$

$$\rho(\partial_t+u\partial_z)\mathbf{v}_2+\nabla p_2=0. \tag{2}$$

The first equation is the usual condition for incompressible fluids, obtainable from the conservation-of-mass equation by setting the convective derivative of the density to zero. The second equation balances the fluid inertia against the pressure gradient.

Assuming that all perturbation quantities depend on $(y,z,t)$ through $\exp[i(k_yy+k_zz-\omega t)]$, Eqs. (1) and (2) imply

$$\mathbf{v}_2=\nabla p_2/(i\omega'\rho), \tag{3}$$

$$p_2\propto e^{-kx}, \qquad x>0, \tag{4}$$

where

$$k\equiv(k_y^2+k_z^2)^{1/2}>0 \tag{5}$$

and

$$\omega'\equiv\omega-k_zu. \tag{6}$$

For fluid 1, at rest in the space $x<0$, Eqs. (3) and (4) are to be replaced by

$$\mathbf{v}_1=\nabla p_1/(i\omega\rho), \tag{7}$$

$$p_1\propto e^{+kx}, \qquad x<0. \tag{8}$$

The boundary conditions are pressure balance

$$p_1=p_2 \quad \text{at} \quad x=0 \tag{9}$$

and the kinematic conditions, namely, that the normal velocity of the fluids, at $x=0$, match the rate of change of the interface displacement $\xi(y,z,t)$. These conditions can be written

$$v_{1x}=\partial_t\xi=-i\omega\xi,$$
$$v_{2x}=(\partial_t+u\partial_z)\xi=-i\omega'\xi, \quad \text{at} \quad x=0. \tag{10}$$

The term $u\partial_z\xi$ in Eq. (10) occurs because the perturbation velocity in fluid 2 at $x=0$ is partly due to the slight angular deviations of the steady flow as it is forced to stream along the actual ripples in the surface, as indicated in Fig. 1.

If one uses Eqs. (3), (4), (7), and (8) in the boundary conditions (9) and (10) and eliminates $\xi$, one finds the dispersion relation

$$\omega^2=-(\omega')^2. \tag{11}$$

Recalling Eq. (6), this has the solutions

$$\omega=\tfrac{1}{2}k_zu(1\pm i), \tag{12}$$

one of which is unstable.

The conclusion is that any relative tangential motion between two uniform regions of a liquid (in the absence of gravity, surface tension, and viscosity) causes the interface to be unstable to all perturbations whose wave vectors have a component along the flow. The surfaces of constant phase, as seen by the $z$ axis, move with the velocity $u/2$.

Esch[5] considered a smooth velocity transition $u(x)$ from the motionless region to the flowing region, also including viscosity. His results indicate that, when viscosity is negligible, only wavelengths larger than the order of magnitude of the shear layer thickness are unstable. He further found that viscosity tends to stabilize somewhat larger wavelengths. (But note the anomalous region in his Fig. 5.)

Menkes[6] considered an inviscid liquid, also with a smooth velocity transition between uniform regions. Like Esch,[5] he found that wavelengths less than the width of the transition region are stable. He included a density gradient as well and found it to have a stabilizing effect.

One might think that water waves generated by the wind furnish an excellent example of the Kelvin–Helmholtz instability between incompressible fluids. However, when one includes surface tension and gravity, which are stabilizing, one finds that the minimum wind speeds necessary to initiate water waves by this mechanism are about 600 cm/sec, whereas observation indicates that water waves are generated at wind speeds of about 10 cm/sec.[4,7] Miles[4,7] developed a theory essentially different from the Kelvin–Helmholtz type, in which the wind-shear profile plays an important role. A matching between the wind velocity at a certain height in the wind profile and the phase velocity of the water waves constitutes a resonance effect which leads to a minimum wind speed of the right order of magnitude. Miles[7] also cites another promising theory of Phillips based on turbulent fluctuations of the wind.

Amsden and Harlow[2] considered a viscous incompressible fluid with a shear layer of initially zero thickness. They solved the nonlinear fluid equations numerically and were able to follow the development of perturbations as they grow to large amplitudes.

Returning to the dispersion relation equation (12), one expects that if there is *no* relative velocity ($u=0$), then there is no growth of a perturbation. However, this is not the case. If one considers again the system of original equations (1) and (2), with the corresponding equations for fluid 1, and the boundary conditions (9) and (10), with $u=0$, so that the mathematics is done in the frame of the fluid, and if one again specifies a space dependence of the form $\exp[i(k_y y+k_z z)]$ but leaves the time dependence open, then one finds that if the interface is given an initial normal velocity $\partial_t\xi(y,z,t)_{t=0}$, the displacement $\xi(y,z,t)$ actually increases linearly with time. This surprising result was thought to be the explanation for the flapping of sails and flags.[1] In this case, the perturbation pressures vanish and the perturbation velocities are localized to within a distance of order $k^{-1}$ of the interface.

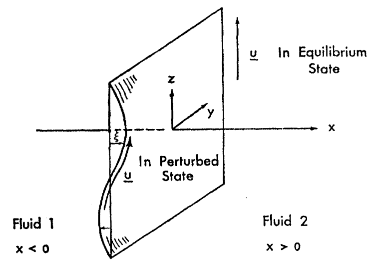


FIG. 1. Geometrical layout for the Kelvin–Helmholtz instability.

Whether or not imaginary surfaces in the air (since with no relative velocity there is now nothing that physically distinguishes fluid 1 from fluid 2) suffer these kinematic distortions that increase linearly with time is a matter that would be hard to settle experimentally. To identify such distortions with the instabilities of wind-blown sails and flags[1] is to lay oneself open to the objections that (a) the mass of the flag has been neglected, (b) the result is physically unrealistic because the rate of distortion proves to be independent of the wind speed.

One can solve in a straightforward way the more realistic problem of an infinite surface with a mass per unit area $\mu$, past which a wind is blowing on both sides with speed $u$. One then finds the phase velocity for a wave number $k$ to be given by $\mathrm{Re}(\omega)/k=u/(1+k\mu/2\rho)$ and the exponential growth rate to be $\mathrm{Im}(\omega)=ku/[(k\mu/2\rho)^{1/2}+(2\rho/k\mu)^{1/2}]$. For realistic densities $\rho$ and $\mu$ and wave numbers $k$, the phase velocity given above is much smaller than the wind speed, whereas in the "imaginary surface problem" the disturbance velocity proved to be equal to the wind speed.

The author has not followed the latest literature on this particular topic, but it seems clear that the time-proportional distortion mentioned in Lamb's book[1] can have little or nothing to do with the flapping of sails and flags.

### B. Compressible Fluids

When both fluids are compressible, there are density perturbations in each of them, denoted here by $\rho_1$ and $\rho_2$. Equation (1) must be replaced by the full, linearized continuity equation, namely, for fluid 2,

$$(\partial_t+u\partial_z)\rho_2+\rho\nabla\cdot\mathbf{v}_2=0. \tag{13}$$

If the adiabatic law holds (thus neglecting production and conduction of heat in the fluid), then the *total* pressure, density, and velocity are related by

$$(\partial_t+\mathbf{V}_2\cdot\nabla)(P_2\rho_2^{-\gamma})=0$$

or

$$(\partial_t+\mathbf{V}_2\cdot\nabla)P_2=\gamma(P_2/\rho_2)(\partial_t+\mathbf{V}_2\cdot\nabla)\rho_2,$$

where $\gamma$ is the ratio of specific heats, assumed to be the same for both fluids. The linearized version of this equation is

$$(\partial_t+u\partial_z)p_2=s^2(\partial_t+u\partial_z)\rho_2, \tag{14}$$

where the speed of sound $s$ is defined by

$$s^2 \equiv \gamma P/\rho. \tag{15}$$

Here $P$ and $\rho$ are the equilibrium pressure and density, assumed to be the same in both fluids. When all perturbation quantities depend on $(y,z,t)$ through the factor $\exp[i(k_y y + k_z z - \omega t)]$ and the velocity and density perturbations are eliminated from Eqs. (2), (13), and (14) in favor of the pressure perturbation, one finds that

$$p_2 \propto \exp(-q_2 x), \qquad x > 0, \tag{16}$$

where

$$q_2 \equiv [k^2 - (\omega')^2/s^2]^{1/2}. \tag{17}$$

If $q_2$ is complex, it must be chosen to have a positive real part to prevent blowup at $x \rightarrow \infty$. If $q_2$ is imaginary, its sign must be determined by proper application of the Sommerfeld radiation condition, as discussed shortly.

A similar procedure applied to fluid 1 shows that Eqs. (16) and (17) are to be replaced by

$$p_1 \propto \exp(+q_1 x), \qquad x < 0, \tag{18}$$

where

$$q_1 \equiv (k^2 - \omega^2/s^2)^{1/2}. \tag{19}$$

Again $q_1$ must have a positive real part. If, however, it is strictly imaginary, the correct sign of the root must again be determined by proper application of the Sommerfeld radiation condition.

If one now uses Eqs. (3), (7), (16), and (18) in the boundary conditions (9) and (10), one finds the dispersion relation

$$\omega^2 q_2 = -(\omega')^2 q_1. \tag{20}$$

As the speed of sound becomes infinite, $q_1$ and $q_2$ reduce to $k$, and this relation reduces to Eq. (11) for incompressible fluids. Although there are many parameters here ($\omega$, $k$, $k_z$, $u$, $s$), it is possible to go to just two dimensionless variables, a phase velocity $\phi$ and effective Mach number $M$, defined, respectively, by

$$\phi \equiv \omega/ks, \qquad M \equiv uk_z/ks \equiv u \cos\theta/s, \tag{21}$$

where $\theta$ is the angle between the vector $k \equiv (0, k_y, k_z)$ and $\mathbf{u}$.

Recalling Eqs. (6), (17), and (19), one can then convert Eq. (20) to the form

$$\phi^2[1-(\phi-M)^2]^{1/2} = -(\phi-M)^2(1-\phi^2)^{1/2}. \tag{22}$$

The most striking difference between the behavior of two uniform liquids and two uniform compressible fluids, as regards the K–H instability, is that the former are unstable for all (except $\mathbf{u} \perp \mathbf{k}$) wave vectors and all (nonzero) relative velocities, whereas the latter are *stable* for all those $\mathbf{K}$ modes whose effective Mach number is larger than a certain critical value. (This statement refers to idealized situations where gravity, surface tension, and viscosity are neglected.)

Suppose $\phi$ is real. If the square roots appearing in Eq. (22) were then real, they would have to be positive to satisfy the conditions on $q_1$ and $q_2$. But then (22) could not be satisfied. Therefore the square roots in Eq. (22) must be purely imaginary and of mutually opposite sign. Thus, when $\phi$ is real, one can write Eq. (22) as

$$F(\phi) \equiv \phi^2[(\phi-M)^2-1]^{1/2} - (\phi-M)^2(\phi^2-1)^{1/2} = 0, \tag{23}$$

where the square roots here are to be taken as positive. This equation is of fifth degree in $\phi$.

The function $F(\phi)$ defined by Eq. (23) is plotted schematically in Fig. 2 for $M$ greater than a critical value that is given below. For values of $M$ larger than about 3, the four outer roots lie very close to $\phi = -1$, $+1$, $M-1$, and $M+1$. These four roots are associated closely with backward and forward sound waves in fluids 1 and 2, respectively. Moreover, there is a new wave uniquely due to the presence of both fluids, represented by the center root of Fig. 2.

When one considers the outer roots of Fig. 2, namely, $\phi \approx -1$ and $\phi \approx M+1$, one finds that they are not acceptable. For all five of these real roots, $q_1$ and $q_2$ are purely imaginary. Therefore, the perturbation, proportional to $\exp(-q_2 x)$ for $x > 0$ and $\exp(+q_1 x)$ for $x < 0$, does not die away as $x \rightarrow \pm\infty$. Thus, a boundary condition at infinity has to be applied. The usual method in such cases is to state that one has only outgoing waves at infinity. Miles[12] has emphasized that this Sommerfeld radiation condition, to be applied in each fluid separately, must be used in a reference frame that moves with each fluid (or is at least subsonic with respect to the fluid). Moreover, one must take care that if the frequency relative to a fluid is negative, then a right-moving wave, for example, is given not by $\exp(ik_x x)$ with $k_x > 0$, but rather by $\exp(-ik_x x)$. When one does this, one finds that only the three center roots of Fig. 2 are valid, which one intuitively expects.

As $M \rightarrow 8^{1/2}$, from above, the two roots adjacent to the center root approach it. When $M$ crosses below the critical value $8^{1/2}$, those two roots disappear into the

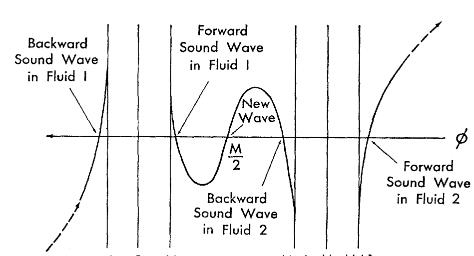


FIG. 2. The roots of the dispersion relation $F(\phi) = 0$ for supersonic relative flow ($M > 8^{1/2}$) between two compressible fluids of equal densities and sound speeds.

complex plane as complex conjugates, and the system becomes unstable to one of those modes.

It is possible to solve the fifth-degree Eq. (22) for $\phi$ exactly. Miles[11] has done so in solving the initial-value problem.

Miles and Fejer[13] have emphasized that the interface is stable only to wave vectors that make a sufficiently small angle with the stream velocity. From the definition of effective Mach number in Eq. (21), one notes that $M$ can be made less than $8^{1/2}$ simply by taking the propagation angle $\theta$ large enough. Thus, those modes propagating almost directly across the stream are always unstable.

Plesset and Hsieh[8] have included gravity $g$ in their treatment of this problem. They claimed that the stratification of the upper fluid, induced by even the slightest amount of gravity, gives rise to instabilities of finite growth rates at all speeds of supersonic flow, for modes propagating exactly in the stream direction. (In fact, they considered *only* that propagation direction.) Their paper[8] has been criticized by Miles,[9] who pointed out that a more likely cause of instability at low gravity is to be found in the ordinary Kelvin–Helmholtz modes that propagate almost transverse to the flow direction. He showed that the growth rates of the unstable gravity-induced modes vanish as $g \rightarrow 0$, in contradiction with Plesset and Hsieh. Miles claimed that their dispersion relation was incorrect because their solution of the differential equation in the upper fluid was wrong. However, Miles asserted that even their incorrect dispersion relation should lead to vanishing growth rates of the gravity-induced modes as $g \rightarrow 0$. He is unable to follow their arguments to the contrary. He concludes by noting that the vertical wavelengths of the gravity-induced modes become vanishingly small as $g \rightarrow 0$, so that these waves should be strongly damped by viscosity in a real fluid.

Also, Gill[10] studied a stratified situation consisting of a jet of compressible fluid shooting through a background of compressible fluid. For both the slab-geometry and cylindrical jet models, he found the flow to be unstable at all Mach numbers.

Chang and Russell[15a] studied the stability of the interface between an incompressible fluid (liquid) and a flowing compressible fluid (gas), including the effects of gravity, surface tension, and viscosity. In the nonviscous case, they also allowed the liquid to have a finite depth. [One should beware of misprints here; for example, their Eq. (2.8 a,b).] In the nonviscous case, they found that the supersonic flow was always unstable, but that the subsonic flow could be stabilized if the product of gravity and surface tension exceeded a critical value. They find these qualitative conclusions are unaltered when the liquid has a finite depth. In the case of a highly viscous liquid, they find the supersonic flow to be stable (when gravity is directed toward the heavy liquid), and that the subsonic flow is again stabilized when the product of gravity and surface tension exceeds a critical value—which is the same critical value as in the nonviscous case. [Their damped solutions $n_1$ to the viscous problem always violate their required inequality (3.14).] All of the stability conclusions of Chang and Russell apply only to waves propagating along the flow direction. This work has recently been extended for arbitrary viscosity by Willson and Chang.[15b]

### C. Incompressible MHD

In reviewing the K–H instability in conducting fluids in magnetic fields, the mathematical details will be omitted, as they are complicated and would make this review too long.

Sen[17] has included gravity and surface tension in his discussion, as well as allowing for different densities and different magnitudes and directions in the uniform magnetic fields on each side of the interface separating the incompressible fluids. As usual in treating this kind of problem, Sen assumes the fluids to have infinite electrical conductivity. He found the conditions under which the interface would be stable to all possible modes of perturbation, that is, stable for all possible magnitudes and directions of propagation vectors. As a special example, consider fluids of equal densities containing identical magnetic fields in the absence of surface tension and gravity. He found the interface to be unstable except when the moving fluid flows exactly along the field lines. The stability criterion for this aligned case is that the Alfvén speed should exceed half the relative flow speed.

In the more general case in which surface tension and gravity may be present, and the densities and fields are discontinuous across the interface, Sen found that two inequalities must be simultaneously fulfilled for a stable interface, rather than just one. In the absence of surface tension and gravity, Todd[20] has derived the same two stability criteria using a physical argument that balances the tension, exerted on the fluid by the bent field lines, against the centrifugal force of the fluid flowing along those curved lines.

Finally, Sen found that surface tension and gravity have a stabilizing effect (with the heavier fluid on the bottom) provided (i) that they are *both* present and (ii) that the fluids have unequal densities. His paper concludes by giving expressions for the wavelength, propagation angle with respect to the flow, and phase velocity of the wave which first grows when the stability conditions are slightly violated.

### D. Compressible MHD

Neglecting surface tension and gravity, Sen[24] has included compressibility in his study of the K–H instability of the interface between two perfectly conducting fluids. The zero-order magnetic field vector, the sound speed, and the density were at first allowed to be discontinuous at the interface. Sen then found that a slight amount of compressibility always destabilized an otherwise marginally stable perturbation.

Specializing next to the case in which the zero-order densities, sound speeds, and magnetic field vectors were identical in both fluids, Sen then studied the effect of large compressibility (supersonic flow). In fact, in this part of his paper he apparently set the sound speed to zero. With the magnetic field exactly parallel to the flow, he found the interface to be stable for all wave vectors, provided the flow speed was less than twice the Alfvén speed. This stability criterion is identical to that for the corresponding situation in incompressible fluids.

For flow speeds exceeding this critical value, the interface becomes unstable. However, those modes propagating at a small enough angle to the flow are still stable. If $u$ denotes the relative flow speed, $\alpha$ the Alfvén speed, and $\theta$ the propagation angle, Sen found the stability criterion to be

$$u \cos\theta > 8^{1/2}\alpha,$$

provided $u \gg \alpha$. This becomes identical to the stability criterion for the nonmagnetic case (Sec. IIB of this paper), namely $M > 8^{1/2}$, if we replace the Alfvén speed by the sound speed.

In the case that the flow is not parallel to the field, Sen found the interface to be always unstable. However, as in the case of field-aligned flow, there were again found to be sets of stable modes within certain propagation angles that depend on the flow speed.

Fejer[22b] had studied the same problem before Sen, solving the dispersion relation numerically for an arbitrary amount of compressibility. He obtained results that differ from Sen's. Both authors consider the case of identical densities, sound speeds, and magnetic field strengths on either side of the interface. They both work in a frame of reference stationary with respect to one of the fluids. They both consider the special case in which the magnetic fields are parallel to the flow vector. Sen[24] found, for this special case and zero sound speed, that the modes propagating directly along the flow are *stable* for *any* ratio of flow speed to Alfvén speed $u/\alpha$. Fejer[22] found, for this special case and zero sound speed, that the modes propagating directly along the flow are *unstable* when $u/\alpha > 1$. This discrepancy is apparently still not resolved. One point favoring Fejer's result is that his numerical calculations show that his stability condition $\alpha > u$ goes over smoothly, with increasing sound speed, into the stability condition known to hold for incompressible fluids, namely, $\alpha > u/2$. The origin of the discrepancy may be that either (or both) of these studies fails to check that the modes whose stability is being considered actually remain bounded at large distances from the interface.

Talwar,[23a] using the adiabatic equations derived by Chew, Goldberger, and Low,[34] has treated the stability of the interface in a fluid characterized by a nonscalar pressure. The uniform magnetic field was aligned with the flow, and he considered perturbations propagating only in this direction. Denoting the parallel and perpendicular sound speeds by $S_{\|,\perp}$, where $S_{\|,\perp}^2 \equiv P_{\|,\perp}/\rho$, Talwar found this class of modes to be overstable, provided both (i)

$$u < S_\perp^2/S_\| \sqrt{3}$$

and (ii)

$$\alpha^2 + 2S_\perp^2 > S_\perp^4(S_\perp^2 + 4S_\|^2)/(S_\perp^4 - 3S_\|^2u^2),$$

where $u$ is the relative flow speed and $\alpha$ is the Alfvén speed. If either of these criteria for instability is violated, Talwar found that these modes are monotonically unstable or stable, according to the following criterion:

$$\text{(iii)} \quad \begin{array}{c} \alpha^2 + 2S_\perp^2 \lesseqgtr S_\perp^4(4S_\|^2 + S_\perp^2) \\ [(\tfrac{1}{4}u^2 - 3S_\|^2)^2 + S_\perp^4] \end{array} \div \left\{ \begin{array}{l} \text{unstable,} \\ \\ \text{stable.} \end{array} \right.$$

In view of the results of Sen,[24] who found that at large flow speeds those modes propagating almost transverse to the flow are unstable (see his Fig. 1), one wonders about the importance of the stability criterion (iii) above, which applies only to those modes propagating along the field and flow.

An attempt at a more comprehensive and realistic treatment of the interface between two magnetofluids in relative motion was made by Lessen and Deshpande.[30] Rather than assume a uniform velocity discontinuity between the two fluids, they undertook first to calculate what the steady-state shear layer would look like if one of the fluids were flowing past a sharp edge, as shown in Fig. 3, the other fluid being at rest far below the edge. They allowed the compressible fluid to have both viscosity and thermal conductivity. Assuming the fluid to be a perfect conductor, they sought a solution characterized by a uniform aligned magnetic field far above the edge, which decreased to zero far below the edge. The numerical solution of the full set of nonlinear equations showed that, for an *incompressible* fluid, the shear layer became infinitely thick as the magnetic field was increased to the point at which $\alpha_0 = u_0$, these quantities being respectively the Alfvén

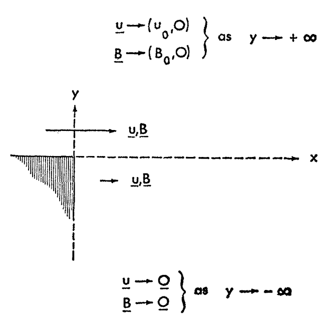


FIG. 3. The MHD half-jet. The flow **u** and the field **B** are vector functions of $(x, y)$, and are assumed to have no $z$ components.

speed and the flow speed far above the edge. A similar result had been derived by Greenspan and Carrier[35] for the incompressible flow past a flat plate, even allowing the fluid to have *any* finite conductivity. Thus the flow appears to be blocked when $\alpha_0 \to u_0$. No calculation was performed for the case $\alpha_0 > u_0$, although the authors of Ref. (35) state that no steady flow is possible in this case.

Allowing the fluid to be *compressible* and denoting the Mach number by $M_0 \equiv u_0/(\gamma p_0/\rho_0)^{1/2}$ far above the edge, Lessen and Deshpande found in this case also that the transition layer becomes infinitely wide when the magnetic field approaches a value determined by

$$\alpha_0^2 = \{-1+[1+2\gamma M_0^2(1+\tfrac{1}{2}(\gamma-1)M_0^2)]^{1/2}\}/\gamma M_0^2,$$

where $\gamma$ is the ratio of specific heats. No experiments have been performed to find the blocking of the flow at the critical magnetic field strengths, as predicted by the above theory.

In their following paper, Lessen and Deshpande studied the stability of the above-mentioned steady state to small perturbations propagating along $x$. Specializing to an incompressible fluid, they found numerically the frequency and wave number for marginally stable modes. The frequency and wave number of the marginally stable modes are eigenvalues obtained from the requirement that the perturbation should vanish properly far above and below the transition layer. They treated only the case where $\alpha_0 < u_0$. In dimensional form, the frequency and wave number are only valid locally; that is, they are dependent on the coordinate in the stream direction. This work has been extended to the stability of compressible magnetofluids.[30b]

A quite different approach was formulated by D'Angelo,[26] which perhaps explains an instability observed by him and von Goeler in a thermally ionized cesium plasma.[27] The basic difference between this theory and the others reviewed here is the neglect of the perturbation magnetic fields associated with plasma perturbation currents. The usual theories may be characterized by a magnetostatic approximation ($\nabla \times \mathbf{B} = 4\pi \mathbf{J}/c$) that neglects the displacement current $c^{-1}\partial_t \mathbf{E}$ and is appropriate when the electromagnetic energy storage resides primarily in the magnetic field. In D'Angelo's theory, on the other hand, the electrostatic approximation was made, $\nabla \times \mathbf{E} = 0$, that neglects the induction $-c^{-1}\partial_t \mathbf{B}$ and is appropriate when the electromagnetic energy storage resides primarily in the electric field. (Both approximations require the speed of light to be large compared to the wave velocity.)

D'Angelo[26] considered a steady state characterized by a uniform longitudinal magnetic field $B_0$ and a transverse plasma density gradient. The thermal motions of the electrons and ions and their extremely different masses prevent them from completely shielding each other, so that there is a transverse space–charge electric field $E_0$ associated with the density gradient. Moreover, the ions are assumed to stream along the field lines with a velocity that has a transverse gradient in the same direction as the density gradient. The space–charge field $E_0$ is assumed to be uniform, and the plasma density inhomogeneity is taken to have an exponential space dependence.

The theory next considered small perturbations of this steady state propagating in a plane perpendicular to the direction of the gradients. Because of the small electron mass, the electron density perturbation is assumed to be in Boltzmann equilibrium with the electrostatic potential perturbation at the ion temperature ($T_e \approx T_i$ in the cesium plasma). Quasineutrality is assumed ($n_e \approx n_i$) and the ion continuity and momentum equations are solved, subject to the relation between ion perturbation pressure and density $p_i = KT_i n_i$, where the ion temperature is supposed to be a given constant, not subject to perturbation. The restriction that the perturbation quantities be independent of the coordinate along the gradients then leads to a dispersion relation $f(\omega, \mathbf{k}) = 0$. Supposing the wavelength of the perturbation is large compared to the thermal ion cyclotron radius and that the perturbation frequency (as seen in a frame moving with the ions) is small compared to the ion cyclotron frequency, the dispersion relation is then solved explicitly for $\omega$, with the following results.

Seen from a frame moving along the field lines with the local ion longitudinal velocity component, the marginally stable modes are found to propagate in a direction perpendicular to $E_0$ and $B_0$ with the crossed-field drift velocity $cE_0/B_0$. These modes become unstable when the ion gradient of longitudinal velocity exceeds $(\sqrt{2}c_i/L)$, where $c_i$ is the ion sound speed, $c_i^2 \equiv \kappa T_i/m_i$, and $L$ is the characteristic length of the density gradient. The density gradient is thus seen to be stabilizing.

As seen from a stationary frame, the marginally stable modes are found to propagate at an angle $\theta$ with the field lines, where $\theta$ is determined by $\tan\theta = L\sqrt{2}/r_i$, where $r_i$ is the thermal ion cyclotron radius.

An experiment was performed[27] in a cesium plasma in which a gradient of longitudinal velocity was imposed. Qualitative agreement with the theory was obtained. The biggest discrepancy between the theoretical model and the experiment seems to be the assumption that the perturbations are independent of the coordinate along the gradients. However, as can be seen from Fig. 9 of Ref. 27 and the accompanying text, the perturbations are actually localized in a domain of about the same length as the density and velocity gradient regions. D'Angelo's theory apparently has not yet been usefully extended to include this effect.

Other, more sophisticated and more complicated theories relating to the Kelvin–Helmholtz instability in a plasma have been formulated by Rosenbluth and Simon[36,37] and by Stringer and Schmidt.[38] These theories again make the electrostatic approximation, but are more general than D'Angelo's theory since they include at once the effects of (a) nonuniform electric fields in the steady state, (b) finite ion cyclotron radius, (c) deviations from quasi-neutrality, and (d) nonuniform steady magnetic fields in the guise of an imposed gravitational drift. Finally, these theories allow the perturbations to depend on the coordinate along the gradients. The end result is a very complicated second-order differential equation for a perturbation quantity, in which this coordinate is the independent variable. For a given wave number in the plane perpendicular to this gradient coordinate, one should, in principle, solve for the eigenfunctions that represent perturbations localized in this direction and for their corresponding eigenfrequencies. By examining all such possible eigensolutions, one should then be able to classify them into stable and unstable modes, with their concomitant growth or damping rates. In practice, this approach has proved to be impossible (although a numerical method might be feasible here, to find the marginally stable modes, as in the work of Lessen and Deshpande[30]). Instead, Rosenbluth and Simon[36,37] have worked directly with the differential equation, and have managed, at least, to obtain sufficient conditions for stability to all those modes of perturbation representing propagation *across* the magnetic field. Their method was an extension of the usual calculus of variations technique applied to self-adjoint differential equations.[39] In some special cases, they were able to obtain necessary as well as sufficient conditions for stability. Their results are too complicated to give here.

## III. CONCLUDING REMARKS

This paper is neither a complete nor an up-to-date review of the literature on the K–H instability. Nevertheless, this survey may be a useful guide for those who are not specialists in the field. It seems clear that there yet remain many unanswered questions connected with the K–H phenomenon.

## ACKNOWLEDGMENTS

The author gratefully acknowledges a helpful discussion with D. Jerde concerning Ref. 20. He also contributed, together with R. Johnson, to the author's understanding of Ref. 30. Several useful talks with R. B. Hall and J. E. Drummond led to the comments at the end of Sec. IIA. Finally, the author would like to acknowledge some very enlightening discussions with M. Lessen.

## REFERENCES

Incompressible

1. H. Lamb, *Hydrodynamics* (Dover Publication, Inc., New York, 1945), Chap. IX.
2. A. A. Amsden and F. H. Harlow, Phys. Fluids **7**, 327 (1964).
3. G. K. Batchelor and A. E. Gill, J. Fluid Mech. **14**, 529 (1962).
4. J. W. Miles, J. Fluid Mech. **3**, 185 (1957).
5. R. E. Esch, J. Fluid Mech. **3**, 289 (1957).
6. J. Menkes, J. Fluid Mech. **6**, 518 (1959).
7. J. W. Miles, J. Fluid Mech. **6**, 568, 583 (1959).

Compressible

8. M. S. Plesset and D.-Y. Hsieh, Phys. Fluids **7**, 1099 (1964).
9. J. W. Miles, Phys. Fluids **8**, 1754 (1965).
10. A. E. Gill, Phys. Fluids **8**, 1428 (1965).
11. J. W. Miles, J. Fluid Mech. **4**, 538 (1958).
12. J. W. Miles, J. Acoust. Soc. Am. **29**, 226 (1957).
13. J. A. Fejer and J. W. Miles, J. Fluid Mech. **15**, 335 (1963).
14. S. C. Lennox and D. C. Pack, J. Fluid Mech. **15**, 513 (1963).

Compressible and Incompressible

15(a). I.-D. Chang and P. E. Russell, Phys. Fluids **8**, 1018 (1965).
15(b). A. J. Willson and I.-D. Chang, Phys. Fluids **10**, 2285 (1967).

Incompressible MHD

16. D. H. Michael, Proc. Cambridge Phil. Soc. **51**, 528 (1955).
17. A. K. Sen, Phys. Fluids **6**, 1154 (1963).
18. Z. Alterman, Phys. Fluids **4**, 1207 (1961).
19. W. I. Axford, Can. J. Phys. **40**, 654 (1962); Quart. J. Mech. Appl. Math. **13**, Pt. 3, 314 (1960).
20. L. Todd, Phys. Fluids **9**, 814 (1966).
21. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Clarendon Press, Oxford, England, 1961).

Compressible MHD

22. J. A. Fejer, (a) Phys. Fluids **6**, 508 (1963); (b) Phys. Fluids **7**, 499 (1964).
23. S. P. Talwar, (a) Phys. Fluids **8**, 1295 (1965); (b) J. Geophys. Res. **69**, 2707 (1964).
24. A. K. Sen, Phys. Fluids **7**, 1293 (1964).
25. T. G. Northrop, Phys. Rev. **103**, 1150 (1956).
*26. N. D'Angelo, Phys. Fluids **8**, 1748 (1965).
*27. N. D'Angelo and S. von Goeler, Phys. Fluids **9**, 309 (1966).
*28. S. v. Goeler, Phys. Fluids **9**, 818 (1966).
29. A. K. Sundaram, Phys. Fluids **8**, 195 (1965).
30(a). M. Lessen and N. V. Desphande, Phys. Fluids **9**, 1960, 1965 (1966).
30(b). M. Lessen and N. V. Desphande, Phys. Fluids **10**, 1968 (1967).
31. K. C. Wang and S. H. Maslen, Phys. Fluids **7**, 1780 (1964).
32. E. N. Parker, Astrophys. J. **139**, 690 (1964).
33. R. Gerwin, Phys. Fluids **10**, 2164 (1967). (There is no relative flow in this paper, but results obtained with flow are briefly stated.)
34. G. F. Chew, M. Goldberger, and F. Low, Proc. Roy. Soc. (London) **A236**, 112 (1956).
35. H. P. Greenspan and G. F. Carrier, J. Fluid Mech. **6**, 77 (1959).
*36. M. N. Rosenbluth and A. Simon, Phys. Fluids **8**, 1300 (1965).
*37. A. Simon and M. N. Rosenbluth, Phys. Fluids **9**, 726 (1966).
*38. T. E. Stringer and G. Schmidt, Plasma Phys. **9**, 53 (1967).
39. R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), Vol. I, Chap. VI.

* Electrostatic approximation.