example of Bell.[9] As Misra has pointed out in particular[8] the quest for hidden variables becomes a meaningful scientific pursuit only if states, even physically nonrealizable states, are restricted by physical considerations.

The example of Bell was useful, because it shows that one of the hypotheses of our theorem [condition (4)°] was not only sufficient but also necessary for the affirmation of the theorem.

A similar remark applies to the example of Bohm and Bub.[10] Here we have the additional objection that they postulate a modification of the evolution of states during the process of measurement. This means according to them that all systems evolve with a Schrödinger equation except those which constitute a measurement.

It is contrary to good scientific methodology to modify a generally verified scientific theory for the sole purpose of accomodating hidden variables.

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Hidden Variables in Quantum Mechanics Reconsidered

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Recently D. Bohm and J. Bub[1] published a refutation of Jauch and Piron's proof[2] that hidden variables can be excluded in quantum mechanics. The objections upon which this refutation is based can, to a large extent, be overcome. This is accomplished by a change in terminology, a reinterpretation of some of the physical concepts, and a weakening of the axiomatic model used by Jauch and Piron. In any case, whether or not the reader agrees with these new interpretations, this theory extends the class of models for which hidden variables are excluded, beyond those considered by Jauch and Piron.

In the author's opinion the main difficulty encountered in hidden variable arguments is an incorrect phrasing of the problem. One should not ask whether a physical system admits hidden variables or not, but only if a particular model used to describe the system admits hidden variables. It is conceivable that there are many mathematical models for a physical system, some admitting hidden variables and some not. The problem then becomes that of finding which model most closely describes the physical situation. In this note it is shown that a quite large class of mathematical models do not admit hidden variables.

However, it has been demonstrated in the papers of Bell[3], and Bohm and Bub[4] that there are mathematical models for quantum mechanics which do admit hidden variables. The question then is whether the additional complications introduced by these models are justified in terms of new results or predictions not obtainable by the use of simpler models. Thus the author agrees with Bell, Bohm, and Bub to the extent that hidden variables cannot be excluded from quantum mechanics in an absolute sense, but only as far as certain mathematical models are concerned. On the other hand, the author has attempted to show that their objections concerning Jauch and Piron's specific model can be overcome.

Bohm and Bub seem to have three basic objections to Jauch and Piron's proof. The first is the use of the word "proposition" to denote the experimental questions concerning a physical system and the confusion this causes because of its similarity to the logic of thought processes. The second is the interpretation of the concept of compatibility of quantum propositions. The third objection is directed against Jauch and Piron's Axiom (4) [or its weakened form (4°)] which says that it two quantum propositions a and b are true with certainty in some state, then the proposition "a and b" is true with certainty in that state. The first two objections can be overcome by a change of terminology and interpretation. The third, however, seems to be much more serious. As is pointed out by Bohm and Bub,[5] if a and b are incompatible, the quantum proposition "a and b" is in many cases the absurd proposition; and this implies that the quantum proposition a and the quantum proposition b can never be true with certainty in the same state. Bohm and Bub demonstrate that this need not happen in all physical cases.[6] (Bell also objects to this postulate in his recent paper.[3]) This third objection is eliminated by weakening the axiomatic model so that the quantum proposition "a and b" need not exist at all.

The Axiomatic Structure. Let S be a physical system upon which we make laboratory experiments. Let $Q_0 = \{a, b, c, \cdots\}$ be the set of *experimental questions* that can be asked concerning the system S. To be quite explicit, a is an experimental question concerning S if it is possible to construct a definite laboratory experiment (or collection of experiments) on S, the outcome of which is capable of giving both a YES and a NO answer to a. This is a slightly different interpretation than is usually given. Experimental questions are usually assumed to be any meaningful questions one may ask concerning the system. Here we insist that the questions be experimentally answerable by an actual, definite experiment or collection of experiments. At the risk of sounding philosophical, it seems reasonable that if a question is not experimentally answerable, it is useless for a physicist to ask it. We next assume that there are two idealized questions 0, and 1. 0 is the question which has a NO answer for every experiment which answers it, and 1 is the question which has a YES answer for every experiment which answers it. Notice that 0 and 1 are not in Q_0. We let Q denote Q_0 together with 0 and 1, and call Q a *question system.*

We now postulate properties that Q should possess. If whenever a has a YES answer b also has a YES answer, we write $a \leq b$ and say "*a precedes b.*" Evidently Q should be a *partially ordered set* under $\leq$ with *first* and *last* elements 0 and 1, respectively: that is,

(Q1) $a \leq a$ for all $a \in Q$;

(Q2) if $a \leq b$ and $b \leq c$, then $a \leq c$;

(Q3) if $a \leq b$ and $b \leq a$, then $a = b$;

(Q4) $0 \leq a \leq 1$, for all $a \in Q$.

The physical interpretation of the equal sign in (Q3) is that, as far as our experiments are concerned, a and b are indistinguishable.

Let $a \wedge b$ be the question: Is the answer to *a and b* YES? Now it is possible that $a \wedge b$ is an *experimental* question: that is, there might be an experiment capable of giving both a YES and a NO answer to $a \wedge b$. For instance, one might be able to carry out an experiment answering a and another experiment answering b which do not interfere with each other, thus giving an answer to $a \wedge b$. However—and this is the point at which this model differs from most others—we do not *assume* that $a \wedge b$ necessarily exists as an element of Q. Similarly, we we define $a \vee b$ as the question: Is the answer to *a or b* YES? And again we do not assume that necessarily $a \vee b \in Q$. The following postulates should be intuitively obvious.

(Q5) If $a \wedge b \in Q$, then $a \wedge b \leq a$, b; and if $c \leq a$, b, then $c \leq a \wedge b$. (And the dual property holds for $a \vee b$.) If $\{a_\alpha : \alpha \in A\} \subset Q$, we define $\wedge a_\alpha$, $\vee a_\alpha$, if they exist, in the obvious way.

We now postulate a map $a \rightarrow a'$ from Q into Q which satisfies:

(Q6) $(a')' = a$, for all $a \in Q$;

(Q7) If $a \leq b$, then $b' \leq a'$;

(Q8) $a \vee a' = 1$ for all $a \in Q$.

The experimental question a' is interpreted as the experimental question which has a YES answer whenever a has a NO answer. Evidently a and a' are answered by the same experiment. If $a \leq b'$, we say a and b are *disjoint*, and we write $a \perp b$. We say that a and b are *compatible*, and write $a \leftrightarrow b$, if there are mutually disjoint elements a_1, b_1, $c \in Q$ such that $a_1 \vee c$, $b_1 \vee c \in Q$ and $a = a_1 \vee c$, $b = b_1 \vee c$. If a and b are experimental questions, answering a may interfere with answering b. If this is not the case, then a and b are supposed to be described by compatible elements in Q. We now indicate briefly why this is so. Disjoint elements are the most obvious kind of noninterfering experimental questions,[7] since, to get a YES answer for one, all we need is a NO answer for the other; thus both experimental questions may be answered using the same experiment; no interference results. The same would be true for three mutually disjoint elements a_1, b_1, and c, and thus $a_1 \vee c$ and $b_1 \vee c$ do not interfere. Conversely, if a and b do not interfere, then $a \wedge b$ should exist as an element of Q and so should $a \wedge (a \wedge b)'$ and $b \wedge (a \wedge b)'$. Now evidently $a \wedge b$, $a \wedge (a \wedge b)'$, and $b \wedge (a \wedge b)'$ are mutually disjoint elements: $a = [a \wedge (a \wedge b)'] \vee (a \wedge b)$ and $b = [b \wedge (a \wedge b)'] \vee (a \wedge b)$; thus a and b are compatible according to our definition.

If a and b are not compatible, one might not expect $a \wedge b$ and $a \vee b$ to exist as elements of Q. Nevertheless, in certain cases (for instance, when the interaction is not too great or the experimental questions not very precise) we may have $a \wedge b$ or $a \vee b$ existing for incompatible a and b. For example, suppose S is the system consisting of a particle p moving in the positive x direction away from the origin 0_0. We might consider the following experimental questions: (a) Is p at the point 10 cm from 0_0; (b) Is the x momentum component of p 10 g cm/sec? Now $a \wedge b$ does not exist in Q_0 since, by the Heisenberg uncertainty principle, there is no experiment capable of answering $a \wedge b$ affirmatively. Also $a \wedge b \neq 0$, since the answer to $a \wedge b$ is not *always* NO for every relevant experiment. Thus $a \wedge b \notin Q$. However, for a less precise question [(c) Is p between 2 and 3 cm from 0_0; (d) Is the x momentum component of p between 1 and 2 g cm/sec?] we see that $c \wedge d$ is an experimental question.

Now it is a well-known and universally accepted fact that in any nontrivial quantum-mechanical system there are processes which interfere with each other. This, of course, is one of the basic differences between quantum and classical mechanics, and even Bohm and Bub[8] accept this. Since we *can* ask experimental questions about such processes, it follows that there *are* experimental questions which interfere with each other, and thus are described by incompatible elements of Q in our mathematical model.

We now give our last two axioms for Q.

(Q9) If $\{a_\alpha : \alpha \in A\}$ are mutually compatible elements in Q, then $\vee a_\alpha$ and $\wedge a_\alpha$ exist in Q.

(Q10) If $a \leq b$, then there is a $c \in Q$ such that $a \perp c$ and $b = a \vee c$.

(Q9) would be obvious if it concerned a finite number of elements. We postulate it for an arbitrary set of elements as a concession to the mathematics. (We need it to prove our theorems.) This additional assumption seems harmless since the a_α are mutually compatible. (Q10) corresponds to Jauch and Piron's[2] Axiom P and is equivalent to the physically intuitive statement that if $a \leq b$, then $a \leftrightarrow b$.

Now the condition of the system S is determined if we know the probability that any experimental question has of getting an affirmative answer. Thus the condition of S may be given by a function m on Q called a *state* which satisfies these conditions:

($M1$) $0 \leq m(a) \leq 1$ for every $a \in \text{Q}$;

($M2$) $m(1) = 1$;

($M3$) $m(a \vee b) = m(a) + m(b)$ if $a \perp b$.

If m_i is a sequence of states and $\lambda_i \geq 0$, $i = 1, 2, \cdots$, and $\sum \lambda_i = 1$, we define the state $m = \sum \lambda_i m_i$ by $m(a) = \sum \lambda_i m_i(a)$ for every $a \in \text{Q}$. We call m a *mixture* of the states m_i. The pair (Q, M) is called a *quantum system* if Q is a question system satisfying (Q1)–(Q10) and M is a set of states on Q which is closed under mixtures and satisfies these conditions:

($M4$) If $a \neq 0$, there is an $m \in M$ such that $m(a) = 1$;
($M5$) if $m(a) = m(b) = 1$, $m \in M$, and if $a \wedge b$ exists, then $m(a \wedge b) = 1$;
($M6$) if $\{a_\alpha : \alpha \in A\}$ are mutually compatible and $m(a_\alpha) = 1$ for all $\alpha \in A$, $m \in M$, then $\text{m}(\wedge a_\alpha) = 1$.

Notice that ($M5$) corresponds to (4°) and ($M6$) is a weakened version of (4) in Ref. 2.

Dispersion-Free States and Hidden Variables. A state m is dispersion-free if m has only the values 0 and 1. In a dispersion-free state our system is completely and precisely determined. That is, every experimental question has a YES or NO answer with certainty, all probabilistic considerations having vanished. When such states exist, this is exactly what we mean physically by a system admitting hidden variables. We thus say that Q *admits hidden variables* if there is a set of states M which are mixtures of dispersion-free states such that (Q, M) is a quantum system. For those who may be bothered by the fact that we are allowing mixtures of dispersion-free states, we may—if we desire—omit mixtures: that is, it is easily shown that Q admits hidden variables if and only if there is a set M of dispersion-free states which satisfies ($M4$), ($M5$), and ($M6$).

Before we present the "hidden variables theorems," we need a few mathematical definitions. The *center* Z of Q is the set of experimental questions which are compatible with all experimental questions. Q is *coherent*, a *Boolean algebra*, or *trivial* if $Z = \{0, 1\}$, $Z = \text{Q}$, or $\text{Q} = \{0, 1\}$, respectively. An element $a \in \text{Q}$ is an *atom* if $a \neq 0$ and $b \leq a$ implies b is a or 0. Q is *atomic* if every nonzero element in Q is preceded by an atom.

Theorem 1. A quantum system (Q, M) has a dispersion-free state if and only if Q has an atom in its center.

Sketch of proof. To prove necessity, let m be a dispersion-free state and let $\text{Q}_m = \{a \in \text{Q}: m(a) = 1\}$. Using Zorn's lemma, one can show that Q_m has a minimal element a_1: i.e., if $b \in \text{Q}_m$ and $b \leq a_1$, then $b = a_1$. One next shows that $a_1 \leq a$ for all $a \in \text{Q}_m$. Now one can show that $a_1 \in Z$ and that a_1 is an atom. To prove sufficiency let $a_1 \in Z$ be an atom and let $m(a_1) = 1$. It now follows that m is dispersion-free.

Corollary. A nontrivial coherent quantum system has no dispersion-free states.

Theorem 2. A question system Q admits hidden variables if and only if Q is an atomic, Boolean algebra.

Sketch of proof. Suppose Q admits hidden variables. If $a \neq 0$, there is a dispersion-free state m such that $m(a) = 1$. As in the proof of Theorem 1, there is an atom $a_1 \in Z$ such that $a_1 \leq a$, and hence Q is atomic. To show that Q is a Boolean algebra let $a, b \in \text{Q}$ and let $\{a_\alpha : \alpha \in A\}$ and $\{b_\beta : \beta \in B\}$ be the atoms preceding a and b, respectively. Let $a_1 = \vee \{a_\alpha : a_\alpha \leq b'\}$, $b_1 = \vee \{b_\beta : b_\beta \leq a'\}$ and $c = \vee \{a_\alpha : a_\alpha \leq b\} = \vee \{b_\beta : b_\beta \leq a\}$. One now sees that a_1, b_1, and c are mutually disjoint, and that $a = a_1 \vee c$ and $b = b_1 \vee c$. Conversely, if Q is an atomic, Boolean algebra, let $\{a_\alpha : \alpha \in A\}$ be the atoms in Q. Define the state m_α by $m_\alpha(a) = 1$ if $a_\alpha \leq a$; otherwise let $m_\alpha(a) = 0$. Then $M = \{m_\alpha : \alpha \in A\}$ is a collection of dispersion-free states which satisfies ($M4$), ($M5$), and ($M6$), and thus Q admits hidden variables.

Now a coherent question system is one which has no superselection rules,[9] and thus the corollary to Theorem 1 tells us that a question system Q which has no superselection rules cannot have even one dispersion-free state. This is much stronger than the statement that Q does not admit hidden variables. As we have pointed out, any question system corresponding to a truly quantum-mechanical situation has incompatible experimental questions; thus Theorem 2 implies that no physically interesting quantum-mechanical question system admits hidden variables.

In the usual formulation of classical mechanics the question system Q is taken to be the subsets of phase space, and thus Q is an atomic, Boolean algebra. Applying a representation theorem for atomic, Boolean algebras, one can conclude (from Theorem 2) that the only question systems which admit hidden variables are those in classical mechanics.

[1] D. Bohm and J. Bub, Rev. Mod. Phys. **38**, 470 (1966).
[2] J. M. Jauch and C. Piron, Helv. Phys. Acta **36**, 827 (1963).
[3] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966). This reference together with those in the two previous footnotes contain a fairly complete list of the works in this area.
[4] D. Bohm and J. Bub, Rev. Mod. Phys. **38**, 453 (1966).
[5] Reference 1, page 472.
[6] Reference 1, page 475.
[7] When we speak of "interfering experimental questions" we do do not, of course, mean that the experimental questions themselves interfere, but that the experiments answering these questions do.
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[9] G. C. Wick, E. P. Wigner, and A. S. Wightman, Phys. Rev. **88**, 101 (1952).