

## QUANTUM THEORY OF RADIATION\*

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### INTRODUCTION

UNTIL a few years ago it had been impossible to construct a theory of radiation which could account satisfactorily both for interference phenomena and the phenomena of emission and absorption of light by matter. The first set of phenomena was interpreted by the wave theory, and the second set by the theory of light quanta. It was not until in 1927 that Dirac succeeded in constructing a quantum theory of radiation which could explain in an unified way both types of phenomena. In this article we shall develop the general formulas of Dirac's theory, and show its applications to several characteristic examples (Part I). In the second part of this work Dirac's relativistic wave equation of the electron will be discussed in relation to the theory of radiation. The third part will be devoted to the problems of the general quantum electrodynamics, and to the difficulties connected with it.

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## PART I. DIRAC'S THEORY OF RADIATION

## §1. Fundamental concept

Dirac's theory of radiation is based on a very simple idea; instead of considering an atom and the radiation field with which it interacts as two distinct systems, he treats them as a single system whose energy is the sum of three terms: one representing the energy of the atom, a second representing the electromagnetic energy of the radiation field, and a small term representing the coupling energy of the atom and the radiation field.

If we neglect this last term, the atom and the field could not affect each other in any way; that is, no radiation energy could be either emitted or absorbed by the atom. A very simple example will explain these relations. Let us consider a pendulum which corresponds to the atom, and an oscillating string in the neighborhood of the pendulum which represents the radiation field. If there is no connection between the pendulum and the string, the two systems vibrate quite independently of each other; the energy is in this case simply the sum of the energy of the pendulum and the energy of the string with no interaction term. To obtain a mechanical representation of this term, let us tie the mass  $M$  of the pendulum to a point  $A$  of the string by means of a very thin and elastic thread  $a$ . The effect of this thread is to perturb slightly the motion of the string and of the pendulum. Let us suppose for instance that at the time  $t=0$ , the string is in vibration and the pendulum is at rest. Through the elastic thread  $a$  the oscillating string transmits to the pendulum very slight forces having the same periods as the vibrations of the string. If these periods are different from the period of the pendulum, the amplitude of its vibrations remains always exceedingly small; but if a period of the string is equal to the period of the pendulum, there is resonance and the amplitude of vibration of the pendulum becomes considerable after a certain time. This process corresponds to the absorption of radiation by the atom.

If we suppose, on the contrary, that at the time  $t=0$  the pendulum is oscillating and the string is at rest, the inverse phenomenon occurs. The forces transmitted through the elastic thread from the pendulum to the string put the string in vibration; but only the harmonics of the string, whose frequencies are very near the frequency of the pendulum reach a considerable amplitude. This process corresponds to the emission of radiation by the atom.

## §2. Analytic representation

Returning to the case of the atom and the radiation field, the first problem which we have to solve is the finding of a convenient set of coordinates to represent the system. The position of the atom may be described by means of any system of general coordinates; if we assume that the atom contains only one electron, we may choose, for instance, its Cartesian coordinates (and eventually also the spin coordinate). The state of the radiation field could be determined by the values of the components of the electric and the magnetic vectors at any point of the space. We could also represent the field by means of a scalar and a vector potential. In this case we must give at any point of space

the values of the scalar potential  $V$  and of the three components  $U_x, U_y, U_z$  of the vector potential. In this representation the field is described by a continuous infinity of variables, which is very difficult to handle; furthermore such representation is inconvenient because the energy of the field expressed in terms of the variables, contains them in a very mixed form, even if we neglect, in a first approximation, the action of the atom on the field.

For these reasons it is often more convenient to represent the field in the following way. Instead of considering the radiation in infinite space, let us consider the radiation enclosed in a cavity of finite volume  $\Omega$  with perfectly reflecting walls. If afterwards we let the cavity become infinite in every direction, we shall get as a limit the properties of radiation in free space.

The electromagnetic vibrations in a cavity of finite volume, just as the vibrations of an elastic body of finite volume, may be represented by the superposition of a discrete infinity of fundamental vibrations each one corresponding to a system of standing waves. The number of standing vibrations whose frequency lies between  $\nu$  and  $\nu + d\nu$  is given, for a very large volume  $\Omega$ , by:

$$dN = \frac{8\pi}{c^3} \Omega \nu^2 d\nu \quad (1)$$

$c$  being the velocity of light.

It is to be noticed that a radiation field only and not a general electromagnetic field may be represented through a superposition of standing vibrations. The general quantum electrodynamics deals with the quantum theoretical representation of a general electromagnetic field; we shall discuss this theory in Part III of this article. At present we shall limit ourselves to the simple radiation theory, that is, we shall consider quantum theoretically only that part of the electromagnetic field which is responsible for the phenomena of radiation. The radiation field may be then represented as a superposition of ordinary plane electromagnetic waves; whereas for instance the Coulomb forces need a more general representation.

The electromagnetic field of a plane standing wave has a vector potential of the form:

$$U = Au(t) \sin\left(\frac{2\pi\nu}{c}(\alpha, X) + \beta\right). \quad (2)$$

The sine factor gives us the amplitude dependence on position;  $X$  is a vector with components  $x, y, z$ ;  $\alpha$  is a unit vector giving the direction of the standing wave;  $A$  is a unit vector giving the direction of vibration of the electric force; since the wave is transversal,  $A$  and  $\alpha$  are at right angles. The factor  $u(t)$  which gives the dependence on the time, is generally a sine function of  $t$ . However, this is not always the case; if there is an atom which either emits or absorbs radiation, the amplitude of the standing vibration may increase or decrease.

Now we represent the radiation field as the superposition of standing waves of the type (2), with frequencies  $\nu_1, \nu_2, \dots, \nu_s$ . The number of frequencies lying between  $\nu$  and  $\nu + d\nu$  is given by (1). The directions  $\alpha_s$  and  $A_s$

of the standing waves and of polarization are distributed at random. We have also:

$$U = \sum_s A_s u_s(t) \sin \Gamma_s = \sum_s U_s \quad (3)$$

where:

$$\Gamma_s = \frac{2\pi\nu_s}{c}(\alpha_s, X) + \beta_s. \quad (4)$$

If there is neither emission nor absorption of radiation, the  $u_s(t)$  are sine functions of the time; but in the general case, they may depend on  $t$  in any way. Now it is evident that if we know, at a given time  $t$  the values of all the  $u_s$  the vector potential throughout the space  $\Omega$  is determined for that instant since this is given by (3). We may therefore take the  $u_s$  as coordinates representing the radiation field at any moment.

### §3. Electromagnetic energy of radiation field

We have now to express the electromagnetic energy of the radiation field in terms of the coordinates  $u_s$ . The electric and magnetic forces, derived from the potential  $U$ , are:

$$E = -\frac{1}{c} \frac{\partial U}{\partial t}; \quad H = \text{rot } U.$$

From these equations we get, using (3) and (4)

$$\left. \begin{aligned} E &= -\sum_s \frac{1}{c} A_s \dot{u}_s \sin \Gamma_s \\ H &= \sum_s \frac{2\pi\nu_s}{c} [\alpha_s, A_s] u_s \cos \Gamma_s \end{aligned} \right\} \quad (5)$$

The electromagnetic energy contained in the space  $\Omega$  is:

$$W_e = \Omega \frac{\overline{E}^2 + \overline{H}^2}{8\pi}$$

where the barred expressions represent mean values. We must calculate the space average of  $E^2$  and  $H^2$ . It is evident that the mixed terms in the squares have the average zero. Remembering that  $\overline{\sin^2 \Gamma_s} = \overline{\cos^2 \Gamma_s} = \frac{1}{2}$  and  $[\alpha_s, A_s]^2 = 1$ , since  $A_s$  and  $\alpha_s$  are perpendicular unit vectors, we find

$$\overline{E}^2 = \frac{1}{2c^2} \sum_s \dot{u}_s^2; \quad \overline{H}^2 = \sum_s \frac{2\pi^2\nu_s^2}{c^2} u_s^2.$$

Thus the electromagnetic energy is

$$W_e = \frac{\Omega}{8\pi c^2} \sum_s (\frac{1}{2} \dot{u}_s^2 + 2\pi^2\nu_s^2 u_s^2). \quad (6)$$

From this expression of the radiation energy we may easily obtain the equations which give the dependence of  $u_s$  on the time in Hamiltonian form. We in-

roduce for this purpose a new variable  $v_s$  canonically conjugated to  $u_s$ , by means of the usual rules

$$v_s = \frac{\partial W_e}{\partial \dot{u}_s} = \frac{\Omega}{8\pi c^2} \dot{u}_s.$$

The energy (6) becomes in the Hamiltonian form

$$W_e = \sum_s \left( \frac{8\pi c^2}{\Omega} \frac{v_s^2}{2} + \frac{\Omega}{8\pi c^2} 2\pi^2 \nu_s^2 u_s^2 \right). \quad (7)$$

From this Hamiltonian we get the canonical equations

$$\dot{u}_s = \frac{\partial W_e}{\partial v_s} = \frac{8\pi c^2}{\Omega} v_s; \quad \dot{v}_s = -\frac{\partial W_e}{\partial u_s} = -\frac{\Omega}{8\pi c^2} 4\pi^2 \nu_s^2 u_s. \quad (8)$$

If we eliminate  $v_s$  from these equations, we get

$$\ddot{u}_s + 4\pi^2 \nu_s^2 u_s = 0. \quad (9)$$

We find, as was to be expected, that  $u_s$  is a periodic function of the time with the frequency  $\nu_s$ . We may also say that the canonical Eqs. (8) are equivalent to the Maxwell equations for the vacuum. It is convenient to avoid complicated factors in the Hamilton function (7) changing by constant factors the canonical variables  $u_s$  and  $v_s$  into two other conjugate variables  $q_s$  and  $p_s$  given by

$$u_s = \left( \frac{8\pi c^2}{\Omega} \right)^{1/2} q_s; \quad v_s = \left( \frac{\Omega}{8\pi c^2} \right)^{1/2} p_s. \quad (10)$$

The energy (7) takes now the form:

$$W_e = \sum_s \left( \frac{1}{2} p_s^2 + 2\pi^2 \nu_s^2 q_s^2 \right) \quad (11)$$

which is the same as the Hamiltonian of a system of many independent oscillators with mass 1 and frequencies  $\nu_1, \nu_2, \dots, \nu_s$ . The vector potential (3) in terms of the new variables  $q_s$ , takes the form:

$$U = c \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_s A_s q_s \sin \Gamma_s. \quad (12)$$

#### §4. Hamiltonian of the atom and the radiation field

We now must write the Hamiltonian for the atom which, added to (11), shall give us the Hamiltonian of the complex system of the atom and the radiation field.

The Hamiltonian function for an electron may be obtained to a first approximation from the ordinary relativistic Hamilton function for a point charge, that is:

$$0 = -\frac{1}{2m} \left\{ \left( mc + \frac{W_a - eV}{c} \right)^2 - \left( \mathbf{p} - \frac{e\mathbf{U}}{c} \right)^2 \right\} + \frac{mc^2}{2} \quad (13)$$

by neglecting the terms in  $1/c^2$ . We shall see later (Part II) how it is possible to use also Dirac's relativistic Hamiltonian of the spinning electron in the radiation theory.

If we neglect the terms in  $1/c^2$  we get from (13)

$$W_a = \frac{\dot{p}^2}{2m} + eV - \frac{e}{mc}(U, \dot{p}). \quad (14)$$

The Hamiltonian of the complex system of the atom and the radiation field is obtained adding (11) to (14) and putting in (14) the expression (12) instead of  $U$ . We obtain in this way:

$$H = \frac{1}{2m} \dot{p}^2 + eV + \sum_s (\frac{1}{2} \dot{p}_s^2 + 2\pi^2 \nu_s^2 q_s^2) - \frac{e}{m} \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_s (A_s, \dot{p}) q_s \sin \Gamma_s. \quad (15)$$

The first and second term of (15) give us the Hamiltonian which describes the motion of the electron if we neglect the effect of the radiation on it. The third term is the Hamiltonian (11) of the radiation field. The last term

$$\mathcal{H} = - \frac{e}{m} \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_s (A_s, \dot{p}) q_s \sin \Gamma_s \quad (16)$$

is the coupling term, since it contains both the coordinates of the radiation ( $q_s$ ) and of the atom ( $\dot{p}$  and  $x$ , contained in  $\Gamma_s$ ).

In some cases, particularly for the theory of dispersion and of the Compton effect, it is necessary to write the Hamiltonian with a little closer approximation. In developing the term  $(1/2m)(\dot{p} - eU/c)^2$  of (13), we have neglected

$$e^2 U^2 / 2mc^2. \quad (17)$$

If we keep this term, and introduce for  $U$  the expression (12), we must add to the Hamiltonian (15) the term

$$H^{(2)} = \frac{4\pi e^2}{\Omega m} \sum_{s\sigma} (A_s, A_\sigma) q_s q_\sigma \sin \Gamma_s \sin \Gamma_\sigma. \quad (18)$$

We shall see later the very peculiar connection of this term with the jumps from positive to negative mass which characterise Dirac's theory of the spinning electron.

### §5. Classical treatment

It is important to notice, that the results of the classical theory of emission of electromagnetic radiation and particularly Larmor's formula can be derived in a classical way from the Hamiltonian (15). This may be seen if we derive from (15) the canonical equations; for instance, if we consider the pair of variables  $q_s, \dot{p}_s$  we get the equations:

$$\dot{q}_s = \frac{\partial H}{\partial p_s} = p_s; \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} = -4\pi^2\nu_s^2 q_s + \frac{e}{m} \left(\frac{8\pi}{\Omega}\right)^{1/2} \sum_s (A_s, p) \sin \Gamma_s.$$

If we eliminate  $p_s$ , we find for  $q_s$ :

$$\ddot{q}_s + 4\pi^2\nu_s^2 q_s = \frac{e}{m} \left(\frac{8\pi}{\Omega}\right)^{1/2} \sum_s (A_s, p) \sin \Gamma_s. \quad (19)$$

This is the equation for the forced vibrations of an oscillator of frequency  $\nu_s$ . If we suppose for instance that at the time  $t=0$  there is no radiation in the field, i.e., that  $q_s = p_s = 0$ ; but that there is an electron moving with non uniform motion, so that its momentum  $p$  varies, we see from (19) that after a certain time  $q_s$  shall be different from zero; this means that there is a certain amount of energy in the  $s$  component of the radiation which has been emitted by the moving charge. The effect is of course bigger, if the motion of the charge is periodic with a period near to  $\nu_s$ . It might actually be shown by this method that the amount of energy emitted per unit time by the moving charge is given, to a first approximation, by:

$$\frac{2}{3} \frac{e^2}{c^3} A^2 \quad (20)$$

where  $A$  is the acceleration of the particle, in accordance with Larmor's result.

This shows us, that the classical treatment of (15) gives us the same results as the ordinary theory of radiation; we must now apply to (15) the quantum mechanical methods.

### §6. Perturbation theory

For this we write down some general formulas of the perturbation theory of wave mechanics which we shall use later. Let

$$H = H_0 + \mathfrak{C} \quad (21)$$

be the Hamiltonian of a system with coordinates  $q$  and moments  $p$ . The Schrödinger equation is:

$$-\frac{\hbar}{2\pi i} \frac{\partial \psi}{\partial t} = H\psi = (H_0 + \mathfrak{C})\psi \quad (22)$$

where  $H$  is an operator obtained from  $H$  with the substitution of  $(\hbar/2\pi i)$   $(\partial/\partial q)$  in place of  $p$ .

We consider now the unperturbed problem corresponding to the Hamiltonian  $H_0$ . The Schrödinger equation corresponding to it is:

$$-\frac{\hbar}{2\pi i} \frac{\partial \Psi}{\partial t} = H_0\Psi. \quad (23)$$

Let

$$\phi_1, \phi_2, \dots, \phi_n \dots$$

be the normalized eigenfunctions of the unperturbed problem and

$$E_1, E_2, \dots, E_n \dots$$

be the corresponding eigenvalues. The most general solution of (23) is then:

$$\Psi = \sum_n a_n \phi_n(q) e^{-2\pi i E_n t / h} \quad (24)$$

where the  $a_n$  are constants. The physical meaning of the  $a_n$  is contained in the statement that

$$|a_n|^2 \quad (25)$$

is proportional to the probability of finding the system in the  $n$ th quantum state. We may also normalize the  $a_n$  in such a way that

$$\sum_n |a_n|^2 = 1.$$

Then  $|a_n|^2$  gives us directly the probability that the system is in the  $n$ th quantum state;  $a_n$  is called the amplitude of probability for the  $n$ th quantum state. The solution  $\psi$  of the perturbed problem (22) can be developed in a series of eigenfunctions of the unperturbed problem; it can therefore be written in the form (24); only the  $a$ 's are no longer constants, but are functions of the time  $t$ . Substituting (24) in (22), we find for the  $a$ 's the differential equations:

$$\dot{a}_n = -\frac{2\pi i}{h} \sum_m \mathcal{H}_{nm} a_m e^{2\pi i (E_n - E_m) t / h} \quad (26)$$

where:

$$\mathcal{H}_{nm} = \int \bar{\phi}_n \mathcal{H} \phi_m dq \quad (27)$$

is the element  $n, m$  of the perturbation matrix, representing the perturbation energy  $\mathcal{H}$ ;  $\bar{\phi}_n$  is the conjugate complex to  $\phi_n$ . From (25) we see that the  $a$ 's vary with time, so that also the probability of the different quantum states, which is given by (25) changes with time. This means that the effect of the perturbation is to induce transition probabilities among the quantum states of the unperturbed system.

### §7. Quantum mechanical treatment

We must now apply these methods to the Hamiltonian (15) of an atom and the radiation field. As the unperturbed Hamiltonian, we take:

$$H_0 = \frac{1}{2m} p^2 + eV + \sum_s (\frac{1}{2} p_s^2 + 2\pi^2 \nu_s^2 q_s^2). \quad (28)$$

The interaction energy (16) is considered as the perturbation energy.

The Hamiltonian  $H_0$  of the unperturbed system is the sum of the terms

$$\frac{1}{2m} p^2 + eV \quad (29)$$

representing the energy of the atom, and terms like:



$$\frac{1}{2}p_s^2 + 2\pi^2\nu_s^2q_s^2 \quad (30)$$

representing the energy of the  $s$ th component of the radiation which is identical with the energy of an oscillator of the same frequency  $\nu_s$ .

Let

$$u_n, u_{n_1}, u_{n_2}, \dots, u_{n_s}, \dots$$

be the Schrödinger functions for the atom (with Hamiltonian (29)) and for each component of radiation (with Hamiltonian (30)). For simplicity of writing we distinguish all these functions only by the index. The Schrödinger function corresponding to the unperturbed Hamiltonian  $H_0$  is then given by the product

$$\phi_{n, n_1, n_2, \dots, n_s, \dots} = u_n u_{n_1} u_{n_2} \dots u_{n_s} \dots \quad (31)$$

The corresponding eigenvalue is the sum:

$$E_{n, n_1, n_2, \dots, n_s, \dots} = E_n + E_{n_1} + \dots + E_{n_s} + \dots \quad (32)$$

Since the Hamiltonians (30) are of the oscillator type we have, as for the oscillator

$$E_{n_s} = h\nu_s(n_s + \frac{1}{2}).$$

We may also neglect the constant energy  $h\nu_s/2$ , which does not affect the phenomena, since the frequencies  $\nu_s$  are constant, and only differences of energy are considered, and write simply

$$E_{n_s} = h\nu_s n_s. \quad (33)$$

Formula (32) now takes the form

$$E_{n, n_1, \dots, n_s, \dots} = E_n + h\nu_1 n_1 + h\nu_2 n_2 + \dots \quad (34)$$

The general form of the field scalar, corresponding to (24) is then:

$$\begin{aligned} \psi(X, q_1, q_2, \dots, q_s, \dots) \\ = \sum a_{nn_1 \dots n_s \dots} u_n u_{n_1} \dots u_{n_s} \dots e^{-2\pi i(E_n + h\nu_1 n_1 + \dots + h\nu_s n_s + \dots)t/h}. \end{aligned} \quad (35)$$

The physical meaning of the  $a$ 's, according to (25), is the following:

$$|a_{n, n_1, n_2, \dots, n_s, \dots}|^2$$

is the probability that the atom is in the quantum state  $n$ ; the first component of radiation in the state  $n_1$ ; the second in the state  $n_2$ . If we have for instance  $a_{300 \dots 0 \dots} = 1$  and all the other  $a$ 's are  $=0$ , we may say that it is certain that the atom is in the third quantum state, and no component of radiation is excited.

If we neglect the effect of the perturbation term  $\mathcal{H}$ , the  $a$ 's are constant; the effect of the interaction term  $\mathcal{H}$  is, according to the general formula (26), that the  $a$ 's vary with the time. If we have, for instance, at the time  $t=0$ ,  $a_{300 \dots 0 \dots} = 1$  and all the other  $a$ 's equal to zero, after a certain time  $t$ , some of the  $a$ 's which were  $=0$ , say  $a_{2,1,0 \dots 0 \dots}$  may have a value different from zero. There is therefore a finite probability of finding that, at the time  $t$  the atom is in the state 2, having jumped down from state 3 to state 2 and that the

first component of the radiation is excited. This is the quantum theoretical mechanism of the radiation of energy.

We have now to write for our case the equations corresponding to (26) to find how the  $a$ 's vary with time. For this we must find the expression of the matrix element of the perturbation  $\mathcal{H}$ , corresponding to a transition of the whole system from a quantum state  $n, n_1, n_2, \dots$  to another  $m, m_1, m_2, \dots$ . This is given, according to (27) (31), since the  $u$ 's are real, by:

$$\begin{aligned} & \mathcal{H}_{nn_1 \dots n_s \dots; mm_1 \dots m_s \dots} \\ &= \iint \dots \int \dots u_n u_{n_1} \dots u_{n_s} \dots \mathcal{H} u_m u_{m_1} \dots u_{m_s} \dots dX dq_1 \dots dq_s \dots \end{aligned} \quad (37)$$

where we must put for  $\mathcal{H}$  the operator (16).

The integral (37) may be very easily calculated taking into account the following relations

$$\int u_{n_s} u_{m_s} dq_s = \delta_{n_s m_s} \quad (38)$$

which expresses the orthogonality of the  $u$ 's;

$$\int q_s u_{n_s} u_{m_s} dq_s = \begin{cases} 0 & \text{if } m_s \neq n_s \pm 1 \\ \left[ \frac{h(n_s + 1)}{8\pi^2\nu_s} \right]^{1/2} & \text{if } m_s = n_s + 1 \\ \left[ \frac{hn_s}{8\pi^2\nu_s} \right]^{1/2} & \text{if } m_s = n_s - 1. \end{cases} \quad (39)$$

These equations may be easily verified, since the  $u_s$  are the well known eigenfunctions of a harmonic oscillator with mass 1 and frequency  $\nu_s$ ; (39) are the elements of the matrix representing the coordinate  $q_s$  for this oscillator. We must remember further that the operator  $p$ , with components  $p_x, p_y, p_z$ , means  $(h/2\pi i)$  grad; we put then

$$\begin{aligned} P_{snm} &= \int u_n \sin \Gamma_s p u_m dx dy dz \\ &= (h/2\pi i) \int \sin \Gamma_s u_n \text{grad } u_m dx dy dz. \end{aligned} \quad (40)$$

We find now easily that the matrix element (37) is always = 0 if more than one of the indices  $m_1, m_2, \dots, m_s, \dots$  of the radiation components is different from the corresponding  $n_1, n_2, \dots, n_s, \dots$ . If only one of the  $m_1, m_2, \dots, m_s, \dots$ , say  $m_s$ , differs from the corresponding  $n_s$ , the result, according to (39) is different from zero only if  $m_s = n_s \pm 1$ . In this case we have:

$$\mathcal{H}_{nn_1 \dots n_s \dots; mm_1 \dots m_s \pm 1 \dots} = - \frac{e}{m} \left( \frac{h}{\pi \Omega \nu_s} \right)^{1/2} (A_s, P_{snm}) \begin{bmatrix} (n_s + 1)^{1/2} \\ n_s^{1/2} \end{bmatrix} \quad (41)$$

where we must use the upper expression  $(n_s + 1)^{1/2}$  if  $m_s = n_s + 1$  and the lower

$n_s^{1/2}$  if  $m_s = n_s - 1$ . Very important is the particular case that the dimensions of the atom are very small compared with the wave-length, so that the  $\Gamma_s$ , i.e., the phases of the radiation components may be considered as constants over all the space where the eigenfunctions of the electron are practically different from zero. In this case we may take  $\sin \Gamma_s$  out of the integral in (40) and we get:

$$P_{snm} = \frac{h}{2\pi i} \sin \Gamma_s \int u_n \text{grad } u_m dx dy dz.$$

Remembering that:

$$\frac{h}{2\pi i} \int u_n \text{grad } u_m d\tau$$

represents the momentum of the electron, it can be immediately proved that:

$$\frac{h}{2\pi i} \int u_n \text{grad } u_m d\tau = -2\pi i m \nu_{mn} X_{nm} \quad (42)$$

where:

$$\nu_{mn} = (E_m - E_n)/h \quad (43)$$

is the frequency corresponding to the jump from state  $m$  to state  $n$ ;

$$X_{mn} = X_{nm} = \int X u_n u_m d\tau \quad (44)$$

is the element  $mn$  of the matrix representing the radius vector  $X$  (observe that the letter  $m$  is used in (42) both as index and as the mass of the electron; but since no confusion is possible we prefer not to introduce a new symbol).

We obtain now:

$$P_{snm} = -2\pi i m \nu_{mn} X_{nm} \sin \Gamma_s \quad (45)$$

(41) becomes then:

$$\mathcal{I}_{C_{nn_1 \dots n_s \dots; imn_1 \dots n_s \pm 1 \dots}} = 2\pi i e \left( \frac{h}{\pi \Omega} \right)^{1/2} \frac{\nu_{mn}}{\nu_s^{1/2}} (A_s X_{nm}) \left[ \begin{matrix} (n_s + 1)^{1/2} \\ n_s^{1/2} \end{matrix} \right] \sin \Gamma_s. \quad (46)$$

Now we may write at once the equations analogous to (26) for the variations of the  $a$ 's as functions of the time. We get:

$$\dot{a}_{nn_1 \dots n_s \dots} = -\frac{2\pi i}{h} \sum_{mm_1 \dots m_s \dots} a_{mm_1 \dots m_s \dots} \mathcal{I}_{C_{nn_1 \dots imn_1 \dots}} e^{-2\pi i (E_{mm_1 \dots} - E_{nn_1 \dots}) t/h} \quad (47)$$

Remembering (46) (43) (34) we get with a few reductions

$$\begin{aligned} \dot{a}_{nn_1 \dots n_s \dots} = & \frac{4\pi^{3/2} e}{(h\Omega)^{1/2}} \sum_{m,s} \frac{\nu_{mn}}{\nu_s^{1/2}} (A_s X_{nm}) \sin \Gamma_s [a_{mn_1 \dots n_s + 1 \dots} (n_s + 1)^{1/2} e^{-2\pi i (\nu_{mn} + \nu_s) t} \\ & + a_{mn_1 \dots n_s - 1 \dots} n_s^{1/2} e^{-2\pi i (\nu_{mn} - \nu_s) t}]. \end{aligned} \quad (48)$$

This is the fundamental equation of the radiation theory. In the applications we shall encounter equations which differ from (48) either because of the use of a higher degree of approximation, or because systems containing more

than one electron are considered. When these cases come up we shall show the necessary modifications of the Eq. (48).

We shall now discuss some applications of the general theory that we have developed, with the chief purpose of showing that this theory may actually be considered as a satisfactory theory of radiation phenomena. For this we shall work out the following examples:

1. Emission from an excited atom and mean life.
2. Propagation of light in vacuum.
3. A case of interference: the Lippman fringes.
4. The Doppler effect.
5. The Compton effect.

For applications to other problems see the bibliography.

### §8. Emission from an excited atom

Let us consider an atom which at the time  $t=0$  is in an excited state; let us suppose that there is no radiant energy in the space surrounding it. We may consider, for the sake of simplicity only two states of the atom, numbered 1 and 2, and suppose that the atom at  $t=0$  is in the state 2. All this may be expressed by saying that for  $t=0$ .

$$a_{200\dots 0\dots} = 1 \quad (49)$$

and all the other  $a$ 's are  $=0$ . We know from experience that after a certain time the atom must go over to the state 1 of less energy and the energy difference must be found in the radiation field.

We will now show how it is possible to study this process with the fundamental Eq. (48).

We put

$$\nu_{21} = -\nu_{12} = \nu; X_{12} = X_{21} = X$$

Eqs. (48) give then:

$$\dot{a}_{1,0\dots 1,\dots} = \frac{4\pi^{3/2}e}{(\Omega\hbar)^{1/2}} \frac{\nu}{\nu_s^{1/2}} (A_s X) \sin \Gamma_s a_{200\dots 0\dots} e^{-2\pi i(\nu-\nu_s)t} \quad (50)$$

$$\dot{a}_{2,0\dots 0\dots} = \frac{4\pi^{3/2}e}{(\Omega\hbar)^{1/2}} \sum_s \frac{\nu}{\nu_s^{1/2}} (A_s X) \sin \Gamma_s a_{1,00\dots 1,\dots} e^{-2\pi i(\nu_s-\nu)t}. \quad (51)$$

We try to solve these equations by

$$a_{20\dots 0\dots} = e^{-\gamma t} \quad (52)$$

where  $\gamma$  is a constant which must be determined.

We substitute (52) in (50) and then integrate with respect to  $t$ , determining the integration constant by the initial conditions  $a_{100\dots 1,\dots} = 0$ ; we find:

$$a_{1,00\dots 1,\dots} = \frac{4\pi^{3/2}e}{(\Omega\hbar)^{1/2}} \frac{\nu}{\nu_s^{1/2}} (A_s X) \sin \Gamma_s \frac{e^{[-2\pi i(\nu-\nu_s)-\gamma]t} - 1}{-2\pi i(\nu-\nu_s) - \gamma}. \quad (53)$$

We substitute (52) and (53) in (51) and multiply by  $-e^{\gamma t}$  and find:

$$\gamma = \frac{16\pi^3 e^2}{\Omega \hbar} \sum_s \frac{\nu^2}{\nu_s} (A_s X)^2 \sin^2 \Gamma_s \frac{1 - e^{(\gamma - 2\pi i(\nu_s - \nu))t}}{-2\pi i(\nu - \nu_s) - \gamma}.$$

The sum may be calculated by the following method; since the phase, direction and polarization of the different radiation components are distributed at random, we may substitute for  $(A_s X)^2 \sin^2 \Gamma_s$  its mean value, taken over all phases, directions and polarizations; we replace then the sum by an integral over  $\nu_s$ , multiplying by the factor:

$$\frac{8\pi}{c^3} \Omega \nu_s^2 d\nu_s \quad (54)$$

which gives, according to (1) the number of radiation components with frequency between  $\nu_s$  and  $\nu_s + d\nu_s$ . We get then, observing

$$(\overline{A_s X})^2 = \frac{1}{3} X^2; \quad \overline{\sin^2 \Gamma_s} = \frac{1}{2}$$

that

$$\gamma = \frac{64\pi^4 e^2}{3hc^3} \nu^2 X^2 \int_0^\infty \frac{1 - e^{(\gamma - 2\pi i(\nu_s - \nu))t}}{-2\pi i(\nu - \nu_s) - \gamma} \nu_s d\nu_s.$$

It may be proved that for small  $\gamma$  this integral has the value  $\nu/2$ ; we obtain thus:

$$\gamma = \frac{32\pi^4 e^2}{3hc^3} \nu^3 X^2 \quad (55)$$

which determines the constant  $\gamma$ . The relation of this constant with the mean life of the state 2 is easily found: the probability of finding the atom in the state 2 as shown by (52) is:

$$|a_{2,0\dots 0\dots}|^2 = e^{-2\gamma t}.$$

By definition of the mean life  $\tau$ , this probability must be  $e^{-t/\tau}$ ; we get then by comparison

$$\tau = \frac{1}{2\gamma} = \frac{3hc^3}{64\pi^4 e^2 \nu^3 X^2}. \quad (56)$$

We may also deduce from this theory the form and the width of the emitted spectral line. For this purpose we observe that, after the emission has taken place, i.e., after a time  $t$  long with respect to the mean life, the exponential  $e^{(-2\pi i(\nu - \nu_s) - \gamma)t}$  becomes negligible; then we get from (53):

$$a_{100\dots 1\dots} = \frac{4\pi^{3/2} e}{(\Omega \hbar)^{1/2}} \frac{\nu}{\nu_s^{1/2}} (A_s X) \sin \Gamma_s \frac{1}{-2\pi i(\nu_s - \nu) + \gamma}. \quad (57)$$

The probability that the emitted quantum belongs to the  $s$  component is therefore:

$$|a_{10\dots 1\dots}|^2 = \frac{16\pi^3 e^2}{\Omega \hbar} \frac{\nu^2}{\nu_s} (A_s X)^2 \sin^2 \Gamma_s \frac{1}{\gamma^2 + 4\pi^2(\nu_s - \nu)^2}. \quad (58)$$

The last factor

$$\frac{1}{\gamma^2 + 4\pi^2(\nu_s - \nu)^2}$$

represents the form of the emitted line; it is identical with the form which may be deduced in the classical theory for exponentially damped oscillators.

### §9. Propagation of light in vacuum

This section and the next one will be devoted to the proof that the results of ordinary wave theory can be applied to the computation of the intensity of light both for the propagation in vacuum and for cases of interference.

This has been proved for a general case by Racah and more recently by a very general and direct method by Heisenberg, who does not use the Fourier analysis but calculates the amplitude of the field vectors directly.

We prefer however to show here by the use of two examples how the phase relations between the different components are effective in determining the propagation with finite velocity and the interference phenomena.

Let  $A$  and  $B$  be two atoms; let us suppose that at the time  $t=0$ ,  $A$  is in an excited and  $B$  in the normal state. After a certain time  $A$  emits its energy which may in turn be absorbed by the atom  $B$  which then becomes excited. Since the light needs a finite time to go from  $A$  to  $B$ , the excitation of  $B$  can take place only after the time  $r/c$ ,  $r$  being the distance between the two atoms. We will show that all this may be deduced from the quantum theory of radiation. We simplify the problem by the assumption that the mean life of the first atom  $A$  is very short, in order that the light be emitted from  $A$  at a very definite time; we suppose further that the mean life of  $B$  is very long. The result is that the line emitted from the atom  $A$  is very broad and might be considered as a portion of a continuous spectrum; on the contrary the atom  $B$  absorbs a very sharp line. We must first modify slightly the fundamental Eqs. (48), for the case of two atoms in the radiation field. We use indices and magnitudes without dash for the first atom  $A$ , and dashed letters for the atom  $B$ .

$$|a_{nn'n_1\dots n_s\dots}|^2$$

is the probability that  $A$  is in the state  $n$ ;  $B$  in the state  $n'$ ; and the radiation components in the states  $n_1 \dots n_s \dots$ ; the equations analogous to (48) for the case of two atoms, may be obtained by the same considerations as Eqs. (48). The right hand side will now consist of two terms, each one analogous to the right hand member of (48) and each one referring to one of the atoms. We get precisely:

$$\begin{aligned} \dot{a}_{nn'n_1\dots n_s\dots} &= \frac{4\pi^{3/2}e}{(\Omega h)^{1/2}} \sum_{m_s} \frac{\nu_{mn}}{\nu_s^{1/2}} (A_s X_{nm}) \sin \Gamma_s \{ a_{mn'n_1\dots n_s+1\dots} (n_s + 1)^{1/2} e^{-2\pi i(\nu_{mn} + \nu_s)t} \\ &\quad + a_{mn'n_1\dots n_s-1\dots} n_s^{1/2} e^{-2\pi i(\nu_{mn} - \nu_s)t} \} \\ &\quad + \frac{4\pi^{3/2}e}{(\Omega h)^{1/2}} \sum_{m'_s} \frac{\nu'_{m'n'}}{\nu_s^{1/2}} (A_s X'_{m'n'}) \sin \Gamma'_s \{ a_{nm'n_1\dots n_s+1\dots} (n_s + 1)^{1/2} e^{-2\pi i(\nu'_{m'n'} + \nu_s)t} \\ &\quad + a_{nm'n_1\dots n_s-1\dots} n_s^{1/2} e^{-2\pi i(\nu'_{m'n'} - \nu_s)t} \} \end{aligned} \quad (60)$$

$\Gamma_s$  and  $\Gamma'_s$  are the phases of the  $s$ th standing vibrations at the places of the first and second atom. We may suppose that the atom  $A$  is at the origin of the coordinates, and the atom  $B$  is on the  $x$ -axis at a distance  $r$  from the origin. We have then from (4):

$$\Gamma'_s = \Gamma_s + \frac{2\pi\nu_s}{c} r \cos \theta_s \quad (61)$$

$\theta_s$  being the angle between the  $x$ -axis and the direction of the  $s$ th radiation component.

As before we consider for each atom only two states, 1 and 2, and we put

$$\begin{aligned} \nu_{21} &= -\nu_{12} = \nu; & \nu_{21}' &= -\nu_{12}' = \nu' \\ X_{21} &= X_{12} = X; & X_{21}' &= X_{12}' = X'. \end{aligned}$$

For simplicity we admit further that both vectors  $X$  and  $X'$  reduce to the only  $y$ -component. At the time  $t=0$ , the first atom is excited in the state 2, and the second atom is in the normal state 1; further there is no radiation in the field; we have thus:

$$a_{210\dots 0\dots} = 1$$

while all the other  $a$ 's are 0 for  $t=0$ . We must find what is the probability that, at the time  $t$ , the first atom has lost its energy and that this energy has been absorbed by the second atom; this probability is given by:

$$|a_{120\dots 0\dots}|^2.$$

We have shown in the preceding section that after a time long with respect to the mean life, the energy of the excited atom  $A$  is transferred to the radiation field according to (57). This formula can be applied also in our case, if we neglect the very small perturbation due to the presence of the atom  $B$ . We may write:

$$a_{1100\dots 1s\dots} = \frac{4\pi^{3/2}e}{(\Omega\hbar)^{1/2}} \frac{\nu}{\nu_s^{1/2}} (A_s X) \sin \Gamma_s \frac{1}{-2\pi i(\nu_s - \nu) + \gamma} \quad (62)$$

We put now in (60):  $n=1$ ,  $n'=2$ ,  $n_1=n_2=\dots=n_s=\dots=0$  and we get:

$$\dot{a}_{210\dots 0\dots} = -\frac{4\pi^{3/2}e}{(\Omega\hbar)^{1/2}} \sum_s \frac{\nu'}{\nu_s^{1/2}} (A_s X') \sin \Gamma'_s a_{110\dots 1s\dots} e^{-2\pi i(\nu' - \nu_s)t}$$

since the other terms are zero. Substituting (62) in this equation we obtain:

$$\dot{a}_{120\dots 0\dots} = -\frac{16\pi^3 e^2}{\Omega\hbar} \nu\nu' \sum_s \frac{(A_s X)(A_s X') \sin \Gamma_s \sin \Gamma'_s}{\nu_s [2\pi i(\nu - \nu_s) + \gamma]} e^{-2\pi i(\nu_s - \nu')t}.$$

We integrate with respect to  $t$  and remember that for  $t=0$ ,  $a_{120\dots 0\dots}=0$ . We obtain thus:

$$a_{120\dots 0\dots} = \frac{16\pi^3 e^2}{\Omega\hbar} \nu\nu' \sum_s \frac{(A_s X)(A_s X') \sin \Gamma_s \sin \Gamma'_s}{\nu_s [2\pi i(\nu - \nu_s) + \gamma]} \frac{1 - e^{-2\pi i(\nu_s - \nu')t}}{2\pi i(\nu_s - \nu')} \quad (63)$$

To evaluate the sum over  $s$  we must transform it to an integral. For this we first substitute in the usual way the mean value for the expression  $(A_s X)$

$(A_s X') \sin \Gamma_s \sin \Gamma_s'$ . Remembering (61), the fact that  $A_s$  and  $\alpha_s$  are perpendicular unit vectors, and that  $X$  and  $X'$  reduce to the only  $y$ -component, we find with some calculation:

$$\overline{(A_s X)(A_s X') \sin \Gamma_s \sin \Gamma_s'} = \frac{X X'}{4} \left[ \frac{c}{2\pi\nu_s r} \sin \frac{2\pi\nu_s r}{c} + \left( \frac{c}{2\pi\nu_s r} \right)^2 \cos \frac{2\pi\nu_s r}{c} - \left( \frac{c}{2\pi\nu_s r} \right)^3 \sin^2 \frac{2\pi\nu_s r}{c} \right]. \quad (64)$$

The average is taken over all values of the phase, the direction and polarization. We suppose now that the distance  $r$  of the two atoms is very large compared with the wave-length; we may then neglect the square and the cube of the very small expression  $c/2\pi\nu_s r$ , and we write:

$$\overline{(A_s X)(A_s X') \sin \Gamma_s \sin \Gamma_s'} = \frac{c X X'}{8\pi\nu_s r} \sin \frac{2\pi\nu_s r}{c}. \quad (65)$$

To calculate the sum (63) we must now substitute this average value, multiply by (54) and change the sum to an integral over  $\nu_s$ . We get thus:

$$a_{120\dots 0\dots} = -\frac{1}{r} \frac{16\pi^3 e^2}{c^2 h} \nu X \nu' X' \int_0^\infty \frac{\sin(2\pi\nu_s r/c)(1 + e^{-2\pi i(\nu_s - \nu')t})}{[2\pi i(\nu - \nu_s) + \gamma]2\pi i(\nu_s - \nu')} d\nu_s \quad (66)$$

The integration can be effected by observing that because of the factor  $(\nu_s - \nu')$  in the denominator the values of the integrand are concentrated in the neighbourhood of the value  $\nu'$  for the variable  $\nu_s$ . Since we have supposed that the mean life of the first atom is very short, the factor  $[2\pi i(\nu_s - \nu) + \gamma]^{-1}$  varies very regularly (that is the line excited by the first atom is so wide that it can be considered as a piece of continuous spectrum). We may therefore take this factor out of the integral putting into it  $\nu_s = \nu'$ . We may also extend the integration from  $-\infty$  to  $+\infty$ , since, for the same reason, the terms that we add are negligible. At last we take  $\xi = \nu_s - \nu'$  as a new variable. We find then:

$$a_{120\dots 0\dots} = -\frac{1}{r} \frac{16\pi^3 e^2}{c^2 h} \frac{\nu X \nu' X'}{2\pi i(\nu - \nu') + \gamma} \int_{-\infty}^{+\infty} \frac{\sin(2\pi r/c)(\nu' + \xi)(1 - e^{-2\pi i t \xi})}{2\pi i \xi} d\xi. \quad (67)$$

The integral may be written:

$$\begin{aligned} & \int_{-\infty}^{+\infty} \frac{(\sin(2\pi r \nu'/c) \cos(2\pi r \xi/c) + \cos(2\pi r \nu'/c) \sin(2\pi r \xi/c))(1 - \cos 2\pi t \xi + i \sin 2\pi t \xi)}{2\pi i \xi} d\xi \\ &= \frac{1}{2\pi i} \sin \frac{2\pi r \nu'}{c} \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \cos \frac{2\pi r}{c} \xi (1 - \cos 2\pi t \xi) \\ &+ \frac{1}{2\pi} \cos \frac{2\pi r \nu'}{c} \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \sin \frac{2\pi r}{c} \xi \sin 2\pi t \xi \\ &+ \frac{1}{2\pi i} \cos \frac{2\pi r \nu'}{c} \int_{-\infty}^{\infty} \sin \frac{2\pi r}{c} \xi (1 - \cos 2\pi t \xi) \frac{d\xi}{\xi} \end{aligned}$$



$$+ \frac{1}{2\pi} \sin \frac{2\pi r\nu'}{c} \int_{-\infty}^{\infty} \cos \frac{2\pi r}{c} \xi \sin 2\pi t \xi \frac{d\xi}{\xi}.$$

The two first integrals are evidently zero, since the integrated functions are odd. The other integrals can be calculated at once by the integral formulas:

$$\int_{-\infty}^{\infty} \frac{\sin qx}{x} dx = \pi; \quad \int_{-\infty}^{\infty} \frac{\sin qx \cos px}{x} dx = \begin{cases} \pi & \text{for } q > p \\ 0 & \text{for } q < p. \end{cases}$$

We find that the integral in (67) is given by:

$$\begin{cases} 0 & \text{for } r/c > t \\ (1/2i)e^{2\pi i r\nu'/c} & \text{for } t > r/c. \end{cases}$$

Substituting in (67) we find:

$$a_{120\dots 0\dots} = \begin{cases} 0 & \text{for } t < r/c \\ \frac{1}{r} \frac{8\pi^3 i e^2}{c^2 h} \frac{\nu X\nu' X'}{2\pi i(\nu - \nu') + \gamma} e^{2\pi i r\nu'/c} & \text{for } t > r/c. \end{cases} \quad (68)$$

The square modulus of  $a_{120\dots}$  measures the probability of finding the second atom excited. This probability is therefore zero if  $t < r/c$ , i.e., until the time necessary for the light emitted by the first atom to reach the second. After that time the probability that the second atom is excited, is:

$$|a_{120\dots}|^2 = \frac{1}{r^2} \left[ \frac{16\pi^3 e^2}{c^2 h} \nu X\nu' X' \right]^2 \frac{1}{4\pi^2(\nu' - \nu)^2 + \gamma^2}.$$

We notice that this probability is inversely proportional to the square of the distance  $r$ ; we conclude therefore that the theory gives correctly the velocity of propagation of light and the decrease of intensity with the distance from the source of light.

### §10. Theory of the Lippman fringes

The Lippman fringes are produced as light is reflected from a mirror perpendicular to the direction of propagation. They consist in the system of standing waves formed by the incident and the reflected waves. We will show now, how this phenomenon may be explained by the quantum theory of radiation.

We must consider a plane mirror  $S$  and two atoms, a first atom  $A$  which emits the light and a second atom  $B$  which absorbs it. We suppose that the atom  $A$  (light source) is very far from the mirror, so that the waves reaching the mirror are very nearly plane waves. On the contrary we suppose the atom  $B$  not very far from the mirror, and we will show that the probability of excitation of  $B$  depends periodically on its distance from the mirror exactly as is to be expected in the classical theory from the position of the maxima and minima of the standing waves.

We simplify the calculations by supposing that the straight line  $AB$  (which we take as  $x$ -axis) is perpendicular to the mirror. As origin of coor-

ordinates we take the intersection of this line with the mirror; the coordinates of the atom  $A$  are  $x, 0, 0$ , and those of  $B$  are  $x', 0, 0$ . Then  $x \gg x'$ . We suppose further as in the preceding chapter, that the mean life of  $A$  is very short and that of  $B$  is very long; and that the vectors  $X$  and  $X'$  which determine the transition probability from the state 2 to state 1 for both atoms reduce to the only  $y$ -component.

We have always considered the radiation contained in a volume  $\Omega$ ; in our case it is convenient to take the mirror  $S$  as one of the walls limiting the space  $\Omega$ . As  $\Omega$  becomes infinite the wall  $S$  remains fixed and all the other walls are taken to infinite distance.

Every standing vibration constituting the radiation field must have  $S$  as a nodal plane. Its  $y$ -component must have therefore the form:

$$B_s = \frac{Y_s}{2^{1/2}} \left\{ \sin \left[ \frac{2\pi\nu_s}{c_s} (\alpha_{sx}x + \alpha_{sy}y + \alpha_{sz}z) + \beta_s \right] - \sin \left[ \frac{2\pi\nu_s}{c_s} (-\alpha_{sx}x + \alpha_{sy}y + \alpha_{sz}z) + \beta_s \right] \right\} \quad (70)$$

where  $1/2^{1/2}$  is a normalization factor;  $Y_s$  is the  $y$ -component of the unit vector  $A_s$ .

By exactly the same method that we used for the deduction of (63) we find now a very similar equation, obtained by substituting in (63) for  $A_s \sin \Gamma_s$  and  $A_s \sin \Gamma_s'$  the values  $B_s$  and  $B_s'$  which are obtained by putting into (70) the coordinates  $x, 0, 0$  and  $x', 0, 0$  of the atoms  $A$  and  $B$ . This formula is:

$$a_{120\dots 0\dots} = \frac{16\pi^3 e^2}{\Omega h} \nu\nu' \sum_s \frac{B_s X B_s' X'}{\nu_s} \frac{1}{-2\pi i(\nu_s - \nu) + \gamma} \frac{1 - e^{-2\pi i(\nu_s - \nu')t}}{-2\pi i(\nu_s - \nu')}.$$

The sum over  $s$  may be effected, as in the preceding chapter by taking first the mean value of  $B_s B_s'$  over all the phases and orientations of the radiation components. We find thus the following formula corresponding to (65):

$$\overline{B_s B_s'} = \frac{c}{8\pi\nu_s(x-x')} \sin \frac{2\pi\nu_s(x-x')}{c} - \frac{c}{8\pi\nu_s(x+x')} \sin \frac{2\pi\nu_s(x+x')}{c}. \quad (72)$$

We substitute this expression in (71), multiply by

$$(8\pi/c^3)\Omega\nu_s^2 d\nu_s$$

and integrate over  $\nu_s$ . We find thus:

$$a_{120\dots 0\dots} = R^- - R^+ \quad (73)$$

where  $R^-$  and  $R^+$  represent two terms equal to the right hand side of (68) where in place of  $r$  respectively  $x-x'$  and  $x+x'$  has been substituted; we get:

$$R^- = \begin{cases} 0 & \text{for } t < \frac{x-x'}{c} \\ \frac{1}{x-x'} \frac{8i\pi^3 e^2}{c^2 h} \frac{\nu X \nu' X'}{-2\pi i(\nu' - \nu) + \gamma} e^{2\pi i \nu' (x-x')/c} & \text{for } t > \frac{x-x'}{c} \end{cases} \quad (74)$$

$$R^+ = \begin{cases} 0 & \text{for } t < \frac{x+x'}{c} \\ \frac{1}{x+x'} \frac{8i\pi^3 e^2}{c^2 h} \frac{\nu X \nu' X'}{-2\pi i(\nu' - \nu) + \gamma} e^{2\pi i \nu' (x+x')/c} & \text{for } t > \frac{x+x'}{c} \end{cases}$$

The two terms  $R^-$  and  $R^+$  clearly represent the effect of the incident and the reflected wave. Let us now consider a time  $t > (x+x')/c$ . Then both for  $R^-$  and  $R^+$  the second expressions are valid and we get from (73)

$$a_{120\dots 0\dots} = \frac{8i\pi^3 e^2}{c^2 h} \frac{\nu X \nu' X'}{-2\pi i(\nu' - \nu) + \gamma} \left\{ \frac{1}{x-x'} e^{2\pi i \nu' (x-x')/c} - \frac{1}{x+x'} e^{2\pi i \nu' (x+x')/c} \right\}. \quad (75)$$

It is evident that this expression has a large value if the two exponential factors have the same phase. The condition of equal phase is:

$$\frac{2\pi \nu'}{c} (x - x') = \frac{2\pi \nu'}{c} (x + x') - 2\pi n$$

where  $n$  is an integer. We get from this:

$$x' = \frac{n}{2} \frac{c}{\nu'} = n \frac{\lambda'}{2}$$

where  $\lambda' = c/\nu'$  represents the wave-length corresponding to the frequency  $\nu'$ . We see therefore that the places where the probability of excitation is small (dark fringes) are planes, parallel to the mirror and spaced by a half wave-length from each other; similarly we find that the places where the probability of excitation is strong (bright fringes) are the planes situated in the middle between two dark fringes. We may conclude that the results of the quantum theory of radiation describe this phenomenon in exactly the same way as the classical theory of interference.

### §11. Theory of the Doppler effect

The change of frequency of the light emitted from a moving source is very simply explained by the wave theory of light. But it finds also a simple, through apparently very different, explanation in the light-quantum theory; it can be shown that the Doppler effect may be deduced from the conservation of energy and momentum in the emission process.

Let us consider an atom  $A$  with two energy levels  $w_1$  and  $w_2$ ; the frequency emitted by the atom when it is at rest is then

$$\nu = (w_2 - w_1)/h.$$

Let us now suppose that the atom is excited and that it moves with velocity  $V$ ; its total energy is then:

$$w_2 + \frac{1}{2}mV^2.$$

At a given instant the atom emits, on jumping down to the lower state, a quantum of frequency  $\nu'$ ; the recoil of the emitted quantum produces a slight change of the velocity, which after the emission becomes  $V'$ ; the energy of

the atom is then  $w_1 + \frac{1}{2}mV'^2$ . We get therefore from the conservation of energy

$$h\nu' = (w_2 + \frac{1}{2}mV^2) - (w_1 + \frac{1}{2}mV'^2) = h\nu + \frac{1}{2}m(V^2 - V'^2). \quad (76)$$

The conservation of momentum gives:

$$m\mathbf{V}' = m\mathbf{V} - \frac{h\nu'}{c}$$

where the bold face letters mean vectors. Taking the square we get:

$$m^2V'^2 = m^2V^2 + \frac{h^2\nu'^2}{c^2} - 2mV \frac{h\nu}{c} \cos \theta$$

$\theta$  being the angle between the velocity and the direction of emission. From this equation and (76) we get, neglecting terms in  $1/c^2$ :

$$\nu' = \nu \left( 1 + \frac{V}{c} \cos \theta \right) \quad (77)$$

which is the classic formula for the Doppler effect to a nonrelativistic approximation.

We will now work out the theory of the Doppler effect with Dirac's theory of radiation. We shall see that the interpretation of the Doppler effect in this theory is very similar to its interpretation in terms of light quanta; and it is due essentially to the changes in momentum due to the recoil of the emitted light. In all the examples which we have worked out till now, we have used the approximation (45), which is obtained by supposing that the portion of space where the electron moves is so small, that the phases  $\Gamma_s$  of the standing vibrations can be considered as constants in it. Now we shall see that this simplification can no longer be made if we wish our theory to represent also the impulse properties of light quanta. So that for the theory of the Doppler effect and the Compton effect it is necessary to consider that the phases  $\Gamma_s$  are actually variables.

We simplify the problem by considering the emitting atom as constituted by a proton (charge  $e$ , mass  $m_1$ , coordinates  $x_1, y_1, z_1 \equiv X_1$ ) and an electron (charge  $-e$ , mass  $m_2$ , coordinates  $x_2, y_2, z_2 \equiv X_2$ ). The Hamilton function for the system consisting of this atom and the radiation field is the obvious generalization of (15); it is:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + eV + \sum_s \left( \frac{1}{2}p_s^2 + 2\pi^2\nu_s^2 p_s^2 \right) - \frac{e}{m_1} \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_s (A_s p_1) \sin \Gamma_{s1} \\ + \frac{e}{m_2} \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_s (A_s p_2) \sin \Gamma_{s2}.$$

We take now as new coordinates: the coordinates

$$\xi = \frac{m_1 X_1 + m_2 X_2}{m_1 + m_2}$$

of the center of gravity, and the relative coordinates of the two particles:

$$X = X_1 - X_2.$$

The momenta conjugated to  $\xi$  and  $X$  are:

$$\eta = M\xi \text{ and } p = m \left( \frac{p_1}{m_1} - \frac{p_2}{m_2} \right)$$

where  $M = m_1 + m_2$  is the total mass of the atom and  $m = m_1 m_2 / (m_1 + m_2)$  is the relative mass.

We make the assumption that the dimensions of the atom are very small compared with the wave-length. We may then substitute for  $\Gamma_{s_1}$  and  $\Gamma_{s_2}$  the value:

$$\Gamma_s = \frac{2\pi\nu_s}{c}(\alpha_s, \xi) + \beta_s \quad (78)$$

of the phase in the center of gravity. The Hamiltonian of the system becomes then:

$$H = \frac{\eta^2}{2M} + \frac{p^2}{2m} + eV(X) + \sum_s \left( \frac{1}{2} p_s^2 + 2\pi^2 \nu_s^2 q_s^2 \right) - \frac{e}{m} \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_s (A_s, p) \sin \Gamma_s. \quad (79)$$

We consider now the last term

$$\mathfrak{H} = - \frac{e}{m} \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_s (A_s, p) \sin \Gamma_s \quad (80)$$

as the perturbation. The unperturbed Hamiltonian is:

$$H_0 = \frac{\eta^2}{2M} + \frac{p^2}{2m} + eV(X) + \sum_s \left( \frac{1}{2} p_s^2 + 2\pi^2 \nu_s^2 q_s^2 \right). \quad (81)$$

The first term of (81) is the Hamiltonian of the translatory motion of the center of gravity; the corresponding eigenfunctions are:

$$\Omega^{-1/2} e^{2\pi i(\eta_n, \xi) / h} \quad (82)$$

where  $\Omega^{-1/2}$  is a normalization factor;  $\eta_n$  represents the momentum of the atom, which can be supposed to assume discrete values, since the atom can move in a finite volume  $\Omega$ .

The second and third term of  $H_0$  represent the Hamiltonian of the internal coordinates of the atom; the corresponding eigenfunctions are  $u_n$ . The last term of  $H_0$  is the Hamiltonian of the radiation components whose eigenfunctions are:

$$u_{n_1} u_{n_2} \cdots u_{n_s} \cdots$$

The eigenfunctions of the unperturbed system are given by the product:

$$\phi_{nn'n_1 \cdots n_s \cdots} = \Omega^{-1/2} e^{2\pi i(\eta_n, \xi) / h} u_n u_{n_1} \cdots u_{n_s} \quad (83)$$

and the corresponding eigenvalue, as in (34) is the sum

$$E_{nn'n_1 \dots n_s \dots} = \eta_n^2/2M + E_{n'} + h\nu_1 n_1 + \dots + h\nu_s n_s + \dots \quad (84)$$

The probability of the state  $nn'n_1 \dots n_s \dots$  is given by the square of the modulus of the quantity:

$$a_{nn'n_1 \dots n_s \dots}$$

The  $a$ 's, according to the general formula (26) satisfy differential equations, analogous to (47)

$$\dot{a}_{nn'n_1 \dots n_s \dots} = \frac{2\pi i}{h} \sum_{mm'm_1 \dots m_s \dots} a_{mm'm_1 \dots m_s \dots} \mathcal{H}_{nn' \dots; mm' \dots} e^{2\pi i(E_{mm' \dots} - E_{nn' \dots})t/h} \quad (85)$$

where  $\mathcal{H}_{nn' \dots; mm' \dots}$  represents, according to (27), the matrix element of the perturbation energy (80) corresponding to the transition from the state with indices  $m, m', m_1, \dots$  to the state  $n, n', n_1, \dots$ ; we have:

$$\mathcal{H}_{nn' \dots; mm' \dots} = \int \dots \int \dots \bar{\phi}_{nn' \dots} \mathcal{H} \phi_{mm' \dots} d\xi dX dq_1 \dots \quad (86)$$

Substituting in (86) the expressions (80) and (83) we see that the integral (86) splits in the product of integrals. We are interested in the factor containing the coordinates  $\xi$  of the center of gravity; this factor is, remembering (78):

$$\frac{1}{\Omega} \int e^{-2\pi i(\eta_n \xi)/h} \sin \left\{ \frac{2\pi \nu_s}{c} (\alpha_s, \xi) + \beta_s \right\} e^{2\pi i(\eta_m \xi)/h} d\xi \quad (87)$$

where  $d\xi$  represents the element of volume. Expressing the sine in terms of exponentials, (87) becomes:

$$\frac{e^{i\beta_s}}{2i\Omega} \int e^{(2\pi i/h) [\eta_m - \eta_n + (h\nu_s/c)\alpha_s, \xi]} d\xi - \frac{e^{-i\beta_s}}{2i\Omega} \int e^{(2\pi i/h) [\eta_m - \eta_n - (h\nu_s/c)\alpha_s, \xi]} d\xi.$$

The integrals have generally a value very near to zero, they are very different from zero only if

$$\eta_m - \eta_n \pm (h\nu_s/c)\alpha_s = 0 \quad (88)$$

since in this case one of the exponentials is equal to unity. Eq. (88) is the condition of conservation of momenta, since  $\eta_m$  and  $\eta_n$  are the momenta of the motion of the center of gravity of the atom and  $(h\nu_s/c)\alpha_s$  is the moment of the emitted quantum; the double sign arises from the fact that the  $s$ -component of the radiation field is a standing vibration which may be considered as resulting from two progressing waves moving in the opposite directions defined by the unit vectors  $+\alpha_s$  and  $-\alpha_s$ .

We see therefore that the conservation of momentum in the emission process follows from the radiation theory. That also the energy principle must hold results from Eq. (85), since only the terms with

$$E_{mm' \dots} = E_{nn' \dots}$$

give an important contribution.

Now we have shown that the formulas for the Doppler change of frequency can be derived if we assume the conservation of energy and momentum. The Doppler formulas can therefore be deduced from the radiation theory. We could also deduce from this theory the formulas for the intensity of radiation in different directions; the results are identical with the results of the classical theory.

### §12. Scattering of radiation from free electrons

The scattering of radiation from free electrons can be considered in two different approximations. The first approximation leads to the classical Thomson formula for the intensity of the scattered radiation with no change of frequency. The second approximation gives the phenomena of the Compton effect where the momenta of light quanta are considered. We shall see also that the theory of scattering carried out by help of Dirac's relativistic wave function for the electron is essentially different from the present non-relativistic theory. In this section we shall always discuss the nonrelativistic theory.

It can be shown that the simple interaction term (16) between the radiation and the electron is not responsible for the scattering from free electrons, not even in first approximation. This is connected with the well-known fact that the free electron has no probability of spontaneous transition between two states  $n$  and  $m$  with different velocity. The interaction term (16) involves the spontaneous transitions; therefore it may be neglected for the case of the free electron.

The interaction term which is responsible for the scattering is the term (17) which we have hitherto neglected, since it is smaller than (16) and only becomes important if the effect of (16) is zero. Let  $m$  and  $n$  be two (translational) states of the free electrons. The corresponding eigenfunctions can be written:

$$\psi_n = \Omega^{-1/2} e^{2\pi i(p_n, x)/h}; \quad \psi_m = \Omega^{-1/2} e^{2\pi i(p_m, x)/h} \quad (89)$$

where  $p_n$  and  $p_m$  represent the momentum of the electron in the states  $n$  and  $m$ ;  $\Omega^{-1/2}$  is the normalization factor.

To get a scattering different from zero, we need consider the interaction term (17). By help of (12) this term can be written in the form (18). We must first calculate the matrix elements corresponding to (18); then we shall substitute them in a formula analogous to (26) in order to find out the variations of the  $a$ 's with the time. This calculation may be carried out to two different approximations. In the first one the supposition is made, that the phase  $\Gamma_s$  of the waves can be regarded as a constant; this approximation yields simply Thomson's formula for the intensity of the scattered radiation and is equivalent to neglecting the momentum properties of light quanta. In order to get the theory of the Compton shift of wave-length it is necessary to consider the dependence of the phases  $\Gamma$  on the coordinates. We shall at present restrict ourselves to the first approximation and consider the  $\Gamma_s$  as constants.

By means of (38) and (39) it may be very easily proved that a matrix element of  $H^{(2)}$ ,

$$H_{n,n_1,n_2,\dots;m,m_1,m_2,\dots}^{(2)}$$

is only different from zero if the following conditions exist:

- (a)  $n = m$
  - (b) the numbers  $n_1 n_2 \dots$  with the exception of only two, say  $n_s$  and  $n_\sigma$ , are equal to the corresponding indices  $m_1 m_2 \dots$ .
  - (c)  $m_s = n_s \pm 1$  and  $m_\sigma = n_\sigma \pm 1$  (where the  $\pm$  signs are incoherent).
- If these conditions are satisfied we get:

$$H_{n_s, n_\sigma; n_s \pm 1, n_\sigma \pm 1}^{(2)} = \frac{e^2 \hbar (A_s A_\sigma)}{\pi m \Omega} \frac{\sin \Gamma_s \sin \Gamma_\sigma}{(\nu_s \nu_\sigma)^{1/2}} \left| \begin{matrix} (n_s + 1)^{1/2} \\ n_s^{1/2} \end{matrix} \right| \left| \begin{matrix} (n_\sigma + 1)^{1/2} \\ n_\sigma^{1/2} \end{matrix} \right| \quad (90)$$

where in the last two terms the upper or the lower expression must be taken according to the  $+$  or  $-$  sign in  $n_s \pm 1$  or in  $n_\sigma \pm 1$ . Putting these matrix elements in the general formula (26) we get the following differential equations for the  $a$ 's:

$$\begin{aligned} \dot{a}_{n,n_1,n_2,\dots,n_\sigma,\dots} &= -\frac{2ie^2}{m\Omega} \sum_{\sigma} \frac{(A_s A_\sigma)}{(\nu_s \nu_\sigma)^{1/2}} \sin \Gamma_s \sin \Gamma_\sigma \\ &\times [a_{nn_1 \dots n_s+1 \dots n_\sigma+1 \dots} [(n_s + 1)(n_\sigma + 1)]^{1/2} e^{-2\pi i(\nu_s + \nu_\sigma)t} \\ &+ a_{nn_1 \dots n_s-1 \dots n_\sigma+1 \dots} [n_s(n_\sigma + 1)]^{1/2} e^{+2\pi i(\nu_s - \nu_\sigma)t} \\ &+ a_{nn_1 \dots n_s+1 \dots n_\sigma-1 \dots} [(n_s + 1)n_\sigma]^{1/2} e^{-2\pi i(\nu_s - \nu_\sigma)t} \\ &+ a_{nn_1 \dots n_s-1 \dots n_\sigma-1 \dots} (n_s n_\sigma)^{1/2} e^{+2\pi i(\nu_s + \nu_\sigma)t}]. \end{aligned} \quad (91)$$

To get the intensity of the scattered light, we make the assumption that at the time  $t=0$

$$a_{n,0,0,\dots,0,\dots} = 1 \quad (92)$$

while all the other  $a$ 's are zero. This means physically that there is a certain amount of radiation in the field; this is the primary radiation of frequency  $\nu_s$  and energy density

$$w_s = n_s \hbar \nu_s / \Omega. \quad (93)$$

We must find the intensity of the radiation scattered in the component  $\sigma$ . The probability that a quantum is scattered in the radiation component  $\sigma$  is given by the square modulus of

$$a_{n,0,\dots,n_s-1,\dots,1_\sigma,\dots}$$

If we limit ourselves to a very short time  $t$ , we might still assume (92) to be valid in first approximation; and we obtain therefore from (91):

$$\dot{a}_{n,0,\dots,n_s-1,\dots,1_\sigma,\dots} = -\frac{2ie^2}{m\Omega} (A_s A_\sigma) \frac{\sin \Gamma_s \sin \Gamma_\sigma}{(\nu_s \nu_\sigma)^{1/2}} n_s^{1/2} e^{-2\pi i(\nu_s - \nu_\sigma)t} \quad (94)$$

since only the third term in the square brackets gives a result different from zero. Integration with respect to time, with the initial condition  $a_{n,0,\dots,n_s-1,-\dots,1_\sigma,\dots} = 0$  for  $t=0$  yields:



$$a_{n_1 \dots n_s - 1 \dots 1 \sigma \dots} = \frac{e^2}{\pi m \Omega} \frac{(A_s A_\sigma) \sin \Gamma_s \sin \Gamma_\sigma}{(\nu_s \nu_\sigma)^{1/2}} n_s^{1/2} \frac{e^{-2\pi i(\nu_s - \nu_\sigma)t} - 1}{\nu_s - \nu_\sigma}.$$

The probability for the scattering of a quantum in the component  $\sigma$ , is the square modulus of this expression, i.e.,

$$\frac{4e^4}{\pi^2 m^2 \Omega^2} \frac{(A_s A_\sigma)^2 \sin^2 \Gamma_s \sin^2 \Gamma_\sigma}{\nu_s \nu_\sigma} n_s \frac{\sin^2 \pi(\nu_s - \nu_\sigma)t}{(\nu_s - \nu_\sigma)^2}.$$

Summing over the index  $\sigma$  we obtain the probability for the scattering of a quantum in any component of radiation. This sum may be transformed into an integral by a method similar to that described in section 8. We first substitute for  $(A_s A_\sigma)^2$ ,  $\sin^2 \Gamma_s$ ,  $\sin^2 \Gamma_\sigma$  their mean values  $1/3$ ,  $1/2$ ,  $1/2$ . Then we multiply by the number of radiation components  $(8\pi/c^3)\Omega\nu_\sigma^2 d\nu_\sigma$  with frequency between  $\nu_\sigma$  and  $\nu_\sigma + d\nu_\sigma$  and integrate over  $\nu_\sigma$ . We obtain:

$$\frac{8}{3\pi} \frac{e^4}{c^3 m^2} \frac{n_s}{\Omega \nu_s} \int_0^\infty \nu_s \frac{\sin^2 \pi(\nu_s - \nu_\sigma)t}{(\nu_s - \nu_\sigma)^2} d\nu_\sigma.$$

The integral can be evaluated by observing that its values are all in the immediate neighborhood of  $\nu_\sigma = \nu_s$ . (The scattered radiation has the same frequency as the primary light.) We can therefore extend the limits of integration from  $-\infty$  to  $+\infty$  and substitute  $\nu_s$  for the first  $\nu_\sigma$ . By the integral formula

$$\int_{-\infty}^{+\infty} \frac{\sin^2 kx}{x^2} dx = \pi k \quad (95)$$

we obtain:

$$N = \frac{8\pi}{3} \frac{e^4}{c^3 m^2} \frac{n_s}{\Omega} t \quad (96)$$

as the number of quanta scattered during the time  $t$ . The scattered energy is:

$$h\nu_s N = \frac{8\pi}{3} \frac{e^4}{c^3 m^2} w_s t. \quad (97)$$

This expression coincides with the well-known formula derived by Thomson from the classical theory of radiation.

The theory carried out to this approximation gives no account of the Compton shift of wave-length, which is due to the momentum of light quanta. A theory of the Compton effect may be obtained if we calculate the matrix element:

$$H_{n, n_1 \dots; m, m_1 \dots}^{(2)} \quad (98)$$

without the assumption, which we have hitherto made, that the phase  $\Gamma_s$  of the radiation components can be considered as a constant over the space occupied by the electron. From (18) and the expressions (89) of the eigenfunctions of the electron we see that the factor in the matrix element (98) depending on  $x, y, z$  is:

$$(1/\Omega) \int e^{-2\pi i(p_m x)/h} \sin \Gamma_s \sin \Gamma_\sigma e^{+2\pi i(p_n x)/h} dx dy dz. \quad (99)$$

We write the sine functions in terms of exponentials and remember that

$$\Gamma_s = \frac{2\pi\nu_s}{c}(\alpha_s, X) + \beta_s; \quad \Gamma_\sigma = \frac{2\pi\nu_\sigma}{c}(\alpha_\sigma, X) + \beta_\sigma.$$

We see therefore that (99) splits in a sum of terms like:

$$\pm \frac{e^{\pm i\beta_s \pm i\beta_\sigma}}{4\Omega} \int e^{2\pi i(p_m - p_n \pm (h\nu_s/c)\alpha_s \pm (h\nu_\sigma/c)\alpha_\sigma, X) / h} dx dy dz.$$

If the coefficient of  $X$  in the exponent is considerably different from zero, the integral over the space  $\Omega$  vanishes, since the exponential is a rapidly varying function with mean value zero. We get a term different from zero only if the coefficient is practically zero. That is:

$$p_m - p_n \pm \frac{h\nu_s}{c} \alpha_s \pm \frac{h\nu_\sigma}{c} \alpha_\sigma = 0. \quad (100)$$

This is simply the condition of conservation of momentum. The double signs arise from the fact that a stationary wave is the superposition of two progressive waves of opposite directions.

From the conservation of momentum, the Compton wave-length change could be deduced with a method very analogous to the one used in the ordinary theory of Compton effect. We will not enter into the details of this theory, from which even intensity formulas in nonrelativistic approximation can be derived.

## PART II. THEORY OF RADIATION AND DIRAC'S WAVE EQUATION

In the second part of this work we shall first show how the general formulas of the previous section can be derived if we take as a basis Dirac's relativistic wave equation for the electron, instead of Schrödinger's equation. After this we shall study the very peculiar role which is played in the theory of light scattered from free electrons by the states of negative energy, characteristic of Dirac's theory of the electron. We shall also discuss the possibility of radiative transitions from states of positive to states of negative energy of a Dirac electron. These transitions in reality certainly do not take place; nevertheless it has some interest to see how they are derived from the present theory, since a correct theory should find some way of preventing them.

### §13. Dirac's wave function of the electron

We shall in this paragraph collect some formulas on Dirac's wave function which will be of use later.

It is well known that in Dirac's relativistic theory the electron at a given time is specified by four coordinates. Three of them,  $x$ ,  $y$ ,  $z$ , are the ordinary

positional coordinates; the fourth coordinate  $\sigma$  will represent some internal degree of freedom; we shall call  $\sigma$  the spin coordinate. While the coordinates  $x, y, z$  have a continuous range of variability from  $-\infty$  to  $+\infty$ , the spin coordinate  $\sigma$  can assume four values only; it is no limitation of generality to call these four values 1, 2, 3, 4. The wave function  $\psi$  will depend on  $x, y, z, \sigma$ :

$$\psi = \psi(x, y, z, \sigma). \quad (101)$$

Since the variable  $\sigma$  takes only a finite set of values, it is often convenient to write it as an index:

$$\psi = \psi_\sigma(x, y, z). \quad (102)$$

The wave function is thus represented by a set of four functions  $\psi_1(x, y, z), \psi_2(x, y, z), \psi_3(x, y, z), \psi_4(x, y, z)$  of the space coordinates only.

In Dirac's theory of the electron two types of operators are to be considered. The operators of the first kind act on the dependence of  $\psi$  on the space coordinates; for instance:

$$x^2, xy, p_x = \frac{h}{2\pi i} \frac{\partial}{\partial x}, \dots$$

A second kind of operators acts on the dependence of  $\psi$  on  $\sigma$ . The most general type of linear operator of this kind is a linear substitution on the four  $\psi_1, \psi_2, \psi_3, \psi_4$ . Therefore these operators are represented by matrices of the linear substitution; they have four rows and four columns. We shall consider chiefly four operators of this kind, *viz.*,

$$\begin{aligned} \gamma_x &= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}; & \gamma_y &= \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}; \\ \gamma_z &= \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}; & \delta &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}. \end{aligned} \quad (103)$$

For instance the effect of the operator  $\gamma_x$  applied to the eigenfunction  $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$  is to change it in  $\gamma_x\psi = (\psi_4, \psi_3, \psi_2, \psi_1)$ . Similarly

$$\gamma_y\psi = (-i\psi_4, i\psi_3, -i\psi_2, i\psi_1); \quad \gamma_z\psi = (\psi_3, -\psi_4, \psi_1, -\psi_2); \quad (104)$$

$$\delta\psi = (\psi_1, \psi_2, -\psi_3, -\psi_4).$$

The  $\gamma$ 's and  $\delta$  satisfy the well-known relations

$$\begin{aligned} \gamma_x^2 = \gamma_y^2 = \gamma_z^2 = \delta^2 = 1; \quad \gamma_y\gamma_z + \gamma_z\gamma_y = 0 \text{ and similar} \\ \gamma_x\delta + \delta\gamma_x = 0 \text{ and similar.} \end{aligned}$$

We will very often summarize  $\gamma_x, \gamma_y, \gamma_z$  by a  $q$ -vector (that is a vector whose

components are q-numbers)  $\gamma$ . It is well known that  $\gamma_x, \gamma_y, \gamma_z, c\delta$  can be considered to transform as the four components of a four vector; our vector  $\gamma$  is the space component of this four vector.

Now we write down the well-known Dirac's relativistic Hamilton function for the electron in the form:

$$W = eV - c \left( \gamma, p - \frac{eU}{c} \right) - mc^2\delta. \quad (105)$$

The product containing  $\gamma$  is to be considered as an ordinary scalar product;  $V$  and  $U$  are the scalar and the vector potential.

The Schrödinger equation corresponding to the Hamiltonian (105) can be written putting in evidence the four  $\psi_1\psi_2\psi_3\psi_4$ . We obtain, remembering the meaning of the operators contained in (105) the four equations:

$$(mc^2 + W - eV)\psi_1 = -\frac{ch}{2\pi i} \left[ \frac{\partial\psi_4}{\partial x} - i\frac{\partial\psi_4}{\partial y} + \frac{\partial\psi_3}{\partial z} \right] + e[(U_x - iU_y)\psi_4 + U_z\psi_3] \quad (106a)$$

$$(mc^2 + W - eV)\psi_2 = -\frac{ch}{2\pi i} \left[ \frac{\partial\psi_3}{\partial x} + i\frac{\partial\psi_3}{\partial y} - \frac{\partial\psi_4}{\partial z} \right] + e[(U_x + iU_y)\psi_3 - U_z\psi_4] \quad (106b)$$

$$(-mc^2 + W - eV)\psi_3 = -\frac{ch}{2\pi i} \left[ \frac{\partial\psi_2}{\partial x} - i\frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_1}{\partial z} \right] + e[(U_x - iU_y)\psi_2 + U_z\psi_1] \quad (106c)$$

$$(-mc^2 + W - eV)\psi_4 = -\frac{ch}{2\pi i} \left[ \frac{\partial\psi_1}{\partial x} + i\frac{\partial\psi_1}{\partial y} - \frac{\partial\psi_2}{\partial z} \right] + e[(U_x + iU_y)\psi_1 - U_z\psi_2]. \quad (106d)$$

It can be proved that the dissymmetry in these equations is only apparent; it arises in some way from the fact, that the spin coordinate has been referred to the  $z$ -axis, which has therefore a different treatment.

The energy  $W$  contains the intrinsic energy  $mc^2$  of the electron; its values are therefore in the neighborhood of  $mc^2$ . As well known, the Hamiltonian (105) has also besides these "normal" eigenvalues, "anomalous" ones, which lie near the value  $-mc^2$ . These negative eigenvalues which have certainly for the electron no physical meaning, would correspond in some way to states of an electron with negative mass. They are supposed to be due to some fault either in the theory or in its interpretation but the tentative assumptions which have been made to get a correct theory cannot at present be claimed successful. In the following sections we shall see the importance of the negative states for the interpretation of actual phenomena, e.g., the scattering of light. Any theory which would try to get rid of the negative states by simply striking them away, should be very careful not to remove the scattering properties of the electron at the same time.

We will now consider for a moment some properties of the normal states with positive energy. Since the energy  $W$  lies near  $mc^2$ , it can be conveniently written:

$$W = mc^2 + E \quad (107)$$

where  $E$  represents the ordinary energy without the term representing the intrinsic energy. For the sake of simplicity we neglect in (106) the terms depending on the vector potential  $U$ . We see now that in Eqs. (106a) and (106b) the coefficient of  $\psi_1$  and  $\psi_2$  is very large:  $(2mc^2 + E - eV)$ , while  $\psi_3$  and  $\psi_4$  in the first side of (106c) and (106d) have a much smaller coefficient:  $(E - eV)$ . From this we infer that  $\psi_1$  and  $\psi_2$  are much smaller than  $\psi_3$  and  $\psi_4$ . From (106a) and (106b), neglecting in a first nonrelativistic approximation  $E - eV$  with respect to  $mc^2$ , we obtain:

$$\left. \begin{aligned} \psi_1 &= \frac{ih}{4\pi mc} \left( \frac{\partial \psi_4}{\partial x} - i \frac{\partial \psi_4}{\partial y} + \frac{\partial \psi_3}{\partial z} \right) \\ \psi_2 &= \frac{ih}{4\pi mc} \left( \frac{\partial \psi_3}{\partial x} + i \frac{\partial \psi_3}{\partial y} - \frac{\partial \psi_4}{\partial z} \right) \end{aligned} \right\} \quad (108)$$

(from these equations we see that  $\psi_1$  and  $\psi_2$  are smaller than  $\psi_3$  and  $\psi_4$  by a factor of the order of magnitude  $v/c$ ). We substitute (108) in (106c) and (106d) always putting the  $U$ 's equal to zero, and obtain both for  $\psi_3$  and  $\psi_4$  the ordinary Schrödinger equation

$$\left. \begin{aligned} (E - eV)\psi_3 + \frac{\hbar^2}{8\pi^2 m} \Delta \psi_3 &= 0 \\ (E - eV)\psi_4 + \frac{\hbar^2}{8\pi^2 m} \Delta \psi_4 &= 0 \end{aligned} \right\} \quad (109)$$

We see therefore that in the nonrelativistic approximation  $\psi_3$  and  $\psi_4$  are eigenfunctions of the ordinary Schrödinger problem, corresponding to the same eigenvalue. Therefore if there is no degeneration in the Schrödinger problem  $\psi_3$  and  $\psi_4$  can differ only by a constant factor, from the normalized eigenfunction corresponding to the eigenvalue  $E$  in Schrödinger's equation

$$(E - eV)w + \frac{\hbar^2}{8\pi^2 m} \Delta w = 0. \quad (110)$$

We can take for instance either:

$$\psi_3 = w, \quad \psi_4 = 0$$

or:

$$\psi_3 = 0, \quad \psi_4 = w$$

These solutions correspond to the two possible orientations of the spin with respect to the  $z$ -axis. From these expressions for  $\psi_3$  and  $\psi_4$ , and (108) we obtain the complete expression of the four components of  $\psi$  in the form:

$$\left. \begin{aligned} \psi_\alpha &= \left[ \frac{i\hbar}{4\pi mc} \frac{\partial w}{\partial z}, \frac{i\hbar}{4\pi mc} \left( \frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right), w, 0 \right] \\ \psi_\beta &= \left[ \frac{i\hbar}{4\pi mc} \left( \frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right), -\frac{i\hbar}{4\pi mc} \frac{\partial w}{\partial z}, 0, w \right] \end{aligned} \right\} \quad (111)$$

$\psi_\alpha$  and  $\psi_\beta$  are the eigenfunctions corresponding to the two orientations of the spin.

We will now write also the expressions of the exact Dirac wave function for the case where there are no forces acting on the electron ( $V=0$ ,  $U=0$ ) and the components of the momentum  $p_x$ ,  $p_y$ ,  $p_z$  are therefore constants. Corresponding to these values of the momentum components,  $\psi$  must contain the space coordinates  $x$ ,  $y$ ,  $z$  in the exponential factor  $e^{2\pi i(p_x x + p_y y + p_z z)/\hbar}$ . The four components of  $\psi$  will therefore be products of four constants  $B_1$ ,  $B_2$ ,  $B_3$ ,  $B_4$  by this factor:

$$\psi = [B_1, B_2, B_3, B_4] e^{2\pi i(p_x x + p_y y + p_z z)/\hbar}. \quad (112)$$

Putting in (105) this expression for  $\psi$  and taking  $V=0$ ,  $U=0$ , we get for the  $B$ 's the following equations:

$$\left. \begin{aligned} (mc^2 + W)B_1 + c(p_x - ip_y)B_4 + cp_z B_3 &= 0 \\ (mc^2 + W)B_2 + c(p_x + ip_y)B_3 - cp_z B_4 &= 0 \\ (-mc^2 + W)B_3 + c(p_x - ip_y)B_2 + cp_z B_1 &= 0 \\ (-mc^2 + W)B_4 + c(p_x + ip_y)B_1 - cp_z B_2 &= 0 \end{aligned} \right\} \quad (113)$$

It can be readily proved that these linear equations have not identically vanishing solutions only if:

$$W^2 = m^2 c^4 + c^2 p^2$$

that is:

$$W = \pm (m^2 c^4 + c^2 p^2)^{1/2}. \quad (114)$$

This is the ordinary relativistic relation between energy and momentum. The + sign corresponds to the ordinary positive values of the energy; the - sign to the anomalous negative energy values. For each of the two energy values (114) there are two linear independent solutions of (113) which correspond to the two possible orientations of the spin. They can be written in the form:

$$\left. \begin{aligned} [B_1, B_2, B_3, B_4] &= \left(1 + \frac{c^2 p^2}{(mc^2 + W)^2}\right)^{-1/2} \left[ -\frac{cp_z}{mc^2 + W}, -\frac{c(p_x + ip_y)}{mc^2 + W}, 1, 0 \right] \\ [B_1, B_2, B_3, B_4] &= \left(1 + \frac{c^2 p^2}{(mc^2 + W)^2}\right)^{-1/2} \left[ -\frac{c(p_x - ip_y)}{mc^2 + W}, \frac{cp_z}{mc^2 + W}, 0, 1 \right] \end{aligned} \right\} \quad (115a)$$

for the positive energy values; and in the form:

$$\left. \begin{aligned} [B_1, B_2, B_3, B_4] &= \left(1 + \frac{c^2 p^2}{(mc^2 - W)^2}\right)^{-1/2} \left[ 1, 0, \frac{cp_z}{mc^2 - W}, \frac{c(p_x + ip_y)}{mc^2 - W} \right] \\ [B_1, B_2, B_3, B_4] &= \left(1 + \frac{c^2 p^2}{(mc^2 - W)^2}\right)^{-1/2} \left[ 0, 1, \frac{c(p_x - ip_y)}{mc^2 - W}, -\frac{cp_z}{mc^2 - W} \right] \end{aligned} \right\} \quad (115b)$$

for the negative energy values.

The normalization factors  $(1 + c^2 p^2 / (mc^2 \pm W)^2)^{-1/2}$  have been so chosen that the sum of the square modulus of the  $B$ 's is unity. For vanishing momentum  $p$ , the  $B$ 's take the very simple expressions:

$$\left. \begin{aligned} [B_1 B_2 B_3 B_4] &= [0 \ 0 \ 1 \ 0] \\ [B_1 B_2 B_3 B_4] &= [0 \ 0 \ 0 \ 1] \end{aligned} \right\} W = + mc^2$$

$$\left. \begin{aligned} [B_1 B_2 B_3 B_4] &= [1 \ 0 \ 0 \ 0] \\ [B_1 B_2 B_3 B_4] &= [0 \ 1 \ 0 \ 0] \end{aligned} \right\} W = - mc^2. \quad (115c)$$

#### §14. Radiation theory in nonrelativistic approximation

We shall restrict ourselves to the case for which the electron of our atom can be considered in an electrostatic field of force, and the only nonelectrostatic forces are those due to the radiation field. In Eq. (105) we may therefore suppose that  $V$  is independent of the time, and represents the electrostatic potential of the atom, while  $U$  vanishes if we neglect the interaction of the atom and the radiation field; if we do not neglect this interaction we put for  $U$  the expression (12) of the vector potential of radiation. The Hamilton function (105) of the electron becomes then:

$$eV - c(\gamma, p) - mc^2\delta + ec \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_s (\gamma, A_s) q_s \sin \Gamma_s \quad (116)$$

where the last term represents the effect of the radiation field.

We get the Hamilton function of the complex system of the atom and the radiation field, adding to (116) the Hamiltonian (11) of the radiation. We get thus:

$$H = eV - e(\gamma, p) - mc^2\delta + \sum_s \left( \frac{p_s^2}{2} + 2\pi^2 \nu_s^2 q_s^2 \right) + ec \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_s (\gamma, A_s) q_s \sin \Gamma_s. \quad (117)$$

The Hamiltonian function (117) can be split up into the sum of an unperturbed Hamiltonian

$$H_0 = eV - c(\gamma, p) - mc^2\delta + \sum_s \left( \frac{p_s^2}{2} + 2\pi^2 \nu_s^2 q_s^2 \right) \quad (118)$$

and a perturbation term:

$$\mathcal{H} = ec \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_s (\gamma, A_s) q_s \sin \Gamma_s \quad (119)$$

representing the interaction energy.

The eigenfunctions of the unperturbed problem represented by  $H_0$  can be very easily written, since  $H_0$  is the sum of a term containing the coordinates of the electron only, and of terms each containing the only variables  $q_s, p_s$  of the  $s$ th radiation component. The unperturbed eigenfunctions are therefore

products of eigenfunctions of the atom and the radiation oscillators as in formula (31)

$$\phi_{n_1 n_2 \dots n_s \dots} = u_n u_{n_1} u_{n_2} \dots u_{n_s} \dots \quad (120)$$

where the symbols are the same as in (31).

We must now calculate the matrix elements of the perturbation (119).

$$\mathcal{H}_{n_1 n_2 \dots n_s \dots; m_1 m_2 \dots m_s \dots}$$

Its calculation is practically identical with the calculation for the derivation of (41). It is found that the only matrix elements which don't vanish identically are:

$$\mathcal{H}_{n_1 n_2 \dots n_s \dots; m_1 m_2 \dots n_s \pm 1 \dots} = ec \left( \frac{\hbar}{\pi \Omega} \right)^{1/2} \frac{1}{\nu_s^{1/2}} (D_{snm}, A_s) \begin{vmatrix} (n_s + 1)^{1/2} \\ (n_s)^{1/2} \end{vmatrix} \quad (121)$$

where the upper or the lower expression of the last factor must be taken according to the two possibilities  $m_s = n_s + 1$  or  $m_s = n_s - 1$ . The vector  $D_{snm}$  has the following meaning:

$$D_{snm} = \int \tilde{u}_n \gamma \sin \Gamma_s u_m d\omega \quad (122)$$

where integration must be extended over all the configuration space for the electron (that is: integrate over the space coordinates  $x, y, z$  from  $-\infty$  to  $+\infty$ , and sum over the four values of the spin variable).

The expression of  $D_{snm}$  can be very much simplified if we make the assumption that the dimensions of the atom are much smaller than the wavelength. In this case we may consider  $\sin \Gamma_s$  as a constant all over the space occupied by the atom and take it out of the integral (122). We get thus:

$$D_{snm} = \sin \Gamma_s \int \tilde{u}_n \gamma u_m d\omega = \sin \Gamma_s \gamma_{nm} \quad (123)$$

where  $\gamma_{nm}$  is the matrix element of the operator  $\gamma$ .

We calculate the last factor by the nonrelativistic approximation (111) for the eigenfunctions. We take first both for  $u_n$  and  $u_m$  eigenfunctions of the type  $\psi_\alpha$  (spin in the positive  $z$ -direction). We have:

$$u_m = \left[ \frac{i\hbar}{4\pi mc} \frac{\partial w_m}{\partial z}, \frac{i\hbar}{4\pi mc} \left( \frac{\partial w_m}{\partial x} + i \frac{\partial w_m}{\partial y} \right), w_m, 0 \right].$$

Let us calculate the  $x$ -component of the vector (123); remembering the meaning of  $\gamma_x$  we obtain:

$$\gamma_x u_m = \left[ 0, w_m, \frac{i\hbar}{4\pi mc} \left( \frac{\partial w_m}{\partial x} + i \frac{\partial w_m}{\partial y} \right), \frac{i\hbar}{4\pi mc} \frac{\partial w_m}{\partial z} \right].$$

We have also:

$$\tilde{u}_n = \left[ -\frac{i\hbar}{4\pi mc} \frac{\partial w_n}{\partial z}, -\frac{i\hbar}{4\pi mc} \left( \frac{\partial w_n}{\partial x} - i \frac{\partial w_n}{\partial y} \right), w_n, 0 \right].$$



We get  $\int \bar{u}_n \gamma_x u_m d\omega$ , summing up the products of the corresponding four components of the two last expressions, and integrating over all space. We get:

$$\begin{aligned} \int \bar{u}_n \gamma_x u_m d\omega &= -\frac{i\hbar}{4\pi mc} \int \left( w_m \frac{\partial w_n}{\partial x} - w_n \frac{\partial w_m}{\partial x} \right) d\tau \\ &\quad - \frac{\hbar}{4\pi mc} \int \left( w_m \frac{\partial w_n}{\partial y} + w_n \frac{\partial w_m}{\partial y} \right) d\tau \\ &= \frac{i\hbar}{2\pi mc} \int w_n \frac{\partial w_m}{\partial x} d\tau - \frac{i\hbar}{4\pi mc} \int \frac{\partial}{\partial x} (w_n w_m) d\tau - \frac{\hbar}{4\pi mc} \int \frac{\partial}{\partial y} (w_n w_m) d\tau. \end{aligned}$$

The last two integrals can be transformed by Green's theorem into surface integrals extended over a surface at infinite distance; since the eigenfunctions  $w_n$  and  $w_m$  decrease very rapidly they vanish. We obtain therefore:

$$\int \bar{u}_n \gamma_x u_m d\omega = \frac{i\hbar}{2\pi mc} \int w_n \frac{\partial w_m}{\partial x} d\tau.$$

Calculating with a similar method the components  $y$  and  $z$  we obtain vectorially:

$$\int \bar{u}_n \gamma u_m d\omega = \frac{i\hbar}{2\pi mc} \int w_n \text{grad } w_m d\tau \quad (124)$$

which shows the affinity between the operator  $\gamma$  of Dirac's theory and the expression  $-p/mc = -v/c$  of Schrödinger's theory (remember that  $p \sim (\hbar/2\pi i) \text{grad}$ ).

From (42) (observe that the  $u$ 's in (42) are the Schrödinger eigenfunctions, so they correspond to our present  $w$ 's) we obtain now:

$$\int \bar{u}_n \gamma u_m d\omega = (2\pi i/c) v_{mn} X_{nm} \quad (125)$$

where  $X_{nm}$  is the matrix element representing the radius vector in Schrödinger's approximation. From (123) and (121) we obtain at last:

$$\mathcal{H}_{n n_1 \dots n_s \dots m n_1 \dots n_s \pm 1 \dots} = 2\pi i e \left( \frac{\hbar}{\pi \Omega} \right)^{1/2} \frac{v_{mn}}{v_s^{1/2}} (A_s X_{nm}) \left[ \begin{matrix} (n_s + 1)^{1/2} \\ n_s^{1/2} \end{matrix} \right] \sin \Gamma_s \quad (126)$$

which is identical to (46) derived previously.

This shows that in the present nonrelativistic approximation the results obtained for the radiation theory with Dirac's wave function are completely identical to those derived in Part I with Schrödinger's wave function.

We notice further that (125) has been derived on the assumption that the states  $n$  and  $m$  have their spins pointing in the same direction. If their spins point in opposite directions, the result is zero. This fact can at first sight seem contradictory, since it means that there are no transitions between states with opposite spin directions. But we must remember that we have made the calculations on the hypothesis that the eigenfunctions of the Schrödinger prob-

lem are not degenerate. The effect of this is that there is no coupling between spin and orbital movement, as in  $s$ -terms; where changes of spin direction are observed, this is only due to the coupling of spin and orbit and if this coupling is loosened, as in Paschen-Back effect, no changes of spin direction actually occur.

### §15. Dirac's theory and scattering from free electrons

The theory of scattering of light from free electrons has some interest, as we have said, because it shows in a very striking way the actual importance of the states with negative energy even for very real phenomena where these mysterious states do not explicitly appear. This theory can of course be carried out either to the approximation of the Compton effect or to the approximation giving simply Thomson's coefficient of scattering and no change of frequency. Since the essential features of the theory are conserved also if we neglect the change of frequency, we shall carry out the theory to this approximation (i.e., we shall not consider the momentum properties of light quanta). The exact theory leads to the intensity formula of Klein-Nishina.

The approximation introduced by neglecting the momentum properties of light quanta is equivalent, as we have often said, to considering the phase  $\Gamma_s$  of light constant over the place occupied by the electron. We may also suppose that the velocity of the electron during the process of scattering is always negligible, since we neglect the recoil of the scattered quanta. We can take therefore the eigenfunctions of the electron in the very simple form (115c) corresponding to velocity zero. We shall indicate the four states (115c) by the indices 1, 2, 3, 4. States 1, 2 are states of positive energy  $+mc^2$ , with spin pointing in the direction  $+z$  and  $-z$ ; states 3, 4 have negative energy  $-mc^2$  with the spin pointing in the directions  $\pm z$ .

We will suppose that at the beginning ( $t=0$ ) the electron is in the state 1 and there are  $n_s$  quanta in the  $s$ -component of radiation (primary radiation). We can put therefore:

$$a_{1,0\dots n_s,0\dots} = 1 \quad (127)$$

while the other  $a$ 's are zero. To find out the amount of radiation scattered into the  $\sigma$ -component, we must find the value of  $a_{1,0\dots n_s-1,\dots 1_\sigma\dots}$  at the time  $t$ . Now the matrix element of the perturbation corresponding to this transition is zero, since two radiation components ( $s$  and  $\sigma$ ) change their quantum number; the transition can therefore occur only through an intermediate state, which can combine both with the initial and the final state. It is easily seen that there are only four such states:

$$(3, n_s - 1, 0), (4, n_s - 1, 0), (3, n_s, 1), (4, n_s, 1) \quad (128)$$

for brevity only the quantum numbers of the electron and of the radiation components  $s, \sigma$  have been indicated. The states  $(2, n_s - 1, 0)$  and  $(2, n_s, 1)$  have not been considered since it is immediately shown by the definition that  $\gamma_{12}=0$  so that these states do not combine.

The intermediate states (128) are states for which the electron has a negative energy; without these states of negative energy no scattering proc-

ess would be possible. We will now show, that the scattering calculated with the intermediate states (128) actually gives Thomson's intensity formula.

For this we must first write the matrix elements corresponding to the transition from the initial state  $(1, n_s, 0)$  to the intermediate states (128), and from these states to the final state  $(1, n_s - 1, 1)$ . From (115c) we immediately find:

$$\begin{aligned} \gamma_{x13} &= 0; \quad \gamma_{y13} = 0; \quad \gamma_{z13} = 1 \\ \gamma_{x14} &= 1; \quad \gamma_{y14} = i; \quad \gamma_{z14} = 1. \end{aligned} \quad (129)$$

Interchanging of the indices changes the corresponding  $\gamma$  into its complex conjugate value: (e.g.,  $\gamma_{y41} = -i$ ). From (121) (123) (129) we obtain the required matrix elements; they are:

$$\left. \begin{aligned} \mathcal{H}_{3, n_s-1, 0; 1, n_s, 0} &= ec \left( \frac{\hbar}{\pi\Omega} \right)^{1/2} \frac{\sin \Gamma_s}{\nu_s^{1/2}} A_{sz} n_s^{1/2} \\ \mathcal{H}_{4, n_s-1, 0; 1, n_s, 0} &= ec \left( \frac{\hbar}{\pi\Omega} \right)^{1/2} \frac{\sin \Gamma_s}{\nu_s^{1/2}} (A_{sx} - iA_{sy}) n_s^{1/2} \\ \mathcal{H}_{3, n_s, 1; 1, n_s, 0} &= ec \left( \frac{\hbar}{\pi\Omega} \right)^{1/2} \frac{\sin \Gamma_\sigma}{\nu_\sigma^{1/2}} A_{\sigma z} \\ \mathcal{H}_{4, n_s, 1; 1, n_s, 0} &= ec \left( \frac{\hbar}{\pi\Omega} \right)^{1/2} \frac{\sin \Gamma_\sigma}{\nu_\sigma^{1/2}} (A_{\sigma x} - iA_{\sigma y}) \end{aligned} \right\} \quad (130)$$

for the transitions from the initial states  $(1, n_s, 0)$  to the intermediate states (128); and

$$\left. \begin{aligned} \mathcal{H}_{1, n_s-1, 1; 3, n_s-1, 0} &= ec \left( \frac{\hbar}{\pi\Omega} \right)^{1/2} \frac{\sin \Gamma_\sigma}{\nu_\sigma^{1/2}} A_{\sigma z} \\ \mathcal{H}_{1, n_s-1, 1; 4, n_s-1, 0} &= ec \left( \frac{\hbar}{\pi\Omega} \right)^{1/2} \frac{\sin \Gamma_\sigma}{\nu_\sigma^{1/2}} (A_{\sigma x} + iA_{\sigma y}) \\ \mathcal{H}_{1, n_s-1, 1; 3, n_s, 1} &= ec \left( \frac{\hbar}{\pi\Omega} \right)^{1/2} \frac{\sin \Gamma_s}{\nu_s^{1/2}} A_{sz} n_s^{1/2} \\ \mathcal{H}_{1, n_s-1, 1; 4, n_s, 1} &= ec \left( \frac{\hbar}{\pi\Omega} \right)^{1/2} \frac{\sin \Gamma_s}{\nu_s^{1/2}} (A_{sx} + iA_{sy}) n_s^{1/2} \end{aligned} \right\} \quad (131)$$

for the transitions from the intermediate states to the final state  $(1, n_s - 1, 1)$ .

From these matrix elements and the general formula (26) we easily calculate the amplitudes of probability for the intermediate states. If we limit ourselves to a very short interval of time we may still suppose (127) to be valid in first approximation, and we obtain from (26):

$$\dot{a}_{3, n_s-1, 0} = -\frac{2\pi i}{\hbar} \mathcal{H}_{3, n_s-1, 0; 1, n_s, 0} a_{1, n_s, 0} e^{2\pi i(-2mc^2 - \hbar\nu_s)t/\hbar}$$

(notice the very big change in the energy due to the transition of the electron

from state 1 with positive energy  $+mc^2$ , to state 3 with negative energy  $-mc^2$ ). Putting  $a_{1,n_s,0} = 1$  and remembering (130) we obtain:

$$\dot{a}_{3,n_s-1,0} = -\frac{2\pi i}{h} ec \left(\frac{h}{\pi\Omega}\right)^{1/2} \frac{\sin \Gamma_s}{\nu_s^{1/2}} n_s^{1/2} A_{sz} e^{-2\pi i(2mc^2 + h\nu_s)t/h}.$$

Integrating and neglecting  $h\nu_s$ , in comparison with  $2mc^2$  in the denominator we obtain:

$$a_{3,n_s-1,0} = \frac{e}{2mc} \left(\frac{h}{\pi\Omega}\right)^{1/2} \frac{\sin \Gamma_s}{\nu_s^{1/2}} n_s^{1/2} A_{sz} e^{-2\pi i(2mc^2 + h\nu_s)t/h}. \quad (132)$$

Notice that the integration constant should have been determined with the condition  $a_{3,n_s-1,0} = 0$  for  $t = 0$ ; instead we have chosen the constant so that the mean value of  $a_{3,n_s-1,0}$  is zero. This corresponds exactly to what is done also in the classical theory of dispersion of light from a harmonic oscillator; one considers in that case the motion of the oscillator to be represented simply by the forced vibrations and one neglects the vibrations with the characteristic frequency of the oscillator which are superposed on them. The justification of this classical proceeding lies in the well known fact that the vibrations of characteristic frequency are very rapidly damped by the reaction of radiation, so that, in the permanent state, only the forced vibrations remain. The justification in our case is quite similar; it could be shown that the effect of an integration constant added to (132) would be very rapidly damped from the reaction of states like  $(1, n_s - 1, 1)$  which has been neglected in our calculations.

The amplitudes of probability for the other intermediate states are deduced with exactly the same method as (132). They are:

$$\left. \begin{aligned} a_{4,n_s-1,0} &= \frac{e}{2mc} \left(\frac{h}{\pi\Omega}\right)^{1/2} \frac{\sin \Gamma_s}{\nu_s^{1/2}} n_s^{1/2} (A_{sx} - iA_{sy}) e^{-2\pi i(2mc^2 + h\nu_s)t/h} \\ a_{3,n_s,1} &= \frac{e}{2mc} \left(\frac{h}{\pi\Omega}\right)^{1/2} \frac{\sin \Gamma_\sigma}{\nu_\sigma^{1/2}} A_{sz} e^{-2\pi i(2mc^2 - h\nu_\sigma)t/h} \\ a_{4,n_s,1} &= \frac{e}{2mc} \left(\frac{h}{\pi\Omega}\right)^{1/2} \frac{\sin \Gamma_\sigma}{\nu_\sigma^{1/2}} (A_{\sigma x} - iA_{\sigma y}) e^{-2\pi i(2mc^2 - h\nu_\sigma)t/h} \end{aligned} \right\} \quad (133)$$

We apply now once more the general formula (26) to the calculation of the amplitude of probability for the final state. We get:

$$\dot{a}_{1,n_s-1,1} = -\frac{2\pi i}{h} \left[ \mathfrak{H}_{1,n_s-1,1;3,n_s-1,0} a_{3,n_s-1,0} e^{-2\pi i(2mc^2 + h\nu_s)t/h} + \dots \right]$$

where similar terms for the other three intermediate states have been omitted. With (132) (133) and (131) we find now:

$$\dot{a}_{1,n_s-1,1} = -\frac{2i}{\Omega} \frac{e^2}{m} \left(\frac{n_s}{\nu_s \nu_\sigma}\right)^{1/2} \sin \Gamma_s \sin \Gamma_\sigma (A_s A_\sigma) e^{2\pi i(\nu_\sigma - \nu_s)t/h}. \quad (134)$$

This equation coincides exactly with equation (94) obtained in the theory of scattering made without using Dirac's wave equation. By exactly the same method used for (94) we deduce from (134) Thomson's formula for the intensity of scattered radiation.

The very profound difference between these two theories of scattering should be emphasized; in the first theory, deduced from the Schrödinger wave equation for the electron, the scattering effect was due to the presence in the Hamiltonian of the term (17). This term is quadratic (and not linear) in the vector potential, and therefore enables transitions for which a quantum jumps in a single act from one radiation component to another. In Dirac's wave equation only terms linear in the potentials are contained; this has the effect that no direct transitions between two states can occur, if more than one radiation component changes its quantum number. Therefore it would seem probable at first sight that Dirac's relativistic free electron has no scattering properties. We have shown however that this conclusion is wrong; the scattering properties come out if one properly takes into account also the negative states. Scattering appears as a sort of resonance, (very far from the resonance line) of the quantum jump (of energy  $2mc^2$ ) between the positive and the negative states.

#### §16. Radiative transitions from positive to negative states

We have seen in the preceding sections that a very great number of phenomena find their natural explanation in Dirac's theory of radiation. We will now briefly discuss some serious difficulties of this theory. They are mainly connected with difficulties in the theory of the electron.

It is well known that the most serious difficulty in Dirac's relativistic wave equation lies in the fact that it yields besides the normal positive states also negative ones, which have no physical significance. This would do no harm if no transition between positive and negative states were possible (as are, e.g., transitions between states with symmetrical and antisymmetrical wave function). But this is unfortunately not the case: Klein has shown by a very simple example that electrons impinging against a very high potential barrier have a finite probability of going over in a negative state.

Dirac has tried with a very keen hypothesis to overcome these difficulties. He postulates that there are in every portion of space an infinite number of electrons which fill nearly completely in the sense of Pauli's principle, all the states of negative energy; a transition from a positive to a negative state therefore occurs very seldom since only a few negative states are unoccupied. Dirac goes further with the hypothesis, as he postulates that the unoccupied, negative places, the "holes", are to be interpreted as protons; in fact it is easily seen that a hole behaves like a positive charge with positive mass. The quantum transition in which an electron jumps from a positive state into a hole would therefore correspond to a hypothetic process of annihilation of an electron and a proton, with radiation of the energy corresponding to their masses.

Oppenheimer, Dirac and Tamm have calculated the probability of tran-

sition from a positive to a negative state with radiation of the energy difference. From the standpoint that the negative states are equivalent to protons, their result gives the rate of annihilation of electrons and protons.

Without carrying out these calculations in any detail, we limit ourselves to some qualitative remarks. Let us discuss the probability of radiative transition between a state (1) in which the momentum of the electron is  $p_1=0$  and its energy  $W_1=+mc^2$  to a state (2) where its energy is negative  $W_2=-mc^2$  and its momentum is  $p_2=0$ .

It is evident that the energy difference  $W_1 - W_2 = 2mc^2$  can not be radiated in a single quantum, since the momentum condition  $p_1 - p_2 = 2mc$  is not verified. It is however possible to obtain a finite probability of transition between the two states 1, 2 with emission of two quanta having both energy  $mc^2/h$  and opposite direction of motion. This process is of course consistent both with energy and momentum conservation.

The process will therefore happen in two steps. First step: a quantum of energy  $mc^2$  is emitted and the electron receives the recoil going over to a state with momentum  $mc$ . The energy of the electron in this state is (114):  $W = \pm 2^{1/2}mc^2$ . This intermediate state does not satisfy also conservation of (unperturbed) energy; as we have often seen in preceding instances, the amplitude of probability for this state can not continuously increase with time, but it can nevertheless be different from zero though having very small oscillating values. From this intermediate state a direct transition to the final state, with emission of a quantum of energy  $mc^2$  and momentum opposite to the momentum of the first quantum is possible; since this last state satisfies energy and momentum conservation, it is actually possible to show that its amplitude of probability steadily increases giving a finite rate of transition from initial to final state.

Carrying out the calculations, the required probability of transition per unit time results:

$$\pi e^4/m^2c^3. \quad (135)$$

If the electron in the negative state has not momentum zero, but has the energy  $W' = -mc^2\alpha$ , ( $\alpha \geq 1$ ) the probability of transition becomes:

$$\frac{\pi e^4}{m^2c^3} \frac{1}{\alpha(\alpha+1)} \left[ \frac{\alpha^2 + 4\alpha + 1}{(\alpha^2 - 1)^{1/2}} \log [\alpha + (\alpha^2 - 1)^{1/2}] - \alpha - 3 \right]. \quad (136)$$

If we assume that all the negative states are empty, formula (136) summed over all negative states would give an infinite probability for the transition from a positive to a negative state: electrons could not remain in a positive state, not even for a very short time.

If we assume on the other hand the "hole" theory of protons, the theory of the transitions becomes very uncertain, since the electron is in that case surrounded by an infinite number of other (negative) electrons. The interaction effects of these electrons are neglected in the theory, though it is evident that they might have enormous effects. Dirac suggests that this interaction might be responsible for the difference in mass of the electron and the proton. If we

tentatively try to apply (136) to the process of annihilation of an electron and a proton, putting for  $m$  some mean value between the masses of the electron and the proton, the rate of annihilation comes out much too rapid; matter would be destroyed in a very short time.

### PART III. QUANTUM ELECTRODYNAMICS

§17. The electromagnetic field, whose interaction with matter we have hitherto considered, is not an electromagnetic field of the most general type, since a field of general type cannot be constructed by simply superposing plane electromagnetic waves. It can be immediately seen that in a plane electromagnetic wave  $\text{div } E=0$  and this equation holds also for any superposition of waves. Instead in a general electromagnetic field we have  $\text{div } E = +4\pi\rho$ ,  $\rho$  being the density of electricity; this shows that no field, where charges are present, can be represented as a superposition of electromagnetic waves.

An electromagnetic field of the most general type is represented by help of a scalar potential  $V$  and a vector potential  $U$  by the well-known relations:

$$E = -\text{grad } V - \frac{1}{c} \frac{\partial U}{\partial t}; \quad H = \text{rot } U. \quad (137)$$

$V$  and  $U$  are classically connected to the density of charge and the velocity  $\dot{X}$  by:

$$\Delta V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -4\pi\rho; \quad \Delta U - \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = -\frac{4\pi}{c} \rho \dot{X}. \quad (138)$$

Further  $U$  and  $V$  are not completely independent of each other; they satisfy the relation:

$$\text{div } U + \frac{1}{c} \frac{\partial V}{\partial t} = 0 \quad (139)$$

which is closely connected to the equation of continuity for the electricity.

A general quantum theory of the electromagnetic field was constructed by Heisenberg and Pauli by a method in which the values of the electromagnetic potentials in all the points of space are considered as variables.

Independently the writer proposed another method of quantization of the electromagnetic field starting from a Fourier analysis of the potentials. Though Heisenberg and Pauli's method puts in evidence much more clearly the properties of relativistic invariance and is in many respects more general, we prefer to use in this article the method of the writer, which is more simple and more analogous to the methods used in the theory of radiation.

We will consider only a region of space of finite volume and we suppose that both scalar and vector potential at a given time can be represented by Fourier series of the type:

$$\left. \begin{aligned} V &= c \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_s Q_s \cos \Gamma_s \\ U &= c \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_s u_s \sin \Gamma_s \end{aligned} \right\} \quad (140)$$

where  $Q_s$  and  $u_s$  are a scalar and a vector function respectively of the time only. The factor  $c(8\pi/\Omega)^{1/2}$  has been put for convenience of normalization as in (12).  $\Gamma_s$  is given by (4). It is convenient to develop  $V$  and  $U$  in series of  $\cos \Gamma_s$  and  $\sin \Gamma_s$  respectively, since in this case Eq. (139) takes a much simpler form. It should finally be noticed that the number of characteristic frequencies between  $\nu_s$  and  $\nu_s + d\nu_s$  is to be taken equal to:

$$(4\pi/c^3)\Omega\nu_s^2 d\nu_s \quad (141)$$

i.e., to the half of (1), since in our case the two possibilities of polarization for the transverse waves are taken in account by the fact that  $u_s$  is a vector.

As variables representing the field at a given time we take  $Q_s$  and the three components of the vector  $u_s$ ; it is convenient however to take these components in directions related to the form of the phase factor  $\sin \Gamma_s$ ; we consider three mutually perpendicular unit vectors:  $\alpha_s$ , which points in the direction of the wave,  $A_{s1}$  and  $A_{s2}$  perpendicular to that direction. Let  $\chi_s$ ,  $q_{s1}$ ,  $q_{s2}$  be the components of  $u_s$  in the directions  $\alpha_s$ ,  $A_{s1}$ ,  $A_{s2}$ ; we have then;

$$u_s = \alpha_s \chi_s + A_{s1} q_{s1} + A_{s2} q_{s2}. \quad (142)$$

As variables describing the field we can take:

$$Q_s, \chi_s, q_{s1}, q_{s2}. \quad (143)$$

They depend only on the time.

It is very easy to deduce from (138) the differential equations that determine the dependence of the variables (143) on the time. Multiply both members of the first Eq. (138) by  $\cos \Gamma_s d\tau$  and integrate over all the space. We suppose that the potential  $V$  vanishes on the very distant surface limiting our space  $\Omega$ , so that certain integrals over that surface can be omitted; we obtain then by obvious transformations:

$$\begin{aligned} -4\pi \int \rho \cos \Gamma_s d\tau &= \int \Delta V \cdot \cos \Gamma_s d\tau - \frac{1}{c^2} \int \frac{\partial^2 V}{\partial t^2} \cos \Gamma_s d\tau \\ &= \int V \cdot \Delta \cos \Gamma_s d\tau - \frac{1}{c^2} \int \frac{\partial^2 V}{\partial t^2} \cos \Gamma_s d\tau. \end{aligned}$$

From (4) we obtain:

$$\Delta \cos \Gamma_s = -\frac{4\pi^2 \nu_s^2}{c^2} \cos \Gamma_s.$$

We have therefore:

$$+4\pi c^2 \int \rho \cos \Gamma_s d\tau = \left( \frac{d^2}{dt^2} + 4\pi^2 \nu_s^2 \right) \int V \cos \Gamma_s d\tau. \quad (144)$$



Putting for  $V$  its expression (140) and remembering that the functions  $\cos \Gamma_s$  are orthogonal and satisfy the relation

$$\int \cos \Gamma_s \cos \Gamma_\sigma d\tau = \frac{1}{2} \Omega \delta_{s\sigma}$$

we obtain:

$$\int V \cos \Gamma_s d\tau = c(2\pi\Omega)^{1/2} Q_s.$$

From (144) we obtain therefore:

$$\ddot{Q}_s + 2\pi^2\nu_s^2 Q_s = c \left( \frac{8\pi}{\Omega} \right)^{1/2} \int \rho \cos \Gamma_s d\tau. \quad (145)$$

This equation takes a much simpler form if we suppose that there are only point charges  $e_1, e_2, e_3, \dots$ , at the points  $X_1, X_2, X_3, \dots$ . The integral in (145) becomes then a sum over the point charges and we obtain:

$$\ddot{Q}_s + 2\pi^2\nu_s^2 Q_s = c \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_i e_i \cos \Gamma_{si} \quad (146)$$

where the sum has to be extended over all the charges;  $\Gamma_{si}$  is the value of the phase  $\Gamma_s$  at the place  $X_i$  of the  $i^{\text{th}}$  charge

$$\Gamma_{si} = \frac{2\pi\nu_s}{c} (\alpha_s, X_i) + \beta_s. \quad (147)$$

By the same method we find a similar equation for the vector  $u_s$ :

$$\ddot{u}_s + 2\pi^2\nu_s^2 u_s = \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_i e_i \dot{X}_i \sin \Gamma_{si}. \quad (148)$$

Remembering (142), we find that the three components of this vector equation in the three directions  $\alpha_s, A_{s1}, A_{s2}$  are:

$$\ddot{\chi}_s + 2\pi^2\nu_s^2 \chi_s = \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_i e_i (\alpha_s, \dot{X}_i) \sin \Gamma_{si} \quad (149)$$

$$\left. \begin{aligned} \ddot{q}_{s1} + 2\pi^2\nu_s^2 q_{s1} &= \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_i e_i (A_{s1}, \dot{X}_i) \sin \Gamma_{si} \\ \ddot{q}_{s2} + 2\pi^2\nu_s^2 q_{s2} &= \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_i e_i (A_{s2}, \dot{X}_i) \sin \Gamma_{si} \end{aligned} \right\} \quad (150)$$

The Eqs. (146), (149), (150) are equivalent to (138).

Take the derivative of (146) with respect to  $t$  and add it to (149) multiplied by  $2\pi\nu_s$ . Then we find:

$$\left( \frac{d^2}{dt^2} + 4\pi^2\nu_s^2 \right) (2\pi\nu_s \chi_s + \dot{Q}_s) = 0. \quad (151)$$

This equation is evidently satisfied if:

$$2\pi\nu_s\dot{\chi}_s + \dot{Q}_s = 0. \quad (152)$$

It is immediately seen that this last equation is equivalent to (139). Eq. (152) does not follow directly from the differential equations (146), (149); though it results from (151) that if at a given time (e.g.,  $t=0$ ), (152) and its derivative with respect to time

$$2\pi\nu_s\dot{\chi}_s + \ddot{Q}_s = 0 \quad (153)$$

are both satisfied, then they are satisfied for all time.

We must now write in Hamiltonian form the equations that describe the motion of the particles and the variation of the electromagnetic field. For this we simply write the Hamilton function and then verify that the canonical equations that can be derived by it actually represent the motion of the particles and the Maxwell equations. The Hamilton function is the following:

$$\begin{aligned} H = & -c \sum_i (\gamma_i, \dot{p}_i) - \sum_i m_i c^2 \delta_i + \sum_i e_i c \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_s Q_s \cos \Gamma_{si} \\ & + \sum_i e_i c \left( \frac{8\pi}{\Omega} \right)^{1/2} \sum_s (\gamma_i, \alpha_s \chi_s + A_{s1} q_{s1} + A_{s2} q_{s2}) \sin \Gamma_{si} \\ & + \sum_s \left[ \frac{1}{2} (\dot{p}_{s1}^2 + \dot{p}_{s2}^2 + \omega_s^2 - P_s^2) + 2\pi^2 \nu_s^2 (q_{s1}^2 + q_{s2}^2 + \chi_s^2 - Q_s^2) \right]. \quad (154) \end{aligned}$$

In this Hamilton function the variables are  $X_i$ , and the spin coordinates, describing the motion of the particles;  $p_i$  are the momenta (vectors) conjugated to the coordinates  $X_i$ ;  $Q_s$ ,  $\chi_s$ ,  $q_{s1}$ ,  $q_{s2}$  are the coordinates describing the field and  $P_s$ ,  $\omega_s$ ,  $\dot{p}_{s1}$ ,  $\dot{p}_{s2}$  are their conjugated momenta.  $\gamma_i$  and  $\delta_i$  represent operators analogous to Dirac's operators  $\gamma$  and  $\delta$  of Eq. (105) operating on the spin coordinate of the  $i^{\text{th}}$  particle. The structure of the Hamiltonian (154) is very simple. Remembering (140), its first four terms can be written:

$$\sum_i \left\{ e_i V_i - c \left( \gamma_i, \dot{p}_i - \frac{e_i}{c} U_i \right) - m_i c^2 \delta_i \right\} \quad (155)$$

which is simply the repetition of Dirac's Hamilton function (105) for all the particles. The last term of (154) represents the Hamiltonian function of the electromagnetic field without interaction with the charges, and is analogous to (11).

From this we see clearly that the Hamiltonian (154) correctly represents the motion of the particles, since their coordinates are contained in (155) which is equivalent to Dirac's Hamilton function. We must show that also the Maxwell equations, or the equivalent equations (146), (149), (150) can be deduced from the Hamiltonian (154). For this we write the canonical equations derived from (154); we obtain:

$$\dot{Q}_s = \frac{\partial H}{\partial P_s} = -P_s; \quad \dot{P}_s = -\frac{\partial H}{\partial Q_s} = 4\pi^2 \nu_s^2 Q_s - \sum_i e_i c \left( \frac{8\pi}{\Omega} \right)^{1/2} \cos \Gamma_{si}. \quad (156)$$

If we eliminate  $P_s$  from these equations, we obtain:

$$\ddot{Q}_s + 4\pi^2\nu_s^2 Q_s = \sum_i e_i c \left(\frac{8\pi}{\Omega}\right)^{1/2} \cos \Gamma_{si}$$

which is identical with (146). Similarly the canonical equations for the pair of conjugate variables  $\chi_s, \omega_s$  are:

$$\dot{\chi}_s = \frac{\partial H}{\partial \omega_s} = \omega_s; \dot{\omega}_s = -\frac{\partial H}{\partial \chi_s} = -4\pi^2\nu_s^2 \chi_s^2 - c \left(\frac{8\pi}{\Omega}\right)^{1/2} \sum_i e_i (\gamma_i, \alpha_s) \sin \Gamma_{si}. \quad (157)$$

Elimination of  $\omega_s$  yields:

$$\ddot{\chi}_s + 4\pi^2\nu_s^2 \chi_s = -c \left(\frac{8\pi}{\Omega}\right)^{1/2} \sum_i e_i (\gamma_i, \alpha_s) \sin \Gamma_{si}. \quad (158)$$

Now we observe that the velocity of the  $i^{\text{th}}$  particle in Dirac's relativistic theory is

$$\dot{X}_i = -c\gamma_i. \quad (159)$$

(This results also from the Hamiltonian (154), since  $\dot{X}_i = \partial H / \partial p_i$ ). Eq. (158) coincides therefore with (149). By the same method it can be proved that also the Eq. (150) for the transverse components of the vector potential can be derived from (154).

Eq. (152) which is equivalent to (139) can be written, remembering (156):

$$2\pi\nu_s \chi_s - P_s = 0 \quad (160)$$

and its derivative with respect to time is  $2\pi\nu_s \dot{\chi}_s - \dot{P}_s$ , which can be written by (156), (157):

$$\omega_s - 2\pi\nu_s Q_s + \frac{c}{2\pi\nu_s} \left(\frac{8\pi}{\Omega}\right)^{1/2} \sum_i e_i \cos \Gamma_{si} = 0. \quad (161)$$

We have proved by (151) that if (160), (161) are satisfied for  $t=0$ , they are satisfied also automatically for any value of time.

§18. In a classical interpretation we could therefore say that electrodynamics and motion of the points can be deduced by integration of the canonical equations corresponding to the Hamilton function (154); the initial values of the variables must satisfy the supplementary conditions (160), (161).

As we go over to the quantum mechanical interpretation, we must first observe that it is in general impossible that two functions of the variables of the system have simultaneously a well determined value, with the exception of the case that the two functions commute; so at first sight it would seem impossible to satisfy simultaneously (160) and (161). This is however, possible in this special case, since the first members of (160) and (161) commute with each other as an immediate verification shows (remember that  $\omega_s$  and  $\chi_s$  are conjugate and therefore  $\omega_s \chi_s - \chi_s \omega_s = \hbar/2\pi i$ ; and similarly  $P_s Q_s - Q_s P_s = \hbar/2\pi i$ , while all the other variables in (160) and (161) commute).

To the classical integration of a system of canonical equations corresponds in wave mechanics the integration of the Schrödinger equation:

$$H\psi = -\frac{\hbar}{2\pi i} \frac{\partial \psi}{\partial t} \quad (162)$$

where  $H$ , given by (154) must be considered in the ordinary way as an operator acting on the function  $\psi$  of the coordinates only:

$$\psi = \psi(t, x_i, \sigma_i, q_{s1}, q_{s2}, \chi_s, Q_s) \quad (163)$$

$\sigma_i$  represent the spin coordinates.

If there were no condition limiting the initial values of the variables, then  $\psi$  for  $t=0$  could be chosen arbitrarily. But we have the conditions (160) (161). We will show that these conditions determine the form of the dependence of  $\psi$  on  $\chi_s$  and  $Q_s$ . Indeed,  $\omega_s$ , conjugate to  $\chi_s$  must have according to (161) the value:

$$\omega_s = 2\pi\nu_s Q_s - \frac{c}{2\pi\nu_s} \left(\frac{8\pi}{\Omega}\right)^{1/2} \sum_i e_i \cos \Gamma_{si}.$$

It results from this that  $\chi_s$  can be contained in  $\psi$  only in a factor:

$$e^{2\pi i \omega_s \chi_s / \hbar} = \exp \left[ \frac{2\pi i}{\hbar} \chi_s \left( 2\pi\nu_s Q_s - \frac{c}{2\pi\nu_s} \left(\frac{8\pi}{\Omega}\right)^{1/2} \sum_i e_i \cos \Gamma_{si} \right) \right]. \quad (164)$$

By the same method we deduce from (160) that  $\chi_s$  must be contained only in a factor:

$$e^{4\pi^2 i \nu_s \chi_s Q_s / \hbar}$$

which is already contained in (164).

We see therefore that the form of  $\psi$  must be:

$$\psi = \exp \left[ \frac{2\pi i}{\hbar} \sum_s \chi_s \left( 2\pi\nu_s Q_s - \frac{c}{2\pi\nu_s} \left(\frac{8\pi}{\Omega}\right)^{1/2} \sum_i e_i \cos \Gamma_{si} \right) \right] \times \phi(t, X_i, \sigma_i, q_{s1}, q_{s2}). \quad (165)$$

If we substitute this expression for  $\psi$  in the Schrödinger Eq. (162) we obtain a new differential equation for  $\phi$ . With some calculations it is found that this equation can be put in the form:

$$R\phi = -\frac{\hbar}{2\pi i} \frac{\partial \phi}{\partial t} \quad (166)$$

which strongly resembles the form of a Schrödinger equation. The operator  $R$  is the following:

$$R = -c \sum_i (\gamma_i, p_i) - \sum_i m_i c^2 \delta_i + \sum_i e_i c \left(\frac{8\pi}{\Omega}\right)^{1/2} (\gamma_i, A_{s1} q_{s1} + A_{s2} q_{s2}) \sin \Gamma_{si} + \sum_i \left\{ \frac{1}{2} (p_{s1}^2 + p_{s2}^2) + 2\pi^2 \nu_s^2 (q_{s1}^2 + q_{s2}^2) \right\} + \frac{c^2}{\pi \Omega} \sum_s \frac{1}{\nu_s^2} \left( \sum_i e_i \cos \Gamma_{si} \right)^2. \quad (166)$$

This operator  $R$  can be considered as a sort of Hamilton function acting on  $\phi$ . By this method therefore the coordinates  $Q_s$  and  $\chi_s$  representing the scalar potential and one component of the vector potential are completely eliminated both from the new amplitude of probability  $\phi$  and from the new Hamilton function  $R$ . Not considering the last term in  $R$ , which we shall discuss later, the operator  $R$  is identical with the Hamilton function of Dirac's theory of radiation (117). (There are only some formal differences: in (117) only one electron, instead of many particles is considered; in (166) the two polarized components are considered separately with the indices  $s_1$  and  $s_2$ , whereas in (117) there is only one index  $s$ ).

We must find out the physical meaning of the last term in (166). This term is:

$$\frac{c^2}{\pi\Omega} \sum_s \frac{1}{\nu_s^2} \left( \sum_i e_i \cos \Gamma_{si} \right)^2 = \frac{c^2}{\pi\Omega} \sum_{ij} e_i e_j \sum_s \frac{\cos \Gamma_{si} \cos \Gamma_{sj}}{\nu_s^2}.$$

The sum over  $s$  can be transformed into an integral. (Take the mean value of  $\cos \Gamma_{si} \cos \Gamma_{sj}$  over all directions of propagation and phases for the  $s$ -component; and then remember (141).) We find at last:

$$\sum_s \frac{\cos \Gamma_{si} \cos \Gamma_{sj}}{\nu_s^2} = \frac{\pi\Omega}{2c^2} \frac{1}{r_{ij}}$$

$r_{ij}$  being the distance between the two points  $i$  and  $j$ . The last term of (166) takes therefore the very simple form:

$$\frac{1}{2} \sum_{ij} \frac{e_i e_j}{r_{ij}} \tag{167}$$

which is the ordinary classical expression for the electrostatic energy of our system of charges.

At this point we meet a very serious difficulty, since the electrostatic energy of point charges is infinite; every charge has an infinite electrostatic selfenergy. We could try of course to avoid this difficulty, as it is very often done in classical electrostatics by simply neglecting in the sum (167) all the terms with  $i=j$  which represent the selfenergy of the charges. We shall see however that even this very crude proceeding is not sufficient to avoid infinite terms of non-electrostatic origin in the self-energy.

The problem which we meet now in quantum electrodynamics is identical with that of radiation theory since our new Hamilton function  $R$  is the Hamilton function of radiation theory plus the electrostatic energy. We have hitherto considered in the radiation theory as unperturbed system, the system obtained by neglecting the interaction between atom and radiation field. The interaction term had then simply the effect of determining transitions between different states of the unperturbed system which have the same or nearly the same unperturbed energy.

But we can ask whether there are quantum states for the complete problem. This problem is mathematically very difficult and can only be discussed by the method of successive approximations. However the second approxi-

mation still yields an infinite perturbation term in the energy levels and it seems therefore probable, that for point electrons there are no quantum states of the unperturbed problem. It could be noticed however that the application of the perturbation method is for this problem extremely uncertain, since the differences between the quantum states of the unperturbed problem are very small in comparison with the perturbation.

To all these difficulties no satisfactory answer has yet been given. One would be tempted to give the electron a finite radius; this would actually avoid infinite terms, as in the classical theory of electromagnetic masses. But this method is connected with serious difficulties for the relativistic invariance.

In conclusion we may therefore say that practically all the problems in radiation theory which do not involve the structure of the electron have their satisfactory explanation; while the problems connected with the internal properties of the electron are still very far from their solution.

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