Phase and Angle Variables in Quantum Mechanics*

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The quantum-mechanical description of phase and angle variables is reviewed, with emphasis on the proper mathematical description of these coordinates. The relations among the operators and state vectors under consideration are clarified in the context of the Heisenberg uncertainty relations. The familiar case of the azimuthal angle variable φ and its "conjugate" angular momentum L_z is discussed. Various pitfalls associated with the periodicity problem are avoided by employing periodic variables ($\sin\varphi$ and $\cos\varphi$) to describe the phase variable. Well-defined uncertainty relations are derived and discussed. A detailed analysis of the three-dimensional harmonic oscillator excited in coherent states is given. A detailed analysis of the simple harmonic oscillator is given. The usual assumption that a (Hermitian) phase operator ϕ (conjugate to the number operator N) exists is shown to be erroneous. However, cosine and sine operators C and S exist and are the appropriate phase variables. A Poisson bracket argument using action-angle (rather J, $\cos\phi$, $\sin\phi$) variables is used to deduce C and S. The spectra and eigenfunctions of these operators are investigated, along with the important "phase-difference" periodic variables. The properties of the oscillator variables in the various types of states are analyzed with special attention to the uncertainty relations and the transition to the classical limit. The utility of coherent states as a basis for the description of the evolution of the density matrix is emphasized. In this basis it is easy to identify the classical Liouville equation in action-angle variables along with quantum-mechanical "corrections." Mention is made of possible physical applications to superfluid systems.

CONTENTS

1. INTRODUCTION

Despite 50 years of vigorous development, the theory[1–3] of quantum mechanics continues to reward its students with new insights. Within the past few years the notion of phase variables in quantum systems has been greatly clarified. In this paper we review recent work in this field. We are especially concerned with the proper definition of phase variables and their conjugate operators. It is of interest that this problem is best viewed from the structure of the uncertainty relations, discovered originally by Heisenberg.[4]

We are not explicitly concerned with the continuing debate on the proper interpretation of the formalism of quantum mechanics. For all practical purposes we subscribe to the orthodox[5–7] "Copenhagen interpretation." We only note in passing that even from the early days of quantum mechanics many distinguished physicists have been unhappy with the probabilistic view of the universe implied by the orthodox view.[3,8] In particular, the famous Bohr–Einstein debates[3,5,6] led to a considerable clarification of the issues at stake.

Challenges to the orthodox interpretation continue. Although detailed mathematical investigations of the quantum theory have been made with the intention of invalidating the uncertainty principle, most of the recent challenges have involved the "hidden variable" concept. We give some pertinent references.[9–15]

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[2] E. Schrödinger, Ann. Physik **79**, 361, 489, 734 (1926).
[3] M. Jammer, *Conceptual Development of Quantum Mechanics* (McGraw-Hill Book Co., New York, 1966).
[4] W. Heisenberg, Z. Physik **43**, 172 (1927).
[5] N. Bohr, article in *Albert Einstein: Philospher-Scientist*, P. A. Schilpp, Ed. (Library of Living Philosophers, Inc., Evanston, Ill., 1949).
[6] W. Heinsenberg, article in *Niels Bohr and the Development of Physics*, W. Pauli, Ed. (Pergarmon Press, Ltd., London, 1955).
[7] W. Heisenberg, *Das Naturbild der Heuligen Physik* (Rowohlt, Hamburg, 1958).
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[10] J. Picht, Acta Phys. Polon. **27**, 25 (1965).
[11] D. Bohm, Phys. Rev. **85**, 166, 180 (1952).
[12] D. Bohm and J. Bub, Rev. Mod. Phys. **38**, 453, 470 (1966).
[13] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).
[14] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Julius Springer-Verlag, Berlin, 1932) [English transl.: Princeton University Press, Princeton, N.J. (1955)], see p. 209, and p. 305ff (English edition).
[15] J. Albertson, Am. J. Phys. **29**, 478 (1961). This reference contains an exposition of von Neumann's proof.

In Sec. 2 we study the general question of uncertainty relations using a new method developed by Jackiw.[16] Equations are given to determine the wave functions minimizing various uncertainty products. In Sec. 3 the "coherent states" of a harmonic oscillator are derived as the minimum-uncertainty position-momentum states. These states were discovered by Schrödinger very early in the history of quantum mechanics. In Sec. 4 we begin our study of angle variables by discussing the relation between orbital angular momentum L_z and azimuthal angle ϕ. It is emphasized that much confusion exists in the literature owing to the improper treatment of periodicity. Well-defined uncertainty relations involving L_z and the periodic variables $\sin\phi$ and $\cos\phi$ are given. It is then shown that these inequalities are nearly satisfied as equalities for a three-dimensional harmonic oscillator in a coherent state. Finally, exact minimum-uncertainty states are given.

Section 5 discusses the extent to which number and phase variables can be used to describe the quantum-mechanical harmonic oscillator. In his original paper dealing with the quantization of the electromagnetic field, Dirac[17] assumed the existence of an Hermitian phase operator ϕ conjugate to the number operator. As shown in an important paper by Susskind and Glogower,[18] this assumption leads to contradictions. However, one can describe the phase by means of two well-defined Hermitian operators C and S, which correspond to $\cos\phi$ and $\sin\phi$ in the classical limit. It is stressed that the absence of a proper phase operator results from the boundedness of the eigenvalue spectrum of the number operator. Uncertainty relations[19] connecting the well-defined operators are derived and discussed.

Section 6 is concerned with the eigenvalue spectra and eigenfunctions of the C and S operators. The eigenvalue spectra of C and S are continuous and range from -1 to $+1$. The eigenfunctions (phase states) are distributions normalized to a delta function; they do not remain eigenstates as time progresses. In Sec. 7 operators are constructed which correspond to the phase difference of two independent oscillators. Here we find the interesting result that the sine and cosine operators representing the phase difference commute with the total number operator. These operators have a point spectrum. We derive the eigenfunctions and the eigenvalue spectrum.

In Sec. 8 the physical properties of the coherent states, the phase states, the minimum-uncertainty states, and the number states are analyzed with emphasis on the uncertainty relations. It is emphasized that for coherent states the proper number-phase uncertainty relations are nearly minimized when the mean oscillator number is greater than unity.[19] Finally the (normalizable) minimum-uncertainty states of Jackiw[16] are given for the simple harmonic oscillator.

In Sec. 9 the usefulness of the number-phase variables for the theory of irreversibility in systems of oscillators is discussed.[20] The equation of motion for the density matrix is given in the coherent state basis; when the transition to "action-angle" variables is given, one obtains equations of the type developed by Brout and Prigogine[21,22] for classical systems, along with quantum-mechanical corrections. We sketch the general derivation of the quantum-mechanical master equation for the energy distribution of a system of coupled harmonic oscillators. Uncertainty relations are given connecting the unperturbed Liouville operator and its conjugate phase angle. Finally, in Sec. 10, we point out some interesting analogies between some of the state vectors studied here and qualitative properties of wave functions representing superfluid systems.[23]

Although we have not treated the problematic energy–time uncertainty relation,[24] we think the present analysis is pertinent to this question. In our opinion, a definitive treatment of the energy–time uncertainty relation remains to be given. In particular, all time operators proposed up to now[25–28] seem to be undefined mathematically. We give a representative list of references on this problem.[27,29–31]

2. UNCERTAINTY RELATIONS

A. Heisenberg's Method and the Direct Method

Variants of the standard Heisenberg derivation of uncertainty relations are given in many text books on quantum mechanics.[32–35]

One starts by defining the rms fluctuation of the Hermitian dynamical variable x by

$$(\Delta x)^2=\langle x^2\rangle-\langle x\rangle^2; \qquad \langle A\rangle\equiv\langle\Psi\mid A\mid\Psi\rangle. \quad (2.1)$$

Then

$$(\Delta x)^2=\langle X^2\rangle, \quad (2.2)$$

[16] R. Jackiw, J. Math. Phys. (to be published).
[17] P. A. M. Dirac, Proc. Roy. Soc. (London) **A114**, 243 (1927).
[18] L. Susskind and J. Glogower, Physics **1**, 49 (1964).
[19] P. Carruthers and M. M. Nieto, Phys. Rev. Letters **14**, 387 (1965).
[20] P. Carruthers and K. S. Dy, Phys. Rev. **147**, 214 (1966).
[21] R. Brout and I. Prigogine, Physica **22**, 621 (1956).
[22] I. Prigogine, *Non-Equibibrium Statistical Mechanics* (Interscience Publishers, Inc., New York, 1962).
[23] P. W. Anderson, Rev. Mod. Phys. **38**, 298 (1966), and references therein.
[24] See, however, the remarks at the end of Sec. **5**.
[25] W. Pauli, *Handbuch der Physik* (Springer-Verlag, Berlin, 1958), Vol. V/1, p. 59–64.
[26] F. T. Smith, Phys. Rev. **118**, 349 (1960); see Eq. (49).
[27] Y. Aharanov and D. Bohm, Phys. Rev. **122**, 1649 (1961).
[28] B. A. Lippmann, Phys. Rev. **151**, 1023 (1966).
[29] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon Press Ltd., London, and Addison-Wesley Publishing Co., Inc., Reading, Mass., 1958), Sec. 44, p. 150ff.
[30] V. A. Fock, Soviet Phys.—JETP **15**, 784 (1962) [Zh. Eksperim. i Teor. Fiz. **42**, 1135 (1962)].
[31] Y. Aharanov and D. Bohm, **134**, B1417 (1964).
[32] D. Bohm, *Quantum Theory* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1951), p. 206.
[33] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1958), p. 45.
[34] E. Merzbacher, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1961), p. 156.
[35] K. Gottfried, *Quantum Mechanics* (W. A. Benjamin, Inc., New York, 1966), p. 215.

where the deviation from the mean is

$$X=x-\langle x\rangle. \tag{2.3}$$

The Schwartz inequality then implies

$$(\Delta x)^2(\Delta y)^2=\langle X^2\rangle\langle Y^2\rangle\geq |\langle XY\rangle|^2;$$
$$(\Delta x)^2(\Delta y)^2\equiv |\langle XY\rangle|^2+R(\Psi). \tag{2.4}$$

R is a remainder term that depends only on Ψ, and is positive semi-definite. One easily sees that the condition for the vanishing of R is

$$R(\Psi)=0 \Leftrightarrow X|\Psi\rangle=cY|\Psi\rangle, \tag{2.5}$$

where c is a constant.

Now, let the commutator of X and Y be given by iA:

$$[x, y]=[X, Y]\equiv iA \tag{2.6}$$

Then the product XY may be written as

$$XY=\tfrac{1}{2}\{X, Y\}+\tfrac{1}{2}iA. \tag{2.7}$$

The expectation value of the anticommutator,

$$\tfrac{1}{2}\langle\{X, Y\}\rangle=\tfrac{1}{2}\langle\{x, y\}\rangle-\langle x\rangle\langle y\rangle, \tag{2.8}$$

is the quantum-mechanical analog of the covariance of two randomly distributed variables.[36] One can consider the variables to be uncorrelated if both sides of Eq. (2.8) are equal to zero.

Combining (2.5)–(2.7) gives us

$$U(\Psi)\equiv(\Delta x)^2(\Delta y)^2=\tfrac{1}{4}P(\Psi)+\tfrac{1}{4}Q(\Psi)+R(\Psi), \tag{2.9}$$
$$P(\Psi)=|\langle[X, Y]\rangle|^2=|\langle A\rangle|^2, \tag{2.10}$$
$$Q(\Psi)=|\langle\{X, Y\}\rangle|^2. \tag{2.11}$$

In the Heisenberg method one considers A to be a constant (c number). Then P goes not depend on Ψ and is just A^2. In this case,

$$U(\Psi)=(\Delta x)^2(\Delta y)^2=\tfrac{1}{4}A^2+\tfrac{1}{4}Q(\Psi)+R(\Psi), \tag{2.12}$$
$$U(\Psi)\geq\tfrac{1}{4}A^2. \tag{2.13}$$

Davidson pointed out[37] that the derivation of (2.13) is not general because it breaks down when $|\Psi\rangle$ is an eigenstate of x or y. In this case x is not Hermitian with respect to $y|x'\rangle$ and y is not Hermitian with respect to $x|y'\rangle$. This is easily seen by contradiction, since, if

$$\langle x'|xy|x'\rangle=x'\langle x'|y|x'\rangle,$$

then $A=0$, contradicting (2.6).

Davidson realized that, at least for the position-momentum case, the general proof is accomplished by an Euler–Lagrange variational principle, and that this leads one to the well-known Gaussian for the state that minimizes $(\Delta x)^2(\Delta p)^2$. As we shall see in the next subsection, the variational principle allows one to construct a proof for general operators x, y, and A.

Equation (2.13) reduces to an equality (i.e., the minimum uncertainty product) if R and Q vanish. Jackiw[16,38] showed that this condition gives an eigenvalue equation for the minimum uncertainty state. One simply substitutes Eq. (2.5) (meaning $R=0$), in Eq. (2.8), setting Eq. (2.8) equal to zero (meaning $Q=0$). One then finds that $c=i\gamma$, where γ is real, and obtains the eigenvalue equation for the minimum uncertainty state:

$$X|\Psi\rangle+i\gamma Y|\Psi\rangle=0 \quad (\gamma \text{ real}) \tag{2.14}$$

or

$$(x+i\gamma y)|\Psi\rangle=\lambda|\psi\rangle,$$
$$\lambda\equiv\langle x\rangle+i\gamma\langle y\rangle. \tag{2.15}$$

γ is easily determined. Multiplying (2.14) by $(X-i\gamma Y)$ gives

$$(X^2+\gamma^2Y^2+i\gamma[X, Y])|\Psi\rangle=0. \tag{2.16}$$

Then, by using (2.6) and (2.13) with the equality sign, we have

$$[X^2+\gamma^2Y^2\mp2\gamma(\Delta x)(\Delta y)]|\Psi\rangle=0, \tag{2.17}$$

where the sign is determined as $A=\pm|A|$ and $\Delta x=+[(\Delta x)^2]^{1/2}$. Taking the expectation value of (2.16) and using (2.2) gives

$$\gamma=\pm[(\Delta x)^2/(\Delta y)^2]^{1/2}$$
$$=\langle A\rangle/2(\Delta y)^2=A/2(\Delta y)^2, \tag{2.18}$$

so that

$$\{[X^2/(\Delta x)^2]+[Y^2/(\Delta y)^2]-2\}|\Psi\rangle=0. \tag{2.19}$$

We then have an eigenvalue equation (2.15) for obtaining the minimum uncertainty state $|\Psi\rangle$. There are three real parameters γ, Re λ, and Im λ, and the normalization condition $\langle\Psi|\Psi\rangle=1$. Following Jackiw,[16] we call this the *direct method*, valid when A is a c number.[39]

B. Analytic Method

The direct method is not a general one for obtaining the minimum uncertainty state when A is a q number (operator).[40,41] For this more general case, Jackiw developed[16] an Euler–Lagrange (EL) formalism.[42]

We start with the subsidary condition $\langle\Psi|\Psi\rangle=1$. Considering the variation of $\langle\Psi|$ to be independent of $|\Psi\rangle$, we obtain the condition for $U(\Psi)$ to be a minimum:

$$\delta U/\delta\langle\Psi|=m|\Psi\rangle. \tag{2.20}$$

[36] E. Parzen, *Modern Probability Theory and Its Applications* (John Wiley & Sons, Inc., New York, 1960), p. 361ff.

[37] E. R. Davidson, J. Chem. Phys. **42**, 1491 (1965).

[38] In this section we are using the notation of Gottfried (Ref. 35) and Jackiw (Ref. 16). The general exposition follows Jackiw. His work is a thorough investigation of this aspect of the subject.

[39] This result does not overcome the objections of Davidson (Ref. 21) for the case when $|\psi\rangle$ is an eigenstate of x or y. However, one can evaluate these cases explicity to see if they minimize the uncertainty relation.

[40] P. A. M. Dirac, Proc. Cambridge Phil. Soc. **23**, 412 (1926).

[41] P. A. M. Dirac, Proc. Roy. Soc. (London) **113**, 621 (1927).

[42] What follows is valid only if the matrix elements of x for eigenstates of y diverge, and vice versa. Otherwise, they would obviously yield zero uncertainty product.

Here m is a Lagrange multiplier. Now from (2.1) we have

$$\delta(\Delta x)^2/\delta\langle\Psi| = x^2|\Psi\rangle - 2x|\Psi\rangle\langle x\rangle$$
$$= X^2|\Psi\rangle - \langle x^2\rangle|\Psi\rangle. \quad (2.21)$$

So, combining (2.9), (2.20), and equations like (2.21), we find

$$[(\Delta y)^2X^2|\Psi\rangle + (\Delta x)^2Y^2]|\Psi\rangle$$
$$= [(\Delta x)^2\langle y\rangle^2 + (\Delta y)^2\langle x\rangle^2 + m]|\Psi\rangle. \quad (2.22)$$

By taking the expectation value of (2.22), we obtain the value of the numerical constant in the brackets on the right-hand side, it being $2(\Delta x)^2(\Delta y)^2$. Putting this back into (2.22) and rearranging, we obtain the Euler–Lagrange equation for $|\Psi\rangle$, for which U is a minimum:

$$\{[X^2/(\Delta x)^2] + [Y^2/(\Delta y)^2] - 2\}|\Psi\rangle = 0 \quad (2.23)$$

or

$$\{[(x-\alpha)/a]^2 + [(y-\beta)/b]^2 - 2\}|\Psi\rangle = 0. \quad (2.24)$$

Equations (2.23) and (2.24) are eigenvalue equations for $|\Psi\rangle$, with the four real parameters α, a, β, b, which are determined self-consistently by having them satisfy

$$\alpha = \langle x\rangle, \qquad \beta = \langle y\rangle,$$
$$a^2 = (\Delta x)^2, \qquad b^2 = (\Delta y)^2. \quad (2.25)$$

We call the states which satisfy (2.24) *critical states*.[16] They could be maximum, minimum, or "points of inflection" of U. One can take a second variation of U to check whether the state corresponds to a minimum of U. By calculating $(\Delta x)^2(\Delta y)^2 = a^2b^2$ for the various critical states, one can see which of them yields the smallest value.

This now gives us a generalization of the direct method that is valid if $[x, y]$ is a c number or a q number. This procedure is called the *analytic method*.[16] We find the relationship between the direct method and the analytic method by attempting to use the direct method for the case where A is a q number. One starts from (2.16), which is still valid, and proceeds as before. This time one obtains

$$\gamma = \langle A\rangle/2(\Delta y)^2 = \pm[(\Delta x)^2/(\Delta y)^2]^{1/2} \quad (2.26)$$

$$\{[X^2/(\Delta x)^2] + [Y^2/(\Delta y)^2] - (2A/\langle A\rangle)\}|\Psi\rangle = 0. \quad (2.27)$$

Comparing (2.26) and (2.27) with the direct method results in (2.18) and (2.19); we see that if A is a q number, the direct method determines a critical state $|\Psi\rangle$ if and only if $|\Psi\rangle$ is an eigenstate of A. These states are not in general all the possible critical states, and so the direct method is not a general method to determine those states which minimize $U(\Psi)$ when A is a q number.

C. "Complicated" Uncertainty Relations and the Direct Method

It is to be noted that, since the EL equation of the analytic method is of higher order than the direct method equation, the EL equation may have solutions although the direct method does not yield a solution. (This still leaves open the question whether or not the solution is a minimum solution.) However, the EL equation is more complicated; and if it were possible (for any given uncertainty relation) to use the direct method, it would certainly be easier to resolve the minimization question.

In fact, by using "complicated" uncertainty relations, this is indeed possible. If we divide (2.9) by $P(\Psi)$, we have

$$\Psi_1(\Psi) = U(\Psi)/\langle A\rangle^2, \quad (2.28)$$
$$U_1(\Psi) = [(\Delta x)^2(\Delta y)^2/\langle A\rangle^2]$$
$$= \tfrac{1}{4} + \tfrac{1}{4}[Q(\Psi)/P(\Psi)] + [R(\Psi)/P(\Psi)], \quad (2.29)$$
$$U_1(\Psi) \geq \tfrac{1}{4}. \quad (2.30)$$

By using the variational principle on (2.28), we obtain the *necessary* condition that

$$\{[X^2/(\Delta x)^2] + [Y^2/(\Delta y)^2] - (2A/\langle A\rangle)\}|\Psi\rangle = 0.$$

Equation (2.31) is the same as (2.27). Thus, by considering the uncertainty relation $U_1(\Psi)$ instead of $U(\Psi)$, we are able to use the direct method to obtain the minimum uncertainty state. [The case when $P(\Psi) = \langle A\rangle^2 = 0$ must be examined separately.] This will be extremely useful to us in the future.

We end this section with an observation. Because of the normalization condition, the set of $|\Psi\rangle$ is not compact. Because of this the uncertainty product may not attain a minimum or maximum, and we will have no normalizable solutions that satisfy our EL equation. Then we have no state that is "as classical as possible," since small uncertainty product states approach a limiting minimum state that is outside our allowed set. We find such cases later.

3. POSITION-MOMENTUM VARIABLES; COHERENT STATES

For the first application of the results of the last section, we consider the familiar position-momentum case. Starting from the commutator

$$[x, p] = i, \qquad p = -i(d/dx), \quad (3.1)$$

we use the direct method [Eqs. (2.15) and (2.18)] to give

$$[x + \gamma(d/dx)]\psi = \lambda\psi, \quad (3.2)$$
$$\lambda = \langle x\rangle + i\gamma\langle p\rangle, \quad (3.3)$$
$$\gamma = 1/2(\Delta p)^2 = 2(\Delta x)^2, \quad (3.4)$$

where the last equality in (3.4) is true because, by assumption, ψ is a minimum uncertainty state. The

solution to (3.2) is

$$\psi \propto \exp\left[(\lambda x/\gamma)-(x^2/2\gamma)\right]. \tag{3.5}$$

Combining (3.2) and (3.5) and normalizing yields

$$\psi(x)=[2\pi(\Delta x)^2]^{-1/4} \times \exp\{-[(x-\langle x\rangle)/2\Delta x]^2+i\langle p\rangle x\}. \tag{3.6}$$

We also could have used the variation[37] method or the EL analytic method,[16] obtaining the eigenvalue equation

$$\left[\left(\frac{x-\alpha}{a}\right)^2+\left(\frac{-i(\partial/\partial x)-\beta}{b}\right)^2\right]\psi(x)=2\psi(x). \tag{3.7}$$

By comparing (3.7) to the Schrödinger equation for the harmonic oscillator, we find that when

$$ab=\Delta x\Delta y=n+\tfrac{1}{2}, \tag{3.8}$$

we obtain normalized solutions of the form

$$\psi_n(x)=\exp(i\beta x)u_n(x-\alpha) =\exp(i\langle p\rangle x)u_n(x-\langle x\rangle), \tag{3.9}$$

where the $u_n(x)$ are the ordinary normalized "number" eigenfunctions with mass (m) of $b^2/2$, force constant (K) of $2/a^2$, and energy 2. In the usual notation they are

$$u_n(x)=[(\eta/\pi^{1/2}2^n n!)^{1/2}H_n(\eta x)\exp(-\tfrac{1}{2}\eta^2x^2),$$
$$(b/a)^{1/2}\rightarrow\eta=mK=m^2\omega^2, \tag{3.10}$$

where the H_n are the Hermite polynomials,[43] and ω is the angular frequency corresponding to the classical harmonic oscillator.

In usual discussions of the harmonic oscillator, the number eigenfunctions of (3.10) are used. They have energy values

$$E_n=(n+\tfrac{1}{2})\omega \tag{3.11}$$

and are orthogonal. However, no matter how large the quantum number n, we have

$$\langle n\mid x\mid n\rangle=\langle n\mid p\mid n\rangle=0. \tag{3.12}$$

Thus, in no way do they go over to the classical harmonic oscillator, which has

$$x_{\text{class}}(t)=A\cos(\phi-\omega t),$$
$$p_{\text{class}}(t)=m\omega A\sin(\phi-\omega t). \tag{3.13}$$

The key to finding the harmonic oscillator states which are "as classical as possible" is given by Eq. (3.8). This shows the well-known result that in the number representation the ground state is the "most classical" of the number states, i.e., $(\Delta x)^2(\Delta p)^2=\frac{1}{4}$. The ground state is special because it is a Gaussian. This means that its Fourier transform (the wave function in momentum space) is also a Gaussian.

Now we have the key, for we know that if a Gaussian is narrow or broad [small or large $(\Delta x)^2$], its Fourier transform is broad or narrow [large or small $(\Delta p)^2$] and that $(\Delta x)^2(\Delta p)^2$ equals a constant (here $\frac{1}{4}$). Also, moving a Gaussian off center, $\langle x\rangle\neq 0$ or $\langle p\rangle\neq 0$, does not affect the Gaussian transform properties. Thus we can have any energy expectation value $\langle H\rangle$ that we desire. This means that we have an uncountable number of states, each with values in the Heisenberg representation like (3.1), i.e.,

$$\langle x\rangle=x_{\text{class}},$$
$$\langle p\rangle=p_{\text{class}}. \tag{3.14}$$

Equation (3.14) can be thought of as a Gaussian wave packet, which at $t=0$ is centered at $A\cos\phi$. As time increases, the packet oscillates between $(+A)$ and $(-A)$, with angular frequency ω, without spreading. This picture has been demonstrated nicely by Henley and Thirring.[44]

The states that we have just described are called "coherent states." By looking at (3.6), we can see immediately that these are just the states we have been seeking. [Multiply (3.6) by $\exp(i\omega t)$ and substitute in (3.14) to obtain the Heisenberg representation.] It is easy to check explicitly that for these states we do indeed have

$$(\Delta x)^2(\Delta p)^2=\tfrac{1}{4} \qquad \text{"coherent" states.} \tag{3.15}$$

The coherent states were first discovered in a very clever manner by Schrödinger,[45] who was looking for wave packets with just the oscillator and non-spreading characteristics we have described.[46] By noting the form of Hermite Polynomial generating function

$$\sum_{n=0}^{\infty}\frac{\xi^n}{n!}\exp(-\tfrac{1}{2}x^2)H_n(x)=\exp(-\xi^2+2\xi x-\tfrac{1}{2}x^2), \tag{3.16}$$

he realized he could get the desired Gaussian form by letting

$$\psi=C\sum_{n=0}^{\infty}\left(\frac{\xi}{2}\right)^n\frac{\psi_n}{n!}, \tag{3.17}$$

where C is the normalization constant.

Physically the coherent states are very important. Besides being the "natural" oscillator states, they are also the states emitted by a classical current source, and are of great usefulness in a quantum-mechanical description of coherent light sources, as was emphasized

[43] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics, Part I* (McGraw-Hill Book Co., New York, 1953), p. 786.

[44] E. M. Henley and W. Thirring, *Elementary Quantum Field Theory* (McGraw-Hill Book Co., New York, 1962), Chap. 2.

[45] E. Schrödinger, Z. Physik 14, 664 (1926).

[46] D. Bohm (Ref. 32, Sec. 2, p. 306) goes through a derivation similar to that of Schrödinger in Ref. 45.

by Glauber[47–51] and by Sudarshan and Mehta.[52–54] They have also been used in investigation of other ordered phenomena, such as superconductivity and superfluidity,[55] and phonons in crystals.[20]

In a previous work[56] the authors have investigated in great detail the relationship between the coherent states and the harmonic oscillator. Also, questions such as the probability distribution of an oscillator in the coherent states, more general forced quantum oscillator, and transition probabilities of a forced oscillator were discussed. For a thorough description of the coherent states in the Heisenberg representation, we refer the reader to this paper and also to Ref. 49.

In place of position and momentum variables, it is often convenient to use the creation and annihilation operators $a^\dagger$ and a. These are related by

$$\begin{aligned} a&=[i/(2m\omega)^{1/2}](p-im\omega x),\\ a^\dagger&=[-i/(2m\omega)^{1/2}](p+im\omega x),\\ [a,\,a^\dagger]&=1. \end{aligned} \tag{3.18}$$

The last equation of (3.18) is equivalent to (3.1). The analysis of the eigenvalue spectrum of the oscillator is most easily done with these complex normal coordinates. The Hamiltonian is

$$H=(a^\dagger a+\tfrac{1}{2})\omega, \tag{3.19}$$

which has eigenvalues (3.11). $a^\dagger a$ is called the number operator. From the commutators

$$\begin{aligned} [a,\,H]&=\omega a,\\ [a^\dagger,\,H]&=-\omega a^\dagger, \end{aligned} \tag{3.20}$$

we see that: (1) the Heisenberg operators $a(t)$ and $a^\dagger(t)$ are $a\exp(-i\omega t)$ and $a^\dagger\exp(i\omega t)$, respectively; (2) a ($a^\dagger$) decreases (increases) the energy of a state by energy ω. For number states $|\,n\rangle$ we have

$$\begin{aligned} |\,n\rangle&=[(a^\dagger)^n/(n!)^{1/2}]\,|\,0\rangle,\\ a\,|\,n\rangle&=n^{1/2}\,|\,n-1\rangle,\\ a^\dagger\,|\,n\rangle&=(n+1)^{1/2}\,|\,n+1\rangle. \end{aligned} \tag{3.21}$$

The position and momentum operators are found by solving (3.18):

$$\begin{aligned} x&=x_0(a+a^\dagger),\\ p&=-im\omega x_0(a-a^\dagger),\\ x_0&=(2m\omega)^{-1/2}, \end{aligned} \tag{3.22}$$

where x_0 is the zero-point rms fluctuation.

The coherent states can be defined as the eigenstates of the annihilation operator, i.e.,

$$a\,|\,\alpha\rangle=\alpha\,|\,\alpha\rangle, \tag{3.23}$$

where α is any complex number. Expansion of $|\,\alpha\rangle$ in number states leads to an easily solved recursion relation for the expansion coefficient:

$$\begin{aligned} |\,\alpha\rangle&=\exp(-\tfrac{1}{2}\,|\,\alpha\,|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}\,|\,n\rangle,\\ |\,\alpha\rangle&=A(\alpha)\,|\,0\rangle,\\ A(\alpha)&\equiv\exp(\alpha a^\dagger-\alpha^* a), \end{aligned} \tag{3.24}$$

where $A(\alpha)$ is the unitary ($AA^\dagger=A^\dagger A=1$) operator that creates the coherent states. The operators a, $a^\dagger$, and A have the properties

$$A^\dagger(\alpha)\,aA(\alpha)=a+\alpha, \tag{3.25}$$

$$A^\dagger(\alpha)\,a^\dagger A(\alpha)=a^\dagger+\alpha^*, \tag{3.26}$$

$$A(\alpha)=A^\dagger(-\alpha). \tag{3.27}$$

As we noted before, the coherent states are not orthogonal. (They could not be, because the countable number basis spans the space, whereas the coherent states have an uncountable number of states, one for every α on the complex plane.) From (3.24) the inner product is

$$\begin{aligned} \langle\alpha\,|\,\beta\rangle&=\exp(\alpha^*\beta-\tfrac{1}{2}\,|\,\alpha\,|^2-\tfrac{1}{2}\,|\,\beta\,|^2),\\ |\,\langle\alpha\,|\,\beta\rangle\,|^2&=\exp(-\,|\,\alpha-\beta\,|^2), \end{aligned} \tag{3.28}$$

and corresponds to the overlap integral of two Gaussians in the Schrödinger wave picture. However, the coherent states are complete, and have the completeness relation

$$1=\pi^{-1}\int d^2\alpha\,|\,\alpha\rangle\langle\alpha\,|, \tag{3.29}$$

$$d^2\alpha=d\,\mathrm{Re}\,\alpha\,d\,\mathrm{Im}\,\alpha. \tag{3.30}$$

We now can very easily calculate expectation values with the coherent states. The expectation value of any polynomial (ordered so that all creation operators stand to the left of all annihilation operators) in a and $a^\dagger$ is then obtained by the substitutions $a\to\alpha$, $a^\dagger\to\alpha^*$. In particular,

$$\begin{aligned} \langle\alpha\,|\,x\,|\,\alpha\rangle&=x_0\langle\alpha\,|\,(a+a^\dagger)\,|\,\alpha\rangle\\ &=x_0(\alpha+\alpha^*)\\ &=2x_0\,\mathrm{Re}\,\alpha. \end{aligned} \tag{3.31}$$

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Similarly,

$$\langle\alpha \mid p \mid \alpha\rangle = 2m\omega x_0 \operatorname{Im}\alpha, \tag{3.32}$$

$$\langle\alpha \mid H \mid \alpha\rangle = \omega\langle\alpha \mid (a^\dagger a + \tfrac{1}{2}) \mid \alpha\rangle = \omega(|\alpha|^2 + \tfrac{1}{2}). \tag{3.33}$$

Equation (3.33) shows us that, as expected, we can have any expectation value of the Hamiltonian.

Combining (2.1), (3.18), and (3.31), it is trivial to see that

$$(\Delta x)^2_{\text{coh}} = x_0^2 = (2m\omega)^{-1}. \tag{3.34}$$

In the same manner,

$$(\Delta p)^2_{\text{coh}} = m^2\omega^2 x_0^2 = \tfrac{1}{2}m\omega. \tag{3.35}$$

Therefore, for all coherent states we have

$$(\Delta x)^2(\Delta p)^2_{\text{coh}} = \tfrac{1}{4}, \tag{3.36}$$

and the coherent states are "as classical as possible" in position-momentum space.

In closing this section we would like to emphasize the importance of the coherent states. This is not only due to their physical significance, but also to the great simplification in calculating physical quantities that often comes with their use. The fact that they are not orthogonal is of small issue and is often completely overweighed by these other considerations.

4. ANGULAR MOMENTUM-ANGLE VARIABLES

A. Uncertainty Relations

A proper description of angle or phase variables requires that periodicity be taken into account. This is especially important as a preliminary to the recognition of a second crucial feature: the nature of the eigenvalue spectrum of the operator that is conjugate to the angle variable.

We begin our discussion by reconsidering the familiar problem of angular momentum in three dimensions. Let ϕ be the azimuthal angle about the z axis; the definition

$$\phi = \tan^{-1}(y/x) \tag{4.1}$$

only defines ϕ modulo 2π. Defining ϕ to be continuous, ranging from $-\infty$ to $+\infty$, we can represent the z component of orbital momentum as a differential operator ($\hbar=1$):

$$L_z = xp_y - yp_x = -i(\partial/\partial\phi). \tag{4.2}$$

L_z and ϕ are commonly regarded as conjugate variables, the relevant commutator being

$$[\phi, L_z] = i. \tag{4.3}$$

However, the use of this angle variable entails many pitfalls for the unwary. In particular, L_z given by (2.2) is Hermitian only in $\{\psi\}_P$, the space of periodic functions with period 2π. Only for this space can be have

$$\langle L_z\psi_1 \mid \psi_2\rangle = \langle\psi_1 \mid L_z\psi_2\rangle, \tag{4.4}$$

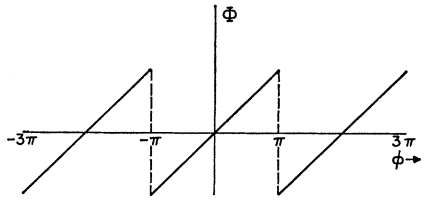

FIG. 1. The periodic phase variable Φ, which is discontinuous.

since, in detail, the equality (4.4) involves the vanishing of a "surface term." But ϕ itself is not periodic, as defined above. Thus L_z will not be Hermitian and one cannot conclude that (4.3) implies the uncertainty relation

$$\Delta L_z \Delta\phi \geq \tfrac{1}{2} \quad \text{(wrong)}. \tag{4.5}$$

Of course (4.5) has long been suspect[57] since one knows that fluctuations in ϕ greater than 2π have little physical meaning. It is easy to choose wave functions for which $\Delta L_z \leq \frac{1}{4}$ so that $\Delta\phi$ is greater than 2π. We can even have $\Delta L_z = 0$ by choosing eigenfunctions of L_z, the spherical harmonics $Y_{lm}(\theta, \phi)$, denoted by $|\, lm\rangle$. (However, one should remember the remarks in Sec. 2 concerning uncertainty relations in states which are eigenfunctions of one of the operators.)

Another aspect of the difficulty concerns a "paradox" which arises from a naive trust in the Hermiticity of L_z. Taking the matrix element of (4.3), we find

$$\langle lm' \mid [\phi, L_z] \mid lm\rangle = i\langle lm' \mid lm\rangle,$$

$$(m-m')\langle lm' \mid \phi \mid lm\rangle = i\delta_{mm'} \quad \text{(wrong)}. \tag{4.6}$$

This equation, if true, would predict that $0=1$, for $m'=m$. The mistake is in supposing that, in

$$\langle lm' \mid L_z\phi \mid lm\rangle,$$

L_z can "operate to the left," as implied by Eq. (4.4). The trouble arises from the non-periodicity of $\phi \mid lm\rangle$, or alternately that ϕ is not Hermitian with respect to $L_z \mid lm\rangle$.

A possible solution is to introduce a periodic coordinate $\Phi(\phi)$ (Fig. 1). Then one can work with spherical harmonics and the periodic variable Φ; L_z is safely Hermitian. However, the commutation rule is no longer given by Eq. (4.3), but rather by[18,58]

$$[\Phi, L_z] = i\{1 - 2\pi \sum_{n=-\infty}^{\infty} \delta[\phi - (2n+1)\pi]\}. \tag{4.7}$$

The discontinuity in Φ could be put elsewhere, but always one of the delta functions in (4.7) would contribute in the physical interval of 2π. Therefore the "uncertainty relation" (4.5) does not hold for Φ either.

Since no one doubts the qualitative content of (4.5), it would be useful to formulate the problem in a mathematically respectable way. One procedure for doing

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[58] D. Judge and J. T. Lewis, Phys. Letters **5**, 190 (1963).

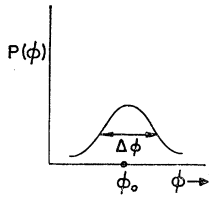

FIG. 2. A well-behaved probability distribution $P(\phi)$ with $\Delta\phi$ small compared to 2π leads to the "usual" uncertainty relation of Eq. (4.5).

this has been proposed by Judge.[59] Judge and others[59–63] studied the uncertainty relation obtained from (4.7) and defined $(\Delta\phi)^2$ to be

$$(\Delta\phi)^2=V(\gamma)=\min\int_{-\pi}^{\pi}\phi^2\mid\psi(\phi+\gamma)\mid^2 d\phi. \quad (4.8)$$

We prefer a different method which seems simpler and also reduces to (4.5) whenever the latter is appropriate, i.e., whenever $\Delta\phi$ is rather less than 2π. The idea is simply to use *continuous* periodic variables to locate the azimuthal position. As Louisell[64] first pointed out, the simplest idea is to give $\sin\phi$ and $\cos\phi$ instead of ϕ. Any other physical (periodic) function can then be constructed using Fourier series.

From the commutation relations

$$[\sin\phi,\ L_z]=i\cos\phi,$$
$$[\cos\phi,\ L_z]=-i\sin\phi, \quad (4.9)$$

we can deduce the uncertainty relations

$$(\Delta L_z)^2(\Delta\sin\phi)^2\geq\tfrac{1}{4}\langle\cos\phi\rangle^2, \quad (4.10a)$$
$$(\Delta L_z)^2(\Delta\cos\phi)^2\geq\tfrac{1}{4}\langle\sin\phi\rangle^2, \quad (4.10b)$$

or

$$(\Delta L_z)^2(\Delta\sin\phi)^2/\langle\cos\phi\rangle^2\geq\tfrac{1}{4}, \quad (4.10c)$$
$$(\Delta L_z)^2(\Delta\cos\phi)^2/\langle\sin\phi\rangle^2\geq\tfrac{1}{4}, \quad (4.10d)$$

since L_z is now Hermitian. (We always assume the periodicity of the wave function; no difficulty occurs for half-integral *total* angular momentum since the wave function occurs quadratically.)

If desired, the pair of relations (4.10) can be rewritten to yield one relation symmetrical in $\sin\phi$ and $\cos\phi$:

$$U\equiv\frac{(\Delta L_z)^2[(\Delta\cos\phi)^2+(\Delta\sin\phi)^2]}{\langle\sin\phi\rangle^2+\langle\cos\phi\rangle^2}\geq\tfrac{1}{4}. \quad (4.11)$$

Any of the relations (4.10)–(4.11) reduce to (4.5) whenever the distribution of ϕ in the wave function is sufficiently localized. It is easy to show this. One simply constructs the probability distribution $P(\phi)$ by integrating out all variables except ϕ. It is assumed that $P(\phi)$ is peaked at ϕ_0 with an essentially symmetrical distribution (Fig. 2). By expanding $\cos\phi$ and $\sin\phi$ up to second order in $\phi-\phi_0$, we obtain

$$\cos\phi=\cos\phi_0[1-\tfrac{1}{2}(\delta\phi)^2]-\sin\phi_0\delta\phi,$$
$$\sin\phi=\sin\phi_0[1-\tfrac{1}{2}(\delta\phi)^2]+\cos\phi_0\delta\phi,$$
$$\delta\phi\equiv\phi-\phi_0, \quad (4.12)$$

[Note that, since $P(\phi)$ is assumed to be localized around ϕ_0, 2π is greater than any significant $\delta\phi$.]

We now find that

$$\langle\cos\phi\rangle=\cos\phi_0[1-\tfrac{1}{2}\langle(\delta\phi)^2\rangle],$$
$$\langle\sin\phi\rangle=\sin\phi_0[1-\tfrac{1}{2}\langle(\delta\phi)^2\rangle],$$
$$\langle\cos^2\phi\rangle=\cos^2\phi_0[1-\langle(\delta\phi)^2\rangle]+\sin^2\phi_0\langle(\delta\phi)^2\rangle,$$
$$\langle\sin^2\phi\rangle=\sin^2\phi_0[1-\langle(\delta\phi)^2\rangle]+\cos^2\phi_0\langle(\delta\phi)^2\rangle. \quad (4.13)$$

Keeping terms up to second order, we have

$$(\Delta\cos\phi)^2\equiv\langle\cos^2\phi\rangle-\langle\cos\phi\rangle^2$$
$$=\sin^2\phi_0\langle(\delta\phi)^2\rangle,$$
$$(\Delta\sin\phi)^2\equiv\langle\sin^2\phi\rangle-\langle\sin\phi\rangle^2$$
$$=\cos^2\phi_0\langle(\delta\phi)^2\rangle. \quad (4.14)$$

Since $\cos^2\phi_0$ is $\langle\cos^2\phi\rangle$ to lowest order, (4.14) reduces either of the relations (4.10) to (4.5) with the natural identification

$$\langle(\delta\phi)^2\rangle=(\Delta\phi)^2. \quad (4.15)$$

To summarize, the use of an angle variable ϕ to locate a geometrical point in the xy plane leads to problems with the Hermiticity of the generator of rotations (L_z) about the z axis, if the traditional commutator (4.3) is to be maintained. It seems preferable to use a pair of continuous periodic variables, the simplest being $\cos\phi$ and $\sin\phi$, and all possible products of these quantities. Then L_z is Hermitian, and the pair of uncertainty relations (4.10) gives a mathematically and physically correct description of the situation.

The eigenvalue spectrum of L_z is discrete and symmetrical, running from $-l$ to $+l$ in integral steps. This is a consequence of the boundedness of L^2. For a truly two-dimensional problem (such as a bead on a circular wire), the spectrum of L_z is discrete but runs over the integers from $-\infty$ to $+\infty$. (Here we assume the absence of double valued representations.) When we consider the phase of a harmonic oscillator, the spectrum of the "conjugate" variable (number) is considerably different, although it too is discrete.

B. Three-Dimensional Oscillator in Coherent States

Since the coherent states are of such important physical significance and are the minimum uncertainty states for position-momentum, it is of obvious interest to find out if they are small uncertainty states for angular momentum angle. This point was previously investigated by one of the authors.[65]

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[63] K. Kraus, Z. Physik **188**, 374 (1965); **201**, 134 (1967).
[64] W. H. Louisell, Phys. Letters **7**, 60 (1963).
[65] M. M. Nieto, Phys. Rev. Letters **18**, 182 (1967).

First, let $|\alpha\rangle$ and $|\beta\rangle$ be coherent states quantized along the x and y axis, respectively. Consider the anisotropic case

$$\omega_x=(k_x/m)^{1/2},$$
$$\omega_y=(k_y/m)^{1/2}, \tag{4.16}$$

where the k_i are the force constants, the ω_i the angular velocities, and m the mass. The root-mean-square positions are now given by (we set $\hbar=1$ here)

$$x_0=(2m\omega_x)^{-1/2},$$
$$y_0=(2m\omega_y)^{-1/2}. \tag{4.17}$$

In the Heisenberg representation we now have

$$\begin{aligned} x(t)&=x_0(a(t)+a^\dagger(t)),\\ p_x&=-im\omega_x x_0(a(t)-a^\dagger(t)),\\ y(t)&=y_0(b(t)+b^\dagger(t)),\\ p_y&=-im\omega_y y_0(b(t)-b^\dagger(t)). \end{aligned} \tag{4.18}$$

Using L_z from (4.2), we obtain

$$\begin{aligned} \langle L_z\rangle&=\langle\alpha,\beta\,|\,(xp_y-yp_x)\,|\,\alpha,\beta\rangle\\ &=4mx_0y_0[\omega_y\,\mathrm{Re}\,\alpha\,\mathrm{Im}\,\beta-\omega_x\,\mathrm{Re}\,\beta\,\mathrm{Im}\,\alpha], \end{aligned} \tag{4.19}$$

where $\langle\alpha,\beta\,|=|\,\alpha,\beta\rangle^\dagger$. Also we have

$$\begin{aligned} \langle L_z^2\rangle=m^2x_0^2y_0^2\{&\omega_y^2[4(\mathrm{Re}\,\alpha)^2+1][4(\mathrm{Im}\,\beta)^2+1]\\ &-\omega_x\omega_y[8\,\mathrm{Re}\,\alpha\,\mathrm{Im}\,\alpha\,\mathrm{Re}\,\beta\,\mathrm{Im}\,\beta+2]\\ &+\omega_x^2[4(\mathrm{Im}\,\alpha)^2+1][4(\mathrm{Re}\,\beta)^2+1]\}, \end{aligned} \tag{4.20}$$

so that

$$\begin{aligned} (\Delta L_z)^2=m^2x_0^2y_0^2\{&\omega_y^2[1+4(\mathrm{Re}\,\alpha)^2+4(\mathrm{Im}\,\beta)^2]\\ &-2\omega_x\omega_y+\omega_x^2[1+4(\mathrm{Im}\,\alpha)^2+4(\mathrm{Re}\,\beta)^2]\}. \end{aligned} \tag{4.21}$$

For the isotropic case, (4.21) reduces to

$$(\Delta L_z)^2=|\,\alpha\,|^2+|\,\beta\,|^2=N_\alpha+N_\beta=N. \tag{4.22}$$

$\langle\sin\phi\rangle$ is given by $\langle y/(x^2+y^2)^{1/2}\rangle$, or

$$\langle\sin\phi\rangle=\langle\alpha,\beta\,|\,y_0(b+b^\dagger)/[\{x_0(a+a^\dagger)\}^2+\{y_0(b+b^\dagger)\}^2]^{1/2}\,|\,\alpha,\beta\rangle. \tag{4.23}$$

Equation (4.23) shows one of the appealing features of the $\sin\phi$ and $\cos\phi$ operators, that they have a meaning in a second quantized formalism. By making use of the properties of the coherent states [cf. Eqs. (3.24)–(3.25)]

$$\begin{aligned} |\,\alpha\rangle&=A(\alpha)\,|\,0\rangle,\\ A^\dagger(\alpha)A(\alpha)&=1,\\ [a+a^\dagger,A(\alpha)]&=(\alpha+\alpha^*)A(\alpha), \end{aligned} \tag{4.24}$$

(4.23) becomes

$$\langle\sin\phi\rangle=\langle 0,0\,|\,y_0(b+b^\dagger+2\,\mathrm{Re}\,\beta)/[\{x_0(a+a^\dagger+2(\mathrm{Re}\,\alpha))\}^2+\{y_0(b+b^\dagger+2\,\mathrm{Re}\,\beta)\}]^{1/2}\,|\,0,0\rangle, \tag{4.25}$$

which is a ground-state expectation value.

If we now transform to the Schrödinger wave picture and use the oscillator ground-state wave functions,[66] we find the result

$$\langle\alpha,\beta\,|\,\sin\phi\,|\,\alpha,\beta\rangle=\pi^{-1}\int_{-\infty}^{\infty}dx\,\exp(-x^2)\int_{-\infty}^{\infty}dy\,\exp(-y^2)\left\{\frac{(y+\mathfrak{B})W}{[(x+\mathfrak{A})^2+W^2(y+\mathfrak{B})^2]^{1/2}}\right\}, \tag{4.26}$$

$$=\frac{1}{\pi W}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,\exp\left[\frac{-(x-\mathfrak{A})^2-(y-\mathfrak{B})^2}{W^2}\right]y(x^2+y^2)^{-1/2}, \tag{4.27}$$

$$W=\omega_x/\omega_y,\qquad \mathfrak{A}=2^{1/2}\,\mathrm{Re}\,\alpha,\qquad \mathfrak{B}=2^{1/2}\,\mathrm{Re}\,\beta. \tag{4.28}$$

Equations (4.24)–(4.27) exhibit the property of the operator $A(\alpha)$ of translating the position of a Gaussian (see Sec. 2). $\sin^2\phi$, $\cos\phi$, and $\cos^2\phi$ are of the same form as (4.27), but with the $\sin\phi$ operator in the integrand [i.e., $y/(x^2+y^2)^{1/2}$] replaced by the other trigonometric operators.

The number W of (4.28) is a measure of the anisotropy of the oscillator. The integrals of the type of (4.27) are easier to do for the isotropic case ($W=1$) and, for our purposes, no important new information is lost by restricting ourselves to this case. For the remainder of this section we will keep $W=1$. (The interested reader is invited to evaluate the integrals for the general case.)

The expressions of the type of (4.27) yield the useful knowledge

$$\begin{aligned} \langle\sin\phi(\mathfrak{A},\mathfrak{B})\rangle&=\langle\cos\phi(\mathfrak{B},\mathfrak{A})\rangle,\\ \langle\sin^2\phi(\mathfrak{A},\mathfrak{B})\rangle&=\langle\cos^2\phi(\mathfrak{B},\mathfrak{A})\rangle. \end{aligned} \tag{4.29}$$

This means that the uncertainty relations (4.10) are the same with $\mathfrak{A}\leftrightarrow\mathfrak{B}$, so only the first need be studied.

Since the trigonometric operators involve only the real parts of α and β, Eqs. (4.10) and (4.22) tell us that the lowest uncertainty product will occur for real α and β. For real α and β, $\langle L_z\rangle=0$ and the "orbit" is

[66] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Co., New York, 1955), Chap. IV.

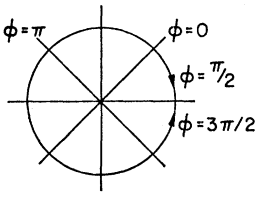

FIG. 3. Four "classical orbits" for $y=y_0\cos\omega t$, $x=x_0\cos(\omega t-\phi)$, with $x_0=y_0$ and various values of ϕ. General ellipses are involved for $x_0\neq y_0$. $\phi=0$ is the case for α and β real. We have included the phase in x. No new orbits are found by allowing β to be complex.

a straight line through the origin, but $(\Delta L_z)^2\neq 0$. If imaginary parts appear in α and β, $\langle L_z\rangle\neq 0$ and the orbits become various ellipses (see Fig. 3).

Since we are trying to construct the smallest uncertainty products, we will keep α and β real from this point on. With α and β real we can define ϵ $(0\leq\epsilon\leq 1)$ such that

$$N=N_x+N_y=\tfrac{1}{2}\mathfrak{A}^2+\tfrac{1}{2}\mathfrak{B}^2$$
$$=\epsilon N+(1-\epsilon)N,$$
$$\epsilon=\mathfrak{A}^2/(\mathfrak{A}^2+\mathfrak{B}^2). \tag{4.30}$$

Using the variables N and ϵ, Eqs. (4.29) are now of the form

$$\langle\sin\phi(N,\epsilon)\rangle=\langle\cos\phi(N,1-\epsilon)\rangle. \tag{4.31}$$

To evaluate (4.27) we change to polar coordinates to obtain

$$\langle\sin\phi\rangle=\frac{e^{-R^2}}{\pi}\int_0^\infty rdr\int_0^{2\pi}d\phi\sin\phi\exp\left[-r^2+2r(\mathfrak{A}\cos\phi+\mathfrak{B}\sin\phi)\right]$$
$$=\frac{e^{-R^2}}{\pi}\int_0^\infty rdr\int_0^{2\pi}d\phi\sin\phi\exp\left[-r^2+2rR\cos(\phi-\delta)\right],$$
$$R^2=\mathfrak{A}^2+\mathfrak{B}^2,$$
$$\cos\delta=\mathfrak{A}/(\mathfrak{A}^2+\mathfrak{B}^2). \tag{4.32}$$

Using the trick of calculating a real and imaginary part gives

$$\langle\exp(i\phi)\rangle=\frac{\exp(i\delta)\exp(-R^2)}{\pi}\int_0^\infty rdr\int_0^{2\pi}d\theta\cos\theta\exp\left[-r^2+2rR\cos\theta\right]. \tag{4.33}$$

The r integration can be done explicitly,[67] giving

$$\langle\exp(i\phi)\rangle=\tfrac{1}{2}[\exp(i\delta)\exp(-R^2)]\int_0^{2\pi}d\theta[\tfrac{1}{2}+\tfrac{1}{2}\pi^{1/2}R\cos\theta\exp(R^2\cos^2\theta)][1+\Phi(R\cos\theta)], \tag{4.34}$$

$$\Phi(x)=\operatorname{erf}(x)=2\pi^{-1/2}\int_0^x\exp(-t^2)dt. \tag{4.34}$$

The first term equals zero because $\cos\theta$ is odd with respect to the interval $[0, 2\pi]$. The error function term also disappears because $\cos^2\theta$ is even while $\Phi(R\cos\theta)$ is odd. This leaves the middle term. After a little algebra, and noticing that the real and imaginary parts of the expectation value are from $\exp(i\delta)$, one finally obtains

$$\left\langle\begin{Bmatrix}\sin\phi\\ \cos\phi\end{Bmatrix}\right\rangle=\begin{Bmatrix}\mathfrak{B}\\ \mathfrak{A}\end{Bmatrix}\frac{2}{\pi^{1/2}}\int_0^{\pi/2}d\theta\cos^2\theta\exp(-R^2\sin^2\theta). \tag{4.35}$$

For $\langle\sin^2\phi\rangle$ and $\langle\cos^2\phi\rangle$, one starts from the expression like the second line of (4.32). Integrating with respect to r gives

$$\left\langle\begin{Bmatrix}\sin^2\phi\\ \cos^2\phi\end{Bmatrix}\right\rangle=\frac{\exp(-R^2)}{\pi}\int_0^{2\pi}d\theta[\tfrac{1}{2}+\tfrac{1}{2}\pi^{1/2}R\cos\theta\exp(R^2\cos^2\theta)][1+\Phi(R\cos\theta)]\begin{Bmatrix}\sin^2(\theta+\delta)\\ \cos^2(\theta+\delta)\end{Bmatrix} \tag{4.36}$$

$$=\tfrac{1}{2}\exp(-R^2)+\frac{R\exp(-R^2)}{2\pi^{1/2}}\int_0^{2\pi}d\theta\cos\theta\exp(R^2\cos^2\theta)\Phi(R\cos\theta)$$
$$\times\left[\begin{Bmatrix}\sin^2\delta\\ \cos^2\delta\end{Bmatrix}\pm\sin^2\theta(2\cos^2\delta-1)\right]. \tag{4.37}$$

In (4.36) the cross terms drop out and we expand the $(\theta+\delta)$ terms. The first integral in (4.37) is done by adding both lines of (4.37) and realizing that the answer is identically equal to one. Combining and changing the form of

[67] W. Gröbner and N. Hofreiter, *Integraltafel* (Springer-Verlag, Vienna, 1961), line (314.5b).

the remaining integral gives

$$\left\langle \begin{Bmatrix} \sin^2\phi \\ \cos^2\phi \end{Bmatrix} \right\rangle = \tfrac{1}{2}\exp(-R^2) + \begin{Bmatrix} \sin^2\delta \\ \cos^2\delta \end{Bmatrix} [1-\exp(-R^2)] \pm \frac{2R}{\pi^{1/2}}(2\cos^2\delta - 1) \times \int_0^{\pi/2} d\theta \cos\theta \sin^2\theta \exp(-R^2\sin^2\theta)\Phi(R\cos\theta). \quad (4.38)$$

The last integral can be evaluated by observing that

$$I = R\int_0^{\pi/2} d\theta \cos\theta \sin^2\theta \exp(-R^2\sin^2\theta)\Phi(R\cos\theta)$$

$$= -\frac{1}{2}\frac{d}{dR}\int_0^{\pi/2} d\theta \cos\theta \exp(-R^2\sin^2\theta)\Phi(R\cos\theta) + \pi^{-1/2}\int_0^{\pi/2} d\theta \cos^2\theta \exp(-R^2). \quad (4.39)$$

The first integral on the right is the one we just calculated, and is $\pi^{1/2}[1-\exp(-R^2)]/2R$. It then easily follows that

$$I = \tfrac{1}{4}\pi^{1/2}(\{[1-\exp(-R^2)]/R^2\} - \exp(-R^2)). \quad (4.40)$$

We finally obtain

$$\left\langle \begin{Bmatrix} \sin^2\phi \\ \cos^2\phi \end{Bmatrix} \right\rangle = \tfrac{1}{2}\exp(-R^2) + \begin{Bmatrix} \sin^2\delta \\ \cos^2\delta \end{Bmatrix} [1-\exp(-R^2)] + \begin{Bmatrix} + \\ - \end{Bmatrix} \tfrac{1}{2}(\cos^2\delta - \sin^2\delta)\left(\frac{1-\exp(-R^2)}{R^2} - \exp(-R^2)\right). \quad (4.41)$$

Simple calculations now give the limits:

$$\left\langle \begin{Bmatrix} \sin\phi \\ \cos\phi \end{Bmatrix} \right\rangle \sim \begin{Bmatrix} \sin\delta \\ \cos\delta \end{Bmatrix}\left(1 - \frac{1}{4R^2} + \cdots\right), \qquad N\to\infty,$$

$$\left\langle \begin{Bmatrix} \sin^2\phi \\ \cos^2\phi \end{Bmatrix} \right\rangle \sim \begin{Bmatrix} \sin^2\delta \\ \cos^2\delta \end{Bmatrix} + \begin{Bmatrix} + \\ - \end{Bmatrix} \frac{1}{2}\frac{\mathfrak{A}^2 - \mathfrak{B}^2}{R^4} + \cdots, \qquad N\to\infty,$$

$$\left\langle \begin{Bmatrix} \sin\phi \\ \cos\phi \end{Bmatrix} \right\rangle \sim \tfrac{1}{2}\pi^{1/2} \begin{Bmatrix} \mathfrak{B} \\ \mathfrak{A} \end{Bmatrix} + \cdots, \qquad N\to 0,$$

$$\left\langle \begin{Bmatrix} \sin^2\phi \\ \cos^2\phi \end{Bmatrix} \right\rangle \sim \tfrac{1}{2} + \begin{Bmatrix} + \\ - \end{Bmatrix} \tfrac{1}{4}(\mathfrak{B}^2 - \mathfrak{A}^2) + \cdots. \qquad N\to 0. \quad (4.42)$$

From (4.10) we define the uncertainty product $S(N, \epsilon)$ for the coherent states by

$$S(N, \epsilon) \equiv (\Delta L_z)^2(\Delta \sin\phi)^2/\langle\cos\phi\rangle^2 \geq \tfrac{1}{4}. \quad (4.43)$$

$S(N, \epsilon)$ was numerically calculated and is plotted in Fig. 4 as a function of N for various values of ϵ. The results agree with the limits for large and small N, which from (4.42) are

$$\lim_{N\to 0} S(N, \epsilon) = [\pi(1-\epsilon)]^{-1} > \pi^{-1} > \tfrac{1}{4},$$

$$\lim_{N\to\infty} S(N, \epsilon) = \tfrac{1}{4}. \quad (4.44)$$

The symmetric uncertainty relation U defined by (4.11) is independent of ϵ for the coherent states. [An explicit use of (4.35) and (4.41) in (4.11) shows this.] In fact, it is given by[68]

$$U(N) = S(N, \tfrac{1}{2}). \quad (4.45)$$

This follows from $\sin^2\delta(\epsilon=\frac{1}{2}) = \cos^2(\epsilon=\frac{1}{2}) = \frac{1}{2}$.

Figure 4 shows that the coherent states do indeed give a low uncertainty product for all N, which is, for practical purposes, the minimum for large N. In a real system we would expect values of ϵ near $\frac{1}{2}$, rather than 0 or 1, on physical and statistical grounds.

Note that the smallest uncertainty product is obtained when $\epsilon=0$, i.e., when $\langle\sin\phi\rangle=0$.

It is possible to understand intuitively why the uncertainty products get better as $N\to\infty$. We know from the last section that minimum uncertainty states

[68] We are indebted to W. Bardeen for aid in proving this at a stage in the work when the result was not so easy to see.

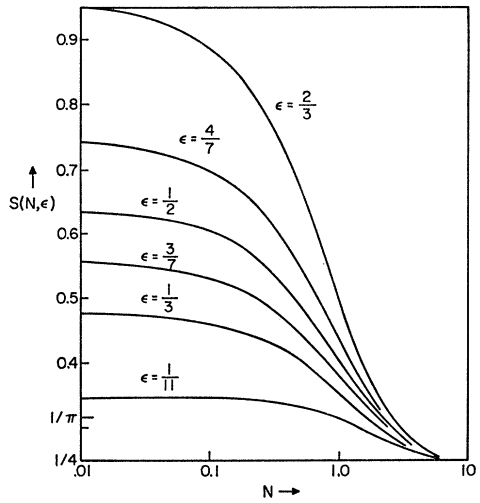

FIG. 4. The uncertainty product $S(N,\epsilon)=(\Delta L_z)^2\times(\Delta\sin\phi)^2/\langle\cos\phi\rangle^2$ is shown as a function of N for various values of the parameter ϵ defined in Eq. (4.30). $S(N,\frac{1}{2})$ is also the uncertainty product $U(N)$ defined in Eq. (4.43). All expectation values are for the two-dimensional coherent states discussed in the text.

for variables whose spectra are the set of all real numbers behave like Gaussians, in both the variable and conjugate momentum space. But here we have a restricted domain. However, as $\langle L_z\rangle\to\infty$, we can "approximate" L_z by a Gaussian, with $\Delta L_z\sim\langle L_z\rangle^{1/2}$. The $\Delta\phi$ is a narrow Gaussian. As $\langle L_z\rangle\to 0$, we can no longer "approximate" $\Delta\phi$ by a Gaussian, for the tails would be cut off. Thus, on these intuitive grounds we expect the uncertainty products to be better as $N\to\infty$. It will become clear in the next subsection why, for $N\sim 0$, the best uncertainty products are for $\epsilon\sim 0$, i.e., $\langle\sin\phi\rangle\sim 0$.

C. Minimum Uncertainty State

Because (4.10c) is the "complicated" type of uncertainty relation discussed in Sec. 2.C, we can use the direct method to find the minimum uncertainty state. From (4.10c), (2.15), and (2.26) we have

$$(L_z+i\gamma\sin\phi)\psi=\lambda\psi,$$
$$[(\partial/\partial\phi)-\gamma\sin\phi]\psi=i\lambda\psi,$$
$$\gamma=-\langle\cos\phi\rangle/2(\Delta\sin\phi)^2. \quad (4.46)$$

The solution for ψ is[69]

$$\psi_{\min}=G\exp(-\gamma\cos\phi+i\lambda\phi). \quad (4.47)$$

The normalization constant is given by

$$G^{-2}=\int_0^{2\pi}\exp(-2\gamma\cos\phi)\,d\phi$$
$$=2\pi I_0(-2\gamma), \quad (4.48)$$

where I_n is the modified Bessel function.[70]

[69] In keeping with the spirit of using sin and cos operators, one can consider $\exp(i\lambda\phi)$ to be $(\cos\phi+i\sin\phi)^\lambda$.

[70] C. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1966), p. 79.

Now (4.46) shows that

$$\lambda=\langle L_z\rangle+i\gamma\langle\sin\phi\rangle, \quad (4.49)$$

where γ is a real number. But we see in (4.47) that if λ is complex, $\psi_{\min}$ is not single valued, for it will have a factor of $\exp(-\phi\,\mathrm{Im}\,\lambda)$. Therefore λ must be real and equal to $\langle L_z\rangle$. Thus, we have the interesting result that the *minimum uncertainty state has* $\langle\sin\phi\rangle=0$.

This illuminates the uncertainty properties of the coherent states, for we recall that the best uncertainty coherent states are for $\epsilon=0$, i.e., $\langle\sin\phi\rangle=0$. Also for $\epsilon=1$, i.e., $\langle\cos\phi\rangle=0$, the coherent state uncertainty product goes to ∞ as $N\to 0$, although even these states are a minimum for large N. This is the analogy of the direct method being ambiguous if $\langle[x,\,y]\rangle=0$. The point to be made is that *the coherent states, which are the minimum uncertainty states for position-momentum, behave very similarly to the critical states for angular momentum angle, including when* $\langle\sin\phi\rangle=0$, *which yields the minimum uncertainty state.* This certainly is a remarkable property, which will also hold in the number-phase case discussed in the next section.

We conclude this section by giving the expectation values for the operators we study. These are easily calculated from the known properties of the modified Bessel functions[70,71]:

$$\langle\sin\phi\rangle_{\min}=0,$$
$$\langle\sin^2\phi\rangle_{\min}=-\langle\cos\phi\rangle/2\gamma=-I_1(-2\gamma)/2\gamma I_0(-2\gamma),$$
$$\langle\cos^n\phi\rangle_{\min}=I_n(-2\gamma)/I_0(-2\gamma), \quad (4.50)$$
$$\langle L_z\rangle_{\min}=\lambda,$$
$$\langle L_z^2\rangle_{\min}=-\tfrac{1}{2}\gamma[I_1(-2\gamma)/I_0(-2\gamma)]+\lambda^2. \quad (4.51)$$

Trivially this implies the equality

$$[(\Delta L_z)^2(\Delta\sin\phi)^2/\langle\cos\phi\rangle^2]\,|_{\min}=\tfrac{1}{4}. \quad (4.52)$$

We finally note that, from (4.45) and our previous discussion in this section, there is no state that allows the symmetric uncertainty relation U in (4.11) to reach its minimum value. This is because we would be doing the same as minimizing U with states that have $\langle\sin\phi\rangle=\langle\cos\phi\rangle\neq 0$, which we found was impossible. This is an example of the type of uncertainty relation discussed at the end of Sec. 2 for which there is no minimum state. We shall come across another example later on.

5. NUMBER AND PHASE OPERATORS FOR THE HARMONIC OSCILLATOR

The quantum-mechanical harmonic oscillator is generally described in terms of the position-momentum variables $(x,\,p)$ or the creation–annihilation variables $(a,\,a^\dagger)$. In terms of the latter the displacement operator is (Heisenberg picture)

$$x(t)=x_0[a\exp(-i\omega t)+a^\dagger\exp(i\omega t)]. \quad (5.1)$$

[71] Reference 67, lines (337.9 a–b).

Here $x_0=(1/2m\omega)^{1/2}$ is the rms zero-point fluctuation, a convenient unit with which to measure the displacement.

In classical mechanics one often uses yet another pair of variables, amplitude and phase:

$$x(t)=2A\cos(\phi-\omega t). \tag{5.2}$$

The relation described below, of A and ϕ to the classically conjugate action-angle variables, leads one to search for quantum-mechanical operator analogs to amplitude and phase variables.

One possible method of approach is suggested by comparing (5.1) with (5.2), the latter written as

$$x(t)=A[\exp(i\phi)\exp(-i\omega t)+\exp(-i\phi)\exp(i\omega t)]. \tag{5.3}$$

Thus, in the classical limit, the operator a is proportional to $A\exp(i\phi)$. The appropriate replacement of the destruction operator by a number is accomplished by taking the expectation value of (5.1) in the coherent state $|\alpha\rangle$, where the phase of α is chosen to be ϕ:

$$\langle\alpha|x(t)|\alpha\rangle=2x_0J^{1/2}\cos(\phi-\omega t), \tag{5.4}$$

$$\alpha=J^{1/2}\exp(i\phi), \qquad J=\bar{N}. \tag{5.5}$$

Thus it might be supposed that there is an operator decomposition of the destruction operator a corresponding to (5.5). Let us suppose that we can factor a into a product of a Hermitian function of the number operator $f(N)$ and a Unitary operator U. The latter then defines (mod 2π) the Hermitian phase operator $\phi_{\rm op}$ by $U=\exp(i\phi_{\rm op})$:

$$a=\exp(i\phi_{\rm op})f(N), \qquad \phi^\dagger{}_{\rm op}=\phi_{\rm op}, \qquad f^\dagger=f. \tag{5.6}$$

The requirement $a^\dagger a=N$, along with the Hermiticity assumptions, yields

$$N=f^\dagger\exp(-i\phi_{\rm op})\exp(i\phi_{\rm op})f=f^2. \tag{5.7}$$

Choosing the positive root ($\phi_{\rm op}$ could absorb any minus sign) gives

$$a=\exp(i\phi_{\rm op})N^{1/2} \quad \text{(wrong)}, \tag{5.8}$$

exactly as one might have guessed from Eq. (5.5). [We remark that the same result (5.8) follows if one inverts the order of f and $\exp(i\phi_{\rm op})$ in (5.6). In that case, $f^2=\exp(i\phi)N\exp(-i\phi)$, giving

$$f=\exp(i\phi)(N)^{1/2}\exp(-i\phi),$$

and $a=f\exp(i\phi)$ is as given in Eq. (5.8).]

The change of variables indicated in (5.8) was introduced even in Dirac's original paper[17] on the quantization of the electromagnetic field. Unfortunately, the result of this formal calculation is incorrect, as discussed in detail below. However, let us momentarily assume its truth in order to review the usual discussion[72] of the number–phase uncertainty relation.

[72] W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, London, 1954), 3rd. ed., p. 65.

In place of the quantization rule

$$aa^\dagger-a^\dagger a=1, \tag{5.9}$$

(5.8) leads to

$$\exp(i\phi_{\rm op})N-N\exp(i\phi_{\rm op})=\exp(i\phi_{\rm op}). \tag{5.10}$$

This equation is solved if N and $\phi_{\rm op}$ obey

$$[N,\phi_{\rm op}]=i, \tag{5.11}$$

since from the latter we find by induction

$$[N,\phi_{\rm op}{}^n]=in\phi_{\rm op}{}^{n-1}. \tag{5.12}$$

Multiplication of (5.12) by $i^n/n!$ and summation from 0 to ∞ recovers Eq. (5.10).

The erroneous assumption that N and $\phi_{\rm op}$ are well-defined Hermitian operators then gives what we call the "traditional" number–phase uncertainty relation:

$$\Delta N\Delta\phi\geq\tfrac{1}{2} \quad \text{(wrong)}. \tag{5.13}$$

However, one could discuss (5.13) in a semiclassical spirit, as has been done by several authors.[73–75]

Equation (5.13) is wrong for two reasons. First of all, there is the periodicity problem already discussed in Sec. 4. This could be repaired by using sine and cosine operators,[18,76]

$$\cos\phi_{\rm op}=\tfrac{1}{2}[\exp(i\phi_{\rm op})+\exp(-i\phi_{\rm op})],$$

$$\sin\phi_{\rm op}=\tfrac{1}{2}i[\exp(-i\phi_{\rm op})-\exp(i\phi_{\rm op})], \tag{5.14}$$

except for the fact that such an operator structure leads to contradictions. The real difficulty lies in the unwarranted assumption of the existence of a unitary operator $U=\exp(i\phi_{\rm op})$.

Susskind and Glogower[18] first exposed the contradictions inherent in Dirac's assumption (5.8). Let us rewrite (5.8) in the form

$$a=UN^{1/2}; \qquad U^\dagger U=UU^\dagger=1, \tag{5.15}$$

and test the unitarity assumptions. Since U is equivalent to an infinite-dimensional matrix, we have to check both unitarity statements in (5.15). Applying (5.15) to a number state $|n\rangle$, we learn that U is a lowering operator for $n=1, 2, \cdots$:

$$a|n\rangle=UN^{1/2}|n\rangle=n^{1/2}U|N\rangle$$

$$=n^{1/2}|n-1\rangle, \qquad n=1, 2, \cdots. \tag{5.16}$$

Thus we find

$$U|n\rangle=|n-1\rangle, \qquad n=1, 2, \cdots. \tag{5.17}$$

For $n=0$, we learn nothing from (5.16). From the completeness of number states we have for the ground state

$$U|0\rangle=\sum_{n=0}d_n|n\rangle. \tag{5.18}$$

[73] R. Serber and C. H. Townes, *Quantum Electronics* (Columbia University Press, New York, 1960), p. 249 ff.

[74] H. Brunet, Phys. Letters **10**, 172 (1964).

[75] J. Harms and J. Loringny, Phys. Letters **10**, 173 (1964).

[76] The use of cosine and sine operators was suggested on physical grounds by W. Louisell, Ref. 64.

To determine the effect of $U^\dagger$ on the number basis we assume completeness and use (5.17) and (5.18):

$$U^\dagger \mid n\rangle = \sum_{m=0} \langle m \mid U^\dagger \mid n\rangle \mid m\rangle$$
$$= d_n^* \mid 0\rangle + \mid n+1\rangle. \quad (5.19)$$

Now we see if $U^\dagger U$ is equivalent to the identity operator. Applying $U^\dagger$ to (5.17) gives (for $n>1$), on using (5.19),

$$U^\dagger U \mid n\rangle = U^\dagger \mid n-1\rangle$$
$$= d^*_{n-1} \mid 0\rangle + \mid n\rangle. \quad (5.20)$$

Equation (5.20) shows that $d_{n-1}=0$ for $n=1, 2, \cdots$, if $U^\dagger U=1$; i.e., $d_n=0$ for *all* n. Thus, from (5.18) we learn that U annihilates the ground state:

$$U \mid 0\rangle = 0. \quad (5.21)$$

Therefore, in the ground state

$$\langle 0 \mid U^\dagger U \mid 0\rangle = 0,$$

which contradicts the unitarity.[77] Note that $UU^\dagger$ really is the identity, since $U^\dagger \mid n\rangle = \mid n+1\rangle$ for all n, from Eq. (5.19). Thus,

$$\langle m \mid UU^\dagger \mid n\rangle = \delta_{mn}. \quad (5.22)$$

Since U is not unitary, we cannot define a Hermitian phase operator ϕ_{op} by $U=\exp(i\phi_{\text{op}})$. Thus, a proper quantum description of the phase variable must be sought elsewhere.

In order to discover the appropriate variables, we consider the Poisson bracket formulation of the oscillator problem. This method of guessing the quantum-mechanical commutators has been championed by Dirac. Application to the present problem was suggested by Lerner.[78]

The action (J) and angle (ϕ) variables are introduced by the transformation

$$p = (2m\omega J)^{1/2} \sin\phi(t),$$
$$x = (2J/m\omega)^{1/2} \cos\phi(t). \quad (5.23)$$

The Hamiltonian is independent of ϕ, since

$$H = (p^2/2m) + \tfrac{1}{2}m\omega^2 x^2,$$
$$H = \omega J. \quad (5.24)$$

Thus J is constant, whereas $\sin\phi$ and $\cos\phi$ depend on time:

$$d\cos\phi(t)/dt = \{\cos\phi(t), H\} = \omega\sin\phi(t),$$
$$d\sin\phi(t)/dt = \{\sin\phi(t), H\} = -\omega\cos\phi(t). \quad (5.25)$$

These equations indicate that the phase angle $\phi(t)$ is

$$\phi(t) = \phi - \omega t. \quad (5.26)$$

J is related to $\bar{N}$ in the classical limit by $J=\bar{N}/\hbar$.

[77] A simpler method of showing this is by combining (5.9) and (5.15). This gives $UNU^\dagger = N+1$, which violates unitarity. We thank Professor W. McGlinn for pointing this out.

[78] E. Lerner, University of South Carolina Report, 1966 (unpublished).

Our experience with periodicity requirements leads us to seek operator analogs of $\cos\phi(t)$ and $\sin\phi(t)$ rather than $\phi(t)$ itself. The classical results (5.25) will be successfully reproduced if we can find operators C and S such that the commutator replacements of the brackets in (5.25) have the same algebraic structure as the latter. According to the general recipe,

$$\{A, B\} \leftrightarrow (1/i\hbar)[A, B]. \quad (5.27)$$

We thus pose the problem of finding C and S such that

$$\dot{C} = (1/i\hbar)[C, H] = \omega S,$$
$$\dot{S} = (1/i\hbar)[S, H] = -\omega C \quad (5.28)$$

hold as operator equations.[79] Writing the quantum-mechanical Hamiltonian as $H=N\hbar\omega$, we simplify (5.28) to

$$[C, N] = iS,$$
$$[S, N] = -iC. \quad (5.29)$$

It will be observed that both C and S obey the normal mode equations

$$\ddot{C} + \omega^2 C = 0,$$
$$\ddot{S} + \omega^2 S = 0. \quad (5.30)$$

In order to solve (5.29), we introduce "exponential" operators $E_\pm$ by

$$E_\pm \equiv C \mp iS. \quad (5.31)$$

Since C and S are presumed to be Hermitian, we have

$$E_\pm = (E_\mp)^\dagger. \quad (5.32)$$

In the classical limit $E_\pm$ corresponds to $\exp(\mp i\phi)$ where ϕ is the classical phase angle. The subscripts on E were chosen because, as we now show, the $E_\pm$ are normalized raising and lowering operators

Equations (5.29) and (5.31) imply that

$$[E_\mp, N] = \pm E_\mp. \quad (5.33)$$

From (5.33) we find

$$N(E_\pm \mid n\rangle) = E_\pm(N\pm 1) \mid n\rangle$$
$$= (n\pm 1)(E_\pm \mid n\rangle), \quad (5.34)$$

except that

$$E_- \mid 0\rangle = 0, \quad (5.35)$$

to avoid negative eigenvalues of N. Thus,

$$E_+ \mid n\rangle = \mid n+1\rangle \qquad \text{all } n,$$
$$E_- \mid n\rangle = \mid n-1\rangle \qquad n \geq 1$$
$$= 0 \qquad n=0. \quad (5.36)$$

[79] Note that, although $[\sin\phi, \cos\phi]$ vanishes, the commutator $[S, C]$ does not vanish, although it becomes negligible in the classical limit. Using Jacobi's identity, Eq. (5.29) can be used to show that $[S, C]$ is diagonal in the number basis. See Eq. (5.42) for the actual form of the commutator. Most of the following is taken from Carruthers and Nieto, Ref. 19); the results are implicitly or explicitly contained in the work of Susskind and Glogower (Ref. 18).

The operator E_- is thus essentially the same as U of Eq. (5.15). However, besides being wrong, that definition

$$U=aN^{-1/2} \tag{5.37}$$

is ambiguous, since the eigenvalue spectrum of N includes the point zero. Instead we use the formally equivalent[80] but unambiguous forms

$$E_-=(N+1)^{-1/2}a, \qquad E_+=a^\dagger(N+1)^{-1/2}. \tag{5.38}$$

The operator $(N+1)^{-1/2}$ is well defined, and the requirements (5.36) are clearly met.

The $E_\pm$ operators are not unitary but are "one-sided unitary." This follows on taking matrix elements of $E_\pm E_\mp$ in the number basis:

$$\begin{aligned}(E_-E_+)_{mn}&\equiv\langle m\mid E_-E_+\mid n\rangle\\&=\langle m\mid (E_+)^\dagger E_+\mid n\rangle\\&=\langle m+1\mid n+1\rangle=\delta_{mn}, \qquad \text{all } m,\ n,\\(E_+E_-)_{mn}&=0 \qquad \text{if either } m,\ n=0\\&=\delta_{mn}, \qquad m,\ n\geq 1.\end{aligned} \tag{5.39}$$

Introducing the projection operator $\mathcal{P}_0$ for the ground state, we can summarize (5.39) in operator notation by

$$\begin{aligned}&E_-E_+=1,\\&E_+E_-=1-\mathcal{P}_0, \qquad \mathcal{P}_0\equiv\mid 0\rangle\langle 0\mid.\end{aligned} \tag{5.40}$$

From this calculation it is clear that *the termination of the eigenvalue spectrum of N at zero is responsible for the nonunitarity of E_-, and hence the nonexistence of a conjugate Hermitian phase variable $\phi_{\rm op}=-i\log E_-$.*

Although $E_\pm$ are not unitary, the operators C and S obtained by solving Eq. (5.31) are manifestly Hermitian:

$$\begin{aligned}&C=\tfrac{1}{2}(E_-+E_+)=C^\dagger,\\&S=(1/2i)(E_--E_+)=S^\dagger,\end{aligned} \tag{5.41}$$

by virtue of Eq. (5.32).

We have now succeeded in explicitly constructing Hermitian operators corresponding to the classical phase variable. Before considering the eigenvalue spectrum and eigenfunctions, we conclude with a few other important results, and state legitimate number–phase uncertainty relations.

From Eqs. (5.28) one learns that C and S are essentially the time derivatives of each other. Thus it is not surprising that C and S do not commute. Explicit calculation from (5.41) shows that

$$\begin{aligned}&[C,\ S]=(1/2i)(E_-E_+-E_+E_-)=\mathcal{P}_0/2i,\\&C^2+S^2=\tfrac{1}{2}(E_-E_++E_+E_-)=1-\tfrac{1}{2}\mathcal{P}_0.\end{aligned} \tag{5.42}$$

The noncommutativity of C and S is intimately connected with the nonunitarity of the operators $C\pm iS$. This is true since, for each operator C and S, we can define independent Hermitian operators ϕ_C and ϕ_S by power series[81]:

$$\begin{aligned}\phi_C&\equiv\cos^{-1}C=\tfrac{1}{2}\pi-\sin^{-1}C\\&=\tfrac{1}{2}\pi-\sum_{k=0}\frac{(-1)^k}{2k+1}\binom{-\frac{1}{2}}{k}C^{2k+1},\\\phi_S&\equiv\sin^{-1}S=\sum_{k=0}\frac{(-1)^k}{2k+1}\binom{-\frac{1}{2}}{k}S^{2k+1},\\\binom{a}{b}&=\frac{a(a-1)(a-2)\cdots(a-b+1)}{1\cdot 2\cdots b}.\end{aligned} \tag{5.43}$$

The expansions (5.43) are legitimate, since the operators C and S have bounded spectra (see the following section).

In terms of ϕ_C and ϕ_S we can indeed define unitary operators of the form

$$\begin{aligned}&U_C=\exp(i\phi_C), \qquad U^\dagger_CU_C=U_CU^\dagger_C=1,\\&U_S=\exp(i\phi_S), \qquad U^\dagger_SU_S=U_SU^\dagger_S=1.\end{aligned} \tag{5.44}$$

However, since ϕ_C and ϕ_S do not commute, i.e.,

$$[\phi_C,\ \phi_S]\neq 0, \tag{5.45}$$

this does not contradict our previous results. The destruction operator, written in terms of ϕ_C, ϕ_S, is

$$\begin{aligned}a=(N+1)^{1/2}\{&\tfrac{1}{2}[\exp(i\phi_C)+\exp(i\phi_S)]\\&+\tfrac{1}{2}[\exp(-i\phi_C)-\exp(-i\phi_S)]\}.\end{aligned} \tag{5.46}$$

It is perhaps worth mentioning that if an Hermitian phase operator $\phi_{\rm op}$ had existed, then one could have given a simple interpretation of the energy–time uncertainty relation for the harmonic oscillator. We could have defined time as the advance in phase of the oscillator, with a time operator $t_{\rm op}\equiv\phi_{\rm op}/\omega$. Then $\Delta N\Delta\phi$ is the same as $\Delta E\Delta t/\hbar$. However, there are two operators ϕ_C/ω and ϕ_S/ω, which precludes the above interpretation. Although a number of time operators have been proposed,[25–28] they all suffer from difficulties of the sort encountered in the attempt to define a single phase operator $\phi_{\rm op}$. In addition to the problem that time is a parameter (rather than an operator) in standard nonrelativistic quantum theory, it will be noticed that the supposedly conjugate variable (H) has a bounded spectrum. In our opinion, the discussions thus far on the energy–time uncertainty relation are not definitive,[27,29–31] and the proper interpretation of an energy–time uncertainty relation remains to be given.[82]

We now give the uncertainty relations which follow from the commutation rules (5.29) and (5.42). Avoid-

[80] For any "analytic function" of the number operator $f(N)$ we have the operator equation $af(N)=f(N+1)a$.

[81] The authors are indebted to Dr. K. Eilenberger for this observation. We also note that on the other hand, R. D. Levine, J. Chem. Phys. **44**, 3597 (1966), Sec. III, champions the view that there is a unique unitary exp ($i\phi$) operator.

[82] The above Refs. 25–31 will give the reader a background on the common understanding of this problem up until the present. In addition, Refs. 27, 30, and 31 constitute a heated debate on the nature of the energy–time uncertainty relation.

ing eigenfunctions of the relevant operators, we obtain

$$\Delta N \Delta C \geq \tfrac{1}{2}\langle S\rangle, \tag{5.47}$$

$$\Delta N \Delta S \geq \tfrac{1}{2}\langle C\rangle, \tag{5.48}$$

$$\Delta S \Delta C \geq \tfrac{1}{2}\langle \mathcal{P}_0\rangle. \tag{5.49}$$

We also can put (5.47) and (5.48) in the form

$$(\Delta N)^2(\Delta C)^2/\langle S\rangle^2 \geq \tfrac{1}{4}, \tag{5.50}$$

$$(\Delta N)^2(\Delta S)^2/\langle C\rangle^2 > \tfrac{1}{4}. \tag{5.51}$$

Whenever appropriate, (5.47) or (5.48) reduce to (5.13), just as for the corresponding equations describing L_z and ϕ. Squaring and adding (5.47) and (5.48) gives a relation symmetrical between S and C:

$$\{(\Delta N)^2[(\Delta C)^2+(\Delta S)^2]\}/(\langle C\rangle^2+\langle S\rangle^2) \geq \tfrac{1}{4}. \tag{5.52}$$

According to (5.49), S and C can be accurately measured simultaneously only when the wave function has negligible overlap with the ground state.

6. SPECTRA AND EIGENFUNCTIONS OF THE SINE AND COSINE OPERATORS

To develop a proper interpretation of the Hermitian operators C and S, it is essential that we investigate the nature of the eigenvalue spectrum and the associated eigenfunctions.[83] The eigenvalues give possible results of measurements of C and S, while the overlap of the eigenfunctions with the state vector gives probabilities. Since C and S do not commute, the eigenfunctions of C will not be eigenfunctions of S, and vice versa.

The results are as follows: Both C and S have continuous eigenvalue spectra lying in the real interval $-1 \leq \cos\theta \leq 1$ and $-1 \leq \sin\theta \leq 1$. Here we introduce the parameter θ to label these eigenvalues in the usual way. Since the spectrum is continuous, the eigenfunctions cannot be normalized, although the inner products are expressible as delta functions. The eigenfunctions of S and C separately constitute complete sets.

Consider the eigenvalue equation

$$C\,|\,\lambda\rangle = \lambda\,|\,\lambda\rangle. \tag{6.1}$$

We expand $|\,\lambda\rangle$ in the number states and take into account the simple effect of C on the number state [cf. Eq. (5.41)]:

$$|\,\lambda\rangle = \sum_{n=0}^{\infty} C_n\,|\,n\rangle. \tag{6.2}$$

The C_n obey the recursion relations

$$2\lambda C_0 = C_1, \tag{6.3}$$

$$2\lambda C_{n+1} = C_n + C_{n+2}. \tag{6.4}$$

Relation (6.4) determines C_2 and all higher C_n.

Provided that λ and p are related by

$$2\lambda = p + 1/p, \tag{6.5}$$

the recursion relation (6.4) is solved by

$$C_n = Ap^n + Bp^{-n} \tag{6.6}$$

for arbitrary A, B. Thus (6.4) takes the form

$$A(p^n+p^{n-2})+B(p^{-n}+p^{-n-2}) = 2\lambda(Ap^{n+1}+Bp^{-n-1}). \tag{6.7}$$

In order to prevent the C_n from becoming unbounded as n goes to ∞, it is necessary that $|\,p\,|$ be unity. Writing p as $\exp(i\phi)$, we get

$$p = \exp(i\theta), \qquad \lambda = \cos\theta. \tag{6.8}$$

Any value of θ gives a solution, but all independent solutions are given by letting θ range over a finite domain. The convenient choices are discussed below.

To complete the solution, we have to find A and B in (6.6). C_0 can be chosen real, so that all C_n are real. For convenience choose C_0 to be $\sin\theta$:

$$\begin{aligned} C_0 &= \sin\theta, \\ C_1 &= 2\cos\theta\sin\theta = \sin 2\theta, \\ C_2 &= 2\cos\theta C_1 - C_0 = \sin 3\theta. \end{aligned} \tag{6.9}$$

It is easy to see that the general solution is

$$C_n = \sin(n+1)\theta. \tag{6.10}$$

From the reality of C_n, we see that $B=A^*$ in (6.6). Writing $A = |\,A\,|\exp(i\psi)$ gives

$$C_n = 2\,|\,A\,|\cos(n\theta+\psi). \tag{6.11}$$

Comparing this with the explicit C_2 shows that $2\,|\,A\,| = 1$, $\psi = \theta - \pi/2$, yielding Eq. (6.10) for general C_n. We have found the eigenfunctions of C:

$$C\,|\cos\theta\rangle = \cos\theta\,|\cos\theta\rangle, \tag{6.12}$$

$$|\cos\theta\rangle = K\sum_{n=0}^{\infty}\sin(n+1)\theta\,|\,n\rangle, \tag{6.13}$$

where K is a normalization constant to be chosen later.

From Eq. (6.13) one sees that all independent solutions are contained in the interval $0 \leq \theta \leq \pi$ (region I of Fig. 5). If θ is in region I, then the state $|\cos(-\theta)\rangle$ corresponding to angle $-\theta$ (in region II) is $-|\cos\theta\rangle$. Of course, $|\cos(-\theta)\rangle$ has eigenvalue $\cos(-\theta) = \cos\theta$. Note that, as θ runs through 2π, the state vectors run over $|\cos\theta\rangle$ and $-|\cos\theta\rangle$. This double-valuedness is reminiscent of the spinor representations of the rotation group.

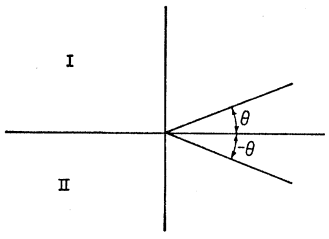

FIG. 5. The independent eigenfunctions $|\cos\theta\rangle$ are contained in the upper-half θ plane, labeled I. The state vector corresponding to $-\theta$ is simply $-|\cos\theta\rangle$.

[83] Many of the results of Secs. 6 and 7 are based on the results of Susskind and Glogower (Ref. 18).

Next we investigate the orthogonality and completeness of the states $|\cos\theta\rangle$. The inner product is

$$\langle\cos\theta \mid \cos\theta'\rangle$$

$$= |K|^2 \sum_{n=0}^{\infty} \sin(n+1)\theta \sin(n+1)\theta'$$

$$= \frac{|K|^2}{2} \sum_{n=0}^{\infty} [\cos(n+1)(\theta-\theta') - \cos(n+1)(\theta+\theta')]. \tag{6.14}$$

In order to deal with distributions of the type encountered here, we quote several useful formulas from Gel'fand *et al.*[84]:

$$\sum_{n=1}^{\infty} \cos nx = -\tfrac{1}{2} + \pi \sum_{-\infty}^{\infty} \delta(x-2\pi n),$$

$$\sum_{n=1}^{\infty} \sin nx = \tfrac{1}{2} \cot \tfrac{1}{2}x,$$

$$\sum_{-\infty}^{\infty} \exp(inx) = 2\pi \sum_{-\infty}^{\infty} \delta(x-2\pi n). \tag{6.15}$$

Hence the right-hand side of (6.14) becomes

$$\tfrac{1}{2}\pi |K|^2 \sum_{-\infty}^{\infty} [\delta(\theta-\theta'-2\pi n) - \delta(\theta+\theta'-2\pi n)]. \tag{6.16}$$

Restricting θ and θ' to the interval $0 \leq \theta, \theta' \leq \pi$, the only possible contributions to (6.16) come from

$$\tfrac{1}{2}\pi |K|^2 [\delta(\theta-\theta') - \delta(\theta+\theta') - \delta(\theta+\theta'-2\pi)]. \tag{6.17}$$

We can now show that the last two terms in (6.17) in fact do not contribute. The second term could contribute only for the possible "end-point singularity" at $\theta=\theta'=0$. To study this set $\theta=0$ and consider θ' near zero. Then the last delta function $[\delta(\theta'-2\pi)]$ vanishes and the first two terms cancel. Similarly, the third term will contribute only for the possible "end-point singularity" at $\theta=\theta'=\pi$. If we set $\theta=\pi$, θ' near π, the middle term in (6.17) vanishes, leaving $\delta(\pi-\theta') - \delta(\theta'-\pi)=0$. Thus, upon choosing $K=(2/\pi)^{1/2}$, we are left with

$$|\cos\theta\rangle = \left(\frac{2}{\pi}\right)^{1/2} \sum_{n=0}^{\infty} \sin(n+1)\theta \mid n\rangle, \tag{6.18}$$

$$\langle\cos\theta \mid \cos\theta'\rangle = \delta(\theta-\theta'). \tag{6.19}$$

There is no singularity at $\theta=0$ or π, as there is no state associated with these points. [Note that all the expansion coefficients in (6.18) vanish if θ is 0 or π.]

The completeness of the $|\cos\theta\rangle$ states is exhibited by the following resolution of the identity:

$$\int_0^{\pi} d\theta \mid \cos\theta\rangle\langle\cos\theta \mid$$

$$= \frac{2}{\pi} \sum_{m,n=0}^{\infty} \mid m\rangle\langle n \mid \int_0^{\pi} d\theta \sin(m+1)\theta \sin(n+1)\theta$$

$$= \sum_{m,n=0}^{\infty} \mid m\rangle\langle n \mid \delta_{mn}$$

$$= 1. \tag{6.20}$$

The construction of the eigenfunctions of S runs along similar lines:

$$S \mid \mu\rangle = \mu \mid \mu\rangle,$$

$$|\mu\rangle = \sum_{n=0}^{\infty} S_n \mid n\rangle. \tag{6.21}$$

The recursion relations for the expansion coefficients are

$$S_1 = 2i\mu S_0,$$

$$2i\mu S_{n+1} = S_{n+2} - S_n. \tag{6.22}$$

The second recursion relation is solved by

$$S_n = Dq^n + Fq^{-n} \tag{6.23}$$

for arbitrary D and F, provided q and μ are related by

$$2i\mu = q - (1/q). \tag{6.24}$$

Again, the only way to keep S_n bounded as $n \to \infty$ is for q to have unit magnitude,

$$q = e^{i\theta}, \qquad \mu = \sin\theta, \tag{6.25}$$

so that the eigenvalue spectrum of S runs from -1 to $+1$.

To discover the expansion coefficients, we conveniently choose S_0 to be $\cos\theta$:

$$S_0 = \cos\theta,$$

$$S_1 = 2i\cos\theta\sin\theta = i\sin 2\theta,$$

$$S_2 = S_0 + 2i\sin\theta S_1 = \cos 3\theta,$$

$$S_3 = i\sin 4\theta. \tag{6.26}$$

The general term is

$$S_n = \cos(n+1)\theta \qquad n=0, 2, 4, \cdots,$$

$$= i\sin(n+1)\theta \qquad n=1, 3, 5, \cdots,$$

$$S_n = \tfrac{1}{2}\{\exp[i(n+1)\theta] - \exp[-i(n+1)(\theta-\pi)]\}. \tag{6.27}$$

The result of our calculation is summarized by

$$S \mid \sin\theta\rangle = \sin\theta \mid \sin\theta\rangle, \tag{6.28}$$

$$|\sin\theta\rangle = \tfrac{1}{2}K' \times \sum_{n=0}^{\infty} \{\exp[i(n+1)\theta] - \exp[-i(n+1)(\theta-\pi)]\} \mid n\rangle, \tag{6.29}$$

where K' is chosen by normalization requirements.

[84] I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic Press Inc., New York, 1964), Vol. 1, p. 331.

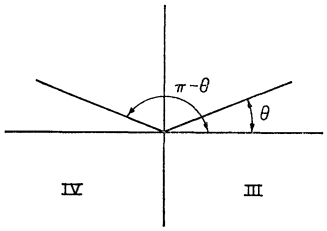

FIG. 6. The independent eigenfunctions $|\sin\theta\rangle$ are contained in the right-hand θ plane, labeled III. The states obtained by reflection in the y axis are the same: $|\sin\theta\rangle = |\sin(\pi-\theta)\rangle$.

From (6.29) we see that

$$|\sin\theta\rangle = |\sin(\pi-\theta)\rangle, \tag{6.30}$$

so that half the θ plane suffices to give all the independent states. We choose $-\pi/2 \leq \theta \leq \pi/2$ (Fig. 6).

Using the identities (6.15) gives

$$\begin{aligned}\langle\sin\theta \mid \sin\theta'\rangle &= \tfrac{1}{2}|K'|^2 \\ &\times \sum_{n=0}^{\infty}[\cos(n+1)(\theta-\theta') - \cos(n+1)(\theta+\theta'-\pi)] \\ &= \tfrac{1}{2}\pi |K'|^2 \\ &\times \sum_{n=0}^{\infty}\{\delta(\theta-\theta'-2\pi n) - \delta[\theta+\theta'-(2n+1)\pi]\}.\end{aligned} \tag{6.31}$$

For θ, θ' confined to the interval $-\pi/2 \leq \theta, \theta' \leq \pi/2$, the possible singularities come from

$$\tfrac{1}{2}\pi |K'|^2[\delta(\theta-\theta') - \delta(\theta+\theta'-\pi) - \delta(\theta+\theta'+\pi)]. \tag{6.32}$$

As before, setting $\theta = \pi/2$, then $-\pi/2$, shows that no contributions come from the special points $\theta = \theta' = \pm\pi/2$. Choosing $K' = (2/\pi)^{1/2}$ gives results similar to Eqs. (6.18)–(6.19):

$$|\sin\theta\rangle = (2\pi)^{-1/2} \times \sum_{n=0}^{\infty}\{\exp[i(n+1)\theta] - \exp[-i(n+1)(\theta-\pi)]\}\,|\,n\rangle, \tag{6.33}$$

$$\langle\sin\theta \mid \sin\theta'\rangle = \delta(\theta-\theta'). \tag{6.34}$$

In analogy with (6.20) we find the completeness relation

$$\int_{-\pi/2}^{\pi/2} d\theta\,|\sin\theta\rangle\langle\sin\theta| = 1. \tag{6.35}$$

The vectors $|\cos\theta\rangle$ and $|\sin\theta\rangle$ are also, respectively, eigenfunctions of the operators ϕ_C and ϕ_S defined in Eqs. (5.43):

$$\begin{aligned}\phi_C\,|\cos\theta\rangle &= \cos^{-1}C\,|\cos\theta\rangle = \theta\,|\cos\theta\rangle, \\ \phi_S\,|\sin\theta\rangle &= \sin^{-1}S\,|\sin\theta\rangle = \theta\,|\sin\theta\rangle.\end{aligned} \tag{6.36}$$

This result is proved by using the power series development of $\cos^{-1}C$ and $\sin^{-1}S$ contained in Eqs. (5.43). Combining (6.36) with the resolutions of the identity [Eqs. (6.20) and (6.35)] gives the following representations for ϕ_C and ϕ_S:

$$\begin{aligned}\phi_C &= \int_0^{\pi} d\theta\cdot\theta\,|\cos\theta\rangle\langle\cos\theta|, \\ \phi_S &= \int_{-\pi/2}^{\pi/2} d\theta\cdot\theta\,|\sin\theta\rangle\langle\sin\theta|.\end{aligned} \tag{6.37}$$

In Sec. 7 we shall give a more complete discussion of the physical properties of these states. Here we wish to introduce the auxiliary states $|\theta\rangle$ defined by

$$|\theta\rangle = \sum_{n=0}^{\infty}\frac{\exp(in\theta)}{(2\pi)^{1/2}}\,|\,n\rangle. \tag{6.38}$$

On superficial inspection it might have seemed that the states $|\theta\rangle$ should be phase eigenstates. To explain why this is wrong, we review the analogous construction for position–momentum variables. The commutator

$$[x, p] = i \tag{6.39}$$

is satisfied by representing p as a differential operator:

$$p = -i(d/dx). \tag{6.40}$$

The momentum eigenfunctions are $\exp(ipx)/(2\pi)^{1/2} = \langle x\,|\,p\rangle$. The position eigenfunction $|x\rangle$ is expended in terms of the eigenfunctions $|p\rangle$ of the conjugate variable p by means of the transformation function $\langle p\,|\,x\rangle$:

$$\begin{aligned}|x\rangle &= \int\frac{dp}{(2\pi)^{1/2}}\exp(-ipx)\,|\,p\rangle, \\ \langle x\,|\,x'\rangle &= \delta(x-x'), \qquad \langle p\,|\,p'\rangle = \delta(p-p').\end{aligned} \tag{6.41}$$

One might attempt to proceed in a similar fashion by attempting to represent the number operator by

$$\begin{aligned}N &= i(d/d\phi), \\ [N, \phi] &= i \quad \text{(wrong)}.\end{aligned} \tag{6.42}$$

The eigenfunctions of N then have the form $e(-in\phi)/(2\pi)^{1/2}$, using periodicity. However, we obtain the wrong spectrum $-\infty < n < +\infty$, whereas we know that the true spectrum runs from 0 to ∞ and that the eigenfunctions are not simple exponentials.

The states $|\theta\rangle$ are not orthogonal. Using Eqs. (6.15) yields

$$\begin{aligned}\langle\theta \mid \theta'\rangle &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\exp[-in(\theta-\theta')] \\ &= (4\pi)^{-1} + \tfrac{1}{2}\delta(\theta-\theta') - (i/4\pi)\cot\tfrac{1}{2}(\theta-\theta').\end{aligned} \tag{6.43}$$

However, we can resolve the identity with $|\theta\rangle$ states:

$$\begin{aligned}\int_0^{2\pi} d\theta\,|\theta\rangle\langle\theta| &= \sum_{m,n=0}^{\infty}|\,m\rangle\langle n\,|\int_0^{2\pi}\exp[i\theta(n-m)]\frac{d\theta}{2\pi} \\ &= \sum_{m,n}|\,m\rangle\langle n\,|\,\delta_{mn} = 1.\end{aligned} \tag{6.44}$$

For two-dimensional angular momentum one can represent L_z in $[\phi, L_z] = i$ by $-id/d\phi$. The eigenfunc-

tions $\exp(im\phi)/(2\pi)^{1/2}(-\infty<m<\infty)$ constitute a complete set with integral m, and the phase eigenstate is

$$|\phi\rangle=\sum_{m=-\infty}^{\infty}\frac{\exp(-im\phi)}{(2\pi)^{1/2}}|m\rangle. \tag{6.45}$$

The wave function for phase eigenstate is

$$\langle\phi\,|\,\phi'\rangle=\delta(\phi-\phi') \tag{6.46}$$

when ϕ, ϕ' are restricted to the interval between 0 and 2π. For three-dimensional angular momentum, $|m|\leq l$, so that all values of l have to be admitted to construct the delta function (6.46). This example shows clearly that it is not the discreteness, but the one-sided nature of the number spectrum which precludes a successful analogy with the $x-p$ problem.

The states $|\theta\rangle$ are not necessarily useless however, and under certain circumstances they may be used as a substitute for the $|\cos\theta\rangle$ or $|\sin\theta\rangle$ states. In particular, we can resolve operator products into matrix elements between these states using (6.44):

$$\langle\theta''\,|\,AB\,|\,\theta\rangle=\int\langle\theta''\,|\,A\,|\,\theta'\rangle\langle\theta'\,|\,B\,|\,\theta\rangle\frac{d\theta'}{2\pi}. \tag{6.47}$$

The $|\cos\theta\rangle$ and $|\sin\theta\rangle$ states are related to $|\theta\rangle$ by

$$|\cos\theta\rangle=-i[\exp(i\theta)\,|\,\theta\rangle-\exp(-i\theta)\,|\,(-\theta)\rangle],$$
$$|\sin\theta\rangle=\exp(i\theta)\,|\,\theta\rangle+\exp(-i\theta)\,|\,(\pi-\theta)\rangle. \tag{6.48}$$

The time development of the θ states is very simple. In fact, just as expected for a "true" phase variable,

$$\begin{aligned}|\theta(t)\rangle&=\exp(-i\tilde{H}t)\,|\,\theta\rangle\\&=\sum_{n=0}^{\infty}\frac{\exp(in\theta)}{(2\pi)^{1/2}}\exp(-in\omega t)\,|\,n\rangle\\&=|\,(\theta-\omega t)\rangle,\end{aligned} \tag{6.49}$$

where $\tilde{H}=H-\frac{1}{2}\omega$ measures the energy relative to the zero-point energy.

The time dependence of the C and S operators and their eigenfunctions is of some interest. To introduce this subject we construct an operator corresponding to the classical quantity $\cos(\phi-\lambda)$.

Since ϕ does not exist as an operator, we cannot use an obvious definition. Let λ be real, or, more generally, any Hermitian operator. Define C_λ, S_λ (Hermitian) by

$$\begin{aligned}2C_\lambda&=\exp(-i\lambda)E_-+E_+\exp(i\lambda),\\2iS_\lambda&=\exp(-i\lambda)E_--E_+\exp(i\lambda).\end{aligned} \tag{6.50}$$

For $\lambda=0$, the operators C_0 and S_0 coincide with C and S. Moreover, the Heisenberg operators $C(t)$, $S(t)$ are in fact of this form, with $\lambda=\omega t$, as we will now show. However, the construction is not appropriate to describe the phase difference of two oscillators. (See the next section for that development.)

From the basic commutation rules, one has

$$\begin{aligned}\exp(iHt)\,a\exp(-iHt)&=a\exp(-i\omega t),\\E_-(t)=\exp(iHt)\,E_-\exp(-iHt)&=E_-\exp(-i\omega t),\\E_+(t)&=E_+\exp(i\omega t).\end{aligned} \tag{6.51}$$

Thus, $C(t)$ is given by

$$C(t)=\tfrac{1}{2}[E_-\exp(-i\omega t)+E_+\exp(i\omega t)], \tag{6.52}$$

whence

$$C(t)=C_{\omega t},\qquad S(t)=S_{\omega t}. \tag{6.53}$$

As a consequence of (6.52), if $|\cos\theta\rangle$ is an eigenfunction of C at $t=0$, it is not an eigenfunction of $C(t)$ for later times. Equivalently, in the Schrödinger picture, $|\cos\theta(t)\rangle=\exp(-iHt)\,|\cos\theta\rangle$ does not remain an eigenstate of C.

In ordinary language one says that the "wave packet spreads." Although

$$C\,|\cos\theta(t)\rangle\neq\cos\theta\,|\cos(\theta(t)\rangle, \tag{6.54}$$

one might suppose that the eigenvalue changes in a simple way, e.g., from $\cos\theta$ to $\cos(\theta-\omega t)$, as for the $|\theta\rangle$ states [cf. Eq. (6.49)].

This is wrong, as follows from (6.49) and the resolution of $|\cos\theta\rangle$ into $|\pm\theta\rangle$ states, Eq. (6.48):

$$\exp(-iHt)\,|\cos\theta\rangle=-i[\exp(i\theta)\,|\,\theta-\omega t\rangle-\exp(-i\theta)\,|\,-\theta-\omega t\rangle. \tag{6.55}$$

Therefore, we also have

$$C\,|\cos\theta(t)\rangle\neq\cos(\theta-\omega t)\,|\cos(\theta(t)\rangle. \tag{6.56}$$

Similar remarks hold for the sine operator S.

The $|\cos\theta\rangle$ and $|\sin\theta\rangle$ states have properties similar to the ordinary position eigenfunction $\delta(x-x_0)$. If at a given time a particle is in such a state, the momentum uncertainty gives rise to spreading. The phase states are distributions like $\delta(x-x_0)$. They provide a useful idealization in making precise the conceptual aspects of phase, but for practical calculations are not as useful as the normalizable number or coherent states.

7. PHASE DIFFERENCE OPERATORS

Although the number and phase variables cannot be simultaneously precise for a single oscillator, it is possible to have a definite phase difference between two independent oscillators for a fixed total number of quanta in the two systems. Labeling the two oscillators by subscripts 1 and 2, we have

$$\begin{aligned}&H=H_1+H_2,\\&N=N_1+N_2,\\&H_1=\omega_1(N_1+\tfrac{1}{2}),\\&H_2=\omega_2(N_2+\tfrac{1}{2}),\\&N_1=a^\dagger{}_1a_1,\ N_2=a^\dagger{}_2a_2,\qquad[a_i,\,a^\dagger{}_j]=\delta_{ij}.\end{aligned} \tag{7.1}$$

For visualization we may imagine space coordinates

x_j for the two oscillators. If we construct a product coherent state

$$|\alpha_1, \alpha_2\rangle = |\alpha_1\rangle |\alpha_2\rangle,$$
$$\langle\alpha_1, \alpha_2 | x_j(t) | \alpha_1, \alpha_2\rangle = 2x_0 |\alpha_j| \cos(\phi_j - \omega_j t),$$
$$\alpha_j = |\alpha_j| \exp(i\phi_j), \tag{7.2}$$

the two motions have phase difference $\phi_1-\phi_2$. From previous experience we know the value of using $\cos(\phi_1-\phi_2)$ instead to describe the phase difference. The trigonometric identities

$$\cos(\phi_1-\phi_2) = \cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2,$$
$$\sin(\phi_1-\phi_2) = \sin\phi_1 \cos\phi_2 - \sin\phi_2 \cos\phi_1, \tag{7.3}$$

suggest the following definition of *operators* C_{12} and S_{12}, corresponding to the cosine and sine of the phase difference:

$$C_{12} \equiv C_1C_2 + S_1S_2, \tag{7.4}$$
$$S_{12} \equiv S_1C_2 - S_2C_1. \tag{7.5}$$

For calculational purposes it is useful to express C_{12} and S_{12} in terms of the raising and lowering operators $E_{i\pm}$[85]:

$$C_{12} = \tfrac{1}{2}(|E_{1-}|E_{2+} + E_{1+}E_{2-}), \tag{7.6}$$
$$S_{12} = (1/2i)(E_{1-}E_{2+} - E_{1+}E_{2-}). \tag{7.7}$$

We next note the important result that C_{12} and S_{12} commute with the total number operator:

$$[C_{12}, N_1+N_2] = 0,$$
$$[S_{12}, N_1+N_2] = 0. \tag{7.8}$$

This follows by cancellation once one uses the operator identity

$$[AB, C] = A[B, C] + [A, C]B \tag{7.9}$$

and the commutation rules

$$[C_j, N_k] = i\delta_{jk}S_j,$$
$$[S_j, N_k] = -i\delta_{jk}C_k. \tag{7.10}$$

As a consequence of (7.8) we can find eigenfunctions of C_{12} or S_{12} having a fixed *total* N. Since the number of such states is finite, the wave function is normalizable. Therefore, the spectra of C_{12} and S_{12} are point spectra, in contrast to the spectra of the constituent C and S operators. In the limit of large N, the spectrum becomes dense, as might be expected.

It is interesting to note that C_{12} and S_{12} do not commute with the Hamiltonian unless $\omega_1=\omega_2$:

$$\dot{C}_{12} = i[H, C_{12}] = (\omega_1-\omega_2)S_{12},$$
$$\dot{S}_{12} = i[H, S_{12}] = -(\omega_1-\omega_2)C_{12}. \tag{7.11}$$

Thus C_{12} and S_{12} are normal mode coordinates whose frequency is the difference of the natural frequencies of the two oscillators:

$$\ddot{C}_{12} + (\omega_1-\omega_2)C_{12} = 0,$$
$$\ddot{S}_{12} + (\omega_1-\omega_2)S_{12} = 0. \tag{7.12}$$

As before, C_{12} and S_{12} do not commute. However, the commutator $[C_{12}, S_{12}]$ is diagonal with respect to the set of states of total number. This is proved by means of the Jacobi identity

$$[[S_{12}, C_{12}], N] + [[C_{12}, N], S_{12}] + [[N, S_{12}], C_{12}] = 0, \tag{7.13}$$

since, according to (7.8), the above reduces to[86]

$$[[S_{12}, C_{12}], N] = 0. \tag{7.14}$$

Thus, if we take states $|N'\rangle$ and $|N''\rangle$, we find

$$\langle N''|[[S_{12}, C_{12}], N]|N'\rangle = (N'-N'')\langle N''|[S_{12}, C_{12}]|N'\rangle = 0. \tag{7.15}$$

Thus, when $N' \neq N''$, the matrix element

$$\langle N''|[S_{12}, C_{12}]|N'\rangle$$

vanishes.

Next we study the eigenfunctions of C_{12} for various product states having total excitation number N:

$$|\lambda_{12}\rangle = \sum_{n=0}^{N} b_n |n\rangle |N-n\rangle. \tag{7.16}$$

In the product state in (7.16) the left-hand $|n\rangle$ refers to oscillator number one, the state $|N-n\rangle$ to oscillator number two. Often we use the shorter notation

$$|m, n\rangle = |m\rangle |n\rangle. \tag{7.17}$$

For $N=0$, the state $|0, 0\rangle$ is an eigenfunction of C_{12} with zero eigenvalue:

$$C_{12}|0, 0\rangle = 0. \tag{7.18}$$

For $N=1$, there are two possible states, $|1, 0\rangle$ and $|0, 1\rangle$. The eigenvalue equation

$$C_{12}(b_0|0, 1\rangle + b_1|1, 0\rangle) = \lambda(b_0|0, 1\rangle + b_1|1, 0\rangle) \tag{7.19}$$

gives the constraints

$$\left.\begin{aligned} b_1 &= 2\lambda b_0 \\ b_0 &= 2\lambda b_1 \end{aligned}\right\} \Rightarrow \lambda = \pm\tfrac{1}{2}. \tag{7.20}$$

The two normalized eigenfunctions are

$$|\pm\tfrac{1}{2}\rangle = (1/2)^{1/2}(|1, 0\rangle \pm |0, 1\rangle). \tag{7.21}$$

We shall adhere to a phase convention in which the coefficient of the component $|m, n\rangle$ with the largest value of m is real and positive. For $N=2$, there are

[85] Note that in the classical limit $E_{i\pm} \to \exp(\mp\phi_j)$, so that Eqs. (7.6) and (7.7) have the expected form.

[86] A simple calculation shows that

$$[S_{12}, C_{12}] = (\mathcal{P}_0^{(1)} - \mathcal{P}_0^{(2)})/2i,$$

where $\mathcal{P}_0^{(j)}$ is the projection operator for the ground state of oscillator j.

three contributing states. The expansion coefficients obey

$$b_1=2\lambda b_0=2\lambda b_2,$$
$$b_0+b_2=2\lambda b_1. \tag{7.22}$$

Therethere are three solutions. If $\lambda\neq 0$, then $b_0=b_2$ and $\lambda=\pm 2^{-1/2}$; if $\lambda=0$, $b_0=-b_2$ and $b_1=0$. The eigenfunctions are

$$\tfrac{1}{2}(|\,2,0\rangle\pm 2^{1/2}\,|\,1,1\rangle+|\,0,2\rangle);\qquad \lambda=\pm 1/(2)^{1/2}$$
$$2^{-1/2}(|\,2,0\rangle-|\,0,2\rangle);\qquad \lambda=0. \tag{7.23}$$

All the eigenfunctions found so far can be represented by an angle θ in the interval $0<\theta<\pi$, $\lambda=\cos\theta$. For $N=0$, $\theta=\pi/2$; $N=1$, $\theta=\pi/3$, $2\pi/3$; $N=2$, $\theta=\frac{1}{4}\pi$, $\frac{1}{2}\pi$, $\frac{3}{4}\pi$. Fig. 7 shows the angles, represented by points on the unit circle $\exp(i\theta)$.

The situation depicted in Fig. 7 suggests the correct answer, namely, that the eigenvalues correspond to discrete values of $\cos\theta$, with θ having $N+1$ values:

$$\theta_{Nr}=\pi r/(N+2),\qquad r=1,2,\cdots,N+1. \tag{7.24}$$

To prove this let us consider the recursion relations satisfied by the expansion coefficients b_n of Eq. (7.16):

$$2\lambda b_0=b_1,$$
$$2\lambda b_N=b_{N-1},$$
$$2\lambda b_n=b_{n+1}+b_{n-1};\qquad 1\leq n\leq N-1. \tag{7.25}$$

We have already solved these equations for $n=0,1,2$. For $n\geq 2$ the last recursion relation is used to compute all b_n lying between b_1 and b_{N-1}. This relation has precisely the same form as in Eq. (6.4), so that

$$b_n=Ap^n+Bp^{-n},\qquad 2\lambda=p+1/p. \tag{7.26}$$

The argument leading to $\lambda=\cos\theta$ is different in the present case. Since λ is real (eigenvalue of an Hermitian operator), we have $p^{-1}=p^*$. If we write p as $A\exp(i\theta)$, we learn that $A=\pm 1$. The minus sign can always be absorbed into θ, so that the eigenvalues of C_{12} are

$$\lambda_{Nr}=\cos\theta_{Nr}. \tag{7.27}$$

In the present problem the termination of the series at b_N restricts θ_{Nr} to certain values. As in the analysis of the eigenfunctions of C, we set $b_0=\sin\theta$, $b_1=\sin 2\theta$, $\cdots$, etc., with $b_n=\sin(n+1)\theta$. The second equation in (7.25) then requires the extra condition

$$2\cos\theta\sin(N+1)\theta=\sin N\theta,$$

or

$$\sin(N+2)\theta=0. \tag{7.28}$$

The latter equation is solved by Eq. (7.24). Changing θ to $-\theta$ does not give extra solutions since the b_n just change sign. Also, the b_n vanish identically for $\theta=0$ or π. The eigenvalue spectrum is shown in Fig. 7(d) for $N=10$. For large N the spectrum becomes dense.

The normalized eigenfunctions are

$$|\cos\theta_{Nr}\rangle=\left(\frac{2}{N+2}\right)^{1/2}\sum_{n=0}^{N}\sin(n+1)\theta_{Nr}\,|\,n\rangle\,|\,N-n\rangle, \tag{7.29}$$

$$\langle\cos\theta_{Nr}\,|\,\cos\theta_{Ms}\rangle=\delta_{NM}\delta_{rs}. \tag{7.30}$$

The possible values of θ_{Nr} are restricted by Eq. (7.24). For explicitness one might prefer to label the eigenfunctions with the number as well: $|\,N,\cos\theta_{Nr}\rangle$. Eigenfunctions having distinct N are automatically orthogonal.

The solution to eigenvalue problem for S_{12} is similar to the preceding analysis.

We write

$$S_{12}\,|\,\mu\rangle=\mu\,|\,\mu\rangle,$$
$$|\,\mu\rangle=\sum_{n=0}^{N}C_n\,|\,n\rangle\,|\,N-n\rangle. \tag{7.31}$$

The ground state satisfies

$$S_{12}\,|\,0,0\rangle=0. \tag{7.32}$$

The recursion relations connecting the expansion coefficients are

$$2i\mu C_0=C_1,$$
$$2i\mu C_N=-C_{N-1},$$
$$2i\mu C_n=C_{n+1}-C_{n-1}. \tag{7.33}$$

For completeness we give the solutions for $N=1,2$, for comparison with Eqs. (7.21)–(7.23):

$$N=1;\qquad \mu=\pm\tfrac{1}{2},\qquad |\,\pm\tfrac{1}{2}\rangle=2^{-1/2}(|\,1,0\rangle\pm i\,|\,0,1\rangle,$$
$$N=2;\qquad \mu=\pm 2^{-1/2},$$
$$|\,\pm 2^{-1/2}\rangle=\tfrac{1}{2}(|\,2,0\rangle\mp i2^{1/2}\,|\,1,1\rangle-|\,0,2\rangle),$$
$$\mu=0,\quad |\,0\rangle=2^{-1/2}(|\,2,0\rangle-|\,0,2\rangle). \tag{7.34}$$

Figure 8 shows the allowed values of $\phi=\sin^{-1}\mu$.

For the case of general N we can take over most of the analysis of the sine operator (Sec. 6). Letting $C_0=\cos\phi$, the general term is given by Eq. (6.27). The second relation in (7.33) provides an extra condition which fixes ϕ:

$$2i(\sin\phi)C_N=-C_{N-1}, \tag{7.35}$$

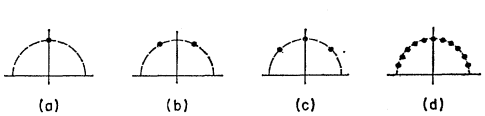

FIG. 7. The eigenvalues of the cosine phase difference operator are represented on the unit circle. In Figs. (a, b, c,) the total number is respectively 0, 1, and 2. As N becomes large, the spectrum becomes dense. This is represented qualitatively in (d). All independent states are obtained by restricting θ to the upper half-plane.

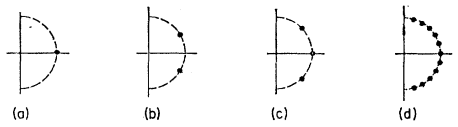

FIG. 8. The eigenvalues of the sine phase difference operator are represented on the unit circle. In parts (a), (b), (c) the total number is, respectively, 0, 1, and 2. As N becomes large, the spectrum becomes dense. This is represented qualitatively in (d). All independent states are obtained by restricting θ to the right half plane.

which simplifies to

$$\exp[i(N+2)\phi]+(-1)^{N+1}\exp[-i(N+2)]\phi=0, \tag{7.36}$$

or

$$\sin(N+2)\phi=0, \qquad N=0, 2, 4, \cdots,$$
$$\cos(N+2)\phi=0, \qquad N=1, 3, 5, \cdots, \tag{7.37}$$

$$\phi_{Nr}=[r\pi/(N+2)]-(\pi/2), \qquad r=1, 2, \cdots, N+1. \tag{7.38}$$

(We could take $+\pi/2$ instead of $-\pi/2$.) Note that $\phi\rightarrow\pi-\phi$ does not change C_n, so that all independent solutions are obtained by restricting ϕ to the interval

$$-\pi/2<\phi<\pi/2. \tag{7.39}$$

In the ϕ plane the S_{12} eigenvalues are represented (Fig. 8) by rotating Fig. 7 by $\pi/2$. The spectra of S_{12} and C_{12} are identical. The normalized eigenfunctions are

$$S_{12}\,|\sin\phi_{Nr}\rangle=\sin\phi_{Nr}\,|\sin\phi_{Nr}\rangle,$$

$$|\sin\phi_{Nr}\rangle=\left(\frac{2}{N+2}\right)^{1/2}\sum_{n=0}^{N}\tfrac{1}{2}\{\exp[i(n+1)\phi_{Nr}]$$
$$-\exp[-i(\phi_{Nr}-\pi)(n+1)]\}\,|n\rangle\,|N-n\rangle$$
$$=\left(\frac{2}{N+2}\right)^{1/2}\sum_{n=0}^{N}(-i)^n$$
$$\times\sin(n+1)\theta_{Nr}\,|n\rangle\,|N-n\rangle,$$
$$\langle\sin\phi_{Nr}\,|\sin\phi_{Ms}\rangle=\delta_{NM}\delta_{rs}, \tag{7.40}$$

where ϕ_{Nr} is always given by (7.38) and θ_{Nr} by (7.24).

8. PHYSICAL PROPERTIES OF THE COHERENT STATES, THE PHASE STATES AND THE MINIMUM UNCERTAINTY STATES

A. Coherent States; Uncertainty Relations for Number and Phase

Among the many types of wave functions which describe the classical limit of a system, the coherent states have a special significance, for it is such states which are produced when an oscillator is coupled linearly to a prescribed classical force.[56] Since this model is the prototype for the radiation emitted by a classical current source, there are many significant applications. It should be emphasized that the coherent states are produced even when the mean number of quanta is so small that the state could not be called classical.

As proved in Sec. 3, the coherent states are minimum-uncertainty wave packets in x–p coordinates. Although they are not minimum-uncertainty states in number–phase coordinates, they are remarkably good in this regard, and for all practical purposes are minimum-uncertainty states when $\bar{N}>1$. In Sec. 8D, we discuss the true number–phase minimum-uncertainty states discovered by Jackiw.[16]

The effect of the C and S operators on the coherent states is not very simple. We have the results

$$C\,|\alpha\rangle=\tfrac{1}{2}\exp(-|\alpha|^2/2)$$
$$\times\sum_{n=0}^{\infty}\left\{\frac{\alpha^{n+1}}{[(n+1)!]^{1/2}}\,|n\rangle+\frac{\alpha^n}{(n!)^{1/2}}\,|n+1\rangle\right\},$$
$$S\,|\alpha\rangle=(2i)^{-1}(\exp(-|\alpha|^2/2)$$
$$\times\sum_{n=0}^{\infty}\left[\frac{\alpha^{n+1}}{[(n+1)!]^{1/2}}\,|n\rangle-\frac{\alpha^n}{(n!)^{1/2}}\,|n+1\rangle\right]. \tag{8.1}$$

We now put α in the form $N^{1/2}\exp(i\phi)$ (where we write $\langle N\rangle$ as N for notational simplicity) so that N is related to α by

$$N=|\alpha|^2. \tag{8.2}$$

With this notation, we find

$$\langle\alpha\,|\,C\,|\,\alpha\rangle=N^{1/2}e^{-N}\psi_1(N)\cos\phi,$$
$$\langle\alpha\,|\,S\,|\,\alpha\rangle=N^{1/2}e^{-N}\psi_1(N)\sin\phi,$$
$$\langle\alpha\,|\,C^2\,|\,\alpha\rangle=\tfrac{1}{2}-\tfrac{1}{4}e^{-N}+\tfrac{1}{2}Ne^{-N}\psi_2(N)\cos2\phi,$$
$$\langle\alpha\,|\,S^2\,|\,\alpha\rangle=\tfrac{1}{2}-\tfrac{1}{4}e^{-N}-\tfrac{1}{2}Ne^{-N}\psi_2(N)\cos2\phi,$$
$$\langle\alpha\,|\,C^2+S^2\,|\,\alpha\rangle=1-\tfrac{1}{2}e^{-N},$$
$$\langle\alpha\,|\,C^2-S^2\,|\,\alpha\rangle=Ne^{-N}\psi_2(N)\cos2\phi,$$
$$\langle\alpha\,|\,CS+SC\,|\,\alpha\rangle=Ne^{-N}\psi_2(N)\sin2\phi,$$
$$\langle\alpha\,|\,CS-SC\,|\,\alpha\rangle=\tfrac{1}{2}ie^{-N}, \tag{8.3}$$

where the functions $\psi_\lambda(N)$ are defined by the power series

$$\psi_1(N)=\sum_{n=0}^{\infty}\frac{N^n}{n![(n+1)]^{1/2}}, \tag{8.4a}$$

$$\psi_2(N)=\sum_{n=0}^{\infty}\frac{N^n}{n![(n+1)(n+2)]^{1/2}}. \tag{8.4b}$$

For future use we calculate the asymptotic limits of ψ_1 and ψ_2. For $N\ll1$, the answer is obtained by simply taking the first few terms of (8.4):

$$\psi_1(N)\sim1+\frac{N}{2}+\frac{N^2}{2(3)^{1/2}}+\cdots, \qquad N\rightarrow0$$

$$\psi_2(N)\sim\frac{1}{2^{1/2}}+\frac{N}{6^{1/2}}+\frac{N^2}{4(3)^{1/2}}+\cdots, \qquad N\rightarrow0. \tag{8.5}$$

For large N, ψ_1 is calculated by using the identity[87]

$$\frac{1}{(n+1)^{z+1}} \equiv \int_0^\infty t^z \exp\left[-(n+1)t\right]dt/\Gamma(z+1), \quad (8.6)$$

where Γ is the gamma function; in our problem $z=-\frac{1}{2}$. Then

$$\psi_1(N) = \sum_{n=0}^{\infty} \frac{N^n}{n!}\frac{1}{\pi^{1/2}}\int_0^\infty t^{1/2}\exp\left[-(n+1)t\right]dt$$

$$= \frac{1}{\pi^{1/2}}\int_0^\infty dt\, t^{1/2}e^{-t}\exp\left(Ne^{-t}\right)$$

$$= \frac{e^N}{\pi^{1/2}}\cdot\int_0^1\left[\ln\left(\frac{1}{1-y}\right)\right]^{-1/2}e^{-Ny}dy, \quad (8.7)$$

where we have used the transformation

$$e^t = 1-y. \quad (8.8)$$

By then using the expansion

$$\{\ln\left[1/(1-y)\right]\}^c = y^c+\tfrac{1}{2}cy^{c+1}+\cdots, \quad (8.9)$$

and defining

$$x = Ny, \quad (8.10)$$

we have

$$\psi_1(N) = \frac{e^N}{(\pi N)^{1/2}}\int_0^N\left[x^{-1/2}-\frac{x^{1/2}}{4N}+\cdots\right]e^{-x}dx. \quad (8.11)$$

In the limit as $N\to\infty$, the integrals become gamma functions, and we obtain

$$\psi_1(N)\sim(e^N/N)\left[1-(8N)^{-1}+\cdots\right]. \quad (8.12)$$

To find $\psi_2(N\to\infty)$ we first observe that if $c\gg1$,

$$\left[c^2+c\right]^{1/2} = c+\tfrac{1}{2}-(1/8c)+\cdots, \quad (8.13)$$

so that

$$\frac{1}{\left[c^2+c\right]^{1/2}} = \frac{1}{c}\left[1-\frac{1}{2c}+\frac{3}{8c^2}+\mathcal{O}\left(\frac{1}{c^3}\right)\right]$$

$$= \frac{1}{c}\left[1-\frac{1}{2(c+1)}-\frac{1}{8(c+1)(c+2)}+\mathcal{O}\left(\frac{1}{c^3}\right)\right] \quad (8.14)$$

But now realizing that

$$\left[(n+1)(n+2)\right] = (n+1)^2+(n+1), \quad (8.15)$$

we have

$$\psi_2(N) = \sum_{n=0}^{\infty}\frac{N^n}{(n+1)!}$$

$$\times\{1-\left[2(n+2)\right]^{-1}-\left[8(n+2)(n+3)\right]^{-1}+\cdots\}$$

$$= (e^N/N)\left[1-(2N)^{-1}-(1/8N^2)+\cdots\right]. \quad (8.16)$$

The leading terms in (5.12) and (5.16) could be anticipated by requiring proper classical limits for the expectation values of the operators. We have, for $N\gg1$, the expected classical behavior:

$$\langle C\rangle\sim\cos\phi, \qquad \langle S\rangle\sim\sin\phi,$$

$$\langle C^2\rangle\sim\cos^2\phi, \qquad \langle S^2\rangle\sim\sin^2\phi,$$

$$\langle C^2+S^2\rangle\sim1, \qquad \langle C^2-S^2\rangle\sim\cos 2\phi,$$

$$\langle SC+CS\rangle\sim\sin 2\phi, \qquad \langle CS-SC\rangle\sim0. \quad (8.17)$$

In working with operator trigonometry, it is clear that, when $N\gg1$, the expectation value $\langle\alpha\,|\,F(C,S)\,|\,\alpha\rangle$ of any function F of C and S goes over into $F(\cos\phi,\sin\phi)$. When $N\gg1$, the ground state component of $|\,\alpha\rangle$ is negligible $\mathcal{O}(e^{-N})$, so that C and S may be treated as commutative variables.

In contrast to the classical limit (8.17) we have the "quantum limit" $N\ll1$ for the coherent states:

$$\langle S\rangle = N^{1/2}\sin\phi; \qquad \langle C\rangle\sim N^{1/2}\cos\phi,$$

$$\langle S^2\rangle\sim\tfrac{1}{4}; \qquad \langle C^2\rangle\sim\tfrac{1}{4}. \quad (8.18)$$

(These relations follow immediately from the fact that $|\,\alpha\rangle\cong|\,0\rangle$ in this limit.)

We are especially interested in testing the coherent states in the uncertainty relations (5.47)–(5.52). If the coherent states are "really" coherent, we might expect the equality to be approached. This is in fact the case.[19]

Recall that in the coherent states the number states are Poisson-distributed[56]:

$$(\Delta N)^2 = \langle N^2\rangle-\langle N\rangle^2 = \langle N\rangle. \quad (8.19)$$

We next want to evaluate the quantities

$$P(N,\phi) = (\Delta N)^2(\Delta C)^2/\langle S\rangle^2, \quad (8.20)$$

$$Q(N,\phi) = (\Delta N)^2(\Delta S)^2/\langle C\rangle^2 \quad (8.21)$$

for various ϕ as a function of $\langle N\rangle$, and compare with the minimum value 1/4. For large and small N we can evaluate (8.20) and (8.21) by means of the asymptotic formulas (8.5), (8.12), and (8.16).

Keeping terms up to order $1/N$ in the brackets in (8.12) and (8.16) gives

$$(\Delta C)^2\sim\sin^2\phi/4N,$$

$$(\Delta S)^2\sim\cos^2\phi/4N, \qquad (N\gg1) \quad (8.22)$$

so that, when $N\gg1$, we obtain the minimum possible values:

$$P(N,\phi)\sim\tfrac{1}{4} \qquad Q(N,\phi)\sim\tfrac{1}{4}. \quad (8.23)$$

When $N\ll1$, we use Eq. (8.18) to find

$$P(N,\phi)\sim1/(4\sin^2\phi),$$

$$Q(N,\phi)\sim1/(4\cos^2\phi) \qquad (N\ll1). \quad (8.24)$$

For intermediate N one has to use numerical evaluation of the functions $\psi_\lambda(N)$ to obtain results. Results of a computer calculation for Q, reported in Ref. 19, are shown in Fig. 9. Notice that if the average excitation number is substantially greater than 1, the coherent states are effectively minimum-uncertainty number–

[87] The authors would like to thank Professor H. Widom for his aid in finding this integral.

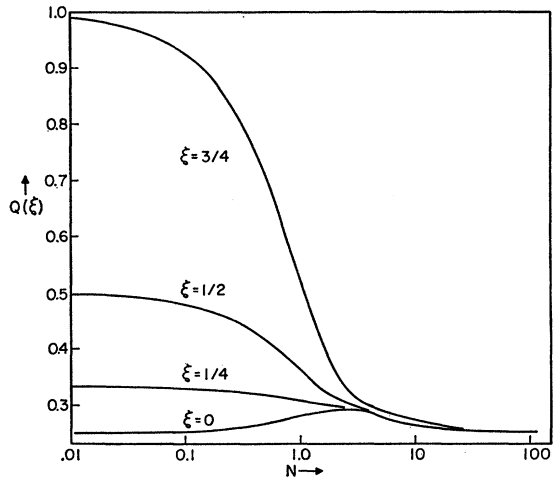

FIG. 9. The quantity $Q(\xi)=S(\xi)/\langle C\rangle^2$ (hence $\xi=\sin^2\phi$) is shown as a function of $(\Delta N)^2=N$ for various ξ, for the coherent states. According to Eq. (5.51), $Q(\xi)$ must be larger than $\frac{1}{4}$.

phase states. The result for $P(N,\phi)$ follows by changing ϕ to $\pi/2-\phi$ in Fig. 9.

The symmetrical uncertainty product, Eq. (5.52),

$$U=\{(\Delta N)^2[(\Delta C)^2+(\Delta S)^2]\}/(\langle S\rangle^2+\langle C\rangle^2), \quad (8.25)$$

is independent of the phase angle ϕ. Explicitly, one has

$$U=[1-\tfrac{1}{2}e^{-N}-Ne^{-2N}\psi_1^2(N)]/e^{-2N}\psi_1^2(N). \quad (8.26)$$

By putting (8.3) and (8.19) into (8.20) and (8.21), and then comparing with (8.25), we see that

$$P(N,\pi/4)=Q(N,\pi/4)=U. \quad (8.27)$$

Thus, U is given by the $\xi\equiv\sin^2\phi=\frac{1}{2}$ curve in Fig. 9. The relation (8.27) is the same type of result we obtained in Sec. 4B for the angular momentum-angle case.

From the asymptotic formulas we see that

$$U\sim\tfrac{1}{2}, \quad N\ll 1,$$
$$U\sim\tfrac{1}{4}, \quad N\gg 1. \quad (8.28)$$

For no value of N is U greater than twice the minimum value.

Finally we consider the uncertainty relation $\Delta S\Delta C$ of Eq. (5.49). In the coherent states this becomes

$$(\Delta S)(\Delta C)\geq\tfrac{1}{4}e^{-N}. \quad (8.29)$$

From (8.18) for $N\ll 1$, the left-hand side is 1/4 and the equality is satisfied. For $N\gg 1$, the left-hand side decreases as $1/N^2$:

$$(\Delta S)^2(\Delta C)^2\sim(\sin^2\phi\cos^2\phi/16N^2)+\mathcal{O}(1/N^3) \quad (8.30)$$

[cf. Eq. (8.12), which is not as small as the right-hand side of (8.29)].

Finally we consider the time dependence of the Heisenberg operators in the coherent states. We have

$$\langle\alpha\,|\,C(t)\,|\,\alpha\rangle=\langle\alpha\,|\exp(iHt)C\exp(-iHt)\,|\,\alpha\rangle$$
$$=\langle\alpha(t)\,|\,C\,|\,\alpha(t)\rangle, \quad (8.31)$$

where $\alpha(t)=\alpha\exp(-i\omega t)=N^{1/2}\exp(i\phi-i\omega t)$. Thus, the previous results carry over with the change $\phi\to\phi-\omega t$. Equations (8.3) become

$$\langle\alpha\,|\,C(t)\,|\,\alpha\rangle=N^{1/2}e^{-N}\psi_1(N)\cos(\phi-\omega t), \quad (8.32)$$

and so on.

It is worthwhile noting the similarity between these numerical results and the numerical results for the angular momentum-angle system in the coherent states (compare Figs. 4 and 9). This gives further insight into the role of the S and C operators.

B. Fluctuation of Phase in the Number States

In the number basis, $\Delta N=0$ and the uncertainty relations (5.47)–(5.52) reduce to the uninformative statement $0\geq 0$. All odd powers of C or S have vanishing expectation value in the number states:

$$\langle n\,|\,C^{2m+1}\,|\,n\rangle=\langle n\,|\,S^{2m+1}\,|\,n\rangle=0; \quad \text{all } m,\ n,$$
$$\langle n\,|\,C^lS^m\,|\,n\rangle=0; \quad l+m=\text{odd}, \quad \text{all } n, \quad (8.33)$$

since the number of annihilation operators has to match the number of creation operators to give a nonzero expectation value.

Let us consider the classical distribution $P(\phi)=1/2\pi$. For such a distribution

$$\langle\cos^{2m}\phi\rangle=\int_0^{2\pi}\cos^{2m}\phi\,\frac{d\phi}{2\pi}$$
$$=\frac{1\cdot 3\cdots(2m-1)}{2\cdot 4\cdots 2m}. \quad (8.34)$$

$\langle\sin^{2m}\phi\rangle$ is the same as (8.34). Since one is accustomed to saying that, when N is known, the phase is "completely undertermined," we now calculate

$$\langle n\,|\,C^{2m}\,|\,n\rangle \qquad m=0,\ 1,\ 2,\ \cdots, \quad (8.35)$$

to show that this is not true.

From the equations

$$C\,|\,n\rangle=1/2(|\,n+1\rangle+|\,n-1\rangle), \qquad n\geq 1,$$
$$C^2\,|\,n\rangle=1/4(|\,n-2\rangle+2\,|\,n\rangle+|\,n+2\rangle), \qquad n\geq 2,$$
$$C^3\,|\,n\rangle=1/8(|\,n-3\rangle+3\,|\,n-1\rangle+3\,|\,n+1\rangle+|\,n+3\rangle), \quad n\geq 3, \quad (8.36)$$

we find

$$\langle n\,|\,C^2\,|\,n\rangle=1/2, \qquad n\geq 1,$$
$$=1/4, \qquad n=0,$$
$$\langle n\,|\,C^4\,|\,n\rangle=3/8, \qquad n\geq 2,$$
$$=5/16, \qquad n=1,$$
$$=1/8, \qquad n=0,$$
$$\langle n\,|\,C^6\,|\,n\rangle=20/64, \qquad n\geq 3,$$
$$=13/64, \qquad n=2,$$
$$=14/64, \qquad n=1,$$
$$=5/64, \qquad n=0. \quad (8.37)$$

Comparison of the classical and quantum results is made in Table I. If the number state $|n\rangle$ is not so small that application of C^m annihilates some of the resultant states, then $\langle C^{2m}\rangle$ agrees with the classical result to $\langle C^6\rangle$, at which point deviations occur. Thus, the first five moments of C are the same for number states and the uniform classical phase distribution, if $n>2$ in the number states $|n\rangle$.

C. Phase States

The eigenfunctions of C (or S) have vanishing ΔC (or ΔS). The number fluctuation of the C (or S) eigenstates is undefined (infinite). Qualitatively, one has the usual result that a completely defined phase leads to a completely undefined number.

We mention also that Jackiw[16] has shown by his analytic method discussed in Sec. 2 that there exist no normalizable states that minimize the C and S uncertainty relation (5.49). One could, however, use unnormalized states such as the $|\theta\rangle$, $|\cos\theta\rangle$, or $|\sin\theta\rangle$ states.

D. Minimum-Uncertainty States

From the discussion in Sec. 2C, we see that, for the complicated uncertainty relations (5.50) and (5.51), we can use the direct method. Jackiw[16] did this and obtained the minimum uncertainty states for (5.50).

If

$$|\psi\rangle=\sum_{n=0}^{\infty}a_n|n\rangle, \tag{8.38}$$

then we have

$$[N+i\gamma C]|\psi\rangle=\lambda|\psi\rangle. \tag{8.39}$$

Taking expectation values gives us the recursion relation

$$\begin{aligned}(\lambda-n)a_n&=(i\gamma/2)(a_{n+1}+a_{n-1}),\\ a_{-1}&=0,\end{aligned} \tag{8.40}$$

where, for self-consistency, we will have

$$\begin{aligned}\lambda&=\langle N\rangle+i\gamma\langle C\rangle,\\ \gamma&=[(\Delta N)^2/(\Delta C)^2]^{1/2}.\end{aligned} \tag{8.41}$$

Now, if we define $a_n=(-i)^n b_n$, we have

$$\begin{aligned}(2/\gamma)(n-\lambda)b_n&=(b_{n-1}-b_{n+1}),\\ b_{-1}&=0.\end{aligned} \tag{8.42}$$

If λ is real, i.e., $\langle C\rangle=0$, the solution is [88]

$$b_n=\nu I_{n-\lambda}(\gamma). \tag{8.43}$$

where ν is the normalization constant and $I_\mu(\gamma)$ is a modified Bessel function of the first kind of order μ. We need $I_{-1-\lambda}(\gamma)=0$, so that λ satisfies $2h+1>\lambda>2h$, $h=0, 1\cdots$. Our state is thus

$$\begin{aligned}|\lambda,\gamma\rangle_C&=\nu\sum_{n=0}^{\infty}(-i)^n I_{n-\lambda}(\gamma)\,|n\rangle\\ \langle\lambda,\gamma|\lambda,\gamma\rangle_C&=1.\end{aligned} \tag{8.44}$$

[88] G. N. Watson, (Ref. 70), Sec. 4, pp. 77, 294.

TABLE I. Classical and quantum (number state) expectation values of powers of the cosine variables are compared. All odd powers vanish in either theory.

n	$\langle C^m\rangle_{\text{classical}}$	$\langle n\mid C^m\mid n\rangle$
0	1	1
2	1/2	1/2 $(n\geq 1)$
4	3/8	3/8 $(n\geq 2)$
6	15/64	20/64 $(n\geq 3)$

This is a complicated state, but is the minimum-uncertainty state. Table II compares the principal qualitative features of the various state vectors.

In a similar manner, one can calculate the minimum-uncertainty state for (5.51). The answer is

$$\begin{aligned}|\lambda,\gamma\rangle_S&=\nu\sum_{n=0}^{\infty}I_{n-\lambda}(\gamma)\,|n\rangle,\\ \lambda&=\langle N\rangle,\\ \gamma&=[(\Delta N)^2/(\Delta S)^2]^{1/2},\\ \langle S\rangle&\equiv 0.\end{aligned} \tag{8.45}$$

The solution (8.45) again makes a comparison to the angular momentum-angle case applicable. All the comments made in Sec. 4 about the closeness of the coherent states to the minimum-uncertainty state again are true.

The coherent states effectively have minimum uncertainty for large N. The coherent state with $\langle S\rangle=0$ was the closest to minimum uncertainty, and such a solution (8.45) gives exactly minimum uncertainty. For $\langle C\rangle=0$, the uncertainty product (8.44) might blow up (as it did for $N\to 0$ in the coherent state case), and here that would mean our direct method is ambiguous.

So once again, we observe that the coherent states are a very coherent set of states for a new pair of variables; here, number and phase. Combined with their physical significance and their coherence properties in position-momentum and angular momentum-angle, this makes them most remarkable states.

Since we would not expect to have both $\langle C\rangle=0=\langle S\rangle$, we would not think that we could obtain a minimum-uncertainty state for the symmetric uncertainty relation (5.52). This is the case, as was shown explicitly by Jackiw[16] (recall the similar result for angular momentum angle). As we already mentioned, Jackiw also has shown that the uncertainty relation (5.49), between S and C, has no normalizable minimum-uncertainty state.

9. STATISTICAL MECHANICS AND IRREVERSIBILITY IN THE COHERENT STATE BASIS

A. General Considerations

The energy distribution in systems of bosons is related to the number operators N_k describing the various degrees of freedom labeled by the variable k.

TABLE II. Important qualitative properties of some interesting state vectors associated with the harmonic oscillator are compared.

Type of state	Number character	Phase character	State vector normalized
1. Number $\mid n\rangle$	$(\Delta N)^2=0$	$(\Delta C)^2=\frac{1}{2}$[a]	Yes
2. Phase $\mid \cos\theta\rangle$	$(\Delta N)^2=\infty$	$(\Delta C)^2=0$	No
3. Coherent $\mid \alpha\rangle$	$(\Delta N)^2=N$	complicated[b]	Yes
4. Minimum—Unc. $\mid \lambda, \gamma\rangle_C$	complicated[b]	complicated[b]	Yes
5. Phase difference $\mid \cos\theta_{12}\rangle$	$(\Delta N_{\text{tot}})^2=0$	$(\Delta C_{12})^2=0$	Yes

[a] Except for the ground state; in which case $(\Delta C)^2=\frac{1}{4}$. [b] See discussion in text.

On the other hand, the time evolution of such systems, in particular, the approach to equilibrium, involves the phase variables in an intrinsic way. In previous sections we have considered idealized "pure" isolated quantum-mechanical systems. In the present section we draw attention to the great utility of the coherent state basis for the description of the time evolution of the density matrix.[20,89] Here we confine our attention to the simple harmonic oscillator; the reader is referred to Refs. 20 and 89 for discussion of a realistic problem involving anharmonically coupled oscillators (phonons in a crystal).

In the coherent state basis the equation of motion for the density matrix, although fully quantum-mechanical, has an obvious correspondence with the classical Liouville equation, as formulated in the action-angle basis by Brout and Prigogine.[21,22]. The classical limit is explicitly present, along with terms which can be regarded as "quantum-mechanical corrections."

The equation of motion for the density matrix is[90]

$$i(\partial/\partial t)\rho=[H,\rho]. \tag{9.1}$$

In the following we suppose that all operators can be represented by a convergent ordered series in a and $a^\dagger$:

$$\tilde{A}=\sum_{mn}A_{mn}(a^\dagger)^m a^n\equiv\tilde{A}(a^\dagger,a). \tag{9.2}$$

Taking matrix elements between coherent states, we find

$$\langle\alpha\mid A(a^\dagger,a)\mid\beta\rangle/\langle\alpha\mid\beta\rangle=\sum_{mn}A_{mn}(\alpha^*)^m\beta^n$$
$$=A(\alpha^*,\beta). \tag{9.3}$$

We call the quantity $A(\alpha^*,\beta)$ the *reduced matrix element* of the operator A. Given sufficiently well-behaved expansion coefficients, $A(\alpha^*,\beta)$ is an analytic function of two complex variables.[91] Therefore, it is sufficient to calculate $A(\alpha^*,\alpha)$, which is a boundary value of $A(\alpha^*,\beta)$. Analytic continuation is used to find the general $A(\alpha^*,\beta)$.

Using the completeness relation [cf. Eq. (3.29)], Eq. (9.1) can be written in the form

$$i\langle\alpha\mid\partial\rho/\partial t\mid\alpha\rangle$$
$$=\pi^{-1}\int d^2\beta[\langle\alpha\mid H\mid\beta\rangle\langle\beta\mid\rho\mid\alpha\rangle-\langle\alpha\mid\rho\mid\beta\rangle\langle\beta\mid H\mid\alpha\rangle]. \tag{9.4}$$

Equation (9.4) seems to indicate that knowledge of off-diagonal elements $\langle\beta\mid\rho\mid\alpha\rangle$ is required to compute $\langle\alpha\mid\rho\mid\alpha\rangle$. However, this is not the case, as is explained below.

Assuming H to have the structure of Eq. (9.2), we encounter integrals with the structure

$$\pi^{-1}\int d^2\beta(\alpha^*)^m\beta^n\langle\alpha\mid\beta\rangle\langle\beta\mid\rho\mid\alpha\rangle, \tag{9.5}$$

where m and n are integers. Using the formula

$$\pi^{-1}\int d^2\beta\,\exp(-\mid\beta\mid^2)\beta^l(\beta^*)^m=\delta_{lm}(l!m!)^{1/2}, \tag{9.6}$$

we find that any function which can be expanded in a Taylor series satisfies the identities

$$\pi^{-1}\int d^2\beta\,\exp(\alpha^*\beta-\mid\beta\mid^2)f(\beta^*)=f(\alpha^*),$$

$$\pi^{-1}\int d^2\beta\,\exp(\alpha^*\beta-\mid\beta\mid^2)\beta^n f(\beta^*)=(\partial/\partial\alpha^*)^n f(\alpha^*),$$

$$\pi^{-1}\int d^2\beta\,\exp(\alpha^*\beta-\mid\beta\mid^2)f(\beta)=f(\alpha),$$

$$\pi^{-1}\int d^2\beta\,\exp(\alpha\beta^*-\mid\beta\mid^2)(\beta^*)^n f(\beta)=(\partial/\partial\alpha)^n f(\alpha). \tag{9.7}$$

The structure of the exponentials in Eqs. (9.7) is dictated by the form of $\langle\alpha\mid\beta\rangle$ given in Eq. (3.28), as indicated in Eq. (9.5).

A trivial example series to illustrate the application of Eqs. (9.7). Suppose H contains the destruction operator a. Then the first term of (9.4) gives rise to a term with $m=0$, $n=1$, in (9.5):

$$\pi^{-1}\int d^2\beta\,\beta\langle\alpha\mid\beta\rangle\langle\beta\mid\rho\mid\alpha\rangle. \tag{9.8}$$

From Eq. (9.3), we learn that $\langle\beta\mid\rho\mid\alpha\rangle$ is the prod-

[89] K. S. Dy, Ph.D. thesis, Cornell University, June, 1967 (unpublished).

[90] A brief summary of the density matrix formalism is given in C. Kittel, *Elementary Statistical Physics* (John Wiley & Sons, Inc., New York, 1961), p. 107.

[91] S. Bochner and W. T. Martin, *Several Complex Variables* (Princeton University Press, Princeton, N. J., 1948).

uct of $\langle\beta \mid \alpha\rangle$ and an analytic function of β^* and α:

$$\langle\beta \mid \rho \mid \alpha\rangle \equiv \langle\beta \mid \alpha\rangle \rho(\beta^*, \alpha). \tag{9.9}$$

Note the result

$$\rho(\alpha^*, \alpha) = \langle\alpha \mid \rho \mid \alpha\rangle. \tag{9.10}$$

Combining Eqs. (9.8), (9.9) with (3.28) results in

$$\begin{aligned}
&\pi^{-1}\int d^2\beta \cdot \beta \langle\beta \mid \alpha\rangle\langle\beta \mid \rho \mid \alpha\rangle \\
&\quad = \exp(-\mid\alpha\mid^2)\pi^{-1}\int d^2\beta \exp(\alpha^*\beta - \mid\beta\mid^2) \\
&\qquad\qquad \times\beta[\exp(\beta^*\alpha)\rho(\beta^*, \alpha)] \\
&\quad = \exp(-\mid\alpha\mid^2)(\partial/\partial\alpha^*)[\exp(\alpha^*\alpha)\rho(\alpha^*, \alpha)] \\
&\quad = (\alpha + \partial/\partial\alpha^*)\rho(\alpha^*, \alpha).
\end{aligned} \tag{9.11}$$

[Equation (9.7b) has been used to go from the second to the third line of (9.11).] In calculations of this nature it is essential that α^* be considered independent of α. (Off-diagonal elements $\langle\beta \mid \rho \mid \alpha\rangle$ are brought into the dynamics in this manner.)

It is not difficult to generalize the preceding calculation to the general matrix element

$$\begin{aligned}
\langle\alpha \mid A\rho \mid \alpha\rangle &= \pi^{-1}\int d^2\beta \mid \langle\alpha \mid \beta\rangle\mid^2 A(\alpha^*, \beta)\rho(\beta^*, \alpha) \\
&= \exp(-\mid\alpha\mid^2)\sum_{mn} A_{mn}(\alpha^*)^m \\
&\quad \times\int \frac{d^2\beta}{\pi} \exp(\alpha^*\beta - \mid\beta\mid^2)\beta^n[\exp(\beta^*\alpha)\rho(\beta^*, \alpha)] \\
&= \exp(-\mid\alpha\mid^2)\sum_{mn} A_{mn}(\alpha^*)^m \left(\frac{\partial}{\partial\alpha^*}\right)^n \\
&\qquad \times[\exp(\alpha^*\alpha)\rho(\alpha^*, \alpha)].
\end{aligned} \tag{9.12}$$

Thus, we find the simple result

$$\begin{aligned}
&\langle\alpha \mid A\rho \mid \alpha\rangle \\
&\quad = \exp(-\mid\alpha\mid^2)A(\alpha^*, \partial/\partial\alpha^*)[\exp(\alpha^*\alpha)\rho(\alpha^*, \alpha)].
\end{aligned} \tag{9.13}$$

From this result we can obtain a useful expression for the ensemble average of the operator A:

$$\begin{aligned}
\langle A\rangle = \mathrm{Tr}\,\rho A &= \pi^{-1}\int d^2\alpha \exp(-\mid\alpha\mid^2)A(\alpha^*, \partial/\partial\alpha^*) \\
&\qquad \times[\exp(\alpha^*\alpha)\rho(\alpha^*, \alpha)].
\end{aligned} \tag{9.14}$$

The method of integration outlined above can be used to convert all relevant physical quantities to operations on diagonal matrix elements. This is an attractive feature of the coherent state representation, which has not seemed simple or obvious in some previous treatments.

Further simplifications occur when ρ is a function of the number operator $a^\dagger a$. In this case $\rho(\alpha^*, \alpha)$ is a function of the product $\alpha^*\alpha$. Suppose the operator $B(a^\dagger, a)$ also depends on $a^\dagger a$ alone:

$$B = \sum_n b_n (a^\dagger a)^n. \tag{9.15}$$

Since B is diagonal in the number basis, we may use Eq. (3.24) to obtain

$$\begin{aligned}
\langle\alpha \mid B \mid \beta\rangle &= \sum_n b_n \sum_{k=0}^{\infty} \frac{k^n}{k!}(\alpha^*\beta)^k \exp[-\tfrac{1}{2}\mid\alpha\mid^2 - \tfrac{1}{2}\mid\beta\mid^2] \\
&\equiv B_1(\alpha^*\beta)\langle\alpha \mid \beta\rangle, \\
B_1(z) &= \sum_{n=0}^{\infty} b_n F_n(z); \qquad F_n(z) = e^{-z}\sum_{k=0}^{\infty}\frac{k^n z^k}{k!}.
\end{aligned} \tag{9.16}$$

Thus, if an operator B is a function of $a^\dagger a$, its reduced matrix element $\langle\alpha \mid B \mid \beta\rangle/\langle\alpha \mid \beta\rangle$ is a function of the product $\alpha^*\beta$.

B. Transition to Action-Angle Variables

In analogy to Sec. 5, an action J is defined to be the mean excitation number of the state $\mid\alpha\rangle$ and ϕ the corresponding phase

$$J = \alpha^*\alpha, \qquad \phi = (1/2i)\ln(\alpha/\alpha^*). \tag{9.17}$$

Noting that α^* is independent of α, we find

$$\begin{aligned}
\partial/\partial\alpha &= J^{1/2}e^{-i\phi}[(\partial/\partial J) - (i/2J)(\partial/\partial\phi)], \\
\partial/\partial\alpha^* &= J^{1/2}e^{i\phi}[(\partial/\partial J) + (i/2J)(\partial/\partial\phi)].
\end{aligned} \tag{9.18}$$

In terms of the action-angle variables, integration over the α plane becomes

$$\int d^2\alpha = \frac{1}{2}\int_0^\infty dJ \int_0^{2\pi} d\phi. \tag{9.19}$$

The most important (and simple) case is that of the free harmonic oscillator:

$$i(\partial\rho/\partial t) = [H_0, \rho] \equiv L_0\rho. \tag{9.20}$$

L_0 is the Liouville operator, defined by (9.20). H_0 is, of course, $\omega(a^\dagger a + \frac{1}{2})$. A short calculation shows that $\rho(\beta^*, \alpha)$ evolves as

$$i(\partial/\partial t)\rho(\beta^*, \alpha) = L_0(\beta^*, \alpha)\rho(\beta^*, \alpha), \tag{9.21}$$

where $L_0(\beta^*, \alpha)$ is the differential operator

$$L_0(\beta^*, \alpha) = \omega[\beta^*(\partial/\partial\beta^*) - \alpha(\partial/\partial\alpha)]. \tag{9.22}$$

Note that if, at $t=0$, $\rho(\beta^*, \alpha)$ is a function of $\beta^*\alpha$, then ρ is independent of time. This corresponds to the trivial remark that $[\rho(N), N]$ vanishes.

Since we only need to compute $\rho(\alpha^*, \alpha)$, Eqs. (9.21)–(9.22) greatly simplify to

$$i[\partial\rho(J, \phi)/\partial t] = L_0\rho(J, \phi), \qquad L_0 = i\omega(\partial/\partial\phi). \tag{9.23}$$

Rewriting Eq. (9.23) in the form

$$[(\partial/\partial t) - \omega(\partial/\partial\phi)]\rho(J, \phi) = 0, \tag{9.24}$$

one sees that any function of the form $\rho(J, \phi - \omega t)$

solves (9.24) exactly as in the case of the classical oscillator.

It is extremely interesting that the Liouville operator L_0 in (9.23) has a form similar to that naively conjectured for H on the basis of N and ϕ being conjugate variables [see the discussion following Eqs. (6.42)]. The implications of this result are discussed further in Sec. 9D.

Generalizing to an arbitrary number of independent oscillators of frequency ω_k, we have

$$L_0=\sum_k i\omega_k(\partial/\partial\phi_k). \tag{9.25}$$

The eigenfunctions of L_0 play an important role in the perturbation expansion of the density matrix, when interaction terms are added to H_0. These eigenfunctions are the same as in the classical analysis:

$$f\{\nu\}=(2\pi)^{-N/2}\exp\left(-i\sum_k\nu_k\phi_k\right),$$
$$L_0 f\{\nu\}=\left(\sum_k\nu_k\omega_k\right)f\{\nu\}. \tag{9.26}$$

Here the ν_k vary over all the positive *and* negative integers; the sum over k covers all N normal modes. The set $\{\nu\}$ should never be mistaken for the occupation numbers N_k; rather, the set $\{\nu\}$ specifies the phase of the oscillators.

The normalization in (9.26) has been chosen according to the inner product

$$(g\mid h)=\int g^*\{\phi\}h\{\phi\}\prod_k d\phi_k \tag{9.27}$$

so that the eigenfunctions $f\{\nu\}$ are normalized as follows:

$$(f\{\nu\}\mid f\{\nu'\})=\delta_{\{\nu\}\{\nu'\}}. \tag{9.28}$$

The Kronecker delta in (9.28) means that $\nu_k=\nu_k'$ for all k. We have used round brackets to denote inner products in the space of the eigenfunctions to avoid confusion with the usual quantum-mechanical inner products (written with sharp brackets).

Finally, we give some results for important functions of the number operator, when $\rho=\rho(a^\dagger a)$. Then $\rho(\alpha^*,\alpha)$ depends on J alone, so that

$$\langle N\rangle=\pi^{-1}\int d^2\alpha\,\alpha^*(\alpha+\partial/\partial\alpha^*)\rho(\alpha^*,\alpha)$$
$$=\int dJ\,J\rho(J)+\int dJ\cdot J(\partial/\partial J)\rho(J). \tag{9.29}$$

Integrating by parts on the second term, and applying the normalization condition gives

$$\langle N\rangle=\int dJ\cdot J\rho(J)-1. \tag{9.30}$$

Therefore, the statistical average of the action variable is

$$\langle\langle J\rangle\rangle=\int_0^\infty dJ\cdot J\rho(J)=\langle N\rangle+1. \tag{9.31}$$

A generalization may be proved:

$$\langle(a^\dagger)^n a^n\rangle=(-1)^n\int_0^\infty dJ\cdot L_n(J)\rho(J)$$
$$\equiv(-1)^n\langle\langle L_n(J)\rangle\rangle. \tag{9.32}$$

Here $L_n(J)$ is the Laguerre polynomial.[92]

For a harmonic oscillator in thermal equilibrium, the density matrix is

$$\rho=\exp(-\beta\omega a^\dagger a)/\mathrm{Tr}\exp(-\beta\omega a^\dagger a), \tag{9.33}$$

where $\beta=1/kT$. Thus, using Eqs. (9.15)–(9.16),

$$\rho(J)=\exp\{-J[1-\exp(-\beta\omega)]\}/[1-\exp(-\beta\omega)]^{-1}. \tag{9.34}$$

From this expression we obtain the expected results

$$\langle\langle J\rangle\rangle=\int dJ\cdot J\rho(J)=[1-\exp(-\beta\omega)]^{-1},$$
$$\langle N\rangle=\langle\langle J\rangle\rangle-1=[\exp(\beta\omega)-1]^{-1}. \tag{9.35}$$

C. Coupled Oscillators; Irreversibility

Suppose a system of oscillators is described by the Hamiltonian

$$H=H_0+H_{\mathrm{int}};\qquad H_0=\sum\omega_k(a_k^\dagger a_k+\tfrac{1}{2}). \tag{9.36}$$

When H_{int} has a structure such that it causes scattering of excitations among various modes in a continuous spectrum labeled by k in (9.36), irreversibility is generally the result. From the structure of the integration formulas (9.7), we see that (9.4) reduces to the following partial differential equation involving the action-angle variables J_k, ϕ_k:

$$i(\partial/\partial t)\rho(\{J_k,\phi_k\},t)=(L_0+L_I)\rho(\{J_k,\phi_k\},t). \tag{9.37}$$

L_0 is given by Eq. (9.23), while

$$L_I=L_I(\{J_k\},\{\partial/\partial J_k\};\{\phi_k\},\{\partial/\partial\phi_k\}) \tag{9.38}$$

depends on the specific problem at hand. (See Refs. 20 and 89 for detailed examples.)

Here we only indicate the formal perturbation solution of (9.37). First define the "interaction-picture" density matrix

$$\rho_I(\{J_k,\phi_k\},t)=\exp(-iL_0t)\rho(\{J_k,\phi_k\},t). \tag{9.39}$$

ρ_I satisfies the equation

$$i(\partial\rho_I/\partial t)=L_I(t)\rho_I,$$
$$L_I(t)=\exp(-iL_0t)L_I\exp(iL_0t). \tag{9.40}$$

The formal solution of (9.40) is

$$\rho_I(t)=T\exp\left(-i\int_{t_0}^t L_I(t')\,dt'\right)\rho_I(t_0), \tag{9.41}$$

where T is the usual time-ordering operator.

[92] *Higher Transcendental Functions*, A. Erdelyi, Ed. (McGraw-Hill Book Co., New York, 1953), vol. 1, p. 188.

$\rho_I(t)$ may be expanded in terms of the eigenfunctions of L_0:

$$\rho_I(t)=\sum_{\{\nu\}}\rho_I(\{J_k,\nu_k\},t)f\{\nu\}. \tag{9.42}$$

The expansion coefficients

$$\rho_I(\{J_k,\nu_k\},t)=(f\{\nu\}\mid\rho_I(\{J_k,\phi_k\},t))$$
$$=\int f^*\{\nu\}\rho_I(\{J_k,\phi_k\},t)\prod_k d\phi_k \tag{9.43}$$

resolve the density matrix into the phase functions $f\{\nu\}$. Inserting (9.39) into (9.43) yields

$$\rho_I(\{J_k,\nu_k\},t)$$
$$=\sum_{\{\nu'\}}\left(f\{\nu\}\mid T\exp\left[-i\int_{t_0}^{t}L_I(t')dt'\right]\mid f\{\nu'\}\right)$$
$$\times(f\{\nu'\}\mid\rho_I(\{J_k,\phi_k\},0)). \tag{9.44}$$

This relation still contains derivative terms of the type $\{\partial/\partial J_k\}$.

Rather than investigate all possible initial conditions, we content ourselves with the usual *initial random phase assumption*

$$\rho_I(\{J_k,\phi_k\},0)=0,\qquad\{\nu\}\neq\{0\}, \tag{9.45}$$

which follows if $\rho(\{J_k\phi_k\}t)$ is independent of ϕ_k at $t=0$. *It is remarkable that, given this one condition, the number distribution retains this property for all time.* As explained before, the independence of ρ from the phase angles $\{\phi\}$ only occurs if ρ is a function of the number operators $\{N_k\}$. This being true, the coefficient

$$\rho_I(\{J_k,\nu_k=0\},t)\equiv\rho_E(\{J_k\},t) \tag{9.46}$$

describes the energy distribution among the various normal modes.

Thus at time t, ρ_E is given by

$$\rho_E(\{J_k\},t)$$
$$=\left(f\{0\}\mid T\exp\left[-i\int_0^t L_I(t')dt'\right]\mid f\{0\}\right)\rho_E(\{J_k\},0), \tag{9.47}$$

where we have set $t_0=0$ for notational convenience. Quantities of the type

$$\left\langle T\exp\left(-i\int_0^t L_I(t')dt'\right)\right\rangle \tag{9.48}$$

often arise in statistical and quantum mechanics. A systematic technique may be used for the evaluation of such quantities.[93] In Eq. (9.47) we have to calculate the average of an "exponential" over the phase angles. As is well known, such an average can be expressed as the exponential of a modified series.

The method of semi-invariants (or cumulants), explained in Ref. 93, offers a rapid and powerful method for exhibiting this exponentiation. For clarity we state a few relations. Consider the moment generating function $\langle\exp(ixt)\rangle$, where the average is taken over some probability distribution $P(x)$. The logarithm of this quantity expands as

$$\ln\langle\exp(ixt)\rangle=\sum_{n=1}^{\infty}\frac{(it)^n}{n!}M_n, \tag{9.49}$$

where the first few semi-invariants are

$$M_1=\langle x\rangle,$$
$$M_2=\langle x^2\rangle-\langle x\rangle^2,$$
$$M_3=\langle x^3\rangle-3\langle x\rangle^2\langle x\rangle+2\langle x\rangle^3,$$
$$M_4=\langle x^4\rangle-4\langle x^3\rangle\langle x\rangle-3\langle x^2\rangle^2+12\langle x^2\rangle\langle x\rangle^2-6\langle x\rangle^4. \tag{9.50}$$

The case of a time-ordered exponential of an operator is not much more complicated.[93] In place of $\langle x^n\rangle$ we have

$$(-t)^n\langle x^n\rangle\rightarrow\left\langle T\left[\int_0^t dt'L_I(t')\right]^n\right\rangle. \tag{9.51}$$

Thus we find

$$\ln\left\langle T\exp\left(-i\int_0^t L_I(t')dt'\right)\right\rangle=\sum_{n=1}^{\infty}\frac{(-i)^nM_n(t)}{n!}, \tag{9.52}$$

$$M_1=\int_0^t\langle L_I(t')\rangle dt',$$

$$M_2=\int_0^t dt_1\int_0^t dt_2[\langle TL_I(t_1)L_I(t_2)\rangle-\langle L_I(t_1)\rangle\langle L_I(t_2)\rangle]. \tag{9.53}$$

From (9.47) and (9.52) we have

$$\rho_E(\{J_k\},t)=\exp\left[\sum_{n=1}^{\infty}\frac{(-i)^nM_n(t)}{n!}\right]\rho_E(\{J_k\},0). \tag{9.54}$$

We now observe that, for long times, the M_n become proportional to t. Also, the term M_1 is often zero, or if it is not, it can be eliminated by an energy renormalization of the ω_k. Thus, the second-order term often dominates (as when the perturbation is weak) and we obtain the Brout–Prigogine equation

$$\partial\rho_E(\{J_k\},t)/\partial t=\mathcal{O}_0\rho_E(\{J_k\},t), \tag{9.55}$$

$$\mathcal{O}_0=-\sum_{\{\nu\}}(f\{0\}\mid L_I\mid f\{\nu\})\delta_+(\textstyle\sum_k\nu_k\omega_k)(f\{\nu\}\mid L_I\mid f\{0\}),$$
$$\pi\delta_+(x)\equiv\pi\delta(x)+i(P/x). \tag{9.56}$$

Here P/x is the principal value.

The density matrix equation (9.55) can be used to derive master equations for the occupation numbers $\langle N_k\rangle$. When spatial inhomogeneities are involved, more complicated components of the density matrix must be used.[21,22,89]

[93] R. Brout and P. Carruthers, *Lectures on the Many-Electron Problems* (Interscience Publ. Co., Inc., New York, 1962), Chap. 1.

D. Uncertainty Relation between Angle and Liouville Operator

The equation of motion for the density matrix

$$i(\partial\rho/\partial t)=L\rho\equiv[H,\rho] \tag{9.57}$$

is formally the same as the Schrödinger equation

$$i(\partial\Psi/\partial t)=H\Psi. \tag{9.85}$$

However, the Liouville operator L is not necessarily bounded on one side, so that the possibility of "phase operators" arises.

We reconsider the simple harmonic oscillator. In the coherent state basis the Liouville operator acting on $\rho(\alpha^*,\alpha)=\rho(J,\phi)$ has the form (9.23). L_0 is Hermitian in the space of periodic functions. Moreover, we have the commutator

$$[l,f(\phi)]=i(\partial f/\partial\phi),\qquad l\equiv L_0/\omega, \tag{9.59}$$

where f is any differentiable function of ϕ. As a special case of (9.59) we have

$$[l,\phi]=i. \tag{9.60}$$

There is no trouble with the existence of ϕ since the spectrum of L_0 runs from $-\infty$ to $+\infty$:

$$L_0 f(\nu)=\nu\omega f(\nu),\qquad f(\nu)=(2\pi)^{-1/2}\exp(-i\nu\phi),$$
$$\nu=0,\pm1,\pm2,\cdots. \tag{9.61}$$

Repeating the analysis of Sec. 4 we find uncertainty relations connecting L_0 and trigonemetric functions of phase:

$$(\Delta L_0)^2(\Delta\sin\phi)^2\geq(1/4)\omega^2\langle\cos\phi\rangle^2,$$
$$(\Delta L_0)^2(\Delta\cos\phi)^2\geq(1/4)\omega^2\langle\sin\phi\rangle^2. \tag{9.62}$$

These expectation values pertain to the space spanned by the eigenfunctions $f(\nu)$ of Eq. (9.61), expect for the special qualifications needed when the quantities in (9.62) are computed with one of the eigenfunctions themselves.

10. ANALOGIES TO SUPERFIELD SYSTEMS

The qualitative description of the physical behavior of various superfluids depends to a large extent on a proper understanding of phase variables.[23] We have seen that the traditional approach, which regards number and phase as conjugate variables, lacks precise meaning for small quantum numbers. In this section we wish to draw attention to similarities between approximate relations used by Anderson[23] to elucidate superfluid behavior and rigorous relations which hold for Bose operators in coherent states. We do not wish to imply that the wave functions for superfluids *are* coherent states of the type written down, although this possibility has been discussed.[55] However, the analogies are so striking that the true wave functions probably have a similar structure.

Consider a Bose system described by the nonrelativistic field operator

$$\psi(x,t)=\sum_k a_k(t)\exp(i\mathbf{k}\cdot\mathbf{x}). \tag{10.1}$$

The a_k obey the usual commutation rules and the $\psi(x)$ satisfy

$$[\psi(\mathbf{x},t),\psi^\dagger(\mathbf{x}',t]=\delta(\mathbf{x}-\mathbf{x}') \tag{10.2}$$

for equal times. For any *number* state the expectation value of (10.1) vanishes. However, as discussed by Anderson,[23] the observed behavior indicates that the true state has expectation value

$$\langle\psi(\mathbf{x},t)\rangle=f(\mathbf{x},t)\exp[i\phi(\mathbf{x},t)]. \tag{10.3}$$

$[f(\mathbf{x},t)]^2$ is the particle density $\rho(\mathbf{x},t)$. Thus, writing

$$\langle\psi(\mathbf{x},t)\rangle=\rho^{1/2}\exp(i\phi) \tag{10.4}$$

is reminiscent of the coherent state of an oscillator

$$\langle\alpha\mid a\mid\alpha\rangle=N^{1/2}\exp(i\phi). \tag{10.5}$$

In fact, we can obtain a completely arbitrary expectation value of $\psi(\mathbf{x})$ using the product coherent state

$$\mid\{\alpha_k\}\rangle=\prod_k\mid\alpha_k\rangle. \tag{10.6}$$

Thus, the α_k are the Fourier coefficients of $\langle\psi\rangle$. However, the preceding is completely devoid of dynamical content.

We consider the equation of motion for a destruction operator of a single mode:

$$i(da/dt)=[a,H]. \tag{10.7}$$

Here H is an arbitrary Hamiltonian. Taking the expectation value of (10,7) in the coherent state α and using the integration formulas of Sec. 9A gives

$$i(d\alpha/dt)=\partial H(\alpha^*,\alpha)/\partial\alpha^*. \tag{10.8}$$

Using Eqs. (9.17)–(9.18) to convert to action-angle variables we find

$$dJ/dt=\partial H(J,\phi)/\partial\phi,\qquad J=\langle N\rangle,$$
$$d\phi/dt=-\partial H(J,\phi)/\partial J. \tag{10.9}$$

These equations play an important role in Anderson's treatment of superfluids.

Although the preceding equations are not literally applicable to real superfluids, we believe such coherent systems will eventually provide the most interesting application of the ideas expressed in this paper. Indeed, using the phase difference operators of Sec. 7, one of the authors has recently proposed a quantitized phase effect for dc Josephson tunneling.[94]

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[94] M. M. Nieto, Phys. Rev. **167**, 416 (1968).