Motion of a Charged Particle in a Uniform Magnetic Field

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The quantum-mechanical study of the motion of a charged spin-$\frac{1}{2}$ particle in a uniform magnetic field has a number of features which are of interest to theorists. Not only can the Dirac equations be solved completely and analytical solutions be obtained, but unlike the Coulomb problem, the expressions for absorption and emission of radiation can be evaluated exactly with these exact Dirac wave functions without resorting to a multipole expansion of the usual sort that is made in the theory of the hydrogen atom. In the ultrarelativistic limit, reasonably simple, closed expressions can be obtained for synchrotron radiation, or photoproduction of electron–positron pairs in a magnetic field. The latter expression allows one to associate an absorption coefficient with the magnetic field *in vacuo*, and through the use of dispersion theory, to obtain the real (dispersive) part of the index of refraction. The use of exact wave functions, rather than plane waves, to describe the electron may simplify the anomalous-moment calculations and shed some light on the computation of the complete power series in α.

I. INTRODUCTION

This paper is divided into two parts: The first part deals with the theory of emission of a single photon by an electron in a uniform **B** field. This leads to a quantum-mechanical expression for "synchrotron radiation" in the extreme relativistic (ER) limit, which can be compared with the well-known classical formulas. The second part deals with e^+-e^- pair production in a magnetic field by a single photon of wave number κ.

The calculation is patterned after the earlier work of Klepikov,[1] but differs from it in that we will assume that the particles undergo circular or helical motion in the magnetic field rather than the linear simple harmonic motion assumed by Klepikov. To this end, we have departed from the gauge choice made by Klepikov and introduced a cylindrical gauge in a cylindrical coordinate system, since this seems to be the natural choice for describing circular or helical motion.

It is shown elsewhere[2] that the energy levels of a charged particle in a magnetic field are hyperdegenerate, i.e., to each eigenvalue of energy, denoted here by l, there is a denumerably infinite number of eigenstates of angular momentum m, where m refers to $m=l$ down to $m=-\infty$. The classical circular motion of a positive charge in a sense opposite to B, corresponds to setting $m=l$. This hyperdegeneracy of the energy levels plays a crucial role in bringing the quantum-mechanical calculation of synchrotron radiation into agreement with the classical result.

A controversy about the validity of the quantum-mechanical calculation arose in the early 1950's, when Parzen[3] produced a formula for synchrotron radiation which differed from the earlier classical results of Schwinger[4] and others (including the original work of Schott published in 1907). Parzen restricted his computation to transitions between circular orbitals and obtained a formula which differed from the classical formula only by a factor of $\exp[-\frac{1}{2}(\omega^2/c^2)(\hbar c/eB)]$, where ω/c is the wave number of the emitted photon and B is the magnetic-field strength. This seems to be a plausible restriction, since in classical physics a charged particle can undergo only circular motion in a uniform magnetic field (neglecting radiation).

After Parzen's work was published in *The Physical Review*, a number of people sent letters to the Editor[5,6] in which it was pointed out that by allowing noncircular orbitals and considering transitions from a given initial state to an infinite number of final states differing in angular momentum, but with the same energy, the total transition probability comes into exact agreement with classical theory. Because the exponential factor $\exp[(-\omega^2/2c^2)(\hbar c/eB)]$ would greatly suppress synchrotron radiation at high frequencies, it was of considerable practical interest to verify by experiment which theory was correct. The early results on the visible radiation spectrum obtained with the 12-in. Schenectady synchrotron[7] were not decisive (the exponential factor being nearly unity) and even showed a slight ultraviolet drop-off, which tended to favor Parzen's hypothesis. In early 1952 Corson and Hartman performed experiments with the high-energy machine at Cornell which provided a crucial test.[8] In fact, one *must* allow an infinite number of orbitals in the calculation, the effect of which is a cancellation of the exponential factor in the final result.

It can further be argued on additional theoretical grounds, that the exponential factor must be spurious. By calculating the rate of photoproduction of e^+, e^- pairs by a beam of photons traversing a strong magnetic

[1] H. P. Klepikov, Zh. Eksp. Teor. Fiz. **26**, 19 (1954).
[2] J. J. Klein, paper read at AAPT meeting, Chicago, Ill., 30 January 1968.
[3] G. Parzen, Phys. Rev. **84**, 235 (1951).
[4] J. Schwinger, Phys. Rev. **75**, 1912 (1949).
[5] D. Judd, J. Lepore, M. Ruderman, and P. Wolff, Phys. Rev. **86**, 123 (1952).
[6] H. Olsen and H. Wergeland, Phys. Rev. **86**, 123 (1952).
[7] F. Elder, R. Langmuir, and H. Pollack, Phys. Rev. **74**, 52 (1948).
[8] D. R. Corson, Phys. Rev. **90**, 748 (1953); P. L. Hartman and D. H. Tomboulian, Phys. Rev. **A87**, 233 (1952).

field and using the dispersion relations of light to compute the real part of the index of refraction from the absorptive part, Toll[9] and Erber[10] obtain the low-frequency limit of $n(\omega)$ which agrees with the well-known nonlinear optical behavior of the vacuum.[11] An extraneous factor of $\exp[(-\omega^2/2c^2)(\hbar c/eB)]$ in the absorption coefficient of the vacuum would spoil the agreement effected by dispersion theory. The mathematical argument which removes this factor is identical with that used in synchrotron-radiation theory.

II. EMISSION OR ABSORPTION OF RADIATION

We use the following wave functions for a charged Dirac particle in a uniform magnetic field B in the z direction. They are derived by writing the Dirac equation in cylindrical coordinates, and assuming that the vector potential has only the cylindrical ϕ component

$$A_\phi = \tfrac{1}{2}\rho B.$$

In Dirac theory, a natural unit of field strength

$$B_0 = m^2c^3/e\hbar$$

occurs, so we express magnetic fields in this unit, just as the energy is expressed in units of mc^2. Omitting the time dependence, we have

$$\Psi = \begin{bmatrix} i\alpha F_{lm}(\rho)\exp(-im\phi) \\ 0 \\ 0 \\ \beta G_{lm}(\rho)\exp[-i(m-1)\phi] \end{bmatrix} \quad \text{for spin-up states,} \qquad (1)$$

$$\Psi = \begin{bmatrix} 0 \\ i\alpha G_{lm}(\rho)\exp[-i(m-1)\phi] \\ \beta F_{lm}(\rho)\exp(-im\phi) \\ 0 \end{bmatrix} \quad \text{for spin-down states,} \qquad (2)$$

where

$$F_{lm}(\rho) = e^{-t/2}t^{m/2}L_{l-m}{}^{m}(t)[(l-m)!B/2\pi l!^3]^{1/2}, \qquad (3a)$$

$$G_{lm}(\rho) = e^{-t/2}t^{(m-1)/2}L_{l-m}{}^{m-1}(t)[(l-m)!B/2\pi(l-1)!^3]^{1/2}, \qquad (3b)$$

with

$$t = B\rho^2/2$$

and

$$\alpha = [(E+1)/2E]^{1/2}, \qquad \beta = [(E-1)/2E]^{1/2}, \qquad (4)$$

where ρ is the cylindrical-coordinate radial vector in the xy plane (in units of $\hbar/mc$), E is the total energy (in units of mc^2) $=(1+2lB)^{1/2}$, $l=1, 2, 3\cdots$ and $m=l, l-1\cdots 1, 0, -1\cdots -\infty$, and B is the magnetic field strength (in units of $m^2c^3/e\hbar$).

These wave functions do not allow for motion of the circulating particle along the z axis; hence in our discussion of emission we assume the photon is emitted at an angle $\theta=\pi/2$. One can allow z motion by suitably modifying the wave functions (see, for example, Sec. III), but since we intend to discuss "synchrotron" radiation, the restriction to emission at right angles to the field is not serious, most of the radiation being emitted in the equatorial plane anyway.

According to quantum electrodynamics, the probability of emission of a single photon, polarized in the x or z direction, involves the evaluation of the matrix elements of α_x or α_z, taken between two states, designated by m, l and m', l' (where $l'<l$), with the photon's vector potential $\exp(i\boldsymbol{\kappa}\cdot\mathbf{r})$ sandwiched between them. (In dipole approximation, this factor is set equal to one.)

We obtain the following expressions for α_x and α_z:

$$i\alpha_x = \alpha'\beta I(l', m'; l-1, m-1) + \beta'\alpha I(l'-1, m'-1; l, m), \qquad (5a)$$

this corresponds to the absence of spin flip and represents electric-dipole radiation;

$$i\alpha_z = \alpha'\beta I(l', m'; l, m) - \beta'\alpha I(l'-1, m'-1; l-1, m-1) \qquad (5b)$$

this corresponds to spin flip and hence to magnetic-dipole radiation.

$$I(l', m'; l, m) = (2\pi)^{-1}\int_0^{2\pi} d\phi \int_0^\infty dt e^{-t}t^{m'+m/2} \times \exp[i\kappa\rho\cos\phi + i(m'-m)\phi]L_{l'-m'}{}^{m'}(t)L_{l-m}{}^{m}(t) \times \{[(l'-m')!/l'!^3][(l-m)!/l!^3]\}^{1/2}, \qquad (6)$$

where κ is the photon propagation vector, y is the direction of emission of the photon, and α_x or α_z is the polarization of the photon.

It can be shown that this can be reduced to the simpler form:

$$I(l', m; l, m) = \left[\frac{(l-m)!}{(l'-m')!}\right]^{1/2}(-q)^{(l'-m')/2}(-q)^{-[(l-m)/2]} \times \sum_{r=0}^{l-m}\frac{(-q)^r}{r!}\binom{l'-m'}{l-m-r}\cdot I(l'; l), \qquad (7)$$

[9] J. S. Toll, Ph.D. dissertation, Princeton University, Princeton, N.J. (1952).

[10] T. Erber, "The Index of Refraction of a Magnetic Field," in *Proceedings of the International Conference on High Magnetic Fields, Cambridge, Mass., 1961* (M.I.T. Press, Cambridge, Mass., 1961).

[11] M. Euler and W. Heisenberg, Z. Physik **98**, 714 (1936); R. Karplus and M. Neuman, Phys. Rev. **80**, 380 (1950); A. Minguzzi, Nuovo Cimento **6**, 501 (1957); J. J. Klein and B. P. Nigam, Phys. Rev. **135**, B1279 (1964). (The refractive indices derived in the last paper are incorrect.)

with the abbreviation $I(l';l)$ for the expression

$$I(l';l)=\frac{\Gamma(l+1)}{\Gamma(l-l'+1)}F(-l',l-l'+1;q)q^{(l-l')/2}\cdot e^{-q}\times\exp[i(l-l')\tfrac{1}{2}\pi]\cdot(l'!l!)^{-1/2},\quad (8)$$

where $q=\kappa^2/2B$.

The following simplification is now possible. We write

$$i\alpha_x=(-q)^{(l'-m')/2}(-q)^{-(l-m)/2}\left[\frac{(l-m)!}{(l'-m')!}\right]^{1/2}\times\sum_{r=0}^{l-m}\frac{(-q)^r}{r!}\binom{l'-m'}{l-m-r}[\alpha'\beta I(l';l-1)+\beta'\alpha I(l'-1;l)]\quad (9)$$

and a similar expression for α_z, except that the last factor in brackets is

$$[\alpha'\beta I(l';l)-\beta'\alpha I(l'-1;l-1)].$$

In order to determine the transition probability for the process $l\to l'$, we square the expression for α_x and α_z, then sum over all final states m', and then average over all initial states m. The terms in brackets are independent of m' and m, and so are irrelevant to the summation on m', or the averaging process. The summation on m' runs from $m'=l', l'-1\cdots$ down to $-\infty$.

The result is remarkably simple:

$$\sum_{m'=l'}^{-\infty}\frac{(l-m)!}{(l'-m')!}q^{(l'-m')}q^{-(l-m)}\times\left[\sum_{r=0}^{l-m}\frac{(-q)^r}{r!}\binom{l'-m'}{l-m-r}\right]^2=e^q.\quad (10)$$

One can then write

$$\sum_{m'}|\alpha_x|^2=e^q[\alpha'\beta I(l';l-1)+\beta'\alpha I(l'-1;l)]^2,\quad (11a)$$

$$\sum_{m'}|\alpha_z|^2=e^q[\alpha'\beta I(l';l)-\beta'\alpha I(l'-1;l-1)]^2.\quad (11b)$$

For mathematical convenience, it is desirable to redefine the expression for I so that the above factor of e^q is absorbed in it. This means that one replaces the factor e^{-q} in the original definition of I, Eq. (8), by $e^{-q/2}$.

Since l' and l are integers, the hypergeometric function above is actually a Laguerre polynomial in q. We now seek an asymptotic formula for $I(l',l)$ valid for $l', l\gg 1$. To this end, we introduce the following integral representation of the confluent hypergeometric function:

$$F(a,c;z)=\frac{1}{2\pi i}\frac{\Gamma(1-a)\Gamma(c)}{\Gamma(c-a)}\oint_c e^{zt}(1-t)^{c-a-1}t^{a-1}dt.\quad (12)$$

(After making the substitution $t=1-t/z$, this yields the familiar derivative expression for the Laguerre polynomials after application of Cauchy's theorem to the pole at $t=z$, assuming a to be a negative integer, and c to be a positive integer.)

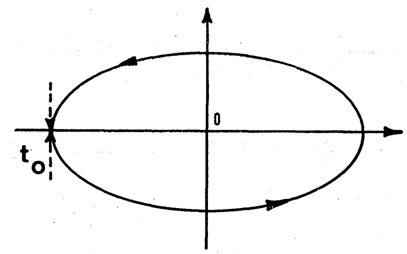

FIG. 1. Contour used in synchrotron radiation theory.

After setting $a=-l'$, $c=l-l'+1$, $z=q$:

$$I(l',l)=\exp[-\tfrac{1}{2}q+\tfrac{1}{2}(i\pi)(l-l')]\cdot q^{(l-l')/2}\cdot(l'!/l!)^{1/2}\times(2\pi i)^{-1}\oint_c e^{qt}(1-t)^l t^{-l'-1}dt.\quad (13)$$

The contour C runs around the multiple pole at $t=0$ (Fig. 1). In the problem of synchrotron radiation, it will be shown that most of the contribution to the integral comes from the vicinity of the point t_0 on the negative real axis, with the contour chosen to lie along the vertical line $t=t_0+i\tau$ in the vicinity of that point. t_0 can be shown to lie halfway between two nearby saddle points which lie on the negative real axis, and can be regarded as a point of coalescence of these saddle points in the extreme relativistic limit.

To obtain an asymptotic formula for $I(l',l)$, it is convenient to change variables by writing

$$q=l(1-\cos\alpha)^2\operatorname{sech}\beta$$

and

$$l'=l\cos^2\alpha,$$

following Klepikov's notation. [In general, there are two independent parameters allowed in this problem, namely, l' and q, which may be related to the energy and direction of the emitted photon. Since we are restricting ourselves to the direction $\theta=\pi/2$, there is only one free parameter l', the conservation-of-energy condition giving

$$\kappa=(1+2lB)^{1/2}-(1+2l'B)^{1/2},$$

which determines q.] $I(l',l)$ now involves the integral

$$(2\pi i)^{-1}\oint_c \exp l[t(1-\cos\alpha)^2\operatorname{sech}\beta+\ln(1-t)-\cos^2\alpha\cdot\ln t]\cdot\frac{dt}{t},\quad (14)$$

where $l\to\infty$. Note that in the ER limit,

$$\kappa\simeq(2B)^{1/2}[(l)^{1/2}-(l')^{1/2}],$$

so that $q=\kappa^2/2B$ is approximately $l(1-\cos\alpha)^2$. Thus, in this limit $\operatorname{sech}\beta$ is nearly unity. The following asymptotic analysis is applied to the integral in question. Let

$$f(t)=t(1-\cos\alpha)^2\operatorname{sech}\beta+\ln(1-t)-\ln t\cdot\cos^2\alpha. \quad (15)$$

The first three derivatives are readily obtained. Instead of setting $f'(t)=0$ as in the usual saddle-point method, we instead set $f''(t)=0$, obtaining

$$t=t_0=-\cos\alpha/(1-\cos\alpha),$$

which lies on the negative real axis.

We have then the asymptotic expression:

$$I(l',l)=\exp\left[-\tfrac{1}{2}q+\tfrac{1}{2}(i\pi)(l-l')\right]\cdot q^{(l-l')/2}\cdot(l'!/l!)^{1/2}$$
$$\times\exp(qt_0)(1-t_0)^{l}t_0^{-(l'+1)}$$
$$\times(2\pi)^{-1}\oint_c\exp il\left[f'(t_0)\tau-\tfrac{1}{6}f'''(t_0)\tau^3\right]\cdot d\tau, \quad (16)$$

where we have set $t=t_0+i\tau$ in the contour integral. The prefactor of the contour integral can be simplified considerably through use of Stirling's formula

$$l!\sim(2\pi l)^{1/2}\cdot l^l\cdot e^{-l},\quad \text{valid for } l\gg1.$$

The result is

$$\exp\left[i\tfrac{1}{2}\pi(l+l')\right]\cdot(1-\cos\alpha)/(\cos\alpha)^{1/2}.$$

Turning now to the integral, we find

$$f'(t_0)=-(1-\cos\alpha)^2(1-\operatorname{sech}\beta)=a, \quad (17a)$$

$$f'''(t_0)=2(1-\cos\alpha)^4/\cos\alpha=2b. \quad (17b)$$

Thus the integral has the form

$$(2\pi)^{-1}\oint_c\exp\left[-il(a\tau+\tfrac{1}{3}b\tau^3)\right]\cdot d\tau. \quad (18)$$

The reason for running the contour through t_0, parallel to the imaginary t axis is now evident. The function $\phi(\tau)=a\tau+\frac{1}{3}b\tau^3$ has the form shown in Fig. 2. The slope a, near $\tau=0$ is very small, since in the ER limit, as noted above, $(1-\operatorname{sech}\beta)\to0$. The integral is therefore a highly oscillatory function having a stationary phase at t_0, provided the contour runs in the direction chosen above. Since l is very large, most of the contribution comes near $\tau=0$. Thus

$$(2\pi)^{-1}\oint_c\exp\left[-il(a\tau+\tfrac{1}{3}b\tau^3)\right]\cdot d\tau$$
$$\sim\pi^{-1}\int_0^\infty\cos l(a\tau+\tfrac{1}{3}b\tau^3)\cdot d\tau. \quad (19)$$

Using the well-known formula

$$\int_0^\infty\cos(tx+x^3/3)\,dx=\tfrac{1}{3}(t)^{1/2}K_{1/3}(\tfrac{2}{3}t^{3/2}), \quad (20)$$

the above integral becomes

$$(1/\pi\sqrt{3})(a/b)^{1/2}K_{1/3}\left[\tfrac{2}{3}l(a^3/b)^{1/2}\right]. \quad (21)$$

If one substitutes the expressions for a and b, and restores the original variables l' and l in place of α and β, one obtains the following asymptotic expression for $|I(l';l)|$ given by Klepikov:

$$|I(l',l)|=\{[l^{1/2}-(l')^{1/2}]^2-q\}^{1/2}$$
$$\times[l^{1/2}-(l')^{1/2}]^{-1}\cdot K_{1/3}/\pi\sqrt{3}. \quad (22)$$

The argument of the Bessel function is the rather cumbersome expression

$$\tfrac{2}{3}(l'l)^{1/4}\{[l^{1/2}-(l')^{1/2}]^2-q\}^{3/2}[l^{1/2}-(l')^{1/2}]^{-2}. \quad (23)$$

It can be shown that

$$|\partial I/\partial q|=(l'l)^{1/4}\{[l^{1/2}-(l')^{1/2}]^2-q\}$$
$$\times[l^{1/2}-(l')^{1/2}]^{-3}\cdot K_{2/3}/\pi\sqrt{3}, \quad (24)$$

with the same argument for $K_{2/3}$ as in Eq. (22).

One may now employ certain identities derived by Klepikov [these are Eqs. (1.26)–(1.31) of his paper], which can be derived from the properties of the confluent hypergeometric functions. We quote three of them here to show that all of the integrals involved in α_x and α_z are expressable in terms of $I(l',l)$ alone:

$$(ll')^{1/2}I(l'-1,l-1)$$
$$=\tfrac{1}{2}(l+l'-q)I(l',l)-q[\partial I(l',l)/\partial q], \quad (25a)$$

$$i(lq)^{1/2}I(l',l-1)$$
$$=\tfrac{1}{2}(q+l-l')I(l',l)+q[\partial I(l',l)/\partial q], \quad (25b)$$

$$i(l'q)^{1/2}I(l'-1,l)$$
$$=\tfrac{1}{2}(q+l'-l)I(l',l)+q[\partial I(l',l)/\partial q]. \quad (25c)$$

In this manner, all of the integrals needed can be expressed in terms of $I(l',l)$ or its derivative with respect to q.

The remainder of the calculation is straightforward. We discuss only the extreme relativistic case, whereby the normalizing constants α, β are equal to $1/\sqrt{2}$, and hence

$$|\alpha_x|^2=\tfrac{1}{4}[I(l',l-1)+I(l'-1,l)]^2, \quad (26)$$

$$|\alpha_z|^2=\tfrac{1}{4}[I(l',l)-I(l'-1,l-1)]^2. \quad (27)$$

In the corresponding classical calculations, the radiation at $\theta=\pi/2$ is completely linearly polarized in the

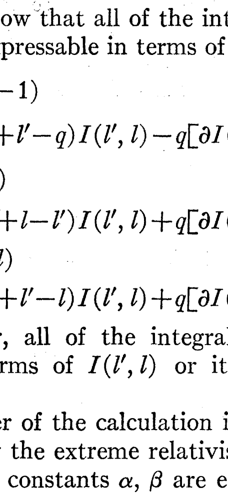

FIG. 2. Behavior of the function $\phi(\tau)$ vs τ.

plane of circulation of the charge, since the electron in the usual classical treatments is regarded as having no magnetic dipole. Here, however, we see that a contribution $|\alpha_z|^2$ arises from the magnetic moment of the Dirac electron.

If the identities (25b) and (25c) are used and the extreme relativistic limit obtained, Eq. (26) reduces to

$$|\alpha_x|^2=\tfrac{1}{4}\{(l-l')/(ll')^{1/2}[\partial I(l',l)/\partial q]\}^2. \quad (28)$$

Thus

$$|\alpha_x|^2=\frac{1}{4}\frac{1}{3\pi^2}\left[\frac{(l)^{1/2}+(l')^{1/2}}{2B(ll')^{3/4}}K_{2/3}\right]^2. \quad (29)$$

A similar calculation for α_z^2 yields

$$|\alpha_z|^2=\frac{1}{4}\frac{1}{3\pi^2}\left[\frac{(l)^{1/2}-(l')^{1/2}}{2B(ll')^{3/4}}K_{2/3}\right]^2. \quad (30)$$

The electric-type radiation intensity polarized in the x direction greatly exceeds the magnetic-type (due to spin flip) polarized in the z direction, in conformity with observation. Summing over polarizations, we obtain

$$|\alpha_x|^2+|\alpha_z|^2=\frac{1}{2}\frac{1}{3\pi^2}\cdot\frac{(l+l')}{4B^2[(ll')^{1/2}]^3}K_{2/3}^2, \quad (31)$$

the argument of the Bessel function being

$$\tfrac{2}{3}[l^{1/2}-(l')^{1/2}/(ll')^{1/2}][1/(2B)^{3/2}].$$

In deriving the radiation law, one starts with the formula (Fermi's golden rule)

$$dw=(2\pi/\hbar)|U|^2\rho_f, \quad (32)$$

where w is the probability/time for emission or absorption of a photon

$$U=(e/V^{1/2})(2\pi\hbar c/\kappa)^{1/2}(\boldsymbol{\alpha}\cdot\mathbf{e}), \quad (33)$$

with $\mathbf{e}$, $\boldsymbol{\kappa}$ as photon polarization and propagation vectors. As usual, ρ_f is the density of final states and may be written as

$$\rho_f=[V/(2\pi)^3](\kappa^2 d\Omega_\kappa/\hbar c). \quad (34)$$

Therefore

$$dw=[4\pi^2/(2\pi)^2]\cdot(e^2\kappa d\Omega_\kappa/\hbar)\cdot\tfrac{1}{2}(\alpha_x^2+\alpha_z^2). \quad (35)$$

The intensity of the emitted radiation is $dI=dw\cdot\hbar c\kappa$:

$$dI=(e^2c/2\pi)\cdot\tfrac{1}{2}(\alpha_x^2+\alpha_z^2)\cdot\kappa^2\cdot 2\pi\sin\theta d\theta. \quad (36)$$

The factor of $\frac{1}{2}$ arises from averaging over polarization directions:

$$(dI/d\theta)_{\theta=\pi/2}=e^2c\cdot(eB/\hbar c)\cdot(\hbar c/eB)\kappa^2\cdot\tfrac{1}{2}(\alpha_x^2+\alpha_z^2). \quad (37)$$

The conservation-of-energy restriction implies that

$$\tfrac{1}{2}\kappa^2(\hbar c/eB)\simeq[l^{1/2}-(l')^{1/2}]^2$$

in the ER limit.

After inserting the previously derived expressions for $|\alpha_x|^2$ and $|\alpha_z|^2$, one obtains

$$\left(\frac{dI}{d\theta}\right)_{\theta=\pi/2}=e^2c\cdot\frac{eB}{\hbar c}\cdot[l^{1/2}-(l')^{1/2}]^2\cdot\frac{(B_0/2B)^2}{3\pi^2}\times\frac{l+l'}{[(ll')^{1/2}]^3}K_{2/3}^2,$$

where the argument of the $K_{2/3}$ function is, in the ER limit,

$$\tfrac{2}{3}[l^{1/2}-(l')^{1/2}/(ll')^{1/2}][1/(2B)^{3/2}]. \quad (38)$$

To compare this result with the classical formula for synchrotron radiation, we define the order n of the harmonic by means of the relations

$$\omega\simeq 2B[l^{1/2}-(l')^{1/2}] \quad\text{and}\quad \omega_0\simeq(2B)^{1/2}/2l^{1/2}. \quad (39)$$

ω_0 is obtained by setting $l'=l-1$. It is identical with the frequency of revolution of the charge, or precession of the particle's magnetic dipole axis, in the magnetic field. ω is the radiation frequency, which, in the ER limit, is predominately a very high-order harmonic of ω_0:

$$n=\omega/\omega_0=[l^{1/2}-(l')^{1/2}]\cdot 2l^{1/2}. \quad (40)$$

We obtain

$$\frac{dI_n}{dnd\theta}=e^2c\cdot\frac{eB}{\hbar c}\cdot\frac{1}{3\pi^2}\frac{n^2}{2l}\frac{B_0}{4l^2B^2}\cdot K_{2/3}^2, \quad (41)$$

but $2lB/B_0\simeq(1-\beta^2)^{-1}$:

$$\frac{dI_n}{dnd\theta}=e^2\cdot\frac{eB}{\hbar}\cdot\frac{1}{3\pi^2}\cdot\tfrac{1}{2}n^2\cdot\frac{2B}{B_0}(1-\beta^2)^3\times K_{2/3}^2\left[\frac{2}{3}\frac{l^{1/2}-(l')^{1/2}}{(ll')^{1/2}}\frac{1}{(2B)^{3/2}}\right]. \quad (42)$$

The argument of the Bessel function can be rewritten as

$$\tfrac{1}{3}n(1-\beta^2)^{3/2}. \quad (43)$$

It can be shown that the following identity follows from a famous formula due to Watson.[12]

As $n\rightarrow\infty$,

$$K_{2/3}[\tfrac{1}{3}n(1-\beta^2)^{3/2}]\sim[\pi\sqrt{3}/\beta(1-\beta^2)]J_n'(n\beta). \quad (44)$$

Using this and writing B_0 as $m^2c^3/e\hbar$, one obtains

$$dI_n/dnd\theta=(e^4B^2n^2/m^2c^3)(1-\beta^2)J_n'^2(n\beta). \quad (45)$$

(note the cancellation of $\hbar$ and $\pi\sqrt{3}$). This agrees with the well-known classically derived result.[13]

III. PHOTOPRODUCTION OF ELECTRON–POSITRON PAIRS IN A MAGNETIC FIELD

Pair production can be thought of as absorption of a photon of energy $\hbar c\kappa$, thereby raising a negative-energy

[12] G. N. Watson, *A Treatise on Bessel Functions* (Cambridge University Press, London, 1944), 2nd ed., pp. 248–252.

[13] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1952), p. 216.

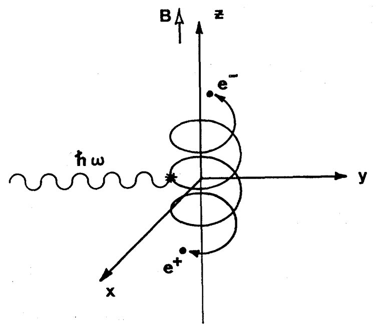

FIG. 3. Pair creation by a photon traversing a magnetic field.

electron across the $2mc^2$-gap into a positive-energy state. We assume the incident photon travels perpendicular to the magnetic field, for example, along the y direction. The positive-energy electron, and the vacancy in the negative-energy "sea" (or positron) acquire momenta of equal magnitude during the photoeffect, travelling in opposite directions along helices whose sense is opposed to that of the field direction B. The photon may be polarized either in the x or z direction (Fig. 3).

However, the motion along the z direction forces us to use Dirac wave functions of a more complicated structure than those used in Part II. The final result is

$$\Psi=\frac{\exp(ik_3z)}{L^{1/2}}\begin{bmatrix} i\alpha AF_{lm}(\rho)\exp(-im\phi) \\ \alpha BG_{lm}(\rho)\exp[-i(m-1)\phi]s \\ i\beta AF_{lm}(\rho)\exp(-im\phi)st \\ \beta BG_{lm}(\rho)\exp[-i(m-1)\phi]t \end{bmatrix}, \quad (46)$$

where, as before, F_{lm} and G_{lm} are given by Eqs. (3a) and (3b).

The normalization constants A, B are

$$\alpha=[(E+t)/2E]^{1/2}, \qquad \beta=[(E-t)/2E]^{1/2}, \quad (47)$$

$$A=[(k+sk_3)/2k]^{1/2}, \qquad B=[(k-sk_3)/2k]^{1/2}. \quad (48)$$

(This B should not be confused with B standing for magnetic field strength.) $t=\pm1$ depending on whether the state in question is assumed to be of positive or negative energy. $s=\pm1$ depending on the electron-spin orientation relative to the field direction. $E^2=1+2lB+k_3^2$ (in natural units), and $k^2=E^2-1=k_3^3+2lB$.

As in Klepikov's work, the resulting expressions which we obtain for α_x and α_z are now rather complicated, e.g.,

$$i\alpha_x=\alpha'A'\beta BtI(l',l-1)+\alpha'B's'\beta AstI(l'-1,l)$$
$$+\beta'A's't'\alpha BsI(l',l-1)+\beta'B't'\alpha A(l'-1,l). \quad (49)$$

After squaring the matrix element of α_x and summing over the initial and final spins s, $s'=\pm1$, one obtains

$$\sum_{s,s'}|\alpha_x|^2=\frac{E'E-t't-k_3'k_3t't}{2E'E}[I(l',l-1)^2+I(l'-1,l)^2]$$
$$+\frac{t'tB(l'l)^{1/2}}{E'E}2I(l',l-1)I(l'-1,l). \quad (50)$$

Similarly

$$\sum_{s,s'}|\alpha_z|^2=\frac{E'E-t't+k_3'k_3t't}{2E'E}[I(l',l)^2+I(l'-1,l-1)^2]$$
$$-\frac{t'tB(l'l)^{1/2}}{E'E}2I(l',l)I(l'-1,l-1). \quad (51)$$

Considerable simplification results in the ultrarelativistic regime, that is, where e^--e^+ pairs have energies greatly exceeding mc^2. One again introduces new variables α and β via the relations:

$$q=l(1-\cos\alpha)^2\cosh\beta,$$
$$l'=l\cos^2\alpha.$$

In contrast to the calculation in Part II, we now have two independent variables l and l' (or α and β) to deal with.

Before proceeding to the ultrarelativistic limit, let us take a look at the contour-integral representation for $I(l',l)$.

$$I(l',l)=e^{-q/2}\cdot\exp[i\tfrac{1}{2}\pi(l-l')]\cdot q^{(l-l')/2}(l'!/l!)^{1/2}$$
$$\times(2\pi i)^{-1}\oint_c e^{qt}(1-t)^{l}t^{-(l'+1)}dt. \quad (52)$$

The point where the integrand has a stationary phase of the sort discussed earlier, i.e., where $f''(t)=0$, is now

$$t=t_0=\cos\alpha/(1+\cos\alpha),$$

which lies on the real, positive t axis between 0 and $\frac{1}{2}$ (Fig. 4).

After some lengthy algebra, one obtains the asymp-

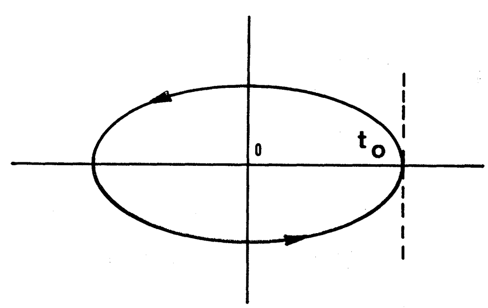

FIG. 4. Contour used in pair-creation theory.

totic formula:

$$I(l',l)\sim\exp\left[i\tfrac{1}{2}\pi(l-l')\right]\cdot\frac{(1+\cos\alpha)}{(\cos\alpha)^{1/2}}\times\frac{1}{\pi\sqrt{3}}\left(\frac{a}{b}\right)^{1/2}K_{1/3}\left[\tfrac{2}{3}l\left(\frac{a^3}{b}\right)^{1/2}\right],\quad(53)$$

with

$$a=(1+\cos\alpha)^2(\cosh\beta-1),$$
$$b=(1+\cos\alpha)^4/\cos\alpha.\quad(54)$$

Having obtained an asymptotic formula for $I(l',l)$, one can then express the squared and spin-summed matrix elements of α_x or α_z in terms of (l',l) by use of Klepikov's identities, integrate over all values l,l' of the positron and electron, hence obtain the absorption probabilities for radiation polarized in the x or z direction. However, l and l' are subject to an important restriction. We note that the incident photon energy must equal the sum of the energies of the electron and positron. That is,

$$\kappa=(1+2lB+k_3^2)^{1/2}+(1+2l'B+k_3^2)^{1/2}\quad(55a)$$

thus

$$k_3^2=\tfrac{1}{4}\kappa^2-(l+l')B-1+\left[(l-l')^2B^2/\kappa^2\right],\quad(55b)$$

which is necessarily ≥ 0.

The domain over which the integral on l,l' must be carried out is rather complicated. Klepikov carries out a further change of variable

$$\cos\alpha=e^{-2w},\quad(56a)$$

$$\cosh\beta=1+(4/\kappa^2)\cosh^2 w\cdot\cosh^2 u.\quad(56b)$$

The domain of integration on $dwdu$ after the transformation:

$$\int dldl'\rightarrow\int J'd\alpha d\beta\rightarrow\int Jdwdu$$

is then simply the first quadrant of the w–u plane (J is the Jacobian).

The expression for $I(l',l)$ is now

$$|I(l',l)|=2\cosh w\cdot(\cosh u/\pi\sqrt{3}\kappa)\times K_{1/3}\left[(4/3\kappa B)\cosh^2 w\cosh^3 u\right].\quad(57)$$

Likewise, one obtains

$$\left|\frac{\partial I(l',l)}{\partial q}\right|=2\cosh w\cdot\frac{\cosh u}{\pi\sqrt{3}\kappa^2}\times K_{2/3}\left(\frac{4}{3\kappa B}\cosh^2 w\cosh^3 u\right).\quad(58)$$

Furthermore, the expressions for

$$\sum_{s,s'}|\alpha_x|^2\quad\text{and}\quad\sum_{s,s'}|\alpha_z|^2,$$

which were derived earlier, can be expressed in terms of the new variables w and u also. One obtains, to lowest order in $1/\kappa^2$, the results

$$\sum_{s,s'}|\alpha_x|^2=\left(\frac{4}{\kappa^2}\cosh^2 w\cdot 2+\frac{8}{\kappa^2}\sinh^2 w\cosh^2 w\cosh^2 u\right)\times|I(l',l)|^2+4\cosh^2 w\cdot 2\sinh^2 w\cdot\left|\frac{\partial I(l',l)}{\partial q}\right|^2,\quad(59)$$

$$\sum_{s,s'}|\alpha_z|^2=\left(\frac{4}{\kappa^2}\cdot 2\cosh^2 w\cosh^2 u+\frac{8}{\kappa^2}\sinh^2 w\cosh^2 w\cosh^2 u\right)\times|I(l',l)|^2+\tfrac{1}{2}\cdot 16\cosh^4 w\left|\frac{\partial I(l',l)}{\partial q}\right|^2.\quad(60)$$

Use has been made of the fact that we need only the lowest-order terms in $1/\kappa^2$, higher terms being neglected in the ER limit. To a good approximation, the cross terms involving products of $I(l',l)$ times its derivative cancel out.

We quote here the final results for the spin-summed, squared matrix elements, after expressing $I(l',l)$ and its derivative in terms of $K_{1/3}$ and $K_{2/3}$.

$$\sum_{s,s'}|\alpha_x|^2=(32/3\pi^2\kappa^4)\left[(\sinh^2 w\cosh^2 u+1)\cosh^4 w\times\cosh^2 uK_{1/3}^2+\sinh^2 w\cdot\cosh^4 w\cosh^2 uK_{2/3}^2\right],\quad(61)$$

$$\sum_{s,s'}|\alpha_z|^2=(32/3\pi^2\kappa^4)\left[(\sinh^2 w\cosh^2 u+\cosh^2 u)\times\cosh^4 w\cosh^2 uK_{1/3}^2+\cosh^2 w\cdot\cosh^4 w\cosh^2 uK_{2/3}^2\right].\quad(62)$$

The most convenient way of describing photoproduction of e^+e^- pairs in a magnetic field is to treat the field as an absorptive medium. The attenuation coefficient is evidently

$$\alpha(\omega)=w\cdot(V/c)\cdot N,$$

where w is the usual transition probability/time given by Fermi's golden rule, c/V is the flux of photons, and N is the number of negative-energy electrons/volume. Therefore

$$\alpha(\omega)=\frac{2\pi}{\hbar}\cdot\frac{e^2}{V}\cdot\frac{2\pi\hbar c}{\kappa}(\boldsymbol{\alpha}\cdot\mathbf{e})^2\cdot\delta(E+E'-\kappa)\frac{1}{mc^2}\cdot\frac{V}{c}\cdot\frac{eB}{2\pi\hbar cL}.\quad(63)$$

Furthermore a summation over l,l' and k_3,k_3' must be performed. We write

$$\sum_{k_3}\sum_{k_3'}\quad\text{as}\quad\iint dk_3dk_3'\left(\frac{L}{2\pi}\right)^2.$$

$$L^{-1}\int_{-L/2}^{L/2}\exp(-ik_3'z)\exp(ik_3z)dz=\frac{\sin(k_3'-k_3)\frac{1}{2}L}{(k_3'-k_3)\frac{1}{2}L}.\quad(64)$$

Thus the dk_3' integral is

$$\frac{2}{L}\int_{-\infty}^{\infty}\frac{\sin^2\eta}{\eta^2}\,d\eta=\frac{2\pi}{L},$$

with the condition $k_3'=k_3$ arising in the limit $L\rightarrow\infty$.

The attenuation coefficient is

$$\alpha(\omega)=\frac{e^2}{\hbar c}\left(\frac{mc^2}{E}\right)\frac{B}{B_0}\frac{mc}{\hbar}\cdot\int_{-\infty}^{\infty}dk_3\sum_{l,l'}^{l'<l}(\boldsymbol{\alpha}\cdot\mathbf{e})^2\delta(E+E'-\kappa).$$

(The quantities to the right of the integral sign are expressed in the customary dimensionless form.)

Using the well-known rule $\delta[f(x)]=[1/|f'(x_0)|]\times\delta(x-x_0)$, where $f(x_0)=0$, we write

$$\delta(E+E'-\kappa)=(EE'/\kappa k_3)\times\delta\{k_3\pm[\tfrac{1}{4}\kappa^2-(l+l')B-1+(l-l')B^2/\kappa^2]^{1/2}\}. \quad (65)$$

The summation on l, l' becomes an integration:

$$\sum_{l,l'}^{l'<l}\rightarrow\int_{l'}\int_{<l}dl\,dl'=\iint du\,dw\,\frac{\kappa^2}{2B^2}\,\frac{\cosh u\sinh u}{\cosh^2 w}. \quad (66)$$

Finally, the attenuation coefficients for photons polarized in the x and z directions (that is, in the directions perpendicular and parallel to B, respectively) are:

$$\alpha_\perp(\omega)=\frac{3}{\pi^2}\cdot\alpha\cdot\frac{mc}{\hbar}\,\frac{B}{B_0}\cdot\left(\frac{4}{3\kappa B}\right)^2\int_0^{\infty}\int_0^{\infty}dw\,du\times[(\sinh^2 w\cosh^2 u+1)\cdot\cosh^3 u K_{1/3}{}^2+\sinh^2 w\cosh^3 u K_{2/3}{}^2], \quad (67)$$

$$\alpha_{||}(\omega)=\frac{3}{\pi^2}\cdot\alpha\cdot\frac{mc}{\hbar}\,\frac{B}{B_0}\cdot\left(\frac{4}{3\kappa B}\right)\int_0^{\infty}\int_0^{\infty}dw\,du\times[(\sinh^2 w\cosh^2 u+\cosh^2 u)\cosh^3 u K_{1/3}{}^2+\cosh^2 w\cosh^3 u\cdot K_{2/3}{}^2], \quad (68)$$

where the argument of the K functions is

$$(4/3\kappa B)\cosh^2 w\cosh^3 u,$$

or in ordinary units,

$$\tfrac{4}{3}(mc^2/E)(B_0/B)\cosh^2 w\cosh^3 u. \quad (69)$$

(An additional factor of $2\times2=4$ has been inserted in the above formulas because k_3 may be positive or negative, and the electron's angular momentum l' may be $>l$, as well as $<l$.)

Although the electron and positron have equal momenta in the z direction, their transverse momenta may differ. If the transverse momenta are equal, then the pair are formed at 180° with respect to each other, and transverse helices of equal pitch. If the transverse momenta are unequal, the helices described by the particles have unequal pitch, and the initial angle between members of the pair deviates from 180° by the amount

$$\arctan[k_3/(2l'B)^{1/2}]-\arctan[k_3/(2lB)^{1/2}]. \quad (70)$$

IV. COMMENTS

(1) Magnetic fields of the requisite strength for pair production are found in nature in the vicinity of the elementary particles. A neutron, for example, has a magnetic-dipole field which could allow pair creation by a photon passing nearby. Within a radius of 6 fermi from the neutron core, there are magnetic fields which equal or exceed the critical field B_0 of 4.4×10^{13} G, required for a reasonable chance of pair production. The incident photon must have energy in excess of $2mc^2$ in the first-order theory given above. The photoproduction process would be analogous to pair creation by a photon in a Coulomb field.

(2) The radius of curvature of the circular (or helical) trajectory in a uniform field can easily be seen to be:

$$R=(\hbar/mc)(B_0/B)(p_\perp/mc),$$

where the momentum at right angles to the field is quantized according to

$$(p_\perp/mc)^2=2l(B/B_0).$$

Hence

$$R=(\hbar/mc)(2lB_0/B)^{1/2}.$$

From a quantum-mechanical point of view the radius would be determined by the maximum in the charge density:

$$F(\rho)^2\sim(\tfrac{1}{2}B\rho^2)^l\exp(-\tfrac{1}{2}B\rho^2).$$

The maximum in $F(\rho)^2$ occurs at $\rho^2=2l/B$, or in the usual units $\rho^2=(\hbar/mc)^2\cdot(B_0/B)\cdot 2l$, in agreement with the classical result.