Collapse Time for the Bohm–Bub Hidden Variable Theory

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Introduction. In a recent paper by Bohm and Bub[1] deterministic equations of motion describing the collapse of the wave packet in a nonrelativistic theory, involving hidden variables, were put forth in order to show that such a theory could logically exist.[2,3] This concern of theirs—the possible existence of a hidden variable theory—has been the subject of many recent papers. The reader may refer to the literature for the arguments for and against the existence of such a theory.[4] It does not seem that the construction given by Bohm and Bub is ruled out by the nonexistence proofs of J. von Neumann or Jauch and Piron. Exactly how the proofs which use the abstract logic of quantum mechanics relate to the Bohm–Bub theory is difficult to see for several reasons. Membership in the question system is hard to decide and this point is critical with respect to claims of circularity in Jauch and Piron's arguments. The concept of a logical model admitting hidden variables is defined without the hidden variables themselves being defined. Hence it is not clear that the various authors are referring to the same things when they use such terms as "hidden variables" and "theory with hidden variables." Though the collapse equations given by Bohm and Bub are coupled, nonlinear, differential equations, they can be solved for a collapse time in the two-dimensional case (and also in the n-dimensional case, if appropriate initial conditions are chosen) by making use of the normalization condition on ψ. Having found the collapse time (call it τ_c), it can be compared with the Δt given by the time–energy uncertainty relation ($\Delta t \Delta E \geq \hbar/2$). If $\tau_c < \Delta t$, the hidden variables must remain hidden or else we have a contradiction with ordinary quantum mechanics. On the other hand, if $\tau_c > \Delta t$, one may be able to devise experiments which detect the hidden variables.

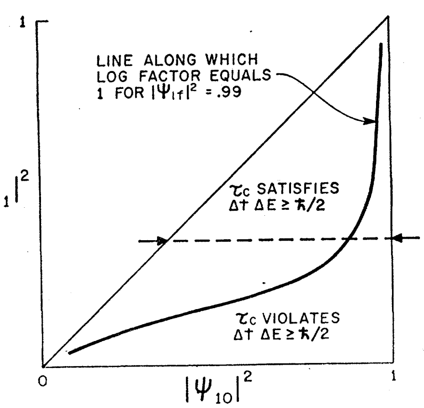

FIG. 1. Domain of the log factor.

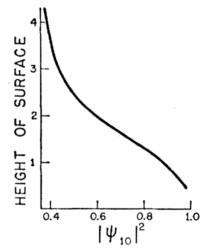

FIG. 2. Cross section of the surface given by the log factor at the dotted line in Fig. 1.

Collapse time vs randomization time. In the paper by Bohm and Bub[5] an *ad hoc* randomization time of h/kT for the hidden variables is suggested. An experiment on the polarization of light by Papaliolios[6] showed this estimate to be too large by a factor of at least 75. Since the collapse time is connected to the Bohm–Bub theory in a fundamental way and the randomization time is not, the outcome of the experiment is of little significance. The relationship of a thermodynamical variable such as T to a system of subquantal variables is difficult to envision.

Solution of the collapse equations. Let $R_i = |\psi_i|^2 / |\xi_i|^2$, $J_i = |\psi_i|^2$

$$\sum_1^n |\psi_i|^2 = 1 \text{ at } t=0.$$

Suppose $\sum_1^n |\xi_i|^2 = 1$ for all t.

Consider the following systems of equations:

(a)

$$\frac{d\psi_i}{dt} = \gamma\psi_i \sum_1^n J_k(R_i - R_k) - \frac{i}{\hbar}\sum_1^n H_{ik}\psi_k;$$

(b)

$$\frac{dJ_i}{dt} = 2\gamma J_i \sum_1^n J_k(R_i - R_k);$$

(c)

$$\frac{dR_i}{dt} = 2\gamma R_i \sum_1^n |\xi_k|^2 R_k(R_i - R_k).$$

Equation (c) is the Bohm–Bub collapse equation where the ξ_k's are the complex hidden variables. γ and the $|\xi_k|^2$'s are assumed to be constant in time—or at least while the differential equations are operative.

Lemma 1: (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow \sum_1^n |\psi_i|^2 = 1$ for all t.

Proof: Using system (a), the fact that

$$\sum_1^n J_k(R_i - R_k)$$

is independent of position and that the measured quantity is conserved (i.e., H is diagonal and Hermitian in the appropriate representation), one sees that

$$\frac{dJ_i}{dt}=\frac{d}{dt}\int \psi_i\psi_i^* d^3x,$$

$$=\int \frac{d\psi_i}{dt}\psi_i^* d^3x+\int \psi_i\frac{d\psi_i^*}{dt}d^3x,$$

$$=\int \gamma\psi_i^*(\sum_1^n J_k(R_i-R_k))\psi_i d^3x$$

$$+\int \psi_i^*\gamma\psi_i(\sum_1^n J_k(R_i-R_k))d^3x$$

$$+\int \frac{i}{\hbar}\sum H_{iK}^*\psi_k^*\psi_i-\frac{i}{\hbar}\int \psi_i^*\sum H_{ik}\psi_k,$$

$$=2\gamma J_i\sum_1^n J_k(R_i-R_k)$$

$$+\frac{i}{\hbar}\sum_1^n\left(\int H_{ik}^*\psi_k^*\psi_i-\int \psi_i H_{ik}\psi_k\right),$$

$$=2\gamma J_i\sum_1^n J_k(R_i-R_k),$$

which is system (b). Notice that we can get the same result by dropping the assumption on H and, instead, assuming an impulsive measurement as Bohm and Bub do in their paper. Now (b) implies (c) since we have taken the $|\xi_k|^2$'s to be constant during the measurement process. Finally,

$$\frac{d}{dt}(\sum_1^n|\psi_i|^2)=\sum_1^n\frac{d|\psi_i|^2}{dt}=\sum_1^n\frac{dJ_i}{dt},$$

$$=\sum_1^n 2\gamma J_i\sum_k J_k(R_i-R_k),$$

$$=2\gamma\sum_{i,k}^n J_iJ_k(R_i-R_k)=0.$$

[N.B. The solution to system (c) lies in an $(n-1)$ dimensional hyperplane through $(1,\cdots,1)$ in $(R_1,\cdots,R_n)$ space. This is easy to see since

$$\sum_1^n|\xi_i|^2R_i=1$$

for all t by the above lemma.] Now let

$$\boldsymbol{\xi}=(|\xi_1|^2,\cdots,|\xi_n|^2),$$

$$\mathbf{R}=(R_1,\cdots,R_n),$$

$$(f)_i=2\gamma R_i\sum_1^n|\xi_k|^2R_k(R_i-R_k).$$

Thus system (c) is $d\mathbf{R}/dt=\mathbf{f}(t,R,\xi)$, and Lemma 1 says $\mathbf{R}\cdot\boldsymbol{\xi}=1$ for all t. Notice that $\mathbf{f}$ is a third-degree polynomial in the R_i's so $\partial\mathbf{f}/\partial R_i$ is of second degree and both are continuous in the R_i's. Now suppose $|\xi_i|^2\neq0$, $i=1,2,\cdots,n$, and let $R_i\neq R_j$ for all $i\neq j$ and $R_i=\max\{R_1,\cdots,R_n\}$ at $t=0$. Then $dR_i/dt>0$ so R_i increases and remains $\max\{R_1,\cdots,R_n\}$. Now

$$0<R_i=(1/|\xi_i|^2)(1-\sum_{j\neq i}|\xi_j|^2R_j)\leq(1/|\xi_i|^2).$$

Hence the terms in $\mathbf{f}$ and $\partial\mathbf{f}/\partial R_i$ are bounded in the R_k's. Also $\mathbf{f}$ is bounded in the $|\xi_k|^2$'s if they are bounded away from zero. From the above observations one can conclude the following.

(1) There exists a unique solution to system (c) for any initial conditions compatible with

$$\sum_1^n|\psi_i|^2=\sum_1^n|\xi_i|^2=1.$$

(2) The solutions are continuous in the initial conditions.

(3) The solutions are continuous in the parameters $|\xi_k|^2$.[7]

The main question is: Under what initial conditions and conditions on the parameters $|\xi_k|^2$ are the limits reached most slowly and how long does it take?

Cases:

(a) If $R_1=1$, $R_2=R_3=\cdots=R_n=0$ at $t=0$, then "$R=1$, $R_2=\cdots=R_n=0$ for all t" is a solution, since $R_i\sum|\xi_k|^2R_k(R_i-R_k)=0$ for all i. This solution is unique by the above result.

(b) If $R_1=R_2=\cdots=R_i=0$, $R_{i+1},\cdots,R_n\neq0$ at $t=0$, then, since $dR_j/dt\propto R_j$, we can take the solution for $R_1,\cdots,R_i$ to be identically zero. And since the terms involving $R_1,\cdots,R_i$ drop out of the other equations, we can treat the problem as an $(n-i)$-dimensional case.

(c) If $R_1=R_2=\cdots=R_n$ at $t=0$, then, since the sums involve differences, they are all zero. Therefore the unique solution is $R_1=\cdots=R_n=\text{constant}=1$. Notice that these initial conditions do not imply $|\psi_i|^2=|\psi_j|^2$ for all i and j, but do imply that $|\psi_i|^2$ $i=1,2,\cdots,n$ is stuck at its initial value if the $|\xi_i|^2$'s are constant.

(d) The nontrivial case is $R_1\geq R_2\geq\cdots\geq R_n$, where none of the R's are zero and they are not all equal at $t=0$. This case is discussed below. (The order of subscripts results in no loss of generality since we can relable.)

Lemma 2:

(a) $R_i>R_j$ at $t=0\Rightarrow R_i>R_j$ for all t;

(b) $R_i=R_j$ at $t=0\Rightarrow R_i=R_j$ for all t.

Proof:

(a) Suppose there is a $t_1>0$ such that $R_j(t_1)\geq R_i(t_1)$. Then since the R's are continuous functions of t, there is a t^* such that $R_j(t^*)=R_i(t^*)$. Now take t^* as the initial time. By the existence and uniqueness of the solutions, we know that $R_j = R_i$ for all t, which is a contradiction.

(b) This case follows from part (a).

Hence any ordering of the R_i's is strictly preserved by the system of collapse equations. Now due to the continuity in initial conditions, the worst (slowest) time behavior can be expected when $R_1 > R_2 = \cdots = R_n$ at $t=0$ [see case (c) above].

Theorem: The slowest collapse time occurs when $R_1 > R_2 = \cdots = R_n$ at $t=0$ and is given by

$$t = (2\gamma)^{-1} \ln\left[\left(\frac{R_{10}}{R_1}\right)^{1-|\xi_1|^2} \left(\frac{1-|\xi_1|^2 R_{10}}{1-|\xi_1|^2 R_1}\right)^{|\xi_1|^2} \left(\frac{R_1-1}{R_{10}-1}\right)\right], \tag{7}$$

where $R_{10} = |\psi_1(t=0)|^2/|\xi_1|^2$.

Proof:

$$\frac{dR_1}{dt} = 2\gamma R_1 \sum_{k=2}^{n} |\xi_k|^2 R_k (R_1 - R_k).$$

Using Lemmas 1, 2, and the normalization condition on the ξ, we get

$$R_2 = \cdots = R_n = (1-|\xi_1|^2 R_1)/(1-|\xi_1|^2) \quad \text{for all } t.$$

Therefore

$$dR_1/dt = -[2\gamma R_1/(1-|\xi_1|^2)](1-|\xi_1|^2 R_1)(1-R_1)$$

which gives the result by an integration.

Notice that the above result is independent of n, so it gives the $n=2$ and $n=\infty$ cases without change. The $n=2$ case involves no special initial conditions since the normalization constraint is enough to uncouple two equations. Also the implicit solution is unique. What one calls the collapse time is somewhat arbitrary. I take τ_c to be the time at which $|\psi_1|^2 = 0.99$, so τ_c is got from the above by setting $R_1 = 0.99/|\xi_1|^2$ in Eq. (1).

In a recent paper by R. K. Wangsness[8] a solution to the spin $\frac{1}{2}$ case is gotten by assuming $2\gamma(R_1 - R_2) =$ constant. It is easy to see that this implies $4\gamma^2(R_1 - R_2) R_1 R_2 (|\xi_1|^2 + |\xi_2|^2) = 0$, which leads only to trivial cases (see above). Hence this supposition is not valid. In the above I have taken $\gamma \equiv$ constant, since letting γ be "nearly" constant is too ill-defined mathematically to yield a solution.

Physical interpretation of γ. In the paper by Bohm and Bub the nature of γ is left rather arbitrary. Since the collapse equations involve the particular experiment in a rather weak way, (the ψ_i's are eigenfunctions of the observable being measured,) it is necessary to put some of the actual physics into the γ involved. One could interpretate γ as a new fundamental constant which couples the subquantum level to the quantum level and quantizes time. Instead I choose to let

$$\gamma = \Delta E/\hbar = 1/\Delta t,$$

where

$$\Delta t = \min_{s} \left\{ \frac{\Delta A}{|d\langle A\rangle/dt|} \right\},$$

where $s = \{A \mid A$ is a Hermitian operator relevant to the given experiment$\}$,[9] and $(\Delta A)^2$ is the variance of A while $\langle A\rangle$ is the expected value of A. This choice, which is made simply by dimensional analysis, allows a comparison with the time–energy uncertainty relation. So the collapse time is or is not in contradiction with ordinary quantum mechanics, contingent on whether or not the log factor in Eq. (1) is smaller than 1. The line along which the log factor is equal to 1, for the $|\psi_1|^2 = 0.99$ case, is given in Fig. 1. In the part of the triangle below this line we get a violation of $\Delta t \Delta E \geq \hbar/2$. Hence we would not expect to observe hidden variables for these choices of initial conditions. As $|\psi_1|^2 \to 1$, the collapse time satisfies $\Delta t \Delta E \geq \hbar/2$ throughout the triangle. This can be seen by looking at the log factor analytically or else by computation. In Fig. 2 we have a cross section of the surface given by the log factor along the dotted line shown in Fig. 1. Notice that the height of the surface does not get far from one, so this interpretation of γ would make it difficult to detect any effects of the hidden variables. It should be noticed that the τ_c involved in the experiment by Popaljolios would be on the order of 10^{-15} sec, (taking $\Delta E = h\nu$) which is less than the times measured by a factor of 1.3×10^{-2}.

Conclusion. It is clear from the Bohm–Bub construction of the collapse equations that the admission of the ξ_i's as hidden variables does not violate the uncertainty relations for noncommuting observables because the equations depend on the fact that the measurements are not made at the same time. The time–energy relation, however, is of a somewhat different nature, both mathematically and epistemologically. The above results show that the hidden variables need not remain hidden, and it is hoped that the validity of the Bohm–Bub theory may be tested in the laboratory.

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