Second-Quantization Process for Particles with Any Spin and with Internal Symmetry*

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In this paper, the second-quantized theory of free particles and antiparticles, with any spin and with internal $SU(2)$ or $SU(3)$ symmetry, is developed. All spins and both statistics are treated in a uniform way in terms of well-defined complete sets of functions that are orthonormal with respect to a Lorentz-invariant positive-definite inner product. Explicit formulas for field operators of energy, momentum, etc., are given, including three-vector, four-vector, and tensor operators for polarization. It is shown that causal space densities of physical quantities exist when the correct spin-statistics connection is used. The field operators for systems self-conjugate under C, G, and GP are treated and self-conjugate multiplets of $SU(3)$ are set up.

I. INTRODUCTION

Recently Joos,[1] Weinberg,[2] and Weaver *et al.*[3] gave a description of the free massive particle with any allowed spin $s=0, \frac{1}{2}, 1, \cdots$, in which there is a close parallel with Dirac's treatment for spin $\frac{1}{2}$. Further developments in the description were made by Mathews[4] and by Williams *et al.*[5]

The Joos–Weinberg formulation uses a $2(2s+1)$-component wave function that must satisfy two manifestly covariant wave equations, one of which is the Klein–Gordon equation. The Weaver–Hammer–Good formulation uses a $2(2s+1)$-component wave function that has a well-defined covariant, but not manifestly covariant, Hamiltonian. The allowed states form a complete set and a positive-definite invariant integral exists.

Each description can be quantized in the usual way, introducing creation–destruction operators for each of the allowed states. Weinberg showed that the usual spin-statistics connection holds for his quantized wave function. Weaver[6] and Mathews and Ramakrishnan[7,8] studied the quantized Weaver–Hammer–Good wave function and Mathews, especially, emphasized that it is not causal for integer spin.

The purpose of the present paper is to carry the theory of the quantization process on further and especially cover these questions:

(1) What is the connection between the two quantization techniques? The answer is that they are equivalent. The Weaver–Hammer–Good field operator is the sum of independent-particle creation and destruction operators times a complete set of functions for the particle and antiparticle. The Weinberg field operator coincides with it for half-integer spin and is related to it by an operator that has an inverse for integer spin.

(2) What are the field operators for energy, momentum, polarization, etc.? Formulas are given that apply uniformly for all spins and both statistics. It is shown that causal space-densities of physical quantities exist when Fermi statistics are used with half-integral spin and Bose statistics with integral spin.

(3) How is the quantization process extended to apply to particles with internal $SU(2)$ or $SU(3)$ symmetry?

(4) What are the field operators for self-conjugate multiplets? States that are self-conjugate under C, and multiplets that are self-conjugate under G and GP are treated. Self-conjugate multiplets of $SU(3)$ are also set up.

(5) What are the quantum numbers of the single-particle states of existence? As well as the momentum, baryon number, and parity, the polarization quantum numbers and the quantum numbers that are introduced in the self-conjugation processes are shown.

Explicit formulas for all the fields considered are given in terms of a well-defined complete set of functions that are orthonormal in the sense of a positive-definite Lorentz-invariant integral. The field operators are appropriate to be substituted directly into an effective Hamiltonian for a reaction. From the point of view developed here, there is hardly any difference conceptually in working with spin s than with spin $\frac{1}{2}$.

Only particles with finite mass are considered; the case of zero mass, any spin, was treated to some extent by Hammer and Good[9] and by Weinberg.[10]

II. NOTATION FOR LORENTZ-GROUP DISCUSSION

The homogeneous Lorentz group is defined by the transformation properties of the coordinates of an event,

$$x_\mu' = a_{\mu\nu}x_\nu,$$

where

$$a_{\mu\nu}a_{\mu\lambda} = a_{\nu\mu}a_{\lambda\mu} = \delta_{\nu\lambda}$$

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and x_4 is it. For a pure Lorentz transformation with relative velocity $\mathbf{v}$:

$$a_{ij}=\delta_{ij}+(\gamma-1)v_iv_j/v^2, \quad (1a)$$

$$a_{i4}=-a_{4i}=i\gamma v_i, \quad (1b)$$

$$a_{44}=\gamma, \quad (1c)$$

where γ is $(1-v^2)^{-1/2}$. For an angular displacement $\boldsymbol{\theta}$,

$$a_{ij}=\cos\theta\delta_{ij}+(1-\cos\theta)\theta_i\theta_j/\theta^2+\sin\theta\epsilon_{ijk}\theta_k/\theta, \quad (2a)$$

$$a_{i4}=a_{4i}=0, \quad (2b)$$

$$a_{44}=1. \quad (2c)$$

Alternatively these two types can be written in exponential form as

$$a=\exp[i(\tanh^{-1}v)\mathbf{v}\cdot\mathbf{t}/v], \quad (3)$$

$$a=\exp[i\boldsymbol{\theta}\cdot\mathbf{s}], \quad (4)$$

where $\mathbf{s}$ and $\mathbf{t}$ are four-by-four matrices with components

$$(s_i)_{jk}=-i\epsilon_{ijk}, \quad (5a)$$

$$(s_i)_{j4}=(s_i)_{4j}=(s_i)_{44}=0, \quad (5b)$$

$$(t_i)_{jk}=(t_i)_{44}=0, \quad (5c)$$

$$(t_i)_{j4}=-(t_i)_{4j}=\delta_{ij}. \quad (5d)$$

The general transformation, continuous with the identity, can be considered to be built from these two types and may be written as

$$x'=\exp[i(\boldsymbol{\theta}\cdot\mathbf{s}+\boldsymbol{\omega}\cdot\mathbf{t})]x, \quad (6)$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$ are the six real parameters.

The matrices $\mathbf{s}$ are Hermitian and $\mathbf{t}$ are anti-Hermitian. The commutation rules are

$$[s_i,s_j]=i\epsilon_{ijk}s_k, \quad (7a)$$

$$[s_i,t_j]=i\epsilon_{ijk}t_k, \quad (7b)$$

$$[t_i,t_j]=-i\epsilon_{ijk}s_k, \quad (7c)$$

and these define the Lie algebra of the group. The algebra can be given in a more concise form by defining

$$s_{ij}=\epsilon_{ijk}s_k, \quad (8a)$$

$$s_{i4}=-it_i, \quad (8b)$$

$$s_{\mu\nu}=-s_{\nu\mu}, \quad (8c)$$

in which case the commutation rules can be written as

$$[s_{\mu\nu},s_{\rho\sigma}]=i(\delta_{\mu\rho}s_{\nu\sigma}+\delta_{\nu\sigma}s_{\mu\rho}-\delta_{\mu\sigma}s_{\nu\rho}-\delta_{\nu\rho}s_{\mu\sigma}). \quad (9)$$

The finite-dimensional representations of the group are found immediately by considering the Hermitian matrices

$$j_i=\tfrac{1}{2}(s_i+it_i), \quad (10a)$$

$$k_i=\tfrac{1}{2}(s_i-it_i). \quad (10b)$$

They satisfy the commutation rules

$$[j_i,k_j]=0, \quad (11a)$$

$$[j_i,j_j]=i\epsilon_{ijk}j_k, \quad (11b)$$

$$[k_i,k_j]=i\epsilon_{ijk}k_k, \quad (11c)$$

so the finite-dimensional representations are simply those of $SU(2)\times SU(2)$. Let $\mathbf{j}$, $\mathbf{k}$ denote also the irreducible angular-momentum matrices for angular momenta j, k. Thus $\mathbf{j}$ is $(2j+1)$-square and has eigenvalues $-j$ to $+j$, where $j=0,\frac{1}{2},1,\cdots$. Also let $\chi_{j_3}{}^j$ denote a column matrix with rows labeled by $j_3=-j$ to j. Then $\chi_{j_3}{}^j\chi_{k_3}{}^k$ form a basis for Lorentz group representations and the transformation rule for these objects is

$$(\chi^j\chi^k)'=\exp[i\boldsymbol{\theta}\cdot(\mathbf{j}+\mathbf{k})+\boldsymbol{\omega}\cdot(\mathbf{j}-\mathbf{k})]\chi^j\chi^k$$
$$=\exp[(i\boldsymbol{\theta}+\boldsymbol{\omega})\cdot\mathbf{j}]\chi^j\exp[(i\boldsymbol{\theta}-\boldsymbol{\omega})\cdot\mathbf{k}]\chi^k. \quad (12)$$

The representations $(j,0)$ and $(0,k)$ have the rules

$$\chi'=\exp[(i\boldsymbol{\theta}+\boldsymbol{\omega})\cdot\mathbf{j}]\chi, \quad (13)$$

$$\chi'=\exp[(i\boldsymbol{\theta}-\boldsymbol{\omega})\cdot\mathbf{k}]\chi. \quad (14)$$

They correspond to completely symmetric spinors with $2j$ upper undotted indices and $2k$ lower dotted indices in the usual notation.

If such a representation (j,k) exists at every event in space-time, the system is described by functions $\psi_{j_3,k_3}{}^{j,k}(x)$ which have the transformation rule.

$$\psi'^{j,k}(x')=\exp[i\boldsymbol{\theta}\cdot(\mathbf{j}+\mathbf{k})+\boldsymbol{\omega}\cdot(\mathbf{j}-\mathbf{k})]\psi^{j,k}(x). \quad (15)$$

These are referred to as the spinor components of the field although some types are just tensors.

Operators $s_{\mu\nu}$ are still well defined by Eqs. (10) in terms of the angular momentum matrices $\mathbf{j}$ and $\mathbf{k}$, considered to act in different spaces, and they satisfy the same commutation rules as the four-by-four $s_{\mu\nu}$. These rules are also fulfilled by

$$l_{\mu\nu}=-i(x_\mu\partial/\partial x_\nu-x_\nu\partial/\partial x_\mu) \quad (16)$$

and by

$$J_{\mu\nu}=l_{\mu\nu}+s_{\mu\nu}. \quad (17)$$

In terms of $\mathbf{J}$ and $\mathbf{K}$ defined by

$$J_i=\tfrac{1}{2}\epsilon_{ijk}J_{jk}, \quad (18a)$$

$$K_i=iJ_{i4}, \quad (18b)$$

the primed spinor functions are found from the unprimed this way:

$$\psi'^{j,k}(x)=\exp i(\boldsymbol{\theta}\cdot\mathbf{J}+\boldsymbol{\omega}\cdot\mathbf{K})\psi^{j,k}(x). \quad (19)$$

For a displacement of the origin

$$x_\mu'=x_\mu-d_\mu,$$

the spinor components are chosen to satisfy

$$\psi_{j_3,k_3}{}'^{j,k}(x')=\psi_{j_3,k_3}{}^{j,k}(x).$$

This means that

$$\psi'(x)=\exp\,(i\,d_\mu p_\mu)\psi(x), \quad (20)$$

where

$$p_\mu=-i\,\partial/\partial x_\mu$$

is the displacement operator. The inhomogeneous Lorentz group, consisting of the homogeneous transformations plus displacements, has the Lie algebra

$$[J_{\mu\nu}, J_{\rho\sigma}]=i(\delta_{\mu\rho}J_{\nu\sigma}+\delta_{\nu\sigma}J_{\mu\rho}-\delta_{\mu\sigma}J_{\nu\rho}-\delta_{\nu\rho}J_{\mu\sigma}), \quad (21)$$

$$[p_\mu, J_{\nu\lambda}]=i(\delta_{\mu\lambda}p_\nu-\delta_{\mu\nu}p_\lambda), \quad (22)$$

$$[p_\mu, p_\nu]=0. \quad (23)$$

From all the functions $\psi'^{j,k}(x)$ that can be produced with the allowed values of $\boldsymbol{\theta}$, $\boldsymbol{\omega}$, and d_μ, one can choose a complete set. Then they form a basis for an infinite-dimensional representation of the inhomogeneous group.

To include space reflection

$$x_i'=-x_i, \qquad t'=t,$$

one introduces the matrix P, with components

$$P_{ij}=-\delta_{ij}, \qquad P_{i4}=P_{4i}=0, \qquad P_{44}=1, \quad (24)$$

so that

$$x'=Px. \quad (25)$$

In terms of the four-by-four matrices one finds that

$$P\mathbf{s}P^{-1}=\mathbf{s}, \quad (26a)$$

$$P\mathbf{t}P^{-1}=-\mathbf{t}, \quad (26b)$$

$$P\mathbf{j}P^{-1}=\mathbf{k}, \quad (26c)$$

$$P\mathbf{k}P^{-1}=\mathbf{j}. \quad (26d)$$

In order to give P meaning where $\mathbf{j}$ and $\mathbf{k}$ are identified with angular-momentum matrices, one considers functions of the type $(j'k')\oplus(k'j')$ and defines P by

$$P\begin{pmatrix}\chi^{j'k'}\\ \Omega^{k'j'}\end{pmatrix}=\begin{pmatrix}\Omega^{k'j'}\\ \chi^{j'k'}\end{pmatrix}. \quad (27)$$

(Questions of phase factors are taken up later.) The matrices $\mathbf{j}$ and $\mathbf{k}$ are now

$$\mathbf{j}=\begin{pmatrix}\mathbf{j}' & 0\\ 0 & \mathbf{k}'\end{pmatrix}, \qquad \mathbf{k}=\begin{pmatrix}\mathbf{k}' & 0\\ 0 & \mathbf{j}'\end{pmatrix}, \quad (28)$$

so that, for example,

$$P\mathbf{j}\begin{pmatrix}\chi^{j'k'}\\ \Omega^{k'j'}\end{pmatrix}=\begin{pmatrix}\mathbf{k}'\Omega^{k'j'}\\ \mathbf{j}'\chi^{j'k'}\end{pmatrix}=\mathbf{k}P\begin{pmatrix}\chi^{j'k'}\\ \Omega^{k'j'}\end{pmatrix}. \quad (29)$$

Thus, representations of the Lorentz group including space reflection are provided by base functions of the type $(jk)\oplus(kj)$.

The most economical way to treat a particle of spin s is to use $(0, s)\oplus(s, 0)$. Then the matrices are $2(2s+1)$-square and are given by

$$\mathbf{j}=\begin{pmatrix}0 & 0\\ 0 & \mathbf{s}\end{pmatrix}, \qquad \mathbf{k}=\begin{pmatrix}\mathbf{s} & 0\\ 0 & 0\end{pmatrix}, \qquad P=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \quad (30)$$

The Lorentz group operators are $p_\mu=-i\,\partial/\partial x_\mu$ and the $J_{\mu\nu}$ of Eq. (17), with $s_{\mu\nu}$ given by

$$s_{ij}=\epsilon_{ijk}\begin{pmatrix}s_k & 0\\ 0 & s_k\end{pmatrix}, \qquad s_{i4}=\begin{pmatrix}s_i & 0\\ 0 & -s_i\end{pmatrix}.$$

III. THE SINGLE-PARTICLE THEORY

In this section, some of the previous results are summarized in the form needed to make the field quantization.

The matrices used are $2(2s+1)$-square and are defined as follows:

$$\boldsymbol{\alpha}=s^{-1}\begin{pmatrix}\mathbf{s} & 0\\ 0 & -\mathbf{s}\end{pmatrix}, \qquad \beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \mathbf{s}=\begin{pmatrix}\mathbf{s} & 0\\ 0 & \mathbf{s}\end{pmatrix},$$

$$\gamma_5=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \qquad C=\begin{pmatrix}0 & C_s\\ -C_s & 0\end{pmatrix}. \quad (31)$$

Here C_s is a unitary matrix such that

$$C_s\mathbf{s}=-\mathbf{s}^*C_s. \quad (32)$$

For spin 0, C_s is defined to be equal to one. Unless otherwise specified, the standard representation for the angular-momentum matrices $\mathbf{s}$, with s_3 diagonal and elements of s_1 real and positive, is used. In that case C_s is $\exp\,(i\pi s_2)$; evidently it is real and C_s^2 is $(-1)^{2s}$. The matrices satisfy the equations

$$C\boldsymbol{\alpha}C^{-1}=\boldsymbol{\alpha}^*, \qquad C\beta C^{-1}=-\beta^*,$$

$$C\mathbf{s}C^{-1}=-\mathbf{s}^*, \qquad C\gamma_5C^{-1}=-\gamma_5^*. \quad (33)$$

Following Ref. 3, one starts from the allowed states of the particle and antiparticle in the rest frame. The Hamiltonian is assigned to be $m\beta$ and the polarization operator to be $\beta\mathbf{s}$ in that frame. Eigenfunctions $v_{R,\epsilon,k}$ exist such that

$$\beta v_{R\epsilon k}=\epsilon v_{R\epsilon k}, \quad (34a)$$

$$\beta\mathbf{s}\cdot\mathbf{e}v_{R\epsilon k}=kv_{R\epsilon k}, \quad (34b)$$

where ϵ is ±1 for the particle/antiparticle, $\mathbf{e}$ is a unit vector in an arbitrary quantization direction, and k ranges from $-s$ to $+s$. The functions may be chosen normalized such that

$$v_{R\epsilon k}{}^\dagger v_{R\epsilon'k'}=\delta_{\epsilon\epsilon'}\delta_{kk'}. \quad (35)$$

The relative phases may be adjusted so that

$$(Cv_{R,\epsilon,k})^*=\epsilon^{2s+1}v_{R,-\epsilon,k}, \quad (36)$$

$$\beta s_\pm v_{R,\epsilon,k}=[s(s+1)-k(k\pm\epsilon)]^{1/2}v_{R,\epsilon,k\pm\epsilon}. \quad (37)$$

Here $s_\pm$ is $\mathbf{s}\cdot\mathbf{f}\pm i\,\mathbf{s}\cdot\mathbf{g}$, $\mathbf{e}$, $\mathbf{f}$, and $\mathbf{g}$ forming an orthogonal right-hand set of unit vectors. One can see that this is a permissible choice of phases because the equations for $\epsilon=+1$ serve to define all the $v_{R\epsilon k}$ from any of them. The equations for $\epsilon=-1$ can then be derived from those for $\epsilon=+1$. For a given spin, this determines the functions $v_{R\epsilon k}$ except for a factor of the form $e^{i\epsilon\delta}$, δ real and independent of ϵ and k. This choice of phases is different from that used originally in Ref. 3; it was suggested by Mullin *et al.*[11] to simplify charge conjugation discussions. There is another consideration which suggests a useful choice of this number δ. Since γ_5 anticommutes with β and $\beta\mathbf{s}$, $\gamma_5 v_{R,\epsilon,k}$ must be proportional to $v_{R,-\epsilon,-k}$. In order to be consistent with Eqs. (36) and (37), the relation must be

$$\gamma_5 v_{R,\epsilon,k}=e^{i\epsilon\delta'}(-1)^{s+\epsilon k}v_{R,-\epsilon,-k}.$$

Let the arbitrary number δ be chosen so this relation is

$$\gamma_5 v_{R,\epsilon,k}=-(-1)^{s+\epsilon k}v_{R,-\epsilon,-k}. \tag{38}$$

The functions $v_{R\epsilon k}$ are then completely determined except for a single over-all plus or minus sign.

The wave function in the rest frame is

$$\psi_{R\epsilon k}=v_{R\epsilon k}\exp(-i\epsilon m t_R)$$

and the equation of motion is

$$m\beta\psi_{R\epsilon k}=i\,\partial\psi_{R\epsilon k}/\partial t_R. \tag{39}$$

The wave function is chosen to be $(0,s)\oplus(s,0)$. Then the function in the laboratory frame is found by Lorentz transforming from the rest frame. Let $\mathbf{q}$ and E be the physical momentum and energy of the particle or antiparticle so that E is positive and $\mathbf{q}/E$ is the velocity. Equations (13) and (14) apply with $\boldsymbol{\theta}$ zero and $\boldsymbol{\omega}$ having the value $(-\mathbf{q}/q)\operatorname{arctanh}(q/E)$, as seen from Eqs. (1) and (3). The transformation is

$$\psi_L=\exp[s\,\boldsymbol{\alpha}\cdot(\mathbf{q}/q)\operatorname{arctanh}(q/E)]\psi_R,$$

and wave functions for a particle or antiparticle of definite momentum $\mathbf{q}$ and polarization k are

$$\psi_L=\exp[s\,\boldsymbol{\alpha}\cdot(\mathbf{q}/q)\operatorname{arctanh}(q/E)]v_{R\epsilon k}\exp(i\epsilon q_\alpha x_\alpha), \tag{40}$$

where q_α is (q, iE). The symbol $\mathbf{p}$ will be used for the operator $-i\boldsymbol{\nabla}$ and its eigenvalue, here $\epsilon\mathbf{q}$. With normalization appropriate for the invariant integral discussed below, the plane-wave eigenstates are

$$\psi_{\mathbf{p}\epsilon k}=(2\pi)^{-3/2}m^sE^{-1/2}\exp[s\epsilon\,\boldsymbol{\alpha}\cdot(\mathbf{p}/p)\operatorname{arctanh}(p/E)]\times v_{R\epsilon k}\exp[i(\mathbf{p}\cdot\mathbf{x}-\epsilon Et)]. \tag{41}$$

The general laboratory-system function is found by summing over all the quantum numbers with arbitrary coefficients, say $E^{-1/2}A_{\epsilon k}(\mathbf{p})d\mathbf{p}$,

$$\psi(\mathbf{x},t)=(2\pi)^{-3/2}m^s\int d\mathbf{p}\,E^{-1}\sum_{\epsilon k}A_{\epsilon k}(\mathbf{p})\times\exp[s\epsilon\,\boldsymbol{\alpha}\cdot(\mathbf{p}/p)\operatorname{arctanh}(p/E)]v_{R\epsilon k}\times\exp[i(\mathbf{p}\cdot\mathbf{x}-\epsilon Et)]. \tag{42}$$

[11] C. J. Mullin, C. L. Hammer, and R. H. Good, Jr. (private communication).

Foldy's wave function[12] is defined by

$$\phi(\mathbf{x},t)=(2\pi)^{-3/2}\int d\mathbf{p}\,E^{-1/2}\sum_{\epsilon k}A_{\epsilon k}(\mathbf{p})v_{R\epsilon k}\times\exp[i(\mathbf{p}\cdot\mathbf{x}-\epsilon Et)]. \tag{43}$$

It satisfies the equation of motion

$$E\beta\phi=i\,\partial\phi/\partial t, \tag{44}$$

where E is the operator $(p^2+m^2)^{1/2}$, the positive root to be taken. The relation between the two functions is written as

$$\psi(\mathbf{x},t)=m^sE^{-1/2}S\,\phi(\mathbf{x},t). \tag{45}$$

An algorithm for calculating S was given in Ref. 3 and results quoted for $s\leq\frac{3}{2}$. Williams *et al.*[5] developed general polynomial formulas for S and S^{-1}. Here their infinite series forms,

$$S=\cosh(\omega\,\mathbf{s}\cdot\mathbf{p}/p)-\gamma_5\beta\sinh(\omega\,\mathbf{s}\cdot\mathbf{p}/p), \tag{46}$$

$$S^{-1}=\operatorname{sech}(2\omega\,\mathbf{s}\cdot\mathbf{p}/p)\;S^\dagger, \tag{47}$$

where ω is $\operatorname{arctanh}(p/E)$, will be used. These are easily derived from the series for the exponential in Eq. (42).

The Hamiltonian for ψ, such that

$$H\psi=i\,\partial\psi/\partial t, \tag{48}$$

is seen from Eqs. (44) and (45) to be

$$H=SE\beta S^{-1}. \tag{49}$$

Detailed results for $s\leq\frac{3}{2}$ are given in Ref. 3 and general polynomial formulas in Refs. 4 and 5. In terms of the hyperbolic series the Hamiltonian is

$$H=E\beta\operatorname{sech}(2\omega\,\mathbf{s}\cdot\mathbf{p}/p)-E\gamma_5\tanh(2\omega\,\mathbf{s}\cdot\mathbf{p}/p). \tag{50}$$

H is evidently a Hermitian matrix. Also $H^2=E^2$ so each of these Hamiltonians gives a square root of $(-\nabla^2+m^2)$.

The positive-definite Lorentz-invariant integral between two functions $\psi^{(l)}$ and $\psi^{(n)}$ may be defined by

$$(\psi^{(l)},\psi^{(n)})=\int d\mathbf{x}\,\phi^{(l)\dagger}\phi^{(n)}. \tag{51}$$

Alternate forms are

$$\begin{aligned}(\psi^{(l)},\psi^{(n)})&=m^{-2s}\int d\mathbf{x}\,\psi^{(l)\dagger}E(S^{-1})^\dagger S^{-1}\psi^{(n)}\\&=m^{-2s}\int d\mathbf{x}\,\psi^{(l)\dagger}E\operatorname{sech}(2\omega\,\mathbf{s}\cdot\mathbf{p}/p)\psi^{(n)}\\&=m^{-2s}\int d\mathbf{x}\,\psi^{(l)\dagger}\tfrac{1}{2}[H,\beta]_+\psi^{(n)}\\&=-\tfrac{1}{2}im^{-2s}\int d\mathbf{x}\left(\frac{\partial\bar{\psi}^{(l)}}{\partial t}\psi^{(n)}-\bar{\psi}^{(l)}\frac{\partial\psi^{(n)}}{\partial t}\right),\end{aligned} \tag{52}$$

[12] L. L. Foldy, Phys. Rev. **102**, 568 (1956).

where $\bar{\psi}$ is $\psi^\dagger\beta$. The eigenstates of Eq. (41) can be written as

$$\psi_{\mathbf{p}\epsilon k}=(2\pi)^{-3/2}m^s E^{-1/2} S v_{R\epsilon k} \exp\left[i(\mathbf{p}\cdot\mathbf{x}-\epsilon Et)\right], \quad (53)$$

and thus they are normalized in the sense that

$$(\psi_{\mathbf{p}\epsilon k}, \psi_{\mathbf{p}'\epsilon' k'})=\delta_{\epsilon\epsilon'}\delta_{kk'}\delta(\mathbf{p}-\mathbf{p}'). \quad (54)$$

From Eqs. (49) and (53) it is seen that the functions $\psi_{\mathbf{p}\epsilon k}$ are eigenfunctions of H with eigenvalue ϵE. It is clear that they make a complete set for expanding a $2(2s+1)$-component wave function at any given time, since H is Hermitian. Also they are eigenfunctions of $\mathbf{O}\cdot\mathbf{e}$ with eigenvalue k, where the polarization operator $\mathbf{O}$ is defined by

$$\mathbf{O}=S\beta\mathbf{s}S^{-1}.$$

With respect to Lorentz transformations, continuous with the identity, Eqs. (13) and (14) give the wave-function transformation rule

$$\psi'(x')=\begin{pmatrix}\exp\,(i\boldsymbol{\theta}-\boldsymbol{\omega})\cdot\mathbf{s} & 0\\ 0 & \exp\,(i\boldsymbol{\theta}+\boldsymbol{\omega})\cdot\mathbf{s}\end{pmatrix}\psi(x). \quad (55)$$

In Ref. 3, the covariance of the system was demonstrated by showing that Eqs. (42) and (55) combine to make

$$\psi'(x')=(2\pi)^{-3/2}m^s\int d\mathbf{p}'\,E'^{-1}\sum_{\epsilon k}A_{\epsilon k}'(\mathbf{p}')$$
$$\times\exp\left[s\epsilon\,\boldsymbol{\alpha}\cdot(\mathbf{p}'/p')\,\operatorname{arctanh}\,(p'/E')\right]v_{R\epsilon k}$$
$$\times\exp\left[i\,(\mathbf{p}'\cdot\mathbf{x}'-\epsilon E't')\right],$$

where the transformation rule for the expansion coefficients is

$$A_{\epsilon l}'(\mathbf{p}')=\sum_k\left[v_{R\epsilon l}^\dagger\exp\,(i\,\boldsymbol{\lambda}\cdot\mathbf{s})\,v_{R\epsilon k}\right]A_{\epsilon k}(\mathbf{p}) \quad (56)$$

and where

$$p_\mu'=a_{\mu\nu}p_\nu.$$

For an angular displacement of the space axes, Eq. (2), $\boldsymbol{\omega}$ is zero and $\boldsymbol{\lambda}$ is $\boldsymbol{\theta}$. For a pure Lorentz transformation, Eq. (1), $\boldsymbol{\theta}$ is zero and

$$\boldsymbol{\omega}=(\tanh^{-1}v)\,\mathbf{v}/v,$$

$$\boldsymbol{\lambda}=2\epsilon\frac{\mathbf{p}\times\mathbf{v}}{|\mathbf{p}\times\mathbf{v}|}\arctan\frac{|\mathbf{p}\times\mathbf{v}|}{[1+(1-v^2)^{1/2}](E+m)-\epsilon\,\mathbf{p}\cdot\mathbf{v}}. \quad (57)$$

For the space reflection

$$x_i'=-x_i, \qquad t'=t,$$

Eq. (30) shows that the matrix P is β, so the suggested wave-function transformation rule is

$$\psi'(x')=\eta_P\beta\psi(x). \quad (58)$$

The equation of motion, Eq. (48), is covariant by this transformation if η_P is a constant or $\hat{\epsilon}$ times a constant where $\hat{\epsilon}$ is the operator $E^{-1}i\,\partial/\partial t$.

For the time reflection

$$x_i'=x_i, \qquad t'=-t,$$

the rule

$$\psi'(x')=\eta_T[C\gamma_5\beta\psi(x)]^* \quad (59)$$

will be used. Equation (48) is also covariant by this transformation.

Two types of charge conjugation may be considered:

$$\psi^{C_1}(x)=\eta_{C_1}[C\psi(x)]^*, \quad (60)$$

$$\psi^{C_2}(x)=\eta_{C_2}[C(H/E)\psi(x)]^*. \quad (61)$$

One finds that

$$(\psi^{C_1})^{C_1}=(-1)^{2s+1}\psi, \quad (62)$$

$$(\psi^{C_2})^{C_2}=(-1)^{2s}\psi, \quad (63)$$

so that the first type has period two for half-integer spins, the second for integer spins. The equation of motion $H\psi=i\,\partial\psi/\partial t$ is invariant to both these charge conjugations.

In addition to the functions $\psi(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$, it is essential to consider the function

$$\tilde{\psi}(\mathbf{x},t)=\left[\tfrac{1}{2}(1-\gamma_5)+\tfrac{1}{2}(1+\gamma_5)\hat{\epsilon}\right]\psi(\mathbf{x},t). \quad (64)$$

For functions with the Klein–Gordon dispersion, $\hat{\epsilon}^2$ is unity and the operator on the right in Eq. (64) has an inverse, itself. Consequently, for physical wave functions, $\tilde{\psi}(\mathbf{x},t)$ contains as much information as $\psi(\mathbf{x},t)$ and serves as an equivalent description of a system with spin s. It can be seen that $\tilde{\psi}$ transforms the same way as ψ with respect to the continuous Lorentz group. If ψ' and ψ are related by Eq. (55) so that A' and A are related by Eq. (56), then $\epsilon A'$ and ϵA also fulfill Eq. (56) and therefore $\hat{\epsilon}'\psi'$ and $\hat{\epsilon}\psi$ satisfy Eq. (55). It follows that $\tilde{\psi}$ transforms according to Eq. (55) the same as ψ and also corresponds to the $(0s)\oplus(s0)$ representation.

For spin 0, the components of $\tilde{\psi}$ are equal and are the ordinary Klein–Gordon function. For spin $\frac{1}{2}$, ψ is the usual Dirac wave function. For spin 1, the components of $\tilde{\psi}$ are closely related to the Proca field components.[13] For spin $\frac{3}{2}$, certain derivatives of the components of ψ are the Rarita–Schwinger components.[14] The wave function used by Joos and Weinberg is of the type ψ for half-integer spin and is of the type $\tilde{\psi}$ for integer spin.

IV. QUANTIZATION OF THE FIELD

In the usual way, one replaces the expansion coefficients $E^{-1/2}A_{1k}(\mathbf{p})$ and $E^{-1/2}A_{-1k}(\mathbf{p})$ by destruction and creation operators $a_{1k}(\mathbf{p})$ and $a_{-1k}^*(\mathbf{p})$, the asterisk denoting Hermitian conjugation in the Fock space. The field operator is then

$$\psi(\mathbf{x},t)=\int d\mathbf{p}\sum_k\left[a_{1k}(\mathbf{p})\psi_{\mathbf{p}1k}+a_{-1k}^*(\mathbf{p})\psi_{\mathbf{p}-1k}\right]. \quad (65)$$

[13] A. Sankaranarayanan and R. H. Good, Jr., Nuovo Cimento **36**, 1313 (1965).

[14] D. Shay, H. S. Song, and R. H. Good, Jr., Suppl. Nuovo Cimento **3**, 455 (1965).

The operators $a_{\epsilon k}(\mathbf{p})$ are postulated to satisfy the rules

$$[a_{\epsilon k}(\mathbf{p}_1), a_{\delta l}(\mathbf{p}_2)]_\pm = 0, \quad (66)$$

$$[a_{\epsilon k}(\mathbf{p}_1), a_{\delta l}^*(\mathbf{p}_2)]_\pm = \delta_{kl}\delta_{\epsilon\delta}\delta(\mathbf{p}_1-\mathbf{p}_2), \quad (67)$$

for fermions/bosons. Also it is postulated that any operator $a_{\epsilon k}(\mathbf{p})$ applied to the vacuum $|0)$ gives zero and that all physical states are produced by applying the operators $a_{\epsilon k}^*(\mathbf{p})$ in any number to the vacuum. According to Eq. (56) the Lorentz transformation rule for the operators is

$$E'^{1/2}a_{\epsilon l}'(\mathbf{p}') = \sum_k (v_{R1l}^\dagger \exp(i\,\boldsymbol{\lambda}\cdot\mathbf{s})v_{R1k})E^{1/2}a_{\epsilon k}(\mathbf{p}), \quad (68)$$

where Eq. (36) was used to express both cases $\epsilon=\pm1$ in terms of the v_{R1k}. One sees here that the transformation of the $a_{\epsilon k}(\mathbf{p})$ depends only on the physical momentum $\mathbf{q}$ and is otherwise the same for particle and antiparticle. The $(v_{R1l}^\dagger \exp(i\,\boldsymbol{\lambda}\cdot\mathbf{s})v_{R1k})$ are simply rotation matrices. The covariance of the commutation rules, Eqs. (66) and (67), follows from Eq. (68) and the fact that

$$E_1'\delta(\mathbf{p}_1'-\mathbf{p}_2') = E_1\delta(\mathbf{p}_1-\mathbf{p}_2).$$

The commutation rules for the field operators are found by a straightforward calculation, with the functions $\psi_{\mathbf{p}\epsilon k}$ given by Eqs. (53) and (46). The results are

$$[\psi_\alpha(x_1), \psi_\beta(x_2)]_\pm = 0, \quad (69)$$

$$[\psi_\alpha(x_1), \psi_\beta^*(x_2)]_\pm = im^{2s}\{\hat{\epsilon}_1\}_F \times[\cosh(2\omega_1\,\mathbf{s}\cdot\mathbf{p}_1/p_1)+\beta\hat{\epsilon}_1-\gamma_5 \times\sinh(2\omega_1\,\mathbf{s}\cdot\mathbf{p}_1/p_1)\hat{\epsilon}_1]_{\alpha\beta}\,\Delta(x_1-x_2), \quad (70)$$

where

$$\Delta(x) = \frac{-i}{(2\pi)^3}\int\frac{d\mathbf{p}}{2E}e^{i\,\mathbf{p}\cdot\mathbf{x}}(e^{-iEt}-e^{iEt}),$$

and $\hat{\epsilon}_1$, ω_1, $\mathbf{p}_1$ act on the x_1 dependence. The notation $\{\hat{\epsilon}_1\}_F$ indicates that the factor $\hat{\epsilon}_1$ is to be included for Fermi statistics but not for Bose. The equal-time commutation rules can be found from these by using the special values

$$\Delta(\mathbf{x},0)=0, \qquad \hat{\epsilon}E\,\Delta(\mathbf{x},0) = -i\,\delta(\mathbf{x}).$$

One can see very easily here which operators commute or anticommute for spacelike separations. The functions $\Delta(x)$ and $\hat{\epsilon}E\Delta(x)$ are zero when x is spacelike, the functions $\hat{\epsilon}\Delta$ and $E\Delta$ are not. For integer spin the operators $\cosh(2\omega\mathbf{s}\cdot\mathbf{p}/p)$ and $E^{-1}\sinh(2\omega\mathbf{s}\cdot\mathbf{p}/p)$ are polynomials in $\mathbf{s}\cdot\mathbf{p}$ and p^2; for half-integer spin the operators $E^{-1}\cosh(2\omega\,\mathbf{s}\cdot\mathbf{p}/p)$ and $\sinh(2\omega\,\mathbf{s}\cdot\mathbf{p}/p)$ are such polynomials. [These results follow from Weinberg's[2] Eqs. (A27–8) and (A31–2) or, in terms of spin-matrix polynomials, from Weber and Williams[15] Eqs. (27–8) and (32–3).] The gradients of functions that are zero at spacelike points are again zero at spacelike points.

[15] T. A. Weber and S. A. Williams, J. Math. Phys. **6**, 1980 (1965).

It is seen that, for Fermi statistics and half-integer spin, the anticommutator in Eq. (70) is causal in the sense that it is zero for spacelike (x_1-x_2), but in no other case is a commutator or anticommutator causal. Mathews[7] considers this to be a criticism of the formulation for integer spins. We take the point of view that the theory is all right and that the above result only shows that ψ is not the appropriate field operator to use in an interaction Hamiltonian. The field operator $\tilde{\psi}$ is well defined by Eq. (64) and, according to Eq. (65), can be written as

$$\tilde{\psi}(\mathbf{x},t) = \int d\mathbf{p}\sum_k [a_{1k}(\mathbf{p})\,\psi_{\mathbf{p}1k} - a_{-1k}^*(\mathbf{p})\,\gamma_5\,\psi_{\mathbf{p}-1k}]. \quad (71)$$

The commutation rules for $\tilde{\psi}$ are found to be

$$[\tilde{\psi}_\alpha(x_1), \tilde{\psi}_\beta(x_2)]_\pm = 0,$$

$$[\tilde{\psi}_\alpha(x_1), \tilde{\psi}_\beta^*(x_2)]_\pm = im^{2s}\{\hat{\epsilon}_1\}_F[\cosh(2\omega_1\,\mathbf{s}\cdot\mathbf{p}_1/p_1)+\beta -\gamma_5\sinh(2\omega_1\,\mathbf{s}\cdot\mathbf{p}_1/p_1)\hat{\epsilon}_1]_{\alpha\beta}\,\Delta(x_1-x_2). \quad (72)$$

These rules are causal for integer-spin bosons. The conclusion is that interactions for half-integer-spin fermions can be constructed using ψ and for integer-spin bosons using $\tilde{\psi}$. The symbol Ψ is used below to denote ψ for half-integral-spin fermions and $\tilde{\psi}$ for integer-spin bosons; it coincides with Joos'[1] and Weinberg's[2] wave function. The commutation and anticommutation rules for Ψ are causal. By combining the results above, one obtains

$$[\Psi(x_1), \Psi(x_2)]_\pm = 0, \quad (73)$$

$$[\Psi(x_1), \Psi^*(x_2)]_\pm = im^{2s}[\{\hat{\epsilon}_1\}_F\cosh(2\omega_1\,\mathbf{s}\cdot\mathbf{p}_1/p_1) +\beta-\{\hat{\epsilon}_1\}_B\gamma_5\sinh(2\omega_1\,\mathbf{s}\cdot\mathbf{p}_1/p_1)]\,\Delta(x_1-x_2). \quad (74)$$

Here the spinor indices have been suppressed. Equation (74) for example means that $[\Psi_\alpha(x_1), \Psi_\beta^*(x_2)]_\pm$ is the $\alpha\beta$ component of the matrix on the right. Weinberg[2] gives Eq. (74) in terms of the covariantly defined matrices.

V. LORENTZ GROUP GENERATORS

The generators and the operator for the excess of particle over antiparticle are defined by

$$\mathcal{P}_\mu = -\tfrac{1}{2}im^{-2s}\int d\mathbf{x}:\left(\frac{\partial\bar{\Psi}}{\partial t}p_\mu\Psi - \bar{\Psi}\frac{\partial}{\partial t}p_\mu\Psi\right):, \quad (75)$$

$$\mathcal{J}_{\mu\nu} = -\tfrac{1}{2}im^{-2s}\int d\mathbf{x}:\left(\frac{\partial\bar{\Psi}}{\partial t}J_{\mu\nu}\Psi - \bar{\Psi}\frac{\partial}{\partial t}J_{\mu\nu}\Psi\right):, \quad (76)$$

$$\mathcal{Z} = -\tfrac{1}{2}im^{-2s}\int d\mathbf{x}:\left(\frac{\partial\bar{\Psi}}{\partial t}\Psi - \bar{\Psi}\frac{\partial\Psi}{\partial t}\right):, \quad (77)$$

where

$$\bar{\Psi} = \Psi^\dagger\beta.$$

Here the dots indicate the normal product of the operators: every term involving $a_{\epsilon k}^*(\mathbf{p}_1)$ and $a_{\delta l}(\mathbf{p}_2)$ is organized so that the a^* occurs on the left of the a, anticommutators for fermions and commutators for bosons being neglected in the process. As will be shown, all the necessary properties do follow from these definitions.

Since $\bar{\Psi}$ and Ψ satisfy the Klein–Gordon equation and $\bar{\Psi}\Psi$ is a scalar, all these operators are time-independent and are Lorentz tensors on the indicated indices. Then $\mathcal{Z}$, for example, is $-i$ times the integral of the fourth component of the current

$$j_\mu(\mathbf{x},t)=i\tfrac{1}{2}m^{-2s}:[(\partial\bar{\Psi}/\partial x_\mu)\Psi-\bar{\Psi}(\partial\Psi/\partial x_\mu)]:.$$

This current is a four-vector with zero divergence so the integral of its fourth component is a time-independent scalar.

Expanded out in terms of the operators $a_{\epsilon k}(\mathbf{p})$, $\mathcal{P}_\mu$ and $\mathcal{Z}$ become

$$\mathcal{P}_i=\int d\mathbf{p}\sum_{\epsilon k}\epsilon p_i\, a_{\epsilon k}^*(\mathbf{p})\ a_{\epsilon k}(\mathbf{p}), \tag{78}$$

$$\mathcal{P}_4=i\int d\mathbf{p}\sum_{\epsilon k}E\, a_{\epsilon k}^*(\mathbf{p})\ a_{\epsilon k}(\mathbf{p}), \tag{79}$$

$$\mathcal{Z}=\int d\mathbf{p}\sum_{\epsilon k}\epsilon\, a_{\epsilon k}^*(\mathbf{p})\ a_{\epsilon k}(\mathbf{p}). \tag{80}$$

These results follow directly from the expressions for Ψ, Eqs. (65) and (71), the orthonormality of the $\psi_{\mathbf{p}\epsilon k}$, Eq. (54), and from the fact that

$$i\int d\mathbf{x}\left(\bar{\psi}_1\gamma_5\frac{\partial\psi_2}{\partial t}-\frac{\partial\bar{\psi}_1}{\partial t}\gamma_5\psi_2\right)=\int d\mathbf{x}\,\psi_1^\dagger[\beta\gamma_5,H]_+\psi_2=0 \tag{81}$$

for any two functions ψ_1,ψ_2 that satisfy the Hamiltonian equation (48). Since $\epsilon\mathbf{p}$ is the physical momentum of a state, Eq. (78) shows that $\mathcal{P}_i$ is the operator for the physical momentum in the field. Also $\mathcal{P}_4$ is i times the operator for the physical positive-definite energy and $\mathcal{Z}$ is the operator for the charge or excess of particles over antiparticles. Furthermore, since the integrands in Eqs. (75) and (77) depend only on Ψ and its derivatives, they can be interpreted as the space densities of the momentum, energy, and charge. The point is that these densities, at points that are spacelike relative to each other, commute, since the rules for Ψ are causal. This shows another aspect of the connection between spin and statistics. For any spin and statistics the total energy, momentum, and charge can be defined by Eqs. (78)–(80) and then they can be converted into integrals over space. Only for the correct spin and statistics relations does this process lead to causal space densities. The integrands in Eqs. (75) and (76) do not give Hermitian operators for the space densities as they stand. However, Hermitian operators would result from more symmetrical definitions, such as

$$\mathcal{P}_i=-\tfrac{1}{2}m^{-2s}\int d\mathbf{x}:\left[\frac{\partial\bar{\Psi}}{\partial t}\frac{\partial\Psi}{\partial x_i}+\frac{\partial\bar{\Psi}}{\partial x_i}\frac{\partial\Psi}{\partial t}\right]:, \tag{82a}$$

$$\mathcal{P}_4=\frac{i}{2}m^{-2s}\int d\mathbf{x}:\left[\frac{\partial\bar{\Psi}}{\partial t}\frac{\partial\Psi}{\partial t}+\frac{\partial\bar{\Psi}}{\partial x_i}\frac{\partial\Psi}{\partial x_i}+m^2\bar{\Psi}\Psi\right]:. \tag{82b}$$

These formulas are found from Eq. (75) by making partial integrations and using the fact that Ψ satisfies the Klein–Gordon equation. Equations (78)–(80) agree with Weaver's assignments.[6] Similar arguments can be made about $\mathcal{J}_{\mu\nu}$ and angular momentum. The forms of the integrals with Hermitian space densities are

$$\begin{aligned}\mathcal{J}_i&=\tfrac{1}{2}\epsilon_{ijk}\mathcal{J}_{jk}\\&=-\tfrac{1}{2}m^{-2s}\int d\mathbf{x}:\left\{\epsilon_{ijk}x_j\left[\frac{\partial\bar{\Psi}}{\partial t}\frac{\partial\Psi}{\partial x_k}+\frac{\partial\bar{\Psi}}{\partial x_k}\frac{\partial\Psi}{\partial t}\right]\right.\\&\qquad\left.+i\left[\frac{\partial\bar{\Psi}}{\partial t}s_i\Psi-\bar{\Psi}s_i\frac{\partial\Psi}{\partial t}\right]\right\}:,\end{aligned} \tag{82c}$$

$$\begin{aligned}\mathcal{K}_i&=i\,\mathcal{J}_{i4}\\&=-\tfrac{1}{2}m^{-2s}\int d\mathbf{x}:\left\{x_i\left[\frac{\partial\bar{\Psi}}{\partial t}\frac{\partial\Psi}{\partial t}+\frac{\partial\bar{\Psi}}{\partial x_k}\frac{\partial\Psi}{\partial x_k}+m^2\bar{\Psi}\Psi\right]\right.\\&\qquad+t\left[\frac{\partial\bar{\Psi}}{\partial t}\frac{\partial\Psi}{\partial x_i}+\frac{\partial\bar{\Psi}}{\partial x_i}\frac{\partial\Psi}{\partial t}\right]\\&\qquad\left.-\left[\frac{\partial\bar{\Psi}}{\partial t}s\alpha_i\Psi-\bar{\Psi}s\alpha_i\frac{\partial\Psi}{\partial t}\right]\right\}:.\end{aligned} \tag{82d}$$

The definitions of the generators and charge correspond to the last expression for the invariant integral in Eq. (52). However, the other expressions for the invariant integral cannot be applied since Ψ does not satisfy the Hamiltonian equation (48). However, the operators can be expressed in terms of ψ and the invariant integral by using the result

$$-\tfrac{1}{2}im^{-2s}\int d\mathbf{x}\left(\frac{\partial\bar{\Psi}}{\partial t}G\Psi-\bar{\Psi}\frac{\partial}{\partial t}G\Psi\right)=\left(\psi,G\left\{\frac{H}{E}\right\}_B,\psi\right). \tag{83}$$

Here G is any of the operators 1, p_μ, $J_{\mu\nu}$, and the H/E factor is to be used for bosons but not fermions. Equation (83) can be proven easily by using Eqs. (65) and (71) to express both sides in terms of the functions $\psi_{\mathbf{p}\epsilon k}$. All the cross terms involving $\psi_{\mathbf{p}1\pm1,k}{}^\dagger$ and $\psi_{\mathbf{p}2\mp1,l}$ are zero: if they include a γ_5, Eq. (81) applies, since $G\psi_{\mathbf{p}\epsilon k}$ is a solution of $H\psi=i\partial\psi/\partial t$; if they do not include a γ_5, there is an orthogonality, since $G\psi_{\mathbf{p}\epsilon k}$ also has quantum number ϵ.

In addition to $\mathcal{P}_\mu$, $\mathcal{J}_{\mu\nu}$, and $\mathcal{Z}$, other operators with Lorentz-invariant meanings can be defined by including another factor of H/E. For example, the number of particles is

$$\mathfrak{N} = :\left(\psi, \begin{Bmatrix} H \\ E \end{Bmatrix}_F \psi\right): = \int d\mathbf{p} \sum_{\epsilon k} a_{\epsilon k}^*(\mathbf{p})\, a_{\epsilon k}(\mathbf{p}). \quad (84)$$

However, it does not have a causal space-density.

The above operators are generators in the sense that

$$[\psi(x), \mathcal{P}_\mu] = p_\mu \psi(x), \quad (85)$$

$$[\psi(x), \mathcal{J}_{\mu\nu}] = J_{\mu\nu}\psi(x), \quad (86)$$

$$[\psi(x), \mathcal{Z}] = \psi(x). \quad (87)$$

The same equations apply for Ψ. These are easily derived from expressions like Eqs. (78) to (80). The operators $\mathcal{P}_i$, $\mathcal{J}_{ij}$, $\mathcal{Z}$ are Hermitian and $\mathcal{P}_4$, $\mathcal{J}_{i4}$ are anti-Hermitian, so

$$[\psi^\dagger(x), \mathcal{P}_\mu] = \mp[p_\mu\psi(x)]^\dagger,$$

$$[\psi^\dagger(x), \mathcal{J}_{\mu\nu}] = \mp[J_{\mu\nu}\psi(x)]^\dagger,$$

$$[\psi^\dagger(x), \mathcal{Z}] = -\psi^\dagger(x),$$

where the minus signs apply for $\mathcal{P}_i$, $\mathcal{J}_{ij}$ and the plus for $\mathcal{P}_4$, $\mathcal{J}_{i4}$. The operators p_i, $-ip_4$, J_{ij}, and $-iJ_{i4}$ were shown to be Hermitian with respect to the invariant integral of Eqs. (51) and (52) in Ref. 3, Sec. VI. Also all the operators G commute with H/E when acting on a solution of $H\psi = i\,\partial\psi/\partial t$. Consequently one finds that

$$[:(\psi, G_1\{H/E\}_B\psi):, :(\psi, G_2\{H/E\}_B\psi):] = :(\psi, [G_1, G_2]\{H/E\}_B\psi):. \quad (88)$$

Therefore the algebra of the operators $\mathcal{P}_\mu$ and $\mathcal{J}_{\mu\nu}$ is the same as the algebra of the operators p_μ and $J_{\mu\nu}$, Eqs. (21) and (22). The charge commutes with both $\mathcal{P}_\mu$ and $\mathcal{J}_{\mu\nu}$.

The c-number three-vector, four-vector, and four-tensor polarization operators are defined by

$$\mathbf{O} = S\beta\mathbf{s}S^{-1}, \quad (89a)$$

$$R_\mu = -\tfrac{1}{2}im^{-1}\epsilon_{\mu\nu\rho\sigma}s_{\nu\rho}\bar{p}_\sigma, \quad (89b)$$

$$R_{\mu\nu} = im^{-1}\epsilon_{\mu\nu\rho\sigma}(H/E)\bar{p}_\rho R_\sigma, \quad (89c)$$

where $\bar{p}_\mu$ is $(\mathbf{p}, iH)$. Here R_μ is the Bargmann–Wigner[16]-type four-vector and $R_{\mu\nu}$ was introduced in Ref. 13. The connection between the three-vector and four-vector polarization operators is

$$\mathbf{R} = \mathbf{O} + [m(E+m)]^{-1}\mathbf{O}\cdot\mathbf{p}\,\mathbf{p}, \quad (90a)$$

$$R_4 = im^{-1}(H/E)\mathbf{O}\cdot\mathbf{p}. \quad (90b)$$

Consequently the Pauli–Lubanski invariant is

$$R_\mu R_\mu = \tfrac{1}{2}R_{\mu\nu}R_{\mu\nu} = \mathbf{O}\cdot\mathbf{O} = s(s+1).$$

[16] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. (N.Y.) **34**, 211 (1948).

Quantized operators may be defined by

$$\mathcal{O}_i = :(\psi, O_i\{H/E\}_F\psi):, \quad (91a)$$

$$\mathcal{R}_\mu = :(\psi, R_\mu\{H/E\}_B\psi):, \quad (91b)$$

$$\mathcal{R}_{\mu\nu} = :(\psi, R_{\mu\nu}\{H/E\}_F\psi):. \quad (91c)$$

Evidently H commutes with $\mathbf{O}$ and therefore with R_μ and $R_{\mu\nu}$. It follows that these operators are all time-independent. Equation (83) again applies so that the four-vector and four-tensor operators may also be written in the forms

$$\mathcal{R}_\mu = -\tfrac{1}{4}m^{-2s-1}\int d\mathbf{x}\,\epsilon_{\mu\nu\rho\sigma}:\left[\frac{\partial\bar{\Psi}}{\partial t}s_{\nu\rho}p_\sigma\Psi - \bar{\Psi}\frac{\partial}{\partial t}s_{\nu\rho}p_\sigma\Psi\right]:, \quad (92a)$$

$$\mathcal{R}_{\mu\nu} = -\frac{i}{4}m^{-2s-2}\int d\mathbf{x}\,\epsilon_{\mu\nu\rho\sigma}\epsilon_{\sigma\tau\kappa\lambda} \times :\left[\frac{\partial\bar{\Psi}}{\partial t}p_\rho s_{\tau\kappa}p_\lambda\Psi - \bar{\Psi}\frac{\partial}{\partial t}p_\rho s_{\tau\kappa}p_\lambda\Psi\right]:. \quad (92b)$$

Their space densities are therefore causal. From the same reasoning as for the other operators one can write

$$[\psi(x), \mathcal{R}_\mu] = -\tfrac{1}{2}im^{-1}\epsilon_{\mu\nu\rho\sigma}s_{\nu\rho}p_\sigma\psi(x), \quad (93a)$$

$$[\psi(x), \mathcal{R}_{\mu\nu}] = \tfrac{1}{2}m^{-2}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\sigma\tau\kappa\lambda}p_\rho s_{\tau\kappa}p_\lambda\psi(x). \quad (93b)$$

Since $\mathbf{O}$ is the operator that corresponds to the quantum number k,

$$\mathbf{O}\cdot\mathbf{e}\psi_{\mathbf{p}\epsilon k} = k\psi_{\mathbf{p}\epsilon k}, \quad (94)$$

it is found that

$$\mathcal{O}_i e_i = \int d\mathbf{p}\sum_{k,\epsilon} k\, a_{\epsilon k}^*(\mathbf{p})\, a_{\epsilon k}(\mathbf{p}). \quad (95)$$

As a consequence of the normal ordering, any of these operators $\mathcal{P}_\mu$, $\mathcal{J}_{\mu\nu}$, $\mathcal{Z}$, $\mathfrak{N}$, $\mathcal{O}_i$, $\mathcal{R}_\mu$, $\mathcal{R}_{\mu\nu}$ applied to the vacuum $|0)$ gives zero. This is consistent with the vacuum being invariant to space–time displacements and Lorentz transformations. For a displacement through d_μ, for example, $\exp(i\,d_\mu\mathcal{P}_\mu)\,|0) = |0)$.

It follows that states of one particle or antiparticle of the type

$$|\mathbf{p}\epsilon k) = a_{\epsilon k}^*(\mathbf{p})\,|0)$$

are eigenstates of some of the operators:

$$\mathcal{P}_i\,|\mathbf{p}\epsilon k) = \epsilon p_i\,|\mathbf{p}\epsilon k), \quad (96a)$$

$$\mathcal{P}_4\,|\mathbf{p}\epsilon k) = iE\,|\mathbf{p}\epsilon k), \quad (96b)$$

$$\mathcal{Z}\,|\mathbf{p}\epsilon k) = \epsilon\,|\mathbf{p}\epsilon k), \quad (96c)$$

$$\mathcal{O}_i e_i\,|\mathbf{p}\epsilon k) = k\,|\mathbf{p}\epsilon k), \quad (96d)$$

$$\mathfrak{N}\,|\mathbf{p}\epsilon k) = |\mathbf{p}\epsilon k). \quad (96e)$$

These results are found by applying the commutation rules

$$[\mathcal{P}_i, a_{\epsilon k}^*(\mathbf{p})] = \epsilon p_i\, a_{\epsilon k}^*(\mathbf{p}), \tag{97a}$$

$$[\mathcal{P}_4, a_{\epsilon k}^*(\mathbf{p})] = iE\, a_{\epsilon k}^*(\mathbf{p}), \tag{97b}$$

$$[\mathcal{Z}, a_{\epsilon k}^*(\mathbf{p})] = \epsilon\, a_{\epsilon k}^*(\mathbf{p}), \tag{97c}$$

$$[\mathcal{O}_i e_i, a_{\epsilon k}^*(\mathbf{p})] = k\, a_{\epsilon k}^*(\mathbf{p}), \tag{97d}$$

$$[\mathcal{N}, a_{\epsilon k}^*(\mathbf{p})] = a_{\epsilon k}^*(\mathbf{p}) \tag{97e}$$

to the vacuum. In consequence of the operator definitions, a many-particle state is, as required, an eigenstate of the same operators with the eigenvalues adding. For example,

$$\begin{aligned}\mathcal{P}_i\, a_{\epsilon_1 k_1}^*(\mathbf{p}_1)\, a_{\epsilon_2 k_2}^*(\mathbf{p}_2)\,|\,0)
&= [\mathcal{P}_i, a_{\epsilon_1 k_1}^*(\mathbf{p}_1)\, a_{\epsilon_2 k_2}^*(\mathbf{p}_2)]\,|\,0)\\
&= \{[\mathcal{P}_i, a_{\epsilon_1 k_1}^*(\mathbf{p}_1)]a_{\epsilon_2 k_2}^*(\mathbf{p}_2)\\
&\qquad + a_{\epsilon_1 k_1}^*(\mathbf{p}_1)[\mathcal{P}_i, a_{\epsilon_2 k_2}^*(\mathbf{p}_2)]\}\,|\,0)\\
&= (\epsilon_1 p_{1i} + \epsilon_2 p_{2i})\, a_{\epsilon_1 k_1}^*(\mathbf{p}_1)\, a_{\epsilon_2 k_2}^*(\mathbf{p}_2)\,|\,0).\end{aligned}$$

The single-particle states are also eigenstates of the Casimir operators $\mathcal{P}_\mu\mathcal{P}_\mu$, $\mathcal{R}_\mu\mathcal{R}_\mu$, $\mathcal{R}_{\mu\nu}\mathcal{R}_{\mu\nu}$:

$$\mathcal{P}_\mu\mathcal{P}_\mu\,|\,\mathbf{p}\epsilon k) = -m^2\,|\,\mathbf{p}\epsilon k),$$

$$\mathcal{R}_\mu\mathcal{R}_\mu\,|\,\mathbf{p}\epsilon k) = s(s+1)\,|\,\mathbf{p}\epsilon k),$$

$$\mathcal{R}_{\mu\nu}\mathcal{R}_{\mu\nu}\,|\,\mathbf{p}\epsilon k) = s(s+1)\,|\,\mathbf{p}\epsilon k).$$

To see this second result for $\epsilon = -1$, for example, one verifies that

$$\begin{aligned}[\mathcal{R}_\mu, [\mathcal{R}_\mu, a_{-1k}^*(\mathbf{p})]] &= [\mathcal{R}_\mu, [\mathcal{R}_\mu, (\psi_{\mathbf{p}-1k}, \psi)]]\\
&= (\psi_{\mathbf{p}-1k}, R_\mu R_\mu \psi)\\
&= s(s+1)\, a_{-1,k}^*(\mathbf{p})\end{aligned}$$

and applies the equation to the vacuum.

VI. REFLECTIONS AND CHARGE CONJUGATION

The subject was studied already by Weinberg.[2] It is carried on further here to show the parity and charge-conjugation quantum numbers of actual states and to show the relations between parity, time-reversal, charge-conjugation, and the other operators defined above. The definitions below agree with Weinberg's, but special choices of the phase factors he left arbitrary have been made.

For a particle–antiparticle system in which the particle has intrinsic parity η, ± 1, the parity operator is defined to be unitary and such that

$$\mathcal{P}\Psi(\mathbf{x}, t)\mathcal{P}^{-1} = \eta\beta\Psi(-\mathbf{x}, t), \tag{98}$$

$$\mathcal{P}\,|\,0) = |\,0). \tag{99}$$

For a spin-zero system, for example, the two components of Ψ are both, redundantly, the Klein–Gordon function and η is -1 for pions or kaons. For fermions, interactions determine the parity of one particle relative to another. Equation (98) is consistent with the idea of choosing the parity of some particular fermion to be $+1$. In principle the parities of the others could be found by analyzing interactions and only the values ± 1 would occur.

Equations (98) and (99) do define an invariance of the system. To see this, one translates Eq. (98) into terms of $a_{\epsilon k}(\mathbf{p})$ and ψ, obtaining

$$\mathcal{P}a_{\epsilon k}(\mathbf{p})\mathcal{P}^{-1} = \eta\{\epsilon\}_F a_{\epsilon k}(-\mathbf{p}), \tag{100}$$

$$\mathcal{P}\psi(\mathbf{x}, t)\mathcal{P}^{-1} = \eta\{\hat{\epsilon}\}_B \beta\psi(-\mathbf{x}, t). \tag{101}$$

These results follow straightforwardly from the property

$$\beta\psi_{\mathbf{p}\epsilon k}(-\mathbf{x}, t) = \epsilon\psi_{-\mathbf{p}\epsilon k}(\mathbf{x}, t) \tag{102}$$

of the plane-wave functions. Equation (100) is consistent with the commutation rules, Eqs. (66), (67), and Eq. (101) corresponds to the space-reflection covariance of the Hamiltonian equation (58). Consequently, the $\mathcal{P}$ transform of the system satisfies all the same differential equations and commutation rules as the original system.

To see in detail what the intrinsic parities are, one considers the states of a single particle or antiparticle at rest, $|\,0\epsilon k)$. Equations (99) and (100) imply that

$$\mathcal{P}\,|\,0\epsilon k) = \eta\{\epsilon\}_F\,|\,0\epsilon k). \tag{103}$$

Thus the particle states, with $\epsilon = +1$, have parity η. Also this equation shows the known fact that, for bosons, particle and antiparticle have the same intrinsic parity, whereas, for fermions, they have the opposite. With the above definition, the parity quantum number is, as required, multiplicative. For example,

$$\mathcal{P}a_{\epsilon_1 k_1}^*(0)\, a_{\epsilon_2 k_2}^*(0)\,|\,0) = \{\epsilon_1\}_F\{\epsilon_2\}_F\, a_{\epsilon_1 k_1}^*(0)\, a_{\epsilon_2 k_2}^*(0)\,|\,0).$$

It is necessary that the operation be defined as in Eq. (98), and not with an additional factor of $\hat{\epsilon}$ on the right, if it is to be identified with the usual space-inversion operation. For example, the energy density, according to Eq. (75), is

$$\mathcal{H}(\mathbf{x}, t) = \tfrac{1}{2}m^{-2s} : \left(\frac{\partial\bar{\Psi}}{\partial t}\frac{\partial\Psi}{\partial t} - \bar{\Psi}\frac{\partial^2\Psi}{\partial t^2}\right): \tag{104}$$

and Eq. (98) yields

$$\mathcal{P}\mathcal{H}(\mathbf{x}, t)\mathcal{P}^{-1} = \mathcal{H}(-\mathbf{x}, t),$$

in agreement with the space-inversion interpretation.

The algebraic properties of this operator are: $\mathcal{P}$ commutes with $\mathcal{P}_4$, $\mathcal{J}_{ij}$, $\mathcal{Z}$, $\mathcal{N}$, $\mathcal{R}_i$, $\mathcal{R}_{ij}$, and $\mathcal{O}_i$; $\mathcal{P}$ anticommutes with $\mathcal{P}_i$, $\mathcal{J}_{i4}$, $\mathcal{R}_4$, and $\mathcal{R}_{i4}$. Equation (98) implies that the $\mathcal{P}$ transform has period two:

$$\mathcal{P}^2\Psi(\mathbf{x}, t)\mathcal{P}^{-2} = \Psi(\mathbf{x}, t).$$

Equations (98) and (99) together imply that $\mathcal{P}^2 = 1$, since $\mathcal{P}^2$ applied to any state vector produced by applying creation operators $a_{\epsilon k}^*(\mathbf{p})$ to the vacuum is the same state vector. The operators $\mathcal{P}$, $\mathcal{J}_{\mu\nu}$, $\mathcal{Z}$ have the same algebraic properties as the four-by-four matrices P, $s_{\mu\nu}$, 1. A consequence of the rules between $\mathcal{P}$, $\mathcal{P}_i$, $\mathcal{Z}$, $\mathcal{O}_i$ is: if a state has quantum numbers $\mathbf{p}$, ϵ, k, then the parity-inverted state $\mathcal{P}\mid \mathbf{p}\epsilon k)$ has quantum numbers $-\mathbf{p}$, ϵ, k. In fact, Eq. (100) gives

$$\mathcal{P}\mid \mathbf{p}\epsilon k)=\eta\{\epsilon\}_F\mid -\mathbf{p}\epsilon k).$$

The time-reversal operation $\mathfrak{I}$ is considered to give a one-to-one correspondence between the state vectors $\mid \xi)$, produced by the operators $a_{\epsilon k}^*(\mathbf{p})$ acting on the vacuum $\mid 0)$, and a conjugate set of state vectors $\mid \mathfrak{I}\xi)$,

$$\mathfrak{I}\mid \xi)=\mid \mathfrak{I}\xi), \qquad \mid \xi)=\mathfrak{I}^{-1}\mid \mathfrak{I}\xi). \tag{105}$$

The sets of states $\mid \xi)$ and $\mid \mathfrak{I}\xi)$ are supposed to remain distinct; the inner product between them is not defined. It is postulated that matrix elements between conjugate states are given by

$$(\mathfrak{I}\xi_1\mid \mathfrak{I}\xi_2)=(\xi_1\mid \xi_2)^*.$$

It follows that the operator is antilinear and antiunitary,

$$\mathfrak{I}[\alpha\mid \xi_1)+\beta\mid \xi_2)]=\alpha^*\mathfrak{I}\mid \xi_1)+\beta^*\mathfrak{I}\mid \xi_2),$$
$$(\mathfrak{I}\xi_1\mid \mathfrak{I}\xi_2)=(\xi_1\mid \mathfrak{I}^{-1}\mid \mathfrak{I}\xi_2)^*.$$

The operator is defined by

$$\mathfrak{I}\Psi(\mathbf{x},t)\mathfrak{I}^{-1}=\eta C\gamma_5\beta\Psi(\mathbf{x},-t), \tag{106}$$

where the parity signature η is included here so that later it cancels out of the $\mathcal{P}\mathcal{C}\mathfrak{I}$ product. For the operator ψ this means

$$\mathfrak{I}\psi(\mathbf{x},t)\mathfrak{I}^{-1}=\eta C\gamma_5\beta\psi(\mathbf{x},-t). \tag{107}$$

By comparison with Eq. (59) one sees that the operator $\mathfrak{I}\psi(\mathbf{x},t)\mathfrak{I}^{-1}$, say ψ_T, satisfies the equation

$$H^*\psi_T=-i\,\partial\psi_T/\partial t.$$

To find the effect of time-reversal on the $a_{\epsilon k}(\mathbf{p})$, Eqs. (36), (38), (46), and (53) are used to show that

$$C\gamma_5\beta\psi_{\mathbf{p},\epsilon,k}(\mathbf{x},-t)=(-1)^{s-k}\psi_{-\mathbf{p},\epsilon,-k}^*(\mathbf{x},t), \tag{108}$$

and then Eq. (106) gives

$$\mathfrak{I}a_{\epsilon,k}(\mathbf{p})\mathfrak{I}^{-1}=\eta(-1)^{s+k}a_{\epsilon,-k}(-\mathbf{p}). \tag{109}$$

The vacuum is considered to be nondegenerate, described by $\mid 0)$ in the original space and by $\mathfrak{I}\mid 0)$ in the conjugate space. Since

$$\mathfrak{I}a_{\epsilon,k}^*(\mathbf{p})\mid 0)=\eta(-1)^{s+k}a_{\epsilon,-k}^*(-\mathbf{p})\mathfrak{I}\mid 0),$$

the operator $a_{\epsilon,k}^*(\mathbf{p})$ is identified as creating a particle in a state with quantum numbers ϵ, k, $\mathbf{p}$ in the conjugate space as well as in the original space. Then the time-reversal of a state with quantum numbers ϵ, k, $\mathbf{p}$, described in the original space, is a state with quantum numbers ϵ, $-k$, $-\mathbf{p}$, described in the conjugate space.

The densities have the expected transformation properties. For the energy density, for example,

$$\mathfrak{I}\mathcal{H}(\mathbf{x},t)\mathfrak{I}^{-1}=\mathcal{H}(\mathbf{x},-t).$$

The algebraic properties of the time-reversal operator are: $\mathfrak{I}$ commutes with $\mathcal{Z}$ and $\mathfrak{N}$, and anticommutes with $\mathcal{P}_\mu$, $\mathcal{J}_{\mu\nu}$, $\mathcal{R}_\mu$, $\mathcal{O}_i$, and $\mathcal{R}_{\mu\nu}$. The period of the $\mathfrak{I}$ transform is found from

$$\mathfrak{I}^2\Psi(\mathbf{x},t)\mathfrak{I}^{-2}=(-1)^{2s}\Psi(\mathbf{x},t).$$

The $\mathfrak{I}$ and $\mathcal{P}$ transforms commute:

$$\mathcal{P}\mathfrak{I}\Psi\mathfrak{I}^{-1}\mathcal{P}^{-1}=\mathfrak{I}\mathcal{P}\Psi\mathcal{P}^{-1}\mathfrak{I}^{-1}.$$

The charge conjugation operator $\mathcal{C}$ is postulated to be unitary and have the properties

$$\mathcal{C}\Psi(\mathbf{x},t)\mathcal{C}^{-1}=[C\{\gamma_5\}_B\Psi(\mathbf{x},t)]^*,$$
$$\mathcal{C}\mid 0)=\mid 0). \tag{110}$$

In terms of $a_{\epsilon k}(\mathbf{p})$ and ψ, this reads

$$\mathcal{C}a_{\epsilon k}(\mathbf{p})\mathcal{C}^{-1}=a_{-\epsilon k}(-\mathbf{p}), \tag{111}$$

$$\mathcal{C}\psi(\mathbf{x},t)\mathcal{C}^{-1}=[C\{H/E\}_B\psi(\mathbf{x},t)]^*. \tag{112}$$

These are found directly from the property

$$[C\psi_{\mathbf{p}\epsilon k}]^*=\epsilon^{2s+1}\psi_{-\mathbf{p},-\epsilon,k} \tag{113}$$

of the plane-wave solutions. Equation (111) is consistent with the commutation rules, Eqs. (66) and (67), and Eq. (112) agrees with the c-number charge conjugation, Eqs. (60) and (61). Therefore, the $\mathcal{C}$-transformed fields satisfy the same commutation rules and differential equations as the original fields, and the charge conjugation is an invariance of the system. From Eq. (111) it is seen that

$$\mathcal{C}\mid \mathbf{p}\epsilon k)=\mid -\mathbf{p}-\epsilon k). \tag{114}$$

Since $\epsilon\mathbf{p}$ is the physical momentum, the effect of $\mathcal{C}$ is only to change particle into antiparticle, leaving other physical properties of the state unchanged.

The algebraic properties of the charge conjugation operator are: $\mathcal{C}^2=1$; $\mathcal{C}$ commutes with $\mathcal{P}_\mu$, $\mathcal{J}_{\mu\nu}$, $\mathfrak{N}$, $\mathcal{O}_i$, $\mathcal{R}_{\mu\nu}$; and $\mathcal{C}$ anticommutes with $\mathcal{Z}$, $\mathcal{R}_\mu$. Also $\mathcal{C}$ commutes with the space densities of $\mathcal{P}_\mu$ and $\mathcal{J}_{\mu\nu}$, and anticommutes with the space density of $\mathcal{Z}$. The $\mathcal{C}$ transform has period two; it commutes with the $\mathfrak{I}$ transform and with the $\mathcal{P}$ transform for bose statistics; it anticommutes with the $\mathcal{P}$ transform for Fermi statistics.

The definitions combine to make

$$\mathcal{P}\mathcal{C}\mathfrak{I}\Psi(\mathbf{x},t)\mathfrak{I}^{-1}\mathcal{C}^{-1}\mathcal{P}^{-1}=[\{\gamma_5\}_F\Psi(-\mathbf{x},-t)]^*.$$

This transform has period two.

To reorganize the system into terms of self-charge-conjugate states, one writes

$$\Psi=\Psi_{(+1)}+\Psi_{(-1)},$$

where

$$\Psi_{(\pm 1)}=\tfrac{1}{2}(\Psi\pm\mathcal{C}\Psi\mathcal{C}^{-1}). \quad (115)$$

This means that

$$\Psi_{(\pm 1)}=\pm\mathcal{C}\Psi_{(\pm 1)}\mathcal{C}^{-1}=\pm[C\{\gamma_5\}_B\Psi_{(\pm 1)}]^*.$$

These two parts can be written as

$$\Psi_{(\pm 1)}=2^{-1/2}\int d\mathbf{p}\sum_k[b_{\pm 1k}(\mathbf{p})\,\psi_{\mathbf{p}1k} \pm b_{\pm 1k}^*(\mathbf{p})\{-\gamma_5\}_B\,\psi_{-\mathbf{p},-1,k}], \quad (116)$$

where the $b_{\rho k}^*(\mathbf{p})$ are operators for creating self-conjugate particles,

$$b_{\pm 1k}(\mathbf{p})=2^{-1/2}[a_{1k}(\mathbf{p})\pm a_{-1k}(-\mathbf{p})]. \quad (117)$$

They have the commutation rules

$$[b_{\rho k}(\mathbf{p}_1),b_{\sigma l}(\mathbf{p}_2)]_\pm=0, \quad (118a)$$

$$[b_{\rho k}(\mathbf{p}_1),b_{\sigma l}^*(\mathbf{p}_2)]_\pm=\delta_{kl}\delta_{\rho\sigma}\delta(\mathbf{p}_1-\mathbf{p}_2). \quad (118b)$$

Since $a_{1k}(\mathbf{p})$ and $a_{-1k}(-\mathbf{p})$ apply for states with polarization k and physical momentum $\mathbf{p}$, the operator $b_{\pm 1k}^*(\mathbf{p})$ creates a particle with $\mathcal{C}=\pm 1$, polarization k, and physical momentum $\mathbf{p}$. The fields $\Psi_{(+1)}$ and $\Psi_{(-1)}$ anticommute/commute with each other, and it follows from Eqs. (73), (110), and (115) that

$$[\Psi_{(\rho)}(x_1),\Psi_{(\rho)}^*(x_2)]_\pm=\tfrac{1}{2}[\Psi(x_1),\Psi^*(x_2)]_\pm, \quad (119)$$

where the function on the right is given in Eq. (74). Thus the anticommutator/commutator of $\Psi_{(\rho)}(x_1)$ with $\Psi_{(\rho)}^*(x_2)$ is causal. Since $\mathcal{C}$ commutes with $\mathcal{P}_\mu$, $\mathcal{J}_{\mu\nu}$, $\mathfrak{N}$, $\mathcal{O}_i$, $\mathcal{R}_{\mu\nu}$, Eqs. (75), (76), (84), and (92) can be rearranged to give separate contributions from the $\mathcal{C}=+1$ and $\mathcal{C}=-1$ types of states. For example,

$$\mathcal{P}_\mu=\sum_\rho(-\tfrac{1}{2})im^{-2s}\int d\mathbf{x}:\left[\frac{\partial\bar{\Psi}_{(\rho)}}{\partial t}\,p_\mu\Psi_{(\rho)}-\bar{\Psi}_{(\rho)}\frac{\partial}{\partial t}\,p_\mu\Psi_{(\rho)}\right]:,$$

$$\mathcal{O}_i=\sum_\rho:(\psi_{(\rho)},O_i\{H/E\}_F\psi_{(\rho)}):,$$

where $\psi_{(\rho)}$ is $\frac{1}{2}(\psi+\rho\mathcal{C}\psi\mathcal{C}^{-1})$. This type of rearrangement cannot be made for $\mathcal{Z}$ or $\mathcal{R}_\mu$. Let terms in this type of ρ sum be written as $\mathcal{P}_{(\rho)\mu}$, $\mathcal{O}_{(\rho)i}$. In terms of the $b_{\rho k}(\mathbf{p})$ they are

$$\mathcal{P}_{(\rho)i}=\int d\mathbf{p}\sum_k p_i\,b_{\rho k}^*(\mathbf{p})\,b_{\rho k}(\mathbf{p}), \quad (120a)$$

$$\mathcal{P}_{(\rho)4}=i\int d\mathbf{p}\sum_k E\,b_{\rho k}^*(\mathbf{p})\,b_{\rho k}(\mathbf{p}), \quad (120b)$$

$$\mathfrak{N}_{(\rho)}=\int d\mathbf{p}\sum_k b_{\rho k}^*(\mathbf{p})\,b_{\rho k}(\mathbf{p}), \quad (120c)$$

$$\mathcal{O}_{(\rho)i}e_i=\int d\mathbf{p}\sum_k k\,b_{\rho k}^*(\mathbf{p})\,b_{\rho k}(\mathbf{p}). \quad (120d)$$

For bosons, the states $a_{\pm 1k}^*(0)\,|\,0)$ have the same parity η, so the self-conjugate particles satisfy

$$\mathcal{P}b_{\pm 1k}^*(0)\,|\,0)=\eta b_{\pm 1k}^*(0)\,|\,0)$$

and have the same intrinsic parity as the particle and antiparticle. For fermions, the states $a_{\pm 1k}^*(0)\,|\,0)$ have opposite parity and the self-conjugate particles do not have definite parities relative to the particle or antiparticle.

The relations between $\mathcal{C}$ and the Lorentz operators are: $\mathcal{C}$ commutes with $\mathcal{P}_\mu$ and $\mathcal{J}_{\mu\nu}$; the $\mathcal{C}$ and $\mathfrak{I}$ transforms commute; the $\mathcal{C}$ and $\mathcal{P}$ transforms commute for bosons. Consequently, it is a covariant notion to consider either of the fields $\Psi_{(+1)}$, $\Psi_{(-1)}$ all alone, provided a special space reflection for fermions is set up. In agreement with Weinberg,[2] one defines the operator $\mathcal{P}'$ to be unitary and such that

$$\mathcal{P}'\Psi(\mathbf{x},t)\mathcal{P}'^{-1}=\pm i\beta\Psi(-\mathbf{x},t). \quad (121)$$

Then, for fermions, the $\mathcal{P}'$ and $\mathcal{C}$ transforms commute and one can take

$$\mathcal{P}'\Psi_{(\rho)}(\mathbf{x},t)\mathcal{P}'^{-1}=\pm i\beta\Psi_{(\rho)}(-\mathbf{x},t)$$

as the space-reflection operation for the self-conjugate fermion fields. The operators $\mathcal{P}_{(\rho)\mu}$, $\mathcal{J}_{(\rho)\mu\nu}$, $\mathfrak{N}_{(\rho)}$, $\mathcal{O}_{(\rho)i}$, $\mathcal{R}_{(\rho)\mu\nu}$ apply for the field $\Psi_{(\rho)}$. From Eq. (119) it is seen that the space densities of $\mathcal{P}_{(\rho)\mu}$, $\mathcal{J}_{(\rho)\mu\nu}$, and $\mathcal{R}_{(\rho)\mu\nu}$ are causal. Since $\mathcal{C}$ anticommutes with $\mathcal{Z}$, the expression

$$\mathcal{Z}_{(\rho)}=-\tfrac{1}{2}im^{-2s}\int d\mathbf{x}:\left(\frac{\partial\bar{\Psi}_{(\rho)}}{\partial t}\Psi_{(\rho)}-\bar{\Psi}_{(\rho)}\frac{\partial\Psi_{(\rho)}}{\partial t}\right):$$

is identically zero. The same remark applies for $\mathcal{R}_\mu$.

VII. ISOTOPIC SPIN

For a particle–antiparticle system with isotopic spin t there are wave-function operators $\psi_\mu(\mathbf{x},t)$, where μ is the isospin component label, ranging from $-t$ to $+t$. In terms of the complete set of functions $\psi_{\mathbf{p}\epsilon k}(\mathbf{x},t)$ the operators are

$$\Psi_\mu=\int d\mathbf{p}\sum_k[a_{1k\mu}(\mathbf{p})\,\psi_{\mathbf{p}1k}+\{-\gamma_5\}_B\,a_{-1k-\mu}^*(\mathbf{p})\,\psi_{\mathbf{p}-1k}] \quad (122)$$

and the commutation rules are taken to be

$$[a_{\epsilon k\mu}(\mathbf{p}_1),a_{\delta l\nu}(\mathbf{p}_2)]_\pm=0, \quad (123a)$$

$$[a_{\epsilon k\mu}(\mathbf{p}_1),a_{\delta l\nu}^*(\mathbf{p}_2)]_\pm=\delta_{kl}\delta_{\epsilon\delta}\delta_{\mu\nu}\delta(\mathbf{p}_1-\mathbf{p}_2). \quad (123b)$$

The different components Ψ_μ anticommute/commute with each other and the commutation rules of a component with itself are as given in Eqs. (73) and (74). The discussions in the previous sections apply to each component separately and the equations can be taken over by adding the appropriate subscripts: Ψ is replaced by Ψ_μ, a_{1k} by $a_{1k\mu}$, and a_{-1k} by $a_{-1k-\mu}$.

The isospin operators are defined by

$$\mathfrak{I}_i=-\tfrac{1}{2}im^{-2s}\int d\mathbf{x} \times\sum_{\mu\mu'}:\left[\frac{\partial\bar{\Psi}_\mu}{\partial t}\,(T_i)_{\mu\mu'}\Psi_{\mu'}-\bar{\Psi}_\mu\frac{\partial}{\partial t}\,(T_i)_{\mu\mu'}\Psi_{\mu'}\right]:. \quad (124)$$

Here T_i are the angular-momentum matrices, the standard representation. Again the space density is causal. Alternative expressions are

$$\mathfrak{I}_i=\sum_{\mu\mu'}:(\psi_\mu,\,(T_i)_{\mu\mu'}\{H/E\}_B\psi_{\mu'}):, \tag{125}$$

$$\mathfrak{I}_i=\sum_{\mu\mu'}(T_i)_{\mu\mu'}\int d\mathbf{p}\sum_k\{a_{1k\mu}{}^*(\mathbf{p})\;a_{1k\mu'}(\mathbf{p}) -a_{-1k-\mu'}{}^*(\mathbf{p})\;a_{-1k-\mu}(\mathbf{p})\}. \tag{126}$$

These are the generators of isospace rotations:

$$[\psi_\mu(\mathbf{x},t),\mathfrak{I}_i]=\sum_{\mu'}(T_i)_{\mu\mu'}\psi_{\mu'}(\mathbf{x},t); \tag{127}$$

this follows directly from Eq. (126). Equation (88) again applies so that

$$[\mathfrak{I}_i,\mathfrak{I}_j]=i\epsilon_{ijk}\mathfrak{I}_k. \tag{128}$$

The $\mathfrak{I}_i$ commute with all the operators previously defined but $\mathfrak{I}$ and $\mathfrak{C}$; $\mathfrak{I}$ commutes with $\mathfrak{I}_1$ and $\mathfrak{I}_3$, anticommutes with $\mathfrak{I}_2$; $\mathfrak{C}$ commutes with $\mathfrak{I}_2$, anticommutes with $\mathfrak{I}_1$ and $\mathfrak{I}_3$.

Since the G parity commutes with $\mathfrak{I}_i$, it gives the decomposition of the particle–antiparticle system of fields into two self-conjugate systems that separately form isospin t representations. As usual, the operator is defined by

$$\mathcal{G}=\mathfrak{C}\exp(i\pi\mathfrak{I}_2)$$

so that here

$$\mathcal{G}\Psi_\mu(\mathbf{x},t)\mathcal{G}^{-1}=(-1)^{t-\mu}[C\{\gamma_5\}_B\Psi_{-\mu}(\mathbf{x},t)]^*. \tag{129}$$

Since

$$\mathcal{G}^2\Psi_\mu\mathcal{G}^{-2}=(-1)^{2t}\Psi_\mu,$$

only integer isospin is considered. Then the fields

$$\Psi_{(\pm1)\mu}=\tfrac{1}{2}(\Psi_\mu\pm\mathcal{G}\Psi_\mu\mathcal{G}^{-1}) \tag{130}$$

have the property

$$\mathcal{G}\Psi_{(\rho)\mu}\mathcal{G}^{-1}=(-1)^{t-\mu}[C\{\gamma_5\}_B\Psi_{(\rho)-\mu}]^*=\rho\Psi_{(\rho)\mu} \tag{131}$$

and still satisfy commutation rules like Eq. (127) with $\mathfrak{I}_i$. This is in agreement with Carruthers' remark that only for integer isospin can there be a self-conjugate system of bosons.[17–21] In terms of the complete set of functions $\psi_{\mathbf{p}\epsilon k}$ the fields are found to be

$$\Psi_{(\rho)\mu}=2^{-1/2}\int d\mathbf{p}\sum_k[b_{\rho k\mu}(\mathbf{p})\;\psi_{\mathbf{p}1k} +\rho(-1)^{t-\mu}\,b_{\rho k-\mu}{}^*(\mathbf{p})\{-\gamma_5\}_B\psi_{-\mathbf{p}-1k}], \tag{132}$$

where

$$b_{\rho k\mu}(\mathbf{p})=2^{-1/2}[a_{1k\mu}(\mathbf{p})+\rho(-1)^{t-\mu}\,a_{-1k\mu}(-\mathbf{p})]. \tag{133}$$

[17] P. Carruthers, Phys. Rev. Letters **18**, 353 (1967).
[18] Y. S. Jin, Phys. Letters **24B**, 411 (1967).
[19] G. N. Fleming and E. Kazes, Phys. Rev. Letters **18**, 764 (1967).
[20] H. Lee, Phys. Rev. Letters **18**, 1098 (1967).
[21] P. B. Kantor, Phys. Rev. Letters **19**, 394 (1967).

The G-parity transform of these operators is

$$\mathcal{G}b_{\rho k\mu}(\mathbf{p})\mathcal{G}^{-1}=\rho b_{\rho k\mu}(\mathbf{p}). \tag{134}$$

It is seen that $b_{\rho k\mu}{}^*(\mathbf{p})$ is the operator for creating a particle with G parity ρ, polarization component k, isospin component μ, and physical momentum $\mathbf{p}$. The commutation rules for these operators are

$$[b_{\rho k\mu}(\mathbf{p}_1),b_{\sigma l\nu}(\mathbf{p}_2)]_\pm=0, \tag{135a}$$

$$[b_{\rho k\mu}(\mathbf{p}_1),b_{\sigma l\nu}{}^*(\mathbf{p}_2)]_\pm=\delta_{\rho\sigma}\delta_{kl}\delta_{\mu\nu}\delta(\mathbf{p}_1-\mathbf{p}_2). \tag{135b}$$

The field commutators

$$[\Psi_{(\rho)\mu}(x_1),\Psi_{(\sigma)\nu}{}^*(x_2)]_\pm$$

are $\frac{1}{2}\delta_{\rho\sigma}\delta_{\mu\nu}$ times the function on the right in Eq. (74). The $\mathfrak{C}$ transform of the $b_{\rho k\mu}(\mathbf{p})$ operators is

$$\mathfrak{C}b_{\rho k\mu}(\mathbf{p})\mathfrak{C}^{-1}=\rho(-1)^{t-\mu}\,b_{\rho k-\mu}(\mathbf{p}). \tag{136}$$

This specializes to the known result that the central state of a multiplet $b_{\rho k0}{}^*(\mathbf{p})\,|\,0)$ is an eigenstate of $\mathfrak{C}$ with eigenvalue $\rho(-1)^t$.

An analogous decomposition can be made using the total parity $\mathcal{G}\mathcal{P}$,

$$\mathcal{G}\mathcal{P}\Psi_\mu(\mathbf{x},t)(\mathcal{G}\mathcal{P})^{-1}=\eta\beta(-1)^{t-\mu}[C\{\gamma_5\}_B\Psi_{-\mu}(-\mathbf{x},t)]^*. \tag{137}$$

Evidently it commutes with $\mathfrak{I}_i$. Since

$$(\mathcal{G}\mathcal{P})^2\Psi_\mu(\mathcal{G}\mathcal{P})^{-2}=(-1)^{2s+2t}\Psi_\mu,$$

only the case of $(s+t)$ integer is considered. Then the fields

$$\Psi_{(\pm1)\mu}=\tfrac{1}{2}(\Psi_\mu\pm\mathcal{G}\mathcal{P}\Psi_\mu\mathcal{P}^{-1}\mathcal{G}^{-1})$$

have the property

$$\mathcal{G}\mathcal{P}\Psi_{(\rho)\mu}(\mathbf{x},t)(\mathcal{G}\mathcal{P})^{-1}=\eta\beta(-1)^{t-\mu}[C\{\gamma_5\}_B\Psi_{(\rho)-\mu}(-\mathbf{x},t)]^* =\rho\Psi_{(\rho)\mu}(\mathbf{x},t)$$

and also satisfy commutation rules like Eq. (127) with $\mathfrak{I}_i$. In terms of the complete set of functions $\psi_{\mathbf{p}\epsilon k}$ these fields are found to be

$$\Psi_{(\rho)\mu}=2^{-1/2}\int d\mathbf{p}\sum_k[b_{\rho k\mu}(\mathbf{p})\;\psi_{\mathbf{p}1k} +\eta\,\rho(-1)^{t+\mu}\,b_{\rho k-\mu}{}^*(\mathbf{p})\{-\gamma_5\}_B\psi_{\mathbf{p}-1k}],$$

where

$$b_{\rho k\mu}(\mathbf{p})=2^{-1/2}[a_{1k\mu}(\mathbf{p})+\eta\,\rho(-1)^{t-\mu}\,a_{-1k\mu}(\mathbf{p})].$$

These also satisfy Eqs. (135) and so are independent particle destruction operators. Since

$$\mathcal{G}\mathcal{P}b_{\rho k\mu}(\mathbf{p})(\mathcal{G}\mathcal{P})^{-1}=\rho b_{\rho k\mu}(\mathbf{p}),$$

the ρ is the $\mathcal{G}\mathcal{P}$ quantum number of the states $b_{\rho k\mu}{}^*(\mathbf{p})|\,0)$. However the states involved here are of an unusual type since they are superpositions of states with opposite physical momenta. If, in addition to $t+s$, t is an integer, then

$$\mathfrak{C}b_{\rho k\mu}(\mathbf{p})\mathfrak{C}^{-1}=\eta\,\rho(-1)^{t-\mu}\,b_{\rho k-\mu}(-\mathbf{p}).$$

This means that

$$\mathcal{C} b_{\rho k 0}^*(0)|0) = \eta_\rho(-1)^t b_{\rho k 0}^*(0)|0),$$

so in the rest frame the central member of a self $\mathcal{G}\mathcal{P}$ multiplet has eigenvalue $\eta_\rho(-1)^t$ of $\mathcal{C}$. Again the field commutators

$$[\Psi_{(\rho)\mu}(x_1), \Psi_{(\sigma)\nu}^*(x_2)]_\pm$$

are $\frac{1}{2}\delta_{\rho\sigma}\delta_{\mu\nu}$ times the function on the right in Eq. (74). These field operators $\Psi_{(\rho)\mu}$ do not transform as spinors under the Lorentz group, since $\mathcal{P}$ does not commute with $\mathcal{P}_i$ or $\mathcal{J}_{i4}$. However $\mathcal{P}$ does commute with $\mathcal{J}_{ij}$, and $\Psi_{(\rho)\mu}$ transforms the same as Ψ_μ with respect to space rotations.

VIII. $SU(3)$ SYMMETRY

An analogous development can be given when the internal symmetry group is $SU(3)$. Let the Lie algebra be specified by the model representation of eight 3×3 Hermitian traceless matrices as given by Pursey[22]:

$$T_1=\frac{1}{2}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\qquad N=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},$$
$$T_2=\frac{1}{2}\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\qquad N'=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},$$
$$T_3=\frac{1}{2}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\qquad M=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\qquad (138)$$
$$U=\frac{1}{3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix},\qquad M'=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}.$$

The T_i are the generators of isotopic spin and U is the generator for hypercharge. The electric charge of a particle is given by $e(T_3+\frac{1}{2}U)$. The four remaining generators connect the multiplets with definite isospin and hypercharge, thereby enlarging the group from $SU(2)\times U(1)$ to $SU(3)$. The extra generators can be organized conveniently into ladder operators:

$$X_+=N+iN',\qquad Y_+=-M+iM',$$
$$X_-=M+iM',\qquad Y_-=N-iN',\qquad (139)$$

[22] D. L. Pursey, Proc. Roy. Soc. (London) **225A**, 284 (1963).

and they have commutation rules:

$$[T_\epsilon, X_\phi]=\delta_{\epsilon,-\phi}X_\epsilon,\qquad [T_\epsilon, Y_\phi]=\delta_{\epsilon,-\phi}Y_\epsilon,$$
$$[T_3, X_\epsilon]=\tfrac{1}{2}\epsilon X_\epsilon,\qquad [T_3, Y_\epsilon]=\tfrac{1}{2}\epsilon Y_\epsilon,$$
$$[U, X_\epsilon]=X_\epsilon,\qquad [U, Y_\epsilon]=-Y_\epsilon,\qquad (140)$$
$$[X_\epsilon, X_\phi]=0,\qquad [Y_\epsilon, Y_\phi]=0,$$
$$[X_\epsilon, Y_\phi]=-4\epsilon\delta_{\epsilon\phi}T_\epsilon+4\delta_{\epsilon,-\phi}(T_3+\tfrac{3}{2}\epsilon U),$$

where ϵ and ϕ can be $\pm$ independently and $T_\pm$ is $T_1\pm iT_2$.

Nelson[23,24] has shown that the two labels needed to specify an irreducible representation can be expressed as j and α where

$$j=\tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots,\qquad |\alpha|\le j-(1/2).$$

For fixed j, α ranges over the specified limits by steps of unity. The labels within a representation are the isotopic spin t, its third component μ, and the hypercharge u. For the purpose of discovering the isotopic spin-hypercharge content of an irreducible representation, it is convenient to introduce the two linear combinations

$$m=(1/2)u+(1/3)\alpha+t+(1/2),$$
$$n=(1/2)u+(1/3)\alpha-t-(1/2).\qquad (141)$$

In an irreducible representation specified by j and α, m and n independently assume all values by steps of unity within the limits

$$\alpha+(1/2)\le m\le j,\qquad -j\le n\le\alpha-(1/2).$$

Pictorially speaking, j defines a ladder with rungs labeled $-j$, $-j+1$, $\cdots$, $+j$; α lies midway between two of the rungs; the labels m and n range over the rungs above and below α, respectively. The isotopic spin-hypercharge content of a representation can be deduced by making a graph with m on one axis and n on the other and writing $(2t+1, u)$, as found from Eqs. (141), at the coordinates allowed by j and α. Figure 1 shows the example $j=\frac{3}{2}$ and $\alpha=0$, the octet representation. Since $2t+1=m-n$ and $u=m+n-(2/3)\alpha$, the point of lowest isospin on this graph is at $m=\alpha+(1/2)$ and $n=\alpha-(1/2)$, where $2t+1=1$ and $u=(4/3)\alpha$; the point of highest isospin is at $m=j$ and $n=-j$, where $2t+1=2j$ and $u=-(2/3)\alpha$. The

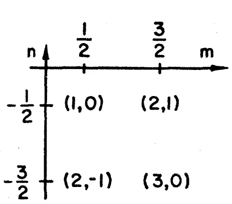

FIG. 1. Isospin multiplicity and hyperchange $(2t+1, u)$ for the $j=\frac{3}{2}$, $\alpha=0$ representation of $SU(3)$.

[23] T. J. Nelson, J. Math. Phys. **8**, 857 (1967).
[24] T. J. Nelson, Nuovo Cimento **52A**, 985 (1967).

rest of the graph may be filled in rapidly. Of course, for fixed t, μ ranges by integer steps from $-t$ up to $+t$, and is independent of the hypercharge u.

The matrix elements of $X_{\pm}$ and $Y_{\pm}$ have been derived in general by Pursey.[22] The nonzero ones, in the present notation, are

$$\begin{aligned} \langle j\alpha;\, m+1,\, n,\, \mu\pm\tfrac{1}{2} \mid X_{\pm} \mid j\alpha;\, mn\mu\rangle &= f_{m,n,\pm\mu}{}^{j\alpha}, \\ \langle j\alpha;\, m,\, n+1,\, \mu\pm\tfrac{1}{2} \mid X_{\pm} \mid j\alpha;\, mn\mu\rangle &= \mp f_{n,m,\pm\mu}{}^{j\alpha}, \quad (142) \\ \langle j\alpha;\, m,\, n-1,\, \mu\pm\tfrac{1}{2} \mid Y_{\pm} \mid j\alpha;\, mn\mu\rangle &= -f_{-n,-m,\pm\mu}{}^{j,-\alpha}, \\ \langle j\alpha;\, m-1,\, n,\, \mu\pm\tfrac{1}{2} \mid Y_{\pm} \mid j\alpha;\, mn\mu\rangle &= \mp f_{-m,-n,\pm\mu}{}^{j,-\alpha}, \end{aligned}$$

where

$$f_{m,n,\pm\mu}{}^{j\alpha} = \left[\frac{2(m-n+1\pm 2\mu)(m-\alpha+\frac{1}{2})(j-m)(j+m+1)}{(m-n)(m-n+1)}\right]^{1/2}.$$

The nonzero matrix elements of $\mathbf{T}$ and U are

$$\langle j\alpha;\, m,\, n,\, \mu\pm 1 \mid T_{\pm} \mid j\alpha;\, mn\mu\rangle = [\tfrac{1}{4}(m-n)^2-\tfrac{1}{4}-\mu(\mu\pm 1)]^{1/2},$$

$$\langle j\alpha;\, mn\mu \mid T_3 \mid j\alpha;\, mn\mu\rangle = \mu,$$

$$\langle j\alpha;\, mn\mu \mid U \mid j\alpha;\, mn\mu\rangle = m+n-\tfrac{2}{3}\alpha.$$

A multiplet of particles carrying the representation specified by j and α has the field operators

$$\Psi_{mn\mu}{}^{j\alpha} = \int d\mathbf{p} \sum_k \{a_{1k;mn\mu}{}^{j\alpha}(\mathbf{p})\, \psi_{\mathbf{p}1k} + \{-\gamma_5\}_B a_{-1k;-n-m-\mu}{}^{j,-\alpha*}(\mathbf{p})\, \psi_{\mathbf{p}-1k}\}, \quad (143)$$

where the $a_{\epsilon k;mn\mu}{}^{j\alpha}(\mathbf{p})$ satisfy the same type of commutation rules as before, Eqs. (123), except with two more δ symbols for the m and n indices. The commutation rules for $\Psi_{mn\mu}{}^{j\alpha}$ are the same as before but with δ symbols for the m, n, μ indices.

The quantized generators of $SU(3)$ transformations are

$$\mathcal{G}_\rho = -\tfrac{1}{2}im^{-2s}\int d\mathbf{x} \sum_{mn\mu,m'n'\mu'} \langle j\alpha;\, mn\mu \mid G_\rho \mid j\alpha;\, m'n'\mu'\rangle \times:\left(\frac{\partial\bar{\Psi}_{mn\mu}{}^{j\alpha}}{\partial t}\Psi_{m'n'\mu'}{}^{j\alpha} - \bar{\Psi}_{mn\mu}{}^{j\alpha}\frac{\partial\Psi_{m'n'\mu'}{}^{j\alpha}}{\partial t}\right):, \quad (144)$$

where $\rho=1$ to 8 and G_ρ are the abstract generators of the group. Alternate expressions are

$$\mathcal{G}_\rho = \sum_{mn\mu,m'n'\mu'} \langle j\alpha;\, mn\mu \mid G_\rho \mid j\alpha;\, m'n'\mu'\rangle \times:(\psi_{mn\mu}{}^{j\alpha},\, \{H/E\}_B\psi_{m'n'\mu'}{}^{j\alpha}):, \quad (145)$$

$$\mathcal{G}_\rho = \sum_{mn\mu,m'n'\mu'} \langle j\alpha;\, mn\mu \mid G_\rho \mid j\alpha;\, m'n'\mu'\rangle \times\int d\mathbf{p} \sum_k [a_{1k;mn\mu}{}^{j\alpha*}(\mathbf{p})\, a_{1k;m'n'\mu'}{}^{j\alpha}(\mathbf{p}) - a_{-1k;-n',-m',-\mu'}{}^{j,-\alpha*}(\mathbf{p})\, a_{-1k;-n,-m,-\mu}{}^{j,-\alpha}(\mathbf{p})]. \quad (146)$$

These are the group generators, since

$$[\Psi_{mn\mu}{}^{j\alpha}(\mathbf{x},t),\, \mathcal{G}_\rho] = \sum_{m'n'\mu'} \langle j\alpha;\, mn\mu \mid G_\rho \mid j\alpha;\, m'n'\mu'\rangle\, \Psi_{m'n'\mu'}{}^{j\alpha}(\mathbf{x},t). \quad (147)$$

The commutation rules among the $\mathcal{G}_\rho$ are the same as among the matrices of Eq. (138).

The discussions of reflections in Sec. VI apply to each $SU(3)$ component of the wave-function operator separately. The equations can be taken over by replacing Ψ by $\Psi_{mn\mu}{}^{j\alpha}$, a_{1k} by $a_{1k;mn\mu}{}^{j\alpha}$, and a_{-1k} by $a_{-1k;-n,-m,-\mu}{}^{j,-\alpha}$. This means that

$$\mathcal{P}a_{\epsilon k;mn\mu}{}^{j\alpha}(\mathbf{p})\mathcal{P}^{-1} = \eta\{\epsilon\}_F a_{\epsilon k;mn\mu}{}^{j\alpha}(-\mathbf{p}),$$

$$\mathfrak{I}a_{\epsilon k;mn\mu}{}^{j\alpha}(\mathbf{p})\mathfrak{I}^{-1} = \eta(-1)^{s+k}a_{\epsilon,-k;mn\mu}{}^{j\alpha}(-\mathbf{p}), \quad (148)$$

$$\mathcal{C}a_{\epsilon k;mn\mu}{}^{j\alpha}(\mathbf{p})\mathcal{C}^{-1} = a_{-\epsilon,k;-n,-m,-\mu}{}^{j,-\alpha}(-\mathbf{p}). \quad (149)$$

By combining Eqs. (146) and (148) one finds that $\mathfrak{I}$ commutes with $\mathcal{G}_\rho$ when the matrix elements $\langle| G_\rho |\rangle$ are real, anticommutes when they are pure imaginary. The reality properties are found from Eq. (142). The result is that $\mathfrak{I}$ anticommutes with $\mathfrak{I}_2$, $\mathfrak{N}'$, and $\mathfrak{M}'$, and commutes with the others. On combining Eqs. (149) and (146) one finds that $\mathcal{C}$ commutes with $\mathcal{G}_\rho$ where the $\langle| G_\rho |\rangle$ are pure imaginary, anticommutes when they are real. Consequently $\mathcal{C}$ commutes with $\mathfrak{I}_2$, $\mathfrak{N}'$, and $\mathfrak{M}'$ and anticommutes with the others.

Fields that are in a sense self-conjugate can be constructed for the representations with $\alpha=0$. To see this, one first verifies that the fields

$$\mathrm{X}_{mn\mu}{}^{j\alpha} = (-1)^{(1/2)(m+n)-\alpha-\mu}\mathcal{C}\Psi_{-n,-m,-\mu}{}^{j,-\alpha}\mathcal{C}^{-1}$$

are the components of a j, α basis. This can be done by taking Eq. (147) as given and calculating the commutators of X and $\mathcal{G}$, the matrix elements of G being known from Eqs. (142). Thus, the transform

$$\mathcal{K}\Psi_{m,n,\mu}{}^{j,0}\mathcal{K}^{-1} = (-1)^{(1/2)(m+n)-\mu}\mathcal{C}\Psi_{-n,-m,-\mu}{}^{j,0}\mathcal{C}^{-1} \quad (150)$$

produces another set of operators that can be added to $\psi_{m,n,\mu}{}^{j,0}$ without disturbing the representation. The transform has period two, so the operators

$$\Psi_{(\rho)mn\mu}{}^{j0} = \tfrac{1}{2}[\Psi_{mn\mu}{}^{j0} + \rho(-1)^{(1/2)(m+n)-\mu}\mathcal{C}\Psi_{-n,-m,-\mu}{}^{j0}\mathcal{C}^{-1}], \quad (151)$$

where ρ is ± 1, have the property

$$\mathcal{K}\Psi_{(\rho)mn\mu}{}^{j0}\mathcal{K}^{-1} = \rho\Psi_{(\rho)mn\mu}{}^{j0}. \quad (152)$$

In terms of the complete set of functions $\psi_{\mathbf{p}\epsilon k}$ these field operators are

$$\Psi_{(\rho)mn\mu}{}^{j0} = 2^{-1/2}\int d\mathbf{p} \sum_k [b_{\rho k;mn\mu}{}^{j0}(\mathbf{p})\psi_{\mathbf{p}1k} + \rho(-1)^{(1/2)(m+n)-\mu}b_{\rho k;-n,-m,-\mu}{}^{j0*}(\mathbf{p})\{-\gamma_5\}_B\psi_{-\mathbf{p}-1k}], \quad (153)$$

where

$$b_{\rho k;n\,n\mu}{}^{j0}(\mathbf{p}) = 2^{-1/2}[a_{1k;mn\mu}{}^{j0}(\mathbf{p}) + \rho(-1)^{(1/2)(m+n)-\mu}\, a_{-1k;mn\mu}{}^{j0}(-\mathbf{p})]. \quad (154)$$

These are independent-particle operators satisfying

$$[b_{\rho k;mn\mu}{}^{j0}(\mathbf{p}_1),\, b_{\sigma l;uv\nu}{}^{j0}(\mathbf{p}_2)]_\pm = 0,$$

$$[b_{\rho k;mn\mu}{}^{j0}(\mathbf{p}_1),\, b_{\sigma l;uv\nu}{}^{j0*}(\mathbf{p}_2)]_\pm = \delta_{\rho\sigma}\delta_{kl}\delta_{mu}\delta_{nv}\delta_{\mu\nu}\delta(\mathbf{p}_1-\mathbf{p}_2).$$

It follows that the fields with different ρ numbers anticommute/commute with each other and that the field commutators

$$[\Psi_{(\rho)mn\mu}{}^{j0}(x_1),\, \Psi_{(\sigma)uv\nu}{}^{j0*}(x_2)]_\pm$$

are $(1/2)\delta_{\rho\sigma}\delta_{mu}\delta_{nv}\delta_{\mu\nu}$ times the function on the right in Eq. (74). The $\mathcal{K}$ transform of the operators b^{j0} is, from Eq. (152),

$$\mathcal{K}b_{\rho k;mn\mu}{}^{j0}(\mathbf{p})\mathcal{K}^{-1} = \rho b_{\rho k;mn\mu}{}^{j0}(\mathbf{p}).$$

Therefore $b_{\rho k;mn\mu}{}^{j0*}(\mathbf{p})|0)$ are eigenstates of $\mathcal{K}$ with eigenvalue ρ. From Eq. (151) it is found that

$$\mathcal{G}\Psi_{(\rho)mn\mu}{}^{j0}\mathcal{G}^{-1} = (-1)^{t-\mu}\,\mathcal{C}\Psi_{(\rho)mn-\mu}{}^{j0}\mathcal{C}^{-1}$$
$$= (-1)^{t-\mu}\,\rho(-1)^{(1/2)(m+n)+\mu}\,\Psi_{(\rho)-n,-m,+\mu}{}^{j0}.$$

If, in addition to $\alpha=0$, one considers $u=0$, then $m=-n$, t is an integer, and

$$\mathcal{G}\Psi_{(\rho)mn\mu}{}^{j0}\mathcal{G}^{-1} = (-1)^t\rho\Psi_{(\rho)mn\mu}{}^{j0}.$$

Therefore the G parity of the states $b_{\rho k;mn\mu}{}^{j0*}(\mathbf{p})|0)$ with zero hypercharge is $(-1)^t\rho$. As a well-known example, since the π's have $t=1$ and negative G parity, the pseudoscalar octet has K parity $+1$, the η must have G parity of $+1$ and C number $+1$. The vector nonet has K parity -1 and the tensor nonet has K parity $+1$.

IX. DISCUSSION

This theory shows the difference between the quantization processes for integer-spin and half-integer-spin particles. In both cases there is a complete set of wave functions and a wave-function operator defined by summing the functions multiplied by independent particle operators. This operator is causal for half-integer spin, but an extra factor of γ_5 has to be put into the negative-frequency terms to get a causal operator for integer spin.

The fact that a uniform treatment of the integrals of motion can be made for all spins, Sec. V, perhaps corresponds to the fact that all the fields have the Klein–Gordon dispersion. The formulas look unfamiliar in the spin-$\frac{1}{2}$ case, but they simplify when $i\,\partial/\partial t$ is replaced by $\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m$. For example, $\mathcal{Z}$ becomes just $\int\psi^\dagger\psi d\mathbf{x}$.

In the definitions of the vector and tensor polarization operators, Eqs. (91b) and (91c), the factors of (H/E) were chosen in such a way that the space densities of $\mathcal{R}_\mu$ and $\mathcal{R}_{\mu\nu}$, shown in Eqs. (92), would be causal. A consequence is that $\mathcal{R}_{\mu\nu}$ commutes with $\mathcal{C}$ but $\mathcal{R}_\mu$ does not. This means that $\mathcal{R}_{\mu\nu}$ is appropriate for discussing polarization of self-charge-conjugate particles but $\mathcal{R}_\mu$ is not. However, if the requirement of having a causal space-density were relaxed, an extra factor of H/E could be inserted in the definition of $\mathcal{R}_\mu$; $\mathcal{R}_\mu$ would then commute with $\mathcal{C}$ and could be used for self-charge-conjugate particles.

Carruthers' point, that self-conjugate isospin multiplets can exist only if t is integral, is seen to be related to the fact that the G-parity operator has period two only for integer isospin. His remark is applied to operators that are Lorentz spinors so the self $\mathcal{GP}$ operators set up in Sec. VII for integer $(s+t)$ are not in contradiction with it. For self-conjugate $SU(3)$ multiplets it is clear right away that α must be zero, since otherwise the conjugate representation is a different representation. The construction made in Sec. VIII shows that the causal self-conjugate multiplets do exist for all j, $\alpha=0$ representations.

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